

**Making Sense of Number:
A Study of Children's Developing Competence**

by

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Abstract

This study investigated young children's construction of meaning for number and explored ways to more comprehensively assess and portray the development of number sense in young children. Greeno's (1991) conceptualization of number sense as situated knowing in a conceptual domain was used to consider both the mathematical tools available to the child and the extent to which the child makes use of these tools.

The data consisted of four videotaped interviews for each of sixteen children between the ages of six and eight. Each of the four interviews involved a different number context: doubling, finding missing parts, sharing, and working with money. Each context involved a task presented in a series of increasingly difficult items, with number size predominantly determining the difficulty level. A dynamic interview format was used to encourage children to work beyond their independent level, or "number comfort zone." Cues and scaffolds were provided to support children's construction of meaning within their "number construction zone" and towards the outer limits of their understanding.

Analysis focused on the strategies children used to make sense of each item, and the cognitive, affective and contextual aspects which enhanced or constrained their mathematical activity within the number construction zone. Results were reported two ways. The first, specific task performance across children, provided a means of describing the diversity of developmentally appropriate ways children made sense of the different tasks and provided a frame of reference for considering individual performance. The second approach to reporting results considered individual children's performance across tasks, and provided a means of focusing on characteristics of emerging competence.

Results of this study illustrate how the nature and use of children's reasoning strategies can provide an indication of developing competence. Results highlight specific conceptual, procedural, functional, and affective characteristics that most directly affected children's capacity to make sense of number situations. No single characteristic alone accounted for children's

success or lack of success, rather the inter-relationships of the different characteristics was apparent, with strengths in one area compensating for weaknesses in another. Though conceptual and procedural abilities appeared to shape to a great extent the nature of the number knowledge available to children, affective considerations and functional competence played a major role in shaping the extent to which children drew on this existing knowledge. Issues of context influenced both aspects of number sense: available knowledge and the nature of its use. Number size, context of the tasks, and presentation of tasks influenced children's mathematical activity in important ways. Children's personal number contexts were considered in terms of how they influenced their approaches to tasks. Overall, dynamic assessment techniques proved to offer a viable alternative for exploring the limits of children's ability to make sense of number situations, and for considering children's construction of meaning for number in developmental terms.

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Chapter 1

Introduction to the Study

I. The Problem

Young children begin to develop mathematical knowledge well before they enter school (Baroody, 1987; Fuson, 1988; Gelman & Gallistel, 1978; Ginsburg, 1983; Resnick, 1983). Research suggests that children actively construct their own number knowledge through restructuring and building on their existing knowledge (Kamii, 1985; Steffe, von Glasersfeld, Richards, & Cobb, 1983).

From a constructivist perspective, recognizing and building on children's early informal and intuitive notions about number is essential to the development of their understanding of number and number sense. Constructivist approaches to teaching require an understanding and appreciation of the full range of young children's formal, informal, and intuitive number knowledge as the basis for instruction, however, the ways and means by which children's mathematical knowledge is formally assessed often fail to adequately represent this knowledge. Consequently, a priority for mathematics educators, especially those involved in the education of young children, should be to find practical, valid, and comprehensive ways of considering the full extent of children's knowledge about number.

While some of what children know is readily apparent, or at least accessible, some aspects of their knowledge are in a more fluid or partial form that is more difficult to recognize. Donaldson (1992) characterized such variations as light and dark knowledge:

Some kinds of knowledge are in the light of full awareness. Others are in the shadows, on the edge of the bright circle. Still others are in the darkness beyond (p. 20). ... our knowledge is not all of a piece. In some cases we can readily say what we know — say it to others or to ourselves. We can discuss our knowledge and reflect on it. ... It is spelled out, available for scrutiny. But what we know is not always ready for inspection. Some of it is kept dark (p. 23).

Most forms of assessment focus on knowledge that is "in the light of full awareness" and represents that which children have "mastered" or "internalized." In this study

an alternative approach to assessing number knowledge is used, that of a dynamic interview method that explores children's partial and fluid understandings through the scaffolding of children's thinking (Collins, Brown, & Newman, 1989). This approach goes beyond a child's "number comfort zone" to investigate the nature and scope of the "number construction zone" (Newman, Griffin, & Cole, 1989) and the child's personal limit or "frontier of understanding" (Hodgkin, 1985) for number. This non-standardized, flexible approach to the assessment of children's capacity to make sense of number offers a comprehensive and respectful estimate or appraisal of a child's personal number domain (Greeno, 1991), and a useful basis upon which to shape instruction.

The original impetus for this study came primarily from my twenty years of personal experience in working with young children who were, for a variety of reasons, not benefitting from or responding to the mathematics program offered in their regular classrooms. These children, many of whom struggled to make sense of school mathematics, often displayed good sense when it came to informal mathematical situations in their daily lives, or showed surprising insight into specific aspects of the mathematics curriculum. However, formal, standardized assessments of their progress invariably provided an inadequate or misleading picture of their number knowledge. This was in part due to the behaviourist, task-analytic approach, and highly procedural focus that characterizes most mathematics achievement tests. Romberg, Sarinnia, and Collis (1990) claimed that the content-by-behaviour matrices which have dominated test design reflect outmoded and incorrect views of both mathematics and of how mathematics is learned. They stated that the hierarchies of behaviour drawn from the psychological theory of behaviourism do not account for recent work that indicates that mathematics is learned more through building complex networks of meaning, and connecting new concepts and skills with those previously learned. Their view was that current testing practices favoured students who had acquired factual and procedural knowledge, but did little to assess the coherence of that knowledge and students' ability to use that knowledge to solve problems.

Similarly, the available tests that were designed to serve a diagnostic function (e.g., *KeyMath Diagnostic Arithmetic Test* (Connelly, Nachtman, & Pritchett, 1971, 1976, 1982); *Test of Early Mathematics Abilities* (Ginsburg & Baroody, 1983); and the *Test of Mathematical Abilities* (Brown & McEntire, 1984), though of a less behaviourist

nature, were framed by a deficit, clinical "diagnostic" model of finding "problems" that could then be addressed with prescriptive modes of instruction. Though both diagnostic and achievement tests measure aspects of children's procedural and declarative knowledge, the tests provide little information about children's conceptual understanding, about their informal everyday mathematics notions, and about their developing "number sense".

A recurring theme in the mathematics education literature continues to be how to effect change in classroom assessment and instruction to reflect a more constructivist approach to mathematics learning and teaching, one that focuses on conceptual understanding, the development of number sense, and the construction of meaning (e.g., Lampert, 1990; Cobb, Wood, & Yackel, 1993). Such changes are part of the overall vision of reform for mathematics education as outlined in the National Council of Teachers of Mathematics' *Curriculum and Evaluation Standards for School Mathematics* (1989), *Standards for Teaching School Mathematics* (1991), and *Assessment Standards for School Mathematics* (1995).

This study provides one means for reinterpreting traditional views of what constitutes mathematical competence, of reconstituting the mathematical environments we provide for children, and of reforming how we go about supporting, monitoring and assessing young children's mathematical development related to number.

II. The Study

The Goals of the Study

This study was designed to address two main goals. The first was to provide a better understanding of the nature of young children's sense-making activity involving number. This descriptive aspect of the study investigated children's construction of meaning in different number contexts in an effort to recognize, understand, and appreciate children's mathematics as seen from their perspective (Steffe & Cobb, 1988). It focused on how children construct meaning for different number situations involving "how many," "how much," and the use of additive structures. Children's activity on specific tasks was studied, however, in the Piagetian tradition, the main interest was not on the tasks themselves. Rather, the

focus was on what performance revealed about the underlying organization of thought (Dean and Youniss, 1991).

This descriptive aspect of the study was designed to investigate the nature and diversity of how young children construct meaning for number. The study considered the multiple aspects which enhance and constrain children's capacity to make sense of number. The study explicated the underlying threads that form the basis of patterns to children's mathematical activity in these different contexts. And finally, the study described the individual and contextual variations that shape children's different ways of knowing, and the extent to which these ways relate to developing competence with number.

The second purpose of the study was exploratory, in keeping with the widely held view expressed by Gelman and Greeno (1989) that it "is important for instructional research to develop and analyze methods of assessment that more adequately capture children's understanding." In this vein, this study investigated alternative ways to assess and portray, in a comprehensive and respectful way, the complexity, depth, and richness of children's construction of meaning for number. It also explored an alternative conceptualization of what constitutes "number sense" in young children, in keeping with Greeno's (1991) interpretation of number sense as situated knowing in a conceptual domain. And finally, it explored an alternative way to consider children's developing capacity to make sense of number over time.

The Research Questions

The focus of this study centred on two aspects of the constructive mathematical activity engaged in by six-, seven- and eight-year-old children in different whole number contexts. The first aspect concerned the nature and range of this activity, and was guided by the following research question:

- What cognitive, affective, and contextual considerations serve to enhance or constrain children's construction of meaning for number?

The second aspect of this investigation explored the extent to which children's mathematical activity reflects their developing competence with number, and was guided by the following question:

- What characterizes developing competence in young children's capacity to make sense of number?

A particular focus in this consideration of children's developing competence was on how children's choice and use of strategies (or methods, or mathematical processes) relates to their increasing competence in constructing meaning for number situations.

Theoretical Perspective

This investigation of children's personal construction of meaning for number drew primarily on a radical constructivist perspective as described by Cobb (1994), Confrey (1994a), and Steffe and Kieren (1994). Children's capacity to make sense of number situations was considered in terms of their active and adaptive constructive mathematical activity. It was assumed that mathematical knowledge is a personal construction shaped by a variety of inter-related factors, including not only cognitive and affective influences, but also contextual influences. It was recognized that the personal construction of meaning does not take place in a vacuum, that it is situated within a social and cultural context that influences mathematical activity in important ways. In keeping with Cobb (1994), both individual and social perspectives were used in a complementary way, the constructivist perspective as a primary lens, and the socio-cultural perspective as the background against which this personal sense-making comes to the fore.

Overview of the Research Process

The methodology employed in the study involved a variation of the Piagetian clinical interview, termed a "dynamic clinical interview." This involved a series of interviews where the focus was on children's construction of meaning for number situations beyond their familiar range, and where interviewer support was provided to enable students to work to the edges of their understanding. This alternative approach to the investigation of children's sense-making activity drew on aspects of constructivist teaching experiment methodology (Steffe & Cobb, 1988) and aspects of

Vygotsky's socio-cultural approach to exploring a child's zone of proximal development (Vygotsky, 1978).

The dynamic interview approach varied from a traditional clinical interview approach in two main ways, the first related to the task presentation. Each of the four tasks was comprised of a sequence of items of increasing difficulty which could be adjusted to suit each student, and the presentation of items within that sequence could be adjusted to suit the representational abilities of each child (i.e., the ability to make sense of increasingly abstract representations of number ideas). The second dynamic aspect involved the interviewer in supporting children's thinking in order to explore the limits to their capacity to make sense. This flexible approach provided the opportunity to move beyond the limits of "mastery" or "internalized knowledge" in order to gain access to children's partial knowledge and the constructive processes that create mathematical meaning in unfamiliar number situations.

Data collection focused on how children made sense of familiar tasks where the number size of items gradually increased beyond their familiar range, or "number comfort zone." This range of partial or fluid knowledge (Donaldson, 1992) where answers are not "known" and constructive activity is required to make sense, is referred to as a "number construction zone" (Newman, Griffin, & Cole, 1989). Drawing on work by Brown, Collins, and Duguid (1989), cues and scaffolds were used to support children's sense-making efforts for items that fell within this range. The collection of data focused on the concepts, skills, procedures, strategic and representational capacities, and personal attitudes that characterized children's mathematical activity. All interviews were videotaped, and these videotape records comprised the data used for the analysis.

The videotapes were used to generate a descriptive analysis of the children's mathematical activity on each task. Guided by the research questions, the analysis focused on two aspects of this activity: the strategies children used to approach and make sense of each item; and the characteristics or features that served to support or constrain their construction of meaning. Descriptive statistics were used to explore relationships and illustrate patterns of mathematical activity within each task sequence, and across three of the four tasks.

A conceptual analysis of each child's mathematical activity across tasks was then used to investigate the child's underlying organization of thought. This aspect of the data analysis focused on what children's mathematical activity implied about their existing conceptual schemes. It further focused on evidence of children's procedural knowledge, their functional, strategic and representational capabilities, and their mathematical disposition.

Data analysis was comprised of two phases. The first involved analysis of the performance of all 16 students for each of the individual tasks. This provided a focus on the diversity of children's approaches and the range of developmentally appropriate mathematical activity typical of the six- to eight-year olds in the study. The second phase involved analysis of individual students' mathematical activity across tasks. This analysis provided a picture of the richness and complexity of each child's thinking about number, and the multiplicity of factors that influenced this thinking. It also provided information on aspects of mathematical activity that differentiated performance and characterized increasingly powerful and competent thinking involving number.

Significance of the Study

This study contributes to the field of mathematics education research in four ways. First, almost all studies that have explored children's strategies and methods related to the use of number have focused on the range of basic number facts, where the goal is automatic recall. This study went beyond that range to explore how children construct meaning for number situations requiring reasoning and number sense and how children draw on their prior knowledge to work beyond the range of automatic responses.

Second, most studies have used a singular rather than global perspective in their interpretation of what shapes children's thinking about number. Steffe et al. (1983, 1988) focused on children's conceptual schemes. Case & Sandison (1992) focused on children's working memory capacity. Gelman & Gallistel (1978) focused on pre-requisite skills, while Lave (1988) focused on the role of social context. This study drew on aspects of all of these interpretations, using an integrated, global approach to consider the richness and complexity of children's mathematical sense-making.

Third, this study provides an alternative interpretation of what constitutes developmentally appropriate number sense, and what might account for “non-sense-making” activity. This alternative perspective has important implications for how we shape mathematical environments for young children, how we support their efforts to make sense of number, and in particular, how we consider these efforts in relation to traditional assessment and evaluation practices.

And finally, this study adds to the range of alternative assessment tools that are currently being developed both better to understand children’s thinking, and more adequately to reflect a constructivist approach to supporting the mathematical development of young children.

III. The Organization of the Chapters

Chapter One presents an introduction, rationale, and overview of the study. Chapter Two provides the theoretical frame for the study along with a review of related literature and explanation of terms used in the study.

Chapters Three and Four deal with the methods used to organize and carry out the study. Chapter Three provides a rationale, supporting research, and a description of the dynamic clinical interview method used in the study. Chapter Four provides a description of all other aspects of how the study was conducted.

The results of the study are provided in Chapters Five and Six. Chapter Five uses each individual task as the unit of analysis, providing a description and analysis of the diverse ways children interpreted and approached each task. Chapter Six uses individual children as the unit of analysis, analyzing children’s mathematical activity across tasks.

Chapter Seven responds to the two research questions which shaped the investigation. Chapter Eight deals with the implications of these results, including recommendations for promoting changes in how educators and parents recognize, support and assess children’s construction of meaning for number.

Chapter 2 Theory and Research

I. Introduction

Number Sense

Current efforts to understand children's potential and capacity for making sense of the domain of numbers and quantities, use the term "number sense" as a referent for discussion. Number sense is widely recognized as a desirable outcome of mathematics education (National Council of Teachers of Mathematics, 1989; National Research Council, 1989; British Columbia Ministry of Education, 1990). However, in spite of the widely held support for an emphasis on number sense, there are wide variations in how it is conceptualized, defined, and described by educators.

Some suggest that number sense is a skill that can be taught through a set of specifically designed activities or teaching objectives. Those who adopt this perspective on number sense attempt to list the component behaviours that characterize number sense. For example, following the 1989 conference "Establishing Foundations for Research on Number Sense and Related Topics", Sowder (1992) compiled a list of characteristics of number sense which were generally agreed upon by conference participants (including Behr, Greeno, Hiebert, Howden, Markovits, Resnick, Silver, Sowder, and Trafton). As summarized from Sowder, 1992, p. 4-6, these characteristics include:

1. the ability to compose and decompose numbers; to move flexibly among different representations; and to recognize when one representation is more useful than another;
2. the ability to recognize the relative magnitude of numbers;
3. the ability to deal with the absolute magnitude of numbers;
4. the ability to use benchmarks;
5. the ability to link numeration, operation, and relation symbols in meaningful ways;
6. understanding the effects of operations on numbers;
7. the ability to perform mental computation through "invented" strategies that take advantage of numerical and operational properties;

8. being able to use numbers flexibly to estimate numerical answers to computations and to recognize when an estimate is appropriate; and
9. a disposition towards making sense of numbers.

Using an alternative approach, Resnick (1989a) described number sense as a form of higher level thinking in the number domain. Using her earlier work on “higher order thinking” (Resnick, 1987), she replaced the term “higher order thinking” with “number sense” to generate the following characterization:

Number sense resists the precise forms of definition we have come to associate with the setting of specified objectives for schooling. Nevertheless, it is relatively easy to list some key features of number sense. When we do this, we become aware that, although we cannot define it exactly, we can recognize number sense when it occurs. Consider the following:

- Number sense is nonalgorithmic.
- Number sense tends to be complex.
- Number sense often yields multiple solutions, each with costs and benefits, rather than unique solutions.
- Number sense involves nuanced judgment and interpretation.
- Number sense often involves uncertainty.
- Number sense involves self-regulation of the thinking process.
- Number sense involves imposing meaning.
- Number sense is effortful (Resnick, 1989a, p.37).

Some authors have conceptualized number sense as a combination of instructional considerations integrated with the creative and generative power of individuals, such as Howden’s (1989) description:

Number sense can be described as good intuition about numbers and their relationships. It develops gradually as a result of exploring numbers, visualizing them in a variety of contexts, and relating them in ways that are not limited by traditional algorithms (p. 11).

Others take a more global, contextual approach to characterizing number sense, recognizing it as the result of a whole range of activities in environments that foster curiosity and exploration, and where teachers and students attend explicitly to

processes of thinking (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Lampert, 1990).

An alternative conceptualization of number sense is Greeno's (1991) environmental metaphor for number sense as situated knowing in a conceptual domain. Greeno interprets number sense as knowing your way around your personal domain of numbers and quantities, knowing what resources are available to you, and knowing how to find and use those resources for understanding and constructive processes of reasoning (p.175).

Learning a domain, in the information-processing framework, is the construction of cognitive structures and procedures that represent the concepts, principles, and rules of inference of the domain... (p. 174). I propose an alternative view. Using a physical metaphor, the domain may be thought of as an environment, with resources at various places in the domain. In this metaphor, knowing the domain is knowing your way around in the environment and knowing how to use its resources.... In the environmental view, knowing a set of concepts is not equivalent to having representations of the concepts but rather involves abilities to find and use the concepts in constructive processes of reasoning (p.197).

Greeno's view of number sense provides a global and inclusive model for considering young children's developing capacity to make sense of number. It accommodates the limited range of resources available to them, instead focusing on the effective use of those resources. It allows looking beyond precocious mathematical activity to respect the developmentally appropriate, creative and ingenious ways young children solve problems with a limited set of resources. How you use what is in your "workshop" becomes the variable of interest, not just what you have in the "workshop." For these reasons, Greeno's conceptualization of number sense is used to frame this investigation of children's construction of meaning for number.

Shades of Knowing

In considering the nature of a child's personal domain of numbers and quantities and his or her sense-making capacities within that domain, this study draws on two important notions. The first of these involves different shades of knowing. While some of what children know is readily apparent, or at least accessible, some aspects

of their knowledge are in a more fluid, partial, or intuitive form that is more difficult to recognize.

Donaldson's (1992) conceptualization of different degrees and shades to knowing and understanding when applied to young children's mathematical activity, highlights the importance of recognizing and valuing the intuitive strengths children bring to their sense-making activity. Resnick (1989b) also recognized the importance of children's early intuitive knowledge to the development of increasingly sophisticated mathematical activity. She identified three proto-quantitative schemas (comparison, increase/decrease, and part/whole) that underlie children's construction of meaning for numbers and quantities, and described how these schemas develop into more formal mathematical knowledge. Recognizing these different levels of knowing in young children's mathematical activity offers the potential of more fully understanding their developing capacity to make sense of number.

This study interprets children's different shades of knowing in terms of different fields, ranges or zones within a personal number domain. These zones are characterized by different forms of knowledge and levels of understanding. Two such zones are of particular interest to this investigation. The first is referred to as a "number comfort zone." This refers to the range of knowledge, skills, and understandings that a child uses comfortably and easily, at an internalized and relatively automatic level. Polanyi's (1958) term "tacit knowledge" might well describe this level of mathematical activity. Tacit knowledge he describes as the underlying mass of experience, attitudes, and skills both physical and intellectual that develop to an automatic level within each individual through their engagement in activities of discovering, making, and judging. Children's knowing within this range is readily available for scrutiny through any number of formal and informal assessment options.

Children's knowing within the next zone is proposed to be much more difficult to access, due to its incomplete, fluid, and formative nature. This range is described in this study as a "number construction zone", beginning at the limits of a child's "number comfort zone" and ending at the child's limit or frontier of understanding. It is proposed that activity within this construction zone is what leads and encourages the development of increasingly powerful and competent number

activity, constituting an instructional zone of sorts. The notion of a number construction zone draws on three sources of research. The first of these is Vygotsky's (1978) notion of a zone of proximal development (ZPD), which describes the child's potential learning space, or the space between independent performance and the outer limits of where the student can work with the support of an adult or more capable peer. Secondly, a similar zone of potential is described by Newman, Griffin and Cole (1989) as a "construction zone", an interactive system or zone within which people work on a problem which at least one of them could not, alone, work on effectively (p. 61). Third, Hodgkin (1985) describes a "potential space" where a learner moves through play, practice, and exploration, towards a dimly perceived learning frontier (p. 24). Hodgkin's work brings in the notion of a limit to that which can be understood at any given time. His terms "frontier of understanding" and "learning frontier" are used to describe a child's personal limit in the construction of meaning.

Figure 1 provides a conceptual model for thinking about children's different shades of knowing and levels of constructing meaning for number.

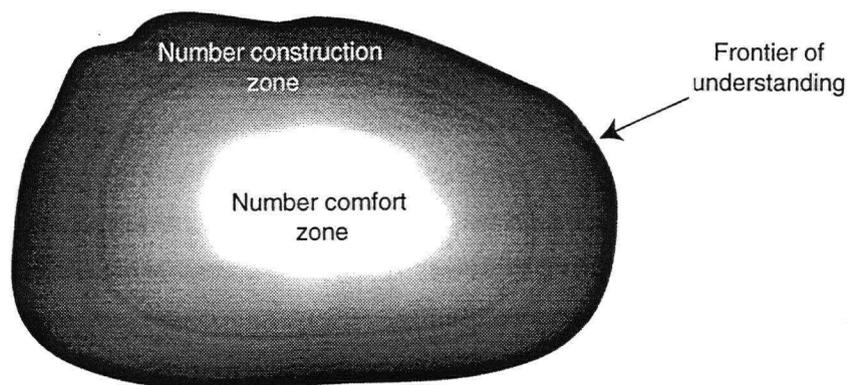


Figure 1. Conceptualization of a child's personal number domain

It is proposed that the nature and scope of the number comfort zone is revealed in good part through children's spontaneous and automatic responses to number situations, while the characteristics and limits of the number construction zone are delineated through children's constructive mathematical activity. Distinguishing between activity within a number comfort zone versus a construction zone can provide useful information for instruction as well as important assessment information. An extensive range of research has focused on what children know. In

contrast, this study focuses on that which children are coming to know, on the ways children go about constructing meaning, and the conditions that support and constrain their sense-making activity within the number construction zone.

Understanding as a Dynamic Process

Donaldson's (1992) conceptualization of "coming to know" eloquently captures the second important notion underlying this investigation, that is, the notion of understanding as an active process of coming to know rather than as a static measure of acquired knowledge. In her book *Human Minds: An Exploration* (1992), Donaldson recognizes two universal human goals, the first of understanding and making sense of things and the second of communicating with one another (p. 7). Donaldson, in characterizing infants and young children as highly competent, sensitive, inquiring, efficient learners, illustrates how children draw on what they remember or what they already know as they attempt to understand a situation. She described our struggle to make sense as coming to know through processes of active interpretation and integration (p. 19).

Within the radical constructivist paradigm, this conceptualization of understanding as an active, integrative process predominates. Von Glasersfeld (1987) defines understanding as a continuous process of organizing one's self-constructed knowledge structures. Knowledge he defines as the schemes of action and operating that are functioning reliably and effectively (1980, p. 81).

This notion of understanding as an active process of coming to know is integral to Pirie and Kieren's (1992) model of mathematical understanding. They describe understanding as a dynamic, non-linear, transcendently recursive process rather than as a single or multi-valued acquisition or combination of acquisitions. Their model of mathematical understanding reflects understanding as a dynamic process rather than as a state of knowing. The model provides one way of considering mathematical understanding by observing a person's mathematical activity as he or she comes to know and understand a particular topic over time. Aspects of this activity are then related to eight potential levels of understanding. "Maps" of this activity over time illustrate children's paths in developing an understanding of mathematical topics. Though Pirie and Kieren's model is not used in this study due to the cross-sectional rather than longitudinal nature of the investigation, aspects of their work have informed the design of the study. In particular, their notion of

“folding back” to an inner level of activity in order to renew the foundation for extending the outer levels of understanding provides a useful model for considering children’s sense-making activity within a number construction zone.

In summary, for this investigation number sense is considered in terms of Greeno’s (1991) conceptualization as knowing your way around your personal domain of numbers and quantities, knowing what resources are available to you, and knowing how to find and use those resources in the construction of meaning. This study uses the terms “making sense of number” and “constructing meaning for number” to describe children’s active participation in processes of “coming to know” or understand how our number system works and how it can be used as a tool for making sense of our environment. The construction of meaning is recognized to be an active process characterized by different levels of understanding and different forms of knowledge. The notion of different ranges or zones of meaning for number is used to conceptualize children’s personal number domains, with a “number comfort zone” characterized by automatic, internalized knowledge and a “number construction zone” characterized by constructive activity involving partial, fluid, or incomplete knowledge, skills, and understandings. The term “frontier of understanding” is used to conceptualize a child’s ever-changing personal limit to making sense of number.

II. Theoretical Perspective

Complementarity of Perspectives

Children’s personal number domains and their individual number contexts, though developed through socially mediated learning processes, ultimately reflect individual perceptions and personal constructions of meaning. In an effort to recognize both individual and social influences on children’s sense-making activity, this study draws on both constructivist and sociocultural perspectives.

In arguing for the complementarity of these two perspectives, Cobb (1994) proposes that mathematical learning should be viewed as both a process of active and individual construction and a process of acculturation into the mathematical practices of wider society. He goes on to explore ways of coordinating constructivist and sociocultural perspectives in mathematics education, recognizing that each tells

half of a good story and that each can be used to complement the other (p. 14). Cobb (1994) describes this complementarity as follows:

For sociocultural research, active individual construction constitutes the background against which guided participation in cultural practices comes to the fore, while for constructivist research, this participation is the background against which self-organization comes to the fore. (p.17)

Driver and Scott (1995) in supporting the complementarity of perspectives state:

Our position is one that is not located exclusively with either personal constructivism or with sociocultural approaches. Indeed, the educational issues that we are concerned about suggest that it is particularly important to adopt a perspective that embraces both positions. (p. 28)

Cobb (1994) proposes that the central issue in research into mathematics learning is that of exploring ways to coordinate constructivist and sociocultural perspectives in mathematics education, and that the particular perspective that comes to the fore at any point in an empirical analysis can then be seen relative to the nature of the questions being asked and the claims being made.

In keeping with Cobb's proposal, in this study a constructivist perspective is used as a primary lens for considering the nature and extent of the conceptual, procedural, and functional "tools" available to the child. This focus shapes the description of what number knowledge appears to be accessible to the child for constructing meaning. For considering how the child goes about making use of those tools, in particular the affective and contextual aspects which influence children's construction of meaning for number, a socio-cultural perspective is used. Both perspectives shape the methodology used to investigate children's sense-making activity, as described in Chapter Three.

Radical Constructivist Perspective

The research questions focus on children's construction of meaning for number. This primary focus on the individual construction of meaning is framed within the radical constructivist theory articulated by von Glasersfeld (1987), Steffe and Kieren (1994), and Confrey (1994a).

A constructivist perspective on learning is grounded in a Piagetian genetic epistemology tradition, or the study of the development (or genesis) of particular concepts over time in children. Piaget's genetic structures were "the models that he made to explain his observations of children's ways and means of operating" (Steffe & Kieren, 1994, p. 716). Though interpretations of Piaget's theory vary, regardless of interpretation, what characterizes research in the Piagetian tradition, is a biological perspective on human development,

that through the process of maturation as defined in relation to experiences in the environment and culture, a child develops certain perspectives and beliefs that are functionally adaptive, and these perspectives and beliefs may or may not correspond well with the views of disciplinary experts. (Confrey, 1990, pp. 10-11)

As researchers attempt to observe and describe the mechanisms that children use as they individually or interactively construct mathematical knowledge, research in the Piagetian constructivist tradition has gradually moved away from a static stage-by-stage view towards a more dynamic approach to issues of development, using the stages as frames for viewing data (Confrey, 1990).

All constructivist-based epistemologies assert that "knowledge is not passively received but actively built up by the cognizing subject" (von Glasersfeld, 1987). Radical constructivism is distinguished from trivial constructivism by the principle that "the function of cognition is adaptive and serves the organization of the experiential world, not the discovery of ontological reality" (von Glasersfeld, 1987). The radical constructivist perspective grew out of a Piagetian epistemology, and has been most clearly articulated by von Glasersfeld, starting when he first presented his "radical" interpretation of Piaget's genetic epistemology to the Jean Piaget Society in Philadelphia in 1975 (Steffe & Kieren, 1994, p. 720).

Confrey (1994a) distinguished radical constructivism from trivial constructivism by its assumption that the individual makes sense of experience in order to satisfy an essential need to gain predictability and control over one's environment. This contrasts with the non-radical goal of reproducing through active participation, the mathematical ideas that dominate modern mathematics. According to Confrey, adopting a radical constructivist perspective on children's construction of meaning

for number requires decentering from an adult perspective in order to imagine how a child's actions and utterances make sense from the perspective of the child (p. 5). To do this, Steffe and others attempt to model children's mathematical realities using the construct of evolving schemes of actions and operations as the basis for their conceptual analysis. Using this approach Steffe, von Glasersfeld, Richards, and Cobb (1983) and Steffe and Cobb (1988) have explicated a model of how children construct the number sequence based on the progressive development of increasingly sophisticated and complex counting schemes.

Drawing on radical constructivist methodology (Steffe et al. 1983, 1988), this study attempts to understand children's construction of meaning for number in terms of conceptual schemes. However, children's conceptual schemes are considered in relation to the procedural, functional, affective and contextual factors that also serve to shape children's mathematical activity. As a result, rather than constructing a conceptual model of children's thinking, a more global and integrated approach is taken to portraying children's sense-making capabilities. In addition to drawing on aspects of the methodology of Steffe et al., (1983), this study also draws on their counting schemes model as a reference point for considering children's counting-related mathematical activity.

Socio-Cultural Perspective

The socio-cultural perspective takes the view that the individual is constituted within a socio-cultural context, and that all intellectual development evolves from the interpersonal (social) to the intrapersonal (individual) (Confrey, 1995). This view is in contrast to the constructivist perspective of a learning child as "an autopoietic (self-regulating) system who forms connections by modeling others as self-regulating individuals" (p. 38). The socio-cultural perspective emphasizes the role of social and cultural context in shaping the intellectual activity and development of the child, particularly in terms of languages and sign systems.

Cobb (1994) in comparing and contrasting socio-cultural and constructivist theories proposed the following characteristics of socio-cultural theory. The first was that socio-cultural theorists link activity to participation in culturally organized practices, rather than to individual student's sensory-motor and conceptual activities (p. 14). Socio-cultural theorists take the individual-in-social-activity as their unit of analysis, as opposed to analyzing thought in terms of conceptual processes located in the

head (p. 14). Cobb recognized that contemporary applications of socio-cultural theory locate learning in co-participation in cultural practices, as opposed to individual conceptual reorganization (p. 14). He suggested the primary explanatory constructs for learning in socio-cultural theory are face-to-face interactions and a process of enculturation into established mathematical practices, whereas constructivists view learning as primarily a process of active individual construction (p. 15).

This study takes the position that no single factor accounts for children's sense-making activity. Rather, the complexity and richness of children's mathematical thinking is attributed to the interaction of multiple factors. Though a constructivist perspective constitutes the primary lens for the study, socio-cultural theory is used to frame the investigation of more socially mediated and situated aspects of mathematical cognition. These include the conceptualization of number sense as situated knowing in a conceptual domain (Greeno, 1991). Considering Greeno's second aspect of number sense, that of how children go about making use of their number knowledge, requires accounting for the social and cultural context within which that activity takes place. Similarly, the approach to the interview, task, and item contexts as described in Chapter Four places particular emphasis on mathematical activity as socially constituted.

III. Multiple Aspects Related to Making Sense of Number

In keeping with a conceptualization of number sense that accentuates both number knowledge and making use of that knowledge, two propositions are put forward. First, it is proposed that conceptual, procedural, and functional aspects of children's mathematical activity provide some indication of the nature and limits of their personal number domain. Second, it is proposed that affective and contextual considerations primarily shape how children make use of the cognitive components of their number domains. This section provides a definition and description of and a rationale for including each of these aspects of children's mathematical activity.

The term "knowledge" in this study is used to describe children's demonstrated competencies, abilities, capacities, capabilities, or understandings. The terms "coming to know" or "making sense of" or "constructing meaning for" all are used to refer to active meaning-making processes of constructively using knowledge in all

its varying forms. Smith (1995) distinguishes between two interpretations, suggesting that constructivists have a primary focus on “knowing” to indicate the subjective meanings of the individual. He proposes that socio-culturalists have a primary focus on “knowledge”, indicating socially negotiated and accepted linguistic and symbolic forms. Driver and Scott (1995) take a position that embraces both, emphasizing that it is the interaction between the personal knowledge (knowing) and knowledge as a social construction that is of central concern to educators.

Adopting Driver and Scott’s position using both terms, knowledge and knowing, supports an active interpretation of sense-making which is in keeping with Greeno’s operational definition of number sense, that it is not just what you know, but how you use what you know. This active interpretation of knowing as opposed to knowledge respects the unevenness and fluid nature of shades of understanding in the development of sense-making processes.

In observing children’s sense-making activity it is clear that some aspects of that activity are cognitive in nature, some relate more to the affective realm, and some relate to contextual or environmental factors. In this study, the cognitive realm has been described in terms of conceptual, procedural, and functional competence. Although for the purposes of analysis these aspects will be examined separately, in practice they overlap and relate to one another in highly complex ways. For instance, effective use of a count-on-with-tally strategy depends on both conceptual knowledge of part-whole relationships and procedural knowledge related to counting. It can only be effective with the representational competence required to tally an unknown, and depends on the support of strategic skills to monitor the tally. Beyond the cognitive realm this activity is fueled by affective aspects such as a willingness to persevere and a desire to succeed. Each of these different factors will be brought to the foreground separately for discussion purposes, with other factors temporarily receding to the background. However, the inter-related nature of these factors that are seen to account for children’s construction of meaning for number needs to be recognized in this discussion.

Cognitive Aspects of Children’s Mathematical Activity

Cognitive development is widely recognized to play a major role in how young children approach and deal with mathematical situations. Since Piaget’s early

research into young children's thinking, researchers in the fields of developmental and cognitive psychology have studied the influence of cognitive development on learning. Traditionally, the study of mathematical cognition has tended to make a distinction between different forms of knowledge, in particular between conceptual and procedural knowledge. There is wide acceptance of the notion that the development of mathematical thinking is characterized by broad advances in conceptual knowledge, by increasing procedural competence, and by a complex interaction between the two.

Conceptual and procedural aspects of mathematical cognition

Conceptual knowledge, according to Hiebert and Lefevre (1986, p. 3-9), involves a rich network of relationships between pieces of information. Conceptual knowledge grows through the construction of relationships between existing or existing and new information. Procedural knowledge on the other hand is seen in terms of two types of skills. One type is composed of the formal language, or symbol representation system of mathematics, not necessarily with a knowledge of meaning. The second type consists of algorithms, rules or procedures for completing mathematical tasks in a specific and linear sequence.

The relationship between conceptual and procedural knowledge is recognized to be close and complex, with considerable debate on the nature of the relationship. Hiebert and Wearne (1986) argue that mathematical competence is characterized by connections between conceptual and procedural knowledge and that incompetence is often due to an absence of connections. Baroody and Ginsburg (1986) speak of the dynamic interaction of early meaningful knowledge (conceptual in nature) and mechanical (procedural) knowledge of facts and procedures, and that this interaction is central to processes of meaning construction. Carpenter (1986) states that "the increase in flexibility (that characterizes the development to more advanced levels of problem solving) is made possible by an increasingly rich conceptual base, more efficient procedures, and the maintenance of links between them."

Information processing models of early number development such as those developed by Riley, Greeno, and Heller (1983) and Briars and Larkin (1984) propose levels of systematic development of early number concepts that are characterized by broad advances in conceptual knowledge and the linking of procedures to this

conceptual knowledge. More current studies that attempt to capture the qualitative changes in conceptual entities (e.g., Steffe & Cobb, 1988) indicate that there is much less coherence and much more complexity to the relationship between conceptual and procedural knowledge than proposed earlier.

In this study, conceptual aspects of mathematical cognition refer to the schemes (Steffe & Cobb, 1988), principles (Gelman & Meck, 1986) and conceptual knowledge (Hiebert & Lefevre, 1986) of relationships that frame children's conceptualizations of number situations. More procedural aspects of mathematical cognition refer to the use of procedural knowledge (Hiebert & Lefevre, 1986) or mechanical knowledge (Baroody & Ginsburg, 1986) in the planning and production of competent or effective behaviours. These include the appropriate use of known facts, patterns, and computational strategies, as well as the use of informal and situated knowledge of a factual or declarative nature.

Functional aspects of mathematical cognition

Besides conceptual and procedural considerations, one further aspect of cognitive activity is used as a focus for this study. Over the course of the pilot study it became clear that the wide variations in how young children supported their own sense-making reflected highly personal preferences and requirements for processing. In some cases, children's inability to monitor their own thinking resulted in faulty reasoning or processing, and masked conceptual capacities that were displayed in other situations. In some cases children drew on highly unique sensory-motor routines to support their processing, enabling them to make up for an apparent lack of procedural knowledge. In other cases, students were able to use materials effectively to support their reasoning as they moved into unfamiliar number territory.

None of these behaviours fell clearly into what could be termed conceptual or procedural competence, yet all seemed to be aspects of overall cognitive competence. All of them dealt with children's individual means of supporting their own constructive processes and their own capacities to make sense of increasingly difficult tasks. These different aspects of mathematical activity were grouped under the label of functional knowledge or competence as a means of including aspects of mathematical cognition that did not fall comfortably under the conceptual or procedural labels.

Three aspects were included under the label of functional competence, the first involving Greeno, Riley & Gelman's (1984) notion of utilization competence. It refers to the interface between conceptualizations and the use of procedures for implementing solutions, and includes the selection of appropriate strategies and the monitoring of their implementation.

Gelman and Meck (1986) applied Greeno, Riley, and Gelman's terms "conceptual, utilization, and procedural competence" to distinguish between three necessary facets of successful performance in their analysis of sources of variability in children's counting. Firstly, conceptual competence refers to conceptual knowledge of counting principles which they believed specified the characteristics that a correct performance required. Their second facet, utilization competence, refers to the capacity to generate a plan for correct performance. And thirdly, procedural competence describes the ability to successfully and correctly execute a plan. Gelman and Meck specify that not one, not two, but all three aspects are required for successful counting performance, and that conceptual competence is easily masked by a shortcoming in the other components of competence. It is the utilization competence that I have included as an aspect of functional competence for this study.

A second aspect of this study's category of functional competence relates to Greeno et al.'s (1984) utilization competence, but extends throughout the action phase, and concerns an aspect of metacognition referred to as self-monitoring or self-regulation. The regulation of cognition refers to planning activities (such as predicting outcomes, scheduling strategies, trial and error, etc.); monitoring activities (such as testing, revising, rescheduling strategies); and checking outcomes through evaluating for efficiency, effectiveness, reasonableness, and accuracy. Even very young children monitor, rehearse, and self-correct their performance, in keeping with Piaget's (1976) argument that any active learning by a structure will self-regulate. Rather than being age-dependent, these capacities are assumed to be task and situation dependent (Brown, 1987).

Piaget (1976) distinguished between three primary types of self-regulation, each differentiated by degrees of consciousness. Autonomous self-regulation refers to the unconscious regulation that is an inherent part of any "knowing act." Active self-

regulation describes trial-and-error sort of behaviour when the learner is engaged in constructing and testing "theories-in-action." According to Piaget (1976):

Action in itself constitutes autonomous and already powerful knowledge. Even if this knowledge (just knowing how to do something) is not conscious in the sense of a conceptual understanding, it nevertheless constitutes the latter's source, since on almost every point the cognizance (consciousness) lags, and often markedly so, behind this initial knowledge, which is of remarkable efficacy despite the lack of understanding. (pp. 346-347)

Piaget's developmental progression for self-regulation moves from unconscious autonomous regulation, through active regulation and the beginning of conscious reflection, to the mature level of reflected abstractions. The earliest level is beyond the capacity of the child to recognize and articulate. The middle level involves some capacity to describe mental processes, albeit sometimes erroneously. The final level includes the capacity to accurately explain actions, theories, and reactions within them, which provides the essence of conscious control of action, or reflected abstraction. Conscious self-regulation refers to the capacity for abstract thought to drive action sequences on a mental plane. Piaget restricted this level of regulation to the stage of formal operations, where the learner could consciously invent, test, modify, and generalize theories and discuss these operations with others. In this study self-regulatory practices are categorized under functional competence.

A third aspect of what I have termed functional competence is related to what Sigel (1991) refers to as representational competence. Sigel defined representational competence as "the understanding and utilization of a fundamental rule to the effect that knowledge presented in various forms (e.g., pictures, words, signs) still retains its intrinsic meaning in spite of variations in form of presentation" (p. 189).

Sigel's notion of representational competence, which integrates and elaborates on Piaget's (1962) notion of representational thinking, has particular application in the area of assessment. He pointed out that children need to have the competence to understand the representational form of a problem before they can proceed to solve it (p. 199). Sigel (1974) illustrated how the representational form of an item involving number concepts for the quantities two, three, and four affected the performance of pre-school children. Teachers claimed that the results of a standardized test were invalid, prompting the researcher to re-present the item

using familiar and motivating objects in a familiar setting rather than illustrations, resulting in far different results. Sigel claimed that such standardized testing assesses the degree to which the child's logic conforms to the adult logic rather than the child's way of thinking. Sigel proposed that:

the challenge is to construct tasks in which the same basic problem is presented in different symbol systems, but with increasing difficulty. In this fashion, the basic competence can be assessed taking into account the influence of representational elements. (1991, p. 204)

Sigel's notion of representational competence was applied to the shaping of all tasks used in this study, through the use of variations in presentation along with increasing number size and through the use of motivating objects in familiar contexts. Furthermore, it provided a means of considering children's capacity to transform problem information into a representational form that made sense. This range of activity included shifting the mode of representation of situational features to make them more accessible, to clarify relationships, or to keep better track of steps in the problem solving process. One frequent example was the switch from mental reasoning to the parallel use of place value blocks to model problem situations. Another was the increase in sensory involvement as number size increased.

Steffe et al.'s (1983) model of counting types includes sensory-motor activity as an aspect of children's conceptual activity and a direct reflection of counting type. Pilot data indicated that some aspects of representational activity were directly connected to children's capacity to conceptualize number. However, many examples of this sort of representational activity appeared to be directly related to self-monitoring rather than to conceptual requirements. For this reason, representational activity has been included as an aspect of functional competence, though the relationship between conceptualization of units and representational activity is acknowledged.

Summary of cognitive considerations

Cognitive aspects of children's sense-making with number were considered in terms of conceptual, procedural and functional competence. Functional competence included utilization, self-regulatory, and representational capacities. Considering children's mathematical cognition from these different vantage points provides a means of accounting for situations where children's activity does not reflect competencies that might have been evident in other situations. Expanding the focus

beyond cognitive activity to include affective and contextual considerations further highlights the complexity and richness of what underlies children's mathematical activity.

Affective Influences on Mathematical Activity

The affective domain is described by McLeod (1993) as "a wide range of beliefs, feelings, and moods that are generally regarded as going beyond the domain of cognition" (p. 576). Research on mathematics education in relation to the affective domain has focused on beliefs, emotions, and attitudes. Reviews of this literature base include Kulm (1980), Reyes (1980, 1984), and Leder (1987).

Compared to research on mathematical cognition, there has been relatively little research on the role of affect in children's learning of mathematics, probably in part because of the complexity and difficulty of adequately dealing with the affective domain. Gardner (1985) cites the desire to avoid complexity as the major reason for the lack of attention to affective issues in cognitive science generally. What research there is comes primarily from the traditional paradigm, focusing on definitions of terms, measurement issues, and relying on questionnaires and quantitative methods (McLeod, 1993).

McLeod (1993) describes an alternative paradigm for research on affect in mathematics learning that is "characterized by its emphasis on theoretical issues, its interest in qualitative methods, its use of interviews and think-aloud protocols, and its attention to beliefs and emotions as well as attitudes" (p. 577). It is from this alternative paradigm that this study's focus on the role of affect in children's mathematical activity was shaped.

Mandler (1989) provides an interesting view of the source of affective factors in mathematics education. Mandler proposes that a physiological arousal is produced when a person's anticipated sequence of actions cannot be completed as planned, with the arousal serving to redirect the individual's attention to an alternative path. As the individual attempts to evaluate the meaning of the interruption, an emotional response is generated. It may be evaluated as a cause for great anxiety, as an annoying event or as a pleasant surprise. How it is evaluated and interpreted depends to a great extent on the individual's knowledge and beliefs as shaped by

prior experience. In this study, children's emotional responses to items and tasks were noted and affective inferences were shaped from these observations.

Mandler's perspective is particularly relevant to this study where students worked with an increasingly difficult sequence of task items, going beyond their personal comfort zone or automatic level to construct responses and eventually to meet a frontier of understanding or potential. Observing how students approached and dealt with these different levels of difficulty across different contexts provided multiple opportunities to interpret children's responses, which in turn provided some sense of the attitudes and beliefs children held about themselves as capable thinkers and mathematical problem solvers.

In this investigation attention was given to the beliefs, attitudes and emotions children displayed as they worked through the tasks in the interviews. Their facial expressions, postures, and actions in conjunction with their verbalizations, the timing of their responses, and the overall feel to responses throughout the interview contributed to observations on the role of affect. In addition, in teacher and parent interviews, questions were asked about children's mathematical disposition, their beliefs about mathematics, about themselves as mathematical thinkers and problem solvers, and about the social context of their mathematics learning generally. The anecdotal information gathered concerning affect served to enrich and provide texture to the descriptions of individual children's mathematical activity. When affect was considered in relation to the use of strategies, to performance across tasks, and to the extent of children's number construction zone, interesting patterns to the important role played by affect were apparent. Of particular interest were characteristics of mathematical disposition as described in the *Curriculum and Evaluation Standards* (NCTM, 1989, p. 233-236), namely confidence, flexibility, perseverance, curiosity, and degree of reflectiveness.

The Role of Context in Children's Mathematical Activity

The study of the role of contextual factors on mathematical learning is receiving increased attention in the literature because of connections with issues of gender and ethnicity (Cole & Griffin, 1987). In part this emphasis has come from the field of anthropology where interest in mathematical setting outside of schools have provided new insights into the situated aspects of children's mathematical knowledge (e.g., Lave, 1988).

Cobb's (1990) socio-constructivist perspective which places increasing emphasis on the role of context on children's mathematical cognition, provides a frame for the use of aspects of context in this study. Four aspects of context were addressed.

The first, drawn from Cobb (1990), defines context as the cognizing subject's own construction as opposed to the situations or settings that the observer might isolate in the subject's environment. Cobb's interpretation recognized that students in the same setting act in different contexts and engage in very different forms of mathematical activity. This notion of personal number contexts corresponds with the global approach to interpreting children's mathematical discourse in this study, and the emphasis on trying to understand children's mathematical activity from their perspective.

The second aspect of context concerned the impact of the interview setting and specific task contexts on children's mathematical activity. The tasks used in this study were designed to tap into children's informal mathematics by using everyday problem solving contexts rather than typical school tasks, and using materials familiar and motivating to young children. However, it was not assumed that the task contexts guaranteed access to children's natural and informal means of constructing meaning. Similarly, the dynamic interview format (described in detail in Chapter Three) contributed to creating a climate that facilitated the exploration of children's sense-making processes. Nevertheless, the interviews were conducted outside classroom routines by a relative new-comer and as such did not constitute "everyday" practice.

The third aspect of context included in this study concerned the impact of item variation on children's sense-making activity. These variations included the changing number size sequences within each task, the changing task presentation from concrete to more abstract forms, and the role played by students' access to concrete materials. The impact of these variations on children's conceptualizations, strategy use, disposition, and willingness to continue are described in the Chapter Five and Six results, and discussed in Chapters Seven and Eight.

The last aspect of context considered from a socio-cultural perspective involved the dynamic aspect of the interview protocols. This interview approach drew on Vygotsky's (1978) emphasis on the importance of social interaction with more

knowledgeable others in the zone of proximal development. It constitutes a variation of the scaffolding used in cognitive apprenticeship (Brown, Collins, & Duguid, 1989) and the negotiation of meaning in the construction zone (Newman, Griffin & Cole, 1989).

Although it is recognized that students' mathematical activity is influenced by their experiential base and their social cognitions, attempting to account for the impact of children's previous mathematical backgrounds, or current mathematical and social settings on their mathematical activity in the interviews was not a focus of this study. However, parent and teacher interviews were conducted to attempt to shape some sense of how these situated aspects might have affected children's mathematical activity in the interviews. Selected anecdotal comments are included to provide richer and more comprehensive information upon which to base the interpretation of children's mathematical activity.

Summary.

This investigation draws on the notion of a personal number domain to conceptualize children's capacity to make sense of number. As a means of considering both the extent of this domain, and the nature of how a child constructs meaning within the domain, children's mathematical activity is considered from multiple points of view. These include cognitive aspects related to conceptual, procedural, and functional competence, as well as affective and contextual aspects. No single aspect is believed to adequately account for variations in children's mathematical activity; rather, it is held that these aspects interact in important and complex ways. The purpose of this investigation is to explicate the nature of these factors as they influence children's mathematical activity and to explore the extent to which they enhance and constrain children's sense-making activity.

IV. Review of the Literature on Children's Mathematical Thinking

Four different tasks involving a range of mathematical concepts and skills related to whole numbers and additive structures are used in the study. As a result, this review of the literature focuses on the specific studies of children's mathematical thinking which served as a basis for these aspects of the investigation, specifically: pre-number knowledge, counting, place value, unitary and multi-unit conceptual frameworks, and children's strategy use across topics.

Pre-Number Research

Research indicates that young children construct important intuitive notions about quantities well before they can systematically label them and apply mathematical operations to them. Piaget's research investigated the emerging logico-mathematico knowledge of pre-operational children between the ages of 2 and 6. He identified such constructs as seriation, conservation, class inclusion, and reversibility, as the cognitive capabilities that emerged as children moved into the stage of concrete operations between the ages of 5 and 7. The impact of this emerging knowledge on specific numerical competencies was not the focus of his work, but it has been the focus of subsequent research efforts, resulting in considerable debate.

The fields of psychology and cognitive science, in the meantime, generated an extensive research base on young children's emerging capabilities with number (e.g., Gelman & Gallistel, 1978). From this research base, Resnick's (1983, 1989b) model of development is described in some detail. This model, though grounded in a cognitive science view, provides a focus on both the cognitive underpinnings and the related mathematical behaviours that characterize very young children's activity, and highlights the importance of the informal knowledge that young children construct prior to formal schooling.

Resnick (1983) proposes a developmental theory of number understanding using student task performances to generate logical theoretical analyses of what knowledge children "must have" to perform the way they did. (This is as opposed to a Piagetian method of hypothesizing a mental structure, then using tasks that might reveal its presence or absence.) Despite the difference in approach, Resnick's analysis shares two important emphases with the Piagetian view, namely an emphasis on part-whole relationships (class inclusion for Piaget), and an emphasis on the importance of coordinating ordinal and cardinal (class inclusion or part-whole) relationships in the course of developing number understanding.

Resnick (1989b) takes a more constructivist stance in providing an overview of the recent reconceptualization of the nature of number knowledge development, focusing on the intuitive, informal knowledge that young children build on as they construct mathematical knowledge. According to Resnick, young children's pre-

number schemas express quantity without numerical precision, through perceptual rather than direct measurement processes. She described three such schemas:

1. the **protoquantitative comparison schema** which demonstrates a form of global estimation or judgement through the use of absolute size labels such as lots, small, etc.;
2. the **protoquantitative increase-decrease schema** which interprets changes as increases or decreases in quantities and implies an intuitive understanding of addition, subtraction and conservation; and
3. the **protoquantitative part-whole schema** which is the informal knowledge about materials that come apart and go together that allows young children to know, for example, that the whole cake is bigger than any of its pieces. This intuitive part-whole schema is the pre-schooler's version of the principle of the additive composition of number, which says numbers are composed of other numbers and any number can be decomposed into parts.

According to Resnick, these three reasoning schemas constitute a major foundation for later mathematical development, and as language develops, preschoolers' implicit protoquantitative reasoning schemas combine with early counting knowledge to generate reliable and generalizable number concepts.

In Resnick's (1983) view, elaborations to the part-whole schema in particular account for changes in mathematical competence. The pre-number informal part-whole schema when systematically applied to quantity (particularly counting) in the early school years, generates a quantitative interpretation of part-whole, while further elaborations characterize subsequent development of an understanding of the place-value system of notation, and related operations (p. 146). Her cognitive science interpretation of number concept development relies on the interaction of procedural knowledge with schematic knowledge as an integral part of this development.

Research describes another pre-number quantification method used by young children, involving the immediate recognition of small sets, a skill sometimes known as subitizing. Most theories hypothesize that children use visual or auditory patterns for quantifying these small numbers (e.g., Klahr & Wallace, 1973; von Glasersfeld, 1982), however, Gelman and Gallistel (1978) propose that subitizing is

only used by children who have previously counted the sets enough times to have an automatic response to the numerosity. This facility with small “perceptual” numbers appears to extend to larger numbers as students begin to see small patterns within larger quantities, what von Glasersfeld (1982) called “figural joining.” Whatever the genesis of the skill of subitizing, this non-counting perceptual process appears to provide an important internal scaffold to moving beyond counting by ones, serving as a precursor to the establishment of the grouping processes necessary for place value and multi-digit understanding.

In the last ten years, neo-nativist researchers have proposed that humans are “pre-wired” with a quantitative schema onto which experience is mapped. Earliest informal knowledge is described in research on infants’ pre-verbal quantitative knowledge, which indicates that even babies can discriminate the numerosity of small sets, can match small quantities, can recognize differences in small quantities and sizes, and make judgements on the basis of comparative rather than absolute size (Starkey, Spelke, & Gelman, 1990). These and other studies on the mathematical abilities of very young children illustrate the wide-ranging informal understandings that children construct well before any formal instruction in mathematics.

This focus on pre-number activity is included as a basis for considering the global, intuitive notions children hold and sometimes apply in number situations that are beyond their grasp, in particular the use of estimation in unfamiliar settings. This pre-number activity is an important foundation for the development of more systematic and elaborated approaches to number situations, and provides an example of pre-schoolers’ developmentally appropriate sense-making activity involving numbers and quantities.

Research on Counting

The role of counting in the development of the concept of number is widely recognized, and has been the focus of considerable research. For the cognitive science community, counting has been seen as the source of the construction of the number concept, whereas Piaget attempted to embed the concept of number in the development of logical thinking (Kilpatrick, 1992, p. 11). This difference in emphasis is reflected in the nature of the investigations of counting within these research traditions.

For cognitive science, the focus has been the development of counting abilities, as they relate to the semantics (meaning) and syntax (grammatical principles) of counting behaviours (Fuson, Richards, & Briars, 1982; Gelman & Gallistel, 1978; Siegler & Robinson, 1982). Gelman and Gallistel's classic study describes the principles that young children hold about counting, and how these build towards accurate counting. These principles are:

1. one-to-one matching, including both tagging individual items with a name, and separating items yet to be counted from those already counted;
2. the need for a stable order of the words used in counting;
3. cardinality (Gelman's version of which simply involves knowing the last number in a counting sequence names the quantity of the set);
4. abstraction of the counting process to count different types of items, some of which cannot be seen (e.g., feelings); and
5. order irrelevance, or the understanding that the order of counting objects will not affect the total number.

Gelman and Gallistel found that young children as young as three or four implicitly know these principles that enable counting to serve as a reliable means of quantification. Fuson (1988) provides a similar cognitive science perspective on the sequence of developmental changes within a child's mental number-word sequence, claiming that changes in conceptual structures are the result of the increasing integration of sequence, counting, and cardinal meanings.

The results of counting research from this cognitive science, logical analysis perspective, indicate a pattern to how young children develop counting abilities. They appear to first use any number to express quantity, as in a global guess with no focus on accuracy. Next, they use a forward count always starting from one, usually with the support of objects to move and count. Next, children develop the ability to count forward or back as in counting on or backward on a mental number line. A further refinement of this abstracted counting facility is the ability to coordinate a double count, using a tally to keep track of the increment or decrement. Finally, children develop an understanding of the additive combination of numbers, providing increasing power to their counting schemes.

Steffe, von Glasersfeld, Richards, & Cobb 's'(1983) model of counting types uses a different approach to the investigation of children's counting. It provides a conceptual analysis of what underlies children's counting performance, rather than a logical analysis as used by cognitive scientists. The work of Steffe et al. (1983) is grounded in a Piagetian epistemology, and implicitly includes Piagetian constructs such as conservation of discrete quantity, invariance, and hierarchical inclusion as features of the model within the context of schemes. According to Steffe (1983, p.110-111), a child's current numerical knowledge can be viewed as the coordinated schemes of action and operation the child has currently constructed. Schemes serve as anticipatory structures through which children interpret mathematical interactions, creating the actual meanings they perceive. Schemes are engendered by actions that are repeatable or generalized through application to new objects (Piaget 1980, p. 24).

Steffe et al.'s (1983) theoretical model of children's counting types describes developmental changes in the nature of the representations that children create while counting. The initial model (Steffe, Richards, & von Glasersfeld, 1978) was developed through a series of teaching experiments, and analysis of videotapes that were generated in an earlier study. The model was subsequently reformulated on the basis of further teaching experiments and with the help of von Glasersfeld's (1981) theoretical model of the construction of units and number.

The 1983 model is comprised of five basic levels of counting, each level characterized by an increase in the flexibility and internalization of the counting process. Steffe et al. argued that the first four levels, which depend on specific sensory-motor signals, constitute a distinctly different stage from the fifth level, which is characterized by internalized, abstract counting capacities.

1. First level counters of **Perceptual Unit Items** can count only physical objects or actions they can perceive directly, and have no way to represent hidden objects. Given the task of determining the hidden part of 7 when only three of the seven are showing, these counters are at a loss.
2. Counters of **Figural Unit Items** can count hidden items, but they must construct very explicit representations of the hidden items, such as touching a specific imaginary pattern to reconstruct the number of hidden items.

3. Counters of **Motor Unit Items** require some physical action such as tapping, to accompany the counting sequence. These counters count the motor action itself, not an imaginary representation of the item. A motor action represents the hidden objects.
4. Counters of **Verbal Unit Items** use the number word itself to represent the counted object. These counters recount sets from one, rather than counting on from the starting number.
5. Counters of **Abstract Unit Items** have a fully operational counting mechanism that allows them to count-on rather than label each item from one, and to maintain a double count as is required to count on from 8 to 14 when faced with the missing addend problem $8 + _ = 14$. This level requires the ability to mentally represent and manipulate number through internalized counting.

Steffe and Cobb (1988) expanded on the earlier theoretical model of counting types to identify a progression of five stages:

1. Stage 1, the **Perceptual Counting Scheme**, included the first two levels of the previous scheme, perceptual unit and figural unit counting. Counting at this level is limited to items that can be perceived (e.g. see, hear, etc.)
2. Stage 2, the **Figurative Counting Scheme**, included levels 3 and 4 from the previous scheme, involving the counting of motor and verbal items. At this stage children can generate sensory input to make countable items, but continue to count from one rather than counting on, as an older child would.
3. Stage 3, the **Initial Number Sequence**, involves the counting of abstract unit items, the highest level of the earlier scheme. This stage makes use of mental representations of number, and uses counting on rather than counting from one, a sign of the construction of numerical concepts, rather than the pre-numerical concepts at the first two stages.
4. Stage 4, the **Tacitly-nested Number Sequence** includes the ability to count-on (for sums and missing addends), count-off-from (8-2 as 8, count back two), count-down-to (8-6 as counting from 8 to 6 and keeping track of the count), and to choose the most appropriate strategy.
5. Stage 5, the **Explicitly-nested Number Sequence** includes the construction of the part-whole operation (Steffe & Cobb, 1988, pp. 150-151), with an explicit awareness of subtraction as the inverse of addition, and the reflective abstraction to extract a part from a whole without destroying the whole.

Steffe and Cobb's counting stage model constitutes a framework for interpreting children's behaviour in terms of the underlying cognitive capacities that their counting behaviours imply, and for planning interactions with children on the basis of those interpretations. Steffe and Cobb argue that the unit types used by young children are closely related to their overall mathematical development, and that their notions of units play significant roles in their understanding of a wide range of elementary school mathematics (Cobb & Wheatley, 1988; Steffe & Cobb, 1988).

Steffe et al.'s (1983, 1988) conceptual analyses of children's counting schemes have provided micromodels of detailed progressions from very early counting to elaborated and refined counting schemes. In considering the impact of children's conceptual understanding on their mathematical activity, references are made to the important ideas upon which these models are based, in particular for interpretation of children's counting perspectives. However, reliance on Steffe et al.'s important work on counting schemes is limited due to the need to account for non-counting conceptualizations and approaches to number situations in this study.

Summary and synthesis of counting research

Looking at the development of counting abilities from Steffe and Cobb's perspective, certain key conceptual understandings appear to provide a foundation for the development of more powerful counting mechanisms. Synthesizing findings from both the logical and conceptual analyses of counting, several connections are important.

Forward counting from one, where students recount from one each and every time there is some perceptual change, does not require the capacity to conserve or mentally hold a quantity (invariance of number in Piagetian terms). This matching process constitutes an ordinal rather than cardinal form of counting, and does not require an understanding of the inclusion relation, since every count is a new count from one. Children approaching number from this perspective can deal with a wide range of mathematical situations, but in a limiting and qualitatively different way from that of more established cardinal counting.

Children approaching number from the perspective of cardinal counting can count on and back from a given number. This depends on some capacity to internally

represent and hold a starting quantity, in order to manipulate it. It also indicates an understanding of the inclusion relation, that seven is part of nine, that changing a group of 7 into a group of 9 does not require rebuilding the group from one. Steffe and Cobb's model of increasingly internalized counting approaches provides a detailed account of this important transition to what they label the "initial number sequence".

Conceptually, once students can mentally represent, hold, and manipulate quantities, they have the potential to store and use chunks of number rather than relying solely on one-by-one counts. This capacity enables a non-counting, qualitatively different approach to number situations that includes the elaboration of part-whole understanding, reversibility of thought, and the additive composition of number.

Research on place value, addition, and subtraction shows a corresponding transition from unitary, one-by-one counting methods to increasing reliance on internalized, non-counting reasoning, such as drawing on the known fact $6+6=12$ to derive the sum of $6+7$, or decomposing 24 and 38 into tens and ones for recomposing. This hurdle of moving beyond unitary thinking (characterized by counting by ones), to using multi-unit thinking (characterized by the use of known facts or relationships as "chunks,") is the focus of the next section on multi-digit research.

Research on Multi-Digit and Place Value Thinking

Once again, different approaches have been used to study children's understanding of multi-digit numbers and place value. The work of Cobb and Wheatley (1988) and Kamii (1986) illustrate research from a constructivist perspective, while studies by Ross (1990), Bednarz and Janvier (1982), and Fuson (1990) represent a more cognitive science approach to this area of research. One way to integrate these two lines of research is to think of constructivist research providing a focus on the conceptual structures that underlie children's understanding of multi-digit thinking while the cognitive science investigations provide detail on the behaviours that characterize these underlying structures.

Cobb and Wheatley (1988) extended Steffe et al.'s (1983) model of unitary counting strategies to include the transition to working with multi-digit numbers, composite or multi-unit notions and place value relationships. They identified three

increasingly sophisticated concepts of the number ten that children construct beyond the first five counting levels.

1. The concept of **ten as a numerical composite** involves a focus on the individual ones that make up the ten rather than on the composite itself as a single entity. Students treat strips of tens as ones, and although they use a new number word sequence to count things (i.e., a 10, 20, 30..... chain), they are simply using a modified form of counting by ones. An elaboration of this first level involves the concept of **ten as an abstract singleton** where students count strips of ten as 10, 20, 30 (but think of them as 1, 2, 3), seeing either ten ones or a single entity sometimes called a ten but not both simultaneously.
2. The concept of **ten as an abstract composite unit** brings into play the first true unit of ten, where students treat tens as single entities while simultaneously maintaining their tensness. Counting by ones becomes organized into modules of ten, students count by tens from the middle of a decade, and coordinate counting by tens and by ones in a single episode. At this stage, it is essential that material of some kind be available, and although students may count tens models by ones, they see the connection, and may respond in tens and ones. However, they cannot simultaneously construct a numerical whole and the units of ten and one that compose it.
3. The concept of **ten as an iterable unit** no longer requires materials for support. Students can mentally add or subtract tens or ones as necessary, (e.g., a missing part is seen as a single entity that can be structured in terms of composite units of ten and units of one). At this stage, students can understand the positional principle of the numeration system and can construct their own algorithms.

Kamii's (1986) interpretation of this same transition is similarly based on a conceptual analysis of what underlies children's performance. Using a Piagetian theoretical framework, Kamii investigated children's understanding of place value and provided a rationale for the difficulties observed. Her explanation is grounded in the belief that number is a relationship created mentally by each individual through a process of reflective, or constructive abstraction, rather than through empirical abstraction from the external world. She claims that children have to

create a system of tens, by reflective abstraction, on the system of ones they have already constructed in their heads. This requires a coordination of ordering and hierarchical inclusion first for ones, then for tens and ones, along with the ability to focus simultaneously on the quantitative part-whole relationships. According to Inhelder and Piaget (1964) until children reach seven or eight years of age, they can think about the whole and the parts successively but not simultaneously.

The results of Kamii's (1986) study showed that children first deal with multi-digit quantities as ones, then as groupings of ones which are handled either as ten ones or as one ten, then finally as coordinated groupings of tens which can be unpacked as ones and repacked as tens and ones. This final ability to coordinate tens and ones first appeared in her study with 39% of the second graders, and with 71% of the third graders. These three levels provide a conceptual analysis of place value from a Piagetian perspective, and serve as the conceptual anchors for other more task-analytic approaches to understanding numeration.

Ross (1990) explored the interaction of developmental factors with instruction in children's ability to understand place value, using her five-stage model for children's acquisition of place-value numeration concepts (Ross, 1985). Her five stages describe differences in performance with printed number and place value skills, and though not explicitly connected, relate to the conceptual levels as described in Kamii's study.

1. Stage One children connect two-digit numbers with the whole they represent, but assign no meaning to the individual digits.
2. Stage Two children can name the value of each digit as tens or ones but assign no meaning to the quantities to which each corresponds. Both these stages correspond to Kamii's first level of thinking of the whole as a grouping of singletons, though the second stage includes some additional verbal knowledge.
3. Stage Three children have what Ross calls "face values", where the two digits are connected to two different kinds of objects such as dimes and pennies, however no ten to one relationship of objects is included in the understanding. This would appear to be the equivalent of Kamii's second level where children can recognize either tens or ones, but not the relationship between them.

4. Stage Four is a transition stage between three and five, characterized by incomplete understanding and unreliable, inconsistent performance in coordinating tens and ones.
5. Stage Five children know that two digit numerals represent a tens and ones partitioning of the whole, and can represent and name non-standard partitionings as two digit numbers (e.g., $30+6$ as $20+16$). This level of performance is rooted in the conceptual understanding of Kamii's third level, and provides a solid foundation for algorithm development and place value extension. (Ross, 1990, p. 1-3)

Stage Five performance, which combines numeration knowledge with underlying conceptual structures, develops gradually, and is typically not achieved until age 9 or 10 (Bednarz & Janvier, 1982; Kamii, 1986; Ross, 1990). Ross's study (1990) concluded that neither instruction nor cognitive development alone accounted for the variation in children's understanding of place value, rather that a strong interaction existed between the two.

Fuson takes a cognitive science, component-skills approach to her extensive research into multi-digit numbers, choosing to focus on children's use of written number marks (symbols) and the spoken system of English number words as evidence of conceptual change (see Fuson 1988, 1990, 1992). Fuson (1990) specified ten structures for multi-unit numbers, only some of which could be considered conceptual. The first four levels deal with marks, positions, and words, providing detail to Kamii's first level. Beyond this, Fuson specified two conceptual structures which must then be constructed and related to the four earlier structures. The first involves the ten-to-one relationship of digit positions and the second involves trading relationships. The last four of Fuson's structures involve advanced multi-unit conceptual structures required for understanding multiplication, division, exponents and scientific notation.

Fuson distinguished between two different kinds of multi-unit quantities that children construct: collected multi-units such as models of ones, tens, and hundreds; and sequence multi-units which are extensions of the unitary sequence counting procedures, involving counting patterns for multiples of ten. With collected multi-units, children reflect on collections of objects, such as groups of tens, ones and hundreds blocks, while with sequential multi-units, children reflect on their own

counting (Fuson, 1992, pp.143-4). This parallels children's development of early counting of collections of objects before counting activity becomes internalized and separated from objects.

Summary and synthesis of multi-digit research

Despite the different research perspectives, several common patterns emerge and can be synthesized as follows. Children's early work with groupings is dominated by perceptual features, and requires physical models for support. Without the models, tenness is not established, and cannot be mentally represented and manipulated (as in mentally decomposing two-digit numbers into tens and ones for mental computation). This reliance on external groupings gradually becomes an internalized process as children construct personal meaning for multi-unit relationships.

Though the role of visual grouping capacities (as in subitizing) in supporting the development of multi-unit understanding has not been established, counting patterns play important roles in naming, connecting ideas, and providing conceptual scaffolding.

Children first use their established unitary structures to work with multi-digit quantities, thinking about 3 tens and 6 ones as a grouping of 36 ones. This thinking is characterized by the counting of tens models as units. Gradually, as Kamii says, children map their new system of tens onto their old system of ones. Before children reach the stage of thinking about 36 as both "3 tens 6 ones" and "36 ones", they go through a period of seeing the 36 as either, but not both simultaneously.

Seeing 36 as any combination of groupings (e.g., 3 tens 6 ones, 36 ones, 2 tens 16 ones, 6 sixes, 4 nines.....) and as interchangeable groupings, requires a fully integrated multi-unit conceptual framework. Conceptually, this requires a rich and flexible understanding of the additive composition of numbers including decomposition-recomposition, reversibility of thinking, and fully elaborated part-whole understanding.

Research on Unitary versus Multi-Unit Conceptual Frameworks

The transition from unitary to multi-unit thinking is an essential component of place value understanding, and necessarily plays a critical role in children's

understanding of multi-digit computation. Considerable attention has been focused on the shift between unitary counting as an approach to number situations and a multi-unit grouping approach. This important shift has various labels depending on the source of the research. Sometimes it is discussed in terms of place value (e.g., Kamii, 1989), sometimes in terms of understanding of units (e.g., Steffe & Cobb, 1988; Cobb & Wheatley, 1988) and sometimes in terms of conceptual frameworks for number (Fuson, 1992). Fuson's interpretation is used for this study because it highlights the underlying conceptual structure of these qualitatively different ways of interpreting number situations, and because it can be applied to a wide range of whole number situations.

Fuson (1992) described how different conceptual structures for numbers affect children's solution procedures with addition and subtraction. She pointed out that unitary structures include both pattern number and counted number, and that though these structures are certainly adequate for single digit addition and subtraction, they can also accommodate multi-digit computation, but only in cumbersome and often inaccurate ways.

Fuson identified three multi-unit conceptual structures that support multi-digit computation: embedded number, triad number, and recomposable triads. Though the labels are different, these closely parallel Kamii's elaboration of part-whole thinking, and Cobb and Wheatley's non-counting-based, multi-unit levels.

Fuson attributed differences in children's approaches to multi-unit addition and subtraction situations to children using a unitary approach to multi-unit situations as a result of inappropriate instruction and language confusion that does not support the construction of tens as multi-units (e.g., "twelve" providing no clues to the ten-and-two aspect of twelveness). Fuson differentiates between unitary and multi-unit approaches to dealing with number by the use of the terms "unitary conceptual framework" versus "multi-unit conceptual framework" (Fuson, 1992). These terms are used in this study (in preference to Kamii's or Steffe and Cobb's terms) to describe the important conceptual role children's understanding of units and composites plays in their capacity to deal with multi-unit number contexts.

Research on Strategy Choice and Use

The four tasks used in this study each involve quite different contexts and mathematical ideas. They were chosen to provide alternative "windows" into how children construct meaning in whole number situations. This choice was based on the belief that there would be underlying patterns to children's mathematical thinking across contexts, and that the nature of strategy use would provide evidence of these patterns. Though strategy use is not the only aspect of children's mathematical activity of interest, it is considered to be an important one. This assumption is in keeping with Resnick (1989b), who, in providing a rationale for exploring what children can do beyond what they verbalize, said, "One way to establish the nature of children's development of number concepts is to examine some of their invented informal strategies for doing arithmetic" (p. 164).

From early in this century (e.g., Brownell, 1928) and up to the present time (e.g., Siegler, 1989), children's solution strategies have been a focus of mathematics education and cognitive science research. Other terms such as "methods", "schemes", or "informal algorithms" are used in the research to describe children's strategies. This study uses the term "strategy" to refer to the action aspect of children's sense-making activity, what children do (both internally and externally) in order to construct meaning and in order to find a solution to the problems they face. Children's strategies, as defined in this study, are not restricted to procedural sequences, but rather are interpreted to reflect the interaction of conceptual, procedural, as well as functional capacities. Some strategies are one-step recognition or retrieval processes, some are procedural sequences of a relatively automatic nature such as established counting, and some are multi-step constructive processes.

Research on young children's strategy use has predominantly focused on one specific content area at a time, rather than across topics or content areas. The majority of this research has focused on children's strategy use for solving addition and subtraction basic fact combinations. A major focus of this basic fact research has been on how children eventually reach a stage of recall or immediate retrieval of sums and differences. For example, Siegler and Shrager (1984) contend that children possess a set of strategies that are organized into a hierarchy of preferences. They believe children prefer to retrieve answers but if they are uncertain or are concerned about accuracy, they will use one of their back-up strategies, usually counting.

Siegler's theory of how strategy choice works involves a personal repertoire of associations of different strengths for different basic fact combinations. If a child has a strong association, the answer is retrieved quickly and the back-up strategies will abort. Each successive correct use of a fact adds a bit of associative strength. Eventually, through repeated use, answers are retrieved so quickly that back-ups are rarely needed, and children resemble adults in their patterns of arithmetic retrieval.

However, not all of arithmetic activity is reducible to one-step associations that students or adults can internalize and retrieve. Rather the set of basic facts for any operation along with high-interest, high-frequency benchmark combinations and patterns make up the pool of such retrieved "facts" or associations. These basic "tools" are used in combination with other tools to construct meaning for a wide variety of situations well beyond the basic fact range. It is the generative power of strategies that is of interest in this study. The focus on strategies provides a means to explore children's capacity to make use of the number knowledge available to them. Strategies in this study are recognized to be applicable across a wide range of situations rather than specific to a certain type of word problem or problem context. Considering strategy choice as a major focus of exploring young children's capacity to make sense of number thus allows for the investigation of children's alternatives to recall or recognition and how they make use of known facts, patterns and relationships. It considers how this knowledge is used to construct new understandings and to solve problems for which no answer is automatically retrievable.

With this proviso in mind, the following studies provided a frame for considering strategy use in this study. Carpenter and Moser (1982) refined a solution strategy classification scheme for use in research on arithmetical word problems. Gray (1993) was successful in using the same scheme for describing the strategies used by children aged 7-12 to solve basic computational problems. This scheme involved the following strategies:

- count-all, using a count from one
- count-on, conceptualizing the value of at least one set and using the appropriate counting procedure to count-up or count-back
- derived fact, involving use of other known facts
- known fact, using retrieval

Considerable research has investigated whether there is a conceptual hierarchy or sequence to children's use of these strategies. Since the use of retrieval does not necessarily require that meaning be attached, it is difficult to establish its place within a sequence. However, there is considerable evidence to establish a developmental sequence for the first two strategies; it has been widely accepted that the use of count-on follows the initial use of count-all (Carpenter & Moser, 1982; Fuson, 1982; Secada, Fuson, & Hall, 1983).

Carpenter and Moser's (1984) longitudinal and cross-sectional study of addition and subtraction problem solving performance over a three year period provides some indication of a developmental shift from counting to use of derived facts and known facts. Carpenter and Moser's study provides a most interesting picture of increasingly sophisticated solution strategies of eighty-eight children who were interviewed three times a year between early first grade and mid-way through third grade, with the focus on how they solved different problems. Though no individual patterns of strategy use are reported in Carpenter and Moser's study, results suggest a developmental sequence to these changes. The data illustrate changes in children's approaches to number situations over time, going from an inability to understand what to do in a given situation, to modelling the problem, to using increasingly efficient counting-by-ones strategies, to recalling answers for a selected group of number combinations, to drawing on known facts and relationships to construct or derive answers without counting. This shift from early counting strategies to more powerful grouping and associative strategies supported by known facts and relationships parallels the shift from unitary to multi-unit conceptual structures as described by Fuson (1992), by Kamii (1986) and by Cobb and Wheatley (1988).

Results from the different tasks in the Carpenter and Moser study also make clear the developmental differences in understanding and approach, and the range of strategies that are typical of and appropriate for young children in the early grades. These developmental differences in children's strategy repertoires can have a powerful impact on achievement across the early grades. Gray (1991) commented on the qualitatively different mathematics children engage in depending on their strategy patterns. His research supports the observation that children who come to first grade using sophisticated grouping strategies for problems have a very different educational experience from those who come to school with limited

number awareness. Similarly, students in third grade who are reliant on modelling and counting-by-ones strategies to deal with problems, have had a very different set of experiences over the grades, and have probably practiced a much more limited set of strategies than those who have had access to a range of ways of approaching number situations.

Studies on young children's strategy choices (Siegler, 1988) indicate that a variety of factors influence strategy choice, not the least of which is availability of a strategy. Some studies propose that the choice of procedures for solving problems and the nature of strategies chosen, directly reflect students' conceptual understandings, and that conceptual change enables the development of this procedural knowledge (e.g., Carpenter, 1986; Riley, Greeno, and Heller, 1983). Schoenfeld (1983) proposes that the effective use of procedures requires conceptually informed decisions at both the tactical and strategic levels. Hiebert and Lefevre (1986) state that conceptual knowledge assists in selection of appropriate procedures and acts as a screening agent to reject inappropriate procedures. (Much of this research equates strategies with procedural knowledge, i.e., procedures. In this study, strategies include procedures of a sequential and automatic nature but also include other kinds of thinking patterns that reflect higher level constructive reasoning processes of a more conceptual nature.)

Others recognize that strategy choice or the flexible use of procedures may reflect non-conceptual as well as conceptual factors. Baroody and Ginsburg (1986) suggest that more efficient strategies are sometimes invented to reduce cognitive demands related to procedural and strategic competence. They also recognize that strategy choice is affected by semantic structure, such as cues from the wording of problems, and from the size of the numbers used in problems. Furthermore, contrary to the widely held assumption that increased conceptual knowledge leads to the development of more sophisticated procedures, Baroody and Ginsburg propose that in some situations the application of procedures leads to conceptual breakthroughs.

Research on children's beliefs about mathematics indicates that children's strategy choices are sometimes guided by their beliefs about what is called for in problem solving situations. Baroody (1983) found that Grade 3 children tended to laboriously compute answers in the belief that they were supposed to calculate rather than look for patterns and relationships, while Grade 2 students used shortcuts that relied on

their conceptual knowledge and informal mathematics. Baroody and Ginsburg (1986) propose that beliefs that are sometimes encouraged by schools can distort children's views of mathematics, interfere with their use of conceptual knowledge and induce an over-reliance on mechanical knowledge. It is suggested that strategy choice may provide an indication of all of these effects.

Most research on children's "strategies" comes from the field of cognitive psychology. Those working from a constructivist perspective do not use the term "strategies" likely due to the procedural connotations of the term and in light of their focus on conceptual analyses. Steffe in a 1983 article entitled "Children's Algorithms as Schemes" made a case for the importance of coming to better understand children's mathematical methods because their early methods serve as a foundation in the construction of more sophisticated methods. Though Steffe did not use the term "strategies" he used "methods", "schemes" and "algorithms" to describe what others might call "strategies". Steffe described children's methods as schemes as did Inhelder and Piaget (1969), and distinguished between schemes that are operative or figurative. An operative scheme must involve mental operations, or construction of interiorized action that can be carried out in thought. Operative schemes are anticipatory and generalized. Figurative schemes, on the other hand, are not attributed to the uniting operation of integration. Steffe suggests they are algorithmic or procedural in the sense that they constitute a finite number of steps carried out to accomplish some end. They are not mental operations, rather are carried out at an external, perceptual level. The use of a counting-all strategy provides an example of a figurative scheme. For counting-all, the child constructs a perceptual model of the situation, and uses unitary counting to count the external objects from one each time. No aspect of the situation is mentally operative, and a shift in the perceptual model requires a re-count from one - it is not an anticipatory or generalized scheme. On the other hand, counting-on would represent an operative scheme resulting from the re-organization of the earlier scheme. Counting-on, according to Steffe, reflects the higher mental plane of numerical structure. Its use reflects the capacity to mentally represent a starting quantity and count-on or back from that number. Steffe states that the organizational patterns in children's mathematical activity (i.e., strategies) that grow out of their progressive organization of actions are observable, constitute schemes, and should serve as a focus for instruction.

Strategies in this investigation provide a means of focusing on children's patterns of approaching number situations. The strategies children use are considered in relation to conceptual, procedural, functional, and affective aspects of their mathematical activity. Strategy use provides a window into children's thinking across all the areas discussed in this literature review: children's early number concepts, counting schemes, and notions about unitary and multi-unit thinking. Across different task contexts, strategy use is used as a focal point for identifying the underlying threads that appear to shape mathematical activity. It is believed that this global approach to the investigation of strategy use is consistent with constructivist principles, and can provide a useful lens for considering a child's overall capacity to make sense of number.

Chapter 3

The Use of Dynamic Interviews in Educational Research

I. Investigating Children's Understanding of Number

The approach to the investigation of children's sense-making with number taken in this study involved a variation on the clinical interview format. It included a dynamic component that provided access to different levels of children's understanding. The following section provides a rationale for and a description of how the dynamic interview approach was used.

In the 1920's Piaget developed the clinical interview as a research method to investigate the cognitive developmental capacities that underlie competence. Since that time, variations of the clinical interview have been developed and used to study a wide range of issues related to children's thinking and problem solving. The clinical interview has been widely used for research framed in a Piagetian individual-cognitive tradition.

Recent studies based on a radical constructivist perspective (e.g., Steffe, von Glasersfeld, Richards, & Cobb, 1983; Steffe & Cobb, 1988; Cobb & Wheatley, 1988) have utilized longitudinal teaching experiments as their means of exploring children's sources of meaning, personal theories, and sense-making constructs. These teaching experiments, originally developed by Soviet educational researchers, utilize the clinical interview in conjunction with an extended period of teaching. The teaching experiment provides opportunities to observe the accommodations children make in their functioning schemes as a result of their experiences over time. Conceptual structures involving schemes of action and operation are inferred from the actual mathematical behaviour of the children involved. Steffe argues that the teaching experiment is the fundamental research methodology for learning how children learn mathematics (Steffe, 1994, p.13).

Clinical interviews, on the other hand, can provide a glimpse of children's current conceptual schemes in action, but they do not provide the same opportunity to study children's constructive processes as provided by teaching experiments. However, in many situations where assessment information is needed, a commitment to a teaching experiment is not possible. In order to explore children's

sense-making constructs and constructive processes involving number, but without a long-term teaching component, this study used a dynamic variation of the clinical interview.

As an alternative constructivist methodology to the teaching experiment, the dynamic interview also provides the opportunity to "explore the limits and subtleties of children's construction of mathematical concepts and operations" (Cobb & Steffe, 1983, p. 83). Using multiple mathematical contexts within the dynamic interviews provides opportunities to test and revise assumptions about the mathematical understanding of the child. Using different and increasing levels of difficulty, and using cues and scaffolding to support children's thinking, the dynamic interview provides opportunities to "observe children's constructive processes firsthand" (p. 85). This does not ensure access to "those critical moments when cognitive restructuring takes place" (p. 84); however, it creates opportunities for the child (and researcher) to reflect on such mathematical activity.

In keeping with Greeno's (1991) conceptualization of number sense, the method used for this investigation had to account for the nature and limits of both the child's personal number context or number domain and how the child made use of that context. Combining aspects of the two research methodologies provides a means of considering both aspects of number sense. The interview method and the conceptual analysis approach were drawn from the radical constructivist approach of Steffe et al. (1983, 1988). Their method is based on the contention that children's mathematical knowledge can be modelled in terms of coordinated schemes of actions and operations (von Glasersfeld, 1980; Cobb & Steffe, 1983). Cobb and Steffe (1983) describe the goal of the methodology as that of specifying children's schemes for the purpose of intervening "in an attempt to help the children as they build more sophisticated and powerful schemes"(p. 88).

The addition of the dynamic component to the interviews provided an opportunity to explore children's construction of meaning for number. The theoretical rationale for utilizing dynamic procedures with clinical interview methodologies in this study is grounded in a social-interactionist, Vygotskian perspective that considers socio-cultural influences in conjunction with cognitive developmental change. Vygotsky (1929) proposed a model of development that recognizes two kinds of development, the natural and the cultural. Acknowledging that natural development "is closely

bound up with the processes of general organic growth and the maturation of the child" (p. 415), Vygotsky focused on the cultural aspects of experience, behaviour, and methods of reasoning. Vygotsky's (1929; 1934/1986) model provides a socio-cultural perspective on how children's thinking changes over time, with an emphasis on the role of the tutor (parent, teacher, capable peer...) performing "the critical function of 'scaffolding' the learning task to make it possible for the child, in Vygotsky's words, to internalize external knowledge and convert it into a tool for conscious control" (Bruner, 1985, pp. 24-25).

The dynamic interview shares the purposes of both constructivist and socio-cultural research. It provides a better understanding of children's thinking in order to shape subsequent instruction, but also it has the purpose of providing a comprehensive and respectful portrait of children's developing competence that reflects the full range of conceptual, procedural, strategic, metacognitive, representational and affective strengths the young child brings to number situations. The dynamic interview focuses on children's conceptual schemes, but considers them in relation to the broader context of children's mathematical activity.

II. Shaping a Dynamic Variation of the Clinical Interview

According to Palincsar, Brown, and Campione, (1991, p.76), the term "dynamic assessment" refers to a number of distinct approaches that feature guided learning for the purpose of determining a learner's potential for change. Dynamic methods are contrasted with more traditional and static procedures that focus on the products of assessment. Where static measures reveal only those abilities that are completely developed, dynamic measures are concerned with how well a child performs once given assistance. Proponents of dynamic assessment believe it provides a prospective measure of performance, indicating abilities that are in the process of developing, and provides a predictive measure of how a child might perform independently in the future.

Feuerstein (1969), generally acknowledged to have coined the term, "dynamic assessment," believed that traditional psychometric devices could not tap a child's ability to acquire knowledge. He developed the "Learning Potential Assessment Device" (LPAD, Feuerstein, 1980) in order to measure low-achieving students' ability to profit from instruction. While administering the LPAD, the teacher-

observer interacts in a flexible and individualized manner with the examinee. "The examiner constantly intervenes, makes remarks, requires and gives explanations, whenever and wherever they are necessary, asks for repetition, sums up experiences, anticipates difficulties and warns the child about them, and creates insightful reflective thinking in the child..."(Feuerstein, 1979, p. 102). Results of this mediated assessment process are considered in terms of a cognitive map. This map helps to specify the nature of the child's strengths and weaknesses in terms of content familiarity, modality preference, phases (the input, elaboration, and output phases of a mental act), strategies, and levels of complexity, abstraction, and efficiency.

Vygotsky recognized the importance of considering both a child's actual level of development as well as the potential level of development thresholds. He credited static tests of development for their capacity to inform us regarding a child's actual development, but recognized that a more dynamic approach to assessment was necessary to tap into the zone of proximal development (ZPD). This zone he defined as "the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance, or in collaboration with more capable peers" (Vygotsky, 1978, p.86).

One thing that dynamic assessment approaches, such as Campione and Brown's (1984) graduated prompting, have in common is an integration of assessment with instruction. While students are being evaluated they are also learning something about how to approach the task. Similarly, the assessment process yields valuable and specific information for shaping future instruction. It also provides observers with information about a child's ability to learn, thus playing an important role in teachers' and parents' expectations (Burns, Vye, Bransford, Delclos, & Ogan, 1987). Overall, dynamic assessment procedures have the potential to reveal a very different picture of individual competence than is possible using static assessment procedures.

This section outlines a rationale and procedure for combining dynamic strategies with the clinical interview method in order to consider children's capacity to make sense of number. This combination provides a powerful methodology for investigating children's constructive mathematical processes. As well as focusing on

the cognitive and affective aspects that enhance and constrain children's mathematical activity, the dynamic interview focuses on the impact of socially mediated constructive processes. Variations to the clinical interview are organized and presented according to: the interview task and protocol; the interview context; the roles of the interview participants; and the focus of the data gathering process.

Variations to the Interview Task and Protocol

Traditional clinical interviews begin with a task and a rough outline of the path of questioning to be followed. How students approach and deal with the task is the focus, with the interviewer attempting through contingent questioning to explicate the child's reasoning processes.

In the dynamic approach used in this study the interviewer presents multiple examples of a task at increasing levels of difficulty. The underlying conceptual level of the task is held constant, however number size increases, and the level of abstraction of the presentation varies. This dynamic aspect to tasks offers several advantages. First, starting with very easy examples ensures children understand the problem, allows them to demonstrate their ability with the task, and builds their confidence for future examples by starting them off with success. It ensures that students are presented with tasks beyond their comfort zone, resulting in the active construction of meaning for at least some examples. The flexible range of examples affords the opportunity to adjust the range to suit the student, and to observe and interact with each student as he or she navigates a path through increasingly challenging number situations.

Early in the sequence of examples for a task, students may "just know" the answers. Easiest to access are the understandings the child has internalized and operationalized to the point of ease of use, or meaningful habituation (Brownell, 1928). This range of understanding is often defined by number size, creating a sort of "number comfort zone" (Kelleher, 1992) where children access responses and solutions without consciously deriving or constructing answers. Besides number range, personal interests and talents can further define this comfort zone, offering a socio-cultural dimension to its shaping. A grasp of baseball statistics, or card game know-how such as cribbage combinations for 15, or money familiarity from playing Monopoly, are all examples of how situated knowledge can enhance the nature and shape of children's personal comfort zone for the domain of number.

However as the difficulty of the task increases, students need to change their approach and begin to draw on alternate resources to construct meaning. The point at which students begin to engage in constructive processes marks what might be called the end of their independent, internalized, or automatic response level and the beginning of their "construction zone" (Newman et al., 1989) or "zone of proximal development" (Vygotsky, 1978). Within this zone a child utilizes partial understandings and fluid knowledge with the support of a more capable peer or adult. As the difficulty level of the task continues to increase, eventually the student will be unable or unwilling to continue, providing some indication of the edge of the child's understanding or "frontier of understanding" (Hodgkin, 1985). The resulting conceptual "map" of the child's capacity to make sense of number situations is shaped by these different shades of understanding. It provides some sense of the child's current number domain, context, or "environment" (Greeno, 1991).

Another task variation that is necessitated by the dynamic format is the provision of a variety of materials for students to use to support their thinking and processing when they move beyond their number comfort zone. Materials provide a further form of support to the learner, beyond that of an adult or more capable peer. Students need assurance that they can use any methods that make sense to them, that as the difficulty increases they may want to use different supports, and that eventually everyone reaches a personal limit. Which supports students draw on, and how these are used, provides valuable information on students' learning strengths and preferences, on their memory limitations, and on their meta-cognitive and strategic processing.

Using one consistent idea or concept across different number contexts provides a focus on the stability of that concept, the extent of the connections the child has established to support the concept, and the impact of number context on children's constructive capacities. The nature and extent of the range of the child's constructive activity is one focus of the dynamic interview.

A dynamic task format allows for some flexibility in the representational mode of presentation of the task in order to adjust it to suit different learners. This might involve using real objects to begin, and working towards the use of numerals for higher examples. It might involve using an explicit format such as a place value mat

to begin and working towards an abstract presentation without cues to indicate place values. It also allows the interviewer or student to adjust these aspects of representation to increase the salience of the task demands as the difficulty increases. The impact of these differences in presentation can provide useful instructional information concerning each student.

Variations to the Interview Context

Any interview situation benefits from a comfortable atmosphere. The nature of the physical setting has an impact on the interview context, and can help to put the child at ease or not. Similarly, the nature of the tasks and their format can go a long way towards establishing a comfortable and positive atmosphere. However, to encourage children to work beyond their comfort zones requires a particularly supportive environment that is respectful of the child's thinking, methods of working, and limits. This requires addressing the power imbalances inherent in adult-student interactions, to create as far as possible a collegial, non-hierarchical environment. Establishing an atmosphere that encourages risk-taking, considers incorrect answers as steps towards finding correct answers, values the reasoned estimate, and supports innovative and personal methods, is essential to creating a supportive context for the dynamic interview.

Variations in the Roles of the Interview Participants

A third variation in the clinical interview that is required for a dynamic interview is a shift in the roles of the participants. Rather than considering the student as a "subject," the student plays a role in helping the interviewer to understand her thinking and reasoning methods. The student takes an active part in the shaping of each interview and in efforts to help the interviewer make sense of each response.

The interviewer's role is similarly redefined to be one of providing direction in the interview, providing encouragement and support as the student attempts to make sense of a challenging task, and providing feedback to help the student reflect on and articulate constructive processes. Once a student reaches a personal limit in working independently, the role of the interviewer becomes one of a facilitator, providing scaffolds and cues to enable the child to continue his or her exploration. The nature of these cues and their impact on student responses becomes a second focus along with the on-going focus on students' thinking processes.

The nature of the interviewer's cues and scaffolds can vary depending on each situation. Ferrara (1987) applied the principles of dynamic assessment in a study which examined the mathematical performance of young children. When the child experienced difficulty, Ferrara used a structured hint sequence involving a maximum of eight hints. Through a qualitative analysis, Ferrara was able to determine the effectiveness of different types of hints for helping individual children to structure problems.

Here, a less formalized approach is taken to the use of, and analysis of hints and cues. In describing cues within the context of this investigation, the distinction between conceptual, procedural, and functional cues is made. Some cues might be of a conceptual nature, such as connecting the underlying part/whole relationship of one task to another previously successful effort. Some cues might be procedural in nature such as providing a missing element of a reasoning process. Functional scaffolds or cues can vary widely, but generally speaking support children in their efforts to manage and monitor their own reasoning processes. This may take the form of organizing the materials differently in order to clarify relationships, or clarifying verbal information by repeating the child's words to aid focus or attentional difficulties. Observing the child's capacity to make use of such cues and supports provides valuable information on the nature of the factors which enable and constrain performance.

Two other categories of cues or supports were used in this investigation. The first involved interventions of an affective nature designed to support children's disposition to persevere. The second involved interventions related to the presentation of items, or item context. Supporting children's performance by adjusting the representation level of an item allowed some children to continue. Adjusting number size also served to support children's capacity to make sense. These affective and contextual aspects of the dynamic interview served to support children's sense-making in important ways, however they were inextricably connected to other aspects of mathematical cognition. In the interests of clarity, this discussion presents scaffolds and cues as discrete, identifiable occasions with identifiable impacts. Since the impact of any given cue is only known by the student, and interpreted by the interviewer on the basis of personal interactions, this belies the messiness and complexity of the actual use of cues in the interviews.

Since the end-point of the interview is not established a priori, the decision to keep going or to stop is shared between the child and the adult. Until that point, support is provided as necessary. Similarly, the path of the interview is determined through a process of negotiation, creating an atmosphere quite different from most traditional interviewer/interviewee situations.

Variations in the Focus of the Data Gathering Process

Non-verbal communication in clinical interview situations constitutes an important aspect of data collection. The one-on-one setting facilitates careful observation, while the flexible questioning protocol allows the interviewer to explore the unique and personal verbal and non-verbal responses of each child. In the dynamic interview, where students work beyond their comfortable levels of performance, and where the use of materials as supports to thinking is crucial, non-verbal aspects of performance, such as gestures and body language take on greater importance.

McNeill's book "Hand and Mind" (1992) explored three important themes related to gesture and its role in communication. He first proposed that gesture and speech cooperate in conveying meaning, each providing an important channel for observation of the mental processes and representations of the mind. His second theme was that gesture connects the spoken language structure with the underlying thought structure, suggesting that the overarching discourse structure can be seen more clearly in the gesture than in the words and sentences (p. 2). His third theme was that gesture itself has an impact on thought:

The gesture supplies the idiosyncratic, the personal, and the context-specific aspects of thought, to be combined with the socially regulated aspects that come from the conventions of language. Such a combination implies a dialectic of gesture and language in which the gesture provides the momentary context of speaking and language carries this individuality to the social plane where it is categorized, segmented, reformatted, and dressed up for the world. (p. 2)

Language limitations often curtail the verbal expression of young children. Gesture therefore plays an important role in their discourse. This is even more the case in the area of mathematics, where children's sensory-motor explorations of concrete situations are believed to play a fundamental role in number and spatial concept development. McNeill (1992) suggests that gesture provides a window into the

imagistic, global, tacit knowledge and analog thinking that underlies developing concepts. Speech on the other hand provides a window into more linear, organized, established knowledge and digital thinking (p. 11).

Goldin-Meadow, Wein, & Chang (1992) applied McNeill's theory to how children's gestures and speech taken together, can provide valuable assessment information:

Thus, gesture, taken in conjunction with speech, may reveal the tacit hypotheses the child in transition is entertaining and, thus, may provide a window into the areas in which the child is ready to profit from instruction, that is, into the child's zone of proximal development. (p. 214)

Attending to what children say, do, and write is standard procedure for clinical interview situations. However, in the dynamic interview, widening the lens to focus on children's less salient non-verbal actions such as ever-so-slight nods or taps, facial and eye movements, posturing, and unconscious verbalizations, provides valuable information on students metacognitive and reasoning processes. Considering these various aspects together, provides a form of triangulation of interview data.

Triangulation is a strategy for increasing the validity of evaluation and research findings through the use of multiple methods of investigating the same phenomena. Mathison (1988) proposed that the value of triangulation lies in providing more and better evidence from which researchers construct meaningful propositions and explanations of the phenomena (p. 15). When the data converge, a clear though not necessarily complete picture of the child's reasoning emerges. When the data are inconsistent or contradictory, the interpretation task of the interviewer becomes more complex.

The use of video as a data recording device becomes essential as a means of capturing the subtleties of non-verbal processing, in particular the role of gesture. Videotaping can also be used effectively as the reason for a child to repeat a process or to articulate thinking (i.e., Can you show the camera how you did that? Can you repeat that loudly enough for the camera to hear?). Although most adults would reject the idea of having videotape capture them as they grapple with a difficult problem, the pilot study showed that young children respond quite positively. When the camera was seen as a tool for helping the interviewer to remember all the

interesting details of the interview, children either ignored the camera or spoke to it as if it was another interviewer.

The change in the role of the interviewer to that of a facilitating participant-observer results in a shift in the focus of the data being collected. When the task is first introduced, and the student is able to answer independently, the student's reasoning and responding processes are the focus. Once the student's independent ceiling is reached, the interviewer intervenes with cues, questions, or suggestions to support the student's efforts with the task. The nature of these supports, and their impact on the student's performance then become another aspect of the data collection.

Summary

The dynamic clinical interview retains the general format of a traditional clinical interview with concrete materials, but without artificial constraints on the wording of the tasks, the nature of interventions, and on the range of examples offered. The initial objective is to ensure the child understands the task. Using a sequence of examples graduated in difficulty, one can observe how far the child can go in working with the concepts and skills involved, given appropriate support. In keeping with a constructivist approach, the major goal is to attempt to see the child's actions from the child's point of view.

The data generated in a dynamic interview format are primarily qualitative, and when considered in relationship to one another, can provide a holistic view of the child's capacity to make sense of number. Considered separately, the data can describe a wide range of aspects that support or constrain overall performance:

- the conceptual understandings demonstrated,
- the procedural skills drawn upon,
- the personal experience drawn upon,
- the strategies employed,
- the self-monitoring techniques used,
- the modes of representation brought into play (sensory involvement, use of blocks or models, use of a mental number line, etc.),
- the disposition of the student before, during, and after (confidence level, interest, curiosity, perseverance, risk-taking, etc.), and
- the nature of the interventions that supported performance.

The dynamic interview format was chosen for this study because it provides the opportunity to explore children's different ways of knowing, to map the extent and nature of their personal number domains, and to access the range of constructive processes that young children employ as they go about making sense of their world. It provides a means for looking beyond children's static, established knowledge base to tap the fluid, partial understandings of the "number construction zone". The dynamic interview provides a means of accessing the conceptual schemes upon which children's mathematical sense-making is based, and considering them in relation to the procedural, functional, and affective aspects of meaning-making, without neglecting the socially mediated and situated aspects of mathematical cognition.

III. Using Multiple Contexts to Investigate Understanding

Four different contexts for applying number concepts and skills were used in this study. The contexts of doubling, determining missing parts, sharing, and using money each provided a practical, relevant, and familiar situation for young children to apply their developing understanding of number. Different task contexts were used as a means of observing which aspects of children's mathematical activity would hold across contexts. These aspects included conceptual understandings, procedural skills, functional capacities (including representational, metacognitive, self-monitoring, and strategic capacities), and affective considerations.

A second reason for using four quite different task contexts was to provide multiple opportunities to observe different aspects of mathematical thinking, as illustrated in Table 1. For instance, children's approaches to each example provided an indication of their grasp of operation concepts, applications, and related skills. Though the missing parts task dealt primarily with additive structures, it was possible to interpret the doubling, sharing, and money tasks from either an additive or a multiplicative perspective. Additive interpretations meant students applied concepts of incrementing, decrementing, joining, separating, adding and subtracting. Multiplicative interpretations involved ideas of duplicating, shrinking, sharing equally, multiplying and dividing (Behr, Harel, Post & Lesh, 1994). These aspects of children's conceptualizations of the problem contexts, were considered across tasks for each child, based on their solution strategies and corresponding rationales.

A related aspect that was considered across tasks was whether a child's approach involved unitary or multi-unit thinking. For instance, in doubling numbers it was possible to think in unitary terms of counting-on by ones from the first number to increment by that number again. Or, it was possible to decompose numbers in place value terms or into familiar parts, doubling multi-units and units separately and recomposing them to find the sum. Providing examples of increasing number size offered the opportunity to see whether children would switch from a unitary to a multiunit approach. All tasks could be handled from either perspective, however using a unitary, count-by-ones approach with large numbers involved complicated and cumbersome procedures that quickly became overwhelming. Unitary approaches necessarily involved additive interpretations, while multi-unit approaches were compatible with both additive and multiplicative approaches.

Each task offered a window into children's understanding of the part-whole relation. All tasks offered opportunities to explore the extent to which children recognized and utilized cardinal number properties, in particular the additive composition of number and the inclusion relation. All tasks offered opportunities to decompose quantities into component parts in order to utilize known relationships such as place value groupings, known facts, or multiple patterns.

Besides these more conceptual aspects of children's sense-making, across tasks it was possible to get a sense of each child's personal store of number knowledge of a more procedural nature. Counting and number fact knowledge was called upon in every task. Using different contexts provided a broader picture of the frequency and consistency with which children drew upon this knowledge. Also, the different contexts helped to stimulate a wider range of the personal number benchmarks that supported children's thinking.

In any one interview, the consideration of affective or functional aspects of performance was expected to be highly dependent on circumstances surrounding that one task or that one point in time or that one day. Some days were better than others for the children (and for the interviewer,) resulting in variations in interest, perseverance, focus, etc. However, across the four interviews it was possible to observe patterns to each child's mathematical disposition, to their general approach to learning situations, and to their metacognitive, strategic, and representational

capacities. Table 1 highlights some of the characteristics that were the focus of the multiple contexts, and illustrates how by using the four tasks together it was possible to repeatedly observe specific characteristics.

Using multiple contexts provided the opportunity for a balanced perspective on the aspects of children's sense-making activity that were of interest in this investigation. Using multiple examples of any one task at varying levels of difficulty provided a form of triangulation of the data as well as an opportunity to observe variations in children's approaches to any one task. Chapter Six describes how using these tasks together as a package provided a means of considering the extent of a child's personal number domain or potential to make sense of number, as well as the nature of how the child went about constructing meaning for number situations within that domain. Chapter Seven then describes how this approach provides an alternative means of considering the development of number sense over time.

Table 1. Aspects of interest to the investigation, illustrated by task

Aspects	Tasks			
	Doubling	Missing Parts	Sharing	Money
Conceptual				
• part-whole relation	•	•	•	•
• decomposition/recomposition	•	•	•	•
• place value concepts	•	•	*	•
• unitary vs multiunit framework	•	•	•	•
• additive vs multiplic. approach	•		•	•
• addition concepts	•	•	•	•
• subtraction concepts	*	•	*	•
• multiplication concepts	•		•	•
• division concepts	*		•	
• inverse of +/-	*	•	*	•
• inverse of x/+	*		•	
Procedural				
• counting skills	•	•	•	•
• number facts	•	•	•	•
• personal benchmarks	•	*	•	•
• algorithms	•	•	*	*
Functional				
• representational capacity	•	•	•	•
• strategic monitoring	•	•	•	•
• memory considerations	•	•	•	•
• language considerations	•	•	•	•
• metacognitive considerations	•	•	•	•
Affective				
• mathematical disposition:	•	•	•	•
- confidence, flexibility,				
- perseverance, curiosity,				
- reflective capacity				
Contextual				
• personal number context	•	•	•	•

- means the task provided a focus on this aspect of children's mathematical activity
- * means the task provided some opportunity to observe this aspect

Chapter 4 The Study

I. The Pilot Study

Pilot work was completed in the spring and fall of 1993, the year preceeding the actual study. This provided an opportunity to try out different tasks, to develop and refine formats, and to make decisions about important aspects of each task and about the study in general.

One assumption behind this investigation of children's sense-making activity involving number is that children, when left to their own devices, use what makes sense to them in order to solve number problems. Pilot work indicated that tasks that were similar to the school work children were doing in class tended to elicit less diversity in, and a more procedural orientation to the approaches used. What appeared to be more personally meaningful sense-making was elicited by situations that were more typical of real life than classroom life. Since the goal was to investigate children's personal construction of meaning rather than to assess classroom learning, tasks for the study were shaped to reflect everyday situations. To do this, engaging materials were selected to support each context, such as real coins and small toys for the money tasks. Similarly, everyday language rather than mathematically accurate wordings were used; for example, the notion of fairness in sharing was used to explore children's capacity to divide a set of materials equally.

Pilot work suggested that the availability of pencil and paper with most tasks affected children's reasoning by triggering a computational response. With access to paper and pencil, students were less inclined to use their personal ways of making sense, less willing to take chances with estimates, less engaged in the real world problem solving aspect, and more inclined to depend on learned procedures and algorithmic thinking which they were often unable to explain. On this basis, the decision was made to provide materials such as place value blocks and different types of counters as aids to thinking, but not to provide pencils and paper. Though this put an extra load on children's short-term working memory, it also encouraged the use of external aids to thinking (such as blocks or fingers) thus providing information on children's representational and strategic monitoring capacities as well as providing a window into the nature of their thinking.

Pilot work provided an opportunity to develop task protocols and appropriate cues and supports to use as children moved beyond their comfort zone. One important aspect of each interview involved the restating of the child's response in order for the child to reflect on it, revise it, or confirm it. This was done to encourage a reflective approach to the construction of meaning, to see if students would recognize and correct discrepancies, and to check their confidence in their answers. For example, in the Missing Parts Task, after a child had provided an answer, I would gesture to the two parts and, using their response, say, "If there are five on this side and six hidden, does that make eleven in all?" Sometimes, upon reflection, students agreed; and sometimes they revised their answers. After each revision I provided the same clarifying question, so that the use of the question was a consistent response to the child's thinking. This process proved to be very useful as a wind-up statement, as a means of focusing children's attention, and as a check on the strength of their response.

II. The Participants

Teachers and students at two small elementary schools in an urban community participated in the main study. Five teachers of students aged six to nine were introduced to the goals of the study and four teachers indicated interest in being involved. These teachers were asked to provide the names of up to ten students as follows:

- a group of students to reflect the range of abilities representative of the class composition;
- a balance of boys and girls;
- a preference for students who used a variety of personal approaches to number situations rather than rule-bound approaches; and
- students who likely would be both willing to share and comfortable with explaining their thinking in the presence of an unfamiliar adult and a video camera.

The names of 31 students were provided, and each of these students took home a letter of introduction along with consent forms (see Appendix A). It was made clear at the outset that not all students would take part in the complete study, and that

final selection of students would be based on the need to compose a small group of children, balanced by age and gender, and representative of a range of developmentally typical ways of thinking about number. The parents of 29 students gave full permission for their children to participate in the study, including consent for pupil interviews, consent for video taping of these interviews, and consent for a parent interview with audiotaping.

Because one goal of the study was to portray the qualitative differences in children's numerical thinking both within and across age groups, a range of ages was used for the study. Ages six to nine represent grades one to four, where both developmental differences and informal mathematical knowledge have an important impact on instruction and learning. The decision to use a group representing a range of ages, abilities, and approaches to dealing with number was made in order to consider results in terms of both diversity and developmental patterns, and to explore the flexibility and scope of the research method.

The choice to start out with a large group and then reduce it to a smaller representative group was made for purposes of manageability. Working first with a larger group provided the opportunity to choose a diverse range of personal approaches to number situations.

The first interview involved a Doubling Task, and was completed with all 29 students. Based on this interview, 16 students were selected for the final group. This group was made up of seven boys and nine girls between the ages of six and eight. Each of these students showed a willingness and capacity to explain personal strategies in ways that corresponded to non-verbal behaviour over the course of the interview. This correspondence of student explanations of thinking with actions such as mouthed counting, use of materials or fingers, and eye and body movements, helped to clarify student approaches to the tasks. Students who were best able to convey their thinking were chosen ahead of those whose explanations were inconsistent with their actions. A further selection criterion involved the issue of diversity. Some students approached the task in much the same ways as others, so some thinning of these clusters was done to ensure a diverse range of approaches and developmental levels across the group.

A final selection decision involved the question of whether to include nine-year-olds in the study. Each of the five nine-year-olds (four from one class and one from another) used the same approach to every example in the doubling task. Their first responses were based on immediate recall of combinations, and as the difficulty increased, they utilized a visualized image of the addition algorithm and computed mentally. I attempted to question these students about the meaning of their procedures and possible alternatives to their algorithmic approach, but was only successful with two students. Due to the lack of diversity in the approaches used by this small group, especially as compared to the flexibility, diversity, and ingenuity apparent with the younger age groups, nine year olds were not used for the study. Characteristics of the main study participants are provided in Table 2.

Table 2. Characteristics of study participants

Name	Grade	Age in Years-Months	Gender	Classroom	School
Adam	3	8-11	B	C	1
Brian	3	8-7	B	C	1
Laura	3	8-6	G	C	1
Candace	3	8-4	G	C	1
Sandy	3	8-1	G	C	1
Chris	2	7-11	B	B	2
Bevan	2	7-10	B	C	1
Nina	2	7-8	G	B	2
Wesley	2	7-5	B	B	2
Bahareh	2	7-4	G	B	2
Samantha	2	7-3	G	B	2
Juliana	1	6-8	G	A	1
Nicole	1	6-4	G	B	2
Cliff	1	6-4	B	A	1
Sam	1	6-3	B	A	1
Maré	1	6-1	G	A	1

III. The Setting

Over a six-week period starting in late January 1994, each student participated in four individual interviews, each of roughly 20 minutes duration. Interviews were conducted in unoccupied offices or workrooms adjacent to each of the classes. Each interview was videotaped. Teachers and administrators were most cooperative in

coordinating these interview times, and students were enthusiastic about taking part.

The parent and teacher interviews were conducted for the most part in the same rooms as the student interviews, however several parents requested that their interviews be conducted in their homes so that both parents could be present and/or to facilitate their childcare arrangements. All parent interviews were audiotaped.

IV. The Student Interviews

Interviews were conducted using the dynamic interview format described in Chapter Three. To ensure a comfortable context for students as they were encouraged to work beyond their comfort zone, extra time in the first interview was spent developing rapport with each student. Care was taken to ensure they understood that the first examples would be easy for them, that at some point they would need to "figure out" answers, and that eventually the examples would be too hard to complete.

In the initial interview, I explained to each student that I was also a student and that I was learning about how children think and work with number ideas. I asked if they would be willing to help me understand their thinking, and asked if I could videotape our interviews to help me remember what they said and did. This approach to the interviews gave the students some responsibility for and stake in how each interview went, and I think served to somewhat equalize the power relationship inherent in teacher-student interactions. The students continually surprised me with their willingness and capacity to explain or clarify their thinking within this context.

Another emphasis that affected the interview context was a focus on attending and responding to the whole child (e.g., Bevan's black eye, Brian's new hockey shirt, Juliana's ballet recital) as appropriate. This often meant taking time at the start or in the midst of each interview to follow the child's lead, leaving the tasks for a time before easing back into them.

Sometimes these diversions were needed for a break, or to connect with a child who was having trouble focusing. Other times they were a student's means of diverting

attention from a difficult problem. On some occasions these diversions provided further insights into the child's number sense. For example, at one point Sam, stumped by a sharing problem, started up a conversation about his dinosaur book collection. He told me that he had eleven books and if he had two more he would have a complete binder of 13 books. Furthermore, the full set included two binders, each with 13 books, 26 books in all. His command of the number relationships involved in this highly personal and motivating collection was impressive given his shaky grasp of lower number combinations. His verbal comments provided me with a window into his potential range for working with part-whole relationships. Overall, this flexible and responsive approach to the interviews provided useful insights, set a more relaxed atmosphere, encouraged children's expression, and made the interviews a pleasure to conduct.

V. The Student Tasks

The first four interviews involved verbal problem solving tasks presented in a series of examples using the dynamic interview format. These dealt with everyday problem solving contexts familiar to children: doubling groups, finding missing parts, sharing groups fairly, and using money. Each of these tasks involved the use of interesting objects to introduce and support the problem contexts and to engage the students. As described in the pilot study description, the use of familiar and engaging contexts for the tasks was designed to attempt to access student's informal number knowledge and personally constructed strategies.

The Doubling Task

For the Doubling Task, students predicted and extended a doubling twos pattern (1, 2, 4, 8, 16....) using a calculator to check their predictions. This task was included to explore each student's store of known facts, the limits of his or her capacity to mentally represent and manipulate quantities, and his or her strategies for mental addition with numbers of increasing size. A sample Doubling Task format is presented in Appendix B.

Each student was introduced to what was meant by doubling, first through the use of verbal examples, supported by concrete models as necessary. Children were asked what they used to help themselves solve similar problems in class, then they were encouraged to use their preferred method to work each doubling example.

Both unifix cubes and place value blocks were available for use, but based on pilot study results, paper and pencil were not provided. A calculator was used to add interest to the doubling task, and to provide memory support for what was being doubled each time.

The calculator was made into a doubling "machine" by entering "2 , X", so that each time the equals button was pressed, the display would double. Students were asked to predict what the display would be each time, then to check by pressing the equals button. If students lost track of their thinking, the display was used to refocus them on what they were doubling.

With the exception of the first few examples, the dynamic nature of the task protocol meant that not all students completed the same set of doubling items. The sequence of doubling two was used as a starting point, and depending on each child's performance, follow-up examples of doubling varied. For young children, this usually meant exploring a limited number range, while for older students, high frequency multi-digit combinations were included such as doubling 25 and 50, along with doubling multiples of tens and hundreds. Sometimes these combinations were verbally presented as questions, but often students themselves drew on such doubles combinations to construct responses. For instance, for doubling 256, some students used known doubles for 200, 50, 6, or 250 as elements in their solutions. The point in the doubling sequence at which children were unable or unwilling to continue provided a rough idea of what range of numbers made sense. The task also provided information on the number concepts and skills students made use of in order to double these numbers, and what aspects of their mathematical activity and the context supported or constrained their performance.

The Missing Parts Task

The Missing Part Task involved naming the hidden part of a given group, a task frequently used in studies of young children's thinking (e.g., Steffe & Cobb, 1988). The purpose was to explore children's understanding of part/whole relationships, their grasp of cardinal number and the inclusion relation, and their personal strategies for applying these concepts in problem solving situations. A sample of the Missing Parts Task format is presented in Appendix C.

The Missing Part Task was introduced by asking each student to make a set of eight, eleven, or fifteen cubes. The size of this starting set depended on performance on the first interview, and subsequent group sizes were adjusted as necessary. All the cubes were then hidden under a small box, and the child was asked how many were hidden, as a means of ensuring the starting set was clearly established. Some of the cubes were then brought out from under the box, and the child was asked how many were still hidden.

Once the task was understood by each student, it was adjusted to suit individual differences as noted from the first interview. Some students worked initially with only numeral cards to represent the whole and the given part, and a box to represent the missing subset. Many of these students gradually switched to using materials to support their reasoning processes as the number size increased. For others, successful performance seemed to depend on the use of materials to model one or both parts. Regardless of the presentation of the task, the issue of how to determine the missing part was the same. It required that the student create some sort of representation of the missing part. Both the diversity of these representations, and the ingenuity of the strategies children used to generate them, were illuminating.

The Sharing Task

The Sharing Task involved determining what would constitute a fair way to share different numbers of counters and real objects among two, three, and four people. The materials included blocks and pennies, along with bagged sets of sticks, spoons, doilies, candles, and balloons. The bagged sets were used to explore which students were capable of mentally manipulating the quantities without the need for a direct model. If students found it necessary to manipulate materials to reason out these bagged examples, blocks were used to represent the contents of each bag.

The Sharing Task was included to explore children's capacity to work with equal parts and remainders, to connect and apply known relationships in a real world context, to manipulate quantities mentally, and to generate and apply problem solving strategies for grouping situations. A sample of the Sharing Task format is presented in Appendix D.

The Money Tasks

The money interview involved four increasingly complex aspects of money use. The task of naming real coins and their values was included first to determine which coins, money relationships and vocabulary were familiar to students. This quick check was followed up by the task of counting up sets of coins that gradually increased in complexity and value. Thirdly, students acted like shoppers, putting out coins to show how to pay for priced toys of increasing value. Lastly, for those who were able, students acted like shopkeepers and made change for different items, given different and increasing amounts. The materials for this task included real coins, small toys and numeral card price tags. This task provided a window into children's capacity to use multi-unit relationships as they apply to coin values, to count-on, to utilize multiple counting patterns in context, to organize coins and values to facilitate problem solving, and for some of the more able students, to apply part-whole relationships in the context of making change. A sample of the Money Interview format is presented in Appendix E.

Though each interview was mainly concerned with completing as much of a task sequence as possible, there were informal opportunities for discussion and to explore the formal skills children had acquired in class or at home. This provided a means of accessing student thinking related to familiar school mathematics exercises and students' developing symbolic abilities as a means of further enhancing my understanding of each child's personal number domain. Due to the highly informal nature of these discussions, no results are reported, however, some specific details are occasionally used to enhance the description and interpretation of children's mathematical activity in the four interviews.

VI. The Parent and Teacher Interviews

Upon completion of the student interviews, informal individual interviews were conducted with the teachers and parents of the students. Most of these interviews took place in the schools, while some of the parent interviews took place in the children's homes at the request of parents. Sample parent and teacher interview formats are presented in Appendix F. The purpose of the parent interviews was primarily to collect anecdotal information on each child's disposition towards mathematics and interest in number related topics in their home environments, as well as to explore possible influences on the child's general mathematical

development. Inclusion of this anecdotal information from parents was used to enrich the descriptions of children's mathematical activity as reported in Chapters Five and Six. A second purpose of the parent interview was to report on results of their child's interviews and to answer questions related to the study. All parents were provided the opportunity to discuss the interview results and to view the videotapes of their child.

The teacher interviews also had a dual purpose. The first was to collect anecdotal information on each child's performance in class, including mathematics-related interests, mathematical disposition and general learning characteristics. The second purpose of the teacher interviews was to view excerpts of the videotapes and to discuss each child's performance in order to compare results of the individual interviews with classroom performance. Where appropriate, references to teacher comments are included in Chapters Five and Six and implications are discussed in Chapters Seven and Eight.

VII. The Data Collection

During the student interviews, data was collected through the use of videotape, supported by some notes written immediately after each interview. To avoid taking notes during each interview, occasionally interviewer reflections were recorded directly onto the videotape in order to highlight certain aspects of interest. Interviewer notes and the videotapes of each student interview comprise the set of data records for student interviews.

The parent and teacher interviews were audiotaped, interviewer notes were made from these tapes, and an interview form was filled out over the course of the interview. Interview notes and completed forms comprise the data from these interviews. A sample of the interview form is included in Appendix F.

VIII. The Data Analysis

One purpose of the study was to explore the unique ways in which young children made sense of number situations, what they accessed or drew on to support their efforts, and what served to constrain or limit their performance. The tasks used in the study were each intended to contribute information to an overall portrait of a

child's current capacity to make sense of whole number situations generally. Another purpose of the study was to explore the range of ways young children approach each of the number tasks used in the study. This was done in an effort to explicate developmental patterns to children's sense-making activity. Each child's performance was then considered in developmental terms relative to the patterns that emerged. Consequently, the data were analyzed from two perspectives: analysis of performance on each task across students; and analysis of each student's mathematical activity across tasks.

Analysis of Task Results Across Students

Analysis of each task involved the consideration of all 16 student interviews for that task. A qualitative approach to the data provided the primary focus, however in some instances, descriptive statistics were applied to highlight patterns of performance across the group.

The data for each task were analyzed focusing on the following aspects:

- the range of examples attempted by students;
- the nature of the strategies used to solve each example;
- the resources students drew upon to support their sense-making activity;
- the nature of the task variations and interviewer interventions used for each task, and how these constrained or supported performance.

This analysis generated classification schemes for each task which qualitatively interpreted and described progressively more elaborate and sophisticated mathematical activity. These analytical schemes for each task differed depending on the nature of each task, but served to organize how the illustrative examples were chosen for each task to present in Chapter Five. The Money Task required a further analysis involving scoring schemes, as described in Chapter Five.

Video records of each interview were not transcribed in their entirety due to the volume of data. Due to the many aspects of non-verbal performance that were important to the interpretation of data, detailed transcripts of interviews were completed only as required for examples. Instead, task records were used to summarize and organize the data of interest. The range of examples attempted by each child was recorded first onto a separate record sheet for each task. The

strategies used, resources accessed, and accuracy levels were then applied to the range. Final strategy categories were then organized from these records, and videotapes were once again analyzed in terms of this categorization. Task Result Tables 4, 5, and 6 in Chapter Five were generated from this second level of analysis.

A third and fourth viewing of the videotapes was used to recheck strategy categorizations and to compile individual records of more detailed task performance. From these, lists of the factors that enhanced and constrained children's sense-making across items within a task sequence were collated. These are reported in the results for each task in Chapter Five. In the rechecking of strategy designations, strategy labels were reassigned to fourteen examples on the first check and five examples on the second check. Appendix G includes sample transcripts for each task, along with a description of how strategy categorizations were determined. These charts, the analytical schemes, a discussion of results related to strategy use for each task, and an analysis of supports and constraints for each separate task are provided in Chapter Five.

Analysis of Student Results Across Tasks

The second part of the data analysis considered each child's overall performance across the four number contexts. Video records were collated for each child across all the tasks. The resulting data records for each child included a listing of examples completed, strategies used, resources accessed to support performance, and characteristics which served to support and constrain the child's mathematical activity. Across all tasks, this pooled information provided a general idea of the nature and extent of each individual student's personal number domain: the child's conceptual understandings, procedural skills, and functional capabilities; the affective characteristics that appeared to influence performance; and the contextual features that supported or constrained mathematical activity. See Appendix H for an example of the data collation record form. Patterns across tasks were analyzed from these data records. These patterns are reported in Chapter Six for the group as a whole, for each age group, and for selected individual examples.

Chapter 5

Task Results

I. Analysis of Strategy Use

In this investigation of children's mathematical activity, the methods or strategies children used to generate their responses to each item were used as the initial basis for analysis. Subsequently, a more finely grained approach was taken to analyze the cognitive, affective, and contextual aspects which enhanced or constrained children's construction of meaning for the different tasks used in the study. Taken together these methods provided a comprehensive view of children's developing capacity to make sense of number.

In this study the term "strategy" is used to describe the action sequence or method used by children as a means of generating their responses to problem situations. The use of strategy choice as a basis for analysis is grounded in several assumptions. The first assumption is that multiple and inter-related factors shape children's choice and use of methods or strategies. It is assumed that when and how children utilize strategies provides some indication of cognitive competence, including conceptual, procedural, and functional considerations. It is assumed that strategy use is not necessarily reflective of the fullest extent of a child's conceptual, procedural, or functional repertoire; but by observing a wide range of items, it is possible to explicate the nature and extent of many of these competencies. Furthermore, it is assumed that strategy choice and use also reflects aspects of affective influences on children's mathematical behaviour. And finally, strategy choice and use is believed to be affected by the context of the task, the situational context, and the child's personal number context.

The initial step in the analysis involved looking at which items in each task series students attempted, and what method or strategy students used to make sense of each item. Several steps were taken in the development of a categorization scheme for strategy use. Prior to the data collection, a search of the literature provided some indication of what sorts of strategies children between six and eight might use for tasks involving additive structures (e.g., Carpenter and Moser, 1984; Siegler, 1989). Considerable research has focused on children's counting and the universality of counting as a developmentally appropriate approach to problem

solving is widely recognized (e.g., Fuson, 1988). Other studies have focused on how early counting proficiency develops into place value understanding (Cobb & Wheatley, 1988; Kamii, 1989). Fuson (1992) and others have interpreted this transition in terms of a shift from a unitary conceptual framework to a multi-unit conceptual framework.

These studies provided an initial frame for guiding the investigation of children's strategy use. Following the collection of data, analysis of the videotapes indicated that a more elaborate strategy framework was required to accommodate the data.

This was in part due to trying to describe a data base which included many examples of non-counting, when the literature directed little attention to non-counting approaches. Also, strategy categories described in the literature failed to adequately differentiate between strategies involving recognition or recall of prior knowledge and strategies involving derived or constructed responses, often lumping these two approaches into a category of "derived or known facts" (e.g., Carpenter & Moser, 1984).

Initially a separate set of categories was developed to describe the data for each task. These different categorizations were subsequently reworked to coordinate categories across tasks in order to provide the analysis of tasks with more internal consistency. This meant giving up some of the detail and specificity of performance on each task, but provided a better means of focusing on the underlying conceptual schemes and strategic behaviours that formed the basis for children's mathematical activity across contexts.

Strategy choice categories were assigned according to how students went about retrieving, constructing, or generating their solutions. The divisions between these strategy categories were created for purposes of clarity only. Although most of the examples of strategy use were relatively easy to categorize, in some cases it was necessary to decide between two or three options and choose the category that seemed to best describe the process. The following criteria were used to guide the categorization of strategy usage.

Strategy Categorization Scheme

The strategy categorization scheme included five categories. These categories are:

1. Recognition strategies
2. Reasoning strategies
3. Counting strategies
4. Algorithm strategies
5. Estimation strategies

Sometimes these strategies were used mentally, with students relying on the internalized construction of meaning and showing no signs of dependence on external perceptual or concrete support. Other times children used the same strategies but in conjunction with reliance on external supports such as concrete models, and with observable perceptual support. This representational aspect to children's mathematical activity is considered to have important implications for considering patterns to strategy use and for understanding children's thinking about number. For this reason, strategies are further described by whether they were used at a more internalized or externalized level. The following descriptions of characteristics used to assign strategy categories are based on internalized use of the strategy, with elaborations to illustrate what that strategy looked like when utilized at externalized levels.

Recognition (X)

- an immediate response of a recall nature, or, a quick response after a brief pause or memory search
- no observable or reported constructive thinking process
- justification based on factual or declarative knowledge e.g. "I just know that $8+8=16$."
- response presented as a fact (versus a guess or a question) or as an insight or "aha" moment
- some Recognition (X) responses categorized as external Recognition (X') appeared to be stimulated by perceptual information (e.g., seeing an array of 8 blocks and immediately recognizing two fours)

Reasoning (R)

- a constructed response utilizing more than one step or procedure (e.g. estimate-check-revise or decompose-recompose strategies)
- drawing on at least one known "chunk" as an element of the constructed response (e.g. drawing on known facts)
- demonstrating the capacity to mentally store, access, and utilize multi-unit groupings

- internal Reasoning involved mentally working with the three characteristics: multiple steps, prior knowledge, and multi-unit relationships
- external Reasoning (R') involved the perceptual support of a visual model (e.g., recognizing a 3X4 array of blocks as 3 groups of 4, or 4 groups of 3), or the use of materials as an integral part of the reasoning process (e.g., using place value blocks to model, then decompose and recompose tens and ones for addition)

Counting (C)

- an incrementing or decrementing procedure involving counting by ones in a sequential and algorithmic fashion
- no involvement of grouping processes as described for Reasoning (R)
- often accompanied by rhythmic movements, verbal counting chains, unitary use of fingers or blocks, etc.
- described with a counting rationale (e.g., Maré saying, "In mind I went nine, ten, eleven.")
- involving either counting-all (directly modelling a problem and enumerating parts or whole by counting from one to the total, e.g., model $4+3$, then count up from one to seven); or
- involving counting-on or back with tally (mentally representing one aspect of a problem, then counting-on or back from that point, tallying the double count, e.g., for $4+3$ starting with 4 and counting-on three times as in 5, 6, 7).
- external Counting involved the active use of unitary models when such use appeared integral to the counting process, rather than incidental or habitual

Algorithm (A)

- a strategy involving the mental visualization of a printed algorithm for solution
- described with directional, procedural language and worked from right to left or ones to tens as in a printed algorithm
- (by definition, this algorithm strategy was used only in an internalized manner)

Estimation (E)

- a relatively immediate, one-step response presented as a question or possibility rather than as a known fact or recalled information
- no reasoning process apparent or offered as a rationale
- often appearing to be based on a global, intuitive sense of what might be reasonable in that situation
- the estimate presented as an end in itself rather than followed by an attempt to determine the accurate answer
- external Estimation (E') involved the input of perceptual information (e.g., looking at a group of 18 sticks to share between two, saying "I'd say we'd each get about ten?"), or physically moving materials to make an estimate

based on global configuration (e.g., moving sticks into three rough piles, saying "I'd say we'd each get about this much?")

Strategy categories illustrate three different types of approaches to generating responses to number situations, as illustrated in Table 3.

Table 3. Children's approaches to items

Approach	Strategies
One-Step Approaches	Recognition Strategies Estimation Strategies
Procedural Approaches	Counting Strategies Algorithm Strategies
Multi-Step Approaches	Reasoning Strategies

The first type of approach involved relatively immediate, one-step responses, as found in Recognition and Estimation strategies. Recognition responses tended to occur at the easiest levels of each task where students "just knew" the answers. While Recognition responses were stated as facts, Estimation responses were presented as questions or suppositions, and tended to occur more towards the limits of children's understanding, though not exclusively so.

A second approach involved the application of a procedural computation method to generate an answer. Both Counting and Algorithm strategies were included in this type of approach. Both offered a rule-bound, linear sequence of predetermined steps that could be widely applied. Both have the potential to become highly routinized, and to be "mastered" without necessarily grasping the underlying conceptual basis of the procedures. For students using unitary Counting, the nature of their counting activity served to clarify the underlying conceptual basis of their activity (Steffe et al., 1983). However, students applying an algorithm to mentally compute their responses were always asked if they could use any other method. This was done in order to explore the meanings that underpinned their use of the procedures.

The final approach, represented by the category of Reasoning, included a rich array of strategies, all of which involved multiple steps of reasoning through applying known concepts and skills in unique and meaningful ways. Any or all of the other strategy categories might be involved in a solution strategy categorized as Reasoning, but always in conjunction with multiple steps or strategies. The use of Reasoning strategies indicated that children were constructing meaning based on their personal knowledge rather than applying a rule-based procedure or simply recognizing that they knew the answer.

An attempt was made to shape each child's set of examples to reflect different levels of performance. First, children were presented with examples that would be easy for them, resulting predominantly in the use of Recognition responses. Second, as the difficulty increased, students moved beyond what they knew, and they found it necessary to construct their own meaning. What characterized children's mathematical activity in that "construction zone" was of primary interest. The third area of interest was the nature of students' activity as the difficulty of the items in each task sequence moved beyond the limits of their capacity to make sense. Children's strategy choices were a focus for analysis across all items in a task sequence.

II. Analysis of Constraints and Supports to Sense-Making

The second focus of the analysis of the data by task was to examine the reasons why students were or were not able to continue as the difficulty level increased, and what eventually curtailed success. These constraints and supports relate to the conceptual, procedural, functional and affective characteristics required to make sense of the tasks, and the impact of the task and item context on performance.

Two general findings were apparent in analyzing these different factors. First, any one factor had the capacity to enhance or to constrain performance depending on the situation. For example, the characteristic of perseverance served to enhance the mathematical activity of some children, while a lack of perseverance appeared to constrain the performance of others. Similarly, availability of reliable and extensive counting chains acted as a support to performance, while weak counting chains constrained performance. Because of this, each factor is discussed in terms of its capacity to constrain or enhance children's mathematical activity.

A second finding that emerged from analysis was that some factors characterized performance across tasks while others were specific to particular contexts. Affective and functional aspects of a child's performance appeared to play a consistent role across tasks. For instance, children with a strong capacity to self-monitor their thinking used that ability to support performance on all tasks. In contrast, conceptual and procedural aspects of a child's performance appeared to be more specific to individual task contexts. For example, the conceptual ability to keep a mental tally was an important aspect of performance in the Missing Parts Task, but was not required to the same extent on other tasks so was not seen to constrain or support children's sense-making activity in those situations. Using multiple contexts in this investigation made it possible to observe these variations.

This second pattern illustrates the complexity and inter-connected nature of what shapes children's mathematical activity. The task context had an obvious impact on performance when it came to changes in number size and representational level. Similarly, the context of each task brought different conceptual and procedural abilities to the fore. Task context also appeared to influence whether children approached the tasks in unitary or multi-unit ways, in additive or subtractive ways, or in linear or global ways. However, it appeared that some children did not have access to a global view of number, or a multi-unit conceptual framework, so were unable to apply such thinking. While context may have stimulated the use of such thinking with some children, it did not do so with others. In addition, the four different task contexts did not appear to shape or affect the functional and affective characteristics of children's performance to any great extent. This observation of the difficulty of trying to understand and account for children's mathematical activity illustrates why taking any one perspective cannot do justice to the richness and complexity of children's thinking, and why multiple perspectives are used in this study.

III. Doubling Task Results

The Doubling Task was the first of the four tasks used in the study. Results for each task are reported in the order in which the interviews occurred: Doubling, Missing Parts, Sharing, and Money. Each set of task results presents a description of the task,

an analysis of strategy use, an analysis of factors that supported or constrained children's capacity to make sense of the task, and a summary.

The Doubling Task interview explored the nature of children's ways of doubling numbers, and the number range within which doubling held meaning for them. (See the description on pages 69-70.) The doubling protocol, which was extensively piloted beforehand, was included in the study because of its potential for tapping into the full range of aspects of interest to this investigation. The first of these aspects was children's range of number familiarity. The increasing number size of the calculator sequence of doubles provided a picture of the extent of each child's "number comfort zone". The capacity to mentally access combinations, the increasing use of external supports to thinking, and the limit to the items the child was able and willing to deal with, provided an indication of each student's potential range of understanding for whole numbers. In addition, affective considerations clearly played an integral role in children's willingness to continue in the face of difficulty.

The numbers involved in the twos' doubling sequence quickly moved beyond what could be handled with a Recognition strategy, thus tapping into the child's strategy repertoire. Having a series of doubling items provided some indication of the range of strategies available to the child, as well as the degree of flexibility to children's strategy choices. The Doubling Task provided a familiar context for exploring children's understanding of the additive composition of number, but was not directly connected to typical school mathematics tasks. The Doubling Task highlighted the child's capacity to construct responses, in particular the ability to decompose and recompose numbers, to apply place value understandings, and to creatively or appropriately draw on known number relationships to derive answers. This task also identified students who readily abandoned their own sense-making efforts in favour of algorithmic approaches.

Since the concept of doubling remained constant while the cognitive demands of memory and strategic monitoring increased with each item, the Doubling Task provided information on how students helped themselves keep track of their own thinking. The availability and use of materials provided a window into how students organized their thinking and what sensory information best supported that thinking. In particular, the use of place value blocks provided information on the

extent to which students conceptualized the number situation in unitary or multi-unit terms. Finally, the Doubling Task was a game-like, motivating context for our first interview and served to establish a positive climate for future interviews. It was structured to ensure success at the start, accommodated multiple solution strategies, provided a non-threatening calculator check, and was respectful of developmental and other differences.

In addition to these reasons for including the Doubling Tasks in the study, there were two important reasons doubling was used as the first of the four interviews. First of all, it provided useful information for selecting the final sample of 16 students from the original group of 29 students recommended by teachers. Besides the range of gender, age and grade wanted, criteria for the final choice included finding students to represent a range of approaches to dealing with number, and finding students who were willing and able to explain their thinking. The Doubling Task provided a quick and efficient means of gathering information from which to evaluate these two criteria.

The second reason for choosing Doubling as the initial interview task was its usefulness in providing an overview of key aspects of performance. Information from this interview was used as a reference to shape the questioning and direction of subsequent interviews. Of the four interviews, the Doubling Task provided the most complete information on each students' grasp of the additive composition of whole number.

Strategy Use on the Doubling Task

Table 4 shows the strategy choices for any items students attempted in the doubling twos' sequence. In many cases, students used a second approach to an item, and both were noted. This occurred when students attempted and then abandoned an approach that proved too cumbersome or complex. Sometimes a strategy choice resulted in an incorrect answer, in which case the child was encouraged to try an alternative strategy. And sometimes students just wanted to check their initial result. Thus, in some cases there are two strategies listed for one item.

Table 4. Strategies used on Doubling Items

	Items from the Doubling Task Sequence										
	2	4	8	16	32	64	128	256	512	1024	2048
Adam	X	X	X	A	A	R, R'	A, R'	A, R'			
Brian	X	X	X	C	R	R	R, R'	R, R'	R	R	R
Laura	X	X	X	R	R	R, R'	R, R'	R'	R	R	R
Candace	X	X	X	A, C'	A, C'	A, R'	R'				
Sandy	X	X	X	A	A	A, R'					
Chris	X	X	X	X, C'	R'	R'	R'				
Bevan	X	X	X	R, R'	A, R	R	R	E, R	R	R	R
Nina	X	C	C	C'							
Wesley	X	X	X	X	C, A	R'	E, R'				
Bahareh	X	X	X	E, C'	C', R'	R'					
Samantha	X	X	C'	C', R'	R'						
Juliana	X	X	X, C	C', R'	R'						
Nicole	C'	C'	C'	C'	C						
Cliff	X	X	C'	C'							
Sam	X	X	X, C'	C'							
Maré	X	X	X	C'							

X= Recognition

R= Reasoning

C= Counting

A= Algorithm

E= Estimation

' denotes use of that strategy in conjunction with the use of perceptual information or concrete materials (see the description of externalized strategy use on page 78)

Table 4 illustrates the range of items completed and strategies used in the Doubling Task sequence. Items early in a child's sequence were usually completed independently, while items towards the end of each child's sequence were usually completed with interviewer support. For the youngest students, strategies were mostly limited to Recognition of known combinations and unitary Counting. The older and more competent students drew on a range of strategies across the series of items, with abstract Reasoning characterizing the most proficient profiles. Though age and experience in school had an influence on performance, clearly many other factors came into play in shaping children's mathematical activity on this task.

In some instances, older students used a mental image of the addition algorithm to double numbers. In these cases, after the initial algorithm response was provided, an attempt was made to explore the child's underlying understanding. Students were

encouraged to model their thinking and to explain how they came up with their answer. This often elicited a further strategy.

Given the dynamic interview format where interviewer cues were provided and where there were variations in each child's set of items as well as in the presentation of those items, quantitative comparisons of accuracy or number of items completed are not appropriate. However a qualitative interpretation of the data reveals interesting patterns to children's mathematical activity as illustrated in the following descriptions.

Children used a range of strategies to deal with the items during the Doubling interview. Strategy choice changed as the difficulty increased, moving from automatic responses to constructed responses, and moving from internalized, abstract processes to more externalized, sensory-dependent responses. Strategy choice also appeared to be related to the mathematical disposition of students, with the less confident children relying extensively on unitary counting processes or algorithmic thinking, and the more confident ones relying on their capacity to make use of known relationships as tools for reasoning.

Recognition. Immediate Recognition of a doubled combination was used by all students but one, at least for some items involving single digits. In all, Recognition was used on 43 of the 128 items, and was the preferred strategy for single digit combinations. Increasingly extensive use of Recognition for doubles combinations was related to age and to overall competence on the Doubling Task as measured by the range of items successfully completed and the nature of the overall performance. Though Recognition as a one-step response was rarely used beyond the basic fact range, children continued to use known facts in conjunction with multi-step Reasoning strategies.

Reasoning. There were 46 examples of Reasoning strategies that involved multiple steps and utilized multi-unit thinking by drawing on known facts or relationships as elements in the construction of meaning. Twenty-three of these involved internal Reasoning such as mentally decomposing the 16s into two tens and two sixes, then combining the $20+12$. This type of constructed response strategy was increasingly prevalent with the higher combinations, and was utilized frequently by the most proficient and confident students.

The other 23 examples fell into the external Reasoning category. In these items, students combined unitary and multi-unit counting as they worked out doubles combinations by using place value models. For 16 doubled, three students built a place value model of two groups of 16, then counted by tens then ones, 10, 20, 21, 22, 23,...,32. These three items involved a multi-unit count-all strategy where a direct model was constructed and enumerated, but chunks of ten were grouped and accounted for as composites. The use of multi-units in a grouping relationship placed this approach into the Reasoning category, rather than in the Counting category. Items of concrete Reasoning illustrate what is proposed to be a critical connecting level in moving from unitary Counting to multi-unit Reasoning with composite units.

Counting. Twelve of the 16 students at some point utilized at least one of a variety of unitary Counting strategies, accounting for 24 of the 128 items. Six of these items involved internalized, mental counting; the remaining 18 items involved modelling with blocks or fingers. The nature of finger use provided useful information on children's understanding and use of number in this study. Some finger use was conscious, explicit, and clearly essential to the counting process. Other finger use was incidental in nature and appeared to be a habitual rather than integral part of the counting process. In addition to these levels of finger use, highly personal variations of finger use were observed. The range of use observed across tasks suggested that the use of fingers plays a role in the internalization of children's counting processes, providing a bridge between external use of materials to represent counting units and abstract, internalized counting procedures.

More important than the representational level of counting processes was the question of whether children used a unitary count-all approach, or counted on to the starting number. Five of the youngest children relied on counting-all, which involved constructing a unitary model of both sets (usually with counters) and counting from one to the total. These children, even when encouraged to switch to multi-unit counting as the number size became unwieldy, were unable to make use of the grouped place value materials, and continued to try to count totals by ones. For example, in the item of 16 doubled, five students constructed two models of 16 ones, then counted from one to 32.

Seven other students who used unitary Counting showed the capacity to count-on from the first set rather than from one, indicating a unitary counting strategy of increasing abstraction and efficiency. For example, two students built both sets of 16, but started at 16 and counted-on from 16 to 32. One student started by saying 16, then successfully counted-on, using his fingers to tally on 16 more to reach 32. One other six-year old appeared to be moving towards thinking of ten as a "chunk" by twice modelling 16 using a ten rod and six ones. For her first attempt, she counted up the total by ones starting from one, but on her second attempt treated the tens rods as a composite of ten, rather than as ten ones. She went on to work out 32 doubled by using place value blocks and multi-unit Reasoning. All seven of the students who used the more abstract count-on strategy with early examples were able to switch and use place value blocks with a concrete Reasoning strategy as the number size increased.

Mental Algorithm. There were 12 examples, mainly by three eight-year olds, where a mental computation strategy was applied and students pictured what the doubled items would look like in addition algorithm form. These students described their thinking working from right to left, from ones to tens, with no reference to place value, as opposed to mental Reasoning items which were always worked from left to right and reflected place value thinking. For example, Sandy, in doubling 16 said, "Sixteen doubled is, well, like six and six is twelve. And I know I go from that side (ones) not that side (tens). So six and six is twelve, take over one then make three, then over here there is a two, so that would make thirty-two." Sandy was unable to explain what the procedure meant in this example, nor did she offer an alternative strategy.

All students who used a mental algorithm were asked to show another way to check if their answers were correct. Some students such as Sandy, above, did not appear to connect their use of the standard addition algorithm with the underlying place value meanings, or with alternative strategies. Others, such as Candace, followed up with a counting strategy. For 16 doubled, Candace made an initial response of 12, successfully adding the two sixes, but ignoring or losing the two tens. Agreeing that 12 was not reasonable, she switched to unitary counting to find 32. Still others followed up their use of the addition algorithm by modelling the items with place value blocks, connecting multi-unit values with their answers. Repeated reliance on the addition algorithm was not related to proficient performance overall. Algorithm

strategies, which tended to be used by the less confident students, usually involved unitary rather than multi-unit thinking, and appeared to supplant the use of personally meaningful reasoned approaches.

Estimation. There were only three examples of Estimation, provided by students who, over the course of all of the interviews, relied relatively frequently on that approach. Bahareh asked if the sum of 16 doubled was 24, then used a back-up strategy of counting-on with fingers to check her accuracy. Bevan made an initial, impulsive estimate of 412 before using mental Reasoning to work out 256 doubled. Wesley also made an estimate of 128 doubled before using materials to work out the actual sum. Others used estimates in conjunction with other steps in a guess-and-test approach, however all such instances were classified as Reasoning.

Factors Affecting Sense-Making on the DoublesTask

Conceptual, procedural, functional, affective, and contextual factors had the potential to either enhance or constrain children's mathematical activity on the Doubling Task. The following section provides a compilation of the factors (or aspects) that served to enable children to continue in the doubling sequence or acted as constraints to performance. In this analysis certain aspects of children's thinking are brought to the foreground for discussion purposes, then returned to the background in order to focus on another aspect. In no way should this focus suggest that any one factor is seen to stand alone in a discrete kind of way. Rather, the interplay of the different factors influencing performance is emphasized.

Conceptual aspects

Conceptual understandings were illustrated through the ways in which students worked out their answers, through their answers to questions, and through the rationales they provided to explain their thinking. Once students were clear from the introductory items of what was meant by doubling, all students were able to apply the doubling concept throughout the interview despite the number size becoming overwhelming. In cases where students were unable to work with the numbers involved in the doubling sequence, all were able to generalize and describe how they would go about finding the doubled quantities through modelling. Thus the conceptualizations involving the relation of the two equal parts to the sum or whole held across difficulty levels. However, the nature of students' conceptualizations differed considerably, and had a significant impact on how

students approached the task. Three conceptual abilities were particularly critical to performance on the Doubling Task. These factors are inter-related, but are described separately for purposes of clarity.

1. Multi-unit reasoning. The first ability involved utilizing multi-units or chunks in reasoning processes as opposed to relying on unitary, linear counting patterns. The ability to use composite units or multi-units made it possible for students to work with larger numbers. This makes sense since the working memory demands of accounting for each and every unit preclude the use of numbers beyond 15 or 20. The ability to hold and manipulate chunks, or composite units – in particular tens, hundreds, and thousands – enabled students to manage the increasing number size of the items.

2. Decompose/recompose. The second related conceptual factor involved the ability to decompose and recompose multi-digit numbers in appropriate ways to facilitate constructive processes. An intuitive grasp of the additive composition of number, an understanding of the need to group like units, and the ability to break numbers up into suitable place value groupings enabled some students to double numbers well into the hundreds and thousands.

3. Mental representation. The third related conceptual capacity was the ability to mentally manipulate number and number relationships. This capacity was indicated by the child's choice and use of strategies, and by the extent to which the child relied on materials rather than on internalized Reasoning across the interview. All students except one began by immediately recognizing doubles combinations for as many of the items as possible. As the items increased in difficulty, six students drew on the use of fingers then on the use of materials as the number size went beyond the manageable range for finger use. All students except for three eventually resorted to the use of concrete materials to support their sense-making, the remaining three working entirely mentally using a mental decomposition/recomposition Reasoning strategy well into the thousands. These three, one aged seven and two aged eight, demonstrated consistently high performance across interviews. In other words, the capacity to mentally represent and manipulate number served to support children's mathematical activity, while the lack of that capacity was increasingly a constraint to pupil performance on the Doubling Task.

Procedural aspects

Though important, conceptual considerations were not exclusively responsible for constraining or enhancing student's potential to succeed. Access to procedural skills, and the appropriate invoking of those skills was important to successful performance on the Doubling Task. Reliable and extensive counting chains were critical elements for students who used Counting strategies. A solid store of known doubles combinations supported students' use of Recognition and Reasoning strategies. Whether or not students had been exposed to the standard addition algorithm affected whether that was a strategy option available to them. The availability of facts, counting patterns or algorithms did not mean they were necessarily drawn upon. For instance, from teacher interviews it was clear that eight students had considerable experience with the standard addition algorithm, yet only four chose to use it for mental computation. In any event, the availability of such supporting skills provided options for students and enhanced the flexibility of their strategy choices.

Functional aspects

Self-monitoring and functional factors played an important role in facilitating performance on the Doubling Task. A student's capacity to monitor and support his or her own cognitive processes often made the difference between successfully completing an item or losing track. The ways students accomplished this were most impressive. Some students talked to themselves throughout each item. Some appeared to summon up mental imagery to clarify their thinking. Some used fingers and materials as mental organizers and reminders the same way they would have used paper and pencil had they been available. And some students capitalized on the organizational cues provided by the interviewer.

Affective aspects

Affective factors played a role in supporting performance in a variety of ways. A positive attitude towards the interview, curiosity about the task, and confidence in one's capacity for success appeared to support students to push the limits of their personal number boundaries. Some students were reluctant to grapple with the more difficult items and needed encouragement to continue, while others had to be encouraged to stop. Some students were quick to say items were difficult, while others gamely attacked items at the outer limits of their range. These personal attitudes and beliefs were not context dependent, rather they characterized

performance across tasks. Their impact on performance cannot be directly measured, however there is no doubt that affective factors played an important role in performance.

Contextual aspects

The doubling concept and the calculator context did not pose a problem for any students after a few introductory examples. Since the presentation of doubling did not vary from the calculator context across the item sequence, no comment on the role of representation of the problem can be made. However, students translated the items into a representation they could manage, as described in the section on Functional Aspects.

Two characteristics of items served to constrain children's performance. The first was the increasing number size which at some point placed the item beyond the child's meaningful number range. Also, if the item involved regrouping, it posed more of a challenge for students. Doubling 32 was easier for some students than doubling 16 since no regrouping was required. Multiple regroupings placed some items out of reach even if they were within the child's range of meaning. For example, unless children recognized and used the benchmark of $250+250=500$, doubling 256 was considerably more difficult for some pupils than doubling 1024. These generalizations held whether students were working mentally or with place value blocks, and appeared to relate more to working memory and functional limitations than to anything else.

Summary

Performance on the Doubling Task was affected by conceptual, procedural, functional, affective, and contextual factors. As for all tasks, the child's age and years in school were clearly factors in proficient performance, due to increased opportunity to learn and to developmental considerations. Three conceptual abilities which supported performance appeared to be developmental in nature.

1. the ability to utilize a multi-unit conceptual framework;
2. the ability to decompose and recompose multi-unit numbers; and
3. the ability to mentally manipulate numbers and number relationships.

Two further characteristics of a more procedural nature which supported competent mathematical thinking on the Doubling Task, also appeared to be age and experience related:

4. access to a store of known number facts, patterns, and relationships; and
5. using a range of concepts and strategies to suit each item.

The remaining characteristics of competent performance on the Doubling Task were functional and affective characteristics that appeared to be unrelated to age:

6. flexibly changing strategies when faced with obstacles;
7. effectively monitoring processing;
8. the ability to know and use what was needed in order to make sense, whether it was fingers, blocks, dot patterns, thinking aloud, etc.; and
9. the inclination to evaluate the reasonableness of responses.
10. curiosity and perseverance to find the doubled sums;
11. confidence in one's sense-making capabilities; and
12. the inclination to make advantageous and creative use of the available number knowledge.

These characteristics of competence clearly showed that students' capacity to make sense of the doubling items wasn't so much determined by what students knew as how they went about using what they knew.

IV. Missing Parts Task Results

The Missing Parts interview explored the child's understanding of the relationship of parts to wholes. The task involved providing a starting set, hiding part of the set, and asking the child to determine how many were hidden. The task was introduced with real materials and followed up with item presentations adjusted to suit the student's ability to work symbolically. Younger students usually continued with the concrete presentation, while older students worked with numeral cards until they began to have difficulty. Then the presentation was made more salient again through the use of materials. This varied presentation appeared at times to influence certain strategy use as described in the specific strategy results. This task

proved to be quite challenging especially for the younger students, resulting in a limited set of items for many students.

The purpose of this task was to explore students' conceptual understanding of part-whole relationships and strategies for generating an unknown part given a whole and one part. It served to illustrate aspects of the child's grasp of cardinal number, the inclusion relation, and the relationship between addition and subtraction. It demonstrated the importance of the capacity to maintain a double count or count-on-with-tally, and the complexity of part-whole subtractive Reasoning as compared to the additive Reasoning required for the Doubling Task. These aspects of the Missing Parts Task provided new information about each child's thinking beyond the information generated by the first task.

Besides providing new information, the Missing Parts Task also provided a second context for looking at aspects of performance that were common to the Doubling Task. It provided a problem solving context for looking at the use of known facts and relationships as "chunks" in the Reasoning process, thus providing another window into the child's access to a multi-unit conceptual framework. It focused on the child's capacity to internally represent and manipulate number and number relationships, and the child's use of materials and external strategies to support this mental representation.

The Missing Parts Task provided another picture of each child's number construction zone. The increased complexity of thinking required by this task as compared to the Doubling Task meant less of a spread in number range was required to differentiate performance. Instead, the difficulty level of each item was affected more by the size of the missing part than by the size of the whole. Missing parts less than five were easier to find than parts greater than five, and students had particular difficulty where decade changes were involved. Similarly, familiar combinations of parts and wholes (such as eight in all, four given; or thirty in all, five given) were easier than less familiar combinations.

The Missing Parts sequence, from which each student's items were selected, was as follows:

whole: 8	given parts: 4, 6, 3
whole: 11	given parts: 8, 6, 2
whole: 15	given parts: 11, 7, 3
whole: 30	given parts: 5, 24, 17
whole: 42	given parts 22, 25, 13

The first given part was designed to be the easiest of the three. For instance, 4 is a familiar part of 8, and 22 as part of 42 encourages the use of chunks of ten. Some combinations of missing parts were designed to see if students would draw on previous items and make connections. For instance, in the item of 5 as a part of 30, some students who identified the missing part of 25 connected that to the next item of 24 as the given part of 30. They used the result of their previous item to help identify the missing part as one more than 5, or 6.

Due to the age range of the students and the dynamic nature of the interview protocol, students completed different sets of items. Seven and eight-year-old students (except for one) started with the second set of items after a brief introduction. How far students were able to continue varied.

Strategy Use on the Missing Parts Task

Table 5 summarizes children's strategy use on the Missing Parts Task including the range of items attempted, the frequency of strategy use, and patterns of strategy choices. The strategy that best describes how a student generated a final response on any one item is listed in the table. Where students used more than one method to solve an item, both strategies are listed. This resulted in the 125 strategy choices across 112 items presented in Table 5. The distinction between internal and external strategy use for this task was clouded by the highly personal and sensory-dependent nature of children's means of representing missing values. Only the use of place value blocks in conjunction with Reasoning strategies clearly constituted external strategy use, consequently no attempt was made to separate the other strategies into internal and external categories.

Table 5. Strategies used on Missing Parts Items

	Missing Parts Items														
	8 = 4+?	8 = 6+?	8 = 3+?	11 = 8+?	11 = 6+?	11 = 2+?	15 = 11+?	15 = 7+?	15 = 3+?	30 = 5+?	30 = 24+?	30 = 17+?	42 = 22+?	42 = 25+?	42 = 13+?
Adam				R', E	X	R', E				R	R	R', E	R'	E, R'	
Brian				C	C	C				R	R	C	R	R, R'	
Laura				R	C	R				R	R	R, R'	R'	R'	
Candace				C	C	R				R'	R'		R'	R'	
Sandy				C	C	C				R'	R	C			
Chris				C	C	C				R	C	C	C		C
Bevan				C	R	R				R	R	R	R	R	R
Nina	X	C	C	E	R'	C									
Wesley				C	X	C	X	R'	R'	R					
Bahareh				E, C	C	C	C	R'	C						
Samantha				C	E, R'	C	E, R'								
Juliana	C	C	C	C	C	C									
Nicole	C	C	C	C	C	C									
Cliff	X	C	R'	R	C	C	X	R'		C	R	E			
Sam	E	E	E	E	E	E									
Maré	C, E	C, X	R', E	C	C, E	C									

X= Recognition
 R= Reasoning
 R'=Reasoning with the support of place value blocks
 C= Counting
 A= Algorithm
 E= Estimation

Recognition. Only 7 examples of Recognition were identified, all involving sets of 8, 11, or 15, some of which were presented concretely and some presented with numeral cards. These items were solved through an immediate Recognition process and were justified with comments such as, "I just know that 4+4=8." One reason for the limited number of Recognition examples is that not all students started with the easiest items. Based on results of the first interview, students were presented with an item thought to be appropriate to their confidence and ability, eliminating some of the more automatic responses, especially from the older students. A second reason for the relatively few responses categorized as Recognition is that the context of the task usually required adapting a known addition combination to the missing part format, thus utilizing more than one step and resulting in a Reasoning categorization. Recognition of known part-whole relationships was frequently used as one part of a multi-step Reasoning process, such as an estimate-test-revise strategy.

Reasoning. Reasoning with known facts and relationships, and applying logical thinking to derive answers accounted for 48 of the 125 strategy choices. These multi-step processes were by far the most interesting approaches to the part-whole task, often involving the creative combination of logical processes along with Counting, Estimation, or Recognition. Reasoning strategies were characterized by multiple steps and by the application of at least one known "chunk", such as drawing on a known addition fact. For instance, Cliff, given the problem of 8 in all, 3 showing, said, "This time it has to be five... because four and four is eight... that extra one went in here." Some items involved Reasoning by building on a previous item (e.g., $25+5 = 30$ so $4+ 26 = 30$). Other problem solving strategies included using an estimate-test-revise process and making an easier example. For example, given the combination of 15 in all, 6 cubes showing, Laura removed 5 of the 6 cubes to reformulate the problem as 10 with 1 showing. She quickly recognized this problem as having a hidden part of 9, and connected that to the original item involving 15.

Most examples of Reasoning involved a whole of 30 or 42. Interestingly, students who demonstrated the capacity to use multi-unit Reasoning with these items in the higher number range frequently approached the lower items through unitary Counting. It causes one to wonder the extent to which the use of greater numbers beyond a comfortable counting range encourage more powerful approaches to problem solving.

Most Reasoning examples occurred when the task was presented with numeral cards. Twenty-five of the 48 items classified as Reasoning were presented with numeral cards only, and they were handled exclusively at an abstract level by high performing students. The remaining 23 items classified as Reasoning were handled externally (R' in Table 5) with the support of a concrete model. Fourteen of these items were presented concretely with one part modelled with cubes and students appeared to use the perceptual information in some way as part of their Reasoning process. An interesting observation was that in cases of such a concrete presentation, if students actually touched the visible blocks, they invariably ended up counting by ones, resulting in their strategy being assigned to Counting rather than to some form of concrete Reasoning. The other 9 of the 23 examples of external Reasoning involved situations where children drew on the use of place value blocks as aids to working out two-digit missing parts. Though counting was used in these

examples, it was of a multi-unit nature, involved multiple steps for solution, and drew on a multi-unit rather than a unitary conceptual framework, thus fitting the category of Reasoning.

Reasoning as a solution strategy was used predominantly by older or high performing students, reflecting both their capacity to mentally manipulate number and their access to a reliable store of known facts and relationships. Interestingly, though, another characteristic that appeared to be associated with students who frequently used Reasoning strategies, was their confidence in their ability to think through to a solution and their positive approach to each item.

Counting. The exclusive use of identifiable unitary Counting strategies accounted for 52 of the 125 items, though incidental counting played a part in many examples of Reasoning. Counting as a strategy choice may well have been encouraged by the fact that introductory items in the Missing Parts Task were presented with blocks in conjunction with numeral cards. Nevertheless, almost half of the 52 counting items involved numeral card presentation without materials, with students counting on or back with a mental tally.

The most prevalent counting strategy was an additive count on with tally. This involved either reading the numeral card or counting blocks to establish the first part, then counting-on to the total, tallying the missing part by using one or more of the following tally methods: a mental tally, a finger tally, a block tally, a verbal tally, or a verbal pattern tally, (3 is one, 4 is two, 5 is...), a visual-spatial tally by tapping out a geometric pattern, or a kinesthetic tally through some body movement. These tally methods represented varied degrees of reliance on sensory information, and as a result were impossible to clearly label as internal or external.

A unique strategy of keeping a pencil mark tally was demonstrated by Juliana. For 8 in all, 3 showing, she drew three lines, placed a dot to separate that given part, then counted on from 4 to 8, drawing five lines to tally her count. She then counted up lines starting from the dot to enumerate the missing part. She reported that she had devised this strategy on her own to help herself keep track, and clearly used it with confidence. Though her strategy did not provide a reliable means of dealing with multi-digit numbers, it provided her with a means of making sense of the part-whole relationships involved in each of the early items. The importance of reliable

means of tallying the missing part was clearly an issue in dealing with this task through the developmentally appropriate strategy of counting.

The other approach to counting involved counting-back-with-tally from a starting number to eliminate the given part in order to be left with the hidden part. This subtractive strategy was used for 8 counting items, most frequently for the item involving a whole of eleven with two of the eleven showing. A subtractive interpretation of the part-whole problem was likely used for other items classified as Reasoning, but only the counting items provided a clearly identifiable view of subtractive thinking.

The versatility of counting as a solution strategy was apparent from its widespread use across items, and the fact that all but two students used counting at least once. Six-year olds relied most heavily on counting as opposed to the use of other strategies, in keeping with their less extensive strategy repertoires. Interestingly, students capable of using Reasoning strategies, as demonstrated by how they dealt with sets of 30 or 42, frequently used unitary counting with items involving a whole of 11 or 15. Moving items beyond a comfortable counting range appeared to stimulate use of multi-unit Reasoning approaches, as was the case for the Doubling Task. This observation puts in question the virtue of the current practice of focusing heavily on number to 20 in the early grades. Perhaps moving students just beyond their comfort zones, thus minimizing the viability of their trusty old counting methods, is what stimulates the development of more powerful mathematical thinking.

Estimation. A fourth category of solution strategies identified in 18 of the Missing Part items, involved an Estimation process. In 7 of these items, the two youngest students provided a perceptually based estimate stimulated by the visual appearance of materials. Such an approach might well be labelled a guess, but in this case responses were usually well within the bounds of reasonableness rather than random in nature. Sam used this approach for all six of his items, falling back on what could be considered his proto-quantitative, part-whole schema (Resnick, 1983). He appeared unable to account accurately for objects beyond his perceptual grasp, having no reliable means of tallying or representing the missing objects. Steffe et al. (1983) would consider this an example of a perceptual counting scheme. Eleven other instances of estimating did not involve materials and were used by

students who provided an Estimation response often in conjunction with a second strategy.

This tendency to take a chance with an Estimation response in the interview setting did not follow a set pattern. Some students appeared confident with their intuitive feel for quantity, such as Cliff's use of an estimate towards his personal limit in the sequence of items, and Sam's use of Estimation as a preferred strategy. Judging from the manner in which he provided his estimates, Adam appeared to use Estimation due to affective factors related to confidence and motivation. While still others, such as Maré, appeared to use Estimation in a flexible, competent, self-talk approach to making sense of the problem.

Factors Affecting Sense-Making on the Missing Parts Task

Various conceptual, procedural, functional, affective, and contextual factors served to support or constrain performance on the Missing Parts Task. Each of these perspectives on sense-making is discussed separately for purposes of clarity, however in practice the distinctions were much less clear, in particular, the distinctions between cognitive aspects of performance.

Conceptual aspects

The Missing Parts Task highlighted the capacity to keep track of a double count by monitoring a tally for the second count, a capacity which involves conceptual, procedural, and functional aspects of cognition. For example, given a whole of 11 with 8 of the 11 showing, counting on 9, 10, 11 enumerates the missing part, and by tallying that count as 1, 2, 3 provides the number of the missing part. Sam, though clear on the need to enumerate the hidden part, was unable to keep a tally of his count, instead making an estimate of how many counts he had made. When presented with the problem of 11 in all, 8 showing, and asked how many were hidden, Sam said, "Oh, I'd say about five." I suggested putting cubes on top of the box to represent the hidden contents so we could find out for certain. Together we counted the first 8, then counted on to 11 as I placed three cubes on top to account for the hidden 9, 10, 11. Looking at the model of 8 cubes showing, and 3 cubes on top of the box, I asked him again how many were hidden. He replied "Oh, I'd say about 4," and was genuinely surprised to lift the box and see 3 hidden cubes. He knew some were hidden, but the salience of the cube tally was not meaningful for him. Having no way to accurately work out the hidden number, he instead relied

on his global estimate strategy. Sam's personal context of "number as a global estimate" appeared to shape his interpretation of that situation despite the perceptual evidence to the contrary. This situation illustrates the maxim, "Children see what they understand, rather than understand what they see."

Having multiple ways to monitor a tally seemed to provide children with flexible ways of approaching the items. Maré used an internalized count for small differences. For 11 in all, 8 showing, she looked up to her right for a moment, then turned back and said "Three!" with the explanation that, "In mind I went nine, ten, eleven." For greater differences she used a verbal count and physical tapping to enumerate the missing subset. There was no indication that she had any other way to keep a record of the count other than to internally store it. Often this meant accurately enumerating the missing part, but tallying it incorrectly. For the item 11 in all, 6 showing, with encouragement Maré used her fingers to keep track of her count (7, 8, 9, 10, 11), but did not recognize that her fingers showed the tally of 5 and instead answered "Four?" Maré's capacity to focus and her use of self-talk to monitor her own thinking, compensated in part for her limited capacity to tally a non-perceptual count. To some extent exposure to tally methods could be the issue, but for the younger students, their conceptualization of units appeared to make the difference. Chapter 6 which looks at individual performance across tasks sheds more light on this question.

At least three further conceptual understandings appeared to play an important role in performance on the Missing Parts items. First, access to a multi-unit conceptual framework enabled students to manage two-digit items such as 30 in all, 17 showing. To monitor a unitary tally of 13 takes considerable focus, while monitoring a tally of ten and three becomes manageable, but requires the capacity to make sense of multi-unit relationships. Second, the capacity to decompose and recompose numbers is needed to support thinking about 13 as ten and three.

The third conceptual capacity that supported performance on the Missing Parts Task was flexibility in thinking about part-whole relationships in additive and subtractive ways. The ability to shift between an additive "eight and how many more will give me eleven in all," and a subtractive "eleven in all, take away this part of two and I'll be left with the other part," provided an affordance to successful performance by enabling students to select the most efficient or manageable solution strategy.

Children's capacity to use and explain both approaches appeared to indicate a grasp of the inverse relationship of addition and subtraction at least within that part-whole context.

Procedural aspects

Two procedural skills played an important part in supporting student performance on the Missing Parts Task. An extensive and reliable forward and backward counting chain was an essential support for successful use of Counting strategies, while reliable access to known facts was an essential support to the use of Reasoning strategies. Interestingly, young children and less confident children tended to rely on counting even when potentially useful number facts were part of their personal repertoires. "Knowing" addition combinations did not mean they would necessarily be applied to the items. Appropriately drawing on known number relationships was characteristic of the more confident and able students.

Functional aspects

The capacity to monitor thinking was a valuable support to performance on the Missing Parts Task. It was necessary to keep track of several quantities at once: the given part which for some children had to be enumerated from one; the value of the missing part; and the size of the whole group. Strategically monitoring one's steps in a Counting or Reasoning strategy, especially when a double count is involved, requires considerable focus especially as the number size increases.

Developmental limitations to short term working memory (Case, 1985) likely play a significant role in accounting for differences in how young children process tasks such as the Missing Parts items. When number size and the degree of abstraction of the task presentation were adjusted to the child's level, all students were successful. Once students understood the question, it did not appear that they lost the conceptual clarity of the part-whole relation. Rather, it appeared that as items increased in size, children became bogged down managing the multiple relationships and quantities, and lost track of their own processing. The memory load exceeded what they could handle. Thus, what appears in some instances to be conceptual confusion may well be a limitation of working memory.

Affective aspects

As for the Doubling Task, confidence and a willingness to grapple with challenge supported performance on this task. Confidence also appeared to play an important role in strategy choice, especially whether or not students would draw on their repertoire of known number relationships to construct a response. All aspects of a positive mathematical disposition (confidence, curiosity, perseverance, tolerance for ambiguity, flexibility, etc.) supported performance in this relatively challenging task. Again, the centrality of mathematical disposition to shaping the nature of children's mathematical activity was apparent.

Contextual aspects

Children's mathematical activity was affected by several aspects related to the context of the task, the first involving number size. In the item sequence, the increasing size of either the missing part or the whole served to constrain performance. Also, whether or not a decade shift was required for solution also increased the difficulty.

Another contextual feature that affected performance involved the manner in which the missing part items were presented. What the dynamic format served to illustrate was that, though grappling with part-whole relationships was relatively difficult for this age-group, the salience of the presentation the problem in less of a constraint to children's performance than how that relationship was made salient for the child to reflect upon. Increasing the available sensory information by providing visual organizers or concrete models of elements of the problem appeared to both support working memory and provide support to children's inclination to grapple with the problem.

Summary

As was the case for the Doubling Task, cognitive, affective, and contextual considerations influenced children's mathematical activity on the Missing Parts Task. All of the aspects that supported performance on the Doubling Task applied here as well (see page 94 for a summary). However, this task served to highlight two other important aspects of mathematical thinking.

The first of these involved the importance of the cognitive capacity to focus, to maintain a double count and to monitor that count. Flexibility in ways of keeping

track of a double count, from a mental tally to use of fingers, materials, and actions, supported success with the Missing Parts Task, while a lack of tallying options constrained performance.

The second aspect highlighted by the Missing Parts Task involved the importance of adjusting the presentation of a problem so that it could be managed by a child. Missing part problems are recognized to provide a conceptual challenge for young children, however, within a child's meaningful number range, representation level played an important role in supporting children's ability to be successful.

All students were successful with grasping the nature of the task, perhaps in good part due to the concrete introduction to the problem. All students were successful with some items, further supporting the notion that the conceptual demands of the task were not what limited performance. Clearly though, some students' understanding of part-whole relationships was more complete and robust than others. All students provided answers well within the bounds of reasonableness, with the exception of some items at the edges of their meaningful number ranges. It appeared that students unable to accurately determine the missing part were still able to draw on their proto-quantitative part-whole schema (Resnick, 1983) to generate a response that reflected their intuitive grasp of the problem.

At the outer limits of performance, students either abandoned reasonableness and began to guess or use incorrect algorithmic thinking, or more often, decided they had gone high enough and opted to draw the interview to a close.

V. Sharing Task Results

The Sharing Task focused on recognizing fair shares or groupings within quantities of increasing sizes. After several concrete warm-up examples to ensure students understood the task, students were presented with transparent, but sealed containers of materials and asked how they could be shared fairly between two, three, or four people. These included a bag of 12 red spoons, a bag of 18 plastic sticks, a box of 24 birthday candles, a bag of 40 doilies, and a bag of 50 balloons. Each of these was clearly marked with the total number of objects. If students were unable to work out the fair shares mentally, counters were available to model the items and work concretely.

Table 6 shows which items students attempted in the series of items, and which strategy students used to deal with each item. In cases where two strategies are listed, students dealt with the item twice before deciding on a response. Because various cues and scaffolds were provided as supports for the increasingly difficult items, students were successful usually up until the last item attempted.

Table 6. Strategies used on Sharing Items

	Sharing Items										
	12/2	12/3	12/4	18/2	18/3	24/2	24/3	24/4	40/2	40/4	50/2
Adam	X	R, C'	R, C'	X	C'	X	R	X	X	X	R
Brian	X	R	R	X	R'	X	R	R	X	X	
Laura	X	R	R	R	R	R	R	R	X	X	R
Candace	X	R	C'	R	R	R	R	R	X	R	R
Sandy	X	R'	R	R'	R'	R	R'	R	R		
Chris	X	C', R	R'	X	R	R		R	X		
Bevan	X	R	R	X	R	X	R	X	X	X	X
Nina	X'	C'	X'	C'	C'						
Wesley	X	R	X	R'	R'	E	E		X		X
Bahareh	X	R'	R'	R	R'						R
Samantha	R	R'	R'	E		R					
Juliana	R'	R'	R'	R	R'	R'	R'	R'			
Nicole	E'	C'	E'	C'	E'	E'			E		R'
Cliff	R	X'	X'	R	E	R	R	E	X	X	
Sam	E'	E'	E'	E'							
Maré	X	E'	E'	R'	E'	R	E'	E'	E'		E'

X= Recognition

R= Reasoning

C= Counting

A= Algorithm

E= Estimation

' denotes use of that strategy in conjunction with the use of perceptual information or concrete materials (see the description of externalized strategy use on page 78)

The organization of the item sequence in this task did not move from easier to harder, so patterns of performance are less obvious than was the case for the Doubling Task. Although the size of the starting set grew gradually across items, the difficulty was determined more by the number of groups it was to be divided into, and whether or not it worked out evenly. Generally speaking, dividing any sized group by two was easiest, by four was next easiest, and by three was most

difficult. In the interviews, dividing into four groups followed dividing into two groups in order to see to what extent students built on the previous item. Items that worked out evenly posed less of a problem than those involving remainders. Where materials were used, remainders were easier to manage; when working mentally without the support of materials, managing remainders proved to be a challenge. Items involving remainders did not appear to be conceptually more difficult, rather the processing demands on short term memory seemed to be the cause of children's problems. This again highlights the role of materials as supports for working memory rather than as conceptual support.

Strategy Use on the Sharing Task

The following description of strategies used on the Sharing Task describes Recognition, Reasoning, Estimation, and Counting approaches (there were no examples of the use of an Algorithm). These strategies were sometimes used mentally or in an internalized fashion, while other times the strategies depended on external perceptual cues or use of materials. Except for the introductory items, the first item of each set size was always presented abstractly with a packaged set labelled to show the number inside. Students capable of manipulating groupings mentally were usually successful in determining how to evenly share the contents without touching or even clearly seeing how many were inside. Internalized strategies refer to children's methods or approaches that were handled at this abstract, symbolic or internalized level.

If students were unsuccessful with the closed packages, the item was re-presented using blocks to model the contents of the package. Simply seeing the whole group of blocks appeared to enable some students to determine fair shares without even touching the blocks. Such responses involved perceptual processes that appeared to be triggered by the visual-spatial arrangement of the concrete materials. Other students appeared to need to handle the blocks in some way in order to determine fair shares. Where concrete materials were an integral part of solution methods, either perceptually or through actual manipulation, strategies were categorized as External.

Recognition. Thirty-nine of 139 examples of sharing involved the application of a known number fact or number pattern. Thirty-five of these were handled internally, mainly for items early in the sequence. Older students used Recognition more

frequently across items. Situations involving sharing among two elicited this strategy more often than for sharing among four or three. The most competent students frequently used known facts, reflecting in part their store of personal number benchmarks, and more importantly their confidence and propensity for applying their existing knowledge to the new sharing context. Recognition was only assigned as a strategy choice for items where an immediate answer was given. Since more than one step was involved in dealing with remainders, usually this category was only used for items without remainders. Items involving the recall and use of known facts and patterns as elements in a multi-step process were assigned to the Reasoning category.

Four examples of Recognition were categorized as external Recognition. In these cases, the physical arrangement of the materials appeared to trigger a known pattern of how the materials could be shared evenly. In these four cases, all or which involved a starting set of 12, the 3X4 geometric array of blocks provided the cue. The same perceptual process was used frequently as an element of a Reasoning strategy with higher numbers, where students moved materials into geometric patterns as they worked. When students were having difficulty, I sometimes moved materials into patterns (such as a 3X4 array) in order to see if students would recognize groupings. This pattern scaffold served to support children's ability to recognize fair shares at least for set sizes below 20. Pattern recognition appeared to provide a salient support to children's sharing activity. Whether or not children provided themselves with pattern cues through their organization of materials served to differentiate problem solving performance. Competence was not just a question of whether or not students had a repertoire of recognizable patterns, but how they went about making use of that repertoire. This was also the case for access and use of known number patterns and facts.

Reasoning. There were 69 examples of the use of Reasoning strategies, 47 at an internalized level of representation and 22 at an external level. Reasoning responses handled in an internalized manner utilized known facts, patterns and relationships in a multi-step process. Frequently students used a decomposition/recomposition process to break a starting set into manageable parts (i.e., parts that corresponded to known groupings.) For instance, for 18 shared among two, Laura found half of 10 and half of 8, then combined the two parts of 5 and 4 to determine that each person would get 9. Other examples of Reasoning strategies included taking half of half for

sharing among four. These multi-step solutions involved at least one known "chunk" in addition to the use of Counting, Recognition, or Estimation processes as elements of the constructed response.

Internalized Reasoning strategies were used predominantly for dealing with items involving higher numbers. Several high performers relied completely on Internalized Recognition and Reasoning for their sharing interviews, showing a preference for and flexibility with internalized processing. Their use of Reasoning also illustrated the importance of estimation to Reasoning processes. They frequently used a guess-and-test approach, mentally trying different possible groupings and revising their estimates based on the results. The capacity to internally represent and manipulate quantities in constructive processes was an essential support to children's capacity to work with groups of increasing size.

Externalized Reasoning was used to describe the same sort of constructed, reasoned responses, but ones that were dependent on concrete models. Externalized Reasoning was predominantly used by younger students who needed or preferred the support of materials to make sense of each item. It involved the use of perceptual recognition of groups, just as Internalized Reasoning involved the use of recall of number combinations. Though counting was sometimes involved as an element in the response, it was not the primary means of solving the problem. Rather the whole was considered in terms of partial groupings first, and counting was used as a check and to name the subsets if necessary. The use of Externalized Reasoning strategies may well indicate a growing propensity for using grouping relationships for thinking rather than relying on unitary Counting. The two most proficient six year olds regularly relied on External Recognition and Reasoning strategies to work with items well beyond their everyday number range. In contrast, lack of or limited use of either level of Reasoning in sharing items was associated with the least proficient performances and least confident students. Three of the four students who did not use Reasoning relied heavily on dealing out quantities by ones, despite encouragement to attempt to use their store of known facts or patterns to generate possibilities.

Counting. Unitary Counting as a means of determining fair shares involved students dealing out the starting set in a linear fashion. It was used for 10 examples, exclusively in conjunction with materials. In most cases, the set was dealt out by

ones, though in a few cases, the process was made more efficient by counting out portions of two or three at a time, but still accompanied by a count-by-ones sequence. Children's strategies were classified as unitary Counting only if they were executed in a procedural, linear, unitary way (as in one for you, one for me....) with no concern for the size of each share until all materials were dealt out. Such use of Counting was very different from that of using groupings in the part/whole manner of partitioning estimated subsets, then counting to check that subsets were even.

In light of Davis and Pitkethly's (1990) finding of young children's preference for using dealing and counting strategies for sharing, it was interesting that so few children chose to deal out materials, and even more interesting to see who chose to do it. Most likely the nature of the task and the context of the interview affected strategy choice. The question posed to each student was "There are ___ in this bag. If we shared them evenly between the two of us, how many *would* each of us get?" (rather than how many *will* we each get). The emphasis on what made sense, what was reasonable, and the lack of emphasis on the "right" answer, may have encouraged children to go beyond the linear, accurate, and safe method of dealing out materials. Similarly, each set (beyond the practice items) was presented first as a bagged set that could not be manipulated, which may have encouraged students to think in more global ways or may have discouraged counting, so that even when students tried the items with counters, they thought about the situation differently.

Estimation. Twenty-one responses which were reasonably accurate, and were presented as possibilities (rather than facts), were classified as Estimation. Almost all of the Estimation responses were provided by the youngest students. Six of these responses appeared to be generated mentally on the basis of number relationships rather than global, perceptual information, so were considered Internalized Estimation. The other fifteen responses, all provided by six-year olds, appeared to be highly dependent on visual-spatial information, so were categorized as Externalized Estimation. A relatively unsophisticated example of these was Sam's use of a global hands-on distribution of groups of blocks so that each share looked about right, at which point he'd announce, "I'd say you'd get about this much". Of the six-year-olds, Sam's global, perceptually-based "feel" for fair shares was qualitatively different from Maré's beginning use of number benchmarks alongside her global estimates, and from Cliff and Juliana's relatively sophisticated use of known groupings and patterns as estimates in a multi-step Reasoning strategy. This

distinction between internal and external forms of Estimation is made only to emphasize the possible importance of visual-spatial recognition of quantity to the development of mental number benchmarks for estimation.

Factors Affecting Sense-Making on the Sharing Task

The Sharing Task provided a useful context for highlighting several aspects of increasingly powerful mathematical thinking, one in particular which was not apparent to the same extent on the previous tasks.

Conceptual and procedural aspects

Aspects which are cognitive in nature included the capacity to conceptualize quantities as groupings rather than as collections of ones. This grouping propensity was apparent in both external and internalized forms. In external form, students tended to utilize perceptual groupings to first visually then physically separate and organize clusters of materials as multi-unit patterns. In internalized form, which could be considered the more sophisticated of the two grouping conceptualizations, students reasoned out possible solutions with multi-units "chunks". Both of these approaches drew on planning and prior knowledge, in contrast to linear, unitary thinking involving dealing out collections in order to determine fair shares.

For items involving concrete materials, this grouping conceptualization was often connected to visual-spatial pattern recognition based on how materials were presented. Many students organized the materials in particular ways such as in patterns and arrays, suggesting a metacognitive self-monitoring process that supported their own learning strengths. This suggests that availability of materials provided a contextual affordance to successful performance. Similarly, students' conceptual capacity to take advantage of the patterns offered by the materials also constituted an affordance. The role of pattern recognition, symmetry, and visual grouping in the overall development of number concepts is an area in need of research. Confrey's (1994b) work on "splitting" as an alternative conceptualization for number systems, may well be related to this aspect of performance on the Sharing Task.

At no time did the concept of sharing appear to constrain any child's mathematical activity. Even when students worked beyond their familiar number range, they knew what they were trying to do, and they could explain the problem. This grasp

of the problem of sharing fairly may well be an aspect of what Resnick (1983) identified as a protoquantitative part-whole scheme, in that at an intuitive level, students appeared to grasp the parts-to-whole relationships inherent in the items. Explicitly, however, there were wide variations in students' capacity to recognize underlying concepts such as the commutative relationship between three groups of four and four groups of three, or the connection between dividing into two and into four.

Another conceptual aspect of mathematical thinking highlighted by the Sharing Task was the ability to flexibly decompose and recompose groupings in different ways. Sometimes this conceptual process was applied along place value lines, such as sharing 18 between two by figuring out half of ten and half of eight. Other times, students demonstrated a capacity to think comfortably and flexibly about the many ways a number could be divided into groups. This was most apparent when students drew on their knowledge of sharing among two to work out how much four people would get. Some students saw those as two very different situations, while others took advantage of the connections, which served to support their performance.

The ability to mentally draw on and manipulate known facts, patterns, and relationships involved conceptual, procedural, and affective aspects of performance. One procedural aspect concerned the availability of a store of number combinations to use as elements in a Reasoning process. The conceptual ability to mentally manipulate number relationships was also involved. And finally, the confidence and ability to apply these known number combinations when and where appropriate was required. Even the youngest children who did not have an internalized store of known facts appeared to access and effectively utilize internalized visual-spatial patterns to support their thinking in a similar way.

Access to a store of personal number benchmarks quite directly affected sharing activity. Students who had their own frame of reference for identifying half of numbers such as 24, 40, or 50, were in a much better position to make sense of the sharing items. Similarly, the availability of multiple counting chains supported thinking while a lack of counting chains constrained performance

Functional aspects

The functional aspect that most frequently influenced performance on the Sharing Task involved working memory and personal limitations on the ability to monitor processing. A strength in this area supported performance, while in cases of weakness, students simply lost track of their own processing. The use of materials helped extend this capacity in many cases, but for all students there came a limit to what could be managed.

Another functional characteristic which affected performance was the strategic capacity to flexibly apply a variety of strategies depending on the situation. The use of both Recognition and Reasoning strategies characterized competent performance. The typical approach involved first checking to see if a known combination applied, and if not, utilizing what was known in order to construct or derive a response. The most competent performances at all ages showed a willingness to apply Estimation strategies in constructing a response. This involved utilizing a guess-and-test approach to trying out possibilities, first estimating, evaluating, then revising responses based on results. Both estimation skills and effective self-monitoring skills were required for students to successfully apply this approach to the Sharing Task. Children who provided the most competent performances on the Sharing Task flexibly, creatively, and effectively drew on a range of Recognition, Reasoning, Estimation, and Counting strategies in conjunction with a propensity for pattern.

Affective aspects

As for the previous tasks, mathematical disposition affected performance in important ways. In some cases a lack of confidence, willingness, motivation, or perseverance to grapple with a challenging item, despite interviewer support, appeared to be the direct cause of lack of success. Similarly, in some instances in this and other tasks, attention and interest flagged towards the end of the interview, thus constraining performance. References to affect are included throughout the discussion of tasks, strategies, supports, and constraints, as it is impossible to do justice to the role of affect in decontextualized terms.

Contextual aspects

All children worked comfortably within the sharing context. Though division as a mathematical operation was not part of the youngest children's experience, the everyday context of the sharing items drew on children's informal and intuitive

experiences with the notion of division as sharing, and supported their ability to figure out fair shares. In addition to the influence of the task context on performance, item contexts had an impact of performance. These contextual features of the different items included the size of the starting set, the number of groups the set was to be shared among, and whether or not a remainder was involved.

A third contextual influence involved the presentation of the item. The availability of concrete materials for manipulation and in some cases for perceptual cues, provided a support especially to the youngest children's mathematical activity.

Summary

The aspects which supported the most competent performances on the Sharing Task were similar in many ways to results on both the Doubling and Missing Parts Tasks. However, in one important way results were quite different. The task itself appeared to stimulate a different conceptualization of number relationships. It did not lend itself to place value interpretations, nor to linear counting interpretations as were typical of the first two tasks. Instead, it appeared to stimulate a global, pattern-based, grouping conceptualization of quantity, one that most students utilized in different ways for most solutions.

VI. Money Task Results

The money interview explored four increasingly complex aspects of money use. The first of these sub-tasks was simply naming coins and stating their values. This was included as a check on which coins were familiar to children in order to better understand performance on subsequent tasks.

The second sub-task items involved counting up sets of coins of increasing complexity and value:

1 nickel, 4 pennies (n,p,p,p,p) = 9 cents

2 dimes, 3 pennies (d,d,p,p,p) = 23 cents

3 dimes, 3 nickels, 3 pennies (d,d,d,n,n,n,p,p,p) = 47 cents

2 quarters, 2 dimes, 1 nickel, 2 pennies (q,q,d,d,n,p,p) = 77 cents

2 loonies, 5 quarters, 2 dimes, 2 nickels, 2 pennies (LLqqqqqddnpp)= \$3.57

This counting task showed the extent to which children were able to organize coins for counting, to recognize and work with the relative multi-unit values of the coins, to apply known multiple values, to connect counting chains with a cumulative count, and to organize coins for counting. Students started counting the set of coins that looked manageable to them. As a result some of the youngest students only completed the first one or two items, while some older students started and were successful with the set of \$3.57, then moved on to the next sub-task. This sequence of items was both efficient and effective for considering coin use, especially when considered in relation to the other sub-tasks.

The third sub-task involved showing different ways to pay for small toys, each of which increased in value (8¢, 10¢, 15¢, 23¢, 36¢, 45¢, 79¢). Once students showed one way to pay, they were encouraged to find other ways. The dynamic interview format was particularly useful for exploring the flexibility and fluency with which students conceptualized values in multiple ways. Success on this sub-task was highly dependent on the underlying conceptual capacity to decompose and recompose values using multi-unit thinking. Performance on the counting sub-task provided me with some indication of an appropriate starting point for this task. Children's first choice of how to pay for the item was often the most revealing. The least sophisticated of the group tended to use pennies exclusively, while the most sophisticated chose efficient combinations of coins. As children constructed alternative ways to pay beyond their first choice, the flexibility and extent of their grasp of coin relationships was readily apparent.

The fourth money sub-task, which involved making change in a hypothetical buying situation, was included to provide an extension to the upper limit of money items. Since making change involves recognizing and applying part-whole relationships to money situations, this considerably increased the conceptual complexity of this fourth money task. Procedural and functional complexity were also increased due to the following conditions: the need to construct the missing part with appropriate coin values; the need to plan the sequence of coin use to take advantage of money benchmarks; and the need for flexible counting chains or mental addition to add the different values.

As a result, the change task proved to be so difficult that five of the least proficient students were unable to make any sense of it. For those who attempted the making change sub-task, the difficulty of each item was tailored to previous performance, so no established sequence of examples was involved.

By analyzing performance across the four sub-tasks it was possible to recognize patterns to children's use of coins. Though familiarity with coin names and values varied, children's use of specific coins was consistent across Money sub-tasks 2, 3, and 4. Consistent use of a coin involved both conceptual and procedural aspects of money use. These appeared to be inseparable, with both being necessary and neither one being sufficient to account for meaningful use of a coin. The conceptual aspect involved the ability to connect the multi-unit value of the coin to the other coins within the child's repertoire. The procedural aspect concerned the facility and fluency with which students connected the coin with known combinations and verbal counting chains in order to operate with these multi-unit values. For example, consistent use of a particular coin meant that the student could comfortably count up sets of coins that included that coin, and could pay for items in different ways using the coin. It involved counting up multiples of that coin using the appropriate counting chain (e.g., for dimes, 10, 20, 30. . .), and integrating that coin into counting patterns for other coins, such as counting on to mixed multiple chunks (10, 20, 30.... 35, 40, 45....46,47,48).

While some six-year olds were limited to the consistent use of pennies with partial use of nickels or dimes, most eight-year olds in the group were consistent in the use of all five coins. Coin use appeared to provide a quick and accurate measure of children's overall understanding of money and the multi-unit relationships involved in money use. Table 7 includes a score for overall coin use for each child in addition to information on levels of performance on the four sub-tasks.

Strategy Use in the Money Interview

Strategy use for the other tasks in the study was described according to the categories of Recognition, Reasoning, Counting, Algorithmic Thinking, and Estimating. This original categorization was inadequate to describe performance on the Money Tasks because working with money necessarily and simultaneously involves Recognition and Counting of coins, as well as Reasoning in order to manipulate the multi-unit values of coins. Because coin use is built upon mental

facility with these multi-units, students did not try to apply Algorithm strategies, although if paper and pencil had been provided this might not have been the case. And finally, the Money Tasks did not elicit Estimation strategies in the same way as was the case for other tasks; instead money seemed to trigger an emphasis on accuracy and systematic quantification.

As a result, almost every item completed by students would have had to be categorized as Reasoning, as that category included the manipulation of multi-units in a multi-step process. For example, making a cumulative count of different composite values in order to count a mixed set of coins involves several steps and draws on multi-unit thinking. Only seven items completed by the very youngest students could have been categorized as unitary Counting, as in these items students thought of each coin in terms of pennies and counted up the value by ones. It was patently clear that the original strategy categorization was inadequate to describe what was important about children's approaches to money situations.

Analyzing Levels of Performance in the Money Interview

Consequently, rather than using the original categorization of strategy use, the focus for analysis was on the qualitative differences in how children combined Counting and Reasoning approaches to deal with money.

Although each child's performance in the Money Interview was unique, there were specific aspects of children's mathematical activity that consistently characterized different ways of dealing with the money sub-tasks. These groupings of characteristics appear to constitute levels of qualitatively different and increasingly sophisticated levels of working with money. These levels of money use are described as Unitary Counting Approaches, Transitional Approaches, and Established Multi-Unit Approaches. Each description of a level is followed by examples of children's approaches that were categorized within that level.

Unitary counting approaches

Four students, Maré, Sam, and Nicole (aged 6), and Nina (aged 7), consistently applied one-to-one counting skills and a preference for working with pennies, to a restricted range of Money items. This unitary approach did not vary across the sub-tasks, and to a great extent corresponded to their approach on other tasks in the study. No multi-unit coin values were solidly established for these students

however with help they were able to count-on to a nickel or dime. A unitary count to represent the value of a nickel or dime was sometimes used in order to combine that coin with pennies. For example, for the item involving counting up the value of a nickel and four pennies, Maré was able to count from one to five in order to consider the nickel in relation to the pennies, then to continue counting up to the value of nine. This corresponds to a count-all, direct modelling strategy in addition situations, (e.g., for adding $5+4$, counting from one up to nine.) Sam, for the same item said, "That's a nickel and four cents," but was unable to reconcile the two different types of coin units. With support, both children were able to count-on to the starting coin, but appeared unable to do so independently.

When asked for the value of a group of coins, some of these children counted the number of coins, such as when Nicole looked at the set of four pennies and one nickel, pointed at each one as she counted them up, then said "Five, its so easy!" When given a choice between five pennies, three nickels, or two dimes, none of these four children chose the set of coins with the highest monetary value. For example, Maré, when given the choice between 5 pennies, 3 nickels, or 2 dimes, chose the nickels, with the rationale of "I like beavers." Children who recognized the multi-unit values of these coins were only concerned about the value, and consistently chose the two dimes ahead of the greater number of coins or the look of the coins.

In buying items, Nina was confident with penny use, but even with help had difficulty accepting and using nickel or dime values. This was consistent with her overall dependence on unitary thinking throughout the interviews. Nina relied on pennies, had few counting chains and money combinations to draw on, and appeared to be limited by her lack of confidence, curiosity, and willingness to persevere.

These children showed either no understanding of the problem of making change, or they recognized the part-whole aspect of making change, but had no idea how to resolve the problem. In all of these ways the mathematical activity of the children indicated the lack of a grasp of the multi-unit values inherent in coins, though in many instances they could tell you the value of different coins. Coin recognition did not correspond to coin use, as can be seen in Table 7.

Unitary/multi-unit transitional approaches

Seven students, Cliff and Juliana (both aged 6), Samantha, Bahareh, Wesley, and Christopher (all aged 7), and Sandy (aged 8) appeared to be at a transition stage between reliance on unitary thinking and access to multi-unit reasoning. Money use characteristic of this transitional group involved the ability to use some but not all coins in addition to pennies.

Beginning indications of this transition, such as the use of pennies in conjunction with nickels and/or dimes, showed some access to multi-unit reasoning. Juliana (aged 6), successfully counted 48 cents (ddd, nnn, ppp), first counting up the dimes by tens, then silently counting on the nickels using a unitary count (e.g., 36,37,38,39,40) to represent each nickel. She added on the three pennies by counting 46, 47,48. Juliana successfully organized the coins for counting, and managed to keep track of her unitary count from 30 to 48, showing how her strong strategic skills compensated for a lack of an applicable fives counting chain. Similarly, Christopher, though hindered by organization and concentration problems, was able to work with pennies, nickels, and dimes interchangeably. His counting patterns were shaky beyond 50 making work with higher values difficult, and he sometimes translated coins into ones to count-on to a value. Both Christopher and Bahareh were able to make sense of some simple change items using unitary counting, and applied not unreasonable estimates to other change items.

Students further along in this transition level were comfortable working with at least four different coins and used these multi-units interchangeably. These students relied on manipulating coins in conjunction with verbal counting chains to solve counting, buying, and change problems, but were not completely fluent in their coin use. For example, Cliff (aged 6), after confidently telling the camera how he organized coins for counting by "starting with the tallest coin first" (meaning greatest value), went on to provide an impressive display of coin use. Given the 77 cent coin set (qq, d, nnn, pp), Cliff silently counted to get 78 cents. In demonstrating his count for me, he counted "25 and 25 is 50 and 10 is 55 and 10 is 65 and 5 is 70 and 1 is 71 and 1 is 72". Despite his one counting error which he subsequently corrected, his familiarity and competence with coins were clear. When it came time to show how to pay 79 cents for a toy, he showed great resourcefulness (not to mention memory) when he remembered where the set of 77 cents had been placed,

recognized and retrieved it, and simply added on two pennies. Cliff was the only six-year-old to successfully handle any change problems.

Children who appeared to be at this transition level between relying on unitary thinking and working comfortably with multi-unit thinking recognized the part-whole aspect of making change, were able to estimate the missing part or to use unitary Counting to find the missing part, but were not able to draw on appropriate coins to make up the change. For example, Sandy, on the item of 50 cents paid, 45 cents spent, counted on her fingers from 46 to 50 in order to find change of 5 cents. She appeared unable to conceptualize change in multi-unit terms, so was at a disadvantage in managing two-digit change items such as 36 from 50.

Multi-unit reasoning approaches

The last five students, Bevan (aged 7), and eight-year old Laura, Brian, Candace and Adam, demonstrated fluency and flexibility in working with the most complex counting, buying, and change-making tasks. They silently and accurately counted the group of coins worth \$3.57, easily dealing with five different coins. They flexibly showed how to pay amounts in several different ways, comfortably managing both the different multi-unit values and the corresponding counting patterns. They organized coins for efficient counting. They showed flexibility in constructing values using different coin combinations. They fluently used appropriate counting chains and coin benchmarks such as knowing four quarters made a dollar. They worked with different counting skills such as adding tens to any number (37, 47, 57....). They demonstrated flexibility in constructing the missing value in making change, using composite units rather than pennies. And overall, they tended to evaluate the reasonableness of their change-making responses, and the accuracy of their addition of coin values.

Although none of the students had much experience making change, the change-making task served to differentiate their relatively proficient levels of money performance. With help, these students correctly worked out roughly half of the change items presented to them. Candace's approach to change-making was the least sophisticated of this group. She counted backwards from 50 to 36 using real pennies to keep track of her tally, to find change of 14 cents. She easily traded a dime for ten of the pennies and connected the recount from 36 to 50 with her display of four pennies and a dime. On her next item of 36 cents from a dollar, Candace

used the same subtractive interpretation, but used dimes to count back by tens to 40. She then mistakenly counted from 31 to 36 to determine the rest of the change. Though her procedures were not reliable, she showed a conceptual understanding of the problem and some facility with using multi-unit coin values.

Laura, on the other hand, used several approaches to making change, in keeping with her flexible and creative approach in other interviews. To find change for 36 cents from one dollar, she modelled \$1.00 with four quarters then removed 36 cents to be left with the correct change. This required making several trades in order to be able to take 36 cents, and these trades were made with ease. Laura's approach indicated a grasp of the inverse relationship of addition and subtraction, a capacity to think flexibly with multi-units, and a positive, competent approach to problem solving with money.

These three levels of money use, Unitary Counting Approaches, Transitional Approaches, and Multi-Unit Approaches, closely corresponded to the children's general capacity to construct meaning for number. How children handled money provided a clearer picture of their capacity to make sense of number than was the case for any other task.

Table 7 summarizes children's performance in the Money Interview. It includes a summary of results for the four sub-tasks, a score for overall coin use as described on page 115, and a total money score.

Table 7. Summary of results for the Money Interview

Names	Task 1 Coin Rec. (max. 5)	Task 2 Counting (max. 5)	Task 3 Buying (max. 5)	Task 4 Change (max. 5)	Coin Use (max. 5)	Total Score (max.25)
Adam	5	5	4	4	5 (pndqL)	23
Brian	4	4	4	4	5 (pndqL)	21
Laura	5	5	5	5	5 (pndqL)	25
Candace	4.5	4	4	4	5 (pndqL)	21.5
Sandy	5	3	3	3	5 (pndqL)	19
Chris	5	3	3	4	3.5 (pnd q)	18.5
Bevan	5	5	5	4	5 (pndqL)	24
Nina	3	1	1	0	2 (p nd)	7
Wesley	4	3	3	3	5 (pndqL)	18
Bahareh	5	3	3	3	3.5 (pnd q)	17.5
Samantha	3.5	2	2	2	3.5 (pnd q)	13
Juliana	4	2	2	0	2.5 (pd n)	10.5
Nicole	1.5	1	0	0	2 (p nd)	4.5
Cliff	5	3	3	4	3.5 (pnd q)	18.5
Sam	3.5	0	0	0	1.5 (p n)	5
Maré	4	1	0	0	2 (p nd)	7

Scoring for Tasks 2, 3, and 4:

- 0 Early Unitary Counting Approach
- 1 Later Unitary Counting Approach
- 2 Early Transitional Approach
- 3 Later Transitional Approach
- 4 Early Multi-Unit Approach
- 5 Later Multi-Unit Approach

Scoring For Task 1:

- 0.5 point for each coin named
- 0.5 point for each coin value given

Scoring for Coin Use, Column Five:

- 1 point for each coin used consistently (pndqL)
- 0.5 point for each coin used partially (pndqL)

All scores are out of a maximum of five. The score for Task 1 involves one half point for each coin named or value assigned (i.e., if the penny and nickel were named correctly and only the value of the penny was given, the score would be 1.5.) Scores for Tasks 2, 3, and 4 are based on the three levels of money performance described earlier. A range of two points is used for each level, the lower number for beginning use of that approach and the higher number for established use of that approach. The fifth column indicates the coins used consistently by each child, each of which were scored as one, and the coins used partially (in italics), which were scored as one-half.

Factors Affecting Sense-Making in Money Situations

Many factors contribute to the ways children make sense of how we use money concepts and skills in North America. It is recognized that there is a significant

socio-cultural aspect to the use of money that has not been the focus of this study. Similarly, affective factors such as children's attitudes or dispositions towards money and money situations were only considered in an incidental manner based on the way in which children approached and worked with the money items. Though differences in awareness of, enthusiasm for, and experience with money were apparent, it was clear that many considerations played a role in shaping children's ability to make sense of money situations and that no single aspect accounted for success or lack of success. This analysis brings certain aspects to the foreground for discussion purposes, however the importance of the non-discreteness and complex inter-relationship of these aspects cannot be overstated.

Conceptual and procedural aspects

Three competencies of a procedural nature appeared to play an important role in working with money. The first of these was the ability to recognize coins and their values, as the more coins reliably identified, the greater the possibility to work with increasing values. Coin recognition alone, however, did not result in the ability to use the coin for counting or buying. A second procedural competence that supported performance was the ability to use appropriate counting sequences to make a cumulative tally of values. Students with unreliable counting chains, or with limited chains were constrained in their performance. But once again, the availability of counting chains was not a sufficient condition to ensure successful use of the corresponding coin value. Several children were quite able to count by tens, however the ten-ness of dimes, and the connection of dimes to a tens counting chain, meant nothing to them at that point. Thirdly, having a mental store of familiar money benchmarks played a role in how students dealt with the different Money Tasks. Knowing that two quarters was 50 cents, or that two nickels made ten cents helped students group, organize and easily recognize values. Students who knew that two twenty-fives was fifty, or who could count by 25s, were much more likely to comfortably use quarters to build values or to make change.

However, having access to any of these procedural skills did not ensure competent performance. Underlying meaningful use of these skills appeared to be the conceptual capacity to mentally manipulate the multi-unit values that most coins represent. Students working exclusively from a unitary conceptual framework were able to deal with some of the Money Tasks by interpreting every coin as pennies and applying unitary counting strategies. This approach which constitutes one early,

developmentally appropriate level of coin use, can be effective in some situations but places a definite constraint on performance. The ability to utilize place value relationships in conceptualizing money situations constitutes one support, however, an understanding of money requires reasoning that goes well beyond the one-to-ten relationships involved in place value. Thinking about values as different possible groupings of coins involves decomposing and recomposing these values in multiple ways. This theoretical assumption was supported in this study, where the ability to accept and work with a variety of multi-unit groupings was recognized to be essential to supporting increasingly powerful thinking involving amounts of money.

Functional aspects

Another category of aspects of money use that served to constrain or support competent performance involved functional, organizational, or strategic capacities. The ability to organize coins for determining their value and to monitor the use of coins through various mental and physical organizations (such as moving coins that have been accounted for), had considerable impact. Similarly, some coin situations required the ability to plan out ahead of time ways to pay or to make change. The capacity to apply a reflective and systematic approach to the coin tasks served to support successful performance, while impulsive and random approaches constrained performance.

Contextual aspects

In addition to the recognition of the general importance of context and affect as influences on children's understanding and use of money, one aspect of the task context in particular served to support performance. The availability of real coins for students to use provided both support for their thinking and a means of better recognizing and understanding that thinking. The support offered by the coins bolstered students' ability to keep track, as in a support to working memory. If no multi-unit relationship existed in the student's mind for a coin value, the availability of that coin made no difference to the student's thinking. In other words, rather than providing conceptual support, coins appeared to offer procedural, affective, and functional (organizational, strategic) support for taking advantage of the student's existing capacity to conceptualize the coin problem. This observation supports similar findings on the role of concrete representations in student thinking and reasoning (Pimm, 1995; Kamii, 1985).

Summary

Since this study was designed to investigate children's construction of meaning for number in familiar, everyday contexts, using money in a dynamic interview context seemed a very obvious choice. It was assumed that the conceptual threads that connect children's understanding of how number applies to different contexts would similarly apply in money situations. As it turned out, not only were the conceptual aspects of children's mathematical activity in the Money Interview commensurate with their activity in previous interviews, but their use of money skills and concepts provided a better window into how number was organized for them than any other task. The Money Interview called upon the full range of conceptual, procedural, and functional competencies available to students. Similarly, it highlighted the important roles played by children's attitudes, by their previous experience, and by aspects of the situation or context. And most importantly, the Money Interview illustrated the complexity and inter-dependence of all of these different considerations in shaping how children make sense of number.

Despite differences in the context and the demands of the sub-tasks in the Money Interview, overall performance was consistent in important ways with performance on the other tasks in the study. Chapter Six provides an analysis of children's mathematical activity across the four interviews, along with analysis and discussion of how performance in the Money Interview related to performance overall.

Chapter 6

Children's Mathematical Activity Across Tasks

This chapter concerns individual children's sense-making activity across the different number contexts used in the study. In keeping with the research questions, there are two purposes to the consideration of overall performance. The first is to explicate children's sense-making patterns across contexts. The second is to highlight characteristics of developing competence in coming to make sense of number.

The first section of this chapter provides an analysis of overall strategy use, including analysis of patterns for each age group along with a brief analysis of individual student's strategy profiles. The second section provides an analysis and summary of the cognitive, affective, and contextual considerations that served to support or constrain children's mathematical activity across the set of interviews.

I. Analysis of Strategy Use Across Tasks

Each of the 16 students in the study completed four tasks, each task in a separate interview. These interviews were shaped to present each child with a set of items that would tap their range of mathematical activity, from independent, relatively automatic performance, through more reflective, constructed responses, and up to a personal frontier or limit to understanding. In order to elicit this range, each of the four tasks involved a sequence of items which moved roughly from easier to more difficult based primarily on number size. Given the age range used for the study, and the diversity within those age levels, items that proved challenging for some were simple for others. For the first task, Doubling, all students started at the same point of doubling two. The Doubling Task results provided some indication of which difficulty level would be appropriate as a starting point for the subsequent tasks of Missing Parts, Sharing, and Money.

As the difficulty increased, interviewer cues and scaffolds were provided for students as a means of supporting their mathematical thinking and as a means of exploring their different levels of understanding. This enabled students to revise and refine their responses and to work well beyond their independent limits. Consequently, the focus was on children's processes for constructing meaning, and

the conditions that supported that construction, rather than on the products of their thinking.

For this analysis of strategy use, children's strategy choices for the first three tasks (Doubling, Missing Parts, Sharing) were pooled to look for patterns across tasks. The fourth task, "Money" did not elicit the same range of strategy use as did the first three tasks, consequently this analysis did not include Money results.

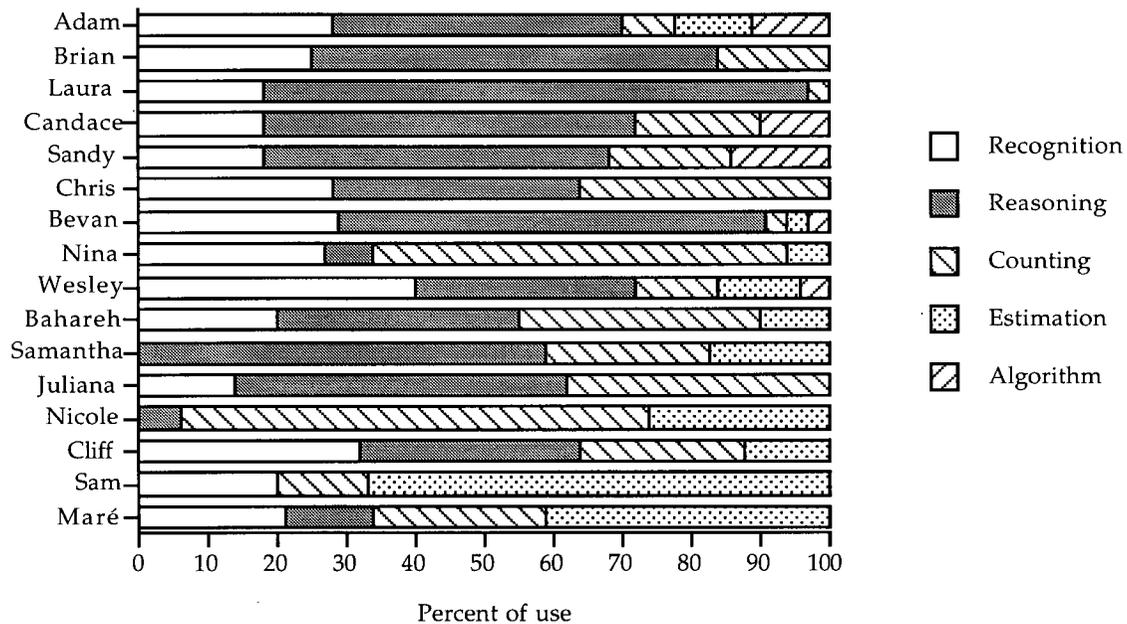


Figure 2. Strategy use across the first three tasks

Figure 2 illustrates the relative frequency of individual students' overall strategy choices by percentage. Percentages are used because each student completed a different number of items, and in some cases used two separate strategies to respond to an item. The average number of items completed on the three tasks was 24, with a range from 15 to 36 items.

Figure 2, organized by age from oldest (Adam, 8 years, 11 months) to youngest (Maré, 6 years, 1 month), illustrates some general developmental trends related to strategy choice and frequency of use. In order to consider this developmental aspect to the choice and use of strategies, the following graph re-presents the strategy choice data from Figure 2, averaged across age groups. Patterns to strategy choices are then discussed in relation to both figures.

Patterns of Age-Related Differences in Strategy Use

Figure 3 presents a comparison of the degree to which each strategy was used by each age group, illustrating changes in strategy use with increasing age.

This measure of strategy use is an average for each age group, derived from the personal strategy use profiles as presented graphically in Figure 2, and broken down by age groups in Tables 8, 9, and 10. For instance, the average for six-year olds' use of Reasoning was 20%, seven-year olds used Reasoning 39% of the time, while eight-year olds used Reasoning an average of 57% of the time.

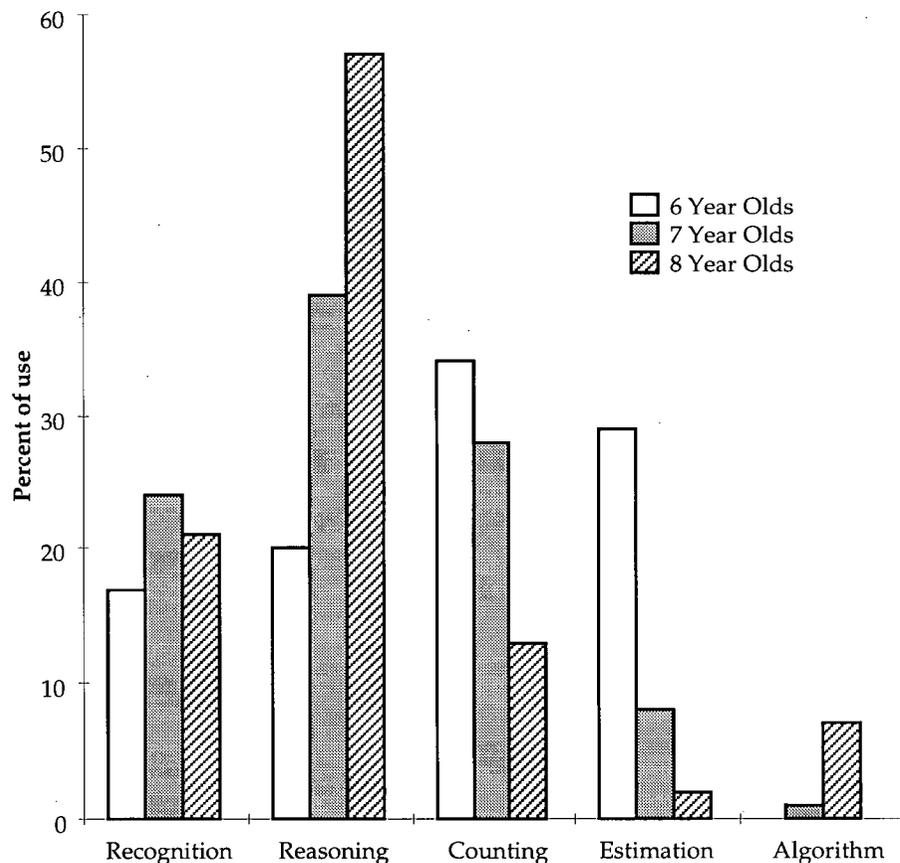


Figure 3. Average strategy use for each age group

This comparison is presented to highlight age-related developmental patterns to strategy use. Results presented in both Figures 2 and 3 are first discussed in terms of individual strategy use. Results are used in the following section to provide a frame of reference within which to consider the impact of individual differences on strategy choice and use.

Recognition strategy use

Overall, the Recognition results which show little or no difference between age groups simply illustrate that individual sets of items included a similar proportion of "easy" items. These results have more to do with the particular set of items presented to each child than with performance patterns typical of the age range. If children had all completed the same items, the older ones, given their experience with number, would have been able to apply Recognition strategies much more extensively than younger ones. Instead, the range of items and the dynamic interview format used in the study provided the opportunity to shape each interview in order to start each child with items within his or her comfort range of known number relationships. This very directly affected children's use of Recognition, making comparisons or conclusions about age-related patterns inappropriate.

In considering individual differences, however, it was interesting to note that only two students did not rely on Recognition strategies for any items. Nicole preferred to count even for the simplest items, while Samantha showed a preference for using a reflective Reasoning approach even for items that likely were part of her potential Recognition repertoire. Wesley's profile showed the most extensive use of Recognition, as he was able to draw on known facts where most students used either Counting or Reasoning.

Reasoning strategy use

The greatest shift in strategy use across age groups was seen in the increasing application of Reasoning strategies as illustrated in Figure 3. Reasoning strategies as defined in this study involved situations where students provided a rationale that included at least one known relationship as a "chunk" in the reasoning process and used more than one step for solution. Reasoning strategy use reflected the capacity to utilize multi-unit relationships rather than exclusively unitary thinking, as well as the capacity to combine multiple steps to construct a response. In particular, it reflected an inclination or disposition to build on previous knowledge through the application of known facts, patterns, and relationships.

In part, the increasing use of Reasoning strategies across ages reflected the fact that older students worked with items in the higher number ranges beyond a

comfortable Recognition or Counting range. Most younger students on the other hand, were unable or unwilling to try items beyond the range where Counting was an adequate choice. Was it because of the unfamiliarity of that higher number range? Or was it the lack of multi-unit thinking that restricted children to the lower items? Is multi-unit reasoning dependent on familiarity and experience with a higher number range? Or is access to a higher number range dependent on multi-unit reasoning? Children in this study showed that there are ways to deal with two-digit values without using multi-units, such as the use of memorization, or counting-on, or computation algorithms. But it is interesting to note that children capable of using Reasoning frequently used Counting, then switched to Reasoning only for higher numbers.

Multi-unit thinking appeared to be important to successfully working with multi-digit items that could not easily be counted. It was required in order to reason with multi-units, in particular place value relationships, and to decompose and recompose quantities on that basis. Where multi-unit thinking was not available to students, the use of Reasoning strategies was limited. Where it was an option, students were able to utilize Reasoning strategies to advantage. Whether or not they did so seemed to depend on factors other than conceptual development, in particular, factors of mathematical disposition, and specifically confidence.

As might be expected, Reasoning strategies were used more by older students than younger students. Older children usually have had more experience working with higher numbers where Recognition and Counting strategies are less effective, and likely have had experience with a wider repertoire of strategy options than young children. They are more likely to have access to a multi-unit conceptual framework and the capacity to decompose and recompose numbers based on multi-unit groupings. They have an increased capacity in working memory (Case, 1985) facilitating the use of multi-step thinking, and likely have a greater store of known facts in long term memory upon which to draw.

Why then were there such wide variations in the use of Reasoning strategies within each age group as illustrated in Figure 2? Part of the reason can be attributed to differences in cognitive development within age groups. However, what stood out in the interviews were differences in children's confidence, willingness, and inclination to construct personal meaning. These affective considerations appeared to

greatly affect the extent to which, and the nature with which children made use of their existing number knowledge. Extensive use of Reasoning strategies indicated a propensity for constructing personal meaning, while limited use of Reasoning indicated a preference for relying on learned procedures such as Counting and Algorithms, or on Estimation. This finding is in keeping with Gray's (1993) results on the qualitative differences in strategy choices of below-average and above-average seven and eight-year-old children (p. 153).

Almost all students showed they had access to some known facts, patterns, and relationships, however some children, such as Cliff (6), Bevan (7), and Laura (8), appeared to have a strong inclination or disposition to reason out items and to grapple with known number relationships to derive new relationships. Their confidence and willingness to persevere with a line of reasoning were evident throughout their interviews.

On the other hand, students such as Sam (6), Nina (7), and to some extent Sandy (8), appeared reluctant to take a chance on Reasoning despite the availability of known facts, choosing instead to use procedural Counting and Algorithm strategies, and in Sam's case, Estimation. Sam and Nina exclusively used unitary thinking while Sandy used both unitary and multi-unit thinking over the course of the interviews. At least for Sandy, the capacity to use multi-unit thinking did not mean she preferred it or that she applied it in every possible instance.

In summary, the use of Reasoning strategies appeared to be supported by a combination of factors: the conceptual capacity to access and manipulate multi-units; the procedural capacity to access and use known facts and relationships to advantage; and the disposition to want to construct responses rather than rely on rules and procedures. Taken together, these characteristics of the use of Reasoning strategies closely match Greeno's (1991) notion of number sense as a reflection of both the availability of mathematical tools and the inclination to use those tools effectively. The increasing use of Reasoning strategies would appear to provide an indication of the development of increasingly powerful ways of thinking about and working with number. Because of this, and because of the diversity and relative complexity of strategies included in the Reasoning category, an elaboration on children's use Reasoning strategies is provided in Appendix I.

Counting strategy use

Though all students used unitary counting at least once across tasks, the use of unitary Counting as the sole solution strategy for an item decreased with age, as illustrated in Figure 3. Counting however, was frequently used as one of several steps in a Reasoning strategy, in which cases the example was categorized as Reasoning. The decrease across ages in the exclusive use of unitary Counting reflected the age-related shift towards use of multi-step, multi-unit Reasoning strategies. Also, older students completed more items in the higher number ranges where unitary counting was a less reliable option. In both the Doubling Task and the Missing Parts Task, older and more confident students tended to switch to Reasoning strategies as the number size of the items increased, even if they had counted up until that point. It is important to note that the use of unitary Counting is a developmentally appropriate and widely applicable strategy that remains a part of children's (and adult's) increasing repertoire of strategy options. However, a decrease in the use of unitary Counting strategies relative to the use of other strategies appears related to developing competence with number.

Estimation strategy use

Figure 3 illustrates how the use of Estimation as a one-step solution process decreased with age. Only younger students or students appearing to lack confidence used Estimation as an end in itself, and this was usually done at what appeared to be the outer reaches of understanding. However, what is not shown by the graph is that estimation continued to be used across ages as one part of a multi-step Reasoning strategy, and was characteristic of flexible and competent mathematical activity.

The use of Estimation by younger students appeared to be related to a lack of concern for accuracy coupled with confidence in their intuitive, perceptually-based sense of quantities. Where a reliable strategy was not available, younger children such as Maré and Sam were quite willing to provide an estimate, and these estimates were usually very reasonable. Although estimation was used by students across age, ability, and confidence levels, some students used estimation from time to time and in various ways, while others did not use it at all. It was almost as if being close was fine for some, while others insisted on accuracy. The analysis of age group and individual children's strategy use explores this observation further.

Algorithm strategy use

Since standard computation algorithms are not introduced in the early grades, it was not surprising that the use of a computational Algorithm strategy was non-existent with six-year olds, and increased for eight-year olds, as illustrated in Figure 3.

Algorithm strategies were rarely used in the study and almost all examples were in the Doubling Task. Three children (Adam, Candace, and Sandy) mentally applied the standard addition Algorithm for the majority of their multi-digit doubling items. However the three children who were most successful with the Doubling Task (Brian, Laura, and Bevan), though able to use a mental addition Algorithm, chose to apply a mental decomposition/ recomposition strategy across the full sequence of doubling items.

Summary

Several interesting patterns to strategy use were apparent from the data as illustrated in Figures 2 and 3. All students used at least three of the strategy categories, six students used four categories, while three students provided examples from five strategy categories. This flexibility in strategy use did not appear to correspond to age or ability patterns. Rather it appeared to be stimulated by variations in item contexts as well as to reflect individual preferences.

Since the use of Recognition was shaped by the nature of the study, and Algorithm strategies were rarely used, neither strategy was considered in terms of patterns across age groups. Reliance on Reasoning strategies increased with age, while reliance on Estimation and unitary Counting as ends in themselves declined with increasing age. However, even within age groups there were interesting variations in who used these different strategies, as is illustrated in the next section.

Patterns to Individual Strategy Use

In order to minimize age differences and to highlight individual differences related to the construction of meaning for number, the results illustrated in Figure 2 are re-presented in this section broken down by age groups.

As shown in Table 8, unitary Counting was the preferred strategy for six-year olds, with some interesting variations in the degree to which individual children relied on Counting. Nicole used Counting most frequently, with no use of Recognition and almost no use of Reasoning. She appeared to enjoy directly modeling each item with

concrete materials, which then usually led to a unitary count-all strategy. When in doubt, she usually chose to apply Estimation.

Table 8. Percentage of strategy use for six-year olds across tasks

Name	(Age)	Recognition	Reasoning	Counting	Estimation	Algorithm	Total # of Examples
Juliana	(6-8)	14	48	38	0	0	21
Nicole	(6-4)	0	5	68	26	0	19
Cliff	(6-4)	32	32	24	12	0	25
Sam	(6-3)	20	0	13	67	0	15
Maré	(6-1)	21	13	25	42	0	24
Average % of use		17	20	34	29	0	

Maré and Cliff used a relatively balanced range of strategies, showing flexibility in approach and a propensity for considering the reasonableness of a response using Estimation and Reasoning rather than relying on Counting. Juliana, though she did not apply Estimation, used a similar range of strategies but with an emphasis on applying Reasoning strategies to construct meaning. All three provided examples of a positive disposition towards their own capacities to make sense of the tasks, though each was at a different level of achievement.

Sam's profile on the other hand shows quite a different pattern. A very young and exuberant six-year old, Sam demonstrated his developing store of known number relationships through the use of Recognition. He recognized geometric number patterns through subitizing, and had committed some basic facts to memory. However, beyond this known number range, Sam had two options. One was to use an unreliable perceptual Counting strategy based on one-to-one matching and counting-all. The other, which he used 67% of the time, was to fall back on his intuitive, global, perceptually based view of quantity and provide an Estimation response. Sam rarely showed an interest in accuracy, but had a "feel" for what might be reasonable, and was willing to count and check when encouraged to do so.

Table 9. Percentage of strategy use for seven-year olds across tasks

Name	Age	Recognition	Reasoning	Counting	Estimation	Algorithn	Total # of Examples
Chris	(7-11)	28	36	36	0	0	25
Bevan	(7-10)	29	62	3	3	3	34
Nina	(7-8)	27	7	60	7	0	15
Wesley	(7-5)	40	32	12	12	4	25
Bahareh	(7-4)	20	35	35	10	0	20
Sam	(7-3)	0	59	24	18	0	17
Average % of use		24	39	28	8	1	

Strategy use profiles for seven-year olds, illustrated in Table 9, show an even greater range of individual differences than was the case for six-year olds. Overall performance within the seven-year-old group ranged from Bevan's impressive performance in working to the limits of each task, to Nina's hesitant performance on a narrow range of items. Bevan relied on Reasoning most of the time, while Nina relied on unitary Counting strategies most of the time, and was unable and/or unwilling to attempt items involving larger numbers. Both students demonstrated access to known facts through their use of Recognition strategies, but only Bevan drew on these to reason out more complicated items. Nina fell back on Counting strategies rather than building on known facts to construct a response. Bevan comfortably manipulated multi-units, and worked almost exclusively at an abstract level. Nina, on the other hand, relied heavily on concrete materials and fingers to count out answers in a unitary, linear, systematic way. The qualitative differences in how these two students approached new items was quite remarkable from conceptual, procedural, functional and affective points of view.

These strategy profiles provide a starting point for considering individual differences in terms of number sense or competence. However, it is by looking beyond strategy use to consider the conceptual, strategic, and affective aspects that constrained and supported mathematical activity that practical implications can be shaped for the classroom. From the interviews it was clear that an over-reliance on unitary thinking and counting by ones dominated Nina's performance, despite some indication that more powerful thinking was available to her. Nina's seeming lack of confidence in her ability to think through a problem may well have led her to fall back on her reliable though immature counting strategies. Based on these

considerations, appropriate support for Nina would include an emphasis on the development of more powerful strategies, on ways to develop self-monitoring skills, and most importantly, on creating a learning climate that encouraged confidence, persistence, flexibility, and other important dimensions of a positive mathematical disposition.

In contrast, Bevan's strategy use profile indicated a high level of use of internalized Reasoning strategies within an extended range of items. This was also the case for six-year-old Cliff. Interviews showed that both were unique in their learning styles, work habits and behavioral patterns. According to their teachers, neither boy was proficient with paper and pencil mathematics, however both had minds and hands that were working at a tremendous pace, providing a challenge for classroom instruction. Their strengths would not necessarily be clear from a traditional written mathematics assessment. However, this alternative perspective focusing on children's construction of meaning illustrated that both boys would be well served by an adapted mathematics environment that supported their unique abilities. Open-ended problems that explored an extended number range, an emphasis on alternative as well as traditional computation strategies, calculator and computer explorations, and active, hands-on problem solving contexts might engage and channel their mathematical energy and precocity. While respecting their capacities for internalized procedures, some attention to systematic recording strategies might help bridge the gap between their preferences and classroom expectations and practices.

Table 10. Percentage of strategy use for eight-year olds across tasks

Name	Age	Recognition	Reasoning	Counting	Estimation	Algorithm	Total # of Examples
Adam	(8-11)	28	42	8	11	11	36
Brian	(8-7)	25	59	16	0	0	32
Laura	(8-6)	18	79	3	0	0	33
Candace	(8-4)	18	54	18	0	11	28
Sandy	(8-1)	18	50	18	0	14	22
Average % of Use		21	57	13	2	7	

Proportion of Reasoning strategy use served to best characterize and differentiate children's mathematical activity within the eight-year-old group. Laura throughout

the interviews consistently applied creative thinking and mathematically sound reasoning to the tasks. Laura, though she did not use Estimation as an end in itself, frequently used an estimate-check-revise Reasoning process. She appeared to have access to both linear and global approaches to number situations, applying global-spatial reasoning to several of the Sharing Task items. Although whole number algorithms were part of her personal repertoire, as demonstrated in the informal questioning, she chose to use her own resources to reason out items rather than to mentally apply an Algorithm. The extensive range of concepts and skills she drew upon in her construction of meaning, her positive mathematical disposition, and her meta-cognitive strengths all contributed to a most impressive series of interviews.

By comparison, Sandy's reliance on Counting and Algorithm strategies showed quite a different approach to the personal construction of meaning. Throughout the interviews, Sandy showed many strengths with number, and took a positive, straight-forward approach. However she appeared to lack both confidence and understanding and was quick to rely on rules. She was anxious to please and appeared very concerned about doing the right thing or doing what I wanted or expected. Though she showed that she could use multi-unit thinking, she frequently fell back on unitary Counting. Her attempts to explain her use of Algorithms showed that she had a very tenuous grip on the meaning behind their use. One such example was her attempt to apply the subtraction algorithm in an informal discussion. Faced with an example where there weren't enough ones and regrouping was required, she explained "You can't do that so you just turn it around." She then reversed the order of the minuend and subtrahend, at which point she gave up and made an estimate, saying, "I'd say about fifteen, ten or fifteen." Sandy's teacher commented, "She's conscientious but again she freezes up so she'll say it's hard, or ask for help, or put it off, or not do it. In a frenzy of wanting to get the one right answer she won't experiment."

Summary

Though strategy use changed with increasing age, even within age groups there were wide variations in who used which strategies. This section looked at these variations within each age group. What was apparent was that overall, what seemed to most clearly characterize the performance profiles of the most competent children was the use of Reasoning strategies, while the minimal use of Reasoning strategies characterized the performance of the least able. Unitary Counting was

widely used across age groups, however heavy reliance on unitary Counting at the expense of other strategy options characterized the mathematical activity of lower achievers.

Reasoning Strategy Use as a Reflection of Competence

The use of Reasoning strategies was more closely associated with increasingly competent and sophisticated mathematical activity than use of any other strategy category in this study. The Reasoning category was very broad, including thinking that involved more than one step in a constructive process, and drew on at least one "chunk" of prior knowledge. The Reasoning category excluded one-step, relatively automatic responses categorized as Recognition or Estimation. It also excluded procedural solution sequences categorized as Counting or Algorithm strategies. It included all other responses, thus covering a wide range of constructive mathematical activity.

Of all types of mathematical activity observed in this study, Reasoning strategies are believed to provide the best indication of children's developing ability to make sense of number. The Reasoning category included several types of strategies. Detailed descriptions and examples of each of the following strategies are provided in Appendix I:

- mental decomposition/recomposition,
- concrete or perceptual decomposition/recomposition,
- building on known facts,
- building on known patterns,
- multiple counting,
- place value counting,
- estimate-check-revise,
- compensation, and
- simplifying.

Two conceptual capacities were characteristic of all of these strategies categorized as Reasoning. The first of these, which was most apparent when dealing with multi-digit numbers, involved the capacity to think about quantities in a multi-unit rather than a unitary way. This was indicated by children's appropriate application of place value groupings and/or known facts and patterns. Since this capacity served

to shape the nature of the Reasoning category, the degree to which children did or did not use Reasoning strategies to some extent reflected their capacity to access both a unitary and a multi-unit conceptual framework.

The second important conceptual aspect underlying children's use of Reasoning strategies was the ability to decompose and recompose numbers to suit particular purposes. Each task drew upon this capacity in a different way. For Doubling it usually involved decomposing numbers into place value groupings before doubling. For Sharing it usually involved decomposing the starting set into parts based on known fact groupings or counting patterns. For the Missing Parts Task both place value and known fact groupings were used as the basis for decomposing and recomposing.

Though the Money Sub-Tasks were not analyzed in terms of the full range of strategies, how children dealt with money items was shaped in good part by these same two characteristics of multi-unit thinking and decomposition/ recombination principles. Competence with Counting, Buying, and Making Change sub-tasks depended on the ability to work comfortably with the multi-unit values of coins, the ability to combine these multi-unit values, and the ability to decompose a value into appropriate component coin values. For example, in order to pay 47 cents, the child needed to think of multi-unit coin values that could be combined to make up that amount. Finding a different way to show 47 cents required decomposing and recomposing the value with different coins.

Summary

The variety of Reasoning strategies used in this study illustrates the ingenuity, flexibility, and scope of children's construction of meaning in number situations beyond their comfort zones, at least for the simple tasks used in this study. Using more complex and integrated tasks might well stimulate use of an even wider range of Reasoning strategies.

Considerable research has focused on the development of children's counting strategies and on the increasing use of recognition and recall at least for basic fact combinations. In contrast to these studies, it is proposed that focusing on children's use of Reasoning strategies would not only provide an indication of growth in children's capacity to make sense of number, but would also serve to place emphasis

on the quality of children's thinking rather than the speed and accuracy of response. It would further serve to highlight alternative constructions of meaning, promote increasingly powerful mathematical thinking, and stimulate mathematical communication in the classroom and at home.

II. Analysis of Constraints and Supports Across Tasks

The second focus of the analysis of children's mathematical activity across tasks involved the different factors which served to support or to constrain children's construction of meaning. Over the course of each interview, certain characteristics were observed to play an important role in children's sense-making activity. These observations for each of the four tasks were reported separately in Chapter Five. Here they are compiled overall in an effort to identify which factors appear to be most important across task contexts and which appear to be task specific.

In observing children's efforts to construct meaning in the four interviews, it was clear that their success or lack thereof could not be attributed to any one factor. Rather, the richness of children's thinking about number was best illustrated by the complex interaction of different factors. Here however, for purposes of clarity these factors or aspects are brought to the foreground separately in the following four categories:

1. cognitive aspects of a conceptual and procedural nature;
2. metacognitive aspects related to functional competence;
3. affective aspects related to disposition; and
4. contextual aspects.

Cognitive Aspects of a Conceptual and Procedural Nature

The concepts and skills that children brought to bear on the different tasks were inextricably related, though the more conceptual aspects of their performance appeared to underlie and direct their mathematical activity overall. The most important conceptual aspect that influenced children's performance in this study was related to whether or not they applied a multi-unit framework to each situation. Students who were able to work with multi-units and who capitalized on their known store of number relationships demonstrated both the capacity and inclination to construct meaning across a much wider range of number situations. Their sense-

making, multi-unit approach constituted a flexible, more powerful approach to number situations, one that was qualitatively different from a unitary approach.

A second conceptual capability that appeared to be important across all four tasks, involved the extent to which a child was able to decompose and recompose numbers to suit specific purposes. The flexibility and extent to which children manipulated numbers in these kinds of ways to suit their purposes was characteristic of active and competent sense-making, even among the six-year olds. The more sophisticated and proficient examples of these conceptual abilities were characterized by the ability to mentally represent number. These internalized Reasoning strategies drew on the capacity to work abstractly with number without the support of perceptual information, to reason with known facts, patterns, and relationships, and to draw on prior knowledge to construct responses. This sort of internalized reasoning was in contrast to more externalized processes involving the same conceptual abilities but in conjunction with the use of models.

Multi-unit thinking and decomposition-recomposition abilities were important across tasks. However, further conceptual and procedural abilities were only called upon in certain contexts, thus appeared to be more task specific. Two were key to the Missing Parts Task. The first involved the capacity to keep a double count or to count-on-with-tally by maintaining a tally of the second count. The second involved the capacity to interpret part-whole situations in either additive or subtractive ways.

Similarly, specific skills of a more procedural nature were required for certain tasks. Certain contexts required reliable and extensive counting chains, multiple chains, and access to a store of known facts. Beyond the basic fact range, students who demonstrated access to a set of personal number benchmarks had their own number frame of reference from which to operate. For example, Chris took 50 cents to cubs every week, usually in the form of two quarters. He knew that $25+25=50$, and used it to advantage not only in money situations, but also in other number contexts. A personal store of recognized perceptual number patterns and groupings provided support for conceptualizing quantities in multi-unit rather than unitary terms, but this was only brought to the fore in the Sharing Task. And finally, the Money Task was dependent on a specific procedural skill, that of recognizing and naming coins and their values. That skill in conjunction with multi-unit relationships of a conceptual nature provided the basis for competent performance with money, thus

illustrating the necessary inter-relationship between knowledge of both a conceptual and procedural nature.

Metacognitive Aspects of a Functional Nature

Functional considerations that supported or constrained children's mathematical activity fell into four categories. The first involved the metacognitive capacity to self-monitor. In every task, the ability to focus, to keep track of thinking, and to effectively monitor processing enabled children to successfully continue. Even with the conceptual and procedural tools necessary to deal with an item, without these self-monitoring strategies, children lost track of reasoning, forgot where they were heading, followed dead-ends or gave up. This capacity to focus and monitor thinking did not appear to be related to age. In fact, two of the six-year-olds, Maré and Juliana, showed the most impressive capacities to manage their own processing. In light of recognized developmental limitations of short-term memory (Case, 1985), the capacity to self-monitor may play an even more noticeable role in supporting younger children's mathematical activity than for older students. Though each task context drew upon different forms of self-monitoring, the degree to which students used self-monitoring strategies was consistent across tasks.

A second related category of functional competence involved the metacognitive executive function of choosing which strategies to use and which knowledge to access. Characteristics of competence in this category involved choosing appropriately from a personal repertoire of concepts, skills, and strategies and creatively applying the available number knowledge. It included the flexible and varied application of strategies depending on the situation and flexibly changing strategies in the face of obstacles. Students who used Reasoning strategies consistently applied this metacognitive decision making in drawing on prior knowledge in the construction of meaning.

The third category in this area of functional competence involved the ability to organize. This capacity was particularly important when it came to the organization of materials, such as managing the blocks that modelled a problem without misplacing some or double-counting one or two. In the Money Task where every child worked with sets of coins, this capacity was highlighted. Children who organized the coins in advance to facilitate counting or grouping were at a clear advantage compared to those who haphazardly attacked the mixed coin sets.

The final category relating to functional competence involved the representational capacity to know what sensory information was required to deal with a situation. At different times children chose to access fingers, blocks, visual dot patterns, thinking aloud, rhythmic patterns, etc. These different modes provided visual, auditory, tactile and/or kinesthetic information to support thinking. The most vivid example of this capacity to know and use what was needed to make sense of situations was Sam's counting-all procedure. In adding $5+3$ in the final interview, he quietly counted out five fingers on one hand and three on the other, each time touching his nose with the finger that represented the counting numbers. Once he had both hands fully engaged in front of him representing the $5+3$, he proceeded to count up all the fingers from one to eight, again each time touching his nose with each finger in sequence as he counted. I told Sam I liked his way of counting and asked him why he touched his nose. He replied, "Well, if I went like this (using a finger from the "three" hand to touch the "five" fingers,) I'd lose track of my five and three." In other words, Sam knew he needed to see, hear, **and** feel his counting procedure, so he came up with a way to keep the $5+3$ in tact and visible while verbally and actively (tactile-kinesthetically) counting it up. Sam's perceptual counting scheme (Steffe, von Glasersfeld, Richards & Cobb, 1983) at that point was dependent on multi-sensory support. Without an internalized representation for number, he needed to account for each and every unit, counting from one rather than counting-on to the five. Though developmentally at an early level in his understanding of number, Sam's performance demonstrated a representational awareness in knowing how to support his own thinking.

The role of this entire area, referred to here as functional competence, is proposed to be an important aspect of early success in mathematics, and warrants future attention as a focus of mathematics instruction and research.

Affective Aspects Related to Disposition

The importance of a positive mathematical disposition to success in mathematics has been the subject of considerable recent research (Mandler, 1984; McLeod, 1992). In this study, several aspects of mathematical disposition and of general learning disposition stood out in terms of supporting or constraining children's mathematical activity.

Across the four interviews, a child's competence and success were related to:

- confidence in one's sense-making capabilities;
- the inclination and confidence to build on prior knowledge;
- a curiosity about both the problem context and the answer;
- a willingness to grapple and persevere with a challenging situation;
- the inclination to evaluate the reasonableness of a given answer;
- a high level of attention, interest, and motivation; and in most cases,
- a reflective, systematic approach.

These differences among children in disposition appeared to be unrelated to age. Some of the most impressive displays of confidence, curiosity, perseverance, and motivation were on the part of six-year-old Maré and Cliff. Disposition clearly influenced the range of items students were willing to attempt, the range of strategies they were willing to employ, the nature of their thinking, the ways in which the tasks were approached and the ways in which solutions were explained. Discussions with teachers and parents, as well as informal observations by the interviewer across students interviews, suggested that children's mathematical disposition, at least for this study, was closely related to previous success with mathematics, and to children's overall sense of themselves as capable thinkers and problem solvers. The relationship between previous success and a positive disposition towards mathematics is of particular interest, since it is one upon which classroom mathematics instruction can have a direct impact. Reconsidering what we value and reward in young children's mathematical activity, and how we convey to parents the nature of these priorities, has important implications which are discussed in Chapter Eight.

Contextual Aspects

In analyzing what provided a constraint or a support to children's mathematical activity in this study, this final section takes into account the context within which this activity was situated. Four aspects of context were considered. The first involved the context of the interviews and tasks. The second involved the context of each item. The third involved the context of the dynamic interview format. And the fourth, and most integral aspect of context for this study concerned children's personal number contexts, or the context which framed their sense-making activity

and differentiated it from others faced with the same situations. This interpretation of context is based on Cobb's (1990) description of a personal context as the cognizing subject's own construction, a view that recognizes that students in the same setting act in different contexts and engage in very different forms of mathematical activity. Because of the perceived pervasive impact of a personal number context on mathematical activity, this fourth aspect of context is discussed in detail in Chapter Seven, while the other three aspects of context are summarized here.

Interview context

The interviews were structured to encourage students to work to the outer limits of their potential range of meaning for number in order to try to get a sense of their strengths, of their prior knowledge as well as their partial, formative knowledge, of their available and developing mathematical tools, and of their ability and inclination to access and effectively utilize the mathematical knowledge available to them. It is argued that this global and integrated approach to considering children's personal number domains can provide a comprehensive picture of children's number contexts and their developing number "sense". It is further argued that the information generated through such an approach provides information that is critical to understanding children's thinking and to shaping future instruction.

Tasks were designed to encourage students to apply their general knowledge and everyday reasoning skills to each number situation. Though there is no way to be certain that this was the case, it appeared that the informal setting of the interviews, the interesting materials, the emphasis on creating an atmosphere supportive to children's unique methods, and the lack of paper and pencil all were conducive to bringing out an extensive range of mathematical activity. There was no evidence that the number contexts of doubling, missing parts, sharing, or money served to constrain children's thinking. Rather, all students were able to make sense of at least some of the item sequence for each task, likely due to their intuitive understanding of these situations. The one task that appeared to offer a particular support for some students was the Money Task, where children familiar and comfortable with money were at a distinct advantage. The use of real coins and other familiar materials in other tasks appeared to support children's thinking through direct associations with everyday activities.

Item context

One of the two most important considerations related to context that served to support or constrain performance concerned how each item was framed. Given the dynamic and flexible format, the presentation of each item was able to be framed to suit the learner. This highlighted the fact that the concepts underlying the tasks were not the problem; rather, the difficulty was shaped by the variable characteristics of the task. The first of these was number size, which served to put solutions out of reach, and the second involved the manner of representation within which the item was framed. Verbal presentations were most difficult, while verbal problems with numerals to represent key aspects of the item were somewhat easier. For example, in the Doubling Task, having the amount to be doubled showing in the display of the calculator appeared to increase motivation, support attention to the task and provide a scaffold for monitoring processing. In the Missing Parts Task, using a small box to represent the missing part provided a manageable format for some students. Others in addition required that one part be modelled with blocks rather than presented with a numeral card. By adjusting the representational level of the presentation of items in these ways, the impact of an unfamiliar number range was often diminished. By adjusting these two variables of number size and representational level, all students were able to be successful to some extent with every task.

For the Sharing Task, this consideration of representational level was built into the task sequence. To begin each set of items, the starting group was presented contained in a transparent bag or container. Children capable of mentally manipulating the number relationships involved for that quantity did not require that the contents of the bags be accessible in order to determine fair shares. Children unable or unwilling to do so were provided with blocks to represent the number of items in the bags. This provided a quick means of recognizing which students were able to abstractly reason with multi-units, rather than requiring concrete materials for perceptual grouping or needing to take a unitary dealing approach to determining fair shares.

Dynamic interview context

The third aspect of contextual supports and constraints that had an impact across tasks concerned how children made use of interviewer cues such as conceptual and procedural scaffolds, organizational suggestions, or strategy alternatives. Due to

the non-standardized and highly individual nature of each interview, no analysis of interviewer cues was completed by tasks, however overall analysis indicated that the following types of cues and supports were provided:

- comments that encouraged, recognized, or celebrated performance,
- comments or questions that re-presented aspects of the problem to provide focus, e.g., "So if there are eleven here in all with six showing, you think there will be three under the box?";
- comments or questions that described actions or sequences to help a student to verbalize thinking or to reconstruct a procedure, e.g., "I saw you use your fingers. How did they help you?";
- comments or questions that encouraged students to take a second look, e.g., "Have you included all the blocks in your count?"
- organizational suggestions, e.g., "What if we put all the blocks together?"
- strategic suggestions, e.g., "What if you started with this group?";
- procedural cues, such as providing a counting or fact correction; and
- conceptual cues, such as re-presenting the item in multi-unit terms.

All of these cues provided the opportunity for the child to self-correct, to go beyond readily accessible knowledge, and to be successful beyond usual boundaries. Where children were reluctant to continue, I provided encouragement. Where they were reluctant to stop, despite my providing most of the input, we negotiated an end to the interview. In trying to determine the limits of the child, this negotiation procedure provided valuable information for future work with the child, and reinforced the importance of considering children's mathematical thinking from multiple perspectives.

In the negotiation of meaning in an interview as in the course of teaching, not every move is met with success and my efforts as an interviewer in this study provide no exception. No standard rubric of cues could necessarily have guided my efforts to support children's mathematical activity in this study. What worked in one situation did not always work in another, or was not even appropriate to try in certain situations. Dynamic interview techniques involve more art than science, but offer the possibility of a better and richer understanding of children's sense-making capabilities than more standardized interview formats can provide.

Chapter 7 Discussion and Conclusions

I. Research Question One

What cognitive, affective, and contextual considerations serve to enhance or constrain children's construction of mathematical meaning?

In this investigation of the features which enhanced or constrained children's mathematical activity, a primary focus was on the cognitive factors which appeared to shape children's perceptions and actions. A secondary focus was on affective considerations such as children's dispositions towards mathematics as inferred from their sense-making activity. A third focus was on contextual considerations and their effects on children's construction of mathematical meaning. A wide range of cognitive, affective, and contextual factors were identified over the course of the study. Table 11 summarizes the factors that most directly and consistently affected children's sense-making activity.

A basic assumption of the study was that no single factor alone accounts for how children approach and deal with number situations, and that only by considering the inter-relationships among these factors can we recognize and appreciate the richness and complexity of children's sense-making activity. However, for discussion purposes, each factor is brought to the foreground separately.

Cognitive Considerations

Conceptual, procedural, and functional aspects of mathematical cognition were considered in this study. Though the findings supported the view that conceptual abilities provide the underlying framework for shaping children's mathematical activity, they also highlighted the inter-connectedness of the conceptual, procedural and functional aspects of cognition, and how strengths in one area served to make up for weaknesses in another. For example, the youngest child in the study, Maré, though limited in her conceptual capacity mentally to represent and manipulate number relationships, demonstrated a high level of metacognitive awareness and strategic monitoring capacity. This functional capacity, along with her inclination to take maximum advantage of her limited store of procedural skills, appeared to mitigate some of the constraints imposed by conceptual development.

Table 11. Key factors that influenced children's mathematical activity

<u>Cognitive Factors</u>	<u>Affective Factors</u>
<p>Conceptual:</p> <ul style="list-style-type: none"> • unitary and multi-unit interpretations • mental representation of number • understanding of parts and wholes • linear and global conceptualizations 	<ul style="list-style-type: none"> • confidence • flexibility • creativity • reflectiveness
<p>Procedural:</p> <ul style="list-style-type: none"> • extensive and reliable counting chains • recognition of visual number patterns • store of basic fact combinations • store of number benchmarks 	<p>Contextual Factors</p> <p>Task Features:</p> <ul style="list-style-type: none"> • salience of presentation • organizational format • number size
<p>Functional:</p> <ul style="list-style-type: none"> • metacognitive skills: <ul style="list-style-type: none"> -choose appropriate strategies -evaluate efficacy of approaches • strategic skills: <ul style="list-style-type: none"> -self-monitoring capacity -attention and focus -redirect thinking • organizational skills • representational skills: <ul style="list-style-type: none"> -ability to interpret different levels of representation -ability to reshape format 	<p>Availability of Materials:</p> <ul style="list-style-type: none"> • conceptual support • functional support • affective support • clarification of thinking <p>Interviewer Interventions:</p> <ul style="list-style-type: none"> • organizational help • procedural support • conceptual support • strategy options

Conceptual aspects

The importance of conceptual abilities in shaping children's mathematical activity has been widely recognized (Kamii, 1985; Piaget, 1952; Steffe et al., 1983; Steffe et al., 1988). In keeping with these previous studies, results of this study highlighted the importance of four conceptual characteristics which appeared to account in good part for qualitative differences in children's mathematical activity related to number.

The first of these conceptual abilities related to children's capacity to interpret units as composites (Steffe & Cobb, 1988; Cobb & Wheatley, 1988) rather than exclusively using a unitary interpretation of numbers as collections of singletons. The ability to recognize and work with composite numbers characterized access to a multi-unit conceptual framework (Fuson, 1992). Closely related to this ability was children's capacity to mentally represent and manipulate numbers and quantities. Children

who appeared unable to mentally represent numbers and operations consistently depended on external strategies and concrete materials. This highly visible aspect of children's interpretation of units is of particular interest as an indicator of developing competence.

In keeping with Resnick (1989b), the third conceptual ability that played a key role in how children interpreted and dealt with the number situations in the study was related to their understanding of the relationship between parts and wholes. Children's understanding of the additive composition of number, their use of decomposition and recomposition principles, and their grasp of inverse relationships, characterized understanding of this part-whole relationship.

The fourth conceptual ability that played a key role in differentiating performance involved whether children conceptualized number situations in linear, additive, counting-based ways or in more global, multiplicative, pattern-based ways. Findings indicated that access to both linear and global conceptualizations provided children with the flexibility to interpret a wider range of situations than any one conceptualization allowed. For each age group, the most able children used both conceptualizations depending on the context of the item. The most highly creative and competent children relied heavily on personal number patterns and intuitive, global responses that would best be characterized by a mental area model rather than the mental number line model (Case & Sandieson, 1992; Resnick, 1983) that is frequently used to characterize counting-based approaches. This finding challenges the view that linear counting interpretations of number dominate children's thinking about number or necessarily characterize competent number performance.

Procedural aspects

Variations in children's store of procedural skills appeared to play an important role in shaping and differentiating their mathematical activity. Procedural skills are generally recognized to contribute significantly to competent number performance. Findings from this study highlighted three such skills. First, how far students were able to work in any task sequence appeared to depend greatly on the flexibility and scope of their counting chains. In several examples of the Doubling Task children used place value blocks to construct a response but were unable to name the number represented by the blocks. They knew what they wanted to do, but the doubled number was out of their naming range. Similarly, for younger children who used

unitary counting to double numbers by building both sets with blocks, limitations in their counting chains curtailed their performance. This finding highlights the importance of developing counting and naming skills well beyond the child's number comfort zone.

The ability to recognize the numerosity of geometric number patterns without having to count (e.g., recognizing three rows of three as nine, or groups of four within eight) appeared to provide procedural support to how children conceptualized and dealt with number situations. For the youngest children, this skill was apparent in their ability to subitize small groupings without counting. The connection between this visual-spatial ability and children's use of global, pattern-based conceptualizations of number situations was clear.

The third procedural skill that served to shape children's mathematical activity concerned the degree to which they had developed a store of internalized basic number facts and number benchmarks beyond the basic fact range. Access to facts did not ensure that they would be used appropriately or to advantage, however the unavailability of these tools precluded certain types of mathematical activity.

Functional aspects

Functional aspects of mathematical cognition played an important role in how children used prior knowledge of a conceptual or procedural nature. The functional label included the metacognitive, strategic, and representational capacities that affected how children went about constructing meaning for the different tasks and items. Relatively little research has been directed at these aspects of mathematical cognition; given the apparent importance of functional competence in this study, further research is warranted.

The metacognitive skills that appeared to make a difference were the ability to choose and implement an appropriate strategy and the ability to evaluate the efficacy of an approach in order to recognize when an alternative approach is required. The strategic skills that played a key role were the capacity to self-monitor the use of strategies, the capacity to maintain attention and focus, and the capacity to shift or re-direct thinking when necessary. Organizational abilities also played a role in supporting children's sense-making activity. In keeping with Brown (1987), no difference among age groups stood out in the interviews. Rather, metacognitive

and strategic strength appeared to characterize the overall performance of particular children regardless of age. However, this finding of metacognitive strength as a personal characteristic is in contrast to Brown's proposal that metacognitive variations are shaped by context.

Representational abilities had considerable influence on children's sense-making activity, in particular, flexibility in interpreting a range of representational levels in order to work with increasing levels of abstraction. As number size increased, those dependent on concrete representations were limited while those able to work abstractly were not. Similarly, the ability to translate situations into a manageable representational form allowed children to reshape the presentation of items. Those unable (or unwilling) to transform items to suit themselves were at a disadvantage. The self-knowledge to know what degree of perceptual information or sensory involvement was required to maximize performance seemed to be a characteristic of certain children and may well relate to an internal locus of control. Those who took less control in reshaping the format of an item were more likely to turn to the interviewer for help. These aspects of representational competence are in keeping with the findings of Sigel (1991).

In summary, the results of this study support the claim that a more complete understanding of the complexity of children's mathematical thinking is obtained by looking beyond conceptual development to consider the wider context of mathematical cognition. The inter-relationship of conceptual, procedural and functional aspects of children's mathematical performance was clear, in keeping with Gelman and Meck's (1986) claim concerning the inter-dependence of components of competence.

Results suggest that the nature and extent of a child's personal number domain, or environment, Greeno's (1991) first aspect of number sense, is shaped primarily by conceptual and procedural abilities. Greeno's second aspect of number sense, the degree to which children made effective use of the mathematical tools available to them, appeared to depend more on functional aspects of children's mathematical thinking, in conjunction with affective and contextual factors.

Affective Considerations

The individual interviews provided opportunities to explore the nature of children's attitudes and beliefs about mathematics, particularly in relation to how children approached increasingly challenging items. Parent and teacher interviews provided further information to support the investigation of personal attitude and belief patterns. The affective considerations that appeared to be most influential in shaping mathematical activity involved certain aspects of mathematical disposition. The characteristic which appeared to have the greatest impact was children's confidence in their capacity as capable thinkers and problem solvers. For almost all children, confidence did not appear to be related to the task, rather it was characteristic of performance across situations. Though number size did affect the manner in which all students approached the more difficult items, relatively speaking, less-confident children appeared substantially affected by the difficulty level, while more-confident children were less concerned. The impact of confidence was pervasive and substantial. It appeared to determine the extent to which children were willing to take risks, the extent to which they would tolerate frustration or mistakes, and the extent to which they would persevere with a line of reasoning.

Two other affective characteristics played important roles in shaping the nature of children's performance. The first was the degree of flexibility and creativity children demonstrated in approaching situations, and the second was the degree of reflectiveness they showed in considering problem situations and evaluating their responses. Again, these characteristics appeared to be unaffected by variations in task contexts. Observations of children's mathematical dispositions across the four interviews were consistent in nature, and were also highly consistent with parent and teacher comments about mathematical activity in general.

Affective considerations appeared to have the greatest influence in determining the range of items children attempted, and the range and nature of strategies children accessed. Results indicate that these two aspects of performance play a key role in shaping the extent of a child's zone of constructive mathematical activity. For example, Juliana, Cliff, Bevan, and Laura displayed a high level of confidence, flexibility and reflectiveness. They attempted an extensive range of items beyond those they "knew," and used a range of strategies. On the other hand, Nicole, Nina, and Sandy engaged in very little constructive activity beyond the items they knew at

an automatic or procedural level. They appeared to lack confidence in their capacity to be successful with the increasingly difficult items, were quick to say items were "hard," and constantly sought my approval for their solutions.

Some students were reluctant to grapple with the more difficult items and needed encouragement to continue, while others had to be encouraged to stop. Some students were quick to say items were difficult, while others gamely attacked items at the outer limits of their range. The personal attitudes and beliefs that appeared to fuel these differences were not dependent on the context of the number situations, rather they characterized performance across tasks. Results of this study highlight the important role played by disposition, and point to the need to find ways to better understand the impact of disposition on children's mathematical activity. This should be an on-going priority for research on learning.

Contextual Considerations

The aspects of context which were within the bounds of this study to investigate, and which appeared to play a major role in how children went about making sense of each task, were related to the nature of the dynamic interview format. These included variations in the task and item features, students' use of materials, and the nature of the cues and supports provided by the interviewer.

Variations in task and item features

Three aspects of the tasks were varied over the course of the task sequences. These included the representation level of the task presentation, the organizational format of the item, and the number size of the item. Presentation of items within each task sequence was adjusted to suit the different ages and preferences of the children. The exception to this was for the introductory examples for each task, which were always presented in conjunction with materials to ensure that children understood what was required. An example of increasingly abstract variations to the presentation of the Missing Parts Task is: first, modelling the whole with blocks, putting one part under a box lid and leaving the other part showing; second, using a numeral card to represent the whole, modelling one part and using a box lid to represent the missing part; and last, using a numeral card to represent both the whole and the given part, with a box lid to represent the missing part.

This last variation is close to what a missing part item would look like printed as a missing addend, but was supported by previous use of the concrete problem solving context and verbal explanations as required. Following the introductory examples of each task, usually the most abstract presentation was used. If students struggled with that presentation, a more salient clue was provided by the interviewer, such as constructing the given part with blocks in place of the numeral card. These adaptations to the representational level accommodated children's different capacities to make sense in their "construction zone". Beyond their frontier of understanding however, the level of representation of the item appeared to be irrelevant.

Generally speaking, the younger the student the more important the perceptual organizers and cues were to keeping the child on task. One reason for this is related to limitations on working memory and the advantage offered by having concrete signs on which to "tack" the problem elements. Using consistent formats also served as a reminder of the problem context in spite of the changing number sizes. Another reason younger or less proficient children relied on perceptual cues was related to their emerging understanding of number. Steffe et al.'s (1983) scheme illustrates how students at an early stage of internalizing numbers and operations depend on the perceptual information afforded by external representations.

Varying the number size in the item sequences appeared to very directly affect the nature of children's mathematical activity. Children's personal limits for both conceptual understanding and procedural skills became more apparent as their strategies changed from more to less automatic and from more internalized to more externalized. Similarly, the role of personal resources of a functional and affective nature came more to the fore as the nature of children's monitoring strategies shifted to be more salient and externalized, highlighting the role of confidence, persistence, and flexibility.

Overall, varying the presentation of items meant that all students could be successful with the task over a wider range of items. This illuminated student understandings that would not have been apparent in less flexible interview situations. As difficulty of the items increased, it was possible to maintain a focus on the strength of the child's understanding of the task and to evaluate the impact of number size and representational levels on the child's ability to be successful.

Results indicated that varying the presentation of items helps to explicate both the nature and extent of a child's personal number domain. This dynamic approach has important implications for improving the validity and scope of the assessment of young children's mathematical knowledge.

Availability of manipulatives

Having materials available for students to use if they so chose, proved to be another contextual aspect that influenced children's performance. The use of materials allowed students to work beyond their capacity for abstract reasoning or internalized processing, thus providing important information on students' partial understandings and formative knowledge. This role for the use of materials appeared to provide two types of support: support for grasping the underlying concepts and functional support for monitoring thinking. For young children at Steffe et al.'s (1983) level of perceptual number, directly modelling examples in a one-to-one correspondence provided them with the conceptual support they needed to make sense of the items. For children with a partial understanding of place value many-to-one correspondences, the place value blocks played a similar role, offering a concrete representation of the many-to-one relationships upon which to apply meaning.

Secondly, the lack of access to paper and pencil placed a significant demand on short term working memory, causing many students to use the materials for what appeared to be an organizational and procedural support to memory rather than a conceptual support. This was particularly the case for students who had a solid grasp of number and place value concepts, but who used blocks in the manner of a recording device. Two characteristics appeared to indicate that materials were being used to support memory, rather than to provide conceptual support. The first was the clarity with which the student organized and used the materials, and the second was the fluency with which the student named the different modelled groupings. Regardless of whether the materials provided conceptual or strategic support, the availability of materials provided information valuable to future instruction.

The availability of materials appeared to offer affective support to the younger children's thinking by providing interest, motivation, and focus. Older and more able students on the other hand, appeared able to persevere without such support.

Finally, the use of materials proved to be invaluable as a means of enhancing and clarifying students' explanations of their thinking. As Cliff put it, "I can't explain it. Watch, I'll show you." Student actions with materials provided a form of triangulation of the data, offering evidence that either supported or conflicted with verbal reports, and providing avenues for follow-up questioning. In summary, results of this study indicate that the availability of materials provided conceptual, functional, and affective support for children's sense-making activity, while the nature of children's use of materials provided valuable assessment information.

Interviewer interventions

The final contextual influence on children's mathematical activity involved the nature of the interviewer interventions. Different sorts of interventions were used, all based on the interviewer's appraisal of a given situation, and all designed to support the children's thinking to enable them to complete as many items as possible.

Organizational help was sometimes provided, such as in arranging materials to clarify relationships. For example, some children had difficulty keeping track of their counters, misplacing or recounting one or two. Helping to create an accurate model allowed students to pursue their line of reasoning to its conclusion without interference from minor procedural or attentional errors.

Procedural support was sometimes provided when a specific skill or fact was the missing link in a child's process of constructing a response. Children puzzling over counting sequences were provided with the number name they needed when it was a case of an isolated difficulty such as a decade shift. Similarly, providing a basic fact sometimes helped students to pursue their line of reasoning to its conclusion.

Sometimes conceptual support was provided to see if students could make use of it in their line of reasoning. This occurred most frequently when students were struggling with unitary reasoning where a multi-unit approach would have been easier to handle. For example, in trying to count-on 32 to a starting set of 32 single cubes, I would suggest thinking of 32 as 3 tens and 2 ones and modelling it that way. Some students were able to use the more powerful conceptualization given that cue, while others were not and reverted to unitary counting.

Another example of conceptual support was related to the provision of strategy options, and was used when students appeared to be at the limit of their understanding. This involved providing an example of a strategy that another student had used successfully, and asking if such a strategy might be of use in this new situation. For example, some children attempted to model every example from one, when counting-on to a starting set could have been used with greater chance of success. For some students asking if it was necessary to recount the first set sometimes cued them into utilizing their counting-on capacity, while for others it had no effect. Similarly, some children depended on one strategy alone, so others strategies were presented to explore the availability of strategy alternatives and flexibility of strategy choice. Whether or not students could adapt the new strategy to the new example was a useful piece of information in considering the conceptual and strategic capabilities of the child. In summary, the dynamic interview protocol provided a valuable means of investigating a child's potential for making sense of number situations beyond his or her number comfort zone. This is in keeping with Feuerstein's (1969) findings concerning potential levels of development, and Vygotsky's (1978) exploration of a zone of proximal development.

Summary

In response to research question one, this section has described the cognitive, affective and contextual aspects that appeared to have the greatest influence on children's mathematical activity across the four tasks used in the study. Results, which are summarized in Table 11 on page 148, support the case for the importance of considering children's thinking and sense-making with number in a global way, rather than attributing performance to any single factor. This global approach provides a comprehensive lens through which to portray the complexity and richness of children's construction of meaning for number.

II. Research Question Two

What characterizes developing competence in young children's capacity to make sense of number?

Traditionally, children's mathematical progress has been measured by their mastery of facts and procedures, their use of mathematical vocabulary, their recognition of patterns and structure, and their ability to solve word problems. The NCTM Assessment Standards for School Mathematics (NCTM, 1995) provide new criteria for shaping the nature of mathematics assessments in order to reflect the NCTM's reform vision for school mathematics (NCTM, 1989) and classroom instruction (NCTM, 1991). The Standards describe a shift in expectations towards using concepts and procedures to solve problems and away from simply mastering isolated concepts and procedures. It defines assessment as the process of gathering evidence about a student's knowledge of, ability to use, and disposition towards, mathematics and of making inferences from that evidence for a variety of purposes (p. 3). Based on the results of this investigation, the following section provides an alternative approach to considering children's developing capacity to make sense of number, one which is consistent with these Assessment Standards.

Research Question One explored the factors that appeared to influence children's mathematical activity in important ways. The second question focused on specific aspects of this activity that appear to best characterize the developing capacity to make sense of number. These aspects are:

1. children's use of strategies
2. unitary versus multi-unit conceptual framework
3. capacity for mental representation of number
4. the nature of a child's personal number domain
5. children's number contexts

The fifth aspect, "Children's Number Contexts" serves as a summary for the first four sections. It provides a scheme for considering children's number contexts within a developmental framework and a rationale for reconceptualizing number sense in a way that accommodates the mathematical strengths and limitations of young children.

Children's Use of Strategies

The results of this study support the contention that strategy use provides an effective and practical means of assessing children's developing ability to construct meaning for number. Focusing on which strategies are used and how they are used over time can provide an on-going means of monitoring progress based on children's conceptual, procedural, and functional development. Attending to children's methods also affords the opportunity to focus on children's attitudes and beliefs about mathematics.

Both the importance of observing and supporting children's own construction of meaning and the key role played by attitude are supported by Hughes (1986). In emphasizing the need to build on children's early and meaningful strategies, he cautioned:

Obviously, we want children to move on eventually to new and more powerful strategies, but if these are forced upon children regardless of their own methods they will not only fail to understand the new ones, but will feel ashamed and defensive about their own. (p. 177)

Eliciting strategy choice patterns within a classroom can indicate qualitative differences in children's mathematical thinking that can inform teacher's instructional decision-making in order to accommodate and support individual differences. For instance, Carpenter and Moser (1984) illustrated patterns of strategy choice typical of students in grades one, two and three. Their results, along with findings by Hughes (1986) support the conclusion that six-year olds who comfortably and confidently apply internalized reasoning strategies demonstrate a qualitatively different approach to number situations than most of their age group. Similarly, eight-year olds who rely exclusively on unitary counting strategies are experiencing mathematics on a qualitatively different level from most of their peers. Results from this study support these conclusions. The implications of these differences for instruction are significant.

Since our assessment practices make clear what mathematics we value, focusing on the use of varied strategies has the potential to shift the emphasis from the mastery of skills and procedures to increasingly powerful ways of thinking about and constructing meaning for number situations. A focus on strategies accommodates both non-linear, qualitative and linear, quantitative conceptualizations, and all levels

of ability within a classroom. An emphasis on strategy development would shift the emphasis away from speed and accuracy towards the exploration and development of an expanding repertoire of strategy options. A more systematic focus on Reasoning strategies in particular could shift the current primary classroom emphasis on counting for short term gain, to more powerful ways of thinking that enable and sustain long term success.

In this study, highlighting external or concrete reasoning provided a means of focusing on the coordination of counting and reasoning as an important step in moving from counting to grouping strategies. This category distinguished early multi-unit reasoning from the conceptually less powerful unitary counting-by-ones strategies. Externalized Reasoning provided a match with the intermediate stage in place value understanding described by Kamii (1986) and Cobb and Wheatley (1988) involving dependence on grouped materials to "hold" the multi-unit relationship of the values. Study results indicate that a focus on children's use of external Reasoning strategies can provide useful assessment information on this important transition from unitary to multi-unit thinking.

Focusing on strategy construction and use has the potential to change established classroom practices in the direction of more constructivist teaching and learning principles. It would serve to encourage respect for alternative approaches to problem solving, to develop a classroom focus on the shared construction of meaning, and to enhance classroom opportunities for developing mathematical communication.

Unitary versus Multi-Unit Conceptual Framework

The one characteristic that appeared to have the most pervasive impact on children's mathematical activity in this study was whether a number problem was conceptualized in unitary or in multi-unit terms. The tasks used in the study could all be operationalized within the limits of either framework, however multi-unit or grouping approaches provided substantially more powerful, and qualitatively different solution strategies from a unitary approach. The ability of the youngest students to apply grouping concepts particularly characterized developing competence. Six-year-old Cliff's use of multi-units was a clear indication of his relatively powerful approach to number as compared to his age-group. Similarly,

with the older students, the lack of use of multi-unit thinking differentiated performance by indicating a comparatively unsophisticated approach.

Capacity for Mental Representation of Number

Closely related to children's use of a unitary versus a multi-unit conceptual frameworks was their capacity to mentally represent and manipulate number. The three children who relied exclusively on unitary thinking also depended on an external unitary representation of the numbers involved in each item, did not utilize known combinations to derive unknown combinations, and appeared unable to mentally try out possible answers using known facts or multiple counting patterns. On the other hand, the three students who regularly used a multi-unit approach frequently relied solely on internalized reasoning processes, mentally manipulating number relationships and testing different possible solutions using their store of prior knowledge. Students working between these two extremes appeared to be at varying stages in the process of internally constructing meaning for number, sometimes working mentally, and sometimes depending on external representations and actions. The difference between internalized and externalized mathematical activity was readily apparent and provided a simple means of considering children's developing capacity to make sense of number (when considered in relation to other indicators.)

Three conclusions about children's capacity to mentally represent numbers and quantities seem warranted. The first is that number size appeared to directly affect the capacity to utilize internal processes. If the number size was within a child's number comfort zone, most children depended on internalized thinking. But, as number size increased, and the familiarity with the number range decreased, thinking became increasingly externalized. For example, in the Doubling Task, the strategy patterns used by children reflected this shift (see page 85, Table 4). Children using a unitary approach (Maré, Sam, Nicole, Nina) moved from Recognition responses to Counting responses. Children using a multi-unit approach (Brian, Laura, Bevan) moved from Recognition through to internalized Reasoning, drawing on external Reasoning for the more complicated items. Children in the process of constructing a reliable multi-unit conceptual framework moved from Recognition through varied strategies, to reliance on place value blocks for Concrete Reasoning, but did not use Mental Reasoning at all.

A second conclusion is that the internalization process for early number concepts and unitary thinking parallels a second sequence of similar internalization for multi-unit thinking. Carpenter and Moser (1984) in a longitudinal study of addition and subtraction in grades one to three showed the gradual shift from external, direct modelling strategies to partially internalized counting-on strategies where a starting number is represented mentally, to internalized derived fact and recall strategies. The same strategies of direct modelling then counting-on, then internalized reasoning were demonstrated by students who were beginning to apply multi-unit reasoning with the support of place value blocks. These children were all capable of utilizing internalized unitary Counting strategies but in coming to understand composite notions, they reverted to the same sequence used to internalize single units. This may well illustrate a developmental sequence of representational thinking that is context specific and may apply to making sense of a variety of other mathematical topics.

The third conclusion about internalization processes is that children at the early levels of working with number appeared to have no other option for how to deal with situations other than relying on concrete materials and external action. On the other hand, children capable of mentally manipulating number had a range of representational options, and for a variety of reasons often chose not to use their most advanced representation capacities. Cliff, a young, active, spontaneous, and mathematically precocious young man, loved to "play" with the materials and literally could not keep his hands off them even though he did not seem to need them to make sense of most of the items he attempted. He constantly fiddled with whatever was close at hand even when his mind and verbalizations were on a completely different track, and delighted in using the materials to show me his thinking. These variations in young children's representational capacities and preferences have important implications for instruction, for assessment, and for our understanding of how different children construct personal meaning. They further support Confrey's (1995b) notion of abstraction as a dialectic rather than a desired goal state at the expense of concrete applications (p. 40).

The Nature of a Child's Personal Number Domain

This study was framed by an assumption that children's personal number domains were made up of different ranges or zones of understanding and competence, and that children's mathematical activity varied relative to these zones. Results indicate

that this notion of shades of understanding provides useful information for shaping instruction. Children working within their number comfort zone appeared to draw on familiar and readily accessible number knowledge. Activity within this zone appeared to be characterized most frequently by Recognition responses, which were applied most often with the lower number items, or for highly familiar number combinations in the higher ranges. This suggests that number size may well provide one of the parameters for shaping a number comfort zone.

Also, some procedures that children used automatically and without effort may constitute activity within a number comfort zone. In this study there were many examples of unitary Counting being used in this effortless fashion, and while no paper and pencil algorithms were applied, certainly computation procedures have the potential to be developed to a level of automaticity. However, in the early stages of working with counting and computation algorithms, these procedures would likely constitute constructive mathematical activity typical of a number construction zone. Consequently, the use of Counting and Algorithm strategies might characterize activity within the outer limits of a number comfort zone or the inner limits of a number construction zone.

The use of Reasoning strategies as defined in this study, characterized activity within a child's number construction zone. This zone refers to the range of potential meaning that begins at the limit of what students are familiar and comfortable with (number comfort zone) and stretches up to their personal frontier of understanding for number. Mathematical activity within this range involved children building upon their existing understandings to construct meaning rather than falling back on recall of prior knowledge or activation of automatic procedures. Mathematical activity within this range frequently involved various external sensory and perceptual cues in addition to internalized reasoning. For most students, activity within the number construction zone involved multiple-step Reasoning strategies that sometimes included use of Counting, Recognition, Estimation or Algorithm strategies.

Data about mathematical activity within the number construction zone can provide valuable assessment information on children's construction of meaning for number, or number sense. Activity within this zone provides information on both the extent of the number knowledge accessible to the child, as well as the inclination of the

child to make effective and creative use of that knowledge. Strategy use within the number construction zone likely involves mathematical processes the child is in the process of constructing, elaborating, and refining. The learning styles and preferences of the child working at this level are observable especially where materials are involved. The kinds of operations and procedures utilized to construct mathematical meaning within this range have the potential to explicate the nature and limits of the child's personal number domain.

The degree to which children engaged in constructive mathematical activity in this study appeared to provide a rough estimate of the extent of their personal number construction zones. For example, the strategy profile of seven-year-old Bevan indicated the use of constructive mathematical activity in 20 of his 31 items, indicating that he worked well beyond his range of familiar number relationships as indicated with Recognition responses for 10 of the 11 other items. Using this strategy profile as a window on his capacity to make sense of number situations, both his ability and inclination to construct meaning were clear.

In contrast, Nina, also seven years old, used constructive activity for only one of the 15 items she attempted. For her other 14 items, she used unitary Counting procedures for 10 items, and Recognition for 4 items. This pattern illustrates that Nina depended on her ability to apply automatic responses and procedures for mathematical meaning and was reluctant to use her existing knowledge to construct responses to unfamiliar items through reasoning. Figure 4 applies the strategy choice patterns for these two children to the hypothetical model for characterizing a personal number domain as described in Chapter Two. The difference in the scope of these number construction zones (as estimated from differences in strategy choice patterns) is striking, with important implications for instruction.

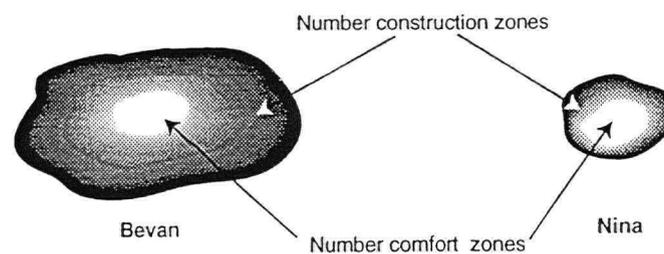


Figure 4. Variations in children's number construction zones

Findings from this study highlighted the relationship between aspects of mathematical disposition and the extent of children's number construction zones as indicated by the items they were willing to attempt beyond what they knew automatically. Children such as Bevan demonstrated confidence in their abilities as capable problem solvers as they attempted to construct meaning across a wide range of items. The performance profiles of others such as Nina indicated a narrow range within which the construction of meaning was attempted. Part of this can be attributed to differences in mathematical background and a lack of appropriate tools, but an important part appeared to be a lack of confidence, an unwillingness to persevere and an unwillingness to take risks. Exploring the range and extent of children's number construction zones may provide an alternative means of considering children's mathematical dispositions as well as an indication of children's inclination to construct meaning with the tools available to them.

Children's Number Contexts

Results from this study support the contention that children have qualitatively different ways of interpreting and dealing with number situations. How young children see or interpret number situations depends to a great extent on their personal number contexts (Cobb, 1990), and the different ways in which they conceptualize number. Steffe and Cobb (1988) used children's progressively more sophisticated counting schemes to qualitatively differentiate children's mathematical activity. Confrey (1994b) used the distinction between counting and splitting conceptualizations of number to account for qualitative differences in performance. This study drew on both of these radical constructivist interpretations of children's thinking to analyze children's performance and to generate a means of characterizing children's approaches to the number situations in this study.

Results from this study support the notion that young children conceptualize number in qualitatively different ways as they develop a progressively more sophisticated and complex understanding of number and its applications. These different conceptualizations of number are proposed to provide a lens or filter through which children interpret and approach number-related situations, however multiple considerations of an affective, functional, and contextual nature are recognized to also influence children's sense-making activity in important ways.

The following four qualitatively different conceptualizations of number characterize the personal number contexts within which children interpreted the problem situations in the interviews:

1. number as a global estimate,
2. number as counting,
3. number as pattern,
4. number as grouping.

The first number context, Number as a Global Estimate, was used least often by the children in this study due to their ages, but is proposed to represent the initial and most prevalent conceptualization used by very young children up to roughly four or five years of age.

Both Number as Counting and Number as Pattern are considered to provide alternative connecting paths between the first and fourth categories or contexts, though these paths are very different one from the other. Ideally, children develop the capacity to apply both pattern and counting conceptualizations for number, rather than developing an over-reliance on either of these two qualitatively different paths. The Number as Counting path is based on a linear, unitary, quantitative approach, while Number as Pattern draws on non-linear, spatial, qualitative conceptualizations for situations involving numbers and quantities.

The fourth level, Number as Grouping, represents the most powerful and widely applicable conceptualization involved in the realm of additive structures, and provides the connecting link with multiplicative structures.

Number as a Global Estimate

What can be referred to as a pre-number conceptualization, this level was characterized by a global approach to number situations without any focus on accuracy, but showing an understanding of the relationships or "big ideas" which provided the underlying structure of the problem. Resnick's (1983) protoquantitative schemas provide the conceptual basis for this approach.

Research indicates that prior to the development of systematic quantification strategies, very young children begin to construct a number concept based on their

experience with perceptual information in their environment. At the same time, with the support of older siblings, parents, and teachers, they begin to make their own sense of our socially and culturally shaped ways of using number. Over time children refine and elaborate their initial global and intuitive approach to number, and develop other more accurate and systematic ways of dealing with numbers and quantities. However, this global approach remains available as an alternative.

Resnick (1989b) described these initial pre-number concepts as protoquantitative schemas which express quantity without numerical precision through perceptual rather than measurement processes. She described three such protoquantitative schemas: the comparison, the increase/decrease, and the part-whole schemas. According to Resnick, these three reasoning schemas constitute a major foundation for later mathematical development, and as language develops, pre-schoolers' implicit protoquantitative reasoning schemas combine with early counting knowledge to generate number concepts.

Research indicates that the roots of another quantification process are established during this pre-number phase, and provide a second path to generating number concepts alongside counting. This is the perceptually grounded capacity to immediately recognize small groupings as specific numerosities. This capacity has been referred to as "subitizing" or "figural grouping" (von Glasersfeld, 1982). Very young children have been shown to be able to recognize specific grouping patterns and to assign number accordingly (Klahr & Wallace, 1973). In fact, the neo-nativist position (Carey & Gelman, 1991) posits that humans are pre-wired with a quantification capacity that even newborns have access to, and that experience is mapped onto this capacity over time. Regardless of the genesis of this capacity, it was clearly related to the non-counting processes used by children in this study.

Once young children begin to apply these early intuitive notions in conjunction with reliable and systematic quantification strategies, they have moved beyond this first and earliest conceptualization for number. Children's reliance on their global, perceptually-based pre-number conceptualizations gradually is replaced by a preference for systematic quantification capacities. However, pre-number thinking is still available to children as a fall-back position in times of need (i.e., when the situation is beyond a child's number comfort zone). Sam's use of a perceptually

grounded Estimation strategy for the Missing Parts items and for Sharing items beyond his comfort range provides an example of such use.

Number as Counting

The linear order of unitary counting is generally recognized as children's initial basis for developing an understanding of whole number concepts and operations (Fuson, 1988; Gelman & Gallistel, 1978; Steffe & Cobb, 1988). Two qualitatively different unitary counting approaches were observed in the study in keeping with similar descriptions in the literature.

An **early counting** approach was characterized by linear, unitary counting involving enumeration by ones and from one with direct modelling of the problem elements, rather than counting-on or internally representing any aspect of the problem. Piaget's ordinal counting, counting-all (e.g., Carpenter & Moser, 1984), and Steffe and Cobb's (1988) perceptual and figurative counting schemes provide the conceptual basis for this interpretation of number situations.

The concept of one-to-one correspondence plays a distinguishing role in differentiating this early counting level from the pre-number level of Number as a Global Estimate. Once students develop one-to-one correspondence, a reliable count-by-ones counting chain to match with objects, and the focus, coordination, and inclination necessary to label and account for elements in a group, they begin to have a degree of accuracy to their counting and related number knowledge (Gelman & Gallistel, 1978).

An **established counting** approach was characterized by evidence of the ability to apply both the inclusion and order relations that are the basis of cardinal number (Piaget, 1952) and to conserve or recognize the invariance of number. Children approaching number situations from a level of established counting made use of mental representations of number in order to count-on or back, and demonstrated the capacity to keep track of a double count for counting-on or back with tally. Fingers and other sorts of tally methods were widely used with this approach. At the most proficient end of this approach, children confidently applied an internalized mental count to solve a wide range of problems. Piaget's (1952) cardinal counting, Steffe and Cobb's (1988) initial number sequence, Carpenter and

Moser's (1984) counting-on provide different labels for describing this approach to number situations.

Number as Pattern

Recognition of visual-spatial pattern-based conceptualizations of number has received relatively minimal attention in the literature as compared to the emphasis on counting. Findings from this study suggest that children's use of non-counting approaches to number situations are more prevalent than is suggested in the literature, but more importantly, are an important link between children's early global-perceptual intuitions about number and the use of increasingly powerful grouping notions. This was particularly evident in the Sharing Task, where children rarely used a count-by-ones, dealing approach to sharing, and instead used perceptual estimates based on either the physical size of the starting set, visual-spatial patterns in arrays, or internalized number patterns and relationships.

This approach appeared to reflect a different way of thinking, that of seeing number from a qualitative (Sharma, 1989), global perspective rather than in a linear, quantitative way. This conceptualization, labelled Number as Pattern, was characterized by the use of visual-spatial pattern recognition as a means of making sense of a problem situation. This approach is in contrast to the use of linear, systematic, increasingly sophisticated unitary counting schemes. Von Glasersfeld's (1982) perceptual numbers, Confrey's (1994b) splitting, and subitizing as described by Klahr and Wallace (1973) and others, provide the conceptual basis for this interpretation of number.

In this study, number as pattern conceptualizations were apparent in use of external Recognition and certain Reasoning strategies. Often the number as pattern conceptualization appeared to be triggered by the perceptual characteristics of a concrete model, especially with the younger children. For example, in the Sharing Task, Cliff looked at a three by four array of blocks and immediately recognized how many blocks three or four people would get.

In explaining to Chris's mother his interesting use of pattern in an interview, she reported how, as a pre-schooler, Chris developed a sense of number as pattern from working with Lego blocks. He would ask for specific blocks based on the dot pattern of the block, as in "I need an eight" when he wanted the block with two

rows of four joiner dots. The patterns of Lego blocks provided him with a frame of reference for specific number patterns, and may well have had an impact on his construction of meaning for early number generally.

In some cases even a verbally presented problem without a corresponding model was interpreted by some children based on internalized patterns, as indicated by their explanations of their thinking. Explanations that made reference to intuitive, visualized patterns for number were usually more difficult to interpret than counting-based explanations, and usually were generated by the more mathematically able. One conjecture is that children with a visual learning preference may draw heavily on number as pattern conceptualizations, while children with auditory learning strengths may rely more on counting-based conceptualizations.

Number as Pattern is interpreted as a complementary path to counting as a means of systematic quantification. In conjunction with counting, number as pattern is proposed to provide a second path between pre-number approaches to number situations and grouping approaches. While exclusive reliance on either a pattern or a counting conceptualization of number characterized the least proficient performances, the most proficient children in the study made use of both of these conceptualizations. In particular, visual-spatial strengths appeared to characterize the performances of children who were the most proficient and creative in their approaches to number.

Mathematics education for young children has traditionally placed a heavy emphasis on counting approaches at the expense of pattern approaches. Confrey's (1994b) work on the parallels between the construction of linear and exponential functions differentiated an alternative to counting. She distinguished her notion of splitting by its conceptual connections to the geometric transformation, similarity. Confrey argued that splitting, with its ties to partitioning, is an alternative basis for the construction of a number system and possesses strong explanatory potential for interpreting children's methods (p. 300). She described the split, an action of creating equal parts or copies of an original, as a primitive operation that is a precursor to a more adequate concept of ratio and proportion and subsequently to a multiplicative rate of change and the exponential and logarithmic functions. Confrey proposed that splitting and counting are complements that they

have their roots in the complementarity of geometry and arithmetic, with splitting structures producing geometric sequences and counting structures producing arithmetic sequences.

Counting models, such as Steffe et al.'s (1983) do not adequately account for conceptions of number that are holistic, symmetry oriented and visual-spatial in nature. Confrey's proposal of an exponential splitting (rather than linear counting) basis for number, provides a possible model to account for the global, spatially intuitive, pattern-based methods used by some students in this study as a basis for interpreting the number situations they faced. These global methods appeared to draw on the perceptually-based capacity to immediately recognize the numerosity of small groups (subitize) as well as the inclination to draw on qualitative visual-spatial approximation strengths. Both of these capacities are independent of a linear counting conception for number, having closer ties to Resnick's (1983) pre-number proto-quantitative schemes and von Glasersfeld's (1982) figurative grouping.

Number as Grouping

The most proficient and potentially productive approaches to number situations used in this study involved varying degrees of use of a multi-unit conceptual framework. Such approaches took into account the grouping patterns and relationships inherent in the problem solving situation. The conceptual acquisitions that appeared to support this new conceptualization included a flexible approach to decomposing and recomposing numbers and an understanding of many-to-one correspondence.

The label **early grouping** approach was used to describe the non-systematic use of grouping concepts in the interpretation of problem situations. This approach for multi-digit situations usually involved some aspect of place value multi-unit grouping. However, it did not include the capacity to consider a number situation simultaneously in both unitary and multi-unit terms. The conceptual basis for this approach relates to Cobb and Wheatley's (1988) "ten as an abstract composite unit", as Kamii's (1986) second level of place value interpretation, and as Ross's (1989) stages three and four which are characterized by unreliable, inconsistent performance in coordinating ones and groupings of ten.

The label **established grouping** approach describes the most sophisticated conceptualization students used for the different tasks. It involved a perspective that highlighted the multi-unit relationships inherent in the problem, and was connected with systematic application of grouping principles. This approach involved the simultaneous recognition of quantities both as a collection of ones and as various multi-unit groupings. This level corresponds to Kamii's (1986) third level of place value interpretation, and Cobb and Wheatley's (1988) ten as an iterable unit. This approach represents a fully operational grasp of the additive composition of number, which in turn can be considered an early form of multiplicative reasoning.

This section has described the schemes, templates, or frameworks that appeared to shape students' interpretations of number situations across contexts. It is proposed that these schemes constitute the child's personal number context or the lens through which number situations are interpreted. With age and experience children expand their repertoire of possibilities to include other ways to conceptualize number. For example, Sam's number comfort zone was characterized by two of these lenses: an early counting framework and his global, visual-spatial intuitive sense of quantity. He frequently applied Estimation to his items and showed early counting proficiency at least for very small numbers. Within his number construction zone, Sam was beginning to develop a more sophisticated counting conceptualization that included the additive composition of number, however this cardinal counting scheme was not fully established. Similarly, he was in the process of developing a systematic approach to number as pattern, based on his ability to recognize small groups without counting. As evidenced by the inability to apply Reasoning strategies or multi-unit thinking to any items, it appeared that grouping conceptualizations for number lay outside Sam's frontier of understanding.

This description of increasingly powerful conceptualizations of number constitutes a synthesis and reworking of the literature into a practical framework for interpreting children's developing number sense. It recognizes that within the constraints of any particular view of number, children can demonstrate effective and creative ways to construct mathematical meaning. Our job as educators is to recognize, respect, and support sense-making both within and across these diverse number contexts.

Chapter 8

Implications and Recommendations

Findings from this study challenge the results from other similar studies in several respects. For example, most studies describe the prevalent use of counting strategies in young children's mathematical activity and emphasize the centrality of counting to children's understanding of number. Findings from this study suggest a much broader base to children's construction of meaning for number, in particular the use of strategies related to grouping and pattern conceptualizations of quantity. In trying to account for this difference, the best explanation I can provide is that the tasks and items used in this study were not limited to the realm of basic facts and instead offered a more open-ended range of possibilities. Similarly, the way my strategy categories were defined was different from other studies. More importantly, my focus was not on correct answers, rather it was on the means by which children constructed meaning for number situations. Tasks were designed to find out what students would or could do to solve the problem, rather than what was the right answer. Items were purposely posed beyond children's "right" zone to investigate their constructive processes and their partial understandings. Given these variations, the different results make more sense.

I. Implications for Curriculum and Instruction

Mathematics curricula are shaped to reflect both the nature of the knowledge we value, and the priorities we place on particular topics, concepts, skills, processes, and attitudes. Instructional strategies are shaped by similar priorities. Findings from this study have implications for what we expect of young children's mathematical thinking, and what emphasis we place on particular instructional methods and goals. The following section provides suggestions for shifting these emphases based on the results of this study.

Encourage an Emphasis on Thinking and Reasoning

Traditionally primary classrooms have placed heavy emphasis on the accuracy of answers rather than on the processes used to generate answers. At a time when children are in the process of constructing increasingly powerful ways of working with numbers and quantities, exploration and risk-taking with new ways of thinking should be a primary focus. An over-emphasis on speed and accuracy at this stage

encourages children to rely on their trusted counting methods. Such an emphasis on speed and accuracy unnecessarily curtails the development, use, and reinforcement of new strategies and ways of thinking about number. This is not a problem for the young children who come to school brimming with confidence and with access to a range of strategies and conceptual tools. Over the course of the school year, these children practice different approaches, and flexibility is reinforced. However, a focus on speed and accuracy at the expense of thinking directly affects young children who have access to a limited repertoire of strategies and who cling to unitary counting as their only reliable means of generating a “right” answer.

The development of reasoning strategies that build on children’s existing knowledge to construct new understandings in unfamiliar number contexts should be a priority. Shifting the classroom culture to value elegant, original, creative, new, and alternative thinking would give children a very different message about what is important. If the construction of meaning, flexibility in conceptualizing and thinking through a problem solution, and the development of progressively more sophisticated and powerful ways of dealing with number are priorities, instruction should recognize and support the corresponding mathematical activity. “Mistakes” within this context take on a new meaning; they provide feedback for re-thinking and are important steps towards more powerful ways of thinking.

Emphasize Children’s Intuitive Number Knowledge

One change in emphasis would be to recognize, support, and build on children’s intuitive, visual-spatial notions about number in more systematic and visible ways. Privileging linear, algorithmic, procedural thinking at the expense of global, holistic, non-linear thinking places many children at a disadvantage both in the short and long term. Very young children have access to global, intuitive ideas about numbers and quantities well before they have systematic quantification skills (Resnick, 1983, 1989b). According to Resnick these early informal concepts and intuitions provide the foundation for future success with making sense of number. These ideas include pattern recognition, perceptual “feel,” visual-spatial sense, and an estimation-type sense of what is reasonable. Some children are quick to abandon these ideas in favour of the more quantitative, culturally-defined aspects of number based on counting. Others hold these ideas alongside their developing counting interpretations of number. Still others, such as Sam, continue to rely on their earliest

notions of number while developing a tenuous grasp of systematic quantitative approaches to number.

Introducing grouping activities similar to the Sharing Task, emphasizing non-counting patterns first through subitizing small groups, then through finding these small groups within larger groups, might build on the perceptual grouping strengths of students like Sam. Similarly, work with geometry and spatial relationships would enhance children's intuitive notions about number and its relationship to space. Informally dealing with area and tiling concepts earlier might be another approach. And finally, emphasizing the interpretation of graphs in terms of relative quantities and comparisons rather than solely as accurate representations of absolute values (e.g., discussing the big ideas behind area representations such as pie graphs) might tap into children's relative notions about numbers and quantities based on a wider range of perceptual information. Such examples of global rather than linear approaches to quantities would capitalize on children's intuitive understanding of quantity and provide a balance to the heavily unitary, linear, counting-based emphasis present in most primary classrooms.

Using these global conceptualizations of number in conjunction with linear, accurate counting-based interpretations of numbers and quantities would serve several purposes. It would provide a more inclusive approach to the understanding of number, accommodating and building on young children's different learning strengths and developmental levels. It would encourage the development of a wider range of strategies for dealing with number situations, thus placing counting strategies within a wider context of strategy options. It would place a higher priority on children's intuitive feel for number, and the meaning behind number and quantity situations. This would serve to support the development of estimation capacities as a respected form of mathematical thinking. It may well serve to support the development of a multi-unit conceptual framework through early exposure and reliance on patterns and groupings. And finally, it would more directly connect with the patterns underlying our number system, thereby possibly providing more direct and reliable access to the development of multiplicative reasoning (Confrey, 1994b).

Recognize and Support Children's Different Approaches to Number

Adapting instruction to accommodate children's different conceptualizations of number requires understanding children's thinking from their perspective, recognizing how children's personal number contexts differ and how those differences affect their interpretation of and approach to number situations. The increasingly sophisticated ways children approached the different tasks in the study provide examples of these qualitatively different interpretations. The description of children's number contexts provides a frame for considering these interpretations.

Providing relevant, flexible and open-ended mathematical situations, and exploring the range of ways that can be used to approach and deal with the task can help parents, teachers and students better understand and respect these alternative approaches. Such a focus provides a meaningful basis for working towards the increasingly efficient, abstract, culturally-defined methods that characterize school mathematics at higher levels.

Support the Shift from Counting to Grouping

As a means of dealing with addition, subtraction, and a wide range of everyday situations, unitary counting is a versatile, developmentally appropriate strategy used by most young children in our culture. However, counting by ones is a highly limiting strategy when used as an exclusive approach to number situations, especially beyond the age of 7 or 8. Children necessarily rely on counting when they look at number as a system of ordered ones (unitary conceptual framework). Once children begin to see number as a system of multi-unit relationships, they have the capacity to look at number situations in a qualitatively different way.

Shifting our attention from computational facility which encourages the use of thinking in unitary terms, to making sense of multi-unit relationships as a basis for problem solving and computation requires a shift in how we assess "progress" in making sense of number. Focusing on underlying conceptual schemes and how they influence mathematical activity could support that shift.

Gray (1993) proposed that some young children "wished to remain at a procedural level" (p. 2) and chose to rely on procedural counting methods. I suggest that young children hold to their primitive and trusted counting methods in order to cope with the demands of their social and educational contexts, that their "choice" is a forced

choice. In other words, the contexts we shape can serve to “freeze-frame” young children at a counting level rather than support and encourage them to move towards more powerful strategies for dealing with number. Children lacking in confidence would be at particular risk in this. As Gray (1991) points out, for some children, procedural counting methods do not encourage the need to remember and that the procedure provides security. Use of deductive or derived methods on the other hand enhance the ability to remember other basic facts. Gray suggests that only through continued successful use does derived fact strategy use become an approach that can take the place of counting as a preferred strategy. Considering patterns to children’s strategy use provides the possibility of recognizing such qualitative differences in children’s mathematical activity, and shaping instruction accordingly.

Use Children’s Thinking as the Basis for Instruction

Results from this study support the contention that children’s thinking and child-generated algorithms should be used as the basis for instruction in the primary grades (Kamii, 1985; Cobb, Wood, & Yackel, 1995; Steffe, 1983). This requires that the introduction of standard adult computation algorithms be delayed until children have made their own sense of multi-digit operations. Activities such as the mental addition Doubling Task can provide evidence of which students have constructed their own meaning for multi-digit addition. Use of place value blocks to support externalized Reasoning, and use of internalized Reasoning strategies involving the decomposition and recomposition of place value groupings can provide indications that children have the necessary conceptual base to make sense of the standard algorithm.

Emphasize Functional Competence

Results from this study illustrated the extent to which metacognitive, strategic, and representational characteristics affected individual children’s self-regulatory activity in their dealings with number situations. The more capable and confident students utilized a wider range of strategies across items and appeared to flexibly choose which of their available strategies was appropriate in a given situation. They showed flexibility in interpreting situations in additive versus subtractive ways, in accurate, quantitative versus global, qualitative ways, in linear versus pattern-based ways, and showed the capacity to change their approach when faced with a hurdle. They flexibly moved between abstract and concrete representations.

On the other hand, less confident and less able students did not demonstrate the same degree of flexibility. Instead they used a more consistent approach across items in a task sequence, and across tasks, both in terms of strategy use and representational level. This conclusion corresponds to findings by Thornton (1990) and Steinberg (1985) who found that less able students were reluctant to change their counting strategy for solving basic number combinations. Similarly, Krutetskii (1976) described how mathematically able pupils demonstrated "flexibility in their mental processes" while less able children found it hard to even switch "from a harder to an easier method if the first is habitual and familiar and the second new and unfamiliar." The degree to which students used a reflective approach also provided an important qualitative difference in children's problem solving performance, and illustrated the centrality of monitoring processes to effective sense-making.

Increased attention to children's personal ways of monitoring their thinking, and to examples of flexible thinking in the classroom would serve to provide a focus on these positive characteristics and a model for less flexible, impulsive learners. Instruction that focuses on alternative ways to model mathematical ideas, and on ways to organize and monitor mathematical activity would support children's development of systematic and reliable mathematical thinking. Such a focus would serve to encourage recognition and awareness of children's different approaches to learning and ways of knowing, and would provide a means of highlighting the role of metacognition in supporting children's mathematical activity.

Emphasize Affective Considerations

The one feature of children's performance that did not appear to be related to age, experience, or conceptual level concerned attitude. Some children's actions in the interviews indicated a strong sense of self, a belief in their capacity to be successful, a confidence in their mathematical abilities, a willingness to try, and a generally positive approach to each task. These personal characteristics appeared to sustain those children through a wide range of constructive mathematical activity beyond their comfort zone. These characteristics were also usually typical of the more able students in each age group. The exception to this was the six-year-old group who all showed positive mathematical dispositions though they demonstrated very different levels of mathematical activity.

Within the two older age groups, some children appeared to have little confidence in their capacity as a capable thinker and problem solver. They were quick to label an item as "hard," they relied on interviewer support, and they seemed to need an inordinate amount of encouragement to continue. By and large, these students engaged in very little constructive activity and appeared to be the comparatively less able within their age groups.

Mathematical topics offer many opportunities to support and sustain the confidence and positive attitudes typical of young children. Within primary classrooms we need to find ways to respect developmental differences while supporting **all** learners through appropriate instructional opportunities. This suggests:

1. using activities that are relevant, fun, active, and varied in order to sustain a positive disposition towards mathematics;
2. using activities that are accessible to multiple levels of understanding to ensure success at some level;
3. attending to alternative ways of thinking and multiple solution paths;
4. attending to individual differences in thinking;
5. using children's thinking as the basis for shaping instruction; and
6. modeling curiosity and a positive problem solving approach.

Attitudes towards mathematics are established at a very early age. After a year or two of struggling to keep up versus a year or two spent being successful, this is not surprising. Finding ways to recognize young children's unique strengths, and sustain positive attitudes in the earliest years when such wide developmental differences and variations in thinking are apparent, should be a priority for the mathematics classroom.

Reconsider the Role of Materials in Supporting Thinking

Study results highlighted the role of concrete materials in supporting young children's construction of meaning for number. They also provided a means of assessing the degree to which children had internalized number concepts. The role of unitary models such as cubes and buttons in providing an external representation for children in the process of internalizing number has been well documented by Steffe et al. (1983, 1988). What this study highlighted in particular, was the role of

place value blocks in providing a model of children's thinking, offering valuable information for understanding how children conceptualized number situations. For instance, when 16 was modeled by the youngest children in the study, some used 16 ones blocks and counted by ones. Some used a ten and 6 ones and counted up each cube from one to 16. Others used a ten and 6 ones, and accepted the ten then counted-on to 16. Similarly, whether students placed the blocks in a linear arrangement as opposed to a side-by-side place value alignment also provided a clue on how they conceptualized number at that point. A linear arrangement suggested a unitary sequence of a collection of ones, while a side by side arrangement suggested a distinction between units and composites, or multi-unit thinking. These variations in interpretation, especially seen in a variety of situations, served to provide a window into the child's development of number concepts generally.

Place value blocks, though sometimes difficult for young hands to manipulate, provided a visual model and mental cue for grouping. Children could interpret them either as a collection of ones (i.e., single blocks glued together) or as composite units, or as both. Thus the blocks appeared to provide conceptual support to children's construction of a multi-unit conceptual framework, an observation which conflicts with the belief that children should not use place value blocks until they have constructed meaning for multi-unit groupings. In fact, the inconvenience of working with the single unit blocks may even encourage children to use chunks of ten (ten rods) even while they count them up by ones, thus providing a catalyst for the construction of meaning for ten as a composite. This gives place value blocks the advantage of acting as an advance organizer for internalizing multi-unit notions as well as acting as a conceptual support to children's understanding of multi-digit number situations.

Place value blocks provided a third means of supporting children's working memory by holding aspects of a problem in place in order to work through a reasoning process. This functional aspect to the use of place value blocks allowed students to work mentally well beyond what their working memory could have accommodated. The blocks replaced what a paper and pencil would have offered, however the blocks stimulated the construction of meaning as opposed to stimulating the use of a procedural algorithm as paper and pencil were found to do in pilot studies. Delaying printed algorithms and replacing them with the use of

place value materials in conjunction with mental computation would support the construction of meaning for multi-digit computation, and would stimulate the development of personally meaningful and creative computation strategies.

Expand Children's Range of Meaningful Numbers

Results from the study highlighted the importance of expanding the range of numbers that hold meaning for children in order to allow the generalization of mathematical ideas to increasingly broad contexts. In particular, the Doubling Task served to illustrate the importance of working with a flexible number range that could be shaped to suit the wide variations of understanding within a primary classroom.

Three recommendations relate to the development of an increasing range of meaningful numbers. What seemed clear was that it was essential that children have a verbal counting chain that extended well beyond the range which held meaning for them. In the case of the Doubling Task, often children were able to construct a model of doubling, were able to describe the part-whole relationship their model illustrated, but were unable to count up high enough in order to name the whole. Encouraging the development of counting and number naming skills clearly supported children's construction of meaning.

Similarly, personally meaningful number benchmarks such as Sam's set of dinosaur books ($13+13=26$) or Chris's dues ($25+25=50$) for Cubs stimulated mathematical connections and provided students with a frame of reference upon which to construct meaning for number. This finding highlights the importance of supporting children in their development of an increasing store of personal benchmarks for number. A mathematics program centred on children's personal interests, everyday experiences, and purposeful problem solving would likely serve to best support the development and use of personal number benchmarks.

The open-ended nature of most of the tasks used in this study offered children the opportunity to apply their number knowledge in practical and creative ways well beyond numbers to 20. What was interesting was that more than half of the students capable of using multi-unit Reasoning strategies used unitary counting strategies for items involving numbers to 20. Not until the items were well beyond this familiar range did they make use of their more powerful grouping capabilities. This

observation suggested that number contexts beyond children's number comfort zones might be more conducive to eliciting more sophisticated thinking while number situations within a comfortable counting range elicited less powerful but more familiar strategies such as counting by ones. This finding suggests that the early grade emphasis on basic fact combinations should be accompanied by experience within a wider number range where new and more powerful grouping strategies would outshine cumbersome counting strategies.

II. Implications for Assessment

"To replace the unmeasurable with the unmeaningful is not progress."
(Achen, 1977, p. 806)

Findings from this study underline the fact that the aspects of children's mathematical activity that are meaningful in terms of indicating progress and potential are not necessarily easy to identify much less measure. This is in good part due to the complexity and inter-relationship of important aspects of children's sense-making activity. It is also due to the fact that affective and metacognitive factors, both of which require qualitative rather than quantitative consideration, play important roles in how children interpret and deal with number situations.

Educators have long considered that having a store of known facts and processes for computing indicates progress or achievement. Though these skills are relatively easy to assess, this study's findings indicate that these traditional measures may not provide an accurate reflection of the development of number sense. They further suggest that **how** children deal with mathematical situations may provide a useful and comprehensive assessment of their capacity to make sense of number. Study results lead to the conclusion that attention to the strategies children develop and use can provide important assessment information to monitor progress and to support instruction.

Results support the contention that conceptual understandings provide the foundation for children's construction of meaning for number, but that these understandings need to be considered in light of other aspects of children's mathematical activity. For example, attention to student's beliefs, attitudes, and emotions can provide valuable assessment information. In addition, children's

representational capacities are important to their understanding of mathematical situations. Metacognitive and strategic skills clearly played a critical role in shaping children's mathematical activity. None of these is simple to assess. The approach used in this study offers an alternative which takes into account the complex inter-relationship of the factors which affect children's construction of meaning for number.

How then can assessment attend to the construction of meaning for number? First, it needs to shift its focus away from assessing mastery of facts and procedures and on to children's construction of meaning for number. It needs to focus on the process as much as the products of thinking. It needs to look beyond what is "mastered" to consider shades of understanding, partial knowledge, and the intuitive, informal schemes that underlie more formal mathematical concepts.

Findings of this study illustrate how a dynamic and global approach involving a range of perspectives on children's mathematical activity can be used assess progress and to consider change over time. Results show how children's use of strategies can provide an indication of increasing sophistication and competence in making sense of number. It explores ways to investigate children's constructive mathematical activity in addition to their independent activity. And finally, it sheds light on a full range of learner characteristics and features of performance that are related to competence with number.

III. Questions for Further Research

Findings from this study raise questions about the nature of the mathematics children experience based on the different strengths and weaknesses they bring to the classroom. In particular, how is learning different for children who come to school with:

- a positive orientation towards mathematics and their ability to learn mathematics as opposed to a less than positive orientation?
- a variety of strategies for approaching number situations, such as Maré's repertoire of recall, reasoning, counting, estimating then checking and revising as opposed to a limited range of options?
- a strong intuitive sense about number?

- solid conceptual understandings?
- a repertoire of counting and basic fact skills?
- strong organizational, metacognitive, or strategic monitoring capacities?

To what extent can a strength in one area make up for a weakness in another? What kinds of instruction might enhance these abilities in the early grades? What kinds of environmental supports might foster the early development of these capacities?

How might a global, dynamic assessment approach be applied to other areas of the mathematics curriculum? Would there be a value to standardizing interviewer cues? In what ways other than number size, level of representation, and degree of complexity might task sequences be developed?

How does a visual-spatial strength in recognizing quantities without counting relate to the development of a multi-unit conceptual framework? How does visual-spatial number sense relate to estimation capacities?

Findings related to estimation capacities were unclear due to the nature of the strategy categorization. Estimation as a one-step response was used by younger or less able students, however estimation as an element of an estimate-test-revise Reasoning strategy was used by the more able students. Investigating the diversity of estimation use across age groups would provide a more accurate picture of the role estimation plays in children's sense-making activity.

This study's findings related to the use of strategies would be worth testing on a wider scale. Using the same starting point in task sequences for children of all ages might provide a clearer picture of the role of Recognition strategies in differentiating performance. Providing items that extended beyond those used in the study would provide a better estimate of the capabilities of high performing students such as Bevan. Comparing children's mental problem solving capacities with paper and pencil algorithm proficiency and understanding might provide more information on the efficacy of using mental number activity as an indicator of readiness for algorithms.

Analysis revealed that the Money Tasks provided information on children's mathematical thinking that was closely related to their capacity to make sense of the other tasks. This relationship warrants further investigation.

And finally, an original aspect of this investigation that was not included in this write-up concerned the possibility of using children's mathematical activity across the four tasks as the basis of creating a portrait or map of a child's personal number domain at that point in time. Such a global portrait would provide a useful alternative assessment option for teachers and warrants further research.

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Appendix A - Letter of Introduction and Consent

January 20, 1994

Dear _____,

This letter is a request for your child _____ to participate in a study called "Making Sense of Number: An Exploratory Study of Developing Competence". The study will be conducted over the next three months, during which time I will be working with several students from each of two schools to study how these children make sense of number situations. I am interested in the different ways young children think about and work with number, and ways to assess number sense. The data I collect will be used to describe the different ways children reason with number, and will be considered in terms of current research on number sense.

There will be three aspects to the study. The first aspect involves each of the selected students working individually with me on a set of number-related tasks. The purpose will be to identify the processes, skills, and concepts students draw upon to make sense of the tasks, and to identify the nature and range of each child's understanding. These interviews will require a total of 3 or 4 hours per child spread over the course of several weeks, and will be videotaped. Sessions will be completed in the school, and will be scheduled to minimize any conflict with the classroom program.

The second aspect of the study involves the observation and videotaping of specific students as they work with number activities in the classroom, in both individual and group situations. I will be planning these number activities with teachers for use with the whole class. These activities will involve approximately 3 hours of class time over several weeks.

The third aspect of the study involves an interview with parents to try to establish some history to the child's current understanding of, and interest in number. No sensitive information is being sought. These interviews will be audiotaped for data interpretation purposes. During this interview, questions can be addressed and videoclips and data will be available to you for discussion. A similar interview with teachers will be conducted at a separate time from parent interviews. A minimum of one hour of your time will be required for this interview. I will contact you by phone to arrange a mutually convenient time for these interviews.

The purpose of this study is to show the range of unique, individual, creative and developmentally appropriate ways that young children make sense of number situations. Videotaping students as they work will be an important aspect of the study, as many indicators of student reasoning are visual, such as the use of fingers, counters and models. To ensure confidentiality, first names only will be used in any

"Making Sense of Number" Project Consent Form

Please return one copy and retain the other for your records.

I _____ have received a copy of Heather Kelleher's description of her study entitled "Making Sense of Number: An Exploratory Study of Developing Competence". I have read the description, and have kept one copy of the consent form for my own records.

Signature _____

I consent I **do not** consent to _____'s participation in the study.

Signature _____

I consent I **do not** consent to participate in the parent interview.

Signature _____

I consent I **do not** consent to my child being videotaped, and to the use of videotape excerpts in the dissertation. I understand that I will have an opportunity to view the excerpts before such use.

Signature _____

I understand that participation is entirely voluntary, and that non-participation or withdrawal from the project will in no way affect my child.

Signature _____

Date _____

Appendix B - Doubling Task Format

This task involved students mentally doubling numbers, with the size of the numbers increasing at each turn. To begin, rapport with the student was established, then the calculator doubling task was introduced in a manner appropriate to the age of the student. The youngest students usually needed clarification of what was meant by doubling and this was done by modeling with materials. Older students usually picked right up on the demands of the task. The calculator was introduced as a device to use for checking their doubling responses. Students were encouraged to use whatever methods made sense to them, and were introduced to optional resources including counters, place value blocks, a hundred chart and a number line. The following interview format was used:

I am going to make the calculator into a doubling machine. (Press $2x=$.)

When I press =, the number in the display will double. Now it says 2.

What is 2 doubled?" Child says 4. If not, clarify what constitutes doubling, using fingers or counters, then return to the question.

Press the = sign to see if you are right. Right...4.

How did you know it would be four? (e.g., If this is 2 and this is 2 then it is four in all.)

Now, if you doubled 4, what would the display say? ...Check to see if you are right.

As the number size increased, students moved beyond the range where they either knew answers or could work the doubled number out mentally, and moved to other forms of reasoning. Interviewer comments varied for each child, but followed along these lines:

I am interested in how you figure out what the number will be.

Can you tell me what you did to get 16?

I saw you use your fingers...how did they help you?

What were you thinking/saying to yourself as you worked?

What did you see when you looked over there? (or other appropriate follow-ups to student actions such as staring to a point in space.)

Do you think you can go any higher?

Could you figure out what 64 would be if you doubled it?

What could you use to help you?

Can you show me how you figured that out?

I saw you using the blocks...how did they help you?

Can you tell me what you said to yourself as you moved the blocks?

What a neat way to do that...can you explain it to me?

Is that as far as you can go?

What if you tried it with these blocks ...would that help?

Could you double any number like you did that one? Show me.

Do you think that is the end for you? What could help you go further?

Appendix C - Missing Parts Task Format

The Missing Parts Task involved students naming a missing part of a set, given the whole and one part. The task was introduced by placing a numeral card on the table and asking the student to put out that many blocks. Younger students and less confident students started with a set of 8, while the other students started with a set of 11. Students closed their eyes while a box lid was placed over one part of the set, then students were asked to open their eyes and name the missing part.

The interviewer adjusted the presentation of subsequent items based on how students managed to deal with the preceding items. Though the younger students for the most part continued to work with materials, confident students, once they understood the task, had their items presented just with numeral cards. The box lid and/or materials were brought back into use once students began to have difficulty with the increasingly challenging item sequence.

The item sequence involved increasing set sizes, with missing parts as follows:

- given 8, missing parts of 4, 6, and 3
- given 11, missing parts of 8, 6, and 2
- given 15, missing parts of 11, 7, and 3
- given 30, missing parts of 5, 24, and 17
- given 42, missing parts of 22, 25, and 13

Cubes and place value blocks were available for students to use to support their thinking. At the outer limits of children's performance, the interviewer used the blocks to model the items to see whether a concrete representation could help the student to continue.

Interviewer directions and comments followed along these lines, with variations as required to support children's different levels of mathematical activity.

You say the missing part is three. If this part is eight (pointing to the given part) and the missing part is three, will that make eleven in all (pointing to the numeral card that shows 11)?

How do you know? Why do you think so?

How did you figure that out? Can you show me?

I saw you using your fingers. How did they help you?

When you were quietly counting, what were you saying to yourself?

Let's check to see how many are hidden. What good thinking - you got it. Now let's try another one. Close your eyes.

This time, how many are showing? And how many are there in all? Do you think you can find out how many are hidden?

What might help you figure it out? What helped you the last time?

Do you think you could find the missing part if we started with a higher number?

Appendix D - Sharing Task Format

The Sharing Task involved students determining how to share different and increasing numbers of materials among two, three, or four people. A set of 8 cubes and a set of 9 pennies were used to introduce the task. Once the students understood what was required, different starting sets of materials were presented in transparent, but sealed containers, including:

- a bag of 12 red spoons,
- a bag of 18 plastic sticks,
- a box of 24 birthday candles,
- a bag of 40 doilies, and
- a bag of 50 balloons.

Students were asked how they would share the set among two, three, or four people. If they were unable to figure out the questions without manipulating the materials, cubes were used to model the number in the bagged set of materials. Interviews were conducted similar to the following:

*Here is a bag of sticks. Can you see the number that shows how many sticks are in the bag? Right, there are eighteen sticks in that bag. If you were to share the sticks with me so that the two of us each had a fair share, how many would we each get? (Using the child's response of nine...) If you get nine and I get nine, will that take care of the eighteen sticks? Is that a fair share?
How did you figure out nine?
I heard you say "ten and ten is twenty." How did that help you?*

*What if Mrs. Locke comes to join us (motioning to show that three fair shares are needed now.) Can you share the eighteen sticks among the three of us? How many will we each get?
(Using the child's response of five...) If you get five, and I get five, and Mrs. Locke gets five, will that take care of the eighteen sticks? No? Do you have another idea? (The child might think out another response using an internalized strategy, and if not:)
What if we put out eighteen blocks and pretended they were the sticks- could you show me how to share them fairly among three? How many will we each get?*

(Once a response is generated from the child's external strategy, the same clarification question followed..) Six, six, and six, will that take care of the eighteen in all? Two of us would get nine each. Three of us would get six each. What if Sally joins us so we need four fair shares? How many do you think we would each get if we shared the eighteen among four of us?

Appendix E - Money Interview Format

The Money Interview was made up of four sub-tasks. Each of these involved the use of real coins, and the last two sub-tasks included the use of price tags and small toys as the objects for buying and making change.

Naming Coins

Given a plate with one of each: penny, nickel, dime, quarter, loon:

Can you tell me the name of some of these coins? (showing a plate of coins)
Do you know some of them and what they are worth? Which ones? How do you pick out the nickels? Can you tell me what a nickel is worth when you use it to buy something?

Counting Coin Sets

Given sets of coins of increasing value and complexity:

9 cents (n, p, p, p, p)

23 cents (d, d, p, p, p)

47 cents (d, d, d, n, n, n, p, p, p)

77 cents (q, q, d, d, n, p, p)

\$3.57 (L, L, q, q, q, q, q, d, d, n, n, p, p):

Can you tell me how much that is in all? (showing a nickel and four pennies)

Can you count out loud for me so I can hear how you figured that out?

What if we put these nickels together to make ten cents. Would that help you count?

Can you show me another way to show nine cents?

What is different about these two ways?

So are these worth the same amount, would you take nine pennies or the nickel and four pennies?

Tell me which you'd rather have. If you had your choice of that (five pennies) or that (three nickels) or that (two dimes), if you could have your choice of one of these piles, which would you choose? Why did you choose the dimes? Which pile is worth the most money

Buying

Given small toys each with price tags: 8¢, 10¢, 15¢, 23¢, 36¢, 45¢, 79¢:

Let's pretend you are at a store. Which toy would you like to buy?

Show me with these coins how you would pay for it.

Can you show me another way to pay for it using different coins?

What if you didn't have any dimes. Could you show me another way?

Making Change

Based on children's performance on the three previous sub-tasks, making change examples were shaped to suit the level of each child. No sequence of change examples was set ahead of time. Given the coins, the toys, and price tags, the task went as follows:

If you have these two dimes and you want to buy this toy for 15¢, would you get any change? How much change would you get? Can you explain how you worked that out?

Let's pretend you are the store keeper and I want to buy this toy. If I give you this much money, is that enough? Will I get any change? How much change?

Appendix F - Parent and Teacher Interview Formats

The following Student Characteristics Form outline was used as the basis for questions asked of all teachers and parents. They were asked to describe the child's typical performance according to the characteristics on the form, using the low to high seven point scale as a rough estimate.

Characteristic	Low.....High						
	1	2	3	4	5	6	7
math achievement							
math potential							
effort in math							
interest in math							
general school achievement							
work habits in math							
awareness of social applications of math							
interest in math games							
interest in calculators							
interest in computers							
interest in winning, competition							
interest in solving problems							
interest in independent exploration of an idea							
creativity							
independence							
concentration							
curiosity, motivation							
tenacity, perseverance							
systematic-ness (vs. randomness)							
reflectiveness, ability to monitor thinking							
assertiveness							
risk-taking							
flexibility							

Parent Interview

The first purpose of the background information interview with parents was to attempt to tap into the contextual variations, environmental influences, and individual patterns of development that might be of help in understanding the child's then-current performance. The second purpose was to report to parents on results of the interviews with their child, to show video clips, and to answer any questions.

The interview was guided first by questions related to the Student Characteristic form. In addition, the sorts of questions asked, included:

Can you describe your child's early interest in number?

When was it first apparent to you?

Can you recall things she did that surprised you or indicated a growing awareness of number? (such as counting steps or objects, setting places at the table for family members, etc.).

Are there any family members or friends who may have influenced your child's interest in number?

Is there an older child at home who may have influenced her/taught her?

Watch what the student did when faced with this task. Would you consider this typical for this student? Is there anything that surprises you? How does this compare to classroom performance?

Is there anything else you could add that might help me better understand her approach to number situations at this point in time?

Teacher Interview

The purposes of the background information interview with teachers were to draw upon the teachers' knowledge of each child's classroom experiences, to compare interview performance with typical classroom performance, and to report on the results of the interviews using video clips to support comments. In addition to questions asked based on the Student Characteristics form, additional questions were based on observations from the interview video data. Teachers were asked to consider the data in relation to the student's ongoing classroom mathematical activity, to relate researcher observations to general classroom performance, and to reflect upon the student's overall mathematical disposition as it may have manifested itself up to that point in time.

Watch what the student did when faced with this task. Would you consider this typical for this student? Is there anything that surprises you? How does this compare to classroom performance?

What sorts of number activities does the student seem to enjoy?

Can you describe any areas of the mathematics curriculum that pose a particular challenge to the student? that are particularly easy for the student?

Are there particular strengths in one of these areas that the student brings to the classroom?

Is there anything else you could add that might help me better understand her approach to number situations at this point in time?

Appendix G - Examples from Student Interviews

Juliana's Sharing Interview

Interviewer:	Juliana:	Interpretation:
<u>12 + 2</u> Here I have a package of twelve spoons (showing the package.) If you and I are going to share them evenly, how many will we each get?	Juliana takes the package and tries to count and move the spoons through the bag without success. Twelve right, kay, (to herself) what number is twelve?	Appears unable to mentally work out twelve shared between two May require materials in order to make sense of the problem.
Here, I'll tell you what, do you want to use blocks and pretend they are spoons?	Takes the 12 blocks, checks to see if there are 12, moves the blocks around, then silently counts out a set of 5, counts out another 5 and picks up the remaining 2.	Considers 5 each to be fair (is this a perceptual estimate or a known grouping?), makes two groups of 5, and removes left-overs.
What are you going to do with those two?	Puts one on each pile of 5 to get 6.	Given cue, comfortably deals with the remainder.
So how many do we each get in all?	Without looking or counting, says Six , immediately recognizing the group of five and one more	Uses known grouping of five and five, adds on one to each group, without counting recognizes groups of six. Categorized as externalized Reasoning .
<u>12 + 3</u> Now what if Miss Jewel came to sit with us, what would we each get? (motioning to show three places) Two and two and two, but what are we going to do with all those (motioning to the six she is holding.)	Starts putting out two sets of 4, then removes two to show 2+2+2 and picks up the rest of the blocks. Two. Looks at her handful, systematically picks up groups of two and drops them at each of the three piles on the table.	Estimates 4 then revises based on perceptual information, changing to groups of two with six remainder. Recognizes that six can be grouped into two for each of the three groups.
So how many will we each get?	Without looking at the blocks, says Four.	Combines groups without counting.
Four and four and four, will that make twelve in all?	Nods her head to agree and points to each group of four.	Using groups of two to decompose and recompose sets through perceptual grouping: categorized as externalized Reasoning
Will there be any left over?	Shakes her head no.	Realizes it divides evenly
<u>12 + 4</u> What about if Miss Keating came to join us. What would we do to make it fair? (motioning to show a fourth group) So we'd each get two. Two, four, six, eight.... what are we going to do with the rest of those? (pointing at her handful) Four people, how many did we each get?	Rearranges a group of four into two twos, then picks two up from each of the other piles of four to be left with four piles of two. Holds four leftovers. Looks briefly at her handful and puts one at each place.	Decomposes and recomposes groupings based on perceptual information. Recognizes how to take care of the remainder without counting so that it works out evenly.
Three and three and three and three, does that make twelve in all?	Without looking says Three. Yup.	Combines groups and leftovers to immediately recognize fair shares without counting. Categorized as externalized Reasoning .

<p><u>18 + 2</u> Good girl, you did a good job with twelve. Let's try the next package. Here are eighteen sticks, eighteen of them. If you and I are going to share them, how many do you think each of us will get. Do you think we'll get as many as ten? Well, do you know what, nine is right. How did you think of nine? What made you think it might be nine? Do you just know that one, or did you think ten and ten is too much? Very clever way to do it, good girl.</p>	<p>Handles the package, pushes sticks around, turns the bag over, looks up to the left, starts to say something but stops. Seems stuck. Still fiddling with the bag, looks away again, looks back, whispers "twenty", looks away and back, and says: Nine? (Then immediately tries to count through the bag to check her response.) Still fiddling with the bag. Cause nine plus nine equals eighteen. Nods her head yes to both questions.</p>	<p>Facial expression and actions suggest that she is trying to reason mentally about the quantity. Uses prior knowledge to try out possibilities. Mentally grapples with the problem. Fiddling with the bag not integral to her reasoning process, rather a check on her answer. Using a known fact (9+9=18), seeming to draw on a related fact (10+10=20) -thinking through the problem in steps. -not dependent on materials -categorized as <u>internalized Reasoning</u></p>
<p><u>18 + 3</u> Now if we were going to divide eighteen among three people? Do you want to use the blocks for that? Here's twelve, and that's eighteen. putting out blocks. Okay, there's nine and nine. Good. Now what are we going to do if Miss Jewel joins us. How many shall we give to her? Or how are you going to give me some then so we have three groups? How many should we give to her? How can we do it? Let's each give her some. Let's each give her two. Will she have enough? She hasn't got as many as we have. (I also move two over.) Okay, I'll give her two also. Oops, she's got too much now. What about if we just take one away. What have we got now? Do you think that looks like three equal groups? Check and see. How many did we each get?</p>	<p>Counts and separates them into nine and nine as for two people. Fiddles with her groups of nine. Moves one group over towards the place for Miss Jewel. Counts out five from her pile of nine, starts to count another pile, but stops and shrugs. She moves two over, as I move two over. Looks at the three piles (4, 7, 7) and shakes her head to show not enough. Agrees, and takes two of her seven to give to Miss Jewel. Laughs, nods in agreement and flicks away some blocks. She takes one, adds it to her pile She arranges each group of six into a neat two by three array, counts up by ones and responds Six.</p>	<p>Does not appear to be clear on the idea of making three groups. Tries an idea but loses her steam. Appearing increasingly disinterested but is willing to respond to questions. Losing it. Attempts to get back into it; off task but responds to the block work and answers the question. Categorized as <u>externalized Reasoning</u> due to her use of perceptual groupings in considering fair shares and the estimate-check-revise process.</p>
<p><u>18 + 4</u> Good. And if Miss Keen came to join us could we balance it up so she had a fair share?</p>	<p>Face resting on her hand, shrugs disinterestedly and says I don't know.</p>	<p>Not clear that she is unable continue, but rather that she is at the end of her ability to concentrate (we continue at a separate time)</p>

Sandy's Doubling Interview

Interviewer:	Sandy:	Interpretation:
<p><u>2+2, 4+4, 8+8</u> After some introductory doubling and calculator explanation, I ask: Two doubled will be.... Four doubled will be.... Eight doubled will be.... Wow, you know these fast. Have you got a lot of them in your head?</p>	<p>Looking at me with her hands folded, she responds immediately with: Four Eight. Sixteen. (looking away very briefly) Agrees. She looks at the calculator display to check each response, and smiles.</p>	<p>All three of these examples classified as Recognition.</p>
<p><u>16+16</u> How about 16. Have you got that one in your head? Now how did you do that? What did you say to yourself? When you say you take a one and put it over there, what is that one? So, if you showed me 16 doubled with blocks, what would it look like? So if you joined them together, what would you get? Do it with the blocks and show me how you get that 3 and 2</p>	<p>Looks all around for several seconds, says I think I should use the blocks... She takes two ten rods, starts counting by ones and puts them back. Well, let's see... Starts talking to herself and tapping on the table in a pattern. Well, 16 doubled, like 6 plus 6 is 12 and I know I go from that side not that side (motions on the table to show right to left adding.) Six and 6 is 12, take over one and make three then over here there is 3, no 2 over there. Yup, so that would make 32. Well, I don't exactly know what that one is and stuff, but I know that you have to take it over because you can't make it over here (in the ten's column) because you can't just put it over there and do the rest of it. Counts out 16 ones, I suggest another way, and she builds two 16s using tens and ones. You get the tens together and that makes 20 and then you take 16 (I help organize her model) 6+6 is 12. This is followed by a long explanation of what to do with the 12, all related to the printed algorithm and her attempts to make sense of the one carried over.</p>	<p>Appears to be tapping out the spatial pattern of the traditional addition algorithm, ones to tens, carrying a one, etc. Interesting that she counted the tens rod by ones before putting it down. Unitary thinking. Mentally using the standard addition algorithm, right to left, described as ones. Right to left, not clear on the place value aspect, following a learned procedure: classified as Algorithm. This is an attempt to explore the meaning that are behind her algorithm strategy. This explanation is not classified as a separate strategy, simply a concrete explanation for me based on her initial algorithm strategy.</p>
<p><u>32+32</u> Could you double 32? Do you think so? Do you think if you doubled 30 you'd get to 40? pointing to the block display of 32. ...</p>	<p>Looks at me, looks to the left, smiles, says 46. Sixty....um hum, 64.</p>	<p>Though not explained, it was apparent she used a mental Algorithm, got her places reversed, said 46 and corrected to 64 with cues. Missed opportunity to check her understanding. Moved on.</p>
<p><u>64+64</u> Is there any way you could double 64? Show me with blocks. Show me what 64 would look like. ...Now double 64. ...Can you build another 64? So you've doubled 64. What is it going to be altogether How would you write 128?</p>	<p>Well, you could take the 4 and make it an 8, then you could take the 6 and bring it, um, to make a 12, and that would be 12 hundred and 8 uses her hand on the table to show the places to write. Moves tens to make 100. A hundred right here (moving another ten over) a hundred and one? two hundred? I help her organize and name the tens, she continues and names the whole. She recognizes the two 64s in her model. One and a 2 and an 8. She checks on the calculator.</p>	<p>Her description is clearly a mental Algorithm. Applies place value groupings to double 64. Unable to name her model of 128 once beyond 100. Once we work out how to name the tens, she is quick to see the 8 ones, to recognize 128, and to connect the model with 64 doubled. Strategy use external Reasoning. Partial knowledge of the place value relationships involved, but counting and naming a constraint here.</p>

Cliff's Missing Parts Interview

Interviewer:	Cliff:	Interpretation:
<p><u>4 + ? = 8</u> Show me eight little yellow blocks. What have you got there? (He affirms there are eight.) Close your eyes. (I hide four under a box.) Part of your eight is missing? What's hidden? How do you know? ...</p>	<p>Looks at the four blocks, looks ahead. Four. 'Cause I know four and four is eight.</p>	<p>Immediately recognizes the other part of eight. Recognition</p>
<p><u>6 + ? = 8</u> Close your eyes. (I hide two.) What's hidden? How do you know that? How does that help you there?</p>	<p>Counts up the six. Looks ahead, nods and moves his lips. Says Three. 'Cause four and four is eight. Because six, I have six here so there <u>has</u> to be three under there. Two. (Covers face.)</p>	<p>Signs of counting with head nods and lips, hands always fiddling, but not appearing to tally. Miscounts. Classification of Counting based on his actions, the appearance of counting on to a starting set of six, and his agreement with my suggestion.</p>
<p>Check and see if you are right. Close, close. What do you think the problem was there? Did you do some counting in your head? Did you go six, seven, eight and come up with three? ...</p>	<p>I was just one over. That's nine Nods his head yes.</p>	<p></p>
<p><u>3 + ? = 8</u> Okay, here we go, close your eyes. (I hide five.) Part of eight is hidden. What is hidden? Why? How does that help you?</p>	<p>Looks at the three, makes a face looks past me, This time it has to be five. Cause, four and four is eight. That helps me because I had six and I thought there were three so if I have three there must be six under there.</p>	<p>No signs of counting, seems to be weighing possibilities. Relates this example to a known fact, but gets carried away with his explanation, referring back to the previous item's first response.</p>
<p>Six? That's not what you told me was missing. You told me five was missing.</p>	<p>I meant five snaps his fingers.</p>	<p>This seems to be what he meant despite his explanation.</p>
<p>And why would five be missing?</p>	<p>Because four and four is eight.</p>	<p>Again, a connection to a known fact that seemed to relate to his thinking.</p>
<p>Okay, (motioning to the model)so if we had four here, now you've only got three. Where did the extra one go? With the other four? Now you've got five? I got ya'. Okay, see if you're right. You got it, good. ...</p>	<p>Nods as I speak, and follows with That extra one went in here. (motioning to the box). Nods. Agrees. He checks and nods.</p>	<p>Follows my wording very closely and uses the model to reinforce it. Classified as external Reasoning.</p>
<p><u>8 + ? = 11</u> Change your eight into eleven. This time we are going to hide them under two boxes. ...I show the 11 card for the number in all, put the 8 card on top of one box, and ask:There's eleven, this part is eight, let's show the eight, what is the other part?</p>	<p>Eleven? Quickly adds on three, saying Nine, ten, eleven. Looks at me. The other part has to be looks to the right, fingers move, lips move, face lights up (aha!) looks back, Three.</p>	<p>This item may well have been handled with internalized Counting, however, Cliff's rationale of a known fact, and the that he drew on his memory of increasing the starting set from eight to eleven by adding three blocks resulted in a Reasoning classification.</p>
<p>How do you know that? ...</p>	<p>Because eight and eight, eight and three, is twelve, eleven!</p>	<p></p>
<p><u>6 + ? = 11</u> This many are under this box (putting the 6 card on top) how many are hidden?</p>	<p>Covers his eyes with his hands, counts silently under his hands, holds his fists up in front of his face. Opens eyes, says There's five. Because I counted in my head. Nods in agreement.</p>	<p>All his actions point to counting on, which he then describes. No explicit finger tally use, rather, an internalized process.</p>
<p>How do you know? Now I heard you say 'Seven, eight, nine, ten, eleven', and then did I hear you say 'seven is one, eight is two, nine is three? ... We check the boxes.</p>	<p>Uses his hand in a circular motion to show the pattern he extended.</p>	<p>Internalized verbal counting pattern used to tally the five. Classified as Counting.</p>
<p><u>2 + ? = 11</u></p>		

<p>Last one with eleven, are you ready for it? Close the old eyes. I hide nine and two. There are the two cards. (He opens eyes.) Two are under this box, eleven in all motioning to the cards. See if you are right. What's under there?</p> <p>I try to clarify his counting process, modeling a finger tally and asking if he ever did it like that.</p> <p>...</p>	<p>Counts three..., four..., five..., eyes closed, fingers moving in fists at his neck, continues counting. Opens eyes, snaps fingers, points at the box and says eight, pauses then says nine. Counts by twos to check and nods head.</p> <p>He responds by showing me how he was using his fingers to tally, and says I was using my brain, my hands and my brain (wiggling his fingers beside his brain.)</p>	<p>Quite clearly used a counting on process with a relatively internalized tally - no explicit external tally. Classified as Counting.</p>
<p>11 + ? = 15 There's eleven. Make it fifteen. Good. What have you got there? ...Close your eyes. OK. (I cover four.) Fifteen in all, there is eleven. What's hidden? How did you know that?</p> <p>Because you added four to make the (eleven into fifteen)?</p> <p>8 + ? = 15 I put eight under the box. Fifteen in all. Here's seven. What's under there? You've got seven. How many more to get fifteen?</p> <p>Why do you think eight? We talk through his 7+7+1 process. (His third example with 15 was impossible to interpret, but I decide to go on to sets of 30 anyway.)</p>	<p>He adds four on, quickly and easily. Fifteen without counting.</p> <p>Immediately says Four. It's easy shrugs and turns hands palms up. Yup.</p> <p>He tries to peek through his hands.</p> <p>He rearranges the set-up..... Looks away to the right, thinks, looks back and says, Seven. No signs of counting. Cuz seven and seven is...I mean, there is, eight. He follows carefully, quietly saying 7+7=14 and there is one more.</p>	<p>Immediate response. Appears that he recalled the part of four, possibly from adding four on to change the eleven to fifteen, or because he knew 11+4=15. Classified as Recognition.</p> <p>Though on his own he is unable to articulate his reasoning, his words and actions support a classification of Reasoning based on use of a known fact, 7+7=14.</p> <p>(3+__=15 unintelligible)</p>
<p>5 + ? = 30 If the whole thing was thirty, and under this box was five (putting out numeral cards but no blocks this time.) what would be under the other box? How did you figure that out?</p> <p>Well, 25 is right. But how did you get it? In your head, like 26, 27, 28, 29, 30, like that? </p> <p>24 + ? = 30 Here's 30, if part of it is 24, what is the other part.</p> <p>24 and 1 would be 25 wouldn't it? Did you use 25 and 5 pointing to boxes. ...</p> <p>17 + ? = 30 Here is the hardest one of all. I show cards for 30 in all, put a 17 card on one box and ask how many are hidden under the other box. How did you figure that out? Boy, you are quite something!</p>	<p>Looks to the right, twirls his fingers beside his face, makes a grimace, looks back and says Twenty-five.</p> <p>I meant (looks away) twenty. No signs of counting, rather he seemed to be working with groupings of five, and an intuitive sense of how 20, 25, and 30 relate to one another.</p> <p>Because I counted 25 here, I said 25 here and I said, I counted five more. Nods. Gives an elaborate rationale about how the materials helped him.</p> <p>Makes a face, looks to the right briefly, (no sign of counting), looks back, points, It has to be, one! I mean it has to be... six more. He smiles and nods enthusiastically.</p> <p>Looks away, no signs of counting, It would have to be...(snaps fingers and looks at me) thirty-eight.</p> <p>Cuz. (Because.) Shrugs and smiles. I bet nobody else knew that answer!</p>	<p>His initial response which appeared to be based on grouping and reasoning was not accompanied by the nodding motion he used in other counting examples. His internalized incrementing procedure was likely a combination of the two strategies.</p> <p>Though it appeared that Cliff was using chunks of five in his thinking, Cliff provided a counting rationale and agreed with my counting description, so his highly internalized incrementing procedure was classified as Counting.</p> <p>Seemed to be confused again with 20, 25, 30 rather than linear ones counting. Self-corrected too quickly to be counting. Classified as Reasoning, as in using the previous example 5+25=30</p> <p>Seems to attempt to reason it out, but grabs at an answer (rather than saying I don't know). Classified as Estimation. I realize it will be impossible to reconstruct his thinking, that he is likely at the end of his rope, and decide to end the interview on a positive note.</p>

Name _____

Appendix H - Sample Record for Collating Data Across Tasks

Doubling	Part/Whole	Build/Change	Sharing	Money
2 2	8/4	3-7	8/2	9c
4 4	8/6	7-12	8/3	23c
8 8	8/3	7-15	8/4	48c
16	11/8	12-9	9/2	77c
32	11/6	15-9	9/3	8b
64	11/2	9-23	12/2	10b
128	15/11	23-28	12/4	15b
256	15/7	28-21	12/3	23b
512	15/3	21-35	18/2	36b
1024	30/5	35-27	18/3	
2048	30/24	27-54	24/2	
	30/17		24/4	
	42/22		24/3	
	42/25		40/2	
	42/13		40/4	
			40/3	

Appendix I - Reasoning Strategy Examples and Descriptions

The use of Reasoning strategies was more closely associated with increasingly competent and sophisticated mathematical activity than use of any other strategy category. Strategies were categorized as Reasoning if they involved more than one step in a constructive process, and drew on at least one "chunk" of prior knowledge. The following examples of Reasoning strategies are provided to illustrate and clarify what sorts of mathematical activity are characteristic of children's increasing competence in making sense of number.

Major Characteristics of the Use of Reasoning

Two conceptual capacities were characteristic of all strategies categorized as Reasoning, as described on pages 136-137. The first of these involved the capacity to think about quantities in a multi-unit rather than a unitary way. The second important conceptual aspect underlying children's use of Reasoning strategies was the ability to decompose and recompose numbers to suit particular purposes. Both of these capacities characterize each of the following Reasoning strategies:

Mental Decomposition/Recomposition

Concrete or Perceptual Decomposition/Recomposition

Building on Known Facts

Building on Known Patterns

Multiple Counting

Place Value Counting

Estimate-Check-Revise

Compensation

Simplifying

Mental Decomposition/Recomposition

This was the label assigned to describe children's internalized reasoning processes as described above, where children mentally manipulated wholes and parts to construct or derive answers. Children almost always looked away as they mentally worked out their answers, frequently looking up and to the right or left. Cliff, one of the youngest students to rely on mental Reasoning strategies, sometimes covered his eyes to think, suggesting an overt attempt to close out perceptual distractions. Frequently students used hand motions, fingers, or movements as they reasoned,

however, this was done in an automatic manner without attention to the motor actions. Even more frequently, children talked quietly to themselves as they worked out their answers.

Once students decomposed a number to provide at least one manageable or known part, they used different ways of recomposing the number. For some children, known facts were applied throughout, such as Bevan's doubling of 64. Bevan's explanation of his reasoning went, "I knew that sixty and sixty was a hundred and twenty, so I just added four and four and got one hundred and twenty-eight." For others, unitary counting was used to deal with the remaining part, such as Brian's doubling of 512. He right away said that $500+500$ was 1000, then counted by ones on his fingers to add the $12+12$. When I asked if he needed to count out $12+12$, he covered his face in disbelief and blurted out, "Oh, NO! I know that, because a day has twenty-four hours and there are twelve of day and twelve of night".

Concrete or Perceptual Decomposition/Recomposition

This strategy described external reasoning processes where materials were used to support the manipulation of groupings. The thinking behind this strategy was the same as for Mental Decomposition/ Recomposition, however children relied on external information to stimulate or support their thinking. Most frequently this involved modeling a multi-digit number with place value blocks, then using the blocks to physically decompose and recompose the values. Sometimes this strategy involved arranging single blocks, then re-arranging or reinterpreting the arrangement another way. For example, Juliana, working with 24 blocks, used a global perceptual estimate to separate the blocks into two roughly equal piles, arranged the two piled groups spatially into 2×6 arrays, and counted to check. When asked to share the 24 between four people, she put the two arrays together to make a 4×6 array, spatially assessed the grouping, and without counting partitioned the array into fourths, and finally counting to check that each share was equal. For both items, Juliana used her intuitive spatial knowledge for her initial attempts at sharing, and her number knowledge was used to check these initial partitions. Such perceptual affordances to sharing enabled the youngest students to be successful even without a store of known number relationships to draw upon.