TEACHERS' INTERVENTIONS AND THE GROWTH OF STUDENTS' MATHEMATICAL UNDERSTANDING

by

Julie Margaret Towers

B.Sc. (Hons.), University of Lancaster, 1985
Post-Graduate Certificate in Education, Didsbury School of Education, 1989
M.Sc., Oxford University, 1994

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Department of CURRICULUM STUDIES

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ABSTRACT

This study explores the ways in which teachers' interventions interact with and occasion the growth of students' mathematical understanding. Two 'cases' were documented, and these form the two strands of my research. The first strand concerns data collected in my own high school classroom at a time when I was a full-time teacher of mathematics in a small, rural secondary school in the United Kingdom. The second strand concerns data collected in a mathematics classroom in a large, urban high school in Vancouver, British Columbia.

The data consist of videotaped lessons in each of the two classrooms, videotaped interviews with students from both strands of the data, copies of students' work from both strands, videotaped interviews with the Vancouver teacher, and my own journal entries.

Analysis of the data, which is described in six stages, resulted in the generation of fifteen themes to describe the teachers' actions-in-the-moment. Three of these themes are distinguished from the others as teaching *styles*, as contrasted with the remaining twelve teaching *strategies*, and a number of the teaching strategies are clustered within the three teaching styles. The notion of a 'continuum of telling' is developed, upon which the three teaching styles lie, and this continuum is explored in order to probe the ways in which teachers' interventions interact with the growth of students' mathematical understanding.

The ways in which teachers' interventions occasion the growth of students' mathematical understanding is probed through an integration of detailed traces of the students' growth of understanding with contemporaneous considerations of the teachers' strategies and styles. Implications to be drawn from these analyses, both for the research community and for teaching and learning, are discussed.

I also share my reflections on my own growth as a teacher and as a researcher that I have experienced as a result of participating in, and conducting, this study.

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LIST OF ABBREVIATIONS

PK: Primitive Knowing
IM: Image Making
IH: Image Having
PN: Property Noticing
F: Formalising
O: Observing
S: Structuring
I: Inventising

iv: image viewing
id: image doing
ir: image reviewing
ise: image seeing
isa: image saying
pp: property predicting
pr: property recording
ma: method applying
mj: method justifying

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CHAPTER 1

ASKING

1.1 Beginning the Journey

In 1990 Stevens, reporting on the deliberations of an International Congress on Mathematical Education discussion group of which he had been a member in 1988, touched upon the problem of "when it is appropriate for a teacher to intervene in order to redirect a child's thinking" (p. 231). Stevens reported that the group was unable to reach agreement, and suggested that the issue was unresolved. The dilemma of when to intervene is an enduring one for teaching, and continues to perplex and challenge practitioners and researchers alike. In this sense it remains "a genuine problem, or question" (Dewey, 1933, p. 13). As a teacher I had long been challenged by this dilemma, and so it was from this perplexity that my journey towards this research project began. It was clear to me, however, as I considered this dilemma that the question of when to intervene was not one that could be addressed by a single researcher in a doctoral dissertation study. It was necessary to consider where my efforts to contribute to this ambitious question could best be applied.

1.2 The Research Question

In previous research work (Towers, 1994) I had considered students' understanding of algebraic processes. During that project I had been drawn to a consideration of the role of the teacher in shaping students' understandings, but a detailed discussion of the teacher's role had been beyond the scope of the study. In reviewing the findings drawn from that

study I began to shape a new study based on the recognition that there are questions that can and should be asked about teachers' classroom interventions before asking *when* teachers should intervene in students' learning. One such question is the one that frames this study. It asks: In what ways do teachers' interventions interact with and occasion the growth of students' mathematical understanding? This question grows not only from my interest in the role of the teacher in supporting student' learning, but also from a theoretical perspective on understanding grounded in the Dynamical Theory for the Growth of Mathematical Understanding (Pirie & Kieren, 1994a), and from my philosophical framework, enactivism. Each of these groundings are explored in the following two chapters.

Framing a question is a defining moment in the life of a research study. As van Manen (1990) notes, in crafting a question the writer must pull the reader into that question in such a way that the reader cannot help but wonder about the phenomenon in the way that the writer does. In choosing to present only one orienting research question for this thesis (rather than sub-dividing the question and thereby hinting at its answer and removing some of its mystery), I have attempted to preserve the wonder that I have experienced in my journey through this study. As Gadamer (1975, p. 266) notes "the essence of the question is the opening up, and keeping open, of possibilities." To reduce a rich and complex inquiry to a series of simpler questions is to close down some of its possibilities. I also intend my choice to reflect the complexity of the environment of my research area. I wished to preserve, as far as possible, the complexity of the space (the mathematics classroom) in which my research is situated by preserving the complexity of the queries

which oriented my data collection and analysis, and I remain convinced that I was best able to do so through a preservation of a single complex question. Responding to such a research question has been both challenging and rewarding.

In shaping my research question, and in refining a strategy for responding to it, I turned to the literature on teaching and learning, and specifically to that on the teaching and learning of mathematics, in order to discover what had been learned since Stevens' call for action. I discovered that though many researchers had contributed to the discussion, concerning themselves with the nature of teachers' behaviour in classrooms (Banbury & Herbert, 1992; Chuska, 1995; Martino & Maher, 1994; Schwartz, 1996; Vacc, 1993) and the nature of students' learning and understanding of mathematics (Cobb, Yackel & Wood, 1992; Confrey, 1994, 1995a, 1995b; Davis & Maher, 1990; Pirie & Kieren, 1994a; Sfard, 1991, 1994; Sierpinska, 1990), most had not attempted to integrate the two areas of research. I recognised the potential for a significant study in this area.

Some researchers have recently turned their attention to the complexity of classroom interactions in an attempt to understand the processes at work in classrooms (see, for example, Cobb, Boufi, McClain & Whitenack, 1997). Such research provides the basis for my investigation, but even these researchers suggest that further "analyses are needed that systematically coordinate accounts of...communal developments [of mathematical understanding] with detailed analyses of individual students' mathematical activity as they participate in, and contribute to, shifts in the discourse" (Cobb, Boufi, McClain & Whitenack, 1997, p. 275). Such statements recognise that there is an absence of studies

which address the nature of teachers' influence on the growth of students' mathematical understanding. Several researchers point explicitly to the value of a study such as mine which integrates a consideration of the role of the teacher with an exploration of the growth of students' understanding of mathematics. For example, Brenner et al. (1997, p. 664) note that there continues to be "a lack of methodologically sound studies on how to promote students' construction of mathematical understanding in a school setting." Further, Doyle (1990, p. 20) recommends research whose purpose is to "understand how meanings are constructed in classroom settings" and suggests that to do this kind of analysis "one must have a powerful language to describe both *events* and the *interpretations* made of these events" (original emphasis). My study attempts to contribute to this call.

In addition, a unique contribution of this particular study lies in the perspectives I offer as both participant and researcher. I discuss later (Section 3.3) the particular nature of my multiple roles in the study, but wish to note here that in studying my own teaching I am able to contribute to Cochran-Smith's and Lytle's (1993) call for significant teacher participation in creating new knowledge about teaching.

1.3 Articulating a Response

In order to address my research question I have designed a study that explores teaching and learning in two classrooms. One of these is my own high-school classroom, the data having been collected at a time when I was a full-time teacher of mathematics in the United Kingdom. The other classroom is in an urban high school in Vancouver, British

Columbia. The detailed descriptions of classroom activities that I offer in Chapters 4 and 5 are narratives that include a large number of what might be termed anecdotes (van Manen, 1990). van Manen (1990, p. 115) distinguished anecdotes as a "special kind of story". He claims that they are not mere illustrations; that they can be understood as a methodological device to make comprehensible some notion that easily eludes us. van Manen suggests that the power of anecdotes lies in their ability to compel by recruiting our willing attention, to lead us to reflect through their tendency to invite us to search for significance, to involve us personally as we actively search for the story-teller's meaning via our own, and to transform us through their capacity to move and teach us.

Borrowing from van Manen, then, the classroom anecdotes included in this study represent far more than 'mere illustrations'. The chapters that follow trace the evolution of my learning about teachers' interventions and their "occasioning" (Simmt, 1996) of students' understanding of mathematics. In Chapters 4 and 5 I interweave stories and anecdotes about the two classrooms with my interpretations of those incidents and with a presentation of the implications I have drawn from my observations and reflections. I begin, though, in Chapters 2 and 3, with a discussion of the philosophical, methodological and analytical perspectives that have framed my investigation.

CHAPTER 2

RE-VIEWING

2.1 Introduction

In this chapter I focus on three main bodies of literature that are relevant to my chosen research area. The first is the theoretical framework which has guided not only my choice of orienting question, but also the methodology which framed my search for understanding of the issues, and the principle which guided my research processes - that being, enactivism. It should be noted that this philosophy has, undoubtedly, also influenced the findings I have generated, and so I encourage the reader to examine critically the chapters that follow in the light of what is revealed in this chapter concerning my "world view" (Gowin, 1981). Contemporary methodological analyses suggest that in our research approaches we are always biased (Peshkin, 1988). As Gadamer (1975) suggests we hear and see what we have been biologically and culturally predisposed to perceive. Rather than regard this prejudice as a deficiency, though, we should interrogate it for what it is able to reveal about us. It is my hope that within this thesis the reader is able to read about the researcher as well as about the research, and that each thread will strengthen the other. The ideas contained in the first section are woven throughout the remaining sections of this chapter, and throughout the rest of the thesis, and so it should not be assumed that all that I have to say concerning my orienting philosophy is contained within this rather brief opening section. Instead, it should be remembered as you read, that the philosophy of enactivism is embodied in all that follows here.

The second body of literature considered here concerns the broad field of research within which my study is located, that being the nature of teacher-student interactions in the classroom. This section considers the existing literature on teacher and student talk in classrooms, and reflects on the separation, in much of that literature, of teacher from learner. Here, drawing on the emerging philosophy of enactivism, I attempt to bring together these discourses. My third focus in this chapter is the considerable body of literature directed towards understanding mathematical understanding. In this section I reflect on several contemporary theories of understanding, and, again drawing on enactivist thought, explain the reasons surrounding my choice of orienting framework of analysis.

2.2 Theoretical Framework

2.2.1 Considering Constructivism

The theoretical framework within which this study lies has its roots in constructivism and, to follow the metaphor, its new growth in enactivism. Constructivism has had many proponents over the years, each of whom has presented a slightly different perspective. Many of these perspectives have acquired distinct names, for example, trivial constructivism, radical constructivism, and social constructivism. As von Glasersfeld (1991), a leading radical constructivist, has indicated, the roots of constructivism can be traced back to Socrates, and include the notion that knowledge is the result of a learner's activity rather than of the passive reception of information. von Glasersfeld (1990) suggests that the beginnings of radical constructivism lie with Neapolitan philosopher Giambattista Vico who published his treatise on the construction of knowledge in the

early 1700s, the main principle of which was that we can only know for certain that which we have created. von Glasersfeld reports that reviewers of Vico's work were shocked by its implications for traditional epistemology. They charged Vico to prove that what had been asserted was true. Present day critics of radical constructivism continue the battle cry, but von Glasersfeld notes that the very thing which they demand is the thing which constructivism must do without. A central part of constructivist theory, that this kind of "truth" can never be claimed for knowledge produced by human reason, is reinforced by von Glasersfeld's statement that "to claim that one's theory of knowing is true, in the traditional sense of representing a state or feature of an experiencer-independent world, would be perjury for a radical constructivist" (1990, p. 19). Radical constructivism maintains that each person's experience is context-dependent, and unique to that individual. It is, therefore, by its very nature inaccessible to others. There can be no such thing as mathematical structure existing apart from an individual's constructed knowledge. Further, radical constructivists hold that there can be no way of knowing that a problem or mathematical concept has the same structure for different individuals, not because it might be found that each person constructs his or her own knowledge differently, but rather because radical constructivist epistemology does not ever permit us to conclude that two individuals have "the same" knowledge (Goldin, 1990).

Lerman (1996) notes that constructivists draw their inspiration from Piaget whilst socio-cultural theorists draw theirs from Vygotsky. Although these theorists have popularly been described as favouring, respectively, the individual and the social context, I hesitate to characterise these philosophies in quite such a simplistic manner, especially in the light

of Confrey's (1994) admonition and DeVries' (1997) recent paper. It is clear, however, that regardless of its roots the essence of constructivism lies in its *privileging* of the individual cognising agent.

2.2.2 Surveying Social Constructivism

Some constructivists, however, acknowledge that construction is a social process as well as an individual one.

> The constructive process is subject to social influences. We do not think in isolation; our choice of problems, the language in which we cast the problem, our method of examining a problem, our choice of resources to solve the problem, and our acceptance of a level of rigour for a solution are all both social and individual processes (Confrey, 1990, p. 110).

Cobb (Cobb, 1988; Cobb, Wood & Yackel, 1990) also expresses concern with the way in which radical constructivism down-plays the importance of social interaction in the learning process. Relying on the work of Vygotsky, Piaget and others (such as Maturana, 1980) Cobb suggests that social interaction is the process by which individuals create interpretations of situations that fit with those of others, and that in doing so they negotiate meanings, resolve conflicts, and construct consensual domains for co-ordinated activity. These compatible meanings are continually modified as individuals strive to make sense of situations while interacting with others. Cobb holds, therefore, that social interaction constitutes a crucial source of opportunities to learn mathematics.

Such a position reflects a growing movement to incorporate socio-cultural notions into radical constructivism, in order, according to Lerman (1996) to incorporate an explanation for intersubjectivity into an overall constructivist position. The resulting

philosophy, usually known as social constructivism, has been adopted by many writers in the mathematics education field, Cobb being perhaps the most notable. Cobb (1994) denies the separation between constructivist and socio-cultural theories, claiming that the two are complementary. This position is not without its critics, however (Smith, 1995; Lerman, 1996). Lerman points out the inconsistency in this assertion, noting that:

> to slip from theory to theory, ignoring the contradictions and disagreements, on the grounds that each offers a richer explanation than the other at different times is ... to do an injustice to theory and also to be in danger of losing the coherence of each and the insights that each theory, when taken in full, can offer (1996, p. 139).

Although Lerman (1996) is scathing in his criticism of social constructivism, and readily suggests that mathematics education would benefit from abandoning constructivism as a view of how people learn, he is less eager to offer an alternative position. Enactivism provides, in my opinion, a viable alternative.

2.2.3 Enveloping Enactivism

The enactive approach was first postulated by Varela, Thompson and Rosch (1991) who draw on a diverse collection of recent and ancient thought, including Buddhism, continental philosophy, biology and neuroscience. They claim that enactivism troubles the cognitivist assumption of a pre-given world that exists "out there" and that can be internally recovered in a representation (p. 150). To support this claim they draw on recent developments in physiological studies of vision and colour perception which have suggested that vision is not a one-way process, and that colour is not a perceived attribute of surfaces. The latest research has demonstrated that there are, in fact, more neural pathways taking information from the brain to the eyes than from the eyes to the brain,

suggesting that the behaviour of the whole system of vision "resembles a cocktail party conversation much more than a chain of command" (Varela, Thompson and Rosch, 1991, p. 96). Not surprisingly, a similar "conversation" has been claimed for the way in which we perceive sound (Jourdain, 1997).

Drawing on areas of "new biology" (Maturana and Varela, 1992) and on recent developments in evolutionary thinking which place an emphasis on natural drift (with the guiding metaphor of viability) rather than the Darwinian notion of natural selection (with the guiding metaphor of optimality), enactivist theorists Varela, Thompson and Rosch (1991) go on to situate cognition not as problem solving on the basis of representations, but as embodied action. These notions are currently being embraced by educational researchers in the fields of curriculum (Davis & Sumara, 1997) and teacher education (Fels & Meyer, 1997), as well as in subject area fields such as mathematics education (Simmt, Calvert & Kieren, 1996).

Enactivism, as a framework, offers a means of incorporating cultural commentary with discussions of individual cognition (Davis, 1995). It parts company with constructivism on the common assumption that constructivism's focus on the individual cognising agent is an adequate unit of analysis either for understanding thought, or for studying education. Enactivism challenges that "in focusing on the individual cognising agent, both the participation of that agent in the larger community and the fluidity of the context are obscured" (Davis, 1995, p.8).

Whilst not denying the insights offered by constructivism, enactivism does not fall into the trap of incoherence laid by Lerman (1996) for the social constructivists. By refusing to privilege the individual, enactivism avoids then having to account for the troublesome social. Enactivists view the individual and the context as co-emerging in mutual specification, rather than as one operating on the already existing other. As Lerman notes (1996), radical constructivism sees the individual as meaning-maker whereas sociocultural theories see meaning first as sociocultural, to be internalised by the subject's regulation within discursive practices. This problem is unresolved in a social constructivist perspective, however, enactivism rejects this chicken-and-egg struggle, denying the twin foci on which came first and which is primary, and instead calling for a recognition that meaning emerges through the co-evolving of the individual-with-the-context. Enactivism troubles the boundaries between knower and known, mind and body, individual and collective, self and other; thereby opening a space for discussions of understanding and cognition which recognise the interdependence of all the participants in an environment (such as a classroom). Such a shift enables understanding to be seen as a continuously unfolding phenomenon, not as a state to be achieved, a distinction which is critical to my later consideration of several recently articulated theories of the nature of mathematical understanding (see section 2.4.3). To emphasise, then, although social constructivist principles recognise the context as well as the individual, the two are held as separate and separable entities. Enactivism can be differentiated from social constructivism by noting that it focuses upon *who* you are (a notion which subsumes the individual and the context), not just *where* you are (which considers the subject as separate from the environment).

The rejection of the modern conception of "self" is central to the philosophy of enactivism, and differentiates it from that of radical constructivism. Individual agents are no longer regarded as fully autonomous but as subsystems of more complex systems. "To use a mathematical comparison, the relationship between our identities and the character of the collective is not one of fragment to whole, but of fractal to completed image. The whole unfolds from the part and is enfolded in the part" (Davis, 1995, p. 7). Abram (1996) confirms that conventional scientific discourse privileges the sensible field in abstraction from sensory experience. Such discourse, he maintains, holds that subjective experience is "caused" by an objectifiable set of processes in the mechanically determined field of the sensible. Abram claims that this view perpetuates the distinction between human "subjects" and natural "objects" and hence reifies the common conception of sensible nature as a purely passive dimension suitable for human manipulation, and avoids the possibility that the perceiver and the perceived are interdependent. This theme of interdependence runs throughout Abram's book, and is embraced by many of those working in the field of education (for example, Bruner, 1996) who recognise that the learner's understandings are intertwined with the understandings of all others present, and who therefore recommend that we study neither the individual nor the cultural context in isolation, but rather the individual with/in the context.

With this in mind, I have begun to consider how a different interpretation of teachers' and students' interactions in the classroom might be enacted. By far the majority of legitimated speaking which happens in most classrooms takes the form of teachers' questions, and so it is here that I have begun my exploration.

2.3 Teacher-Student Interaction

2.3.1 Introduction

There has been a great deal published on the topic of teachers' questioning strategies. Much of this literature was produced in the 1970s and 1980s, and reflected the fact that the focus of educational research in those years was on the teacher (Hunkins, 1972; Hyman, 1979; Winne, 1979; Redfield & Rousseau, 1981; Wilen, 1987a, 1987b; Dillon, 1988). Some of this literature was concerned with the categorisation of teacher questions (into higher and lower order cognitive questions) and their differing effects on student achievement (Winne, 1979; Mahlios & D'Angelo, 1983), however, some of the early material did highlight the predominance of teacher talk in classrooms, and suggested that there was much to be gained from encouraging more student talk, and particularly more student questions (Dillon, 1987; Edwards and Mercer, 1987).

In the early 1980s there began a shift in focus from the teacher to the student. Several researchers initiated extended projects (which are still continuing) focusing on student learning and student understanding (Pirie, 1988; Maher & Davis, 1990; Cobb, Yackel & Wood, 1992). Such attention produced a flurry of interest in promoting discussion in classrooms. Many writers began to concern themselves with the value of student-student discussion (Pirie, 1991), small group work (Hoyles, 1985), and problem solving activities (McGlinn, 1991), and this fuelled the fire of constructivism, and reinforced the push in Europe and North America in the 1980s and early 1990s for reform of teaching practices (Cockroft, 1982; National Council of Teachers of Mathematics, 1989). There has also been a counter-response to the shift from a teacher to a student focus, with educational

researchers, such as Richards (1991), commenting on the constructivist neglect of the role of the teacher. Running throughout these debates has been a concern about curriculum content, reflected most obviously in new curriculum documents and initiatives in the UK (the National Curriculum), Canada (the Atlantic Consortium documents and the Western Canada Protocol documents), the US (in the guise of the Standards documents), and elsewhere.

More recently, some researchers have begun to view the classroom more holistically. Contemporary literature reflects this shift, and many recent works focus explicitly on the culture of the classroom (Cobb & Bauersfeld, 1995a; Bruner, 1996; Bauersfeld, 1996, Yackel & Cobb, 1996). This is not to say that in the 1960s and 70s there was no published research which addressed cultural issues, interaction, discussion and group work, nor to say that recently there has been a paucity of papers focusing on teacher questioning, merely that it is possible to identify a shift in the collective concerns of the mathematics education community[1]. I suggest that in order to more fully understand the complexity of classroom interaction we can no longer heed only one or other of these discourses. I am proposing a conflation of ideas and methodologies. Following the lead of Lerman (1996) and others, I suggest that rather than "watering-down" existing philosophies (such as that of constructivism) to incorporate other notions (such as socio-cultural perspectives), which invariably creates an inclusive but inconsistent position, we shift our attention to a new philosophy of human learning. Such a shift would necessarily

[1] For example, for early material on cultural perspectives see Hammersley, 1977, and Bruner, 1986; and for recent material on teacher questioning strategies see Vacc, 1993; Martino & Maher; 1994; Chuska, 1995; and Brogan & Brogan, 1995.

mean a re-orienting of our research agendas to include studies which, whilst still addressing manageable research questions, are more exploratory in nature, and reflect the complexity of classroom life. I believe that mine is one such study.

As my research focuses on the role of the teacher I must study the talk and behaviour of the teachers who are part of my project in great detail. I am, however, interested in the ways in which teachers occasion students' understanding, and I cannot address this problem without an intimate knowledge of the students' understanding of the mathematics they are learning. I must, therefore, also study the talk and behaviour of the students as they learn. This requires me to be familiar with the existing literature on both teacher talk and behaviour, and student understanding. Introducing an enactivist framework enables me, in fact, compels me, also to address these elements simultaneously - to study the *inter*actions between the participants, not merely their actions. As Abram (1996, p. 75) notes, "meaning sprouts in the very depths of the sensory world, in the heat of meeting, encounter, participation". This is how I believe understanding, as well as meaning, evolves, and so the literature that has informed my enquiry is that which reflects my belief in the co-evolving nature of mathematical understanding in classrooms.

Let us look, then, at some of the emerging discourses in more detail. The traditional tendency in educational research to separate the teacher from the student, what is taught from what is learned, and what is meant from what is understood, means that, in terms of my focus, the relevant literature is sparse. Enactivist literature which refers directly to

educational issues such as teacher and student interaction, whilst growing in importance, is still in its infancy, and so in addition to the writings of those researchers who explicitly claim this orientation I must look to others who, whilst not using the label "enactivist", offer ideas and theories which are in keeping with an enactivist philosophy. Many of these are mathematicians (Waldrop, 1992), ecologists (Abram, 1996), and linguists (Lakoff & Johnson, 1980), as well as educational philosophers and psychologists (Bruner, 1996).

To situate my later comments and ideas, to expose the breadth of the field, and to acknowledge the contribution of research literature to my current thinking, I now propose to present an overview of pertinent research findings reported in recent years. The relevant literature falls into three main categories, which were summarised by the opening three paragraphs of this section. The first, characterised by a focus on the teacher, encompasses issues of teacher talk, behaviour and questioning strategies. The second, characterised by a focus on the student, encompasses issues surrounding the use of discussion and group work. The third, characterised by a focus on the interplay between the previous two, encompasses issues of culture and communication.

2.3.2 Questions and questioning

I will begin, then, by considering some of the literature that concerns itself with teacher questioning, by far the most pervasive type of teacher talk. For many years a great variety of authors have been pointing to the preponderance of teachers' questions in the classroom whilst bemoaning the dearth of student ones (Postman and Weingartner, 1969;

Dillon, 1987, 1988; Chuska, 1995). Dillon (1987, p. 56) laments the fact that whilst there is a book called *Questions Children Ask*[2] "there is no book called *Questions Pupils Ask* because children do not ask enough questions in school to fill a book". Research has shown that teachers commonly ask as many as 50,000 questions a year and their students as few as 10 each (Watson and Young, 1986). Dillon (1988, p. 25) goes on to claim that this deplorable state could be remedied, suggesting that "the single most effective act [teachers] can take....is to *stop asking questions*". However, research has shown that there are powerful reasons why teachers may be reluctant to cease questioning, or even to modify their questioning strategies. Edwards (1980) points out that like lawyers, doctors, and police officers, teachers learn to ask questions, indeed may have been trained to ask questions, which restrict the scope of the answers so as to get only as much as is required. (What is *required* is open to debate, of course, but in this sense is taken to mean *required -in-order-to-keep-the-locus-of-control-firmly-in-the-hands-of-the-teacher*). Edwards and Mercer (1987) report that interviewers, therapists, barristers and others whose job it is to ask questions are typically advised that asking strings of direct questions is the surest way of shutting up the interviewee. It should come as no surprise to us, then, that for many teachers one function of questioning is to maintain control of the classroom. Further to this, Edwards (1980, p. 241) notes that the teacher's power is pervasive, "not only because he [or she] asks so *many* questions, but because the asking of questions by someone who 'already knows' allows the questioner to retain the initiative by evaluating the response".

[2] See Hughes, 1981.

A moment ago I mentioned that Dillon suggests that teachers should stop asking questions. It is important to note that I do not take this to mean that teachers should eradicate all questioning strategies from their repertoire, but rather that they should change the emphasis of their techniques, adopting instead a different kind of questioning strategy which relies less on the rapid-fire elicitation of rote-responses, which is the dominant form of questioning in mathematics classrooms (Davis, 1996), and more on the type of questioning characterised by Gadamer (1975) as *hermeneutic*. Hermeneutic questions are ones for which the questioner does not know the answer and is sincere in the wish to learn it. This is how Davis interprets such questioning:

> The questioner is oriented toward gaining a fuller understanding and is thus vigilant to the fallibility of interpretation of any response given.....The questioner participates in the questionability of what is questioned; there is some indeterminacy.....It is this sort of question that might "break open" a lesson - but only if the person who has posed it is genuinely participating in the question by listening to the actions it provokes (Davis, 1996, p. 250-252).

Such questions can be contrasted with Gadamer's *rhetorical* questions, which lack both questioner and answerer, and *pedagogical* questions which lack a questioner, in that the person asking already knows the answer. The latter category of questions, characterised some time ago by Barnes, Britton and Torbe (1969) as pseudo-questions, will be familiar to those who have studied classroom interactions and observed that teachers rarely ask questions to which they do not already know the answer.

2.3.3 Teacher Talk

Of course, questioning is not the only form of teacher talk. I contend that researchers should be paying attention not only to teachers' questions in the classroom, but also to

other types of interventions that teachers make. Studying the rest of teacher talk reminds us that the teachers' authority is maintained through almost all of the discourse. As Edwards (1980, p. 241) notes, in many classrooms "the teacher tells the pupils when to talk, what to talk about, when to stop talking, and how well they talked". The power relation is most clearly displayed through the teacher maintaining control over turn-taking, and especially by his or her assuming, and usually exercising, the right to a turn after each pupil turn. This results in the familiar teacher-student-teacher-student speaking sequence in classrooms. In fact, as Ball (1991, p. 44) notes:

> without explicit attention to the patterns of discourse in the classroom, the long-established norms of school are likely to dominate - competitiveness, an emphasis on right answers, the assumption that teachers have the answers, rejection of nonstandard ways of working or thinking, patterns reflective of gender and class biases.

Although research has tended to focus on teacher questioning strategies (presumably because this tends to account for the majority of teacher talk) there is clearly a great deal to be learned by studying the remainder of teacher talk, too. In the fields of discourse analysis and linguistics there is also a great deal of published literature which deals with the micro-analysis of classroom discourse. In mathematics education, the emphasis generally has not been upon the details of the language used in classrooms (though there are some exceptions, see for example Pimm, 1987), but upon a broader analysis of talk. Such research has tended to focus on the merits of student-student discussion and group work, so it is to a consideration of student talk that I now turn.

2.3.4 Student Talk

As Pirie (1998) notes in her recent review of the use of talk in mathematics classrooms, among the earliest publications to address the value of classroom discourse was that of Barnes, Britton and Torbe (1969). Since then a great deal of attention has been paid to student-student talk, and the process of group learning. Hiebert and Wearne (1993) suggest that it seems reasonable to believe that the nature of the discourse present in a particular classroom influences the learning. They note, however, that although the theoretical support for this conjecture is considerable, the empirical data are less compelling. Several researchers would disagree (Ball, 1991; Cobb, Yackel & Wood, 1992). These, and other, researchers attest to the value of group discussion, in particular as it applies to the negotiation of meaning in the mathematics classroom. In her discussion of the process of such negotiation, Wood (1990) notes that the typical discourse pattern in classrooms which employ whole-class discussions has been commonly described as one where the teacher controls, directs and dominates the talk. Wood claims that this need not necessarily be the case during whole-class discussions, and that such settings can involve the negotiation of taken-to-be-shared mathematical meanings and the mutual construction of social norms that constitute the obligations and expectations crucial for the development of a classroom setting in which learning can take place. This mutual adaptation of teacher, learner and mathematics is reflected in much of the emerging enactivist literature, and is a theme to which I will return later.

What actually constitutes a discussion has been a concern of many researchers and teachers. Healy, Hoyles and Sutherland (1989) note that students sometimes simply co-

work, that is, they sit side-by-side and may speak to one another as they work, but don't actually engage in shared discussion. Pirie and Schwarzenberger (1988) define mathematical discussion as purposeful talk on a mathematical subject in which there are genuine pupil contributions and interaction. In their study of four classrooms, however, Pirie and Schwarzenberger discovered that discussion as they had defined it was hard to find, even though they had selected as their sample four teachers who had been recommended on the basis that they used 'discussion' as an intentional component of their teaching. Dillon (1994) has elucidated a difference between what he calls recitation, and true discussion. He suggests that we can tell the difference by considering three aspects of the talk - the characteristics of the talk, the perceptions of the participants, and the concepts of discussion. Dillon elaborates on each of these aspects at length, concluding that the use of "true" discussion in classrooms is still scant, and that what usually passes for discussion is really recitation, a form where the teacher is still the predominant speaker, the speaker pattern follows the familiar teacher-student-teacher-student sequence, the pace is fast with many changes of speaker, the majority of questions are asked by the teacher (and the purpose of asking those questions is to determine if the listener knows the answer), and the evaluation of answers is done solely by the teacher. Although all of these features are pertinent to a consideration of the place of discussion, and its common usage in classrooms, Dillon's treatment is somewhat simplistic and naïve. He suggests that the concepts of discussion are the only aspect which require inference from the data - that the characteristics of the talk can be easily seen and heard, and that the perceptions of participants can be easily discovered by simply asking. This unproblematic approach to the analysis of classroom discourse would be challenged by

many researchers, myself included, who have worked intensively with classroom video and audio data[3]. Dillon (1994, p. 105) also notes that discussion is "time-consuming, kaleidoscopically unpredictable in process, and uncertain of outcome as much as unsure of success", all of which contributes to its under-use in classrooms. One of the major factors limiting its use in school appears to be the fact that teachers have usually had no personal experience of true discussion.

> They may be worried about the waste of time in discussion, the not getting anywhere, the curriculum material not being covered, the topic first being opened up and then not being closed down, the immaturity and incorrectness of student opinion, the rise of conflict and emotion in class, the loss of direction and control over the whole process - all finally leaving them with the debilitating sense that they don't even seem to be teaching during discussion. These are powerful reasons against using discussion (Dillon, 1994, p. 106).

Such reasons are also reflected in case reports of beginning teachers' concerns about student-centred teaching methods (Ball, 1990). In addition, Burbules (1993) bemoans the anti-dialogical practices of the school, including curriculum as content coverage, aims as tested outcomes, teaching as management and control, classroom interaction as question-answer recitation, and overcrowded and competitive classrooms, all of which militate against the introduction and sustaining of discussion in classrooms.

Despite all of these problems there is evidence to suggest that encouraging discussion in classrooms is worthwhile. Holden (1993) reports that some studies suggest that girls talk less than boys in whole-class discussions, and that work in smaller groups would therefore have the potential for counteracting such trends. Such claims must be taken cautiously, however. Holden also reports that other studies have found that when students

[3] See, for example, Pirie, 1996a.

are working in small groups and disagreements take place, boys' answers prevail. Further, Holden notes that one study she reviewed considered the ratio of boys to girls in groups and its subsequent effect, concluding that when boys outnumber girls, or vice versa, the girls were disadvantaged because in predominantly boy groups the girls were ignored, and in predominantly girl groups the girls deferred to the only boy present. Such findings suggest that teachers (and researchers) cannot take for granted that small group work *per se* is the answer to providing equal opportunities for girls; however, Holden (1993) did conclude from her own research that the talk by girls in co-operative group work in mathematics is far greater in both quality and quantity than the talk elicited from them in whole-class discussions. There are other reasons why encouraging student-student discussion in classrooms, and particularly in mathematics classrooms, might be important. Dillon (1994, p. 112) reveals the purpose of discussion in this evocative paragraph:

> We engage in discussion for the very practice of essential goods. We discuss for the experience of community and inquiry in the lived moment, for participation with our fellows in communal reflection, discovery and deliberation. Discussion is a good way for us to be together. We use it to face our common perplexities about what to think and how to act...So discussion is a way for adults and children to be together in a fundamental human relation and essential educative activity. That is the good of discussion, its *raison d'être*.

Dillon suggests that postmodern discourses have recovered this traditional view of the purpose of discussion. Burbules (1993) also reminds us that dialogue is a relation we enter into. This notion of mutual exchange with other human beings brings me to the third area I intend to discuss - the interplay between teacher and learner.

2.3.5 The Complexity of Classroom Interaction

Recently there has been an increasing amount of attention paid to forms of teacher talk and behaviour other than questioning. Studies of teachers' body language and other forms of non-verbal communication (Banbury & Herbert, 1992), and their strategies for praising students (Schwartz, 1996), have revealed that there are numerous hidden messages in teachers' communication patterns. Students learn to read these messages as clearly as they do the spoken ones. As Bruner (1996) notes, children actually learn along more fundamental paths, and learn deeper lessons, than those we think we are teaching them. Studies of patterns of communication in classrooms are widespread[4], however, most focus primarily on verbal communication. Many of these studies base their findings on detailed and painstaking analyses of audio and video data. The researchers claim that such analyses reveal a range of features of classroom discourse which would not readily be noticed without the aid of the video and audio tapes and/or careful transcriptions of those tapes. There are those, however, who challenge the growing reliance on linguistic data. Stubbs (1986, p. 64, original emphasis) claims that there is a large volume of work which uses linguistic data in "*ad hoc* and *unprincipled*" ways, and simply scratches the surface of the linguistic data it relies upon for its claims. Whilst this is an interesting point, and bears scrutiny in some cases, it is fair to say that research reports which include significant excerpts of the captured dialogue, and detailed attempts to set those excerpts within a context for the reader, go a long way towards answering Stubbs' challenge by offering the reader an opportunity to critically assess the basis of the claims made.

[4] See, for example, Maher & Davis, 1990; Pirie, 1991; Cobb, Yackel & Wood, 1992; Vacc, 1993.

Recently, the notion of the *culture* of the classroom has become a widely used "catchword" (Bauersfeld, 1996). Social constructivists have turned our attention away from the individual construction of meaning by the student, and towards the influence of the social environment. Recent studies of classroom interactions, grounded in a social constructivist theoretical framework, have focused on socio-mathematical norms (Yackel & Cobb, 1996), the negotiation of mathematical meaning (Wood, 1990), cultural tools and mathematical learning (Cobb, 1995), and taken-as-shared understandings (Cobb, Yackel & Wood, 1992). Such work attempts to co-ordinate psychological and sociological perspectives in mathematics education (Cobb & Bauersfeld, 1995b), drawing on von Glasersfeld's (1987) characterisation of students as active creators of their ways of mathematical knowing, and on the interactionist view that learning involves the interactive constitution of mathematical meanings in a classroom culture. As Voigt (1995, p. 199) clearly states, "the interactionist approach mediates between individualism and collectivism". Enactivism, however, challenges the need to mediate between the individual and his or her environment by suggesting that the two are always and already inseparable. Enactivism contends that, as in the radical constructivist paradigm, the social constructivist approach continues to privilege the individual cognising agent, and merely notices the presence of an environment that the individual acts *within*. Enactivism rejects this separation of individual and environment, suggesting instead that the two co-evolve in mutual adaptation.

The theme of adaptation, though not always so named, surfaces in the writing of several researchers (Hargreaves, 1984; Wood, 1990). Hargreaves' (1984) paper presents an early

reference to adaptation, a phenomenon that is now very much at the forefront of research in a number of diverse fields, such as "new" biology, cognitive science, and immunology[5]. In a challenge to the dominant perspective at the time, Hargreaves noted that students' behaviour influences teachers' questioning strategies just as much as teachers' questioning strategies influence student behaviour. He recognised that ethnographers were emphasising the degree of mutual adaptation between teacher and students far more than were the educational researchers who were employing systematic observation techniques. From an enactivist perspective there can be no doubt that teacher and learner each influence the behaviour of the other in profound ways. This mutual adaptation is described elaborately in Davis' (1996) recent study of one high school teacher. Davis focuses on the change in the teacher's mode of listening from evaluative listening, through interpretive listening, to hermeneutic listening, the latter being a mode of attending characterised by a problematising of the boundary between the role of the teacher and that of the student. Davis suggests that this manner of listening is more negotiatory, engaging, and messy, involving the hearer and the heard in a shared project. Such a description of the processes of teaching and learning might be informed by a consideration of the etymologies of the words we use to describe what happens in classrooms when teachers and students interact. My attention was first drawn to the distinctions between such words as communication, conversation and discussion, and their differing Latin roots, by Richards (1991). Communication is from the Latin *communicare*, to make common; conversation is from the Latin *conversari*, to associate with; and discussion is from the Latin *discutere*, to investigate. Richards defines a

[5] A detailed discussion of these developments is given by Varela, Thompson and Rosch, 1991.

discussion as a conversation in which communication occurs, however, returning to the roots of these words might enable us to re-consider discussion as an association, or relation, in which commonalities unfold. Such an orientation to discussion, a practice which, as I have indicated earlier, is currently being encouraged in mathematics classrooms, reminds us of our relation to one another, of our interdependence, and of our responsibility as teachers and learners of mathematics to participate in the bringing forth of a world (Maturana & Varela, 1992).

The following sections of this chapter detail the analytical frameworks I considered as I designed my study, and reflect my continuing search for theories and methods of analysis which are in concert with my philosophical beliefs.

2.4 Understanding Understanding

2.4.1 Introduction

I'd like to begin this section by preparing you for what is ahead. I intend to address in a number of ways the issue of my adoption of the Dynamical Theory for the Growth of Mathematical Understanding (Pirie & Kieren, 1989, 1994a) as a theoretical tool for analysis. Firstly, it will be necessary to outline briefly the recent history of developments in understanding understanding, and to make explicit the ways in which my interpretation of those developments has been coloured by my philosophical framework. In the light of these revelations I intend to examine some of the competing and complementary models of mathematical understanding proposed by other researchers, such as Sfard, Sierpinska, and Confrey. In addition to a consideration of these alternative models of understanding, I

also want to open the discussion surrounding the ways in which an enactivist research agenda can be put into practice, an issue which I believe remains under-addressed in the emerging literature on enactivism.

2.4.2 A Historical Perspective on Mathematical Understanding

Beginning with Skemp's (1976) landmark paper, the last two decades have seen a flurry of interest in mathematical understanding. Skemp proposed two types of understanding - relational and instrumental, the former characterised as knowing what to do and why, and the latter as being in possession of a rule and knowing how to use it but not knowing why it works. He suggested that it is instrumental understanding that is usually taught in schools. This understanding/not understanding dichotomy provoked several researchers to propose subsequent models of understanding that expanded upon the number of "levels" of understanding[6]. As early as 1980 Byers proposed, however, that although a satisfactory model of understanding of mathematics must take into account the existence of *levels* of understanding, an adequate description of understanding would not be provided by a linear model (such as many of those proposed in the late seventies, including Skemp's). Despite this forewarning, several researchers persisted with their agendas and a number of models of understanding appeared in the 1980s and 1990s that were still linear in nature (see, for example Hiebert, 1984, Sfard, 1991). This shortcoming will be considered again later when I address some of the more recent models in more detail.

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[6] See, for example, Bergeron and Herscovics (1989).

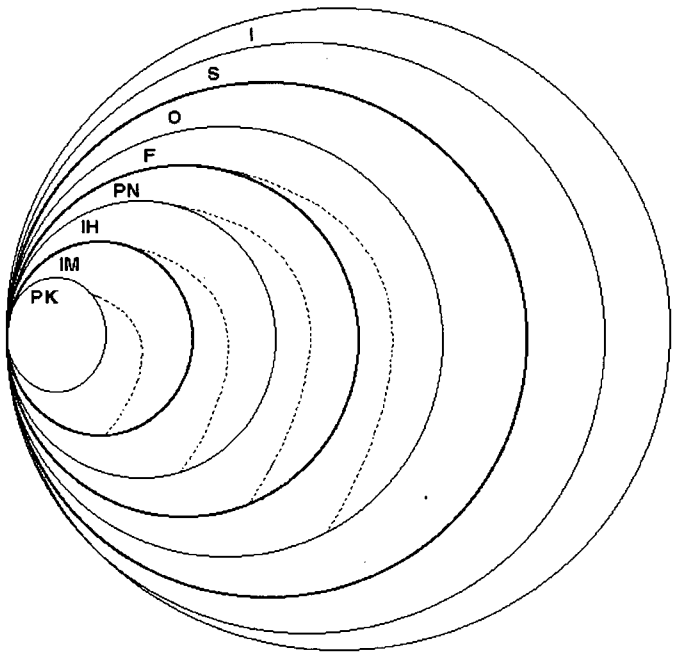
As a belief in constructivism as a theory of knowing became widely held in the mathematical education research community, an increasing amount of attention was paid to the nature of mathematical understanding. At this time, several researchers (not all of whom associated themselves with the constructivist movement) established research programmes, many of which are still thriving, that were designed to investigate the nature of children's mathematical understanding[7]. Only a few researchers, though, have worked systematically to produce the kind of manageable, coherent model capable of applying to all of school mathematics envisioned by Byers (1980). The most notable are the theories of Sfard (1991, 1994), Sierpinska (1990, 1995), Confrey (1994, 1995a, 1995b), and Pirie and Kieren (1989, 1994a), and I intend to examine each of these later.

The domination of constructivist philosophies has recently, however, been challenged. Convergent streams of thought are emerging in disciplines as diverse as ecology, biology, physics, mathematics, cognitive psychology, and continental and pragmatist philosophy. These ideas have been embraced by researchers working in the area known variously as deep ecology - a term which recognises the "fundamental interdependence of all phenomena and the fact that, as individuals and societies, we are all embedded in (and ultimately dependent on) the cyclical processes of nature" (Capra, 1996, p. 6), or enactivism - following Varela, Thompson and Rosch (1991).

[7] I'm thinking in particular of the work of Terry Wood *et. al.* at Purdue University, Robert Davis and Carolyn Maher at Rutgers University, Susan Pirie, then at Oxford University, and Tom Kieren at the University of Alberta, Jere Confrey at Cornell University, Deborah Ball, then at Michigan State University, and many others.

2.4.3 Contemporary Models of Mathematical Understanding

Though I leave until the next chapter a detailed description of my chosen tool of analysis, the Dynamical Theory for the Growth of Mathematical Understanding (Pirie & Kieren, 1994a), it is important to present here some of the salient features of the theory in order to situate some of the comparisons I make between it and the other models of understanding I have considered. The Dynamical Theory for the Growth of Mathematical Understanding considers mathematical understanding as an on-going process in which a learner responds to the problem of reorganising his or her knowledge structures by continually revisiting existing understandings. Pirie and Kieren have termed this process "folding back" (Kieren & Pirie, forthcoming; Pirie and Kieren, 1991). The theory considers understanding in terms of a set of embedded levels or modes of knowledge-building activity. These modes are illustrated in diagrammatic form in Figure 1, and I describe them more fully in Chapter 3. Pirie and Kieren stress, however, that it is not the modes themselves which define the growth of mathematical understanding, but rather "it is the non-linear pathways of students' behaviours, which can be tracked through the modes, which illustrates such dynamical growth" (Kieren, Pirie & Reid, 1994, p. 50). Pirie and Kieren maintain that growth in understanding involves multiple and varied actions of *folding back* to inner less formal understanding in order to use that "thicker" understanding as a springboard to the construction of more sophisticated outer level understanding. This is supported by the findings of several reported research projects (Pirie & Kieren, 1992a, Kieren, Pirie & Reid, 1994; Towers, 1994).



Key:

PK	Primitive Knowing
IM	Image Making
IH	Image Having
PN	Property Noticing
F	Formalising
O	Observing
S	Structuring
I	Inventising

Figure 1: The Dynamical Theory for the Growth of Mathematical Understanding

In addition to the modes of understanding there is one more feature of the theory that is relevant to my discussion here - the "don't need" boundaries. The theory claims that one of the strengths of mathematics is the ability to operate at a symbolic level without reference to basic concepts. The notion that students of mathematics are able to operate at a symbolic level without reference to basic concepts is reflected in the "don't need" boundaries of the theory, illustrated in the diagram by the bold rings. Beyond these boundaries the learner is able to work with notions that are no longer obviously tied to

previous forms of understanding, although these forms are embedded in the new level of understanding and readily accessible if needed. These rings are called "don't need" boundaries in order to convey the idea that beyond each of these boundaries the learner does not need the specific inner understanding that gave rise to the outer knowing. This does not mean that the learner cannot return to the specific background understanding, in fact, quite the contrary is claimed, as evidenced by the notion of *folding back*. The boundaries simply point to the fact that one does not need to be constantly aware of inner levels of understanding.

Each of the alternative models of understanding I am about to critique has similarities to the others, which is not surprising as each has a basis in constructivism. There are also, however, some significant differences between the various proposals. I will begin with the model of understanding proposed in several papers by Sfard (1991, 1994, Sfard & Linchevski, 1994). Sfard (1991) suggests that abstract notions can be conceived in two fundamentally different ways: structurally - as objects, and operationally - as processes. She argues that the ability to see a concept, such as number, as both a process and an object is indispensable for a deep understanding of mathematics. Sfard stresses the duality of her distinction between structural and operational, and sets it apart from what she sees as the dichotomies of other models (such as Skemp's (1976) notions of instrumental and relational understanding, and Hiebert's (1984) conceptual and procedural knowledge). In distinguishing between a duality and a dichotomy Sfard (1991, p. 9) claims that unlike *conceptual* and *procedural*, for example, the terms *operational* and *structural* refer to "inseparable, though dramatically different, facets of the same thing"; that whereas other

classifications decompose knowledge into two separate components, her complementarian approach stresses their unity. However, rather than investigating the blurring of the boundaries between these two "facets", in declaring her intent to "pin down the edges" of the operational-structural distinction, Sfard (1991, p. 8) reinforces that very distinction. Further, this notion of a duality is again emphasised in Sfard's later papers (1994; Sfard & Linchevski, 1994), but not as something different from a dichotomy. This omission is unfortunate as it invites an interpretation of Sfard's *operational* and *structural* understandings as dichotomous, not dualistic. In either case, however, enactivism rejects the whole notion of dichotomy/duality, preferring to conceptualise understanding as a co-emerging process, not as a shift from one state to another.

Sfard (1991) proposes a stage model of concept formation, which centres on the transition from computational operations (operational) to abstract objects (structural), a process known as reification. She suggests that this transition is accomplished in three steps: interiorization, condensation, and reification. She acknowledges that this process is cyclical, in that what is conceived operationally at one level is conceived structurally at a higher level. She suggests that this hierarchy emerges in a long sequence of reifications, each one starting where the former ends, and each one adding a new layer to the complex system of abstract notions. There are some similarities in this depiction to the view of the growth of understanding presented by the Pirie-Kieren theory, but the previous statement also highlights one of the major differences. In contrast to the Pirie-Kieren theory, Sfard's model (at her own admission (1991, p. 16)) is unidirectional and does not adequately

account for the intricate and complex pathways to understanding exhibited by learners of mathematics. To be fair, in some of her later work Sfard (1994) documents the struggle to reach reification, and discusses reification's "discontinuous, almost chaotic" nature, implying a sensitivity to the "non-linear, non-monotonic" growth of understanding characterised by Pirie's and Kieren's model (Kieren, Reid & Pirie, 1995), which was not prevalent in her earlier writing. One similarity between Sfard's model and that of Pirie and Kieren has been noted by Kieren (1992), who suggests that Sfard's notion of reification might be related to the feature of the Pirie-Kieren model known as a "don't need" boundary. Beyond each of these boundaries the learner is able to work with notions that are no longer obviously tied to previous forms of understanding, but these forms are embedded in the new level of understanding and readily accessible if needed. Reification, then, as an "act of creation of...abstract entities" (Sfard, 1994, p. 53), and in its manifestation as an "instantaneous quantum leap" (Sfard, 1991, p. 20) may reflect part of the notion of a "don't need" boundary, particularly that between *observing* and *structuring* (see, Pirie & Kieren, 1994b), but Pirie and Kieren stress that this is not the only occasion where such a metamorphic change occurs in mathematical understanding.

In an alternative model, Sierpinska (1990) has conceived of understanding as an act (of grasping the meaning) and not as a process or way of knowing. Sierpinska (1990, p. 26) regards understanding as "an act, but an act involved in a process of interpretation, this interpretation being a developing dialectic between more and more elaborate guesses and validations of these guesses". Sierpinska suggests that in gaining an understanding of a concept these acts of understanding become linked in a series of acts under the categories

of *identification* (of objects that belong to the concept), *discrimination* (between two objects, properties or ideas that were confused before), *generalization* (becoming aware of the non-essentiality of some assumption, or the possibility of extending the range of applications), and *synthesis* (grasping relations between two or more objects, properties, or facts, and organising them into a consistent whole). Sierpinska's categories of *identification*, *discrimination*, *generalization* and *synthesis* are reminiscent of (but clearly not the same as) Pirie's and Kieren's *Image Having*, *Property Noticing*, *Formalising* and *Structuring* modes of understanding, and as such suggest a conception of understanding as a process. Further, then, I am troubled by Sierpinska's denial that gaining understanding is a process in light of her description of gaining understanding as a *series* of occurrences.

Sierpinska (1990) also suggests that some, but not all, acts of understanding are acts of overcoming epistemological obstacles, asserting that in many cases overcoming an epistemological obstacle and understanding are just two ways of speaking about the same thing. Again, Kieren (1992) has noted a similarity between the notion of overcoming epistemological obstacles and the Pirie-Kieren concept of *folding back*, whereby learners create a "thicker" understanding of the concept each time they *fold back*. Sierpinska views coming to understand as starting with a guess which is refined, improved or rejected. The new guess is then subjected to justification and validation. Sierpinska sees this as a spiral process which continues until the thing to be understood is considered to have been appropriated. This image of a spiral growth of mathematical understanding with its implicit suggestion of unidirectional progress is one which Pirie and Kieren

would reject. Further, the suggestion that understanding can be considered to have been appropriated implies that understanding is an achievable state. This claim is rejected in an enactivist formulation of the growth of understanding, which instead suggests that understandings co-emerge in the context and are always changing, moving, flowing. This notion causes some concern, of course, for those of us interested in studying such understandings, however, I will return to these issues later.

The view of understanding which in many ways comes closest to the views of Pirie and Kieren, and to an enactivist perspective, is that articulated by Confrey in her three part essay on intellectual development (1994, 1995a, 1995b). Despite the many interesting insights offered in these essays, Confrey stops short of offering an explicit model of understanding, and does not explicate clearly a formal theory of how understanding grows. Confrey does propose that a less incremental view of knowledge development than that offered by constructivist proposals is desirable, and she offers some useful recommendations in her call to re-cast theories of learning from a feminist perspective through the metaphor of reproduction, claiming that neither Piaget nor Vygotsky paid adequate attention to the importance of nurture and reproduction in human development. In alignment with the emerging thinking in many academic disciplines (several of which I have mentioned in this chapter), such a shift would emphasise evolutionary biology as a metaphor for the growth of understanding, replacing the mechanistic metaphors (and therefore thinking) which have predominated since the time of Descartes.

In terms of my philosophical framework, however, there remain some inconsistencies and shortcomings in Confrey's notions. The first of these is Confrey's reluctance to step beyond constructivism's focus on the individual. Despite her willingness to describe what she sees as the limitations of constructivism, (such as an assumption of an incremental view of knowledge construction, and the lack of an adequate theory of instruction), and whilst celebrating some of the insights of social constructivism, Confrey refrains from taking these insights one step further and rejecting a focus on the individual cognising agent. Confrey warns against simplistic interpretations of Piaget (as an individualistic theory) and Vygotsky (as a theory about society), suggesting that such descriptions seem woefully inadequate. Confrey tries to move beyond the tensions she suggests exist between these theories to "create a bridge between them" (1995b, p. 43). Enactivism goes one step further in denying that there is, in fact, anything to bridge. Enactivism troubles the notion that the individual can be viewed (as he or she often is in the social constructivist formulation) as being *in* a context. As Davis, Sumara and Kieren note:

> the concern is not with how the cognizing agent comes to know the world, but with how the learner-and-learned, knower-and-known, self-and-other co-evolve and are co-implicated. Context is not merely a place which *contains* the student; the student is literally part of the context (1996, p. 157).

Although in discussing some alternative models of the growth of mathematical understanding I have already touched upon some of the strengths of the Dynamical Theory for the Growth of Mathematical Understanding, I feel I need to emphasise what I see as the major advantages of this model *for the purposes of my study*. The first advantage is that this model is a far more detailed framework, being more explicitly formulated than any of the others I have described above and having been specifically

developed as a practical tool to analyse complex classroom data. I have videotaped children in classrooms, and attempted to analyse their interactions with their teachers to determine how the teachers' interventions may have occasioned the growth of the students' mathematical understanding. In order to say something about the teachers' *influence*, I cannot merely do a descriptive piece, (and, in fact, I question whether any piece of research can claim to be purely descriptive). I must interpret the teachers' actions and relate them in some way to the students' actions. To do this it is necessary to have a framework for my interpretations; some means of characterising the teachers' actions, and linking those characterisations to the actions of the children. Of the alternative theories I described above, Confrey's is least explicit in terms of how it would be applied to the analysis of real classroom data, (Sierpinska's, perhaps, being the most explicit), and, in fact, none of these models have been as comprehensively used, tested, developed and reported as that of Pirie and Kieren. It is interesting to note, however, that it is difficult for me to say whether I have chosen to use Pirie's and Kieren's model because of the detailed framework it provides which suits the data I have (complex, rich, video data), or whether I have chosen to collect video data because only this kind of data can provide the complexity and richness necessary to fully exploit the Pirie-Kieren analytical framework. The fact that it is so difficult for me to say which of these decisions came first suggests that neither one preceded the other; that the decisions are interdependent and inseparable, a notion consistent with an enactivist philosophy.

It is also significant to note that Pirie and Kieren consistently emphasise the dynamical nature of their model. This is realised most clearly in their description of their theory as a

model of the *growth* of understanding. I see this as a critical departure from other models, which claim to be theories about *understanding,* not about the growth of understanding (Skemp, 1976; Sierpinska, 1990; Sfard, 1994). From an enactivist perspective, this orientation represents a significant movement towards a more fluid, dynamic, constantly evolving view of understanding. It is also important to note that the Dynamical Theory for the Growth of Mathematical Understanding has itself evolved over time, blending the theoretical with the experiential, and reflecting the influence of the diverse body of learners (students of all ages, teachers, and researchers) who have collaborated in its development. These developments have been informed not only by deep analyses of collected data, but also by significant developments in the fields of education, philosophy, psychology, mathematics and many other disciplines. This evolution is evident not only in the increasing complexity of the model (through the introduction of notions such as the complementarities of acting and expressing, and the categories of teacher interventions, see Chapter 3 Section 3.5), but also in the language Pirie and Kieren have developed around the model, and in the language they use to describe their thinking about their theory. To illustrate this last point, I refer the reader to a sequence of Pirie's and Kieren's papers which reflect the development of their ideas, and which demonstrate the way in which the language they choose reflects their evolving philosophy (Pirie & Kieren, 1989; Pirie & Kieren, 1992a; Kieren, Pirie, Davis & Mason, 1993; Pirie & Kieren, 1994a; Kieren, Pirie & Reid, 1994; Kieren & Pirie, 1996; Kieren, Pirie & Calvert, 1997).

Finally, the Dynamical Theory for the Growth of Mathematical Understanding specifically attributes a role (in the development of students' understanding) to the

teacher, identifying three types of teacher intervention: invocative, provocative and validating. It was upon these categories that I initiated my analysis, and from here that I began to investigate the complex web of interactions that interweave to support the growth of understanding in a classroom. Herein, though, lies a problem - the tension between maintaining a belief in the enactivist philosophy that the parts of any environment are interdependent and, ultimately, inseparable, and engaging in research whose purpose is to try to make sense of that environment. This tension is the subject of the next section of this chapter.

2.4.4 An Enactivist Understanding

As Bauersfeld (1996, p. 1) has noted, "the most promising trend in mathematics education at present appears to be the shift from specialization to integration, from applying limited monolithic theories toward a combining of insights and findings from different academic disciplines", and nowhere has this shift been rendered more transparent than in the emerging philosophical framework of enactivism. This notion of integration, of considering environments holistically, focuses on the dynamic interdependence of individual and environment rather than on their autonomous constitution (Davis, 1996), and sees the individual and environment as bound together in reciprocal specification (Varela, Thompson & Rosch, 1991). This position, which draws on the evolutionary metaphors of Darwin rather than the analytic and reductionist thinking of Descartes, recognises the futility of separating what we do (as teachers and learners together) from who we are and what we know. As Davis, Sumara and Kieren (1996) note, knowing, being, and doing are not three things, they are one. This circularity,

this connection between action and experience, this inseparability between a particular way of being and how the world appears to us, tells us that "*every act of knowing brings forth a world*" (Maturana & Varela, 1992, p. 26, original emphasis).

Following this thinking, in his new book, Capra notes that:

> nature is seen as an interconnected web of relationships, in which the identification of specific patterns as "objects" depends on the human observer and the process of knowing. This web of relationships is described in terms of a corresponding network of concepts and models, none of which is any more fundamental than the others (1996, pp. 40-41).

Capra (1996, p.41) also notes, however, that this new approach to science immediately raises an important question. If everything is connected to everything else, how can we ever hope to understand anything? Capra has an interesting answer for this question which invokes the concept of approximate knowledge. This concept recognises that, contrary to the Cartesian belief in the certainty of scientific knowledge, science can never provide any complete and definitive understanding. Capra's question, however, echoes the concerns of many who, as Davis (1996) points out, have profoundly misinterpreted one of the precepts of postmodernism - that is, that the quest for new groundings is doomed to failure - as suggesting that we can say very little about anything. Enactivist theorists suggest that, on the contrary, we can and should reflect upon notions such as cause and effect. Although enactivist theorists begin and end their analyses with an acknowledgement of the fundamental inextricability of all things (Davis, Sumara & Kieren, 1996), this is not to say that we cannot or should not make distinctions, for, after all, we cannot pay attention to everything at once. However, we must bear in mind that the distinctions we make are merely conveniences and therefore we are obligated to

acknowledge those things which at any one time we push to the background. We can also ensure that those things do not always appear as background, but sometimes are the focus of our attention.

Enactivism, then, gives us a new language with which to explore such notions as cause and effect. Maturana and Varela, who have laid the ground work for much of the current enactivist discourse, note that:

> the perturbations of the environment do not determine what happens to the living being; rather, it is the structure of the living being that determines what change occurs in it. This interaction is not instructive, for it does not determine what its effects are going to be....[T]he changes that result from the interaction between the living being and its environment are brought about by the disturbing agent but *determined by the structure of the disturbed system* (1992, pp. 95-96, original emphasis).

It follows that the growth of students' understanding can be interpreted as being *dependent on*, but not *determined by*, the actions of the teacher. Such a re-formulation will be important as I begin my analysis and interpretation of teacher and student actions in the classroom.

Traditional thinking placed teachers in a dominant role in classrooms, assuming when the desired end product of "understanding" was reached that the teaching *caused* the learning. Enactivism rejects this formulation, and, as I described earlier, not only challenges the view of understanding as a possibly achievable end-state, but also challenges the assumption that teaching *causes* learning. The issue of causality in learning is one that has been recently taken up by enactivist writers Davis, Sumara and Kieren (1996). They suggest that Varela's analogy of a wind chime is useful in thinking about causality.

Varela asks the reader to imagine a wind chime made with thin pieces of glass dangling like leaves off branches.

> Clearly, how the [wind chime] sounds is not determined or instructed by the wind or the gentle push we may give it. The way it sounds has more to do with ... the kinds of structural configurations it has when it receives a perturbation or imbalance. Every [wind chime] will have a typical melody and tone proper to its constitution. In other words, it is obvious in this example that in order to understand the sound patterns we hear, we turn to the nature of the chimes, and not to the wind that hits them (Varela, 1992, p. 50).

I do not want to suggest that the teacher is rendered powerless in an enactivist formulation, for, like the wind hitting the wind chime, the teacher certainly has a part to play in occasioning students' understanding. As I have chosen to base my doctoral research around a consideration of how the teachers' interventions occasion and interact with the growth of students' understanding, I clearly believe that the teacher does exert some influence. Further, I have taken it upon myself to investigate how teacher-student interactions evolve in the classroom, and how those interactions result in the growth of students' mathematical understanding. To clarify, then, I reiterate my belief that the growth of students' mathematical understanding is dependent on, but not determined by, the teacher, and my position is therefore consistent with that of Varela.

2.5 Summary

I believe that the issues I have addressed in this chapter are significant, touching as they do on fundamental questions of human cognition and understanding. As I hope was made clear in my summary of the evolution of the current discourse on enactivism, some of the claims I have made here about the embodied nature of human cognition are not universally accepted in the field of education. I am, however, committed to a belief in the

co-emergence of understanding in classrooms, and, as back-drop to that, to a belief that understanding is historical, situational, dynamic, intersubjective, and consensual (Davis, 1996).

CHAPTER 3

ACTING

3.1 Introduction

Before I describe the specific data collection and analysis methods I have chosen to enable me to address my research question, I want to turn to a consideration of my methodology. I have chosen to present case studies of two teaching situations, and so the next section of this chapter deals with issues pertinent to the use of case study as a methodological approach to qualitative research. Within this chapter I have also included a section which considers methodological issues concerning research studies in which, like this one, the researcher plays a significant part, and a section on the use of video as a data gathering and analysis tool as this tool has not only guided my choice of participants and data collection methods but also structures and enables my analysis, and has therefore had a significant influence on my findings. Additionally I describe in more detail the theory which frames my analysis, the Dynamical Theory for the Growth of Mathematical Understanding, and I introduce the students and teachers who were participants in this study and describe the nature of my data collection methods.

3.2 Methodology

3.2.1 Introduction

In order to carry out my research it has been necessary to study the interaction between teachers and students in considerable depth. From an enactivist standpoint it was essential that I gain a deep and thorough understanding of the collective. The detailed analysis involved in this search made it entirely appropriate to study just a small number of

teachers and students. This qualitative case study approach has provided a rich and detailed analysis, and a deep and comprehensive description and interpretation of the processes of classroom interaction.

3.2.2 Case Study Methodology

Walker (1993) recognises that case study is a tradition that finds inspiration in the methods and techniques of social anthropology. Adelman, Jenkins and Kemmis (1976, p. 140) also reveal that case study uses a variety of techniques, often sociological in character, stating that "case study is an umbrella term for a family of research methods having in common the decision to focus an enquiry around an instance" and that "case study itself is about moving between the general and the particular." This statement reflects one of the common definitions of a case study as "the study of an instance in action" (Adelman, Jenkins and Kemmis, 1976, p. 141). However, Atkinson and Delamont (1993, p.206) in their critique of case study research state that:

> such definitions (if they deserve the term) seem to be of symbolic value to the research network of devotees precisely insofar as they are vacuous and commit its members to remarkably little.
>
> Confusion in this area is compounded when it is recognised that the unit of analysis ('case') can, in practice, mean just about anything.

The main objection appears to be founded on the premise that case studies commonly focus on a statistically insignificant sample. Instead of trying to explain and excuse this fact, however, qualitative case study authors, particularly ethnographers (and those like myself who, whilst not attempting to produce an ethnography, are borrowing ethnographic techniques) openly acknowledge what they are attempting to do. They claim that their forte lies in knowing their cases extremely well, and in recognising a distinction

between generalising and over-generalising (Wolcott, 1985). Similarly, Merriam (1988, p. 173) suggests that "one selects a case study approach because one wishes to understand the particular in depth, not because one wants to know what is generally true of the many." My research study has adopted this kind of emphasis. However, although my intention has been to create deep and detailed analyses and understandings of two particular situations, I also hope to offer some implications for others, both teachers and researchers, and in this way I might be interpreted as attempting to generalise from my data.

Firestone (1993) points out that whilst generalisability is clearly not a strength of qualitative research, many past reservations have been overstated. He claims that generalising from data is always problematic, requiring extrapolation that can never be fully justified logically, and that different research traditions have developed their own arguments to justify such extrapolation. Further, Firestone and others (see, for example, Walker, 1993) note that qualitative studies claiming case-to-case translation achieve generalisability, in part, by passing the responsibility for application from the researcher to the reader. In this sense, broad applicability is achieved by providing the reader with sufficient information about the theoretical underpinnings of the research, the methods of data collection and analysis, and the interpretation of the findings to enable him or her to assess the claims made. One way in which it is suggested that this may be achieved is by the provision of rich, "thick" description (Geertz, 1973).

As Lincoln and Guba (1985) point out, however, generalisation is an appealing concept, but one which oozes determinism and "seems to place the entire world at the feet of those persons who can unlock its deepest and most pervasive generalities" (p. 111). They go on to identify what they see as the major problems with the classic concept of generalisability, their argument centring on the assumptions made by those researchers who claim generalisation to be the "be-all and end-all" of inquiry (Lincoln & Guba, 1985, p. 110). Although it is widely recognised that the aim of much scientific research is generalisation, tensions are currently surfacing concerning what can and cannot be generalised *from.* Lather (1991, p. 51) contends that we are currently in a period of dramatic shift in our understanding of scientific enquiry. It is increasingly acknowledged that facts are theory-dependent; they are social constructions as much as they are theories and values, a recognition that is at the heart of the enactivist philosophy.

Yin (1989) draws our attention to a fundamental aspect of case study, which is the use of multiple sources of evidence. Yin refers to six sources of evidence, and attests that each calls for slightly different skills and methodological procedures. Two of the sources are participant and direct (or non-participant) observation, variations of both of which feature in my study. It is generally thought that the method of observation will have an impact upon the character of the study. As will become clear in the next section of this chapter, there are interesting aspects associated with the methods of data collection of this study that revolve around the issue of teacher-as-researcher. Observation methods have been the focus of much criticism in terms of their effect on the validity, reliability, and even generalisability, of case study reports. Criticism is mainly levelled at the fact that

observers, however practised, must rely on their own perceptions, which bias their interpretations of everything they see and hear. In order to counteract this claim, case study researchers have employed the technique of triangulation. This is a multi-method approach advocated by many qualitative researchers (Yin, 1989; Merriam, 1988) to enhance the internal validity of research. This technique commonly recommends the use of multiple investigators, multiple data sources or multiple methods to confirm the emerging findings (Merriam, 1988). Such an approach, though, implies that there is only one truth to be told, an assumption rejected by Walker (1986). This rejection has been echoed more recently by Richardson (1994, p. 521) who suggests that "there is no such thing as 'getting it right', only 'getting it' differently contoured and nuanced". She contends that "in traditionally staged research we valorize 'triangulation'... [but] in postmodernist mixed-genre texts, we...*crystallize*. We recognise that there are far more than 'three sides' from which to approach the world" (p.522). Richardson proposes that the central image for validity of texts should not be the rigid, fixed, two-dimensional triangle, but the crystal which "combines symmetry and substance with an infinite variety of shapes, substances, transmutations, multidimensionalities, and angles of approach" (p. 522). This position is in harmony with the enactivist perspective on issues of validity, reliability, and generalisability which underlies this research. Davis (1996, p. 103) suggests that

> conventional concerns for validity, reliability, rigor, and generalizability - notions that are founded on a belief that reality is "out there" awaiting our efforts to capture it in our language - are replaced with an acknowledgement of the contingency of interpretation.

I follow Davis further in suggesting that the concerns of this research:

> lie more with issues of viability, reasonableness, relevance, and applicability - that is, more with the qualities of an emerging tale which implicates teller and listener alike than with an account that is sanitized of any particular human involvement (Davis, 1996, p. 103).

Nevertheless, I have shared my analyses with interested others, including those in my immediate academic field, in an attempt to address the issue of the validity of my findings. I discuss these efforts in the next chapter (Section 4.7).

Goetz and LeCompte (1984, p. 40) also remind us of the importance of an orienting theoretical perspective when they recognise that research designs are "improved radically - in applicability and generalizability, in credibility and validity, and in precision and reliability - by explicit attention to the influence of theory throughout the design and implementation process". This view is in harmony with the first of Sanjek's (1990) canons of ethnographic validity - theoretical candour. This notion affirms the importance of making explicit in the research report the theoretical framework on which choices made by the researcher (for there are always choices made) have been based. Such choices include the problem addressed, the methods used and the interpretations made. It is my hope that woven throughout this report is a detailed enough outline of my theoretical framework that the reader is able to critically assess how that orientation may be reflected in my findings.

The "presence" of the researcher in all aspects of research, from choice of topic and participants, through data collection and analysis, to the final written report, is seen by its critics as a major challenge to the worthiness of case study research. Atkinson and

Delamont (1993, p. 210) for instance, claim that "the case study research tradition is seriously deficient due to both inadequate methods and a lack of methodological self-awareness". My approach, however, is that since the effect of the researcher cannot be eliminated, it must instead be rendered as transparent as possible through an explicit, reflective and truthful description of the research process, the participants, the setting, and the relationships between them. Lather (1991), however, cautions writers to be aware that there is a traditional reluctance to accept research in which the "ethnographer's path" (Sanjek, 1990) is laid open to scrutiny:

> Research programs that disclose their value-base have been typically discounted...as overly subjective and, hence, "non-scientific." Such views do not recognize that scientific neutrality is always problematic; they arise from an objectivism premised on the belief that scientific knowledge is free from social construction (Lather, 1991, p. 52).

I want to stress that Lather is not suggesting that researchers should refrain from laying open to scrutiny their "world views", but rather that qualitative researchers should be aware of the potential for criticism from those who believe such details to be evidence of a subjective bias in their work. Due to the nature of my research methods the issue of bias is one to which I must pay particular attention. A significant part of the data for this study was collected in my own classroom, and so it is to a consideration of the promise and perils of such research that I now turn.

3.3 Being a More-Than-Participant Researcher

3.3.1 Introduction

I can clearly recall the day I first took a video camera into my classroom to begin my first research project. Perhaps my memory of the events of that lesson has been enhanced by

the fact that I have a permanent record of those events in the form of videotapes to which I have returned many times over the years. As I write this I am reminded of my consternation whenever I am asked to recall a significant childhood memory. At these times I often recall an event that was captured at the time by cine-film or photograph - the way that the family dog used to pull me on the garden swing using a dangling length of rope, or a particularly hot summer's day spent in the paddling pool. It is now difficult for me to separate the true memory of those events from the images I have seen many times since, and I wonder whether this effect also influences my memories of events captured by my video camera during research sessions. Nevertheless, certain thoughts and feelings are vividly evoked whenever I recall my first foray into classroom research. Because I was making constant notes about my thoughts and feelings at the time of that research I am now in a better position to separate 'then' from 'since' than I am for my childhood memories, but the fact remains that every re-viewing of an event adds a new layer of interpretation. In self-study, as in the movies, nothing is quite as it seems. The subjectivity which is an inherent part of self-research has been a major source of criticism of the genre. The following sections explore not only issues of bias surrounding the participation of the researcher as a subject of the research, but also the attendant ethical issues. Before considering these issues in detail, however, it is worthwhile exploring for a moment the advantages and disadvantages associated with participating in one's own research study.

Some of the advantages often claimed for research generated from teachers' own observations are that such research extends what 'counts' as research (Anderson, Herr &

Nihlen, 1994), interrupts traditional assumptions about knowers, knowing and what can be known about teaching (Cochran-Smith & Lytle, 1993, p. xiv), and makes visible not only teacher thinking (Flower, Wallace, Norris & Burnett, 1994) but also the ways that teachers and students co-construct knowledge and curriculum (Cochran-Smith & Lytle, 1993, p. xiv). The main advantage I see for teacher-research is the potential of such research to provide access to otherwise unobtainable research data, and hence to strengthen the methodological basis of the research. By this I mean that such research enables the researcher to access, and to critically assess, aspects of teacher thinking which might otherwise (even through stimulated recall interviewing techniques) remain inaccessible. When interviewing others one is necessarily limited to what they are prepared to reveal (or can remember). Adept interviewing might encourage the revelation of more information than the interviewee might have intended (although this practice is ethically questionable) but usually an interviewee is able to control how much of his or her story is told. It is less easy to hide things from oneself, however (although there still remains the issue of how that 'truth' will be presented to the reader). I know a great deal about what the teacher was thinking when I *am* the teacher. Although I have access to those thoughts, of course, it is less easy to critically assess them. The problems for researchers studying their own practice become ones of 'how much' of the vast story to tell, and how to assure readers that the story has been critically assessed. The issue of subjectivity in this kind of research is addressed later in this chapter (section 3.3.3), but there are other issues associated with this mode of research which critics (see, for example, Huberman, 1996) have seized upon as challenging the reliability of such studies.

Though for me the personal rewards associated with engaging in this research project have been great, the challenges have also been manifest. Like Wilson (1995), however, I hesitate to refer to the challenges that have arisen for me in my roles as teacher and researcher as conflicts. The concepts of conflict and tension (often tied to issues of duty or responsibility) are ones that spring up frequently in the teacher-research literature (see, for example, Wong, 1995; Fecho, 1993; Goldstein, 1996). As Wilson (1995) comments in her response to Wong's analysis of what he sees as the conflicts inherent in researching one's own practice, one does not enter the classroom one part teacher and one part researcher. During both strands of my research I was one person, "moved at once to help students learn and intensely curious about teaching and learning" (Wilson, 1995, p. 20). My differing roles in the two strands simply meant that I was sometimes more actively involved in helping students learn than at other times. This anomaly brings me to an important question, which is the focus of the next section, concerning how this piece of research should be situated theoretically and methodologically within the literature.

3.3.2 Is this Teacher-Research?

In critiquing research generated primarily by university-based, not school-based, researchers, Cochran-Smith and Lytle (1993) claim that the voices of teachers themselves are missing, and therefore that those most directly responsible for the education of children have been disenfranchised. They claim that through teacher-research a recognition of the unique perspectives of teachers would emerge. But what exactly is teacher-research, and have I been doing it? Though the term teacher-research has been used as an umbrella term to describe a wide range of activities it tends to be most closely

linked with those that can be traced to the action research paradigm. Such research is characterised by comparative studies of the effects of various forms of social action. In teacher-research studies the action research paradigm commonly materialises as investigations of the effects of introducing something different into the classroom (a new curriculum, an alternative assessment routine, increased attention paid to the teacher's own questioning strategies etc.) It is important to recognise that not all teacher-research is action research. Some, like my own, is concerned not with investigating change, but with seeking meaning.

The quick answer, then, to the question that heads this section is: it depends how you define teacher-research. Cochran-Smith and Lytle (1993, p. 7) suggest that it is "systematic and intentional inquiry carried out by teachers". On the surface it appears that I have indeed been engaging in such practices. But here I must pause and ask, can I still be called a teacher? I hope that I will always be designated as such, and I certainly remain one in my own mind, but I am no longer employed as a teacher, and no longer have a classroom and students of my own. I have become a university-based researcher. Would Cochran-Smith and Lytle still classify me as a teacher-researcher? Probably not. And yet there is a further twist to this tale. Part of the data upon which this thesis is based was collected in my own classroom at a time when I *was* a full-time teacher. I have been engaged in a self-study of a teacher at work, so I must ask the question again. Is this teacher-research? To answer the question this time we must look deeper into the teacher-research paradigm.

In a consideration of the issues that divide research on teaching and teacher-research, Cochran-Smith and Lytle compare the foundations which underpin the two genres. In a number of these areas I find that my research does not sit comfortably under the heading 'teacher-research'. I will give just a couple of examples to illustrate my predicament. Cochran-Smith and Lytle suggest that the research questions generated by teacher-researchers generally emerge from problems of practice, and are referenced to the immediate context. As I described in the opening chapter, my orienting research question emerged from a troubling aspect of my teaching practice - the dilemma of when to intervene in students' learning, so this, at least, is consistent with Cochran-Smith's and Lytle's formulation of teacher-research, however, the analysis of the data I collected has been done 'at a distance' from the immediate context of that classroom, and my findings are no longer to be addressed to that particular situation as I am no longer a teacher there. In addition, the theoretical framework underpinning this piece of research is derived from a wide range of academic disciplines related not only to teaching and learning but also to psychology, philosophy, neuroscience, complexity theory, and deep ecology. This is in contrast to the theoretical frameworks which Cochran-Smith and Lytle claim are common in teacher-research studies, and which are derived primarily from the knowledge of professional practice, supplemented by theories of learning, teaching and schooling. So, is this teacher-research? Yes, and no. I believe that I am uniquely placed to do research on teaching, and that this thesis is uniquely placed as a contribution to research on teaching and teacher-research. It interweaves my own classroom teaching experiences with my one-step-removed researcher perspective, so that at once it is and is not teacher-research. The added complexity of the two strands of data running throughout this work further

complicates matters, making me simultaneously a not-only-participant-observer, and a not-quite-non-participant-observer. The complexities of these dual roles are unravelled in sections 3.3.5 and 5.3. My presence as a subject of my own study, whether this work is classified as teacher-research or not, will be cause for suspicion by some readers, and so I turn now to a consideration of the issue of bias in self-study research.

3.3.3 In Search of my Subjectivity

There can be no doubt that there is bias in the work of those who research their own practice, but I question whether that bias is necessarily any more significant than the bias present in other forms of research. As Peshkin (1988) notes, we should begin from the premise that subjectivity in all research is inevitable. Peshkin calls for researchers to systematically search out their subjectivity, not retrospectively when the data have been collected and the analysis completed, but while the research is actively in progress. As this research study has combined two strands of research, with two differing roles for me as the researcher, I feel that I am in a strong position to be able to comment on the ease (or otherwise) with which each role lends itself to actively seeking out subjectivity. My experience has been that in my role as teacher-researcher (or participant researcher, which I think better describes my role) I found it much easier to identify moments of subjectivity during data analysis than I did when analysing data from the second strand where I had tried to maintain the position of a non-participant observer during data collection. Whilst watching the 'Vancouver' video tapes I often became aware that I was judging the standard of teaching and learning that was occurring. It is, of course, very hard not to judge. As any teacher knows, it is easy to sit at the back of a classroom and pass

judgement on someone else, even when most of what you are seeing is, by your own standards, excellent. And no matter how great the teaching is, it is almost always possible to find something wrong. It seems to be a sort of defence mechanism that is evoked whenever you see someone else doing what you do. It somehow feels better if you can say to yourself "I would have done this", or more usually, "I wouldn't have done that". My point, however, is that although I made judgements about the teaching in the 'Vancouver' tapes, and although I sometimes noticed that I was judging, I less frequently noted that down as I was analysing than I did when watching my own tapes. In addition, it was not until I turned to reviewing my own tapes after spending several months analysing the 'Vancouver' data that I began to notice how my readiness to judge was impacting the research findings I was generating. In other words, then, I have become much more aware of my place and presence in the findings of this research by virtue of the fact that I am not only the researcher.

Such, I believe, is the value of self-study. A heightened awareness of one's presence in self-study research brings with it a heightened awareness of one's presence in all research. As Peshkin (1988, p. 20) writes we are all "in the subjective underbrush of our own research experience" and therefore our own prose. Self-study appears to force us into the open. Once in the open, though, Peshkin (1988, p. 20) warns that we should take care not to produce a study that is "blatantly autobiographical". This is somewhat difficult for self-study research reports, as an important dimension of the research is the presence of the researcher. As Bresler (1996) claims, subjectivity is the inevitable consequence of having values, dispositions and beliefs. Rather than attempting to hide or extinguish these

characteristics we should value them for what they are able to tell us about the researcher, and explore them for their potential (especially in self-study where we have such ready access to them if we allow ourselves to listen when they nudge us) to lend an aura of humanity to research reports. So, it is not enough to simply declare our presence, we must, as I am attempting to do in these opening chapters (and, indeed as I try to do throughout this writing), try to lay open to scrutiny not only our theoretical and methodological biases, but also our personal ones.

The perceived problems of dealing with the subjectivity of the researcher in studies such as mine constitute one of the greatest challenges to the legitimacy of the kind of research I have undertaken, and are frequently offered as evidence of the unreliability of the findings. There is often a suspicion that being close can mean being *too* close and losing perspective (Huberman, 1996). The kind of research in which I have been engaged is often called "introspective" (Huberman, 1996, p. 128). Such emotive descriptors immediately raise concerns that it is full of "preconceptions, distortions, and self-delusions" (Huberman, 1996, p.128). Such labels can be hard to shake, even when, as is the case here, there is more to the study than mere 'navel-gazing'. It is hoped that through an explicit rendering of the philosophies guiding my self-analysis, and through some of the specific methods adopted during data analysis which I discuss later in this chapter, any such concerns will be alleviated.

There are other important issues, however, that critics believe impact upon the reliability of the data and that make participating in this sort of research a challenge. These are considered in the next section.

3.3.4 Ethical Issues

Ethical issues concerning the researcher's participation in the research study form what some critics see as perhaps the greatest challenge to the reliability of the study. Doing research involving human subjects is always a matter for ethical concern. Some would suggest that doing research involving participants with whom one has a potentially coercive relationship (such as a teacher with her or his students) should be cause for additional concern; concern that might lead us to doubt the reliability of any findings. There is no easy remedy to this predicament. One approach that I can take is to assure readers of the professional nature of my approach to both the teaching and the research that is discussed. Such assurances will not, and should not, be taken on face value, and so in addition I have, as I crafted this report, attempted to include descriptions of the ethical and professional dilemmas I faced and the solutions I enacted. In this section I want to consider some of the broad methodological issues around the concept of ethics in educational research that have been raised recently in the literature.

One of the abiding principles that has guided the procedures of this research project has been that of avoidance of harm, a principle adapted from the overarching doctrine of the medical profession, and one which has prompted me to be more often guided by the

ethics of *teaching* than those of *research*[8]. This principle influenced many of the decisions I made, both as teacher and researcher. Most often, it surfaced as a concern to respond to an expressed need of a particular student. For example, in the 'UK' case I tended to be far more 'teacherly' in my style when interviewing students. By this I mean that I more readily answered questions, re-directed students when they started to go off track, and corrected incorrect answers than I did in the 'Vancouver' case, where, as my mandate was to investigate and not to teach, I found it easier to refrain from explicitly teaching or even helping (although this was not always as easy as it sounds. Teachers, even those who are now researchers, love to teach!) My suggestion here is not that 'teacherly' behaviours such as correcting wrong answers and re-directing flailing students are *necessarily* advantageous strategies for promoting the growth of students' understanding, and that by refraining from such behaviour I was succeeding in 'not teaching'. The complexities of teaching and learning are far more subtle. In fact, were the route to excellent teaching so easy to formulate there would be no need for this thesis. What I do mean to say, however, is that at the time of making those interventions I was acting on the principle of avoidance of harm, and doing what I expected the parents of the students in my study would have wanted me to do - that is, continuing to respond to their children in the same manner as I would have done had the camera not been present. This undoubtedly made for differences in the approaches I took during interviewing in the two strands of this research. Such differences have contributed to the findings of this study. However, far from generating inconsistencies, these differences in style have helped to generate some interesting conclusions (see chapter 5).

[8] See Bresler (1996) for an interesting discussion of this tension.

Other writers would promote a more radical approach to ethical research. Lovat (1994) suggests that due to their special relationship to their research participants teacher-researchers, above all others, should adopt an ethical stance based on what he terms bioethics. This orientation is also borrowed from the medical profession but goes beyond the dictum *primum non nocere* (above all do no harm) and extends to encompass other principles such as teacher autonomy (which might be seen to threaten student privacy), justice (which ushers in the concept of due care), and beneficence (which extends the principle of avoidance of harm into one of actively seeking to do good.) All of the above are clearly appropriately applied to the practice of teaching, and need careful attention in the practice of research. In particular, the concept of beneficence is one which is potentially the most difficult to claim for many, if not most, research methodologies. In fact, it is the explicit aim of many research methods to maintain as little impact as possible on the participants in order to avoid tainting the data. It is obvious that as the researcher in this study I was operating with two different ethical principles in terms of beneficence. In the 'UK' case I was usually actively trying to do good throughout (privileging my role as teacher over my role as researcher). In the 'Vancouver' case (with no mandate to "teach") I was tending more towards the 'do no harm' principle. I believe my decisions were ethical in both cases.

3.3.5 Becoming an Observant Participator

In summarising the challenges of undertaking research in which one is participating on various levels, I intend to invoke the concept of the observant participator, a term which I have adapted from Erickson's (1993) "observant participant", and which I distinguish

from the more common "participant observer". I feel that the term observant participator is a better descriptor than any other for the stance I adopted in both strands of this study, though I invoke it in two different ways, as I am about to explain. By considering my role in the 'UK' strand as an observant participator rather than a participant observer we are immediately alerted to the participatory nature of my involvement. We are reminded that the most important feature of this strand is that I was a participant, not that I was an observer, but also that I was a participant with an additional role to the ones usually attributed to teachers[9]. Likewise in the 'Vancouver' strand, though I was less of a participant, and in fact, as I discuss later (see section 3.6.3.1), tried to maintain a 'distance' from the activities happening in the classroom, the term reminds us that the researcher is always a participant in the research, and alerts us to the futility, in research involving observational data of any kind, of trying to claim non-participant status.

The final sections of this chapter move us towards more practical aspects of my methodology by blending theoretical and practical considerations concerning my major data collection and analysis tool - video.

3.4 Video as a Research Tool

3.4.1 Introduction

In contemporary usage the word video most commonly refers to an image on a screen. An etymological search in the Oxford English Dictionary (1994) reveals that the word video

[9] I would like to suggest here that it is, in fact, often completely appropriate to attribute this descriptor to teachers who are not explicitly researching in their own classrooms. One of the things we would hope that all teachers are is observant.

derives from the Latin *vide¯re*, to see. The dictionary also defines videotape as "magnetic tape on which can be recorded moving visual images such as television programmes (as well as sound)". Although this definition stays true to the etymology of the word, in relegating the contribution of sound in this medium to parentheses, the dictionary reinforces the misconception that the primary power of videotape is its ability to portray the visual. My claim is that the strength of videotape in educational research lies in its ability to *combine* sound with pictures, resulting not in the creation of dominant and minor characteristics, but in the coalescence of complementary elements. As Hanson (1987) notes, the availability of tremendous amounts of media in our daily lives has conditioned us not to examine the processes by which we communicate. We take them for granted, and in choosing to turn away from an examination of them, we lose the ability to understand the real impact and potential of those forms. In the sections that follow I intend to respond to this challenge by investigating the implications of adopting video as a tool for data gathering and analysis in educational research.

Video data have been collected for a number of purposes and in a variety of ways for educational research projects in recent years. One such use is that of video-stimulated recall. Stimulated recall (originally from audio- not videotape) was devised as a method of evoking records of students' in-class thought processes (Bloom, 1953). The common theme in stimulated recall studies is the recording of practice (usually within classrooms) and the subsequent re-playing of the audio- or videotape in the presence of the

participant(s) in order to stimulate recall of the events[10]. The aims or foci of stimulated-recall projects differ widely, and include: video as a tool to improve teaching skills (Kagan, 1984; Trang & Caskey, 1981; Perlberg, 1984), assessing students' thoughts (Marland, 1984; McConnell, 1985; Marland & Edwards, 1986), improving students' study skills (Main, 1984) and investigating teachers' thoughts and decision making (Calderhead, 1981). Video-stimulated recall is, of course, only one of many techniques available to researchers who collect video data as an integral part of their work. Video is commonly used as a data gathering tool in classrooms without the subsequent use of stimulated recall methods.

Many educational researchers testify to the use of videotaping as a data collection tool (Wilkinson & Brady, 1982; Wood, Cobb & Yackel, 1991; Frid & Malone, 1994; Maher, Martino and Davis, 1994). Wood, Cobb and Yackel (1991) in their report of a year-long teaching experiment in a second-grade classroom reveal that they used several video cameras within one classroom. One or two cameras remained focused on particular pairs of students, and another was used to record all whole-class discussions. These video recordings, together with ethnographic field notes, copies of the children's work and records of open-ended meetings with the teacher formed the data source. The field notes, records of the meetings, and selected pieces of the videotapes were transcribed for further analysis. In a similar approach, Maher, Martino and Davis (1994) used multiple cameras to collect video data of children's development of fraction ideas in a fourth-grade

[10] For reports of video-stimulated recall in particular see, for example, Calderhead, 1981; Trang & Caskey, 1981; Kagan 1984; Main, 1984; Marland, 1984; Perlberg, 1984; McConnell, 1985; Marland & Edwards, 1986.

classroom. They recorded a total of twenty sessions, and also used the children's written work and journals as supplementary data. Artzt and Armour-Thomas (1992) in their study of small-group problem solving strategies, collected videotaped data of students working together and developed a framework for protocol analysis, incorporating data collected in individual video-stimulated recall interviews with the students. In their ongoing development of a theory for the growth of mathematical understanding, Pirie and Kieren (1989, 1994a), rather than undertaking longitudinal analyses of single classrooms, work with many teachers and students in a variety of contexts, at various age levels, and in at least two countries. Their video data are usually collected using only one camera focused on a pair or small group of students, sometimes in a classroom setting and sometimes in interviews.

3.4.2 Advantages of Video as a Research Tool

Although all of these groups of researchers discuss their methods of data collection (albeit to varying degrees of depth), fewer comment on the use of video as a tool of analysis. This is perhaps due to the fact that there are a limited number of researchers who commonly use the videotapes they collect as a primary data source. Instead, the videotapes (or parts thereof) are usually transcribed, and the resulting transcriptions are used from that point on as the main source of data for analysis. Analysis is therefore performed using the written text of the videotapes rather than the tapes themselves, an approach which I believe negates the very advantages of using videotape in the first place. In an insightful paragraph, Eisner (1991, p. 19) elucidates a critical disadvantage of transcribed conversation:

> Almost any phrase - such as "Did you do that?" - can take on a very wide array of meanings depending upon which words are emphasized. Emphasis is a matter of controlling qualities. "*Did* you do that?" "Did *you* do that?" "Did you *do* that?" "Did you do *that*?" Any written transcript of classroom discourse that omits such emphasis is likely to miscommunicate. Only one part of the meaning of any set of words resides in their so-called literal meaning. To understand what goes on in schools and classrooms requires sensitivity to how something is said and done, not only to what is said and done. Indeed, the what may very well depend on the how.

The use of video as a data *analysis* tool appears to address these concerns. Occasionally I have found it necessary to transcribe parts of my videotapes (for instance to take to conferences at which I am giving a presentation using video, in the event that the video should fail, or to insert sections of dialogue as examples in articles or papers). When I have worked for months with the original videotapes, I notice that when I read the transcripts I see and hear in my head the portions of videotape being replayed. I see the nuances of expression on the faces of the participants and hear the lilt of their voices. I constantly struggle, however, with how to convey those subtleties on paper to readers who have not seen the tapes. There are, of course, other researchers who work with video as a primary tool of analysis, many of whom struggle with this same issue. Notable amongst these are Goldman-Segall, whose work I discuss later, and Pirie. Pirie (1996b) speaks passionately for the retention of videotapes as the primary source of data, claiming that there will always be a loss of data when working with transcripts. She acknowledges, however, that videotaping does not capture everything, noting that "we rationalise as best we can the value of the data we gather and the worthlessness or irrelevance of that which we do not" (Pirie, 1996a, p. 553).

Other researchers, in a variety of disciplines such as medicine and anthropology, also comment on the danger of assuming that video can capture all that occurs in a particular environment (Bottorff, 1994; Balikci, 1995). There are, however, for the purposes of my proposed research, obvious advantages of using videotape over, for example, audio-tape, some of which are succinctly described by Wilkinson and Brady (1982). These include additional visual information such as certain types of body language (gaze direction, touching, head orientation etc.) which can denote an intended listener, and the ability to sometimes identify a speaker when this would have been impossible with the sole use of audio-taping. There are many other advantages of video as a data gathering tool, and as a data analysis tool. In terms of its superiority as a data gathering tool, video "permits coverage of the stream of activity in the natural setting in much of its complexity over relatively extended time periods" (Schaeffer, 1995, p. 255). It also "preserve[s] the 'natural coloring' of an event in ways that prose, anecdotes, and audio-tapes cannot" (Wilkinson & Brady, 1982, p. 1). Further, in celebrating the richness of video data, Bottorff notes their density and permanence (1994, p. 245).

Video also has advantages as a tool of analysis. Perhaps the greatest such advantage, and one noted by a number of researchers (Bottorff, 1994; Pirie, 1996b), is the ability to delay decision-making in the interpretation of an action, interaction or verbalisation. In contrast to many formal observational techniques wherein an observer is required to make instantaneous categorisations or interpretations in the field (taxonomies of teachers' questioning styles are an extreme example of this, but field notes may also be considered to fall into this grouping), video permits the researcher to reflect more deeply before

interpreting. Schaeffer (1995) suggests that the quality and reliability of statements concerning the activity may, as a result, be increased, which in turn strengthens scientific rigor. This is not to say, however, that the recording of video eliminates this initial layer of interpretation, for, as I shall note later, the researcher must still make choices about what is captured by the video, and this in itself is a form of bias. The point I am trying to make, however, is that the ability to view and re-view many times a particular episode captured on videotape affords the researcher a certain distance, a breathing space in which to ask the question "What is it that I am seeing and hearing?", and perhaps more importantly, "Why do I think so?", more than just once in the heat of the action in the field. This opportunity to reflect at leisure, and usually in a different physical space, is an opportunity to experience the moment from a different perspective. Video absorbs the viewer in a way that reading field-notes, or even listening to audio-tapes, does not. When viewing the tapes (especially for the first time) one has the overwhelming feeling of almost being back in the classroom. Almost, but not quite, and herein lies the source of the altered perspective. However absorbed one might be by the events unfolding on the screen, it is still different to actually "being there".

Other researchers also note that the person who recorded the data may experience that data quite differently while sitting in front of the television analysing it (Goldman-Segall, 1993; Wilkinson & Brady, 1982). Wilkinson and Brady (1982), for example, note that videotaped data afford scrutiny in a different setting, but do not expand upon what they see as the value of this exercise. When I watch videotapes of classrooms I notice different things than those I noticed when I was present in the classroom at the time that the video

was recorded. This is always the case, even when at the time the video was recorded I was sitting right beside the camera and concentrating hard on the activities of the students being taped. Somehow, the live classroom absorbs sound differently, so that you hear things differently. Also, there are others around you in the classroom, and, as a teacher, I am attuned (in any classroom) to keeping an 'eye' and an 'ear' on everyone in the room, not just those upon whom my gaze is directed, so I also see things differently, and see different things, when I sit down later with the tapes. This is even more pertinent when I watch videotapes that I have recorded in my own classroom, where, as the teacher, I was almost always somewhere else in the room, away from the students being taped. The fact that when I later watch these tapes I am always surprised by what I see and hear, reminds me that at those times I am not simply reviewing an episode of my life, but re-viewing it through a different lens.

For several researchers the ability of videotape to address the problem of abstraction of interpretations from the raw data is important. Schaeffer (1995, pp. 255-256) notes that

> videotape records can be employed to establish connections between abstractions and inferences and the observed phenomena upon which they are based. Videotape records as a constant reference point, a point of departure, permit grounded, well considered conclusions during interpretation of original field data.

Wilkinson and Brady (1982, p. 1) also suggest that "videotaped records of classroom activities address a major problem in educational field research: the increasing abstraction of the original data from raw materials to coded form to summarized findings, making alternative interpretations impossible." It is significant, however, that neither of these papers describe exactly *how* videotape addresses the problem of abstraction of

interpretations from the raw data, in view of the fact that most video data is still re-presented in written form in journals and research reports. Nor do these authors indicate how videotape is superior in this regard to audio-tape records or field notes. Only in presentations at conferences, where videotapes are able to be shown, do we see the possible defence of this bold claim, however, certain researchers are raising the issue of alternative forms of scholarly representation of findings. For example, Pirie (1996b) asks when we might expect to see video clips included in articles published on the internet. Of course, such inclusions would raise additional questions regarding the ethical procedures surrounding the use of videotape. Pirie (1996b) notes that our current promises to participants to protect their identities by use of pseudonyms in research reports make little sense if we are then to show their faces in video playback at conferences (even with their explicit permission). Schaeffer (1995) suggests that some control of confidentiality may be extended to participants by allowing, for example, tape destruction. One suggested method for dealing with participants' concerns about video data is to give participants the opportunity to erase parts of the tape with which they are uncomfortable, but from the researcher's point of view Pirie (1996b) questions whether this, in fact, compromises the data. These struggles concerning the ethical issues surrounding the gaining of informed consent to videotape form one of the two main areas I identify as being problematic in terms of video as a data gathering tool, the other being the effect of the camera.

3.4.3 Disadvantages of Video as a Research Tool

Several researchers comment on what is seen as a major disadvantage of video - the effect of the camera on the participants (Bottorff, 1994; Wilkinson & Brady, 1982), whilst

others note the disturbance that the presence of cameras creates in their work but do not comment further on what effect this may have on the data (see, for example, Marland & Edwards, 1986). Although cameras in the classroom undoubtedly have some effect on participants, there are measures which may be taken to reduce their impact. In part of the data included in this study, I was both the researcher and the classroom teacher and so I had the additional concern that students might say or do things which would be captured on videotape that they would not normally say or do if I was within earshot or watching them (such as swearing, name-calling, or using a calculator when they had been asked not to - all of which occurred). I wanted to be sure that the students could feel comfortable acting as normally as possible even though the camera was recording them, so we agreed that anything I did not hear or see at the time would be ignored if I saw it later on tape. In this way, the students were able to feel that the video was not a tool for 'spying'. All of the participants in this study were, in fact, fully apprised of the reasons for the presence of the camera, an aspect of data collection that is essential if the effects of the presence of a camera are to be minimised. The agreement to treat as "dead" any issue or behaviour that I had not noticed in my role as teacher (rather than researcher) worked very well, although as I look back on it now it raises concerns for me about what I would have done had, for instance, a pair of students discussed, on tape, some unlawful behaviour. This sort of occurrence, which I know has happened to other educational researchers, raises important ethical concerns and needs to be at the forefront of the minds of anyone who promises confidentiality to students who are being audio- or videotaped.

Other steps which can be taken to enhance the acceptance of the camera, and reduce reactivity to the recording process are given by Bottorff (1994). She suggests that in addition to full disclosure of the purposes of the videotaping, researchers are advised to videotape for up to a month before actual data collection is due to begin in order to habituate participants to the presence of the camera, and to minimise the intrusiveness of the camera. Minimising intrusiveness is a tricky issue in a classroom, as most high-quality video cameras are bulky pieces of equipment which are also often mounted on a tall tripod (although zoom lenses ease this concern by allowing the camera to be as far away as possible). The need for good sound recording usually requires individual radio-microphones or a separate microphone to be placed on the students' desk, and these further draw attention to the recording process. It has been my experience, however, and that of others doing similar work (for example, Pirie, 1996b) that students soon acclimatise to the presence of the equipment. The recording of the same participants for extended periods also helps to ensure that the data gathered are as authentic as possible, as it is unlikely that participants would maintain atypical behaviours for the entire time they were being videotaped. Other researchers refer to more extreme measures for minimising the intrusiveness of the camera, from covering the red recording light which is a feature of many cameras and which might draw a participant's attention to the fact they are being taped (Pirie, 1996a), to mounting the camera on the wall and monitoring it from a different room (Bottorff, 1994). Pirie (1996b) reminds us, however, that the choice of participants, the position of the camera(s), the focus of the camera(s), and the number

of cameras all affect the data collected[11].

There are a number of other disadvantages of using video as a data gathering tool. Bottorff (1994), in somewhat contradictory statements, notes both that there is an "absence of contextual data beyond what is recorded" (p. 247), and that "more often than not, more information is captured...than can be used and interpreted easily" (p. 246). In response to the former statement, I claim that video is probably the most complete classroom recording device we currently have at our disposal, and that many researchers, such as those I described earlier, and myself included, take precautions to maximise the amount of contextual data they gather by taking additional field-notes, collecting students' work, and interviewing teachers and students. In response to the latter statement - I couldn't agree more, which brings me to the disadvantages of video as a data analysis tool. The first of these is that the viewing and re-viewing of video data is an extremely time-consuming activity (Bottorff, 1994). Additionally, Pirie (1996b) notes that with video data one cannot "skim" (at least initially) as one can with text such as field-notes, so that one is forced to watch hours and hours of videotape, many times, waiting to see or hear something pertinent. Although I have experienced this frustrating activity, I also see this as a valuable aspect of video, perhaps even an advantage, for it forces one to pay attention to *all* of the data. Once transcriptions of audio- or video tapes are made the temptation to "skim" is much greater (for it is easier to scan a page of transcript than to scan a videotape). Simply repeatedly watching complete videotapes, however, does not

[11] The important issue of validity of data collected using video, and the validity of subsequent claims made from those data are considered in the next section.

guarantee that one will necessarily see something new every time. As Pirie (1996b) cautions, a problem with video is the danger that only what is looked for may be seen, and what has been concluded, confirmed. At such times one can see the value of a triangulated approach, as discussed earlier, involving others in the analysis of one's data. In many cases, such an approach enables researchers to make stronger claims about the validity of their analysis; however, the concept of validity of qualitative research, can be problematic.

3.4.4 The Issue of Validity

As Eisner (1991) notes, one of the persistent sources of difficulty for those using qualitative methods of research pertains to questions about the validity of their work. In commenting on validity Eisner suggests that in matters as complex and subtle as the description, interpretation, and evaluation of teaching and life in classrooms, we are not seeking a purchase on reality 'as it really is'; that we will always have to be content with a mind-mediated version of what we take to be the case. Such subjectivity, however, should not be seen as a liability, but a way of providing individual insight into a situation (Eisner, 1991). As Peshkin (1985, p. 280) notes, "by virtue of subjectivity, I tell the story I am moved to tell. Remove my subjectivity and I do not become a value-free participant observer, merely an empty-headed one." Although we are always stuck with judgements and interpretations, it does not mean that we can have no basis for assessing the soundness of those judgements. Although there has been concern expressed by some critics of qualitative research concerning the trustworthiness of the analyses which lie at the heart of qualitative research reports, Cobb and Whitenack (1996) suggest that the

convention for qualitative analyses has been to illustrate general claims and assertions by providing sample episodes from the data. However, interpretations of these samples frequently do not seem justified when considered in isolation from the rest of the data. I sympathise with their dilemma, but suggest that *their* claim would have appeared more trustworthy had they described their method of analysis (as the title of their paper, and their abstract imply). In fact, Cobb and Whitenack describe how they collected their data, and what they analysed *for*, but not *how* they did the actual analysis, a problem Pirie (1996b) notes is typical of much qualitative research.

In judging the quality of qualitative research, then, the issue turns on what counts as evidence. Artzt and Armour-Thomas (1992) suggest that a major problem with the type of research in which they and I are engaged is that thought processes are only accessible through verbalisations. Elsewhere, however, I have disputed whether this is, in fact, necessarily true (Towers, 1996). The problem of accessing participants' thought processes might be solved to some extent through the use of stimulated recall techniques, as described earlier, but Pirie (1996b) cautions that this method is not without dilemmas. For example, in stimulated recall situations when questioned about their in-class thinking students may offer what they think they *ought* to have been thinking, rather than what they actually were thinking. The issues I have faced in attempting to assess the growth of students' understanding by studying videotape of their in-class verbalisations, interactions and body language are no less problematic. Whether my findings and claims are to be trusted ultimately depends on the reader's assessment of the validity of my data and interpretations. So, those in the qualitative research community who create narratives

from video, must devise ways by which others who read those narratives can recognise the scholarly contribution, and be assured of the validity of the data and the interpretations taken from them.

There has been a lengthy struggle for acceptance of qualitative work in the educational field and qualitative researchers have responded in a number of ways to criticisms of their art. Maxwell (1992) cites examples of two approaches: denying the relevance of the quantitative or scientific paradigm for what qualitative researchers do, or arguing that qualitative research has its own procedures for attaining validity that are simply different from those of qualitative approaches. He points out, however, that explicit attention to *how* qualitative researchers conceptualise validity issues has been slow to develop. Maxwell (1992, p. 284) does raise a critical point when he says that validity is not an inherent property of a particular method, "but pertains to the data, accounts, or conclusions reached by using that method in a particular context for a particular purpose." Maxwell suggests that qualitative researchers are concerned with five categories of validity; descriptive validity, interpretive validity, theoretical validity, generalizability, and evaluative validity. Many of these categories overlap with the categories or aspects of validity discussed by other qualitative researchers (for example, Geertz, 1973; Lather, 1991; Lincoln & Guba, 1990; Sanjek, 1990), which I commented upon earlier in this chapter (section 3.2.2). Here, I intend to concentrate on the problems associated with those categories, or aspects of categories, which I feel can be addressed specifically by video data as opposed to other forms of qualitative data.

Perhaps the greatest advantage of video lies in its ability to address some of the concerns associated with the category of descriptive validity. This category deals with what Maxwell calls the "factual accuracy" of an account and, as such, concerns itself with errors of omission as well as commission. As an example Maxwell cites the use of interview transcripts which omit features of the informant's speech, such as stress and pitch, that are essential to the understanding of the interview. Although no account can include everything (and as I noted earlier, not even video captures all), it does seem that the use of video as a tool of analysis (and particularly the use of videotapes, not transcripts thereof, as primary sources of evidence) can aid in this respect. Maxwell's term "factual accuracy" implies, however, that there is only one truth to be told in every research situation, a suggestion that I strongly reject. Notwithstanding my earlier comments about the subjectivity of all evidence, I do believe that video is superior to many other tools of data collection in its ability to preserve the complexity of an environment. As mentioned earlier, video preserves much of the "colour" and density of an environment, and in this way helps to alleviate concerns that the researcher may be making up or distorting the things he or she saw or heard. As I have said, however, there remains considerable leeway for "distortion" between the evidence on the videotape and the evidence as presented in written form for the reader. Although it would be a fascinating diversion, a discussion of how we, in fact, draw meaning from what we see, hear or read is beyond the scope of this thesis. Suffice to say that how any interpretation I make from my data is understood, by a reader, has as much to do with the point of view of that reader, as with the meaning I think I am conveying when I carefully choose the words I use. In choosing and using words I am operating within a language, and a

language is simply a medium through which I represent what I have come to know. As Eisner (1991) notes, however, one feature of a medium is that it mediates, and anything that mediates changes what it conveys. It is essential, therefore, that my point of view is foregrounded in my writing, so that a reader can critically assess that point of view in terms of his or her own, and will be better situated to evaluate the basis of my interpretations and claims. I hope that I have achieved this aim in the opening chapters of this work.

3.4.5 Interpreting Video Data

Goldman-Segall (1993) has suggested that interpreting video is fundamentally different from interpreting text.

> As linguists and other scholars will testify, analyzing text is filled with endless webs of intricate paths that intertwine in many directions. Now, imagine the complexity involved in analyzing the same five minutes of a video stream...To build valid interpretations about these subtle events, we need tools for sharing with others what we think is significant about what we see in what was recorded (Goldman-Segall, 1993, p. 263-264).

Goldman-Segall works with videotape as a primary source of data for analysis and also involves the perspectives of multiple viewers in that analysis. She has created the research tool "Constellations" as a method of layering interpretations from several viewers (Goldman-Segall, 1993). Chunks of videotape are selected and multiple interpretations are made possible by facilitating the viewing of these chunks by multiple researchers. Each is able to read the others' comments, add their own interpretations, and refine the analysis by selecting "chunks within chunks", but is not able to alter the interpretations made others. In my research I am concerned with tracing the growth of

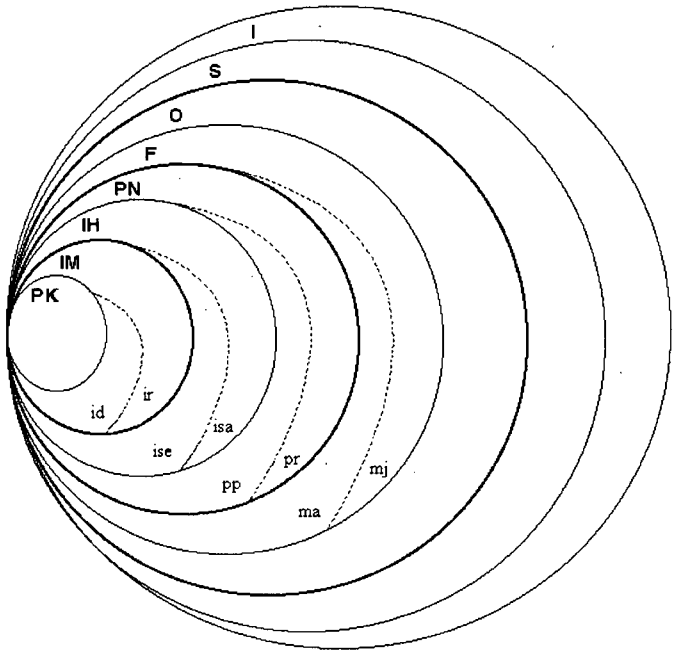
students' mathematical understanding over time, and with the role of the teacher in occasioning that growth. The significant loss of data that, in Goldman-Segall's method, results from the original whole tapes no longer being the focus of analysis, means that such an approach would be insufficient for me to create a thorough analysis which would foreground the importance of context, and emphasise the complexity of the interactions between students and teachers. Clearly Goldman-Segall's technique is appropriate and successful for certain investigations, but from an enactivist point of view her technique of "chunking" data has some limitations. Rather than pulling apart my data I have attempted to celebrate the complexity of the environments I have studied by striving to understand them in their entirety. This is, of course, by no means an easy task, but to continue to avoid an attempt to develop methods that may begin to address this issue is to reject the notion that the essence of understanding lies in the interaction between participants, not in individual minds. It was my hope in embarking upon this research project to explore one way in which the complexity of classroom interactions might be understood. I see video as the best research medium available to assist in the task.

The preceding sections have summarised a number of important issues surrounding the use of video as a data gathering and analysis tool in qualitative educational research. In concluding this section I suggest that although video presents methodological challenges for researchers, it also opens up a world of exciting possibilities to those willing to explore its potential.

3.5 Framework of Analysis

Although I outlined briefly in the previous chapter the framework within which the analysis of this piece of research was set, I present here a more detailed consideration of the relevant aspects of the theory. To reiterate briefly, the Dynamical Theory for the Growth of Mathematical Understanding (Pirie & Kieren, 1994a) considers mathematical understanding as an on-going process in which a learner responds to the problem of reorganising his or her knowledge structures by continually revisiting existing understandings, a process known as *folding back* (Kieren & Pirie, forthcoming; Pirie & Kieren, 1991). The theory considers understanding in terms of a set of embedded levels or modes of knowledge-building activity. These modes are illustrated in diagrammatic form in Figure 2. I intend to provide, here, a description of only those levels of understanding that are directly relevant to the discussions of student understanding which follow in the next chapter. For a full description of the Dynamical Theory for the Growth of Mathematical Understanding the reader is referred to the original writings of Pirie and Kieren[12]. The theory claims that the process of understanding begins with *Primitive Knowing*. It is important to note that the word primitive is not meant to imply low-level mathematics, or even low-level understanding, rather it is the starting place for growth. *Primitive Knowing* is assumed to be everything that the student knows and can do already, outside the topic under consideration.

[12] In particular, see Pirie & Kieren, 1994a.



Key:

PK	Primitive Knowing	id	image doing
IM	Image Making	ir	image reviewing
IH	Image Having	ise	image seeing
PN	Property Noticing	isa	image saying
F	Formalising	pp	property predicting
O	Observing	pr	property recording
S	Structuring	ma	method applying
I	Inventising	mj	method justifying

Figure 2: The Dynamical Theory for the Growth of Mathematical Understanding

In the second mode, *Image Making*, the student is observed to be engaged in activities aimed at helping her or him develop particular images. The learner is being asked to make distinctions in previous knowings and use them in new circumstances or to new ends. After engaging in the actions of *Image Making* the student can replace these actions with a 'mental plan'. This is *Image Having*, which frees the student from the need for particular actions. At the next level, *Property Noticing*, the student can manipulate or combine images to construct context specific, relevant properties. Such understanding involves being able to predict how a situation within one's image works, and being able

to express that prediction. Growth to the next level of understanding, *Formalising*, involves building and expressing methods in terms which allow the student to understand a part of mathematics in a sense 'for all'. This requires consciously thinking about related properties and abstracting some common method of *acting* that is seen to work for (some conception of) all cases. The student is able to explain or justify this method.

In addition to the modes of understanding there are a number of features of the theory that are relevant to my discussion. The first of these features is the concept of the "don't need" boundary which I briefly mentioned in the previous chapter and which is illustrated in Figure 2 by the bold rings. I refer the reader to Section 2.4.3 for a fuller description of this feature, but as a reminder, you should recall that beyond each of these boundaries the learner is able to work with notions that are no longer obviously tied to previous forms of understanding. The first of these boundaries occurs between *Image Making* and *Image Having*, highlighting the notion that when a person has an image of a mathematical idea they do not need the actions or specific images of *Image Making*. In contrast, there is no "don't need" boundary between *Image Having* and *Property Noticing* because the latter is defined as the result of working with existent images to notice general properties, and access across the *Image Making/Property Noticing* boundary is therefore essential. The next "don't need" boundary occurs between *Property Noticing* and *Formalising*. A person who has a formal mathematical idea does not need a specific image.

A feature of the theory which I have already mentioned briefly, but which is crucial to an understanding of the findings I present in Chapter 5, and therefore warrants further

attention here, is *folding back.* No matter how sophisticated their understanding, whenever a student finds his or her situation incoherent or incomprehensible she or he is prompted to *fold back* to an inner level of activity (often observed to be less sophisticated activity) in order to then extend her or his current understanding. This returned-to level is not the same as the original inner level understanding as it is shaped into a "thicker" understanding by the previous more sophisticated, outer level understandings. In this way, growth of understanding is seen as a recursive process.

A third important feature of the theory are the complementarities of *acting* and *expressing* that structure the levels themselves. Each level beyond *Primitive Knowing* is composed of complementarities of *acting* and *expressing*, each of which is necessary for growth. For the relevant levels these complementarities are shown on Figure 2. Growth is observed to happen through the back-and-forth movement between these complementary aspects. At any level, *acting* encompasses all previous understanding, providing continuity with previous levels, whilst *expressing* gives distinct substance to that particular level. It should be noted that *acting* can encompass mental as well as physical activities, and *expressing* is to do with making overt to others or to oneself the nature of those activities. It is also important to note that *expressing* is not intended to be synonymous with reflecting. Reflection is frequently a component of *acting*, since it incorporates the process of looking at *how* previous understanding was generated. *Expressing* is concerned with articulating *what* was involved in those actions.

Image Making, for example, is seen to consist of two aspects - *image doing* (the *acting* complementarity), wherein the student is engaged in activities associated with constructing an image of the situation, and *image reviewing* (the *expressing* complementarity) wherein the student reviews previous work and attempts to make sense of the situation. The next level, *Image Having*, consists of *image seeing* (wherein the student is able to demonstrate that he or she has now formed an image (though not necessarily a complete or correct one) and *image saying* (wherein the student is able to express aspects of that image). In *Property Noticing*, the complementarities of *property predicting* and *property recording* are distinguished by the student's ability to connect previous images and recognise a pattern, and by the student's ability to express that they have 'organised' these schemes. It is important to clarify, here, that the use of the word *recording* is not intended to imply that a written record be produced, however, *property recording* must involve articulate expression of some form. At the *Formalising* level, the complementarities are *method applying* (wherein the student is able to apply their understanding of a particular concept in appropriate generalised circumstances), and *method justifying* (wherein the student is able to justify a generalisation).

Complementing these features of the theory, and of particular relevance to my study, are the three forms of teacher intervention which Pirie and Kieren (1992b) have identified. These are *invocative*, *provocative* and *validating* interventions, which may influence students' learning and which Pirie and Kieren see as crucial to the teacher's task of encouraging the growth of students' understanding. An *invocative* intervention is one which prompts the student to *fold back* to an inner level of understanding. A *provocative*

intervention is one which points the student toward outer or more sophisticated understanding. A *validating* intervention is one which establishes that a student is working within some level of understanding, but results in no movement.

It is important to note here that although many interventions made by teachers take the form of questions, I do not intend to imply that only *questions* can be categorised as *validating*, *invocative* or *provocative*. Teachers make many other types of interventions in students' learning, some of which are more explicit than others. Such interventions might take the form of instructions to individuals, groups, or the whole class, or such subtle 'hints' to the student as a nod, a frown, a smile, a raised eyebrow, or even a well-placed moment of silence. In fact, I am not suggesting that only *teachers'* interventions influence students' learning. It is often possible to characterise students' interventions in each other's learning as *validating*, *invocative* or *provocative*, an area which is the subject of current research by Pirie and Kieren. My specific interest is in the role of the teacher in occasioning the growth of students' understanding, though, so I will confine my analysis and discussion to that aspect of the data.

Though there is much more that can be said about my specific methods of data analysis, I feel that it is more appropriate to discuss these as they arise in relation to the unfolding of the stories I am about to present in the next chapter, and after I have detailed, in the next section, the specific data gathering methods I employed, and have discussed in more detail the particular contexts in which I collected those data.

3.6 Data Gathering

3.6.1 Introduction

As part of this study two 'cases' were documented, and these form the two strands of my research. In order to develop an understanding of the classroom environment, I have engaged in detailed analyses of video-recorded lessons, field-notes, copies of students' work, my own journal entries, and video-recorded interviews with the 'Vancouver' teacher and with the student participants in both strands of the study.

3.6.2 Strand 1 - Mossbrook School[13], Oxfordshire, United Kingdom

3.6.2.1 Overview

The first strand concerns data collected in my own classroom at a time when I was a full-time teacher of mathematics in a small, rural, British secondary school. The data consist of a series of videotaped lessons in each of three mixed-ability classes - a Year 7 class and two Year 8 classes (Canadian equivalent Grades 6 and 7), together with copies of the students' work, journals kept by myself during and after the data collection period, and a single, videotaped, individual interview conducted at the conclusion of the data collection period with each of five of the seven students who participated in this strand of the study[14].

In each of the three classes I videotaped, a pair of students was the focus of the video-recording. The three pairs were chosen to reflect the range of abilities of the students at

[13] All school names and participant names (with the exception of my own name) used in this study are pseudonyms.
[14] I discuss in Section 3.6.2.3 why two of the students were not interviewed.

the school. There were also logistical considerations that guided my choice of classes and individuals, for instance the participant students were chosen, in part, because they could be relied upon to speak to each other whilst they were working. This was deemed necessary because the analysis of the data relies primarily (although by no means exclusively) upon an interpretation of the students' words. The desks in my classroom were always arranged in groups (usually six students to a group) so the students were used to working together and used to talking about their work. Other considerations guiding my choice of participants included the student's willingness to participate, his or her regular attendance record, the obtaining of parental consent, and time-tabling issues.

A static camera focused on only one pair of students was used for each class, together with a separate directional microphone placed on the desk of the pair of students. As I was the teacher as well as the researcher in this strand, it was not possible to redirect the camera during the lessons (for instance to capture what I was drawing on the board), however at the end of each lesson I made a copy of the work on the board, if necessary. (In fact, I made very little use of the board, but more will be said about my particular teaching style as it pertains to this study later). This constant focus on just one pair of students in each class was, however, an advantage rather than a disadvantage as it enabled me to have a much more complete record of their actions and interactions during the lesson than if I had set up the camera with a wider focus.

A sequence of lessons of several weeks in duration was videotaped for each pair of students. At the end of the sequence of lessons, a semi-structured one-to-one interview

was conducted with each student. These interviews were also videotaped. In their individual interview each student was asked to work through a series of problems similar to those attempted during the sequence of class lessons. As the interviewer, I consciously attempted to elicit the student's understanding, without falling into the role of teacher, an approach which, as I mentioned earlier (Section 3.3.4), was problematic. The data from this (UK) strand were analysed, in part, before data collection and analysis began on the second (Vancouver) strand. In this way, the first strand informed the decisions made about data collection and analysis in the second strand. The UK strand was not, however, intended to be a pilot study for the Vancouver strand as the data from both strands play equal parts in the findings of the thesis and ultimately in the claims made.

Each of the three classes in this strand were videotaped during a module of work on algebra. Much of the content of the algebra module was based on the work of the Strategies and Errors in Secondary Mathematics (SESM) project reported by Booth (1984), which in turn had been based on the work of the Concepts in Secondary Mathematics and Science (CSMS) team described by Hart (1982). This work, although algebraic in nature, is set within the context of perimeter and area and I adapted and extended it for the purposes of this study. An example of the kind of problem that was used is shown in Figure 3. For such a problem the student would be required to generate an expression for the perimeter and/or the area of the shape.

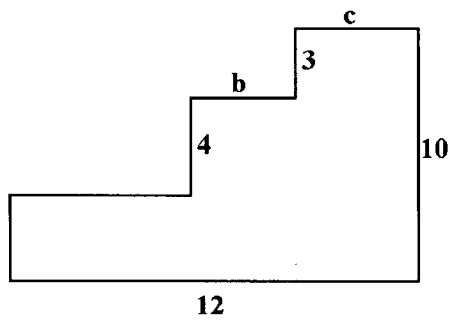


Figure 3: Example Problem

The specific aim of the series of lessons captured on videotape was to teach the students how to formulate and manipulate simple algebraic expressions. Due to the nature of the context of the work, it was necessary for the students to have an understanding of the basic concepts associated with finding the perimeter and area of a variety of plane figures, and, for the relevant students in each class, their work on this preparatory topic was also video-recorded. The videotaped series of lessons for each class consisted of lessons in which the students worked in pairs to complete relevant worksheets, a practical lesson in which the students were introduced to algebraic terms and the means by which such terms can be manipulated (referred to throughout as the 'Hats' lesson for reasons which will be made clear in a moment), and a number of lessons of a more investigational nature, in which the focus was upon formulating algebraic generalisations. The three classes were also given a test in their second lesson (after the preparatory lesson on perimeter and area and before any algebraic work was introduced) in order to assess the students' *Primitive Knowing* of mensuration and of algebraic techniques.

The 'Hats' lesson was designed as a concrete introduction to the notion of collecting like terms in an algebraic expression. The students in each of the three classes spent some

time at the beginning of the lesson designing and making a hat from large sheets of coloured paper. I then offered them a selection of labels to stick onto the hats, each label showing either a number, a letter, or a combination of both. Metre sticks were laid on the floor in a variety of arrangements and the students were asked to come to the centre of the room in various combinations (depending on the label on their hat) and arrange themselves around the shape created by the metre sticks. I encouraged the students, as a group, to create expressions for the perimeters of the shapes (mostly rectangles and squares) they were forming (by summing the algebraic values on their hats), and I worked with them to introduce the notion of collecting like terms in an expression. I will be saying more in the next chapter about my teaching strategies in these lessons.

Later in the series of lessons I included an investigational lesson which I shall refer to throughout as the 'Ponds' investigation. This lesson was concerned with investigating the patterns to be found in the relationship between the size of a square (and later other shapes) drawn on grid paper (the 'pond') and the number of squares ('paving stones') needed to surround it, as shown by the family of 'Ponds' seen in Figure 4.

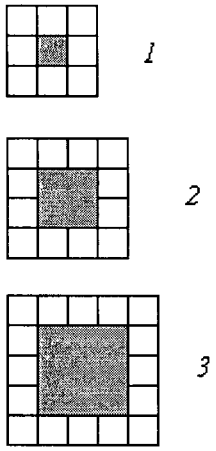


Figure 4: 'Ponds'

3.6.2.2 Introducing Jo

I feel that there should barely be need for this introduction, but for the sake of completeness I include here some details which the reader may feel are relevant to the discussion. As I have mentioned, I was the classroom teacher as well as the researcher in this strand of the study. At the point of beginning the data collection I had been teaching for three years, although I had only been teaching at the school in which I based the study for just over a year. I was a full-time teacher of mathematics, teaching Years 7 to 13 (equivalent to Canadian Grades 6 to 12).

3.6.2.3 Introducing Donny, Sonya and Lucy

Donny, Sonya and Lucy were, at the time of videotaping, Year 8 (equivalent to Canadian Grade 7) students. A number of criteria were used for the selection of these participants. Donny was chosen because he was eager to participate, he maintained regular attendance

at school, he was always willing to ask for help in class, both from me and from his friends, and he was usually willing to talk about his work. Sonya was chosen because she always worked closely with Donny, and I thought that the most natural data possible under these circumstances would be obtained by keeping the classroom routines as normal as possible. Sonya was absent on one day during videotaping, and on that day Lucy (a student who also often worked with Donny) was asked to sit in Sonya's place so that Donny would have somebody to talk with during the lesson. Absentee students is one of the greatest problems associated with data collection methods involving small groups of participants. In order to get worthwhile data, it is imperative that the students are able to interact with one another. This problem is highlighted when only a pair of students are the focus of the investigation in any one classroom. This is also the reason that in the second strand I began by videotaping a group of three students in case on any particular day one student happened to be absent. As I will report below, though, (section 3.6.3.3) even this solution is not infallible. Lucy was not interviewed because she was not considered a major participant, having only replaced Sonya on the day Sonya was absent. In fact, I also do not consider Sonya to be a major participant, and so she was not interviewed either. The primary reason for this is that data collection in this class was, in many ways, a 'pilot' for the remaining classes. I undertook the video-recording in this class several months before that in the other two classes of this strand, and had anticipated it being very much a test-run of my methods. I was primarily focusing on just one student (Donny) in order to have a trial run of the experience of videotaping on a day to day basis in my own classroom. I had not anticipated including this data in the full study, but became captivated by its richness, and decided to include it after all. By the

time I made that decision it was too late to return to interview Sonya, so this class remains somewhat different to the others. Donny, Sonya and Lucy were all eager and hard-working students, but they each, and Donny in particular, found mathematics difficult, and often struggled with some of the material.

3.6.2.4 Introducing Kayleigh and Carrie

At the time of videotaping Kayleigh and Carrie were high-achieving students in a Year 7 class (equivalent to Canadian Grade 6). Year 7 is (usually) the first year of secondary school for British students, so these students were relatively new to the school, and to me. We had been meeting together three times a week for mathematics lessons for only a few months when the study began. Again, these students had been chosen on the basis of their regular attendance, willingness to participate, and willingness to talk in class. In many ways Kayleigh is the qualitative researcher's dream as she verbalises her thinking constantly, and so the data from this part of this strand of the research is particularly rich.

3.6.2.5 Introducing Kerry and Graham

Kerry and Graham were Year 8 students (equivalent to Canadian Grade 7). Again, these students were chosen for similar reasons to those outlined above. Kerry often struggled with mathematics, and, as her teacher at the time, I was particularly interested in spending some time looking in detail at her in-class mathematical activity when I was elsewhere in the room and not directly engaged in helping her. Graham was a more able student, but he was keen to participate and although these two students had not worked as a pair very

often, they often formed part of a larger group that worked together, so I believed that they would work co-operatively.

3.6.3 Strand 2 - Jameston School, Vancouver, British Columbia, Canada

3.6.3.1 Overview

The second strand concerns data collected in another secondary mathematics classroom, this time in a large urban secondary school in Vancouver. A single mathematics teacher and a group of her students are the focus of this strand. In this second strand, I videotaped a sequence of eighteen lessons with a Grade 9 class. I focused on just one class in this strand (in contrast to my focus on three of my own classes in the UK strand) in order to cause as little disruption as possible for the teacher, and to simplify the procedures for gaining permission to videotape. In fact, there were very similar 'amounts' of data collected in each of the two strands in terms of the total number of lessons. Again I used a static focused camera and a separate directional microphone in this strand, and again focused the camera on a small group of students rather than the whole class. Similar criteria to those employed in the first strand were used in the selection of participant students in this setting. The students in this strand were engaged in work on the topic of geometry. This topic had two strands, one of which focused on solid geometry and the other on angle relationships. For the solid geometry topic the students worked primarily on problems assigned from a text-book, and for the angle relationships topic they were given worksheets.[15]

[15] These materials are discussed in more detail in the next chapter, where examples from each are presented.

During the lessons I sat away from the students, and made notes of any board work or other visual activities that were relevant to the group of students being videotaped, but off camera. Though I attempted to maintain a 'distance' from the activities of the Vancouver classroom, it was obvious that the students considered me a part of the classroom, sometimes making quiet comments to me which I heard only when I listened to the videotapes later, and often looking in my direction when something happened in the room (such as a disruption caused by another student). Clearly they were aware of my presence, and yet sometimes, as their conversation wandered through a variety of non-mathematical topics, they appeared almost oblivious to the fact that I would be later "listening in" on their conversations. Such conversations are, of course, not relevant to the focus of my study, and have therefore been 'ignored' in terms of my data analysis.

In addition, I carried out interviews with the teacher and the students. These took the form of open clinical interviews (with the teacher), and semi-structured interviews (with the students) where I asked the students to work on a series of problems related to those they were doing in class. The students' interviews in this strand differed in two ways from those in the first strand. Firstly, in this strand rather than conducting just one interview for each of the students at the conclusion of the data collection period, I conducted four interviews with the students spread throughout the data collection period. I did this in order to gain additional information about the *growth* of their understanding. In addition, in a departure from the one-to-one interview method used in the first strand the students in this strand were interviewed together. The decision was taken to interview the students together for the first interview (which occurred immediately before the classroom data

collection began) because I thought it might be less intimidating for them than a one-to-one interview with a person who was not particularly well known to them, and who had been introduced to them as a "researcher" from "the university". Despite the fact that I had visited their class in the weeks leading up to the start of the project, and had talked with the class about my background and the reasons for my interest in their mathematical understanding, and had conducted videotaping in their lessons to acclimatise them to the presence of the equipment, I still felt like an outsider in their worlds in a way that I had not felt when researching my own classroom. I decided to continue to interview these students together because the students in this strand were given fewer opportunities in lesson time to work together than those studied in the first strand, and I believed that to continue to interview the Vancouver students together throughout the study would ultimately provide a richer source of data about their mathematical understandings. These four semi-structured interviews took place outside lesson time and were videotaped.

In the case of the teacher in the second strand, an initial interview focused on the way in which she was preparing for the videotaped lessons. The teacher was also interviewed at intervals during the sequence of lessons. This is another instance where the data collection methods differ between the two strands. In the first strand, as I was the teacher as well as the researcher, it was not possible to interview myself (although journal entries and notes made during data analysis could be thought of as a form of self-interrogation). The interviewing of the teacher in the second strand is an attempt to parallel the notes made by the teacher in the first strand. Having had the experience myself of keeping written records of my reflections, I believed that it would be less demanding of the

Vancouver teacher's time to conduct interviews rather than ask her to keep a journal throughout the data collection period.

Although whilst planning the data collection phase of the second strand I had considered the possibility of employing video-stimulated recall methods to try to discover more about the teacher's thinking during her interactions with the students in the lessons, I eventually decided against it. Although this video-stimulated recall technique may have been a fruitful source of data in order to establish the intents of the participants at key moments, it is a method I have found to be time- and equipment-intensive. It requires video playback equipment to be available in the school (as it is impractical to transport it), and also requires a great deal of setting-up time to organise video *recording* of the video-stimulated recall interview. It is also most profitably done as soon after the lesson as possible so that the teacher or student can recall as clearly as possible their thoughts and understandings, and this can not always be arranged conveniently. As I was conscious of the participant teacher's busy schedule, I decided that this technique would only be used if it became apparent during data collection or analysis that further specific information was desirable. In the event, sufficiently rich data was generated without the use of video-stimulated recall techniques.

There are a number of interesting features of this study which result from the similarities and differences between the participants and settings. Firstly, there are some dissimilarities between the participants in terms of contexts in which they teach (one in the UK and one in Canada). These contexts will be discussed more fully in the next

chapter, although it should be noted that the purpose of this study is not to compare and contrast these two educational systems. The grade levels of the students in each of the two strands is different, and the specific topics taught were also different. However, as the aim is to understand the processes at work in the classrooms, rather than, for instance, to compare student achievement, these differences are not a matter of concern to me. In fact, the Grade 9 class in the Vancouver strand was suggested by their teacher as being a class who usually talk about their work, which fits the mandate set for the selection of students for this study (see earlier discussion, section 3.6.2.1). The focus on a Grade 9 class, however, also further strengthens the study by providing data relating to a wider age range of participants.

It will have been noted that the methods of data collection have been kept as consistent as possible between the two strands, but this is mainly to facilitate rich data (as discussed above) rather than to provide a basis for comparison. There are, then, a number of similarities and differences between the contexts, teachers, classrooms, and students that form this study, however, I reiterate that this study is not intended to be a comparison study of British and Canadian teachers, students or schools. The purpose of the study is to gain a deeper understanding of the complex processes at work within mathematics classrooms, and in particular to investigate the ways in which teachers' interventions occasion and interact with the growth of the students' understanding of mathematics.

3.6.3.2 Introducing Karen

Karen is an experienced teacher, having taught mathematics for over twenty years. She is currently completing a Masters Degree in Education, and, although the focus of this thesis is not her area of research specialisation, she was interested in the question this study addresses and was keen to participate.

3.6.3.3 Introducing Abby, Tasha, Dale and Serena

Abby, Tasha, Dale and Serena are students in Karen's Grade 9 class (equivalent to UK Year 10). Abby and Tasha are the main participants in this strand of the study. Serena participated on just one day when Tasha was absent (and so she was not interviewed at all), and although Dale began the study as a full participant (and was therefore present in the first interview) he withdrew after a few days as he was uncomfortable being on camera. Although permission was given to use the existing data for this student, I have not traced his understanding and have restricted my reference to him to only those moments when without it the interaction I am describing between the other participants would not make sense to the reader. Abby and Tasha are high-achieving students and were eager to participate in the study. They were chosen by Karen following the criteria I have described for the participants in the first strand.

Unfortunately, although she usually had excellent attendance, Tasha was absent on several occasions during the data collection period. As I mentioned above, Serena took Tasha's place on one day, but was also absent on the other occasions. The class contained an unusually high number of students who wanted to participate in the research but whose

parents had not given permission, and a number of students who did not particularly want to participate, but whose parents *had* given permission. In terms of the university and school board requirements it had not been necessary for me to gain written permission from the students themselves, but as a matter of personal ethics I had done so, so we were aware of each student's preference. Neither Karen nor I would have been comfortable persuading a reluctant student to appear on camera, and so there were a couple of occasions on which Abby appears on video alone. This is unfortunate, but preferable to cajoling unwilling students, and should be anticipated as a potential problem in any research which relies on small numbers of participants.

3.7 Summary

In this chapter I have critically reviewed my chosen methodology, considered the methodological issues surrounding my own participation in the study, discussed in detail the specific observation and analysis tools which were significant in my work, and introduced the particular contexts in which my study is situated. In the next chapter I begin the journey through the analysis and interpretation of my data.

CHAPTER 4

ANALYSING

4.1 Staging the Analysis

Data analysis involving videotapes is notoriously time-consuming, and this study was to prove no exception. I spent over five months viewing and re-viewing the videotapes that form the basis of my data. This chapter details the analysis process which I have arranged in six stages, described below. Though the UK data were collected before the Vancouver data, and I had perused the UK tapes before embarking upon data collection in Vancouver, I began the detailed analysis of the data for this study with the Vancouver tapes. I chose to do this because the experience of being in the Vancouver classroom was still fresh, and I was interested in re-viewing some of the incidents that had captured my attention at the time of recording. In fact, I claim later (Section 4.6) that this fortuitous choice has strengthened my findings.

4.2 Stage 1: The First Run-Through

Analysis of the data for this study began with what I call a "first run-through" of the video tapes and supporting documentation for the Vancouver strand. I began by watching each tape very closely and making hand-written notes. This run-though served a number of important purposes. My aim when analysing data is to become very familiar with those data. Of course this takes a great deal of time, however, the first run-though excites me as I always notice something that I did not notice during the actual lesson. During the first run-through of the Vancouver data I made a note of the precise time (using the VCR counter) of any moments which seemed particularly significant, such as a particularly

intensive intervention stream[16], or a moment of eloquence by one of the students which might later contribute to my tracing of their growth of understanding.

At this stage of my analysis I tried not to dwell too long on any one excerpt, preferring to gain an overall picture of the students' progress and the teacher's style. I refrained from using a marker pen to highlight anything I wrote in my first run-though notes as I wanted to be sure that I would read everything I had written when I reached the end and before beginning a second pass. I also want to point out here that I feel it is significant that I chose to hand-write these (and the later) notes. I did this, rather than use computer software designed specifically for this purpose (such as C-Video[17]) or even a simple word-processor on a laptop computer, because I feel that I can write much more freely by hand. When I write using a keyboard I have the feeling that I must write only what makes sense. When writing free-hand I feel free to 'doodle' in words, to let my ideas flow from the end of the pen. Perhaps it is because I am not a competent touch typist that I feel restricted by having to concentrate on the act of typing in a way that I do not when engaged in the act of writing, and yet there is more to it than this. My hand-written notes are informal, colloquial, tentative and speculative. These notes include things I wasn't then sure about, ideas, questions, conjectures, possibilities, interlineations and corrections. As Berry (1990, p. 193) notes, a hand-written page is to some degree a palimpsest; "it contains parts and relics of its own history - erasures, passages crossed

[16] I define an intervention stream as period of activity involving extended exchanges between students and teacher, or an extended period of exposition by the teacher.
[17] C-Video is a specially-designed software package created as a tool to facilitate the analysis of video. By linking one's computer and video cassette recorder C-Video enables the user to 'annotate' the video.

out, interlineations - suggesting that there is something to go back to as well as something to go forward [from]", a description which has added significance for me as I begin to explicate here the notions of *folding back* and extending with respect to understanding mathematics.

By the time I had completed the first run-through of the Vancouver tapes I had forty-nine pages of notes which referred not only to interactions and gestures captured on the videotapes but also to the copies of the teacher's board-work that I had made during the lessons, and to the copies of the students' written work that I had collected. My hand-written notes, therefore, begin to integrate all aspects of the data.

Before progressing further with this chapter I want to comment on its structure. In order to help the reader to navigate through the essential aspects of my analysis I have elected to present my analysis of the teachers' interventions separately from my analysis of the students' growth of understanding. Although a description which leads the reader back-and-forth from teacher-to-student-to-teacher-to-another-student (both within and between the two strands) would better represent the messiness of the process in which I was engaged for many months, it probably would be a confusing and frustrating document to read. I have therefore resisted the temptation to present an integrated and holistic analysis at this stage, but wish to note here that each of the threads that are created in this chapter are woven together in the next chapter as I respond to my research question.

4.3 Stage 2: Searching for the Growth of Students' Understanding

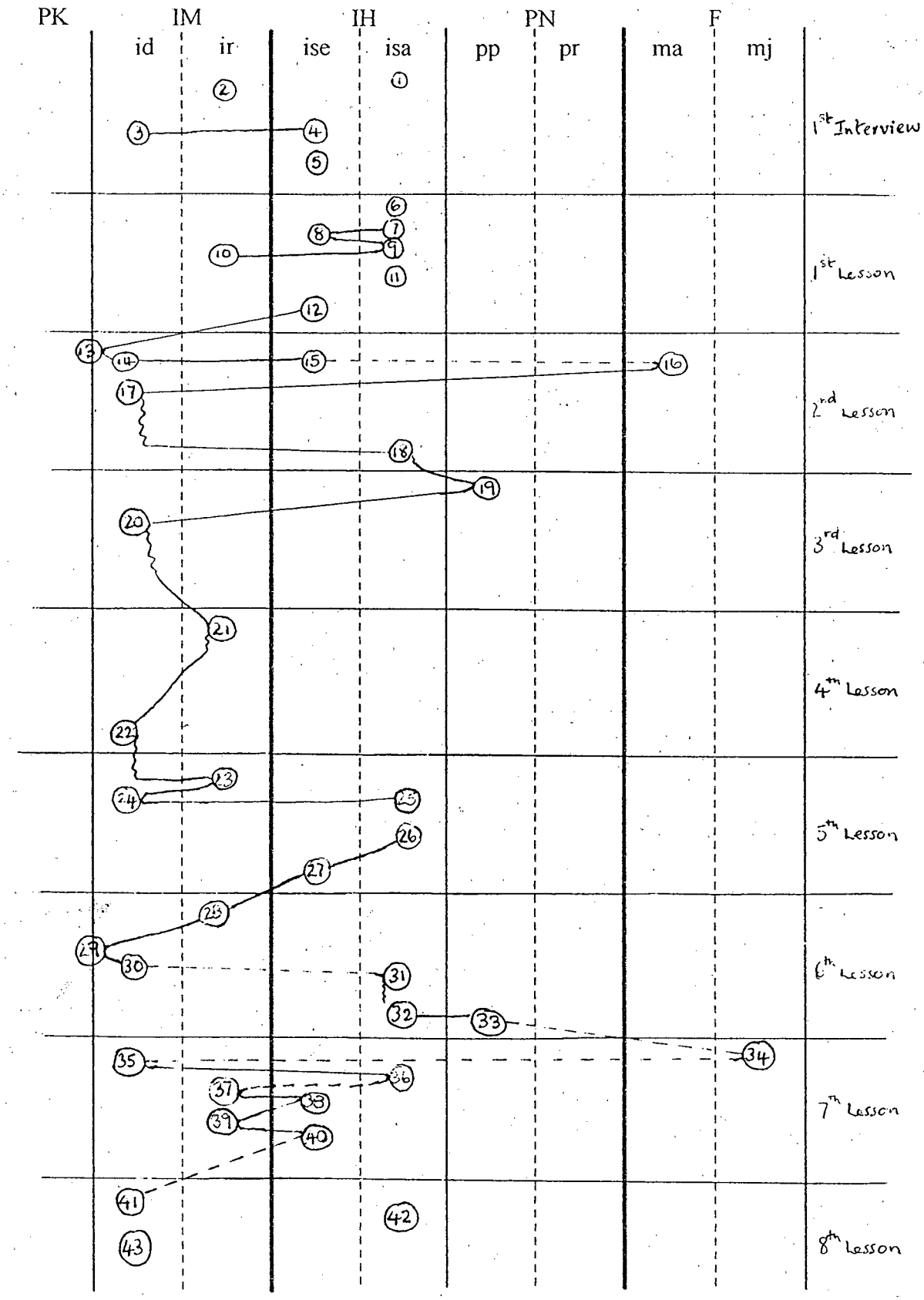
4.3.1 Introduction

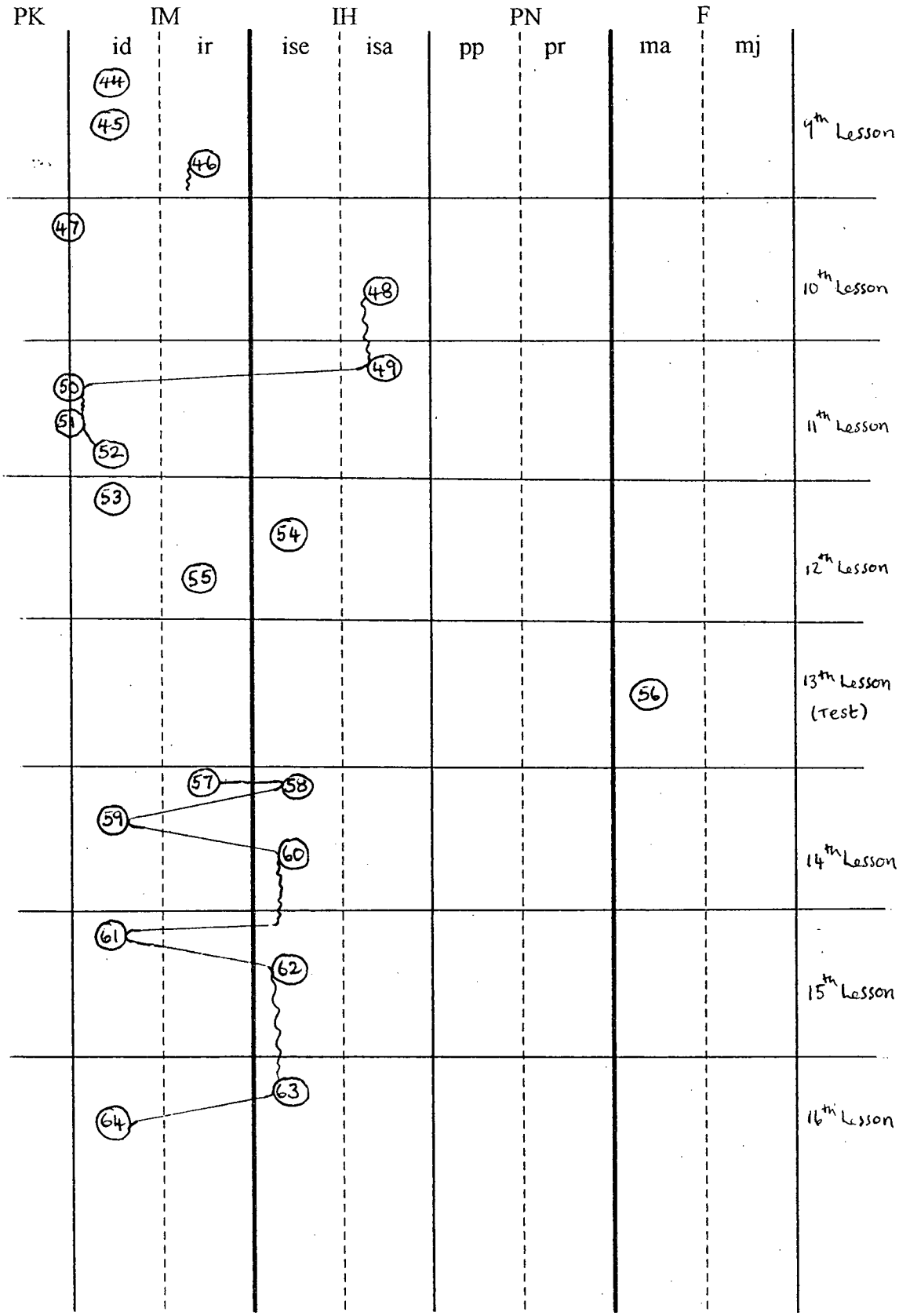
This stage consists of a number of sub-stages. Not only did I make several more passes through the complete set of videotapes, but I also made more hand-written notes (ninety-three pages in all). From these notes, and the video-tapes, I created a mapping diagram[18] for each of the student participants. Due to the extensive nature of the mappings it would be unreasonable to expect a reader to read a description of how I mapped each and every point[19] for each of the student participants of this study. I have elected, therefore, to describe in the text of this thesis the most interesting and significant features of the mappings, and to include in the appendices notes about each point on every mapping diagram should the reader wish to gain a more thorough sense of the flow of the students' growth of understanding (see, for example, Appendices 2 and 4 for details concerning Abby's and Tasha's growth of understanding).

[18] A mapping diagram, or mapping, is the name given to the trace of the student's growth of understanding on a diagram of the embedded modes of understanding presented in the Dynamical Theory for the Growth of Mathematical Understanding.

[19] For the purposes of this study, the word 'point' refers to a numbered item on a student's mapping diagram. (See, for example, 'points' 1 to 96 on Abby's mapping diagram, Figure 5). Each point represents an instance where I was able to distinguish that the particular student concerned was engaged in actions or verbalisations consistent with one of the modes of understanding identified by the Dynamical Theory for the Growth of Mathematical Understanding. Readers are referred to the List of Abbreviations (page xii) for information concerning the abbreviations on the mapping diagrams.

FIGURE 5: ABBY'S MAPPING DIAGRAM

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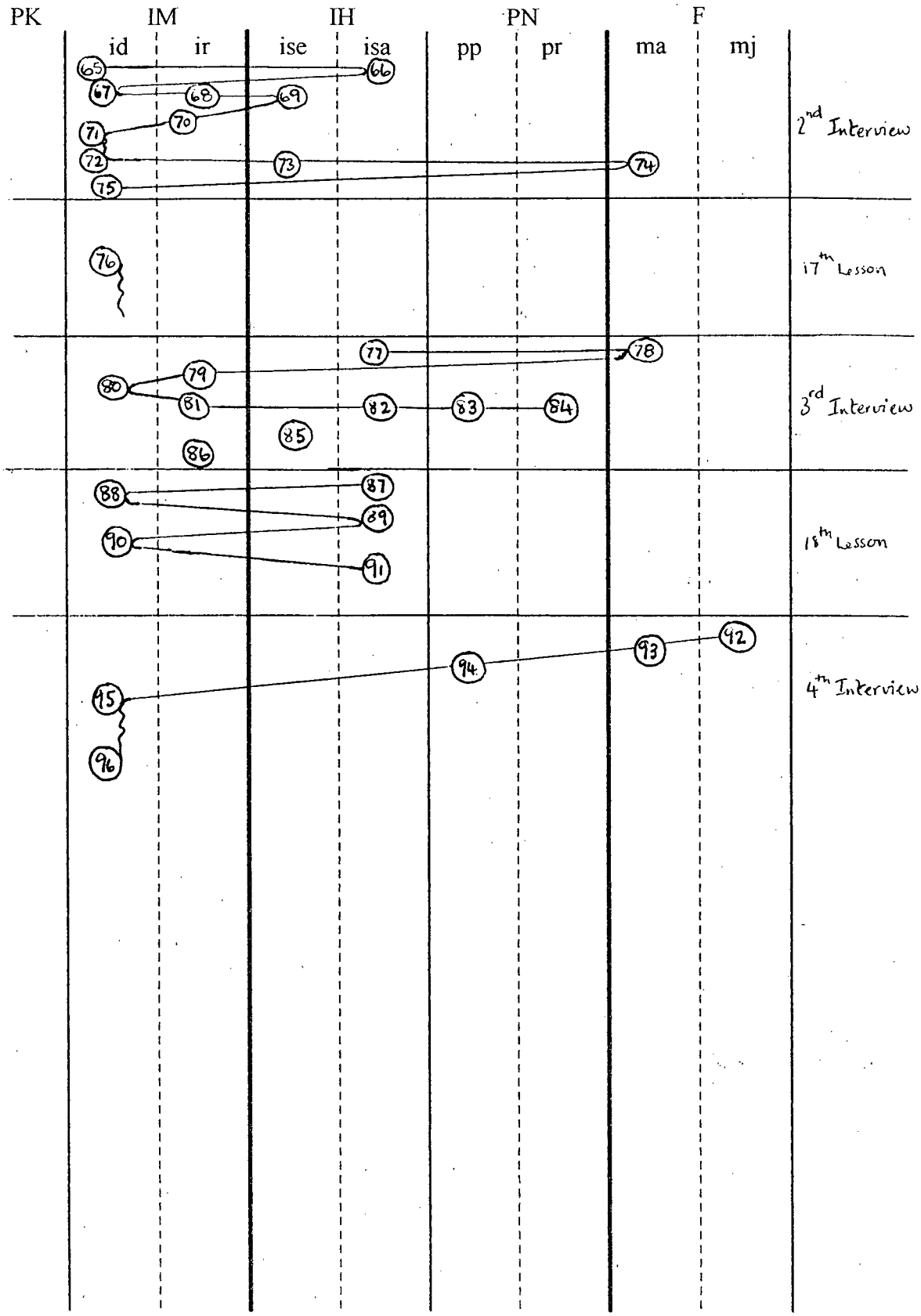
11th Lesson
12th Lesson
13th Lesson (Test)
14th Lesson
15th Lesson
16th Lesson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2nd Interview
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17th Lesson
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In choosing which points to describe here I have been guided by three main concerns. The first was to help the reader to gain a sense of each student's initial understandings of each topic (so the reader will note that, for example, I have described in detail all of the points concerning Abby's initial interview with me, and many in the opening lessons on each new topic that was introduced). My second concern was to explicate in detail some of the main 'turning points' in each student's growth of understanding, and to give the reader a sense of the capabilities and developing understandings of each student as the data collection period progressed. The final concern was to help situate the later comments I will make regarding how Karen's interventions occasioned the growth of her students' understanding. These same concerns also structured the choices I have made in presenting the stories of growth of understanding for the students in the UK strand (see Section 4.5).

I began with Abby. The notes I made refer to specific moments on the videotapes, and to particular evidence in Abby's written work which enable me to claim that Abby was engaged in activities consistent with a particular level of the model. These notes form the basis of my mapping of Abby's understanding (see Figure 5). All of the points below refer to this mapping diagram, which is also given in Appendix 1. A complete description of the ninety-six points on Abby's mapping diagram is given in Appendix 2. It will be noted that my mapping diagrams are of a different form to the 'standard' model diagram used by Pirie and Kieren (which is a series of embedded rings). Although I began Abby's mapping on a standard diagram I soon found that format to be unsuitable for the detail I was including. Even using a large printout of the standard diagram there simply was not enough room to squeeze all of the points into some of the inner rings. In addition, I began

to find that it was interesting to look at the students' progress in specific periods of time (such as a particular lesson) and the factor of time is not easily represented on the standard mapping diagram. I therefore developed a more tabular layout, which, although it facilitates analysis, admittedly does not invoke the same images for the viewer concerning the embedded nature of the levels of understanding, however, it does not alter *my* conception of these levels as embedded.

The following descriptions are detailed stories of the growth of Abby's and Tasha's mathematical understandings over time. The main classroom resources used during this strand of the study were a Grade 9 textbook produced by Addison-Wesley entitled Mathematics 9 (Kelly, Alexander, & Atkinson, 1987), and a collection of worksheets called Geometry 9: Focus on Reasoning, produced by the Ministry of Education in British Columbia.

4.3.2 Abby's Growth of Understanding

Abby participated with enthusiasm in the initial interview I conducted with her and her classmates, Tasha and Dale. When asked to explain what she understood by the term "area", Abby gave two responses, firstly "length times width" and then "it's what covers over a space or whatever. It, I don't know, fills it, or whatever." The second of these responses indicates that Abby already has an image for area, and has been categorised as *image saying* (1), Figure 5. Further evidence that Abby was *Image Having* for this

concept was her accurate calculation of the area of the shape shown in Figure 6, accomplished by splitting the shape into two smaller rectangles[20].

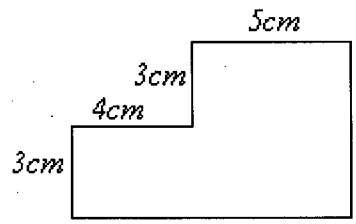


Figure 6: A 'Deviant' Rectangle Problem

The three students were then given a soup can and asked how they would find out its surface area. Abby again led the way giving a description of the need to find the area of an "end" and double it, and the need to "roll out" the body of the can "like a rectangle", work out the length times the width and then "add everything together". This has been mapped as evidence that Abby is also *image saying* for the surface area of a cylinder. Abby did not give any details about how the area of the circular ends would be calculated, or how the circumference of the circular ends contributes to the calculation of the area of the body of the cylinder (though this is implied by her description), and so this is not yet evidence of a *Formalised* understanding.

When asked for the volume of the can, none of the students could initially offer a response. Abby's tentative suggestion "Ah, I don't remember. I don't know. How tall it is? I don't know" suggests that she is *Image Making* by attempting to search for relevant information (such as the height of the can). This has been categorised as *image reviewing*

[20] Shapes such as this one feature repeatedly in this thesis, and shall hereafter be referred to as 'deviant' rectangles. I acknowledge here my thanks to Dr. Gaalen Erickson for this inspiration.

(2) because it was later revealed that Abby had studied volume of a cylinder previously, and appeared to have forgotten what to do.

Abby showed that she was initially *Image Making* (*image doing*, 3) for angle properties associated with parallel lines as she initially made an incorrect suggestion for an angle equal to the angle marked *a* on Figure 7. Later in the interview she was able to describe, in informal language, why angle *c* which was suggested by Dale is not equal to *a*, and she also suggested that angle *d* is equal to angle *b* because it is "the reverse of it". Though this is not the recognised terminology, it does suggest that Abby had an image for some angle properties, and hence this has been mapped as *image seeing* (4).

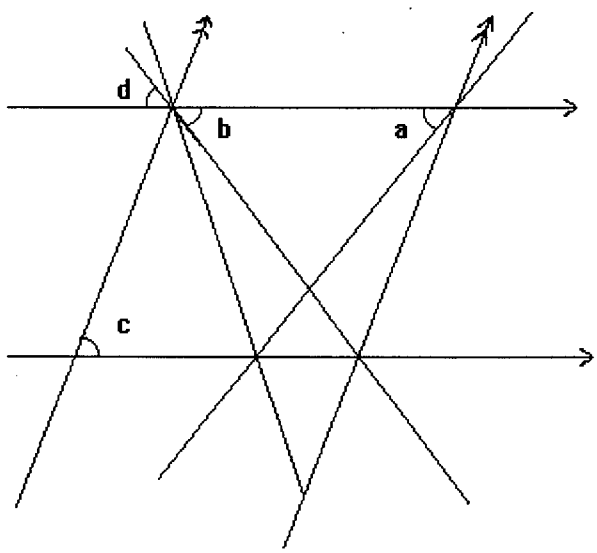


Figure 7: An Angle Relationships Problem

When I asked how they would find the volume of the solid in Figure 8, Abby again led the way, noting that the two parts, which she called the "bottom" and the "roof" need to be calculated separately. When asked how to find the volume of the "roof" Abby

suggested that "this is like one layer. You have to calculate this [*pointing to the triangular end of the triangular prism*] and then just times it how many". Again this has been mapped as *image seeing* (5).

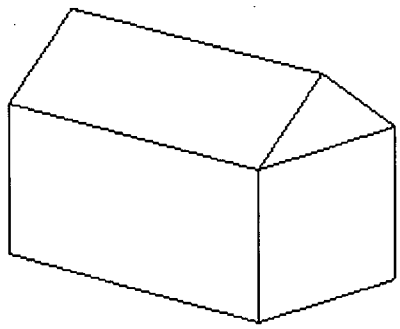


Figure 8: A Combined Solid Problem

During the first lesson the class members were encouraged by Karen to give definitions for the term "area". Abby's "it takes up space" was the only offering that did not specify an operation (such as "base times height" or "length times width") and yet it was rejected by the teacher on the grounds that space is a concept associated with volume. As we learned from Abby's initial interview, however, Abby had used the word "space" in conjunction with the words "cover" and "fill" suggesting that she did understand the area concept, and so this has been mapped as *image saying* (6), strengthened also by other evidence of Abby's understanding of this concept related to her knowledge of the correct units to use, and her definitions of mathematical objects, such as a plane. Similar evidence for Abby's understanding of the concept of volume was also mapped as *image saying* (7). Point (8) represents Abby's level of understanding of properties of plane figures, and in particular I have noted Abby's understanding of the method of finding the

area of a parallelogram (9) and a trapezoid[21] (10). Abby was able to describe the method of calculating the volume of a right rectangular prism (11), and she correctly calculated its surface area (12).

During Karen's introduction in the second lesson it became clear that Abby, and most of the rest of the class, did not know Pythagoras' theorem. Karen briefly explained the theorem whilst the class watched and listened. I initially struggled with mapping this incident (and others like it). The main difficulty lay in the nature of Abby's (and the other students') involvement in the episode. Although Abby was clearly watching and listening to Karen's presentation, and therefore, presumably, forming an image of Pythagoras' theorem, I remained reluctant to categorise Abby's actions as *image doing* due to the lack of observable active participation. I was reluctant to categorise the episode as *image reviewing* because the information was obviously new to her (Abby had at one point mentioned to Dale that she'd never seen this theorem before). As can be seen on the mapping diagram (13) I have mapped this (and other similar incidents) on the boundary of *Primitive Knowing* and *Image Making* and I refer to these incidents as instances of *image viewing*. This concept is a novel addition to the Dynamical Theory for the Growth of Mathematical Understanding, and was previously unaccounted for by Pirie and Kieren.

Some time later in the same lesson Abby expressed some understanding of the conditions which render it necessary to drop a perpendicular from the upper vertex of a triangle, and

[21] Throughout this thesis I will use this term as it is used in North America (meaning a plane quadrilateral), and not as it is used in the UK (to describe a three dimensional solid).

employ Pythagoras' theorem to calculate the area of the triangle.

> Abby: 'Cause to find this see, whatever. 'Cause there's no right angles, right? Or I mean right triangles or whatever. So you have to find this [*indicating the perpendicular height*].

Though this description is still too internal and idiosyncratic to be termed *image saying*, it has been mapped as *image seeing* (15).

Later Abby was able to explain to Dale that when calculating the surface area of a cube one only needs to calculate the area of one side and then multiply by the number of sides. This has been mapped as *method applying* (16). The line connecting points (15) and (16) has been shown dotted to indicate that whilst the two points refer to the same very general topic area (both were concerned with aspects of finding the surface area of solid figures) the understanding relating to Pythagoras' theorem (point (15)) did not grow to a *Formalised* understanding. Point (16), in fact, was concerned with a 'simpler' solid.

This decision reflects a long period of struggle concerning how to represent on one mapping diagram the intertwining understandings of several areas of mathematics, both within one topic area (such as surface area) and between more disparate topic areas (as the teacher in this strand often switched between, and even within, lessons from one topic area to another). My final decision has been to link with a solid line any points which appear to represent changes in a student's understanding of a general topic (such as surface area, volume, or angle properties in Abby's case), and to have no line at all connecting adjacent points which jump from topic to topic. Dotted lines represent those

movements that are within one broad topic area but seem from the data to be concerned with different sub-topics (as in the example I have just described). 'Wiggly' lines (I can think of no more descriptive term, and I am sure the reader will recognise this description) represent times when the student spent an extended period (extending in some cases through several lessons) at one level.

Point (17) shows Abby *folding back* from *Formalising* to *Image Making* as she asked Dale to explain to her the meaning of the term "prism". Abby remained at the *image doing* level for some time as she endeavoured to calculate the surface areas of a variety of solids including a triangular-based pyramid drawn as a net. Later Abby was able to describe to another student how to use Pythagoras' theorem to solve this problem and correctly worked it through on paper. I have mapped this as *image saying* (18) rather than as *Formalising* because Abby received help from the teacher, Karen, with this question, so further evidence is required that she really has a generalised understanding of how to apply Pythagoras' theorem appropriately.

During the third lesson Abby was able to explain how to use Pythagoras' theorem to Tasha who had been absent from Lesson 2, and Abby was also able to justify to another student why Pythagoras' theorem is *not* required to calculate the height of a triangle when two sides of that triangle are perpendicular to each other (*property predicting*,19). Such evidence might be initially assumed to justify mapping Abby's understanding at the *Formalising* level, however, the video-tapes show that later (in the seventh lesson whilst attempting to find the volume of the triangular prism shown in Figure 9) Abby appeared

to forget her own advice and initially attempted to use Pythagoras' theorem (to find the height of the triangular base) in a case where its use is not necessary. This led me to conclude that Abby had not, after all, *Formalised* her image at point (19), but that her image was strengthened from its level at point (18). I have therefore mapped Abby's understanding at point (19) as *property predicting*, to reflect the fact that she was, at that point, able to articulate a distinction in the properties applicable to this concept for a specific example. Towards the end of this lesson, however, Abby *folded back* again to *Image Making* (20) as she struggled to apply her understanding of surface area to a new situation, that of a combined solid composed of a rectangular-based pyramid and a right-rectangular prism (Figure 10).

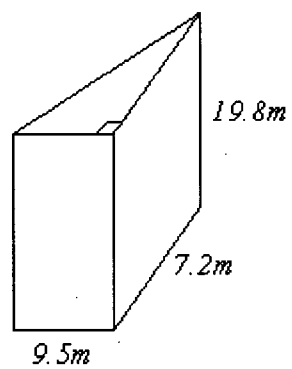


Figure 9: A Volume Problem

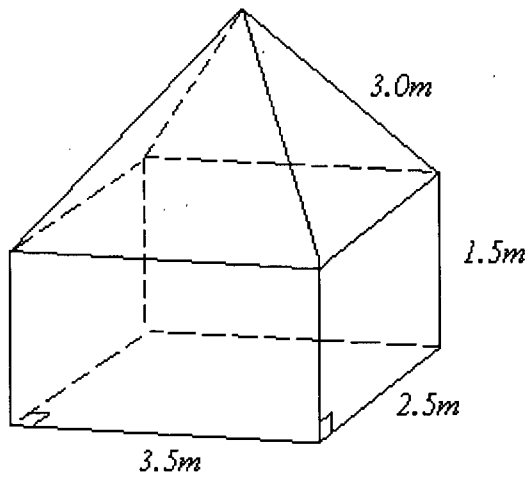


Figure 10: A Combined Solid Problem

In the fourth lesson, although Abby had previously demonstrated formal competence in calculating the surface area of a right-rectangular prism, she returned to *image reviewing* (21) when asked to apply that understanding to a real-life situation involving the calculation of the amount of paint required to cover the walls and ceiling of a room, given its dimensions (and the dimensions of areas to be avoided such as windows), but given no diagram. Such an intervention demonstrates how fragile is understanding. Though Abby's understanding of surface area of a right-rectangular prism and area of plane shapes appeared strong and formal in one situation (when presented with a de-contextualised mathematical image such as a line drawing of a hollow solid in a text-book) she was forced now to review that understanding to make sense of the room-painting situation, and she made several errors before reaching consensus with Tasha and Dale.

Abby later *folded back* further to *image doing* (22) as she pondered the surface area of a cylinder, and, in lesson 5, a half-cylinder. During this lesson Abby was also caused to reflect on her image of Pythagoras' theorem, which she then had not needed to use for

several days (*image reviewing*, 23). Her use of the theorem was more stilted again, and she seemed unsure about whether the formula itself can be re-arranged even though she earlier re-arranged numbers after substituting them into the standard formula.

Karen opened Lesson 6 by checking students' familiarity with the term "volume", and then demonstrated (*image viewing* for Abby (29)), using water and small, hollow, clear plastic solids, that the volumes of a square-based pyramid and a cone are one third of the volumes of the related cube and cylinder, respectively. After the demonstration Abby worked tentatively on a number of textbook questions relating to pyramids and cones (*image doing,* 30).

Later, on a question concerning a triangular prism whose bases were isosceles triangles, Abby was able to explain to Tasha that Pythagoras' theorem is not only applicable to right-triangles. She clarified what she meant by indicating that in this situation (i.e., an isosceles triangle) you can drop a perpendicular through the triangle, thereby creating two identical right-triangles and enabling you therefore to use Pythagoras' theorem to find the height of the original triangle. I have mapped this incident as *property predicting* (33) because it demonstrates that Abby was attempting to make sense of the situation by combining several properties of triangles (her knowledge of isosceles triangles and how the base of the triangle is halved by dropping a perpendicular to it, and her growing understanding of Pythagoras' theorem and its applications). During the next lesson, however, Abby was prompted to *fold back* once more to *image doing* (35) when, as I mentioned earlier, she attempted to apply Pythagoras' theorem in a situation in which it is

not needed (to find the height of a triangle with two perpendicular sides). After a discussion with the teacher Abby was able to explain how to do this same question to her classmate, Serena (*image saying*, 36). Abby's mapping diagram (Figure 5) also shows that between points (33) and (35) she demonstrated *Formalised* understanding. This growth concerned solids for which she did not need to employ Pythagoras' theorem (a right-rectangular prism and a cylinder) and therefore I have joined points (33) to (34) and (34) to (35) with dotted lines to indicate that Abby's understanding concerning *Pythagoras' theorem* (at point 34) did not extend to the *Formalising* level (point 35).

Points (43) to (46) show that Abby remained at *Image Making* for an extended period during lessons 8 and 9, as she struggled to understand and work with the constant ratios associated with fitting a sphere inside a cylinder of equivalent diameter and height. Karen showed the class that for a sphere of given radius that fits exactly inside a cylinder, the ratio of the volume of the sphere to that of the cylinder will be 1 : 1.5. She then showed, using the formulae, that this is a general result and the ratio will remain constant for any such spheres and cylinders. The students were then asked to construct similar proofs for the ratios of the volumes and surface areas of other combinations, such as a sphere inside a cube, and a cone inside a cylinder.

Points (48) to (52) represent Abby's responses to Karen's introduction of a new topic area, angle geometry. Abby began at *image saying* (48) and (49), responding to Karen's checking of the class' knowledge of the geometrical terms that were to be used throughout the topic. There followed a period where Karen digressed into a discussion of

non-Euclidean geometry, and then rapidly reviewed some properties of quadrilaterals, through which Abby sat passively, occasionally commenting to Tasha that she wasn't "getting it", and which I have categorised as *image viewing* (50) and (51). Finally, when they were assigned some work on this topic which involved calculating the missing angles in various figures and giving reasons for those answers (based on knowledge of angle relationships such as corresponding and alternate interior angles), Abby began *image doing* (52).

At this stage, Abby, like many of her classmates, was usually able to calculate correctly the value of missing angles, but was unable to assign a reason for her answer that satisfied her teacher's strict requirements concerning accurate terminology. I struggled here (and in later similar instances) with whether to categorise Abby's responses as *Image Having* or *Image Making*. Her correct calculations suggest the former would be appropriate, and yet her continuing wrestle with coherent justifications and accurate terminology suggests the latter. I finally decided that *image doing* would be an appropriate choice for Abby's initial responses, as her justifications at that time were often hazy. Later, although her justifications became more forceful and coherent she sometimes still did not use accepted terminology so I categorised her responses as *image seeing* (58), (60) and (63) rather than *image saying*.

During the second interview I asked Abby to design several different ways of packaging four tennis balls. During this and the next interview I extended this problem to an investigation of the amount of wasted space within each of the designs, and also spent a

brief period of time considering with the students the amounts of packaging materials that would be required to manufacture the designs. Tasha was late arriving for the scheduled meeting and so Abby had to begin this investigation on her own. She rapidly drew four diagrams, see Figure 11, explaining to me that the fourth design had "one [tennis ball] on top".

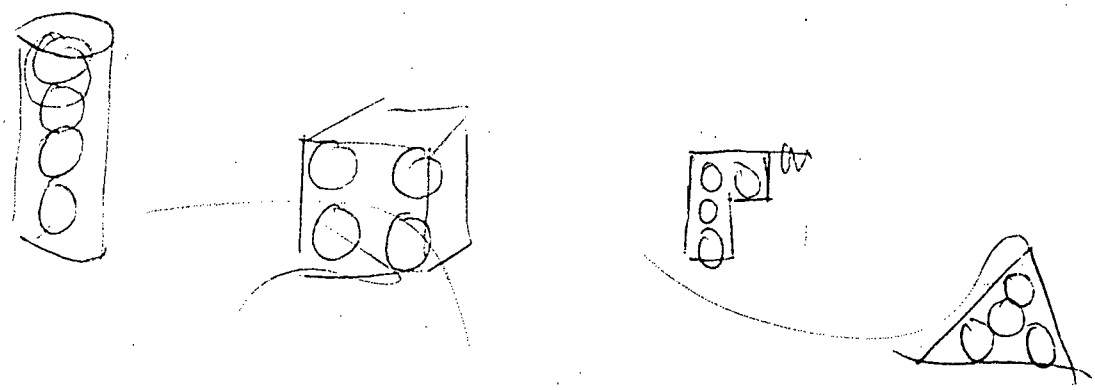


Figure 11: - Abby's Tennis Ball Packages

I then told her that I wanted to find out what measurements would be needed on these packages in order to actually build them. I had provided a range of equipment, including rulers, string and a tape measure, as well as tennis balls. Abby 'measured' the diameter of a ball with a ruler (by roughly aligning the ruler with the circumference of the ball and estimating its diameter), re-drew her diagrams more neatly, and started to annotate the diagrams with measurements, based on a value of 8cm for the diameter of the ball. Figure 12 shows the new diagrams and the annotations she added. Due to the fact that these were later changed several times in discussions with Tasha, and the diagrams therefore became very untidy, I have chosen to redraw them here. (Brave readers are referred to Appendix

19 for a copy of the students' actual written work). I have categorised this as *image doing* (65).

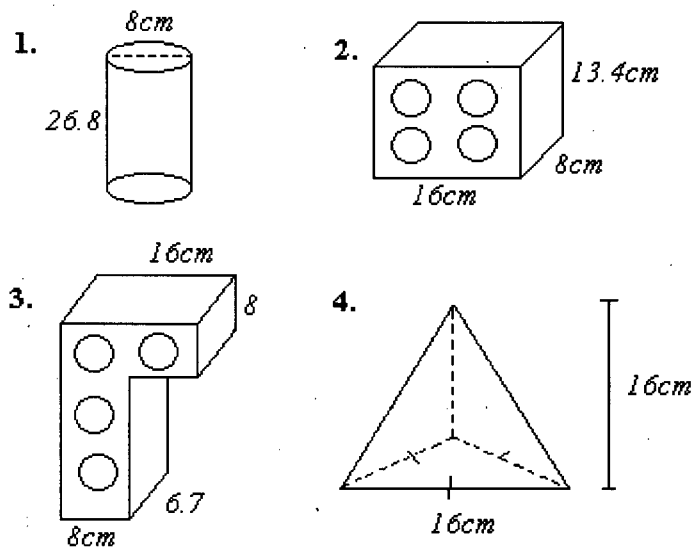


Figure 12: Abby's Annotated Diagrams

Abby then moved between the *Image Making* and *Image Having* levels several times in her attempt first of all to explain the problem to the late-arriving Tasha, and then to make sense of the issues surrounding calculating the volume of the tennis balls, and the amount of wasted space in the first design (the cylinder) (66 to 71). A similar pattern of movements characterises Abby's sense-making of the calculation of wasted space in the second design (the right-rectangular prism) (72 to 75), though it was clear that Abby had *Formalised* her method for calculating the volume of a right-rectangular prism (*method applying*, 74).

During the third interview with me (77 to 86), there is further evidence that although Abby was earlier able (in a test situation (56)) to use many of the formulae she was being

asked to recall and apply here, she was unable to demonstrate a *Formalised* understanding when asked to apply her understanding to a practical situation. Abby and Tasha had applied a number of strategies in response to my request that they establish which of their designs wasted the least space within the package. Interestingly, they had not chosen to pursue Tasha's initial method (for the cylinder) of simply subtracting the volume of the four tennis balls from the volume of each package. Instead, for each design, and led by Abby, they had calculated the ratio of the volume of the balls to the volume of the package as a fraction, and also converted this to a percentage (a method that I have assumed was inspired by their earlier class work on ratio).

It was clear, however, that at first they did not understand that this percentage represented the amount of space taken up by the balls within each package, not the space remaining within each package (*image reviewing*, 81). When I asked exactly how much space was wasted in the packages it was Abby who suddenly recognised the true nature of the percentages (*image saying*, 82), and who then, without calculating actual wasted space, was able to use the percentages to predict which of the designs would waste the least space (*property predicting*, 83). Abby then expressed some concern about how low the percentages that they had calculated were (ranging between 38% and 49%), noting that not even fifty percent of the available space was taken up in any of the designs (*property recording*, 84).

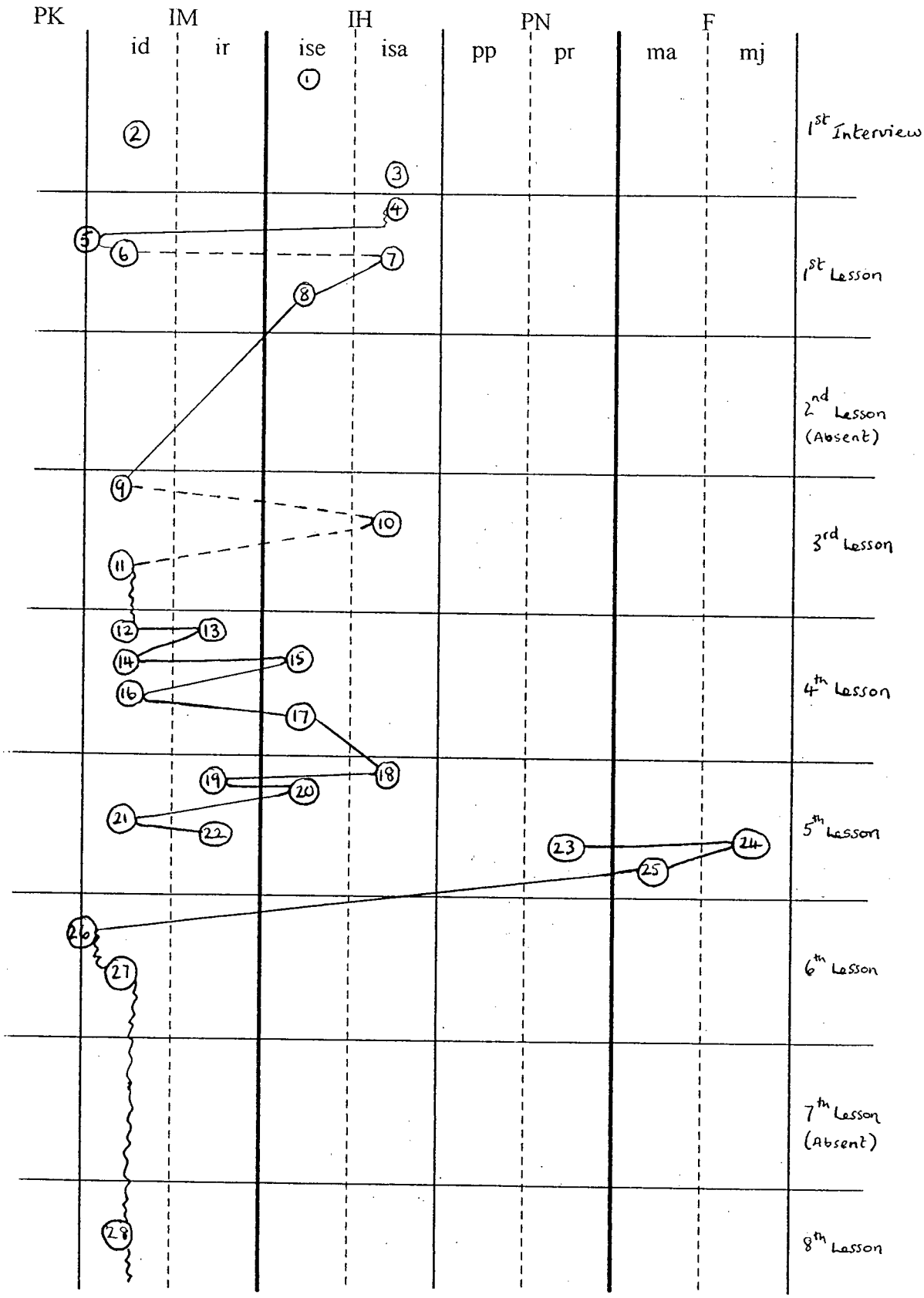
In the final interview with me, Abby again demonstrated that although she was able to justify why some of the basic angle properties she had learned were legitimate (*method*

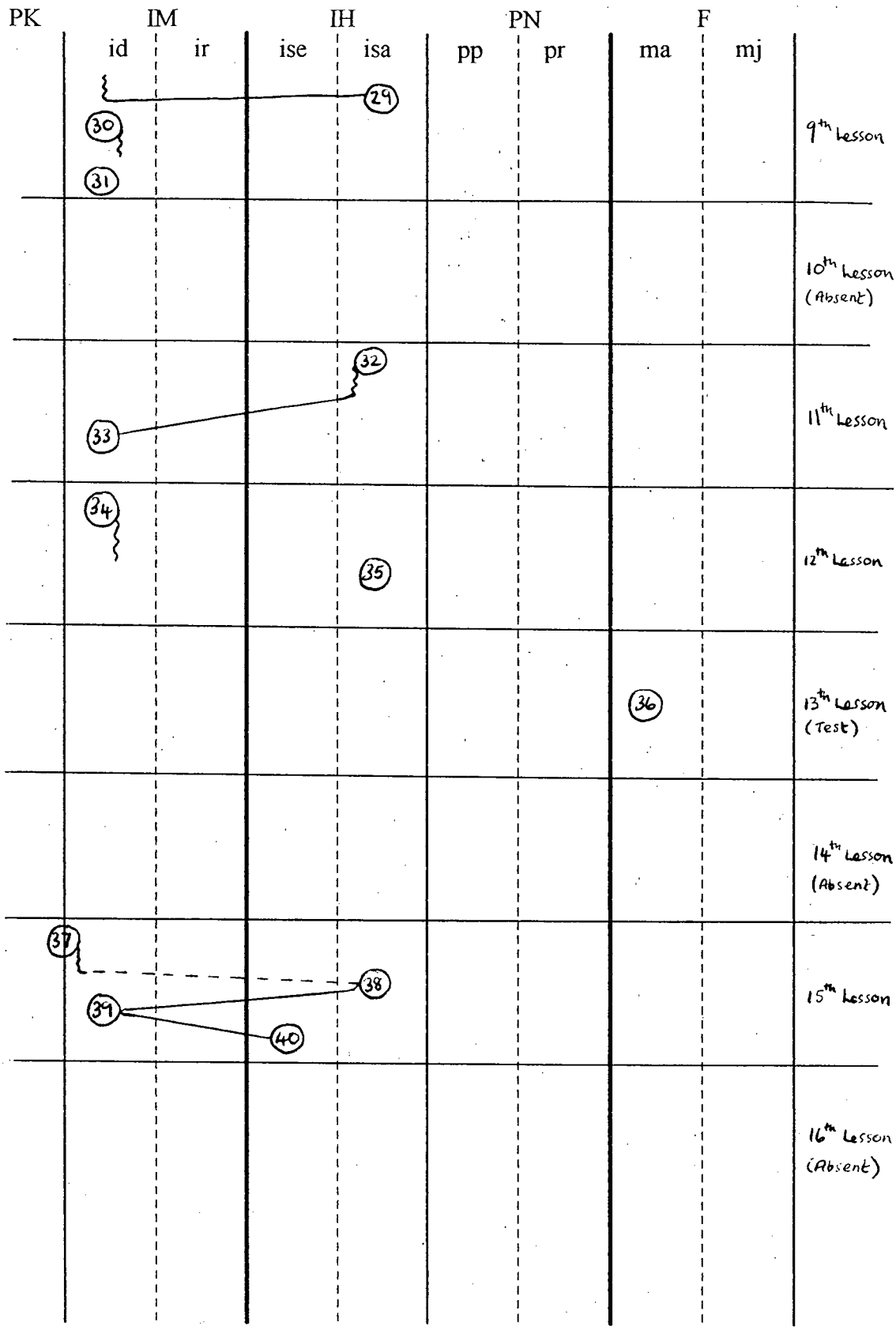
justifying, 92), when she was asked to apply that knowledge in a slightly different context, and to prove a new postulate using those properties, she *folded back* further and further (93 to 96).

4.3.3 Tasha's Growth of Understanding

A complete description of the points on Tasha's mapping diagram (Figure 13) is given in Appendix 4. The following description of Tasha's growth of understanding covers only the significant points.

FIGURE 13: TASHA'S MAPPING DIAGRAM

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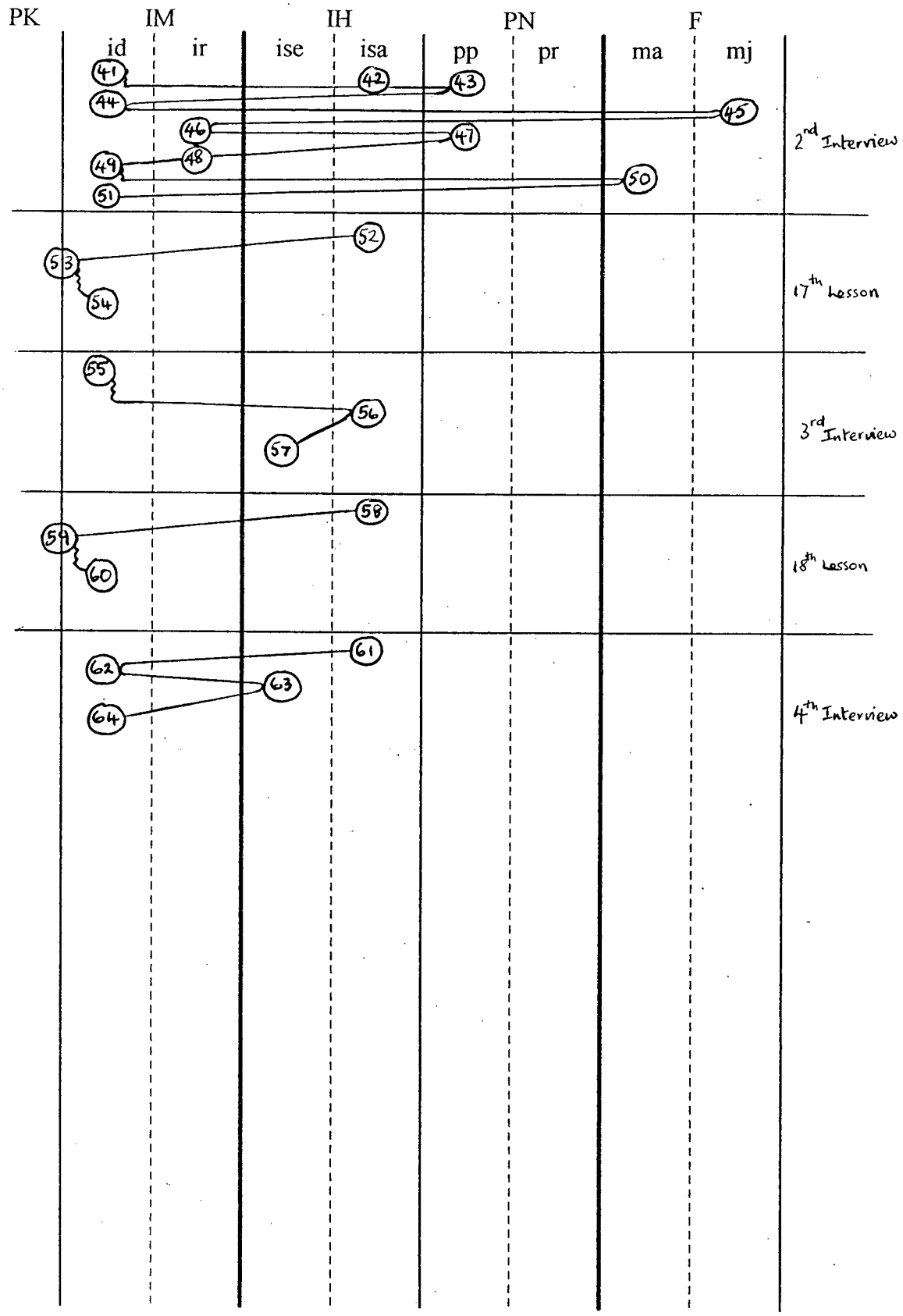
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18th Lesson
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4th Interview

Tasha was less vocal than Abby in the initial interview situation, but nevertheless was able to demonstrate that she knew how to calculate the area of the shape shown in Figure 14, and she began an explanation of how to find the surface area of the soup can, but fell silent as Abby took over the description, though she did not disagree with Abby's rendering of the solution. The data reveal that Tasha and Abby often objected to one another's solution attempts, so I have mapped these incidents as *image seeing* (1) in an attempt to recognise that Tasha had some understanding at this stage of the methods of solution of these kinds of area problems. I have mapped Tasha's tentative attempts to find angles equal to angle *a* in Figure 15 as *image doing* (2).

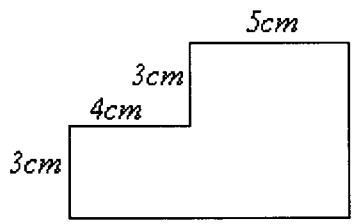


Figure 14: A 'Deviant Rectangle Problem

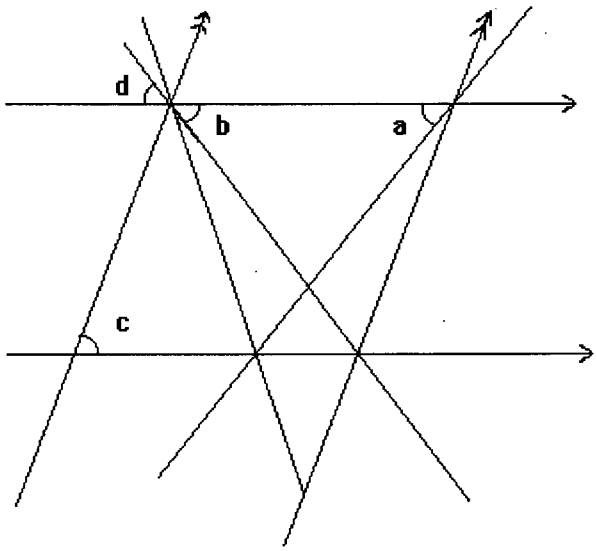


Figure 15: An Angle Relationships Problem

Although Tasha did not offer any method for calculating, mathematically, the volume of the soup can, after some time she picked up the can and began looking at the label and noticed that the volume was written there, a fact which rendered the three students helpless with giggles for a few moments, and which seemed to help to establish a good working relationship between us as I laughed with them and praised Tasha for her ingenuity. As a note of significance, I did not press them to continue to try to calculate the volume after this incident, allowing them, instead, to feel that they had done what was required of them. In fact, I had noticed beforehand that the can stated the volume of the contents, but had chosen to leave the label on the can, wondering if any of them would think to look for it. Establishing trust between researcher and participants is a curious process, and this seemed to me to be an unusual beginning to our relationship, but one I feel was instrumental in establishing me as an inquisitive, but ultimately humane, 'stranger'. The atmosphere was noticeably more relaxed in the remainder of the interview, and the students seemed to become more eager to 'play' with the problems I posed in this and later interviews. In no sense did I feel that the students thought I had presented the soup can problem to trick them, though in hindsight I can see that they might have done. Fortunately, they appeared to recognise that I was laughing with them, and not at them.

In addressing the volume of the solid shown in Figure 16 Tasha demonstrated that she knew how to calculate the volume of a right-rectangular solid (*image saying*, 3), but as she was beginning to develop a verbal solution for the triangular prism, Abby again took over the explanation. Instances such as this caused me, both at the time and later, to

ponder my decision to interview the students together rather than separately. I have concluded, though, that the advantages outweigh the disadvantages. One advantage is clearly seen in the example I have just described concerning the soup can. Another is that it affords the interviewer the opportunity to withdraw from a prominent role in the interview which allows the students the freedom to explore. It is my experience that students in one-to-one interviewing situations tend to respond far more as though they were in a test situation, required to find the answer that the interviewer wants, than do those students who are interviewed in a group situation.

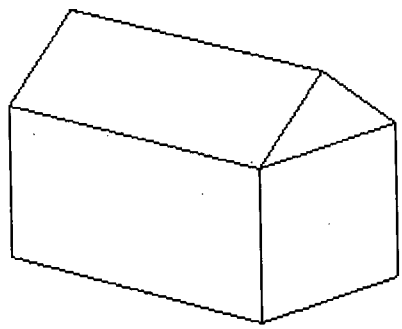


Figure 16: A Combined Solid Problem

During the first lesson Tasha spent a great deal of time at the *image saying* level (4), responding to many of Karen's questions about area and volume and the properties of various plane figures, including a trapezoid. Tasha was prompted to *fold back* to *image viewing* (5) as Karen described the method of calculating the area of a trapezoid, mentioning to Abby that she didn't think they had done this before. Soon Tasha progressed to *image doing* (6) as she began copying the derivation of the formula from the board and trying to make sense of it by asking Abby for clarification.

Tasha was absent from the second lesson, so began the third lesson at *image doing* (9) for Pythagoras' theorem, making notes as Karen worked through some examples on the board and asking Abby to clarify various concerns she had. Although Tasha was able to express her image concerning the area of various figures (*image saying*, 10), she did not progress beyond *image doing* (11) for any question which required knowledge of Pythagoras' theorem.

Like Abby, although Tasha had appeared confident about calculating the surface area of a right-rectangular prism when it was given as a line diagram in a text-book (*image saying*,10), she was less successful when asked to apply this knowledge to a practical situation (the painting a room problem), and for that problem she remained at *image doing* and *image reviewing* (12 to 14).

During the fifth lesson Tasha again demonstrated that she understood, this time in more general terms (*property recording*, 23), how to find the volume of a prism. In response to a question from Karen about how to find the volume of a hexagonal prism or a cylinder, Tasha said to Abby "base times height, isn't it?". Abby disagreed, so Tasha gave further information, saying "you find the bottom [*swirling her finger around in the air*]. You times it by the height". A few moments later as Karen explained to the class how to find the volume of any prism Tasha turned to Abby and said "see, I told you so" and explained that she had meant surface area of the base ("you find the bottom") times the height (*method justifying*, 24).

In the sixth lesson Tasha folded back to *image viewing* (26) as Karen demonstrated with the water pouring activity why the volume of a square-based pyramid (or a cone) is one third that of the equivalent cube (or cylinder), and then remained at *image doing* (27) for the remainder of the lesson as she worked on problems concerning these relationships.

In the area and volume test given by Karen in lesson 13, Tasha demonstrated that she had *Formalised* (36) her image for the volumes of right-rectangular prisms, cones and spheres, and for the surface areas of right-rectangular prisms and spheres. For administrative reasons, however, some students were given an alternative version of the test, so Tasha's was not the same test taken by Abby and it did not contain any questions involving Pythagoras' theorem, with which Tasha had struggled during the preceding classes. I was therefore unable to determine how Tasha responded to a question involving Pythagoras' theorem in a test situation.

During Lesson 15 Tasha began at *image viewing* (37) as Karen introduced various terms concerning angle properties, and, whilst she was later able to demonstrate some understanding of how the various angles were calculated (*Image Having*, 38 and 40), Tasha, like Abby, continued to struggle to produce justifications for her answers that were phrased in terminology acceptable to Karen (*image doing*, 54).

When Tasha arrived (rather late) at the second interview she spent three minutes clarifying the problem with Abby (*image doing*, 41) before suggesting that the cylinder would waste least space, and setting about calculating its volume (*image saying*, 42). She

later predicted that all the designs might be the same in terms of wasting space because they "all have the same measurements and all that. And the balls are all the same size". Though this prediction is false, it does show that Tasha was thinking about the nature of the measurements that they were making, and recognising that there would be some consistency if they always used the same values (*property predicting*, 43). (For instance, they had agreed to have a "gap" of 1cm around the balls in each design so that the balls would slide in and out of the package smoothly).

Tasha was able to describe to Abby how to work out the volume of a cylinder (*method justifying*, 45), but then *folded back* to *image reviewing* (46) expressing concern about the value she had calculated being too high (their value for volume of the cylinder was approximately twice the volume they had calculated for four balls). After some discussion, however, Tasha concluded that perhaps the value was all right, recalling that in class they had considered the ratio of the volumes of a sphere and a cylinder into which the sphere fits exactly. Tasha was unclear about the exact ratio, though, mentioning that the volume of the cylinder was three times or two times that of the sphere. In fact, neither of these are correct, and I believe that she was confusing the true ratio in this case ($^2/_3$) with the earlier volume demonstrations that had been carried out by Karen to show that the volume of a cylinder is three times that of a cone with equal height and radius. Nevertheless, Tasha was attempting to combine her images of the volumes of multiple solids, and their relationships to one another, and so I have mapped this as *property predicting* (47).

During the seventeenth and eighteenth lessons Tasha was prompted to fold back to *image viewing* (53 and 59) as she listened to Karen's explanations about various new angle properties and the method of completing a standard two-column proof, and *image doing* (54 and 60) as she attempted to produce acceptable (to Karen) justifications for her answers to various problems.

4.4 Stage 3: Shifting Attention

4.4.1 Teacher Interventions

Though I have represented this part of the analysis as a single stage in this description, it was actually threaded more evenly throughout the time I spent analysing my video tapes than any of the other aspects of the analysis. Although at any one time I might have been focusing my attention on one or other of the participants of the study, I was always subtly aware of my overarching goal, which was, and is, to investigate the role of the teacher in occasioning the growth of students' mathematical understanding. As such, I was continually noticing and remarking upon interesting interventions made by the teachers (both as side-bars in the notes I happened to be making at the time, and in longer reflections in a journal I kept to help me reflect upon the research process itself). At first, these notes were structured by references made to one of the three types of intervention identified in the Dynamical Theory for the Growth of Mathematical Understanding (*provocative*, *invocative* or *validating* interventions). Soon, however, I began to feel that these categories, rather than facilitating my understanding of the processes at work in the classrooms I was studying, were, instead, constraining my thinking. I found myself struggling to assign certain incidents to one of the three categories, and began to consider

rejecting the categories as a tool of analysis. I returned to the initial run-through notes I had made and realised that I had repeatedly used certain phrases to describe my initial impressions of the interventions made by the teacher. I realised that in fact what I had been doing, without realising it, was developing themes for the teacher's interventions. I began to make lists of those emerging themes by shortening the phrases I picked out to (usually) one word. Some of these initial themes were such activities as *telling*, *checking* and *managing*. As soon as I identified what I thought might be another theme I would add it to my list and pay attention for similar occurrences throughout the data. The list soon grew. Though my description here makes this generative process sound unproblematic, it was not so, and I also struggled at times with categorising incidents within this new categorisation system.

Though I had not set out explicitly to generate themes from my data, nor to employ the constant comparative method championed by Glaser and Strauss (1967), it is clear that my method is, in fact, reminiscent of that approach. Glaser and Strauss (1967, p. 107) note that the researcher will discover two kinds of categories, those he or she has constructed him- or herself, and those that have been "abstracted from the language of the research situation". It appears that by this Glaser and Strauss mean 'abstracted directly from words spoken by research participants', and as such I confess that I have no categories so abstracted. I do however, have a number of themes whose names are abstracted directly from the written and spoken language *I* initially used to describe the data (such as *telling*, *checking* and *praising*).

4.4.2 Developing Themes

Table 1 gives a brief description of the themes I developed from the Vancouver strand of my data. The themes are arranged in the approximate order in which they were developed. In order to explicate each theme, I intend to provide here an example of each within the context of the Vancouver strand. I wish to emphasise, though, that each of the themes I describe occurred to greater or lesser extent in *both* strands of this study, and so it should not be assumed that the themes I develop using examples from the Vancouver data were peculiar to Karen's teaching. Likewise, the themes I develop in Section 4.6 when discussing my own teaching, were also to be found in the Vancouver strand. I deliberately have chosen to discuss only those themes that occurred in both strands of the data, as I believe this draws attention to the similarities in the practices of the two teachers I have studied. However, I do not want to imply that I have ignored aspects of teaching which were not present in *both* strands. There was, in fact, only one distinct strategy that was evident in one strand but completely absent from the other (a strategy employed by Karen which I discuss briefly later in this section). I therefore feel that my decision represents the data fairly.

Checking	The teacher is checking for student understanding.
Showing and telling	An extended stream of interventions often involving the giving of new information but usually without checking that the students are following the explanation.

Leading	An extended stream of interventions aimed at directing the student towards a specific answer or position, often involving step-by-step explanations. Differs from *showing and telling* by its attempts to involve the students in the explanation through frequent questioning.
Reinforcing	Giving further emphasis to a significant point (often one already made by a student).
Inviting	Suggesting of a new and potentially fruitful avenue of exploration. More open-ended than *clue-giving* (see below).
Clue-giving	A deliberate attempt to point the student to the correct answer or preferred route.
Managing	Including disciplining, keeping students on task, giving instructions etc.
Enculturating	Inducting students into the language, symbolism and practices of the wider mathematics community.
Blocking	Preventing a student from following a certain path (sometimes preventing a student from *folding back* to *Image Making* activities).
Modelling	The teacher explicitly models her own thought processes.

Table 1: Vancouver Strand Intervention Themes

The first theme to emerge was that of *checking*. Karen spent a great deal of time in the first lesson, and a significant amount of time throughout the data collection period, in fact, determining that students were paying attention, checking their understanding and confirming their knowledge of certain mathematical facts. I define *checking* as questions

and other interventions where the teacher is engaged in checking the students' understanding of the mathematics. Karen asked a great number of questions throughout the data collection period, and during the first lesson she asked many questions that I have categorised as *checking*, including "do you all know what the word perimeter means?", "what does perimeter mean?" and "how many agree with [this] definition [for area]?"

Throughout the data collection period Karen engaged in practices which I have come to call *showing and telling*. I define this theme as an extended intervention stream, often involving the giving of new information, but usually without checking that the students are following the explanation. Though I first noticed this theme in the very first lesson, the example I offer here is taken from the eighteenth lesson. Although it is a lengthy excerpt, I include it in full in order to better demonstrate the nature of the intervention streams I have collected under this theme. Here, Karen is beginning a new topic, congruence, and starts by explaining to the class the difference between congruence and similarity.

Karen: Basically, similar means the same in shape, but congruent means the same in size and shape. So that, for example, if I took a triangle like this and a triangle like this *A B C* and *P Q R* [*drawing the triangles in Figures 17 and 18*]. These two triangles would be congruent if [*pause*]. What? [*Pause*] They're the same size and shape which would mean their angles, corresponding angles, would have to be the same. And this side would have to equal this side [*pointing to AB and PQ*]. The corresponding sides would have to be equal. In that case you say triangle A B C is congruent to, the congruent signs, sign, is an equal sign with a squiggle over top of it [*writing* $\Delta ABC \cong \Delta PQR$]. And we would have to say it's congruent to triangle *P Q R*. You have to have the angles mentioned in the corresponding order so that those two angles are equal [*annotating the diagrams to show that the angles at A and P are equal*]. Angle *B* and angle *Q* are equal and angle *C* and angle *R* are equal [*writing a list of the equal angles*]. And when you

say they are congruent... [*pause to discipline some students who are not paying attention*]...And when you say they are congruent the angles are the same and the corresponding sides are the same so the side *A B* would have to equal the side *P Q*. The side *B C* would have to equal the side *Q R* in length. And the side *A C* would have to equal the side *P R*. So *A C* would equal *P R* [*writing a list of the equal sides and annotating the diagrams*]. And this concept of congruence can hold for any shapes. Two rectangles are congruent if they're exactly the same size and shape. Two pentagons are congruent if they're exactly the same size and shape. And when we say they're congruent we have to be sure you list them in the congruent order. That you have the same sizes in the same order, OK? Now, if they are similar, what would be the difference between, now if I wanted to draw a triangle similar to this one [*pointing to the triangle in Figure 17*]. Does anybody know? [*Pause*] Using one word. Could you tell me what the difference would have to be? [*Pause*] One word. Or there's two different words you could say, basically. The similar triangle would have to be either bigger or [*pause*] smaller than this one. So that you'd have a triangle, let's call it *X Y Z* [*drawing Figure 19*] where the angles are equal but the sides are either shorter or longer. And in this case we could say triangle *A B C* is similar, and you just put the squiggly line for similar [*writing* $\Delta ABC \sim \Delta XYZ$]. Is similar to triangle *X Y Z*. Now again you have to have the orders correct. This would mean angle *A* equals angle *X*, angle *B* equals angle *Y*, and angle *C* equals angle *Z*. But there's also a relationship between the sides. It's not just that it's smaller. They have to be proportional to each other. Remember when you were taking ratios and proportions with [the student teacher]? The ratio of the corresponding sides would have to be the same. That is, if you took the ratio of side *A B* in the first one to side *X Y* in the second one that would have to be the same as if you took the ratio of *B C* in the first one to *Y Z* in the second one, or *A C* in the first one to *X Z* in the second one. So it's like it's, it's shrunken in the same proportion or ratio that it was. Erm, you couldn't have for example, if this one had sides of ten, twelve and seven just for the fun of it [*annotating triangle ABC as shown in Figure 20*]. This one maybe would have sides five, six, could this be four? [*Annotating triangle XYZ as shown in Figure 21, and pointing to side XY*]. No. What would this side have to be if it was similar? [*Pause*] Ten divided by five is two. Twelve divided by six is two. So seven divided by [*pause*]

Tasha: Three point five

Students: Three and a half

Karen: Three and a half. So that would have to be three and a half. And so that's the difference between congruence and similarity. OK?

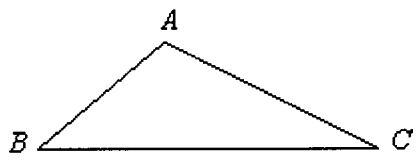


Figure 17: Triangle *ABC*

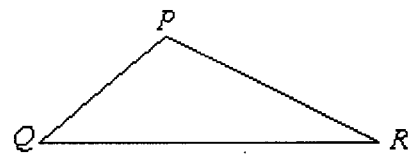


Figure 18: Triangle *PQR*

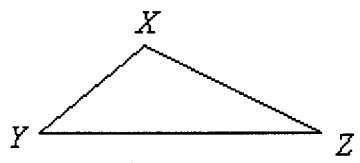


Figure 19: Triangle *XYZ*

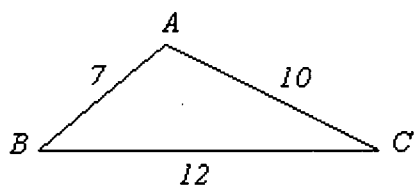


Figure 20: Annotated Triangle *ABC*

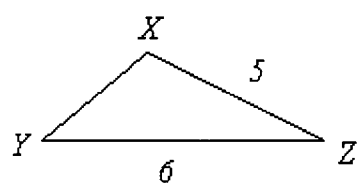


Figure 21: Annotated Triangle *XYZ*

Sometimes Karen made an intervention which served to confirm that (at least some) students were paying attention to her explanation. For example, Karen often asked fill-in-

the-gaps questions, which are actually not questions at all, but unfinished statements. Examples included:

Karen: Angle three equals [*pause*] [*Lesson 12*]

Karen: Angle four is vertically opposite to angle [*pause*] [*Lesson 12*]

In this classroom, such 'questions' were usually rewarded with a chorus of 'answers' from the class. The class usually recognised these statements as requiring a response although they did not always respond as can be seen from a close scrutiny of the lengthy *showing and telling* transcript above where similar fill-in-the-gaps statements appear. When Karen was engaged in *showing and telling* the students often did not respond to fill-in-the-gaps statements, whereas when Karen was involved in step-by-step explanations (such as if she was going over a problem on the board that the students had previously attempted for homework) the students generally did chorus a response[22]. This difference may rest on the issue of whether the students actually knew the answers to the 'questions', however, I suggest that this is probably too simplistic an explanation. It *did* appear, though, that the students did not respond to Karen's questions and fill-in-the-gaps statements in the lengthy *showing and telling* transcript above because they did not know the answers.

In fact, whether students did or did not respond in these situations seemed to hinge upon the general mode of teaching at the time. In order to explain this statement I introduce here the next theme, *leading*. I define *leading* as an extended stream of interventions

[22] Step-by-step explanations of this sort do not fall within my definition of *showing and telling*. They are associated with the next theme I will be presenting (*leading*).

aimed at directing the students towards a specific answer or position, often involving step-by-step explanations. *Leading* differs from *showing and telling* by its attempts to involve the students in the explanation through frequent questioning (taking various forms, including fill-in-the-gaps statements). The following excerpt from the twelfth lesson is an example of what I might term fairly strong *leading* by Karen[23]. In response to appeals from the students to go over one more example before the test which occurred in Lesson 13, Karen drew the diagram shown in Figure 22.

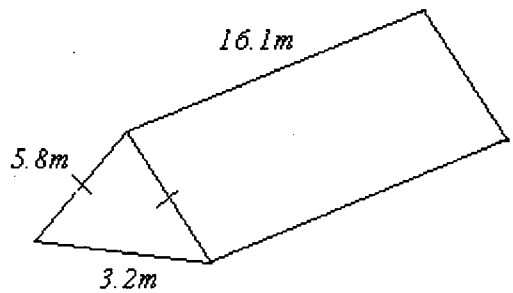


Figure 22: Triangular Prism

Karen: The surface area of this thing which is, what would be the name given to this shape? [*Pause*] What's it called?

Abby: Triangular prism. Triangular prism

Karen: Triangular prism. We have something like this which doesn't look like a prism but in actual fact it is because it has a base, a flat base that's triangular and the sides are vertical to this. This is actually a right-triangular prism because the sides are at right angles to the base. So in order to find the surface area of this thing we have first of all the flat bottom part which is, if we look at the bottom it's a rectangle. How long is the base, or is the rectangle?

Student: Sixteen point one

Karen: Sixteen point one metres, right? And what's this width of the rectangle? Of the bottom rectangle? It would be [*pause*]

Student: [Inaudible]

Karen: Three point two, wouldn't it? The width of the rectangle would be this part here. So it's sixteen point one by three point two. And we would have only one of those. Now. We have two of these sides [*the sloping*

[23] As with all of the themes I have developed there are a range of interventions that fall within the descriptions I offer.

rectangular faces]. And they are, so let's call them. What will we call those, the the the [*sic*] sides. Sounds good enough to me. The sides are both rectangles. Are they the same size?

Students: Yes

Karen: Yes. So there's two of these. They are again sixteen point one metres long, and what's this distance [*the other dimension of the rectangles*]

Student: Five point eight

Karen: Five point eight metres. For that part. And then you also have our two ends, or the top and bottom if we stood it up that way [*pointing to the triangular end of the prism*]. And so we have two of these. And what do we know about those ones? What would the base of this triangle be?

Student: Three point two

Karen: Three point two metres. And what about the sides? Because this is an isosceles triangle each of those is [*pause*]

Student: [Inaudible]

Karen: Five point eight metres. So there it is divided up into its parts. Any questions there? [*Pause*] OK. We may as well find these areas first because they're very easy to do the rectangular ones. So the area here [*pointing to the base*] is sixteen point one times three point two. Should be somewhat over fifty.

Students: [*Calling out answers*]

Karen: Fifty-one point five two. How many agree that it's fifty-one point five two? [*Pause, one or two students raise their hands*] Metres squared. OK. For the sides again we have a rectangle. Area equals base length times width. Sixteen point one times five point eight. Which is? [*Pause*]

Student: [Inaudible]

Karen: Ninety-three point three eight. How many agree? [*Pause. A few students raise their hands*] OK. Metres squared. And then we have two of those so we have to multiply by two. So a hundred and eighty-six point seven six metres squared. Any questions there? [*Pause*] Now. We have the two ends to worry about. And we know that the area of a triangle is one half the base times the height. We take a look at our triangle. We have to drop a perpendicular from the vertex down to the base so that each of these parts would be one point six because the base would be divided into half. And to find the height of this triangle we will need to find the height of that yellow triangle [*see Figure 23*] using the Pythagorean theorem. *a* squared plus *b* squared equals *c* squared. Any question? [*Pause*] OK. Now. The five point eight is the hypotenuse or the *c* because it's opposite to the right angle. So we would get five point eight squared here. Our value can be one point six. This part. And then we need to find this part [*the unknown side*]. We'll call that our *b* squared. So that we need to have *b* squared equals five point eight squared minus one point six squared. [*Pause as Karen*

works this out on her calculator]. Oh that's interesting. *b* equals five point seven metres. How many got that? Five point five seven, pardon me.

Abby: Oui. [*Waving her hand in the air*] Moi.

Students: [Inaudible]

Karen: Five point six if you do it to one decimal place. Yes.

Abby: [*Whispers to Tasha*] I'm on the right track

Karen: Five point five seven which rounds to five point six. And we probably should round it to one decimal place because all the other numbers we're dealing with are one decimal place numbers. OK. Now that we have the height of the triangle we can find the area of the triangle. So area is one half the base of the triangle which is three point two, times the height of the triangle which is five point six. That gives me [*pause*] five point six times [*pause*] equals. Area of each triangle is eight point nine six metres squared. Everybody agree? [*Pause*] And then we have two of those.

Tasha: [*Quietly*] Oh, sheesh!

Abby: [*Quietly*] Hang in there, Tasha. I know you'll make it through [*laughing*]

Tasha: [*Smiling*] Shut up

Karen: Seventeen point nine two metres squared for the two triangles. We have here the bottom, the two sides and the two ends [*indicating her calculations on the board*]. Anyone have any questions? Note that I've left my calculations to the second decimal place at this point. It's probably best to do that and then just round them off at the end if we want it to one decimal place. So, to find the total surface area of this thing. The surface area is going to be the area of the bottom which is fifty-one point five two. Plus the area of the two sides which is a hundred and eighty-six point seven six. Plus the area of the two ends which is seventeen point nine two. We add those together [*pause*] two hundred fifty-five point two metres squared. Now. You might differ just slightly if you had rounded off along the way.

Student: [Inaudible]

Karen: Let's see. Nine [*pause*] oh, yeah. 'Cause I carried a two. I didn't carry a one. Thank-you. Good. [*Karen alters her answer to 256.2*] So that's not too bad is it?

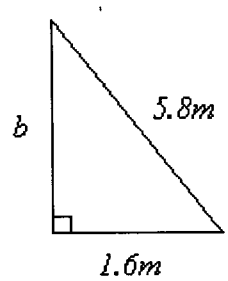


Figure 23: Right-angled triangle

In *leading* episodes such as the one above the students appeared to understand that they had an obligation to respond to direct questions and to fill-in-the-gaps statements. When Karen was engaged in what I have called *showing and telling*, there appeared to be an understanding that they were being shown how to do something and that their participation was not explicitly required. Heap (1992) notes a similar phenomenon in his study of a second-grade class. He describes instances where, although the teacher paused, she had not "given up the floor". How such classroom norms are negotiated and enacted are beyond the scope of my study, but it is clear that the students in this strand of my study differentiated between certain teaching acts and altered their actions accordingly.

In the excerpt below, Karen presses the students for information about what constitutes the *base* of a figure. The last section of speech here serves to exemplify the next theme, *reinforcing*. I noticed that Karen often repeated what students had said in class[24] and also frequently reinforced a point that seemed to have been grasped by the class already. Karen appeared to use this technique as a 'safety net' to make sure that the point was explicitly

[24] This is a common teaching technique often employed to ensure that important points are reinforced by the weight of the teacher's authority, and also a common management technique employed to ensure that students are paying attention to the main points being developed in a discussion by enabling them to focus on just one voice - that of the teacher.

stated every time.

Karen: Base times height. What do we mean by base? [*Checking*]
Students: The bottom.
Karen: The bottom. So, I could not find the area of something like that [*drawing Figure 24*] because it doesn't have a bottom.
Abby: You have to cut it in half.
Karen: [*Sounding surprised*] You have to cut it in half? Why?
Abby: Because it doesn't have a base. Or, no. You can just do the side. Do it on the side. And then…
Karen: [*Interrupting*] So the base doesn't have to be at the bottom?
Abby: No.
Student: Something horizontal.
Karen: So we couldn't find the base of this [*Figure 24*] or the area of this because it doesn't have a bottom?
Abby: You can.
Karen: [Inaudible] …this point. Is that what you're telling me?
Abby: It doesn't have to be the bottom.
Students: It doesn't have to be the bottom.
Abby: A flat line thing.
Karen: Flat?
Student: As long as you know both sides.
Karen: As long as you know both sides. As long as I know this side and this side? [*pointing to two opposite sides on Figure 24*]
Student: No, that side and that side [*pointing to two adjacent sides*]
Karen: Oh, that side and that side! The base is basically one of the sides, right? It doesn't have to be at the bottom. Technically. We usually use the base because we set something down, but if you're talking about a geometric figure you can choose any side you want to be the base. A triangle [*drawing Figure 25*]. Is this the base? [*pointing to the side I have marked a*]. Normally, if we look at this we would choose this as the base [*side a*], but we could choose this as the base [*side b*] if we wanted, depending on how we were looking at it. [*I would describe all of Karen's closing statement here as reinforcing*]

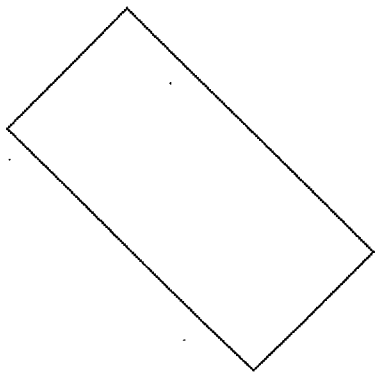


Figure 24: Rectangle

Figure 25: Triangle

In much of the above excerpt Karen was taking the role of devil's advocate in her interactions. Though I recognise this as a distinct teaching strategy I have not included it as a theme in this study because it was not a strategy that occurred in my own teaching, and, as I described earlier (Section 4.4.2), the themes that I have chosen to describe in detail are those that occur in the teaching in both strands of this study.

The following excerpt is representative of many interactions between the teachers and their students in both strands of the study when the teachers were working with a small group of students (rather than the whole class). It introduces two new themes, *inviting* and *clue-giving*, and shows what I have found to be a common progression from *inviting* through *clue-giving* to *reinforcing* and/or *leading* and/or *showing and telling* in rapid succession. I have identified *inviting* as those moments when a teacher suggests a new and potentially fruitful avenue of exploration. *Clue-giving* consists of deliberate attempts

to point the students towards the correct answer, or the preferred solution route. In this episode, which occurred in the second lesson, Dale and Abby had called Karen over to help them to find the area of the shape shown in Figure 26. Karen began by *inviting* them to look at each triangle separately, and for a few minutes Dale and Abby attempted to use the information given on the diagram to pursue this route. There followed a discussion about the lengths of the sides of various triangles, and Dale appeared to become confused about how Abby and Karen were viewing the diagram:

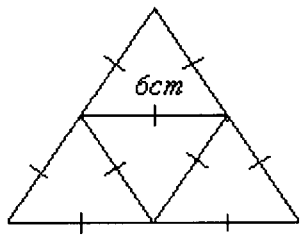


Figure 26: Net of a Triangular-Based Pyramid

Dale:	Are we looking at the whole triangle, or just this [*pointing to the upper small triangle*]
Karen:	Depends which way you want to look at it [*pause*]. Could we look at the whole thing at once? [*Inviting*]
Abby:	Yeah, you can.
Dale:	[Inaudible]
Karen:	And if you did that, do you know how long this base is? [*pointing to the base of the large triangle*] [*Clue-giving*]
Abby:	Yeah.
Karen:	How big would that be?
Abby:	Twelve.
Karen:	Twelve. Could you drop a perpendicular from this vertex…
Abby:	Ohhh.
Karen:	…down to that base? [*Clue-giving*]
Abby:	Yeah
Karen:	Yeah, and do you know this side? [*pointing to the right hand edge of the large triangle*] [*Clue-giving*]
Abby:	Yeah. OK. I get it.
Karen:	And you'd know this would be six [*pointing to the right-hand half of the base of the large triangle*]

Dale: And that would be twelve [*pointing to the right-hand edge of the large triangle again*]

Karen: And that would be twelve. So you could do it in one instead of in four parts [*Reinforcing*]

At this point in the interaction it appeared from Karen's body language and verbalisations that she believed that the students now understood how to proceed with the problem, however Abby continued to describe what she was intending to do, and it became necessary for Karen to repair Abby's image of how to solve this particular problem. Karen continues the interaction by *leading* Abby towards a solution route:

Abby: So you would times that by two, right? To make the whole thing.

Karen: [*Cautiously*] It depends what you're finding.

Abby: Well, no. After this. 'Cause you split it down the middle, right? [*It is difficult to know, here, exactly the thrust of Abby's question, but it appeared that she was concerned that dropping a perpendicular splits the large triangle into two halves. It would be possible to reach a solution by using Pythagoras' theorem to find the length of the perpendicular bisector, then finding the area of one of the right-triangles so formed, and doubling it, however, I do not think that Karen intended for the students to do this. I believe she expected them to find the length of the perpendicular bisector of the large triangle and then multiply the base of the large triangle by its height and halve the result.*]

Abby: So you'd be getting one half separately [*I think Abby is thinking of the two right-triangles*]

Karen: You're gonna get the height that way. Right? [*I think Karen is still thinking of the method I described above*]

Abby: Yeah.

Karen: [Inaudible] the height the triangle is.

Abby: So you're not really doing anything, just trying to find the height.

Karen: Right.

Abby: Oh great. It's not that useful.

Karen: And then you have to find the area after you find the height [*Leading*].

Abby: Ahhh.

I will return to consider in more detail these kinds of patterns of interaction in the next chapter (Section 5.2.1.3).

Throughout all of the lessons in this strand there were occasions on which the teacher managed the behaviour, noise level, or progress of the students, or was engaged in giving instructions to the class, or engaged the class in administrative tasks such as taking the register or discussing test dates. I clustered all of these activities in the theme *managing*.

I also noticed a group of interventions consisting of incidents where Karen alluded to, or directly invoked, the authority of the wider mathematics community. I have labelled such interventions as *enculturating*, defining them as moments where the teacher attempts to induct the students into the language, symbolism and practices of the wider mathematics community. As examples of this theme I offer here several phrases used by Karen, which, although they are out of context, serve to demonstrate the frequency with which this theme occurred, and the variety of forms it took. In addition, I offer an excerpt from my third interview with Karen, demonstrating that a concern to induct students into the practices of the mathematics community lay at the root of the activities in which Karen asked her class to participate during the latter few lessons of the data collection period (when Karen emphasised the comparison by ratio of the volumes of various solids):

Karen: That's a way we often compare things [*lesson 8*]

Karen: Sometimes in mathematics…[*lesson 8*]

Karen: In mathematics, three words we assume everyone knows the meaning of are point, line and plane [*lesson10*]

Karen: I want to be sure we're all talking the same language here. That's what we're doing [*lesson10*]

Karen: You're going to have to get more and more strict about giving the official rule [*lesson 12*]

Karen: Notice our little symbol for parallel? The parallel lines with the little *l* sort of like an exponent *l* up there [*lesson 14*]

Karen: Why is that bad mathematical language I'm using? [*lesson 14*]

Karen: In mathematics there is no such thing as a *z* angle [*lesson 14*]

Karen: What I've sort of done on this section now is diverge from the direct stuff as it is in the curriculum, which is just the surface area and the volumes, to try to bring in the ratios from the previous [topic] and show them how it worked and show them that there are constants in mathematics [*third interview*]

I would like to pause for a moment here to reconsider Karen's statement to the class that "in mathematics there is no such thing as a *z* angle", because this statement, and many similar ones made by Karen, heralds my next theme, *blocking*[25]. This statement was made during the fourteenth lesson when the class were being introduced to angle geometry as a precursor to formal two-column proofs. Throughout all of the lessons on this topic Karen was at pains to establish what was and was not an acceptable justification for each statement that the students made. In establishing these justifications Karen frequently invoked the concept of "traditional authority" (*enculturating*) as she insisted on certain terminology (including a number of "official" stock phrases such as "angles on the same side of a transversal add up to one hundred and eighty"), but also specifically *blocked* students' use of mnemonics (such as the commonly used z-angle, x-angle or f-angle to describe angles associated with parallel lines).

During the sixteenth lesson Karen was challenged by a number of the students who said that their friends in other classes had been shown the z-angle and other mnemonics. They

[25] I am indebted to Dr. David Pimm for suggesting the name for this theme.

wanted Karen to show them those mnemonics in order to help them recognise the angles. Karen objected, saying that such descriptors are not recognised by the markers of the provincial exam, so students might as well learn the correct terminology from the start. The students continued their appeal, promising that they wouldn't use the mnemonics as authorities on any examination papers, just as an aid whilst they were learning. Karen continued to *block* this approach, finally closing down the discussion with her statement "do not learn incorrect terminology and use it as a means of trying to learn the mathematics". *Blocking*, then, is defined as preventing a student from following a certain path, and (as in the example I have just described), can manifest itself as *blocking* students own attempts to *fold back* to build "thicker" images for the concept. In this case I see the students' appeals to be shown the angle mnemonics as an attempt by them to *fold back* to *Image Making* activities, and I see Karen's refusal as *blocking* those attempts to *fold back*. As I shall demonstrate later (Section 4.6), I also *blocked* my own students' attempts to *fold back* to *Image Making* activities.

In the ninth lesson I noticed Karen using a strategy which, although not common in her teaching (or mine), I have retained as a separate theme due to its distinct nature. I have called this theme *modelling*, and I define it as moments where the teacher makes explicit her own thought processes in solving a problem as an example for students to follow. As an example, consider this exchange between Abby and Karen, concerning the ratio of the surface area of a cube to the surface area of a sphere which fits exactly inside it (i.e., the diameter of the sphere is equivalent to the dimensions of the sides of the cube). Before this episode the class had been asked to find the ratios of volumes and surface areas of

various combinations of solids. In this case, Abby had generated an incorrect expression for the surface area of the cube (as can be seen in the reproduction of her working below) and hence created an incorrect ratio. In her working Abby used the letter s to stand for side for side, r for radius, SA for surface area, and subscripts of c and s to stand for cube and sphere, respectively:

$$\begin{aligned} SA_c &= s^2(6) \\ &= (2r)^2(6) \\ &= (4r)(6) \\ &= 24r \\ SA_s &= 4\pi r^2 \end{aligned}$$

$$24r / 4\pi r^2 \quad = \quad 6r / \pi r^2 \quad = 6r^{-1} / \pi \quad = \quad 6 / \pi r$$

Karen: So with what you have here it would mean that you've got the surface area of the sphere divided by the, no, the cube divided by the sphere and that depends on the radius of the sphere, right? 'Cause you've got your answer is involving r. Every other one we did worked out to a number, didn't it?

Abby: Yeah.

Karen: Which meant that there was a constant ratio between the volume of a sphere and the volume of a cube [*pointing, as an example, to Abby's earlier ratio calculations for the volume of these two solids*]

Abby: Not here because you need the radius.

Karen: Which would make *me* question if I made a mistake or not 'cause every other case turned out to be a constant. Do you think maybe you made a mistake somewhere? [*Modelling*]

Abby: Who me?

Karen: Yeah.

Abby: I did?

Karen: Do you think you did?

Abby: I don't know.

Karen: No, I'm just saying that if I was doing questions like [*Modelling*]

Abby: [*Interrupting*] No, but it's

Karen: I've done a few of them already, right? And I always ended up with a constant of some sort. Now I've done a similar question and I did not get a constant, I got a variable type answer. If I was doing the question and that happened I'd say "is there something different in this one that would make it happen that I'd get a variable, or was some of my manipulations wrong?" So I'd go back and I'd look over it to see. It may be right. It may be wrong. [*Modelling*]

Abby: Oh geez!
Karen: But it's a question to think: If something all of a sudden is different than you've had before you might question it.
Abby: OK. [*Pause*] Tasha, what did you get?
Karen: [*Laughing*] No, no, no, no.

4.5 Stage 4: Re-viewing the UK Data

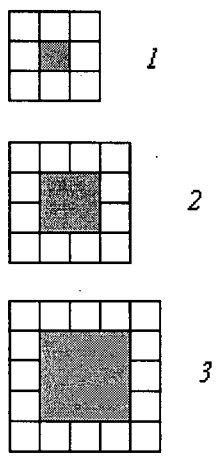
4.5.1 Introduction

Although I had viewed parts of the UK tapes during the time period between their recording and my return to them at this point in the analysis, it had been some time since I had viewed them in their entirety. I therefore began my re-viewing of the UK data with a "first-run through", similar to that described in Section 4.2. Again I have chosen to present below the main highlights of the growth of the students' mathematical understandings in this strand.

Mapping the growth of students' understanding of several topic areas onto one mapping diagram continued to be a source of difficulty for me, but I persisted with my method of joining with solid lines on the mapping diagrams any points which were connected conceptually, and joining with dotted lines any points which were within a broad topic area but referred to separate sub-topics. 'Wiggly' lines again refer to those periods when a student remained at a particular level for an extended period of time.

One activity in particular needs further explanation here before I begin to describe the students' understanding. This is the investigational activity I called 'Ponds'. During this activity the students were introduced to a 'family' of square garden ponds, that required a

border of paving stones around them, as shown in Figure 27. In each class I drew the first three ponds on the board (with their borders) and with the help of the class each time constructed and completed the first three rows of a table of results (as shown). The students were encouraged to draw as many of the sequence of ponds as they felt they needed in order to generate enough data to enable them to suggest a link between the size of the pond and the number of paving stones needed to surround it. Though all students were able to participate in the search in some way (by drawing additional diagrams, filling in the table, suggesting patterns in the columns of the table, or expressing a link verbally) many students were not able to generate an algebraic expression for this relationship.



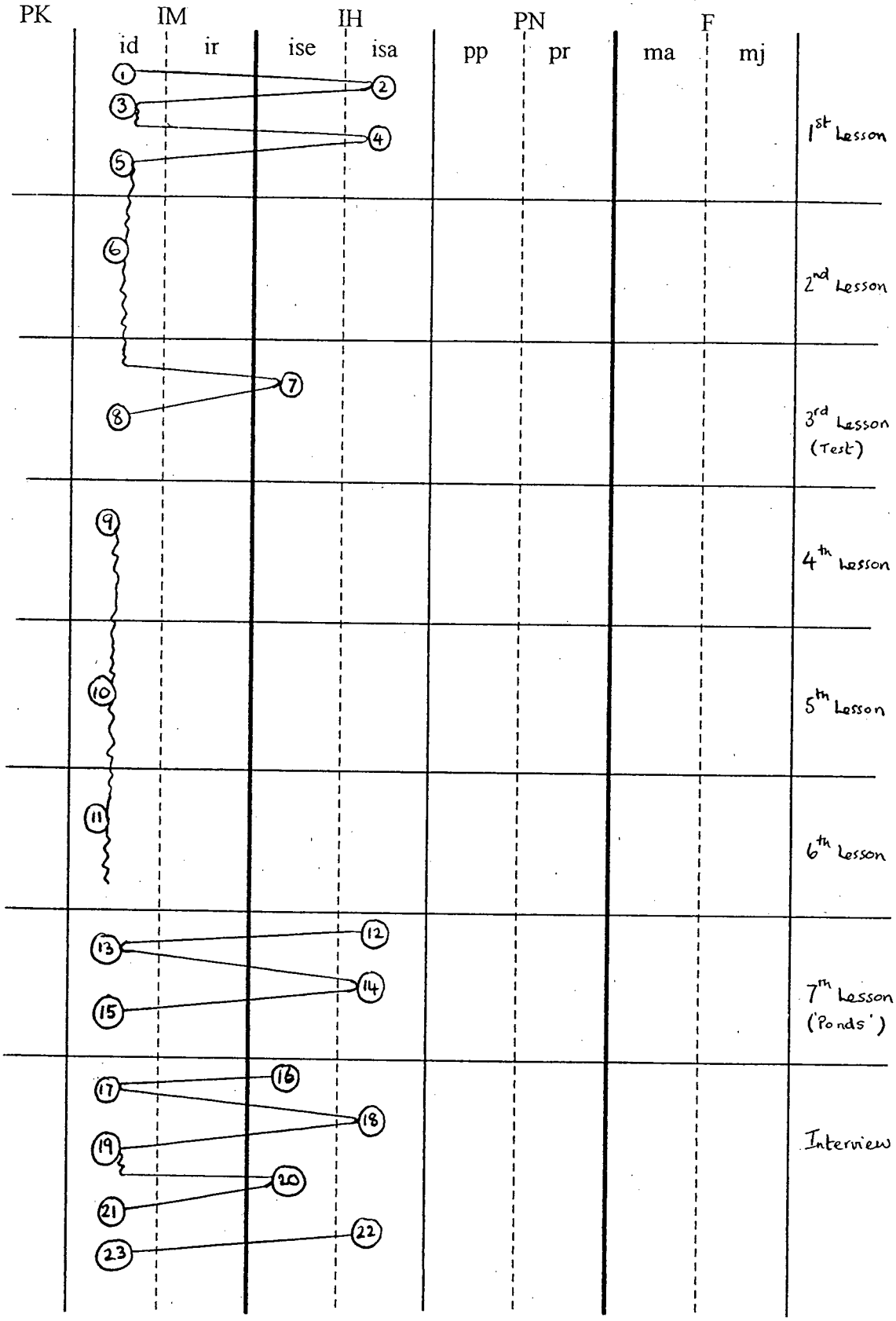
Pond Size	Number of Paving Stones Needed
1	
2	
3	

Figure 27: 'Ponds' Investigation

4.5.2 Donny's Growth of Understanding

The reader is referred to Appendix 6 for a more complete description of each of the points on Donny's mapping diagram, which appears in Figure 28. Point (1) on Donny's mapping diagram represents Donny's initial, brief, response (*image doing*) to the topic of perimeter of plane figures in the first lesson, as he listened to my introduction and chatted briefly about the first question with his partner, Sonya. As he began work on the assigned questions, Donny was able to demonstrate his ability to work out the perimeter of a rectangle *(image saying*, 2) and a square *(image saying*, 4) given all sides of the figure, however, each time he was faced with a figure with missing values (see, for example, Figure 29) he folded back to *image doing* (3 and 5).

FIGURE 28: DONNY'S MAPPING DIAGRAM



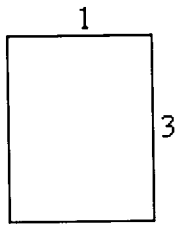


Figure 29: Rectangle With Missing Values

This is how Donny and I explored the perimeter question, shown in Figure 30. I have mapped Donny's understanding as *image saying* (2) during this interaction:

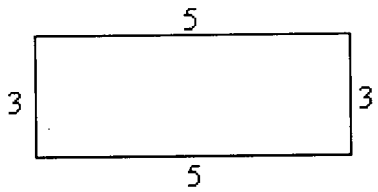


Figure 30: Perimeter Problem

Jo:	You tell me what you think the perimeter of the first one is. What do you do?
Donny:	Sixteen centimetres altogether.
Jo:	Good. Now, how did you get that?
Donny:	'Cause I added those two [*pointing to the sides marked 5*] that makes ten, and then add those threes.
Jo:	Right. That's excellent! Now then, how do we write down the 'Donny' method?

When we moved on from perimeter to area, it became clear that Donny was unable to work out the area of any shape unaided (*image doing*, 6). During the test in the third lesson Donny was able to demonstrate *image seeing* (7) for the perimeter of rectangles and squares, with and without missing values, but still could not find the area of any shape. During the following three lessons the concept of using letters to represent lengths was introduced and the students were asked to generate expressions for the perimeter

(and, for the simpler geometric figures, area) of shapes such as those shown in Figures 31 and 32.

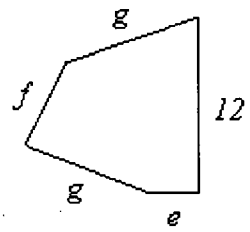


Figure 31: Perimeter Problem

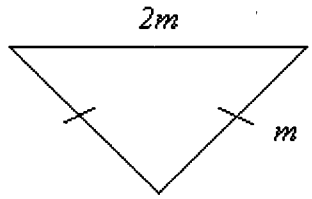


Figure 32: Perimeter Problem

Throughout this time Donny remained at *image doing* (9, 10 and 11), as the following comments from Donny demonstrate:

Donny: [*Referring to Figure 31 during lesson 5*] Two *g* [*long pause*] Two *g* [*pause*] Oh no. Do you add up the twelve as well as the two *g*?

Donny: [*Referring to Figure 32 during lesson 5*] Is it only two *m* and *m*?

During the seventh lesson Donny moved out to *image saying* (12 and 14), and *folded back* (13 and 15) to *image doing* a number of times as he tried to find patterns in the data he generated during the 'Ponds' activity, and to use those patterns to generate a general 'rule' relating the number of paving stones to the size of the square pond. For instance,

although after generating the table shown in Figure 33 he was able to describe the pattern in the right-hand column as "add four" (*image saying*, 14), he was unable, even with assistance, to suggest a link between the two columns of his table[26], and appeared never to fully understand what was required of him (*image doing*, 15).

Pond Size	Number of Paving Stones Needed
1	8
2	12
3	16
4	20
5	24

Figure 33: Donny's 'Ponds' Data

During the interview at the end of the data collection period, Donny extended to *image seeing* (16 and 20) or *image saying* (18 and 22) and *folded back* to *image doing* (17, 19, 21, and 23) several times, reflecting his ability to deal more competently with numerical questions posed about perimeter than algebraic ones, and also, interestingly, his ability to deal more competently with non-contextual algebraic questions requiring him to collect like terms (such as $x + 9y + 3z + 4m + 3z + 2x \quad =$) than with contextual questions requiring the formulation of an algebraic expression before collecting like terms (such as finding the perimeter of the rectangle in Figure 34).

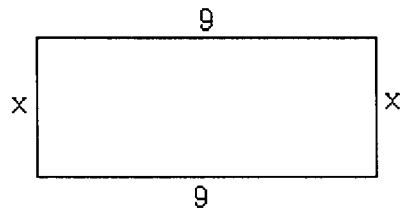


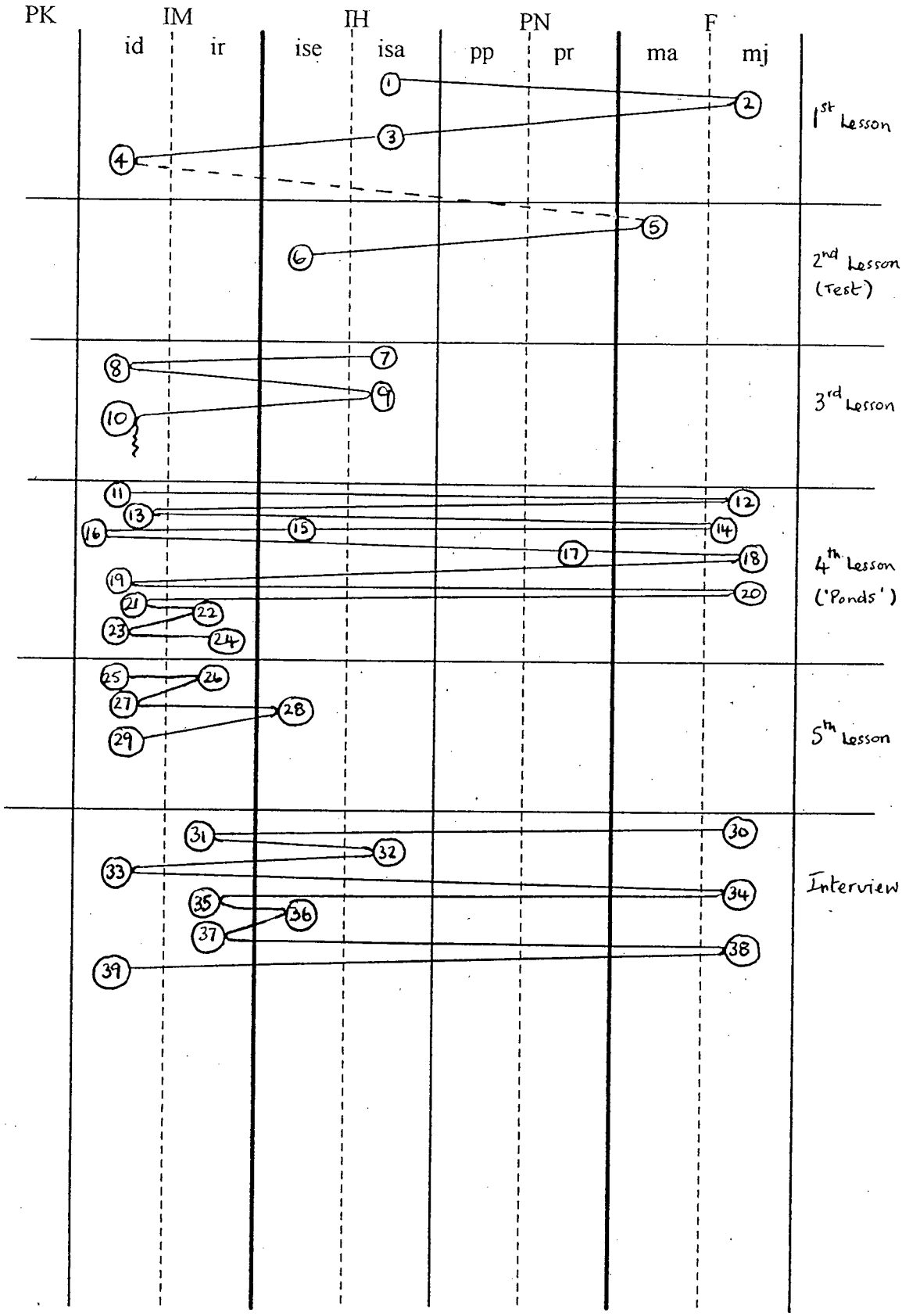
Figure 34: Perimeter Problem

[26] One such link is 4n + 4 where n is the length of one side of the inner 'pond', otherwise expressed as the perimeter of the inner 'pond' plus four.

4.5.3 Kayleigh's Growth of Understanding

For this section, the reader is referred to Appendix 8 for a more complete description of each of the points on Kayleigh's mapping diagram, which appears in Figure 35. Kayleigh began the first lesson by demonstrating that she already had some understanding of some of the areas to be addressed during the data collection period. She competently completed all of the questions relating to the perimeter and area of various plane figures, explaining her method as she proceeded. For instance, she pointed out to her partner, Carrie, that for a regular shape such as that in Figure 36 it is possible to just multiply the length of a side by the number of sides in order to find the perimeter (*method justifying* (2) on Kayleigh's mapping diagram, Figure 35).

FIGURE 35: KAYLEIGH'S MAPPING DIAGRAM



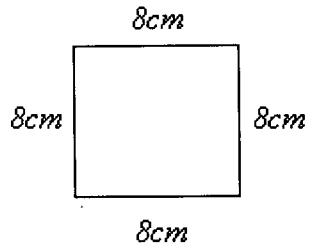


Figure 36: Perimeter Problem

Some questions did cause difficulties for Kayleigh, however, for example when she was asked to find the area of the combined figure shown in Figure 37 (*image doing*, 4). On this problem Kayleigh initially 'completed' the shape to give a large rectangle (6 by 8), and then made mistakes calculating the areas of the two triangles which needed to be subtracted.

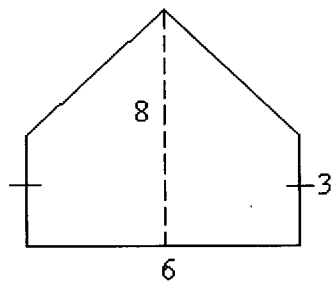


Figure 37: Area Problem

During the test in the second lesson Kayleigh was able to demonstrate that she had a *Formalised* image for the perimeter and area of rectangles and squares (*method applying*, 5), for non-algebraic problems. Her answers when asked to provide expressions for the perimeter and area of shapes with algebraic values representing lengths were interesting. For example, Kayleigh's response for the perimeter of the rectangle shown in Figure 38 was 14 + (m x 2), which is sound, but mathematically untidy.

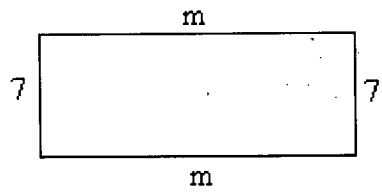


Figure 38: Perimeter Problem

Her written response of (y = x 2) + (x = x 2) to the next problem (the perimeter of Figure 39) clearly shows that she was attempting to express the notion of two lots of y added to two lots of x. She adopted the same algebraic system for the next problem (Figure 40), but then (perhaps as an example) assigned numerical values to the letters p and t and produced as her answer: (p = 9) (T = 10) 38 (p = x 2) (T = x 2). (Though the diagram had lower case letters Kayleigh was not consistent in this respect, and I have reproduced here exactly what she wrote).

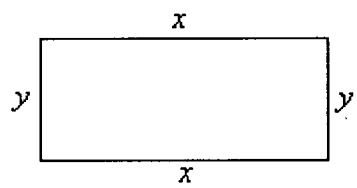


Figure 39: Perimeter Problem

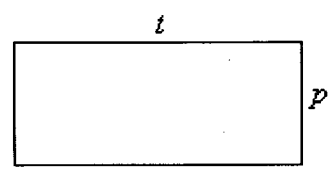


Figure 40: Perimeter Problem

Kayleigh was able to produce 'correct' (but mathematically untidy) answers to many of the algebraic *area* questions, too, such as m x 7 for the area of Figure 38, and y x x for the

area of Figure 39, as well as 'correct' expressions for shapes other than rectangles, such as her answer of (e x 6) for the perimeter of the hexagon in Figure 41.

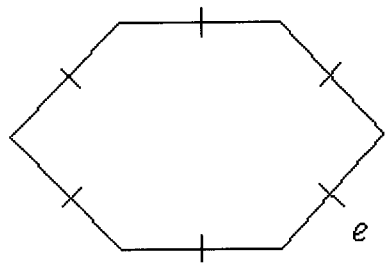


Figure 41: Perimeter Problem

I have categorised Kayleigh's understanding for generating algebraic expressions as *Formalising* (*method applying*, 5) to reflect Kayleigh's consistent approach (as described above), and suggest that Kayleigh's work shows that she is perhaps operating without an *enculturation* into the ways of knowing of the mathematical community which might (had she been *enculturated*) have prompted her to produce more conventional expressions. Though her use of numerical substitution might lead one to believe that she has not yet crossed the "don't need" boundary between *Property Noticing* and *Formalising*, I contend instead that she offered those substitutions as examples *for her reader* (me) to help explicate her algebraic system, not because she was in any way relying on them for her own understanding.

In the following lesson Kayleigh needed some guidance to refine this image, and, although she was soon able to produce mathematically correct answers for expressions involving the perimeter and area of simple figures (such as regular geometrical figures, and rectangles), and appeared to be competent when asked to collect like terms in an

expression she had generated, she continued to struggle somewhat with the generation of expressions for more complicated shapes, such as the 'deviant' rectangle shown in Figure 42. This varying competence is reflected in Kayleigh's mapping diagram (7, 8, 9 and 10).

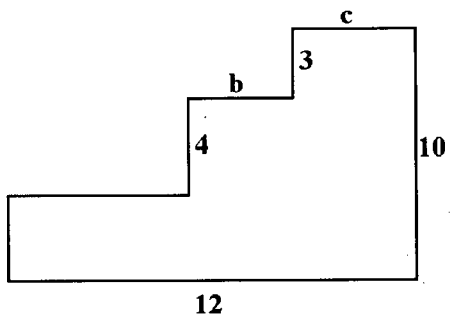


Figure 42: 'Deviant' Rectangle Perimeter Problem

In the fourth lesson, Kayleigh's class were introduced to the 'Ponds' problem, and as can be seen on the mapping diagram, this resulted in a period of intense fluctuation for Kayleigh in terms of her understanding. On many occasions during this lesson Kayleigh was prompted to *fold back* to *Image Making* activities, (13, 16, 19, 21 etc.), in order to generate new diagrams or tables of results, which helped her to build thicker understandings of the problem and to then move outwards again to more sophisticated understandings (14, 18 and 20). This was clearly an important lesson in Kayleigh's growth, so I will be discussing this lesson in more detail later (Section 4.6). For the moment, the reader is referred to Appendix 9 for a brief description of the activities in which Kayleigh was engaged at each of these points on her mapping diagram.

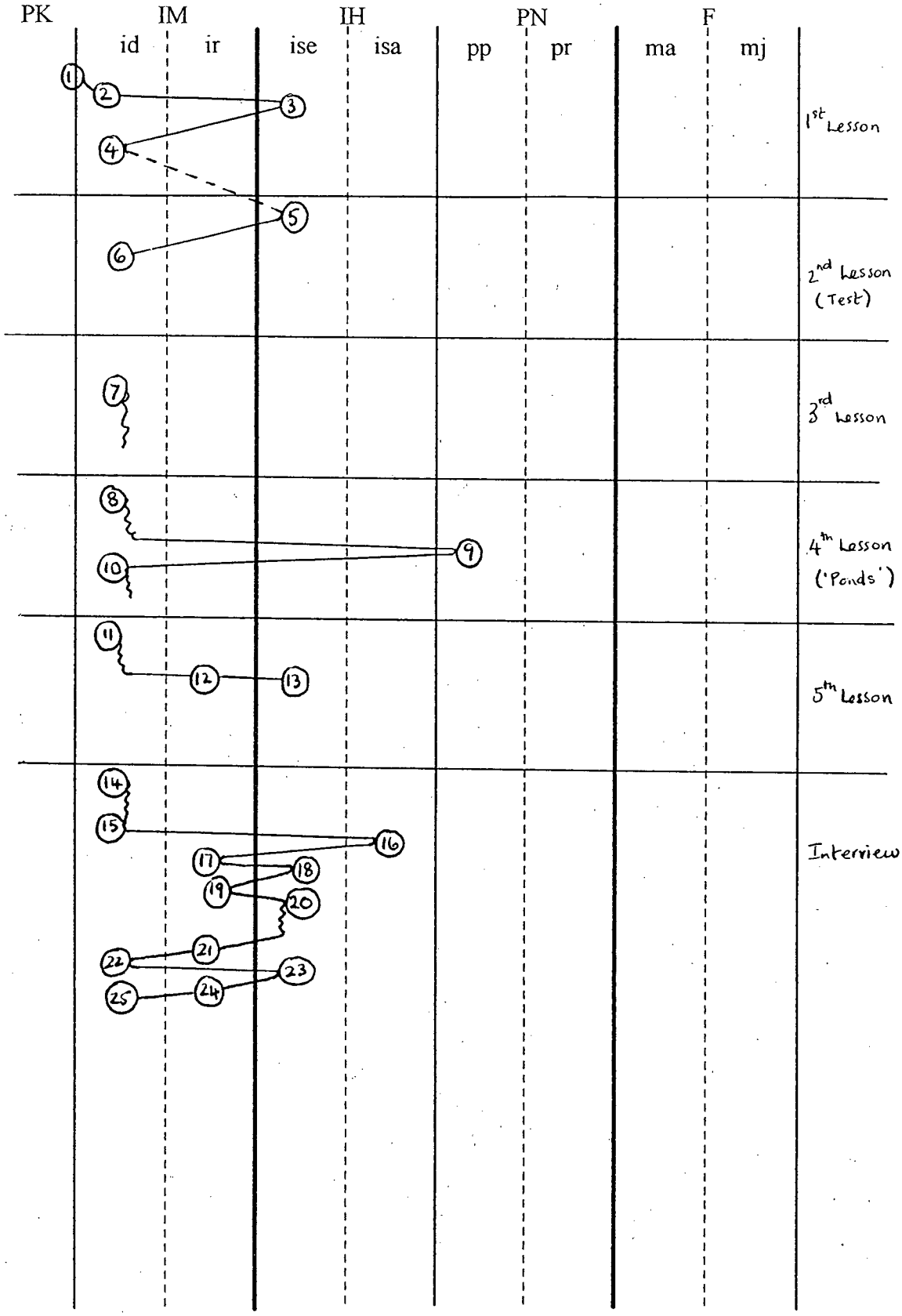
During the interview, Kayleigh was able to demonstrate that she continued to have a *Formalised* image for perimeter and area in numerical situations (*method justifying*, 30), but that she also had *Formalised* her image in some algebraic situations, too, both within

the context of perimeter and area and without that context (34 and 38). For example, when presented with the following question: $x + 9y + 3z + 4m + 3z + 2x \ = \ ,$ Kayleigh produced the answer $3x + 9y + 6z + 4m$.

4.5.4 Carrie's Growth of Understanding

For this section, the reader is referred to Appendix 10 for a more complete description of each of the points on Carrie's mapping diagram, which appears in Figure 43. Points (2) and (4) on Carrie's mapping diagram (Figure 43) show Carrie predominantly within the *Image Making* level during the first lesson when the topic of perimeter and area of plane figures was introduced, reflecting the fact that, for instance, Carrie had trouble with questions involving the area of combined shapes such as the figure shown in Figure 37. Nevertheless, Carrie was able to demonstrate her ability in some aspects of the topic, for example she correctly identified the missing numerical values on the sides of the shape shown in Figure 44 and was therefore able to calculate the perimeter (*image seeing*, 3). Unfortunately, Carrie was strongly led by the more vocal Kayleigh throughout the data collection period, and so it was often difficult for me to assess her true understanding by studying the video tapes. This is partially reflected, I feel, in the limited movements shown on Carrie's mapping diagram.

FIGURE 43: CARRIE'S MAPPING DIAGRAM



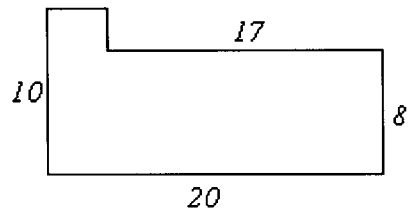


Figure 44: 'Deviant' Rectangle Perimeter Problem

In the test in the second lesson, Carrie was able to correctly answer the non-algebraic perimeter and area questions. This competence might suggest that she had a *Formalised* image for the perimeter and area of plane figures within purely numerical contexts, however, in the interview with me at the end of the period of study, she made several mistakes on non-algebraic perimeter and area questions, and appeared rather hesitant. I have therefore mapped her performance on the test during the second lesson as *image seeing* (5), reflecting the unstable nature of her image at that point. During the test Carrie was also unable to construct correct responses to any algebraic question, although she did attempt the questions and was obviously trying to formulate an image for such situations (*image doing*, 6). Throughout this test her preferred solution method for algebraic problems was to assign numerical values to the letters (according to their place in the alphabet) and proceed to sum the values so produced.

Though Carrie remained at *image doing* (8 and 10) for most of her work on the 'Ponds' activity, she was able to demonstrate that she was working, for at least some of the time, at an outer level. For instance, when discussing with Kayleigh the link between the 'pond' size and the number of paving stones needed to surround it for the family of rectangular ponds of fixed width three shown in Figure 45, Carrie pointed to the top and bottom of

one of the 'ponds' and said "Well, it will go up two, won't it?" in response to Kayleigh noticing a pattern in the table of results. Carrie had realised why two more paving stones were needed for each subsequent 'pond'. I have mapped this as *property predicting* (9).

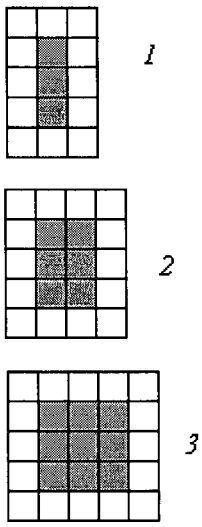


Figure 45: A Family of 'Ponds'

During the interview Carrie seemed to initially have lost some of the competence she had earlier demonstrated. For example, she asked for confirmation that she needed to add up all around the shape when I asked her to find the perimeter of the rectangle shown in Figure 46, and initially produced an incorrect answer for the perimeter of the 'deviant' rectangle shown in Figure 47 (*image doing*, 14 and 15).

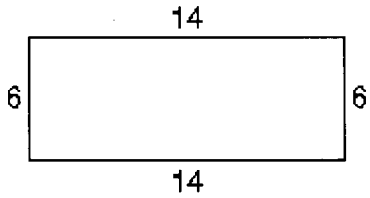


Figure 46: Perimeter Problem

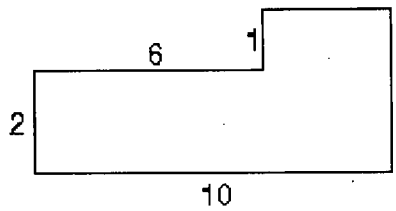


Figure 47: Perimeter Problem

However, this may have been due to nervousness under the pressure of what may have been felt by Carrie to be a test situation, although I had taken pains to explain to all of the participants that the interviews were not tests, and that I was genuinely interested in *how* they did the questions, not *whether* they could do them. It must be remembered that I was Carrie's teacher and, therefore, ultimately, responsible for not only teaching Carrie, but also assessing her mathematical competence. Such recognition is probably at the back (or front) of the minds of all students who participate in research carried out by their own teachers, no matter what the teacher says about why she or he is asking the questions.

Carrie's attempts at some of the algebraic questions were similarly tentative. For example, in finding the perimeter of the shape shown in Figure 48 Carrie wrote $6b + 2a$ and asked me if it was correct. I shook my head and asked how she had got $6b$. Carrie said "that was up there, but..." [*pointing to the bottom edge of the shape and then pointing up to the upper regions of the shape*]. After a long pause I reminded her that we were trying to find an expression for the distance all around the edge of the shape and Carrie silently produced the expression $4a + 2a + 3b$ and then shortened it to $6a + 3b$. I have mapped this as *image reviewing* (19).

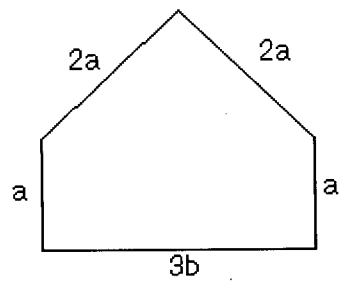
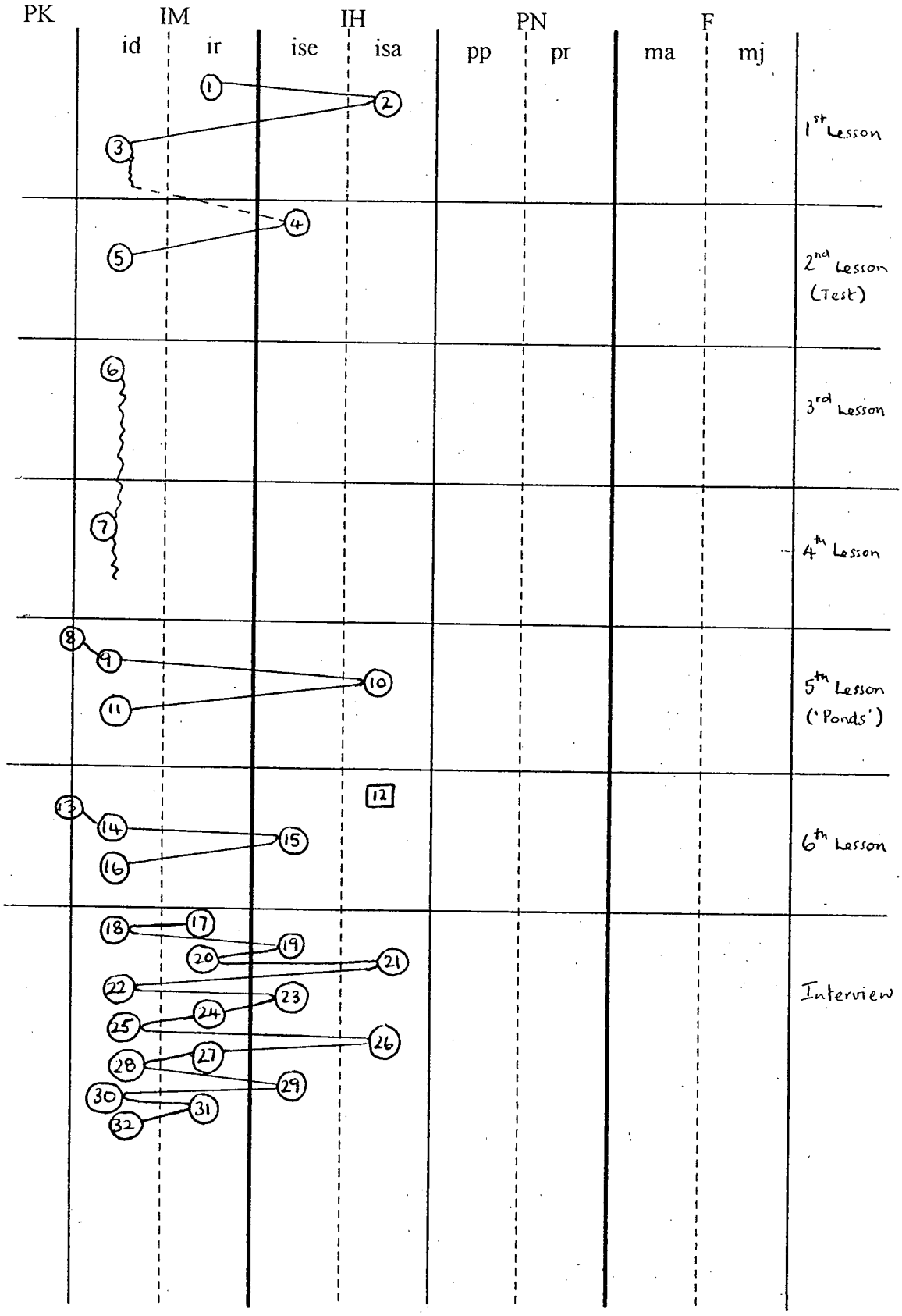


Figure 48: Perimeter Problem

4.5.5 Kerry's Growth of Understanding

For this section, the reader is referred to Appendix 12 for a more complete description of each of the points on Kerry's mapping diagram, which appears in Figure 49.

FIGURE 49: KERRY'S MAPPING DIAGRAM



In the first lesson Kerry described her strategy for finding the perimeter of the rectangle shown in Figure 50, which I have mapped as *image saying* (2), but, like Donny, Kerry was also confused by the 'deviant' rectangle question (Figure 51) and had difficulty with all the area questions (*image doing*, 3).

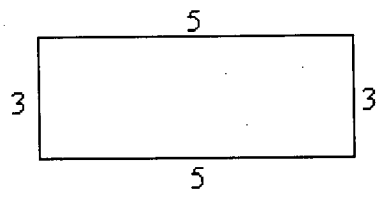


Figure 50: Perimeter Problem

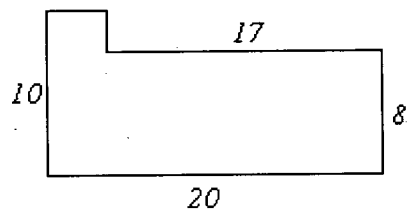


Figure 51: 'Deviant' Rectangle Perimeter Problem

As Kerry's teacher I had noticed for the first time through the videotaping of this lesson that Kerry worked out multiplication problems exclusively by repeated addition, even when she used a calculator. For instance, when faced with the problem of finding the perimeter of a shape with eighteen sides all of length 2cm (no diagram was given), Kerry called me over and asked if she should do eighteen twos. I praised her and left, assuming that she was going to multiply, but the video shows that Kerry picked up her calculator and proceeded to laboriously add up eighteen twos. "Eighteen twos" clearly meant something different to Kerry than it does (or at least did) to me. When she reached an answer, her partner, Graham, attempted to show her the quicker way to use a calculator to

calculate eighteen twos, but Kerry leaned over and repeatedly pressed the clear button as he was explaining, showing that she was not interested in hearing his suggestion. Episodes such as this one clearly show the power of videotape for teachers.

Despite this multiplication technique, Kerry demonstrated during the test in the second lesson that she had an image for the perimeter and area of simple plane figures such as rectangles and squares (*image seeing*, 4) and although she attempted one or two of the algebraic questions (*image doing*, 5), sometimes by substituting values for the letters, she obviously did not understand how to formulate algebraic expressions in these situations. In fact, Kerry remained at *image doing* (6 and 7) throughout the next two lessons as she worked to understand how to generate expressions, and collect like terms in those expressions. Similarly, during the 'Ponds' activity Kerry remained predominantly at the *Image Making* level (9 and 11), and although her partner Graham was the first to declare that the downwards pattern in the second column of their table (Figure 52) was "add four", Kerry was able to describe to me how the pattern worked and predict the next value in the table (*image saying*, 10).

Number of Squares on Side of Pond	Number of Paving Stones Needed
1	8
2	12
3	16
4	20
5	24

Figure 52: Kerry's 'Ponds' Data

During the sixth lesson, Kerry was able to demonstrate *image saying* for expressing the perimeter of a figure algebraically (12). As can be seen on Kerry's mapping diagram (Figure 49) I have enclosed this point on the diagram within a square, rather than a circle, in an attempt to differentiate it from the other points. I have done this to represent the fact that Kerry had, in fact, not answered the questions I had set correctly. I had begun this lesson by asking the students to attempt individually to find the *area*, not the perimeter, of some plane figures similar to those with which they were already familiar. I had not taught them how to generate algebraic expressions for area, and was hoping to see whether any of them could extend their image for perimeter to accommodate this new situation. Despite my stressing several times that I wanted them to find the *area* not the perimeter of the figures, Kerry (and her partner Graham, and, in fact, several other students in the class) misheard or misunderstood my directions and proceeded to find the perimeter. When I stopped the class a few minutes later to give them some indication of how to solve area problems such as those they were facing, Kerry and Graham both declared that they had thought they were supposed to be finding the perimeter. Upon later scrutiny of the answers they recorded during this opening part of the lesson, I noticed that they had each correctly formulated expressions for the perimeter of the figures, and so I have recorded this on each of their mapping diagrams as *image saying*, but enclosed the point number within a square to make it distinct.

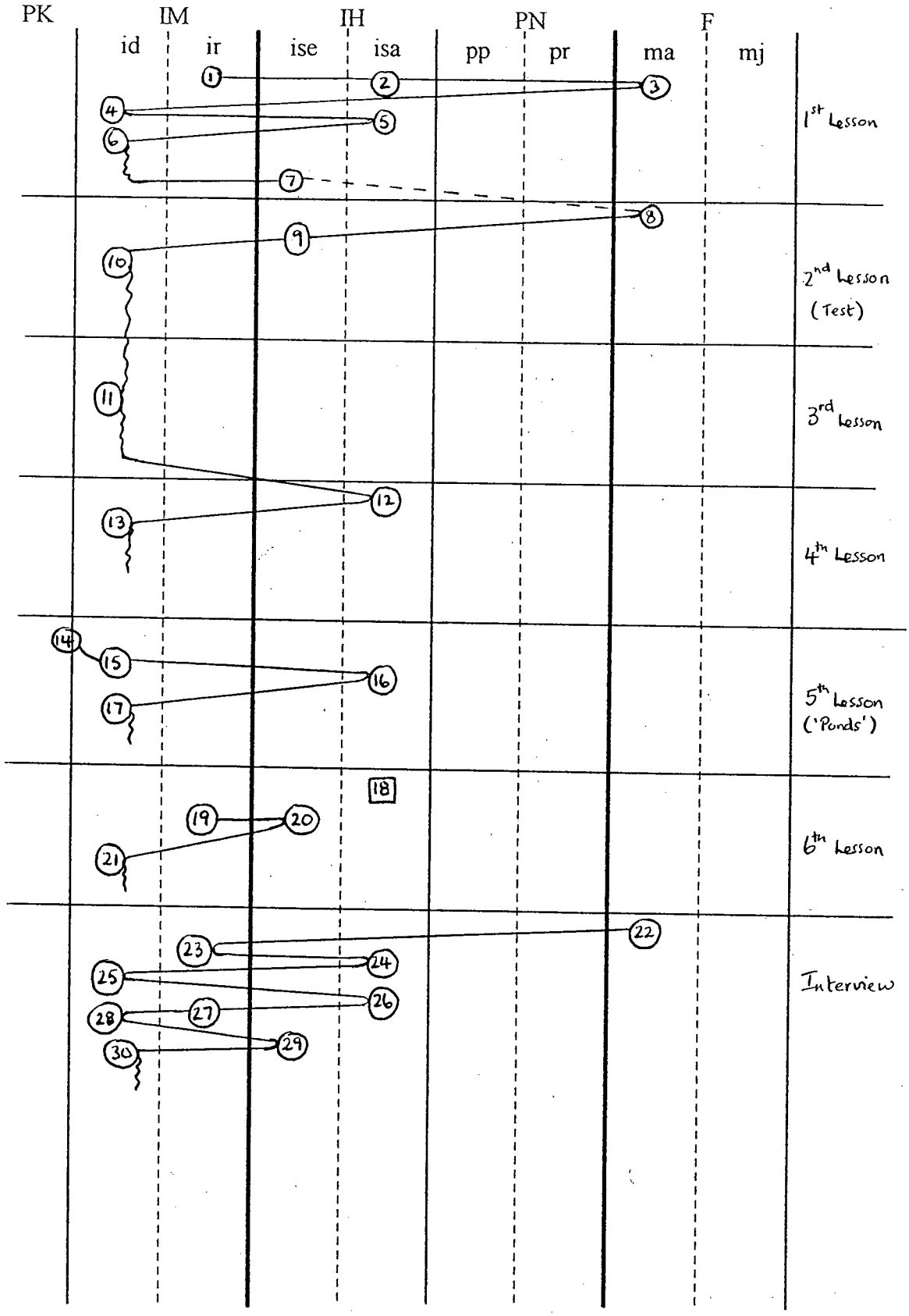
During the interview, Kerry demonstrated that she had reached the *Image Having* level for formulating algebraic expressions and collecting like terms within the context of the perimeter of plane figures (*image seeing*, 19 and 23, and *image saying* 21), although she

continued to have difficulty finding the perimeter of 'deviant' rectangles (both algebraic and non-algebraic) (*image doing*, 18 and 22). Kerry also had trouble with non-algebraic area problems (*image doing*, 25), although after being helped with these area problems she was able to solve the algebraic area problems (for shapes other than 'deviant' rectangles) without help (*image saying*, 26), and was able to demonstrate her ability to apply this understanding to non-contextual questions. For example when she was asked to work out $3pd \times 5ab$ Kerry produced the answer $15pdab$, even though she had never been shown how to deal with terms with more than one variable (*image seeing*, 29).

4.5.6 Graham's Growth of Understanding

A complete description of the points on Graham's mapping diagram (Figure 53) is given in Appendix 14.

FIGURE 53: GRAHAM'S MAPPING DIAGRAM



Graham was able to demonstrate early within the first lesson that he understood how to calculate the perimeter of plane figures, including 'deviant' rectangles (*method applying*, 3), and although he *folded back* to *image doing* (4) when faced with the first area questions, after checking with me how to find the area of a rectangle he was able to complete questions relating to the area of squares and rectangles (*image saying*, 5). In the test in the second lesson Graham demonstrated that he could find the perimeter and area of non-algebraic rectangles and squares (with and without missing lengths) (*method applying*, 8). I have mapped Graham's attempt to formulate algebraic expressions for the perimeter of plane shapes during the test as *image seeing* (9) to reflect the fact that Graham did produce 'correct' but mathematically untidy expressions. For example, for the rectangles shown in Figures 54 and 55 Graham produced the expressions $7 + 7 + m + m$ and $x + x + y + y$ respectively. For the shapes shown in Figures 56 and 57 Graham wrote $d + d + d + d + d$, and *x times 14*, respectively.

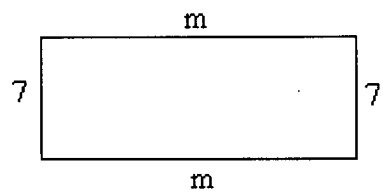


Figure 54: Perimeter Problem

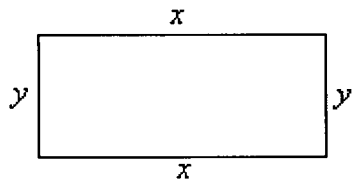


Figure 55: Perimeter Problem

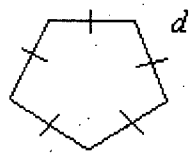


Figure 56: Perimeter Problem

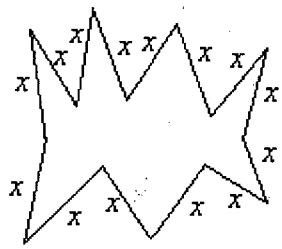


Figure 57: Perimeter Problem

During the interview Graham showed that his image for formulating and manipulating algebraic expressions had grown during the preceding lessons, as he was able to demonstrate that he knew how to manipulate the algebraic expressions he generated. For example, Graham produced correct expressions for perimeter for (among others) the shapes shown in Figures 58 and 59, his responses being $2x + 2y$, and $2a + 2a + a + a + 3b = 6a + 3b$, respectively. He also applied his knowledge in non-contextual situations, for example he correctly collected like terms in the expression x + 9y + 3z + 4m + 3z + 2z.

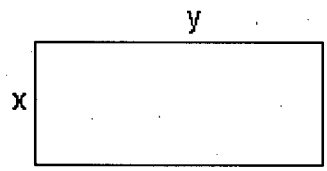


Figure 58: Perimeter Problem

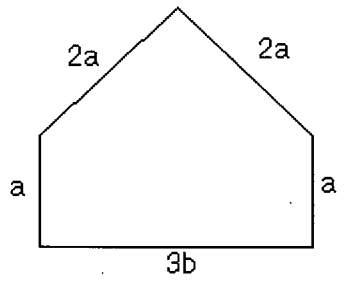


Figure 59: Perimeter Problem

These competencies have been mapped as *image saying* (24 and 26), rather than at the *Formalising* level, because Graham was sometimes hesitant in his approach, and also his responses to some of the later problems posed in the interview suggested that he had not *formalised* his image in the sense of being able to extend his image to fields other than just that of perimeter.

4.6 Stage 5: Looking Again at Teacher Interventions

As I turned to a consideration of my own interventions as teacher in the UK strand, I was already aware that I had also made many of the same types of interventions as had Karen in the Vancouver strand. This awareness had evolved during my re-viewing of the tapes as I mapped my students' understandings. I had been fascinated to realise that I made certain types of interventions that I had not noticed until I closely studied another teacher at work (Karen). For instance, for the first time I noticed that I made many references to the wider mathematical community (particularly to what 'mathematicians' do), an intervention type I have called *enculturating*. Though I had watched portions of my own (UK) tapes before embarking upon a close analysis of Karen's teaching I was now listening differently to my interventions, and noticing new things. I recorded my

deliberations and thoughts about this growing awareness as I was mapping the (UK) students' understanding, and used these notes as a starting point for this penultimate stage of the analysis.

The additional themes which I created during this stage of the research are shown in Table 2.

Praising	Praising individual students, groups or the whole class.
Shepherding	An extended stream of interventions directing a student towards understanding through subtle nudging, coaxing, and prompting.
Rug-pulling	A deliberate shift of the student's attention to something that confuses and forces the student to reassess what he or she is doing. Often results in a return to *Image Making* activities.
Retreating	A deliberate strategy whereby the teacher leaves the student(s) to ponder on a problem.
Anticipating	Preventing students from falling into common pit-falls, trying to prevent mistakes before they happen, protecting students from error, or removing the challenging aspects of a task.

Table 2: UK Intervention Themes

The first new theme I created is called *praising*, and, as its name suggests, it is defined as those moments when the teacher praises an individual, a group of students, or the whole

class. In reviewing my own teaching I noticed that I frequently praised individual students in my interactions with them, both for their work (such as its standard of presentation or its detail) and for their thinking (see, for example the short interaction between Donny and myself in section 4.5.2). I began to wonder why I had not created this theme earlier, as it seems to be high on the list of what we, as educators, might expect to see and hear teachers doing in classrooms. I recalled specific instances where Karen had praised the whole class (for instance by telling them they were smart), but could not recall many instances where she had praised the individual students who were the focus of the study. However, Karen worked with the class as a whole group far more than I did with my classes, and as she did less individualised work with students we might therefore expect to hear fewer occasions where she gave individual praise.

The reader may have noticed that thus far I have resisted the urge to make evaluative comments about the teaching interventions I have described, however, the issue of praise is a complicated one, and I do not wish to leave the reader with the impression that I believe that I was necessarily being a 'better' teacher in these episodes because I frequently praised my students. As Schwartz (1996) has suggested, praise takes many forms, and *excessive* use of such phrases as "nice job" (though, incidentally, this was not a phrase I used) can become meaningless for students, and can, in fact, diminish the development of their academic autonomy by placing the teacher in the sole position of authority in determining the worth of children's efforts.

Also in this first lesson I began to be uncomfortable with the broadness of the interventions I was categorising as *leading*. After creating this category I had assigned many of Karen's interventions to it, and I began to realise that there were certain intervention streams that seemed to fall somewhere beyond the description of *leading* I had created. A particular incident in the first lesson with Donny, which I describe below, made me reconsider these categorisations. I moved back-and-forth through the data after coming to this realisation, looking again at all the instances of *leading* I had identified in the Vancouver strand, and re-considering my characterisations of those episodes. I should point out, here, that I deliberately chose to present an instance of *leading* in Section 4.4.2 that exemplifies my revised thinking about the theme *leading*.

I include the following exchange as an example of the new theme of *shepherding*, which I define as an extended stream of interventions directing a student towards *understanding* through subtle nudging, coaxing, and prompting. This episode concerned finding the perimeter of the shape shown in Figure 60. As I have already mentioned (section 4.5.2) Donny had considerable difficulties finding the perimeter of any shape with missing values. This question proved no exception, but it was the nature of the interventions I made which caught my attention:

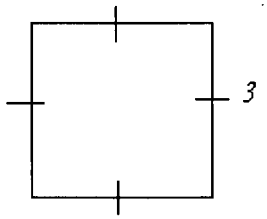


Figure 60: Perimeter Problem

Donny:	How do you work out this one?
Jo:	Well, what sort of shape is it?
Donny:	Small square.
Jo:	It's a square. A small square. And what do you know about a square?
Donny:	It has four sides.
Jo:	Mmmm. And what about the four sides? There's something about them, isn't there? [*Pause*] Something special about them.
Donny:	They all [*pause*]. They all add up together.
Jo:	[*Tentatively*] Ermm, no. There's something special about the four sides of a square, Donny. What is it?
Donny:	They're all equal.
Jo:	They're all equal! So what does that tell you they all are for this one?
Donny:	They all add up to [*pause*]. They're all three.
Jo:	Mmmm. So now you can do it, can't you?
Donny:	[*Starts work immediately*].

Though I can offer other examples (and will, in a moment present a longer example from Karen's classroom which better exemplifies *shepherding*) I include the above example because it was this interaction that alerted me to the need to differentiate between those instances where the teacher appeared to be "teaching" in a more subtle way, and in which there seemed to be some concern that the student *understand*, and those where there was no such concern evident. I want to draw out a critical difference in the definitions of the three themes *showing and telling*, *leading*, and *shepherding*, and that is the inclusion of the word understanding in the definition of *shepherding* (and its deliberate omission in the other two definitions).

To explain what I mean by this I describe below an incidence (of *leading*) concerning Donny's struggle to find the perimeter of the 'deviant' rectangle shown in Figure 61. This 'deviant' rectangle incident became a critical focus for my attention and I rely heavily on

it later in my discussion, so it seems appropriate to produce it in its entirety here. Indented transcript indicates that those words were spoken simultaneously:

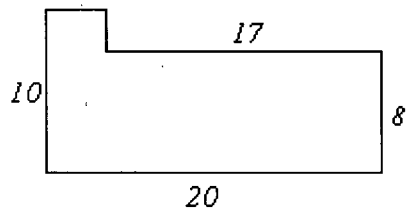


Figure 61: 'Deviant' Rectangle Perimeter Problem

Donny: Miss Towers?
Jo: Yes, Donny?
Donny: What do you put for…what do you put for this bit? [*pointing to the horizontal line section with the missing value*]
Jo: Which one?
Donny: What do you put for this on the top?
Jo: Ah, well now. We've got to work that out.
Donny: Do you think that's ten, or twenty you put?
Jo: No, it's neither of those. Tell me the whole length of the rectangle. The whole thing, of the picture, the whole length.
Sonya: Which one?
Sonya: You've already got the answer down, Donny.
Donny: Eight?
Jo: No, well, no. [*Pause*] Tell me the length of the whole shape.
Donny: [*Picks up his ruler*]
Jo: No. [*Blocking*]
Sonya: No, you don't need your ruler, Donny, you've got it written down.
Jo: Read it off. [*Pause*] What's this distance? [*Pointing to the bottom of the shape*]
Donny: Twenty centim…
Jo: Twenty. Twenty something. We don't know what, but twenty. OK and what's this part? [*Drawing her finger along the section marked 17*]
Donny: Eight.
Jo: No.
Donny: Seven.Seventeen.
Jo: Seventeen. So what's this part? [*Pointing to the horizontal line section with the missing value*]
Donny: Seventeen.
Jo: [*Shakes head*]
Donny: Twenty!
Jo: No, the whole thing is twenty. That much is seventeen. [*Pointing to the length marked 17*]

Donny: Oh, thirty seven.
Jo: [*Shakes head*]
Donny: Is it ten, then?
Jo: No.
Donny: Eight?
Sonya: Can I tell Donny the easy way of working it out?
Jo: No, no, don't guess, don't guess. [*Blocking*]
Sonya: Can I tell him the easy way of working it out?
Jo: No. I want him to get there for himself. The whole distance is twenty to there, do you agree? And the distance to there is seventeen, so how much must that be?
Sonya: Oh, God!
Jo: The whole distance is twenty.
Jo: No, it's all right, Sonya. Donny'll get there.
Donny: Forty. Forty altogether.
Jo: No don't. You're not adding. Don't add. Don't add. Listen, listen to what I say. The whole distance is twenty. That much [*Donny glances away*] Look, look, look, that much of the line is seventeen, so what's that bit? [*Pause*] [*Donny shakes his head*] No idea? Er. Let me see. Supposing we were going, do you agree that that is above here, and that line there joins up to there? [*Drawing in the dotted lines, see Figure 62*] Yeah? So, if that's twenty [*pointing to the bottom of the shape*], what's all that? [*pointing to the top of the newly created rectangle*].
Donny: Well, that's er seven. Seven. Seventeen.
Jo: To there, yes, I agree. So what's this bit?
Donny: Eight!
Jo: No. [*Pause*] It's all right Donny, look again. Watch. All of that from there to there is twenty. And there to there is seventeen. So how much have we got left of the line, here? We've used up seventeen and it's only twenty centimetres long, so how much have we got left?
Donny: Three.
Jo: Three!
Jo: Yeah!
Sonya: Yeah!
Jo: OK.
Sonya: Got it at last, Donny.
Jo: OK. Now, can you do the same for that length? [*pointing to the vertical section with the missing value*]
Sonya: Oh, I didn't…I didn't do that one.
Jo: Try and find what that length is.
Jo: [*To Sonya*] Oh, well, you've missed out that on your perimeter, then, haven't you?
Donny: Ha…ha
Sonya: That means I've got that wrong, then, doesn't it?

Jo: You need a bit more adding on, don't you?
Donny: That must be a three, as well.
Jo: No, not necessarily. Look how much it is. Look, it's ten to there. This bit of it is eight, so what's that bit of it?
Donny: Must be te… ten again.
Sonya: I know. I know what it is.
Jo: Well, altogether the whole bit is ten, isn't it, from there to there.
Sonya: I know what it is, it's…
Sonya: I know what it is.
Jo: Do we agree?
Jo: [*Sonya shows her answer to the teacher*] Yes.
Donny: It's a five. It's a five.
Jo: [*To Sonya*] Oh, hang on. No, no, no, no, no, no, no, no. I didn't see.
Sonya: It's not five, Donny.
Jo: [*To Sonya*] I didn't see. Show me again. Which one?
Sonya: Oh, is that the right one? Yeah.
Jo: Yes. Yes. I can see what you've added on. Yes, that's right.
Sonya: Now, I'd better add that on to that one.
Donny: Is it six?
Jo: No, don't guess. [*Blocking*] We'll do the same again, Donny. Watch the picture. You agree, don't you, that from there to there is ten? [*pointing to the left-hand edge of the shape*] So, from there to there must be ten as well [*pointing to the right-hand edge*].
Sonya: [*Carrying on with a later question*] How do I work this one out? Twenty-four. Twenty-four times twelve.
Jo: [*To Donny*] The whole thing is ten, yeah? From there to there is eight [*pointing to the right-hand edge of the shape*]. So do we agree that from there to there is eight? [*drawing in the dotted line shown in Figure 63*].
Donny: Yes.
Jo: [*Looking at watch and speaking to whole class*] OK folks. Can you be packing away quietly please. Quietly.
Donny: So the answer must be [*pause*] eight.
Sonya: No.
Jo: This bit is eight, I agree [*pointing to the dotted line*].
Donny: Not, necess-"celery"
Jo: [*Laughing*] Celery, yeah. So from here to here is ten. From here to here is eight. So, what's this little bit left over?
Donny: Nine.
Sonya: No!! Donny, how many more numbers do you need to make it up to ten? [*Pause*] From eight?
Donny: Two.
Jo: Two!
Sonya: Right. That's it. You've got it.
Jo: Do we agree?

Donny: Yes.

Jo: I'm not convinced that you're absolutely convinced, but we'll try another like that next time. So, you quickly add the perimeter and then you can pack away.

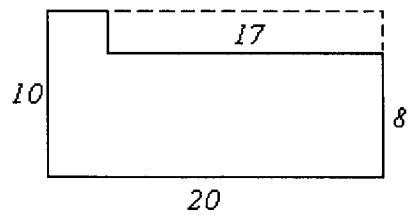


Figure 62: Jo's Annotated Diagram

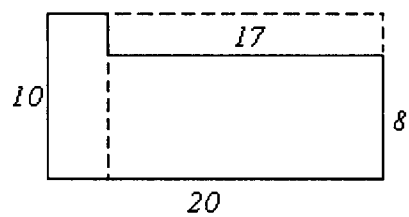


Figure 63: Jo's Annotated Diagram

I also want to note here that this episode shows that, like Karen, I sometimes *blocked* students' attempts to *fold back* to *Image Making* activities (as in, for example, in my refusal to allow Donny to use his ruler to measure the shape, though at the time this was done for a good (or so I thought) reason in that the diagram was not drawn to scale and I was worried that that might confuse Donny further).

As the following excerpt from the Vancouver strand shows, it is often still possible to identify instances of *clue-giving* within interactions where the teacher is *shepherding*, but the overall flavour is one of unhurried progress towards understanding rather than (as might be observed with *leading*) a forceful push towards the right answer. This

interaction occurred during the seventh lesson as Abby and Serena were trying to find the volume of the solid shown in Figure 64.

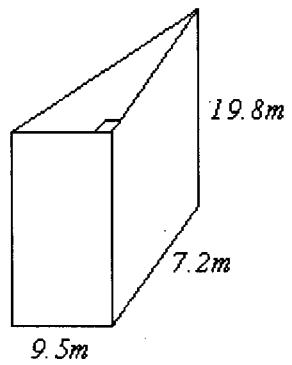


Figure 64: Triangular Prism

Abby had copied the diagram and annotated it (as shown in Figure 65) and produced the following calculations:

$$
\begin{aligned}
b^2 &= h^2 - a^2 \\
&= 7.2^2 - 4.75^2 \\
&= 51.84 - 22.5625 \\
&= 29.3 \\
&= 858.49 \\
&=
\end{aligned}
$$

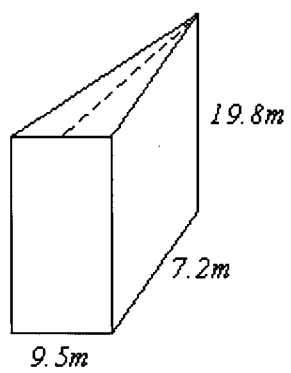


Figure 65: Abby's Diagram

Abby then called the teacher over to help her:

Abby: Wait a minute. Erm Miss Barrett. What are you supposed to like OK I erm squared this whatever. I was trying to get that, right? [*pointing to the dotted line, Figure 65*]

Karen: Uh-huh.

Abby: Am I supposed to get the square root of this or times it? [*pointing to the 29.3 in her calculations*]

Karen: If that is *b* squared then you would have to take the square root of it, yes.

Abby: Of this? [*pointing to the 29.3*]

Karen: Mmmm.

Abby: OK. Wrong answer then. So I have to get the square of that?

Karen: Square root.

Abby: Or whatever. Yeah. OK.

Karen: Can you tell me what you're doing now?

Abby: I'm getting, OK, I have to get the height of this. I'm getting the height for that [*pointing to the triangular face on Figure 65*]

Karen: The height for the triangle?

Abby: Yeah.

Karen: OK. How do you know what's the height of a triangle?

Abby: By doing the py… py… py… something 'gorean theory.

Karen: Pythagorean theorem?

Abby: Yeah.

Karen: Now, draw me a right triangle all by itself. Pretend it's the base of the triangle you're dealing with.

Abby: The base of it?

Karen: Well. Pretend it's the triangle of that prism [*Figure 65*]. Draw me the triangle that's involved in that prism.

Abby: [*Draws Figure 66*]

Karen: OK. Label it.

Abby: What do you mean by label?

Karen: The sides [inaudible] the sides.

Abby: [*Annotates her diagram, see Figure 67*]

Serena: The bottom's the base, right? [*Nobody answers her*]

Karen: OK. Is there anything missing on your diagram that we've got on the blackboard? [*Figure 64*] [*Clue-giving*]

Abby: [*Squinting at the board, then turning to Serena*] I don't know. Is there?

Serena: Is there anything missing?

Abby: Oh, that! [*further annotating her diagram, see Figure 68*]

Karen: What does "Oh, that" thing mean?

Abby: The box.

Karen: What does the box mean?

Abby: It's a right triangle.

Karen: Right. [*Pause*] Otherwise for one thing this wouldn't work, right? [*pointing to Abby's Pythagoras calculations*]

Serena: Oh, man.

Karen: Now. What does that angle being a right angle tell you about [*Karen pulls out a chair and sits down*] Hang on my back's hurting. Erm [*pause*]

Abby: I don't have to use this [*pointing to her calculations*]

Karen: Why not?

Abby: Because I already know the height.

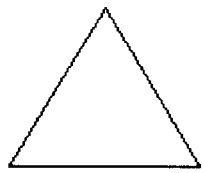


Figure 66: Abby's Triangle

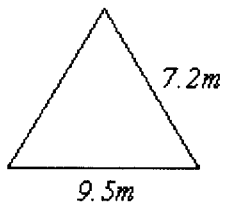


Figure 67: Abby's Annotated Triangle

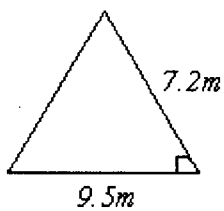


Figure 68: Abby's Annotated Triangle

The above extract serves to demonstrate, I hope, that there was a <u>search</u> for <u>understanding</u> entwining the participants in this interaction that was not evident in the examples of *leading* and *showing and telling* I have presented.

The above excerpt also draws attention to an aspect of the data to which I have so far not alluded in any detail, and that is the agency of the students in influencing the teachers'

patterns of acting. I emphasise, here, that I have not ignored such influences, merely that in addressing my research question I am foregrounding in this analysis the role of the teacher in occasioning mathematical understanding. I want to point out, however, that it is in episodes of *shepherding* that the students' 'voices' are mostly clearly heard, and this may reflect the more attentive listening being enacted by the teachers at these times.

During my analysis of one particular lesson involving Kayleigh and Carrie I noticed one more theme. The lesson was the investigational 'Ponds' lesson, and I have called the first of these new intervention themes *rug-pulling*. During the 'Ponds' lesson Kayleigh and Carrie had advanced quickly beyond the initial starting case of a 'family' of square 'Ponds' and, led by Kayleigh, had quickly produced a correct algebraic expression linking the pond size[27] and the number of paving stones required to surround it. I had then *invited* them to investigate a similar relationship for the family of rectangular ponds with a fixed width of one (see Figure 69), to which they had responded very quickly, and, without drawing any diagrams had created a table of values and another correct algebraic relationship. I next asked Kayleigh and Carrie to investigate rectangular ponds of *any* dimension, and drew a three by seven pond with its border of paving stones as an example (see Figure 70). This resulted in them *folding back* to the use of diagrams again to support their *Image Making*, before creating a table of values and establishing a new expression. I have categorised this intervention as *rug-pulling* to reflect its (initially) destabilising influence on the students' understanding.

[27] Pond size in this context was taken to mean the pond number that I had written beside the first few ponds in the family as I had drawn them on the board, see Figure 27.

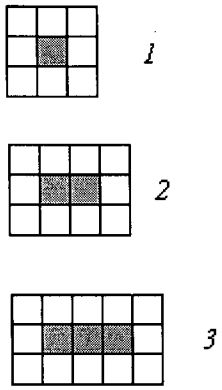


Figure 69: A Family of 'Ponds'

Rug-pulling interventions are those which deliberately shift the focus of the students' attention to something that confuses and requires them to reassess what they are doing. In this case I was deliberately trying to shift attention to the fact that more than one variable was needed to solve this problem (i.e., both the length and the width of the 'pond' were now significant).

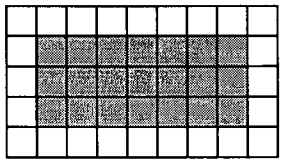


Figure 70: Rectangular 'Pond'

In reviewing the lessons involving Kerry and Graham I noticed an additional theme which I have called *retreating*. I have defined it as a deliberate strategy whereby the teacher leaves the student(s) to ponder on a problem. It is important to note that I would not consider instances where, for example, the teacher is called away by other students, or leaves to deal with a discipline problem on the other side of the room, to be instances of *retreating*. Rather, *retreating* manifests itself as a deliberate choice made by the teacher. As an example, consider my interaction with Kerry, below. This interaction occurred

during Kerry's and Graham's first lesson as part of the study, and concerns Kerry's attempt to find the perimeter of the 'deviant' rectangle shown in Figure 71. Kerry had needed help to work out the length of the vertical line section with the missing value, and after a lengthy discussion with me she had concluded (correctly) that the length was two. I was very unsure that she really understood why it was two, so, in my opening statement below, I *reinforced* the method I had talked her through, and then asked her about the length of the horizontal line section with the missing value:

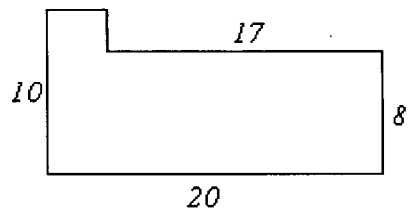


Figure 71: 'Deviant' Rectangle Perimeter Problem

Jo:	Yeah. So we had that part was eight and the whole thing was ten, so the missing bit was two. Now in this case we've got the whole length is twenty and that part is seventeen, so what's the missing bit?
Kerry:	Oh no! Can I add them together?
Jo:	You don't want to be adding them together.
Kerry:	Ermm. [*Pause*]
Jo:	Shall I leave you to think about it for a moment?
Kerry:	Yeah.
Jo:	And I'll come back.

My suspicions were thus confirmed, and it was clear that Kerry did not understand how the lengths of a diagram such as this related to one another, so I decided to leave the problem with her for a while. When I did return sometime later, Kerry told me that the horizontal line section with the missing value was three, but the video reveals that Graham had told her the answer, and when she queried it, had explained to her how to work it out by counting on from seventeen to twenty with his fingers.

The final theme I have developed from my data is *anticipating*. This theme describes the way in which the teachers in this study frequently anticipated the students' thinking and attempted to prevent problems before they occurred, to shield students from error, and commonly, to remove the challenging aspects of a task. For example, during the first lesson Kerry and Graham had reached the questions on area, specifically area of a triangle. I had explained to them how to work out the area of a triangle and left them to complete the first question (Figure 72). I returned a few moments later and asked them to explain to me how they had done it, which Graham did. Then instead of leaving them to try the next question themselves (Figure 73) I proceeded to suggest that they turn the page upside-down to do the next question. There was no evidence that they were struggling to do the problem (as they had not even been given the opportunity to try it), so such an intervention from me was decidedly premature, and (though we will never now know) possibly unnecessary.

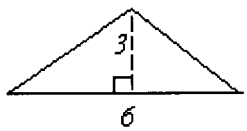


Figure 72: Area Problem

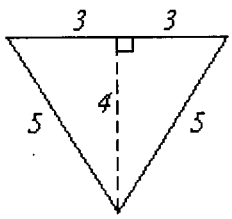


Figure 73: Area Problem

To prevent repetition I have decided not to include here another example of each of the earlier themes to show that they all occurred in the UK strand of data as well as the Vancouver strand, nor to return to the Vancouver data to include another example of the themes developed in this, the fifth, stage of analysis. I want to stress, though, that each theme that appears in my complete listing of themes (which I am about to present) appeared to greater or lesser extent in *both* strands. It is significant for me that each of the themes generated from the Vancouver data also occurred in my own teaching, as captured by videotape during this study. What is fascinating for me is the fact that I had not realised that I made many of these types of intervention until I closely studied another teacher. I believe that my findings are strengthened through my having begun analysis with the Vancouver data, as the majority of my themes have been developed from analysis of another teacher teaching.

A question must be raised here, though, concerning the limit of my themes. How can I know that I have exhausted the pool of possible intervention themes? Clearly, I cannot know that I have, based on a sample of just two teachers, no matter how closely I have studied their behaviour. I am not claiming, however, to have produced an exhaustive list of all possible interventions types. (In fact, the themes I present here are the culmination of a long process of noticing, discarding, and refining themes). I am claiming to have created a collection of themes that describe the nature and variety of interventions made by the two teachers in this study, and I remind the reader that these teachers worked in very different environments. Earlier (Section 3.6.3.1) I made it explicit that I was not creating a comparative study of two teachers, or two educational systems, one Canadian,

one British. In fact, what has surprised me in this study, and what lends further significance to its findings and conclusions, are the similarities, not the differences. The extent to which each of the themes occurs in each strand (in other words, the similarities and the differences) has led me to draw interesting conclusions about the ways in which these teachers occasioned the growth of the students' mathematical understanding. The next chapter opens the discussion about this relationship. Before turning to this discussion, though, there is one more significant stage of the analysis to report here.

4.7 Stage 6: Clustering the Themes

Table 3 shows the complete list of themes I have developed. I have arranged them in the order in which they were introduced in the previous sections of this document, with the exception that I have listed as the first three themes *showing and telling*, *leading*, and *shepherding*. I have done so to begin to draw a distinction between these three themes and the remainder.

Showing and telling	An extended stream of interventions often involving the giving of new information but usually without checking that the students are following the explanation.
Leading	An extended stream of interventions aimed at directing the student towards a specific answer or position, often involving step-by-step explanations. Differs from *showing and telling* by its attempts to involve the students in the explanation through frequent questioning.

Shepherding	An extended stream of interventions directing a student towards understanding through subtle nudging, coaxing, and prompting.
Checking	The teacher is checking for student understanding.
Reinforcing	Giving further emphasis to a significant point (often one already made by a student).
Inviting	Suggesting of a new and potentially fruitful avenue of exploration. More open-ended than *clue-giving* (see below).
Clue-giving	A deliberate attempt to point the student to the correct answer or preferred route.
Managing	Including disciplining, keeping students on task, giving instructions etc.
Enculturating	Inducting students into the language, symbolism and practices of the wider mathematics community.
Blocking	Preventing a student from following a certain path (sometimes preventing a student from *folding back* to *Image Making* activities).
Modelling	The teacher explicitly models her own thought processes.
Praising	Praising individual students, groups or the whole class.
Rug-pulling	A deliberate shift of the student's attention to something that confuses and forces the student to reassess what he or she is doing. Often results in a return to *Image Making* activities.

Retreating	A deliberate strategy whereby the teacher leaves the student(s) to ponder on a problem.
Anticipating	Preventing students from falling into common pit-falls, trying to prevent mistakes before they happen, protecting students from error, or removing the challenging aspects of a task.

Table 3: Intervention Themes

Showing and telling, *leading*, and *shepherding* differ from the others in two main ways. Firstly, they are extended intervention streams whereas the others tend to be one-off, brief interventions (although *modelling* can also manifest as an extended stream of interventions). *Showing and telling*, *leading*, and *shepherding* also are intervention streams which I suggest represent broad teaching styles, rather than strategies (and I suggest *modelling* does not satisfy this description). Teaching styles, as I identify them here, then, are broad practices that appear extensively within a particular teacher's activity. The teachers in this study tended to draw upon one of these styles predominantly and the others less frequently. I, for instance, exhibited mostly *leading* interventions in my teaching, whilst Karen tended to draw upon *showing and telling* and *leading* in fairly equal proportions. A teaching strategy, on the other hand, is usually a brief intervention, and a teacher might have a repertoire of many of these upon which to draw.

By this distinction I am aiming to draw attention to the fact that the three intervention types I call teaching styles (*showing and telling*, *leading*, and *shepherding*) could be used to describe all of the teaching I studied, and many of the other themes fall within their

descriptions. For instance, I often noticed *clue-giving* within a stream of interventions I had characterised as *leading*, and I often noticed *enculturating* interventions within episodes that I had characterised as *showing and telling*. From this point of view I was able to cluster many of the themes under the broad teaching styles of *showing and telling*, *leading* and *shepherding*. Table 4 shows eight of the themes clustered within the three teaching styles I have identified. It will be noticed that *clue-giving* appears in more than one cluster. This theme was used extensively by the teachers in this study, and took a variety of forms from gentle hinting to strong suggesting, the former of which would be associated with *shepherding* and the latter with *leading*.

Showing and Telling	**Leading**	**Shepherding**
Reinforcing	Clue-giving	Clue-giving
Enculturating	Blocking	Inviting
Modelling	Anticipating	Retreating

Table 4: Theme Clusters

It is the remaining themes which appear in only one cluster that have drawn my attention, for it is these which help to differentiate the three teaching styles. For example, instances of *inviting* and *retreating* (associated most often with episodes I have characterised as *shepherding*) may serve to demonstrate that the teacher was trying to reduce the amount of direct *telling* in which she was engaging. Certainly, these strategies were associated with episodes in which it appeared that the teacher was concerned that the students should understand the mathematics, not simply be able to reach correct answers by following

proceduralised rules. On the other hand, episodes of *enculturating* (associated with the *showing and telling* teaching style) seemed to reflect a desire to directly *tell;* to impart information. I enlarge upon these notions in the next chapter.

It will also be apparent that not all of the themes I have developed are clustered within the three teaching styles I have named. This deliberate omission will be addressed in the next chapter when I consider the occasioning of mathematical understanding (Section 5.2.2).

In a form of triangulation I have shared descriptions of the themes I have developed with other researchers and colleagues. In particular I asked these people to consider whether the themes resonated with their own experiences of teaching, and of watching and studying others teach. I offered a number of transcribed excerpts from my data and asked these volunteers to familiarise themselves with the descriptions of the themes (and in particular the three teaching styles, *showing and telling*, *leading*, and *shepherding*) and try to characterise the excerpts according to my descriptions. This activity proved valuable in refining the definitions of the themes and in deciding, in the early stages, which themes were distinct and which could be subsumed within others. In addition, it is significant to note that when, towards the end of the study, I chose transcribed excerpts that I felt were representative of my three teaching styles, all of the people who were asked to try to differentiate between those examples (given only definitions of the three teaching styles as a guide) were consistent in their decisions, and those decisions reflected precisely the choices I had made in characterising the excerpts. This independent

verification, though not a systematic "testing" of my model, does offer some validity to the categories I have created.

Throughout the analysis I had maintained an awareness of my orienting analysis tool, the Dynamical Theory for the Growth of Mathematical Understanding and its three-fold categorisation of teacher interventions, and I had made attempts to associate my themes with the *provocative*, *invocative* or *validating* intervention types. In a significant article, Pirie and Kieren (1992b) suggest that their three categories of interventions are defined by the outcome of the intervention (in other words, by what students are observed to do and say after the intervention) not by the *intent* of the intervention. In fact, it is recognised that many classroom decisions are made at an instinctive level, on the spot, in the heat of the moment and without prior deliberation (Pimm, 1987). Nietzsche, too, cautions us that a person's conscious intention may play a subsidiary role in their action:

> [The suspicion has arisen] that the decisive value of an action lies precisely in that which is *not intentional,* and that all [that in it which is intentional] belongs to its surface or skin - which, like every skin, betrays something, but *conceals* still more (1967, p. 47, original emphasis).

In this formulation teaching is seen as a not-completely-conscious activity, and so for me to put undue emphasis on teacher intent may cloud the essential aspect of what it is that I am engaged in here - the study of how teachers' interventions (as they are enacted in the classroom) occasion the growth of students' mathematical understanding. The significance of the themes I have developed is that they encourage the categorisation of teachers' interventions by *action in the moment*, rather than later outcome. Further, they do not depend on knowledge of (or inferences about) the teacher's intent. Though I have

been privy to the intentions of one of the teachers in this study (myself), and the themes I have developed do, therefore, rest on detailed knowledge of intent (my own), and though as I watch interactions unfold in Karen's classroom and observe interventions that I describe as, for instance, *blocking*, *clue-giving* or *shepherding*, I am acutely aware of why, when I myself make those same types of interventions, (as the data reveals that I do) I choose that path over another, I recognise that I cannot project my own intentions onto others. In developing my themes, then, I have been concerned to provide a starting point in the form of a new language for teachers and researchers to discuss notions of teaching and learning from the point of view of the action in the moment as it can be observed and heard. This is not to devalue my own role as participant, as well as researcher, in this study, but rather to recognise that, as Heap (1992) notes, such a perspective takes as its starting point not what a participant intends or claims to intend, but what a hearer can understand from the participant's actions and words.

The part of speech that I have chosen for naming my themes is also a point of significance. The use of the active verb brings us closer to the lived experience of the teachers, not, I am claiming, because the teachers were in any sense thinking *at the time* that they were engaged in, say, *leading* or *rug-pulling*, but because these terms resonate with the actions and interactions that were captured on tape, and also with my own feelings about some of those actions and interactions as they occurred for me at the time and which I documented in journals and research notes. Such situatedness is one advantage of being as closely integrated into one's own research study as I have been in this one.

Attempting to associate my themes with the categories of *provocative*, *invocative* and *validating* interventions provides yet another layer of detail, enabling us to begin to project from teachers' *actions* the mechanisms by which students come to understand mathematics. Such mechanisms warrant further investigation involving many more teachers than I was able to include in this study, and preferably those willing to engage in self-study, for it is in that situation that it is easiest to gain access to teachers' thinking about their practice. In the next chapter I reveal the progress that I have been able to make during this study in contributing to such a discussion.

CHAPTER 5

RESPONDING

5.1 Introduction

In this chapter I summarise my findings, respond to the research question I posed in Chapter 1, consider what I, personally, have learned as a result of studying my own, and another teacher's, practice, address the issue of what further research my study prompts, and consider the implications of this study for research, teaching and learning.

In Chapter 1 I presented my research question: In what ways do teachers' interventions interact with and occasion the growth of students' mathematical understanding? This is an important question for the field of mathematics education. However, in an area as complex as the study of human interactions one cannot expect to be able to derive a list of clear strategies which teachers can employ to *guarantee* the growth of mathematical understanding. What my research shows (as does that of other professionals working in this field, for example Davis (1996), and Martino & Maher (1994)) is that the situation is not at all simple. It ought to be possible through a study such as this one, however, to develop detailed ways to describe the action of teaching, to discuss how those actions might participate in the learning process, and to make suggestions regarding where further research efforts might be most profitably directed. I have attempted to engage with each of these concerns during this study.

5.2 Responding to the Research Question

5.2.1 Interacting With Understanding

5.2.1.1 Situating the Intervention Themes

There is considerable evidence within the data collected during this study that teachers' interventions do indeed *interact with* students' understanding. It would have been a rather shocking finding had I concluded that they do not, for after all we expect and hope that teachers do influence students' learning. My concern here, though, is the *ways* in which interventions interact with understanding. My research question also contains a further obligation - that I attend not just to the question of how teachers' interventions *interact with* the growth of students' understanding, but also how they *occasion* that growth. This aspect of the question is demanding of more than a description of the complex processes at play in a classroom. It speaks directly to the role of the teacher, requiring a judgement of the strengths and weaknesses of the teaching actions in relation to the actions and evolving understandings of the students, and I address this aspect of the question later (Section 5.2.2).

The themes I have developed to describe the interventions made by the teachers in this study present a summary of the ways in which these teachers *interacted with* their students. These themes are not intended to be an exhaustive list. As I discussed in Chapter 3, the value of this study lies in the richness of its descriptions of two particular classrooms, and I fully expect that by observing other teachers at work in other classrooms I would generate further themes. Further, this study relies, at least in part, on the reader to make connections with situations in their own experience.

In developing themes to describe the interventions made by the two teachers I have studied, I claim to have contributed to Chazan's and Ball's (1995) call for a better language for describing the moves that teachers make in the classroom.

5.2.1.2 Developing a Continuum of "Telling"

Chazan and Ball (1995 p. 16) claim that the term "telling" is insufficiently precise. In doing so they are, I claim, appealing for the kinds of refinements I have been able to develop during this study. They claim that:

> to say merely that the teacher "told" students something is an insufficient description to understand what the teacher did. We need to understand what kind of "telling" it was, what motivated the "telling", and what the teacher thought the telling would do (Chazan & Ball, 1995, p. 23).

In this section I elaborate upon the three teaching styles I have identified (*showing and telling*, *leading*, and *shepherding*) and describe how they contribute to Chazan's and Ball's appeals to elucidate the notion of "telling".

The styles of teaching which lean most heavily towards what Chazan and Ball and others (see, for example, Smith, 1996) refer to as "telling" are those I have called *showing and telling* and *leading*. These styles of teaching, to greater or lesser degree, were dominant in both strands of the data. It is not difficult to see why this might be so if one takes a look at the pattern of interactions which typify *showing and telling* and *leading* compared with, say, *shepherding*, which occurred much less frequently. Typically, *showing and telling* and *leading* involve patterns of interaction whereby the teacher is the predominant speaker, and all exchanges are mediated through her or him. In *showing and telling* the

teacher asks few questions (other than those which affirm that the students are paying attention). The predominant impression is that the teacher is 'delivering' information. *Showing and telling* exemplifies the 'information transfer' model of teaching. In *leading*, the teacher asks frequent questions of the students, but they are generally low-level questions usually of the form Barnes, Britton and Torbe (1969) call pseudo-questions, in other words, those for which the questioner already knows the answer. Here the usual communication pattern is one of initiation (by the teacher) - reply (by the student(s)) - evaluation (by the teacher), a pattern that is recognised to be a common one in mathematics classrooms (Gregg, 1995).

The students have few opportunities to contribute to the development of ideas in either of these situations (when teaching favours *showing and telling* or *leading* styles). Control (of both behaviour and mathematical ideas) is, of course, much easier to maintain through *showing and telling* and *leading* than through *shepherding,* which entails a much more subtle form of teaching, and is associated with less formal approaches to classroom organisation and management. This style of teaching is approaching the "inquiry mathematics" style which Cobb, Wood, Yackel and McNeal (1992, p.577) refer to as orchestrated to help the students "learn with what is typically called understanding" (p. 573). For the two teachers in this study, though, *shepherding* appeared to be an unnatural and difficult form of teaching.

Smith (1996) comments on the enduring nature of the 'teaching by telling' paradigm, noting that one of the major reasons for its longevity, despite reform efforts, is that reform

efforts undermine the base for teachers' sense of efficacy that teaching by telling provides. He suggests that teaching by telling allows teachers to build and maintain a sense of efficacy by defining a manageable content base and by providing clear descriptions of how they should present that content. Reform efforts remove both of these supports. *Showing and telling* and *leading* may, therefore, appeal to teachers as safe options. Nevertheless, there is evidence in this study that for both of the teachers there was a desire *not* to tell. This desire seemed to be at the root of the *leading* and *shepherding* intervention streams. It was far more successful in the *shepherding* intervention streams, and in this sense I see *showing and telling*, *leading*, and *shepherding* interventions on a continuum, moving from more to less telling.

As examples of this continuum I refer the reader to the three excerpts from Karen's classroom presented in the previous chapter (the *showing and telling* episode concerning congruence and similarity (Section 4.4.2), the *leading* episode concerning the triangular prism shown in Figure 22 (Section 4.4.2), and the *shepherding* episode concerning the triangular prism shown in Figure 64 (Section 4.6).

In the *showing and telling* excerpt Karen's only questions to the students (which Karen often answered herself when answers were not forthcoming quickly) required them to 'guess what the teacher is thinking' (for example with her requirement that they search for the "one word" that differentiates similarity from congruence), or were prefaced with so much information that there was little doubt as to the required answer, (see, for example, the sequence leading up to Tasha's offer of "three point five").

In the second excerpt, Karen takes a step towards *leading* rather than *showing and telling*, by asking questions which the students *are* expected to answer. Here the students are expected to follow the presentation but also work things out as they go along (though they are still told what to work out and when to work it out). Though I characterise this episode as an example of *leading*, it is (if I was to further sub-divide my themes) perhaps an example of very strong *leading*.

This *leading* episode shows Karen's attempts to involve the students in the presentation. She is still the predominant speaker, the teacher-student-teacher-student speaking pattern is still preserved, and Karen is strongly directing the path taken to a solution of the problem. She asks some 'testing' questions (to check students knowledge), for example in her opening statement, but she also asks several questions that might be termed 'procedural' which structure the solution (such as "And what's this width of the rectangle?", and "The sides are both rectangles. Are they the same size?"). In a way she is still *showing* the students how to get the answer, but there is evidence that she is concerned to ensure not just that they are listening and paying attention, but that they are, ostensibly, participating in generating the solution.

I ask the reader to return briefly to the example of *shepherding* from Karen's classroom which I presented in the previous chapter (Section 4.6) which completes the continuum I am describing. In this episode, though the teacher-student-teacher-student speaking pattern is still preserved, Karen no longer makes extended presentations of information. She gives clues about critical information (for example "Is there anything missing on your

diagram that we've got on the blackboard?" and "What does that being a right-angle tell you...?") but does not actually *tell*. At the end of the episode Karen checks that Abby understands why Pythagoras' theorem is not needed to solve the problem by asking "Why not?" when Abby says "I don't have to use this".

As Yackel and Cobb (1996) note, teachers have interpreted reform efforts as a call to reduce the amount of direct telling they do, reflected, for example, in Karen's concerns about 'standing and teaching' which she articulated in the first interview in response to a question from me regarding why she had chosen to participate in this study:

> Karen: I think in many ways I'm a very traditional teacher and I sort of often do the stand and teach whatnot and I thought it might sort of get me to do, to try a few different things. And I also thought it might be interesting to see how I really am interacting with the kids. So it was a lot of interest in what I'm doing. You know, I've never watched myself before [*laughs*] so this might be a way of watching myself [*1st interview*].

Certainly, reform efforts convinced me of the value of trying not to tell. Unfortunately, such beliefs can be problematic. After watching the videotapes of myself I was, for a long time, left with the unpleasant feeling that I was failing to embody many of the practices which the reform literature celebrates. The predominance of what I have come to call *leading* interventions in my teaching made me acutely aware that believing in something, and making it a reality in the classroom are two different things[28]. Clearly, the urge to tell tempered by the desire not to constitutes a dilemma for teachers. The evidence I have collected during this study suggests that 'not telling' is very difficult. In many of the *leading* intervention streams the teachers in this study appeared to be trying hard not to

[28] See Davis and Sumara (1997) for a discussion of this dilemma.

tell, but were reluctant to allow students to struggle. They often solved this dilemma by *clue-giving*, *anticipating*, and *blocking* in order to 'help' the students to 'see the answer' as soon as possible.

As I noted earlier (Section 4.4.2) both Karen and I often began interactions with our students with a fairly open question or suggestion (*inviting*), and sometimes *shepherded* briefly, but then rapidly closed-down the students' responses by *anticipating* (particularly prematurely removing challenging aspects of a task), and *blocking* (when it looked as though students were about to make an error of strategy). Heavy *clue-giving* was ubiquitous, and both teachers often quickly resorted to *leading* and then explicit *showing and telling*. Often such patterns were initiated and completed within a couple of minutes - certainly not allowing time for the students to explore and discover. Consider, for example, the interaction below, which is taken from the sixth lesson with Donny and Sonya. Sonya has called for my help to find the perimeter of the shape shown in Figure 74. Donny listens in and joins in, but was not at that time actually attempting this question:

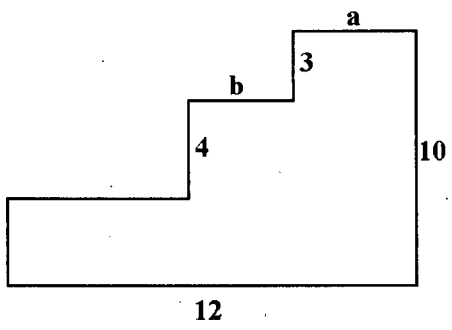


Figure 74: 'Deviant' Rectangle Perimeter Problem

Jo: Tell me something about the *a* and the *b* and the bit that's missing [*pointing to the horizontal line section with the missing value*]. [*The open-ended nature of this first gambit led me to characterise this as an inviting intervention. There followed a long pause, and then I immediately started to narrow down their thinking.*] How long are they altogether? [*Here I'm clue-giving already. There followed another shorter pause before I stressed again the three parts to consider.*] The *a* and the *b* and the bit that's missing.

Sonya: It's got to make up twelve.

Jo: It has. [*Looking backwards and forwards between Donny and Sonya*] So you really, do you need the *a* and *b* and the bit that's missing at all? [*I was about to tell them that they didn't need the three horizontal line sections at all, but I turned it into a question. However there is really only one answer to this leading question.*]

Sonya: No.

Jo: You just call it [*pause*]. [*Another of those fill-in-the-gaps statements.*]

Jo: Twelve.

Sonya: Twelve.

Donny: Twelve.

Jo: So you've got twelve there [*pointing to the upper three horizontal line sections. I'm leading*].

Sonya: And then.

Jo: And you've got twelve at the bottom.

Sonya: Then would you have ten at the other side as well? [*This appears to suggest that Sonya is moving forward in realising that the three vertical line sections can be 'ignored' and treated together as ten, but I'm so intent on leading them through the solution I hear the word "ten" and assume she's talking about the ten shown on the diagram.*]

Jo: And ten.

Donny: And ten.

Jo: And the three little downward bits must also make [*pause*].

Donny: Ten.

Jo: [*Turns to Donny and smiles.*] Ten! [*I was surprised that he seemed to have made the connection. Later evidence reveals that he did not understand why it was ten, but was offering the most likely number based on our conversation to that point. As several of the transcripts of my interactions with Donny show, Donny, was adept at using the teacher's cues to generate answers.*]

Sonya: So that must be…

Donny: Ten add ten add [*pause*].

Jo: Twelve add twelve [*I'm telling!*]

Sonya: [Inaudible] …Forty four, mustn't it? Or have I added it up wrong? No.

Donny: Hundred and forty four.

Jo: Write down what you did.

Sonya: No it won't be a hundred and forty-four, Donny.

Jo:	Not a hundred and forty-four, Donny [*I emphasised the word "hundred". Clue-giving.*]
Sonya:	No. I know what it is.
Donny:	Right.
Jo:	Write the answer to [this question] while, whilst we've just done that together. Remember what I said.
Donny:	Ten.
Jo:	You've got ten and ten on the two sides. [*Telling*]
Donny:	[*Writing*] Plus ten.
Jo:	And what's at the bottom? [*Leading*]
Donny:	[*Writing*] Add twelve add twelve.
Jo:	Right. [*Pause*] Now add that up. [*Telling*]
Donny:	Twenty-four. Forty-four. Forty-four.
Jo:	Write it down then.
Donny:	[*Writing*] Equals forty-four. [*Whispers to Sonya*] Forty-four.
Sonya:	Forty-four. I already know that, Donny!

This episode shows how reluctant I was to allow the students to struggle with the problem. They had called me over for help, and I felt compelled to do something. Unfortunately, the something that I did was, I claim, not the best thing that I could have done. My 'help' in the form of the question "So you really, do you need the *a* and *b* and the bit that's missing at all?" was premature. Donny especially (as we saw in the extended episode given in the previous chapter (Section 4.6) had not understood the relationship between the various lengths of a 'deviant' rectangle in numeric terms, and so certainly was unlikely to be ready for the kind of pushing I was doing in the episode above, though, of course, I was primarily directing my interventions to Sonya who had asked for help.

There is further evidence throughout the data that the teachers in this study were making some efforts not to directly tell students how to solve the problems. For example, the instances of *retreating* in the data, although fairly infrequent, suggest that the teachers were attempting to return control of the learning to the students and to offer them the

space to reach conclusions without direct intervention from the teacher. I offered one such example in the previous chapter (Section 4.6). The fact that instances of this theme occurred so rarely in this study lends further weight to my claim that moving to a style of teaching in which telling (in one form or another) is reduced is a difficult process.

There is evidence that the teachers in this study may have been struggling with the question of whether (and how much) to tell students *as they interacted with them* in the classroom. Consider, for example, this interaction between Karen and her students Abby and Tasha concerning the problem shown in Figure 75. The problem required the students to find the angles marked 4, 5 and 6. The students had been working for several minutes on their worksheets:

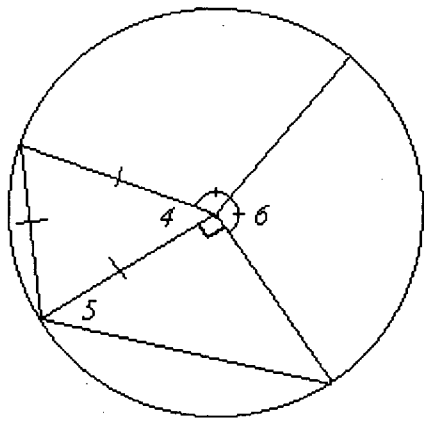


Figure 75: Angle Problem

Abby: Miss Barrett? How do you do these circle things?

Karen: How do you do these circle things? What do you mean how do you do them? [*Checking*]

Abby: I don't know how to do them.

Tasha: You just have to find the angles.

Karen: Uh-huh. So you take a look at it, and what do you notice here? [*Drawing the outline of the equilateral triangle marked on the diagram with her pencil.*] [*Shepherding*]

Abby: It's erm.

Tasha: Right triangle.

Abby: It's er. Yeah. No. Equilateral.

Karen: This one here. It's equilateral, isn't it? They're all the same size, right?

Abby: Yeah.

Karen: So what do you know about the angles in an equilateral triangle? [*Shepherding*]

Abby: Oh! Sixty.

Karen: Yeah.

Abby: OK [*This may have been an appropriate moment to retreat, but Karen continued with her explanation even though Abby seemed to have made progress and the two students may well have been able to complete the problem without further intervention.*]

Karen: And so each of those guys is sixty. [*Pointing to the angles of the equilateral triangle.*]

Abby: And the ninety, and then divided by two. OK. And then…

Karen: [*Interrupting*] OK. And so you can get that one. [*Pointing to angle 6.*] And then what do you know about this point? [*Pointing to the centre of the circle.*]

Abby: Three sixty.

Karen: Three sixty. You know those two. [*Pointing to the marked right-angle and angle 4.*] You know those two are equal. [*Pointing to angle 6 and the angle marked as equal to angle 6.*]

Abby: Uh-huh.

Tasha: Divide by two.

Abby: Yeah. And then just divide by two.

Karen: So the circle's just there to confuse you. Except for one thing. Why do you need to know that it's a circle? [*This may have been an attempt to check that the students had understood, but I suggest that it is, instead, anticipating the problems that the students might have with finding angle 5 which requires recognition that it is within an isosceles triangle.*]

Abby: Three sixty? [*Pause*] No. You don't need that.

Karen: 'Cause it wouldn't matter. But which angle do you need to know that it's a circle for? It didn't matter for this one, did it? [*Pointing to angle 4.*] [*Anticipating an incorrect answer that might be forthcoming.*] For the angle four because this was an isos, an equilateral triangle. So you didn't need it there. How about this angle five? [*Telling the answer to her own question.*] How did you get angle five?

Abby: I don't know.

Tasha: Ninety [*pause*]. [*Tasha is starting to articulate a response but Karen doesn't capitalise on this.*]

Karen: Well what size is angle five? Did we jump too quick there? [*An indication that Karen is considering that she may have left the students behind in her explanation.*]

Abby: No I don't know what number angle five is. Errr.

Tasha: Isn't it like one eighty and then ninety and then divide by two?

Karen: Why is it divide by two? [*Checking, but it seems to make Tasha believe she is wrong.*]

Tasha: Oh no it's not [inaudible] thing.

Abby: I don't know how to get, how to get five.

Karen: [*To Tasha*] I will agree with you. You're right. But why is it divided by two?

Tasha: Because erm this is one eighty.

Karen: Right.

Tasha: And that's ninety, so that's ninety left.

Karen: Right.

Tasha: And then you divide by two.

Karen: Now how do you know you can divide by two? [*I think this is becoming more than just checking. It seems Karen is trying to get Tasha to explain her thinking in order to help her clarify her ideas, and also in order to help Abby who has declared that she does not know the size of angle 5, and so I would characterise the continued effort to persuade Tasha to articulate her thinking as shepherding. Karen now, though, starts to lead.*] We only really know that these two [*angle 5 and the unmarked angle in that triangle*] add up to ninety.

Tasha: Yeah. That's why [inaudible].

Karen: What's this figure? [*Drawing the outline of the circle.*]

Abby: Circle.

Karen: And what do we know about radii? [*Pause*] This would be a radius, wouldn't it? [*Pointing to the left-hand side of the triangle.*] [*Leading*]

Abby: It would be a ninety degree angle…

Karen: [*Interrupting*] No. What do you know about each of these lines here? [*Indicating the two radii that form part of the triangle.*]

Tasha: They're a radius.

Abby: They're half of the diameter.

Karen: And how are they related to each other?

Abby: In degrees? I don't know.

Karen: No. Not in degrees. The radii is a line segment. A radius is a line segment, right?

Abby: Yeah.

Karen: And if I draw a radius here and I draw a radius here [*drawing two additional radii onto the diagram*] what could you tell me about the lengths of them?

Abby: They're the same.

Karen: They're the same. So therefore this radius is equal to [*pause*] this radius. [*Pointing to the two radii forming part of the triangle containing angle 5.*] [*Telling*]

Abby: Uh-huh.

Karen: Therefore those two angles…

Abby: Are the same.

Karen:	...have to be equal. Therefore you can divide the ninety by two. [*Telling*]
Abby:	Oh!
Karen:	So anytime you have a circle you're gonna be
Abby:	[*Interrupting*] So they're all...
Karen:	Yep.
Abby:	...They're all, oh! OK. I get it now.
Tasha:	So.
Abby:	So now they're all like erm what's that word again? Erm that word where the two sides are the same.
Karen:	I i i start with an i. [*Clue-giving*].
Abby:	Isosceles!
Karen:	Yes!
Abby:	Yeah. They're all isosceles. OK. I get it now.

Though in this episode Karen was still predominantly either *leading* or *telling* there is evidence that she was concerned that the students may not have followed her explanation. She attempted briefly to elicit the required information from them (by pressing Tasha to explain her thinking) but then could not seem to resist the urge to 'straighten them out' and make things 'perfectly clear' when Tasha was unable to adequately (for Karen) articulate her understanding.

I claim, though, that reform efforts can succeed in changing teachers' classroom practice if they succeed in problematising teachers' own practice from the point of view of the teachers' themselves. In other words, reform efforts stand the greatest chance of success if the teachers themselves identify problems with their current practices. In achieving this first step, I want to stress the value of watching oneself teach through use of videotaping. My own interest in considering the value of research to my own practice was awakened initially by my participation in someone else's study. I was intrigued enough by that participation to consider studying my own practice, and then to progress to studying the

practice of others. Through this came a growing belief in the applicability of research findings, something that has continued to prove a difficult 'sell' to teachers. In considering for a moment the influence of participating in research on one's practice, I offer here an interaction between myself and Kayleigh and Carrie concerning the same question as I presented above in the interaction with Donny and Sonya. For convenience I reproduce below the diagram to which this interaction relates (Figure 76)[29]. This interaction occurred several months after the interaction with Donny and Sonya, during which time I had watched the videotapes of my teaching. I suggest that that there is a qualitative difference in my approach in the two episodes, and that I am beginning to show evidence here of a more subtle form of teaching:

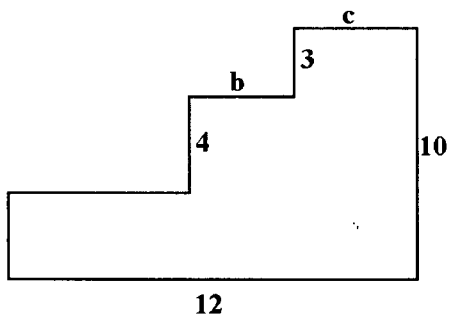


Figure 76: 'Deviant' Rectangle Perimeter Problem

Kayleigh: Here we've got. Ah. Problemo. We need to take *c* and *b* off twelve. No it's not, sorry, don't worry about me I'm being totally too complicated.
Carrie: What happens if we go ten…
Kayleigh: [*Interrupts*] We don't know that length so we don't know what to do. [*Pointing to the horizontal line section with the missing value.*]
Carrie: We do 'cause that's twelve and then that's like what you said. Oh, yeah I see what you mean, see what you mean. Oh, it's just [*pause*].

[29] It will be noted that the letter a on Figure 74 has been replaced by the letter c in Figure 76. Some children in Donny's and Sonya's class had (because the worksheets were hand-drawn) mis-read the letter a as a number 9. In reponse to this difficulty (and other similar mis-readings of, for instance, the letter z and the number 2) I had revised the worksheet for subsequent classes. The essence of the problem, is, however, the same in both versions.

Kayleigh: Well, how are we supposed to know what *c*, what the value of *c* and *b* is? [*Both raise their hands.*] Miss Towers! We know what we have to do but we're stuck. [*I approach their desk.*]

Carrie: Because it doesn't say how much.

Kayleigh: Because we know we've got to take *c* and *b* from the value of twelve to get that [*pointing to the horizontal line section with the missing value*] but how do we write that 'cause we don't know the value of *c* and *b* to take away from twelve.

Jo: Say that again, you know you have to [*pause*]. [*Checking*]

Kayleigh: Take *c* and *b* from the value of twelve.

Carrie: Take *c* and *b* from the value of twelve.

Jo: You do. [*Confirming tone.*]

Carrie: [*Interrupts*] But we don't know how much.

Kayleigh: But how do we know how much *c* and *b* are?

Jo: Stop there. Write down for me the sum that you would do then if you happened to know what *c* and *b* were. [*Note that I do not, as I did with Donny and Sonya, ask them how long the b and the c and the bit that's missing are altogether. Instead I start from their statement that they want help finding c and b and begin to assist them to find out how they might express c and b in terms of twelve. Though this is clearly going to lead to a complicated expression for perimeter of the whole shape, I allow them to proceed, whilst hoping for opportunities to shepherd them to the more efficient method.*]

Kayleigh: Twelve take away *c b*. [*Pause*] *c* and *b*.

Jo: *c* and *b*. Write that down.

Kayleigh: So, if we did all the other sides first. So if we added it all together and then we put take away *c* and *b*. So, if we make that all twelve there. [*Pointing to the three upper horizontal line sections.*]

Jo: It certainly is! It certainly is all twelve, isn't it? The *c* and the *b* [*pause*]. [*My emphasis here was intended to be a hint (clue-giving) that this was a significant statement, but as the rest of the interaction shows the students did not appear to recognise this significance, and I pursue this route only once more.*]

Kayleigh: [*Interrupts*] Wait a minute, if we did twelve add ten add four add three is what? Is sixteen. Nineteen.

Kayleigh: Twenty-nine.

Carrie: Twenty-nine.

Kayleigh: So, if we had twenty-nine add…

Jo: Mmmm.

Kayleigh: …Add one. Well, let's just say add twelve. No, that doesn't make sense, does it? We need to make, add another twelve to [*pause*].

Jo: Twelve to where? Which bit's twelve? [*This was again intended as a clue, but the students interpreted it as a check*].

Kayleigh: Along there [*pointing to the upper three horizontal line sections*].

Carrie: Along there [*pointing to the upper three horizontal line sections*].

Carrie: 'Cause if you'd pushed it all up. [*The students obviously recognise that the three horizontal line sections are equivalent in length to the base of twelve, but it seems that they do not see this information as 'sufficient' and are using this fact only to calculate the missing horizontal length.*]

Kayleigh: And then we're going to take away *c* and *b*.

Jo: OK, yes, that's fine, I'm happy. [I *abandon my attempts to get them to 'see' that they don't need to work out the missing length.*]

Kayleigh: Twelve and twenty-nine. And twelve and twenty-nine is going to give us our grand total which is?

Carrie: Forty-one. [*This is 12 + 10 + 3 + 4 + 12*].

Kayleigh: Forty-one. Ah, and then you have to take away *c* and *b*. So, it's forty-one take away one *c* plus [*pause*]. [*Writes 41 - 1c + 1b*] Does that make sense?

Jo: Well, you've taken away *c* but added on *b* at the moment. [*We hadn't come across this kind of manipulation before and I had no idea whether Kayleigh and Carrie would be able to respond appropriately. Though I am telling them that their expression is wrong, I resist the urge to tell them what to do about it.*] [*Kayleigh adds brackets to her expression to give 41 - (1c + 1b)*] Yeah, that's more like it. Can you see what she's done?

Carrie: Oh, yeah, yeah.

Jo: I'm fairly happy so far but what about this little length here? [*Pointing to the vertical line section with the missing value.*]

Kayleigh: Well, we've just taken that away.

Jo: No, what you've found [*Despite my best efforts I'm starting to lead.*]

Kayleigh: [*Interrupts*] But how are we supposed to find out that length?

Jo: Which, what, where? [*I was pretty sure Kayleigh hadn't noticed that there were two missing lengths on the diagram, so I claim that this is shepherding.*]

Kayleigh: That length along there. [*Pointing to the horizontal line section with the missing value.*]

Jo: That length? [*Pointing to the vertical line section with the missing value.*]

Kayleigh: No, that. Oh, right, I see! *That* length.

Jo: That little length.

Kayleigh: Well, you have to take four and three away from ten.

Carrie: Oh, yeah!

Jo: Yes, you do.

Kayleigh: And four and three is…

Kayleigh: Seven.

Carrie: Seven.

Kayleigh: Take away from ten is…

Kayleigh: Three.

Carrie: Three.

Kayleigh: So, that would be three. This is not a scale drawing! Three. And then we need to add three to our total.
Carrie: Yes, my dear!
Kayleigh: Add three. Add three? [*To me*].
Jo: Hang on, let me look.
Carrie: Because seven…
Jo: [*Interrupting*]. Yeah, that bit's certainly three. OK, so you've added together all the numbers that you can see plus another three. So, you could add the three on, couldn't you? [*Leading*] [*I was trying to get them to collect together some of the numbers in their expression which now read 41 - (1c + 1b) + 3*]
Kayleigh: What, so add ten and three's thirteen add seventeen add twelve is what? Twenty-nine. And we need to add another twelve to get that [*pointing to the horizontal line section with the missing value*] and take away *c* and *b*, and four.
Jo: Gives you that.
Kayleigh: Gives you that. And you have to add three, and that's your total.
Jo: But you've not added on the *c* and the *b* yet, have you? [*Telling*]
Carrie: Yeah!
Kayleigh: Yeah, but if we had…
Jo: [*Interrupts*] No hang on, let's think. All you've done is you've added all the number bits that you could…
Kayleigh: Oh, right. I see.
Jo: …including this three, and then you've found that missing length [*pointing to the horizontal line section with the missing value*] which is the twelve take away the *b* and the *c*. [*Strongly leading*]
Kayleigh: So, we need to add. And now we need to add…
Jo: So we still need…
Jo: The *b* and the *c*.
Kayleigh: The *b* and the *c*. [*Kayleigh completes her expression to give 41 - (1c + 1b) + 3 + 1b + 1c*].

For me, there are many questions which emerge from this episode, including the issue of my acceptance of their final expression as correct and my (and Kayleigh's) 'silencing' of the quieter Carrie, however, these issues are of less importance to the point I am trying to make here (though they are important in terms of my teaching practice, and the students' learning)[30]. The central issue here is the dilemma of trying not to *tell*. The reluctance to

[30] I have addressed these issues elsewhere (see Towers, 1996).

let students struggle for long (or at all) without 'helping' characterised much of the teaching in both of the classrooms I studied. Edwards and Mercer (1987, p. 126) describe this dilemma as one of having to "inculcate knowledge whilst apparently eliciting it", a tension which is evident in these interactions with Kayleigh and Carrie.

Though in this episode the interactions still mostly take the familiar form of teacher-student-teacher-student, notice how much more of the talking is done by the students (albeit one student in particular). If this was a play the role of 'teacher' would not be the leading one. Our attention focuses on Kayleigh (particularly when watching the video as opposed to reading the transcript, which I have annotated here to redirect attention to the interventions made by the teacher). It is Kayleigh who does much of the 'explaining' in this episode. Berry (1990, p. 49) talks of one of his former teachers as one for whom "when he spoke to the class the class felt spoken to. You did not feel that he was glancing at himself." This image reflects the difference in positioning I see between the teacher who is *showing and telling* (and to some extent, the teacher who is *leading*) and the teacher who is *shepherding*. The focus of the viewer shifts in these opposing situations from the teacher (in *showing and telling*) to the students (in *shepherding*).

My own worries about the predominance of *leading* interventions in my teaching, and Karen's articulated concerns about her "traditional" style, suggest that some teachers are concerned about their practice in the light of current ideals, and clearly want to find new ways of supporting students' efforts to learn with understanding. Such desires manifest themselves in *shepherding* intervention streams, and in some of the organisational

practices evident in the UK strand of this study. For instance, it was common practice in my own classroom for the students to work in groups, and an analysis of the tapes shows that their talk-time in class far out-weighed my own. Strictly speaking I ought to point out that I did talk for the majority of most lessons, but my talk was usually with individuals or with two or three pupils at a time. I did not often speak to the whole class at once, and when I did it was usually to introduce the day's task (such as in the 'Ponds' activity). Though I do not claim that such a style, in and of itself, produces students who understand the mathematics they are learning, I do suggest that such a classroom organisation is more conducive to creating an atmosphere in which *shepherding* intervention streams can thrive, and, as I shall demonstrate later (Section 5.2.2.6), wherever it manifested, the *shepherding* teaching style led to growth of students' understanding.

It is not without significance that I found no evidence of *shepherding* interventions when the teachers in this study were interacting with the whole class at once. This is not to say that it cannot be done. The inquiry classrooms described by Cobb and his associates in numerous articles (see, for example, Cobb, Yackel & Wood, 1992; Wood, Cobb, & Yackel, 1991) and by Ball (1990) and Lampert (1990) show that practices similar to *shepherding* are possible with a large group, though it is interesting to note that none of the exemplary classrooms described in these studies are secondary school classrooms[31]. The patterns of discourse in the classrooms described in the studies reported by Ball,

[31] I am aware of a small number of similar studies done in secondary classrooms, and on college-level courses, such as those recorded by Romagnano (1994) and Schoenfeld (1991), but reform efforts which discourage *telling* seem to have been less successful in the secondary arena than in the elementary one.

Lampert, and Cobb *et al.* are, I would suggest, perhaps even further along the continuum I have described. I would characterise them as representing a style which gives an even less prominent role to *telling* than does my theme of *shepherding*. Hence, I am not suggesting that my themes *showing and telling* and *shepherding* are at the extremes of the continuum. I suggest, though, that for teachers who are attempting to move from a traditional teaching style in which *showing and telling* predominates to a style closer to that advocated in recent reform documents and research literature, then a re-orientation of their teaching activity to incorporate the kind of style I have called *shepherding* would represent a significant step along the continuum. Here I find the notion of the theme clusters I have developed to be useful. Studying one's own teaching using the themes I have developed, and identifying which of the clusters best represents one's teaching would, I claim, enable teachers to notice their oft-used moves, and recognise their predominant teaching style, and also provide the support for a re-orienting of their teaching style towards (and beyond) *shepherding*. A study which assesses the value of my themes for other teachers who are interested in reforming their practice, and which also assesses teachers' success in changing their teaching style in the classroom would constitute an important extension to my study.

5.2.2 Occasioning Understanding

5.2.2.1 Understanding Occasioning

In this section I respond to that part of my research question which asks "In what ways do teachers' interventions *occasion* the growth of students' mathematical understanding?" Though it should be clear by this point that there are many ways in which the teachers'

interventions have *interacted with* the students' growth of understanding, it is a much more difficult task to try to account for how they have influenced understanding. I have chosen to ask how the interventions *occasion* understanding, rather than how they influence it, due, in part, to the philosophical orientation which underpins this thesis, and which I described in Chapter 2. I see occasioning as a situation in which a growth of understanding is *allowed for*, not caused. To investigate the *occasioning* of understanding, then, foregrounds my conviction that teaching does not cause learning, or, for that matter, understanding. The episode I described earlier (section 4.6) concerning Donny's search for the missing values on the 'deviant' rectangle shown in Figure 61 stands as testament to this statement. In that extract, though most observers would, I think, agree that I was engaged in the act of teaching, however it might be defined[32], Donny was clearly not learning (or at least not learning with understanding). Lampert (1988) and others have reminded us that teaching (whether intentionally or not) is often concerned not merely with content but with teaching what a lesson is and how to participate in it, and I note that in the episode I have just mentioned Donny was demonstrating that through my "teaching" he had "learned" how to participate in my lessons - you make suggestions (which should not be called guesses) until eventually you hit upon the one that satisfies the teacher.

Many studies which address what might, on the surface, appear to be questions of how the teacher shapes the growth of understandings in the classroom (for instance, many of the studies reported by Cobb and his collaborators, see for example, Cobb, Yackel &

[32] See Moran (1997) for a discussion of the varying understandings of the term "to teach".

Wood, 1992; McClain & Cobb, 1997) tend to focus on shared and collective understandings of the participants, and pay little attention the crucial role of specific interventions made by the teacher and how they might occasion growth of understanding for particular pupils. Such studies of classroom cultures are valuable in terms of contributing to our understanding of how the mathematics is shaped, and how meanings are constructed, but they rarely address the particular focus of my interest here. In this section I attempt to draw together my findings concerning how the teachers in this study contributed to individual students' growth of understanding.

I have addressed this aspect of my question primarily through an integration of the students' mappings with my detailed notes concerning the teachers' interventions, and have also returned many times to the raw data of the videotapes to confirm my own understandings of the events as they evolved. I began by considering those moments (as shown on the students' mappings) when the students *folded back* or extended their understanding. In other words, I focused on *changes* in the students' understandings. I then considered the events leading up to those changes, and attempted to synthesise from all of these events some further patterns.

5.2.2.2 The Unreasonable Elusiveness of *Formalising*

It is clear from the students' mappings (Appendices 1, 3, 5, 7, 9, 11, and 13) that many of the students reached the *Formalising* level infrequently, or not at all. Relying on inner levels of understanding, though, the students were able to answer many of the problems set on the formal written tests in both strands. For example, Donny and Kerry were both

able to answer questions based on finding the (non-algebraic) perimeter of rectangles and squares during the tests they were given, suggesting that their understanding perhaps could have been mapped as *Formalised* for this aspect of mensuration. Given only their (correct) written responses to the test, in fact, one might assume that their understanding was adequate. Indeed, it *was* adequate to answer the questions as presented. When, however, these students were asked to apply their understanding in a new area (such as finding the non-algebraic perimeter of a 'deviant' rectangle) their level of understanding was found to be inadequate.

Conversely, students (such as Kayleigh and Graham) who in their early work on perimeter demonstrated a *Formalised* understanding were better placed to apply that understanding to the 'deviant' rectangle situation. A level of understanding limited to *Image Having* is therefore seen to be adequate for the correct completion of *familiar* problems in a 'standard' paper-and-pencil assessment of the form used most often in formalised testing regimes in high school. *Formalised* understanding is needed, however, if the assessment requires students to *apply* their knowledge in an unfamiliar situation.

Limiting the scope of school assessments to a test of students ability to use standardised procedures in familiar situations (which dictates that teaching be nothing more than information transfer) reduces the value of the assessment, obviates the need for more subtle (and therefore difficult) teaching strategies, such as *shepherding*, and, ultimately, impoverishes the students' mathematical education. Despite this, the evidence in this study is that the teachers' aims were to help students to reach the *Image Having* level (so

that they could be successful in school examinations), not to strive for other levels of understanding. Such evidence, I claim, is manifest in the prevalence throughout the data of the *showing and telling* and *leading* teaching styles, and of strategies which I have collected under the themes of *anticipating* (in particular when this concerned removing challenging aspects of a task), *blocking* (of students sense-making efforts), and in some cases *modelling* and *enculturating*.

5.2.2.3 *Inviting* Interventions and the Growth of Mathematical Understanding

There are a number of occasions on which an *inviting* intervention appears to occasion for a particular student a movement outwards through levels of the Dynamical Theory for the Growth of Mathematical Understanding. For example, in the 'Ponds' investigations I asked students to look for patterns in the data they were generating. This request, which I have characterised as an *inviting* intervention, resulted in movements to outer levels of understanding for Kerry (points 9-10, Appendix 11), Graham (points 15-16, Appendix 13), Carrie (points 8-9, Appendix 9) and Kayleigh (points 11-12, Appendix 7). I suggest, therefore, that *inviting* interventions may be *provocative* in nature.

This might be accounted for by recognising that *inviting* interventions, like Simmt's (1996) "variable-entry prompts", are open-ended interventions that welcome a variety of interpretations by the hearer, and therefore provide scope for a range of movement in terms of growth of understanding. For instance the *inviting* intervention I have just described (asking students to search for patterns in their data) provoked a range of responses from the students, including relatively conservative outward movements for

Kerry (*image doing* to *image saying*), and Graham (*image doing* to *image saying*), and more expansive movements for Carrie (*image doing* to *property predicting*) and Kayleigh (*image doing* to *method justifying*). I recognise such expansive movements (inwards or outwards) to be significant features of students' mappings, and have defined them as *jumps in understanding* when the movement is through at least one level (so a movement from *image doing* to *image saying* would not be counted as a *jump*).

Jumps in understanding occurred in both outward and inward directions. Kayleigh made the most *jumps*, whilst Donny and Kerry made none at all. Outwards jumps such as the ones I have just described tend to be associated with the providing of opportunities (through *inviting*) for students to demonstrate specific competence at higher levels. For instance, outward *jumps in understanding* sometimes represent moments where students are able to demonstrate that they have a strong understanding of one part of a topic area (such as Abby's strong understanding of how to find the volumes of certain solids such as prisms) within a wider topic in which the student's understanding is still growing (i.e., volumes of all solids). It is significant that the most able students (Abby, Tasha, Kayleigh) tended to exhibit *jumps in understanding* (both outward and inward) more frequently than the less able students (Donny, Kerry), with students of average ability (Carrie, Graham) placed somewhere between the two extremes.

A second feature of the students' mapping which I want to draw out here because of its link to *inviting* interventions is a feature I have called *incremental extending*. I define *incremental extending* to be occasions on which the mapping shows that the student could

be identified at each of a series of stages of the model without (or with little) *folding back* to inner levels[33]. Abby was the only student who clearly exhibited this type of growth (though it occurs in a limited way in several of the other students' mappings). For Abby this occurred in a pronounced manner during the third interview whilst she worked on the development of her ideas about wasted space within the packages (80-84, Appendix 1). Though only Abby clearly exhibited this pattern of extending I note it because it appears to highlight a very particular mode of growth of understanding. The viewer's impression of this area of Abby's mapping is that the points of growth are 'stepping stones' to more sophisticated understanding.

With only one example of this feature in the data it is difficult to suggest firm connections between the 'teacher's' activity (this was an interview situation) and the student's progress. Almost my only intervention (other than to suggest to Abby and Tasha that they use a clean piece of paper) in this whole sequence was at the beginning when I suggested they explore the *actual* wasted space within each package, as up until then their investigations had centred on the percentages, and they were clearly misinterpreting what the percentages meant. I would characterise my intervention here as *inviting*. This intervention initiated the chain of thinking which resulted in Abby's growth of understanding concerning the meaning of the percentages she had already generated, and resulted in her claims concerning the quantity of packaging involved in the designs, and the surprisingly (for her) large amount of wasted space. Further research would be needed

[33] I want to note that I have also identified a corresponding pattern of *incremental folding back*, which occurs most noticeably in Abby's mapping (points 26-29 and 92-95), however, as I was unable to clearly account for this pattern in terms of the teacher's interventions I have elected not to discuss it here.

to establish the significance of *inviting* interventions in relation to *incremental extending* growth patterns. In particular, studies of other mathematically-talented students may be revealing in this respect[34].

I claim, then, that *inviting* interventions predominantly promote the growth of students' mathematical understanding.

5.2.2.4 *Rug-pulling* Interventions and the Growth of Mathematical Understanding

Rug-pulling is the intervention theme which most clearly appeared to contribute to students *folding back*. Perhaps the strongest evidence of *rug-pulling* in the data is the episode I described earlier (Section 4.6) during Kayleigh's and Carrie's exploration of the 'Ponds' problem, however there are many more, such as the following example from the interview with Kayleigh. It will have been noted from the lengthy episode transcribed above concerning Kayleigh's and Carrie's solution method for the question shown in Figure 76 (Section 5.2.1.3) that Kayleigh understood that the three upper horizontal line sections added up to twelve, and further that she did not see that information as sufficient. In order to problematise for Kayleigh this inefficient solution route I offered the following problem (Figure 77) in the interview, asking Kayleigh to suggest an expression for the perimeter. I suggest that posing this problem was a *rug-pulling* intervention. As the following transcript shows, Kayleigh was initially confused by this question and thought it was not possible to solve it with the given information:

[34] Pirie and Kieren (1994a) have also documented (though without identifying a novel label for this phenomenon as I have done here) a similar pattern of growth for one student (a mathematics graduate).

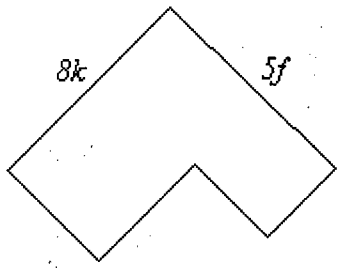


Figure 77: Interview Perimeter Problem

Kayleigh: Erm. [*Pause*] Oh. [*Pause*] How do we know how much that is? [*Pointing to one of the line sections with a missing value.*] [*Pause*] It doesn't tell me how much that is so I don't know how much to take off.

Jo: You can work it out. [*Shepherding*]

Kayleigh: Can I?

Jo: [*Pause, realising that with my last statement I might have led Kayleigh to believe that she can work out how much she needs to take off, which, of course, she can't.*] Do you need to take anything off? [*Clue-giving*]

Kayleigh: [*Pause*] Not really. No. 'Cause you could do ten *f* add sixteen *k* [*writing 10f + 16k*]

Jo: OK. Good girl. Well spotted. Good. Good. How did you know it was ten *f*? [*Checking*]

Kayleigh: Because, erm, there would be two sides. This, these two sides would equal the same as five *f*. [*Pointing to the two line sections opposite to the side marked 5f*]

Jo: Good.

Kayleigh: These two sides would equal the same as eight *k*. [*Pointing to the two line sections opposite to the side marked 8k*]

Jo: Brilliant! Good thinking!

Kayleigh was clearly initially stumped by the problem her teacher had set. Such problems, if chosen carefully and delivered at critical moments, have the potential to encourage students to *fold back* to work at inner levels of understanding in order to build a 'thicker' image of the concept (and hence *folding back* is recognised as contributing to

growth of understanding[35]). In other words, *rug-pulling* interventions appear to be *invocative* in nature and may support the fostering of deeper understandings. I suggest, therefore, that *rug-pulling* has the potential to be a powerful pedagogical strategy in the mathematics classroom.

5.2.3.5 *Blocking* Interventions and the Growth of Mathematical Understanding

I wish to voice my concern, here, regarding *blocking* interventions, which were commonly associated with the *leading* teaching style. As was demonstrated in the episode where I was seen to *block* Donny's attempts to use his ruler to measure the diagram (Figure 61, section 4.6), such *blocking* stifles students' sense-making efforts (recall that this was not the only one of Donny's strategies that I *blocked* in that episode. I also discouraged Donny from guessing, and I *blocked* his use of units, thereby discouraging him from (making sense of the situation by) associating the numbers on the diagram with "real" measurements). As I have mentioned already, I view such *blocking* by the teacher as *blocking* students' own efforts to *fold back*. In light of the discussion above concerning *rug-pulling* interventions (which prompt *folding back* and have been seen to support the growth of understanding), I question the practice of *preventing* students from *folding back*.

I suggest that further research is needed which explores the phenomenon of *blocking*, in particular as it relates to the inhibiting of students' growth of mathematical

[35] Kieren, Pirie and Reid (1994) have noted the significance of *folding back* in patterns of growth of students' mathematical understanding, and further work is currently being done on this feature of the Dynamical Theory for the Growth of Mathematical Understanding by Martin (1998).

understanding, as it is clear from the interaction with Donny that, even by the end of the exchange, he did not understand why the 17 and the missing horizontal line section combined to make 20, or why a similar relationship held for the vertical line sections. Though with my and Sonya's "help" Donny had (eventually) articulated correct responses to our questions, a close reading of the transcript ought to assure the reader that Donny had not learned how to do these problems, and certainly had not *understood* the geometrical relationships of the shape. Later, when I offered him graph paper to attempt a similar problem he was able to make some progress by re-drawing the shape to scale, and measuring the missing lengths. This shows that Donny did have images that might have enabled him to solve the first 'deviant' rectangle problem, but in *blocking* his use of his ruler and his reference to units of measurement I prevented Donny from accessing those images. I am claiming, therefore, that my (*blocking*) interventions during the lesson not only did not support Donny's growth of understanding, they actually inhibited it (by closing down avenues of *folding back* to *Image Making* activities for Donny).

5.2.2.6 The *Shepherding* Teaching Style and the Growth of Mathematical Understanding

Though I have already discussed how three themes (*inviting*, *rug-pulling*, and *blocking*) promote or inhibit understanding, I now want to draw particular attention to one of the teaching styles I have identified. This teaching style is *shepherding*. There are a number of occasions (some of which I have described in detail in this and the previous chapter) where a stream of *shepherding* interventions by the teacher appear to lead to an outward movement for the students through the levels of the Dynamical Theory for the Growth of Mathematical Understanding. For example, Karen *shepherded* Abby to an understanding

of why Pythagoras' theorem was not needed to find the volume of the triangular prism shown in Figure 64 (described in Section 4.6) (Points 35-36 on Abby's mapping diagram, Appendix 1), and I *shepherded* Donny to an understanding of how to find the perimeter of a square which has missing values such as that shown in Figure 60 (described in Section 4.6) (points 6-7 on Donny's mapping diagram, Appendix 5). I suggest, therefore, that *shepherding* interventions may be *provocative* in nature. In fact, whenever I identified the *shepherding* teaching style, there was evidence of an associated growth of understanding for the students. I claim that *shepherding* interventions promote the growth of understanding because they encourage the student to *make sense of* the mathematics they are being asked to consider, rather than (as with *showing and telling* and *leading* interventions) simply requiring the student to learn how to produce an acceptable (to the teacher) answer.

5.2.3.7 Spontaneous Growth of Mathematical Understanding

In some cases the students themselves appeared to initiate their own instances of *folding back*, or appeared spontaneously to extend their image to a new level. In such cases it is possible that they had been influenced in some way by earlier interventions by the teacher, but I am not able to say for sure. For example, during the 'Ponds' investigation I had left Kayleigh and Carrie with the problem of finding the relationship between the size of the 'pond' and the number of paving stones needed to surround it for *any* size of rectangular pond (after they had successfully found expressions for square 'ponds' and for the family of rectangular 'ponds' of fixed width one). As an example I had drawn on Kayleigh's paper the diagram shown in Figure 78. Kayleigh had initially assumed

(incorrectly) that I had requested an expression for a family of such 'ponds' with fixed width of three (see Figure 79).

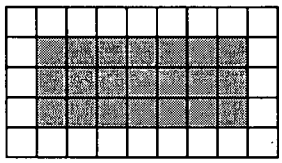


Figure 78: Jo's Rectangular 'Pond'

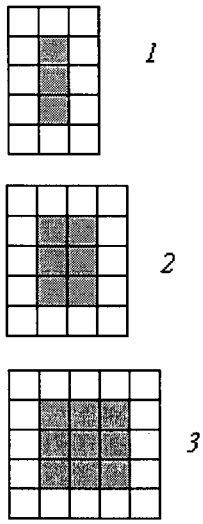


Figure 79: Family of 'Ponds'

Although Kayleigh had not drawn this family she had created an expression which would befit such a sequence. She raised her hand to call me over to show me her work and during the moment's pause before I noticed her raised hand she suddenly continued:

> Kayleigh: Or, it could otherwise be written as [*pause*]. Now this may be 'cause she might do. If you had more than three on the thing it would be [*pause*] if you had more than three in the width of the rectangle inside it would be…

She went on to produce an expression involving two variables for the width and length of the inner 'pond', however, her expression was mathematically incorrect due to an absence of appropriately placed brackets, an omission which Kieran (1992) notes is a common one for beginning algebra students. I clarified this omission with her when she next called me over, but the point I am making is that this sudden growth of understanding (the move to extend her formula to incorporate rectangular 'ponds' of *any* dimension), though it appeared spontaneous, may have rested in part on my earlier request that they find an expression for *any* size of rectangular 'pond'. Kayleigh's brief reference to me ("...'cause she might do...") suggests that I was implicated in some way in the revision she made to her expression, but without asking her, it is clearly not possible to say exactly how I was implicated. We cannot know whether Kayleigh was recalling what I had actually said (recall that I had asked for the more encompassing expression in the first place), or in her mind pre-empting where I might be going next. Such a moment remains on the borderline of what I might call a *spontaneous movement*, and suggests that it is not necessary for the teacher to be closely monitoring and guiding the students' growth of understanding all the time. This finding should enable teachers to feel more confident about moving away from a teaching style that relies on strong *leading* or *showing and telling*.

5.2.2.8 *Repeated Folding Back* and the Growth of Mathematical Understanding

I define instances of *repeated folding back* as those episodes when the student *folds back* to an inner level and then extends to an outer level repeatedly during a short period of time (such as one lesson). Such a pattern clearly represents a period of growth of understanding for the student concerned. For every student the interviews provided

environments where *repeated folding back* flourished. The interviews took rather different forms in the two strands of the research, though. For instance the final interviews in the UK strand involved the students being presented with a sequence of problems, some algebraic, some not, most (but not all) contextualised within the over-arching theme of perimeter and area. My presentation in these interviews of increasingly difficult problems (including some, like the one I have described above for Kayleigh, Figure 77, which were specifically intended to cause consternation) are, I claim, *rug-pulling* interventions. This is a situation in which we might reasonably expect to see a pattern of repeated *folding back* as the students reach the limit of their understanding in one area and *fold back*, but are later able to extend further in a related area.

In contrast, in the second interview in the Vancouver strand the students were offered only one (albeit complex) problem (*inviting*) - to design several packages for tennis balls, and to find out which design wasted the least space within it. This problem is more reminiscent of the 'Ponds' investigation from the UK strand, where, it should be noticed, Donny (Appendix 5) and Kayleigh (Appendix 7) showed a similar pattern of *repeated folding back* (and each of the other students *folded back* at least once), and where I employed a range of interventions including *inviting* and *retreating*, which are both within the *shepherding* cluster, as well as (on occasions) *rug-pulling*. Note also that in the practical room-painting problem in Lesson 4 in the Vancouver strand, Tasha (Appendix 3) also exhibited a pattern of *repeated folding back* associated with an *inviting* (initial) intervention. So, while it is not possible to associate *repeated folding back* with any one type of intervention, it is significant that the majority of the interventions associated with

it are of the forms that I have identified in this study as occasioning the growth of understanding (i.e., *inviting*, *shepherding*, and *rug-pulling*). Further, if for a moment, we consider the environments (in other words the particular problems or situations facing the students at times when they exhibited this pattern of growth) it seems that practical and/or investigative situations may also provide increased opportunities for *folding back*, and therefore growth.

Very similar patterns of growth, then, were generated by very different situations. Two of these were teaching situations ('Ponds' and the room-painting problem) the other two were assessment situations (interviews), and yet they generated similar patterns of activity in terms of the growth of understanding for the students. The fact that different "situations" generated similar patterns of growth is less surprising, however, if we consider these situations from the point of view of the interventions that were predominant, and notice that (in almost every case) *inviting* and *retreating* (and therefore *shepherding*) and *rug-pulling* interventions predominated.

5.2.3.9 Summary

It was not possible to establish clear and consistent links between the remaining themes and teaching styles, and the students' growth of understanding, so I leave to further studies a consideration of how these aspects of the teacher's role might contribute to the shaping of students' understanding of mathematics. I do not think, however, that my creation of these other themes was superfluous. At the time of developing my themes I was engaged in characterising teachers' interventions in terms of their action in the

moment, and only later did I begin to draw together my conclusions about how those interventions occasioned the growth of the students' understanding. The themes stand as a detailed new language to describe teachers' actions and verbalisations in classrooms, and as such have value as a tool to explore the relationship between teaching and learning. In clustering the themes I have been able to respond to Chazan's and Ball's (1995) call to create a more precise way of describing the "telling" that teachers do. I expect that the development of the themes themselves, and the clusters of themes within teaching styles, will be of use to researchers and teachers.

5.3 Personal Growth

In this section I want to reflect on the growth in my own understanding of teaching and learning that has resulted from my participation in this study. My role in this study has been as both teacher (participant) and researcher (observer). In Chapter 3 (Section 3.3.5) I described my role as one of observant participator, and I hope that it is now clear why I chose to re-work the conventional descriptor of participant observer. I feel that positioning myself as observant participator rather than participant observer more clearly foregrounds the participatory nature of my involvement, without obscuring the essential aspect of my research focus. This re-working (to observant participator) is intended to re-focus the reader's attention on my involvement and to re-mind that I have been (and remain) thoroughly implicated in all that I have studied.

Re-casting myself as an observant participator in this study, rather than a participant observer, has foregrounded a number of features of the research. The first is the problem

to which I alluded in Chapter 3 (Section 3.3.3) of subjectivity in my reporting. Bias is present in all forms of research, and it seems to me that research such as this falls prey just as easily as any other form of research. Though I claimed in Chapter 3 (Section 3.3.1) that participating as a teacher in this research has the advantage of enabling me (as researcher) to know a great deal about the thoughts and feelings of one of the participating teachers, I recognise now the difficulties that this brings. Although I still claim to know a great deal about what I was thinking and feeling during my interactions with my students, I want to acknowledge how difficult it was at times to separate what were pertinent issues for me personally about those interactions from what were pertinent issues to the research question at hand. I have tried to represent fairly each classroom, and each teacher, but even this has raised challenges, for it is clear to me that another person accessing the same data might see things very differently. In the end, I have, after all, had ultimate choice over what is and is not explicitly included in the reporting of this study. It is my hope, though, that the subjectivity inherent in this type of research will have been clearly laid open to scrutiny through the detail with which I have described both the growth of the students' understandings and the flow of the teachers' intervention patterns, and also through the many transcribed conversations I have included which, I hope, enable readers to judge for themselves the strength of my claims.

My positioning of myself as observant participator in this study also highlights my struggle with the ethical positioning of this research. In Chapter 3 (Section 3.3.4) I commented on the way in which I behaved in a more 'teacherly' manner during the interviews with my own students than I did in my interviews with students in the

Vancouver strand of the data. The videos show that I was far more able in the Vancouver strand to sit back and allow Abby and Tasha to grapple with a problem than I was with any of my own students in their interviews (or their classroom lessons). I claim, however, that there are a number of reasons for this difference. Firstly, as I mentioned in Chapter 3, I did have an ethical responsibility to 'teach' my own students, and this certainly played its part. The data show that for me, at that time, to 'teach' meant mostly to *tell*, or at least to *lead*. It should be noted that although I was not intending to 'teach' in the Vancouver interviews, I did make (predominantly) *inviting* interventions. I was therefore participating in *shepherding* the students to understanding, and therefore still 'teaching', however, the 'teaching' in these interviews was much more subtle.

There are also other reasons for the differences between my stances in the two sets of interviews. The first of these is that my interviewing in the Vancouver strand occurred some time after the data collected in my own classroom, and during that time I had not only had the opportunity to watch my own interviewing and teaching styles, but had participated in additional graduate study where my awareness of the difficulty of remaining an interviewer (and only an interviewer) in interviews had been heightened. Not only this, I had, during those intervening years, been immersed in reform literature which, for the most part, problematises teachers eagerness to 'tell'. It is not surprising, therefore, that I was able to maintain a different stance in the Vancouver interviews than I had displayed in the UK interviews. That I was able to re-orient my stance does, however, make a positive statement both for the value of watching oneself on video, which I have

already mentioned, and also for the value of graduate study and professional development.

It is significant to note that even though I had watched the video tapes of my own classroom before beginning to watch the Vancouver data I had collected, it was not until after I had noticed the theme of *enculturating* emerging in the Vancouver strand of the data that I realised that I, too, often made reference to the wider mathematical community in my own teaching. This realisation was a powerful one, and points to the value (for teachers) of studying other teachers teaching. Though observing others teaching is a mainstay of most teachers' *pre-service* training, it is often neglected in (and sometimes completely absent from) their later professional development opportunities. My work suggests that teacher professional development strategies should capitalise on the power of this technique, and encourage teachers to learn from each other in more formal ways than simply swapping notes in the staff-room.

The fact that I noticed my own *enculturating* interventions only after noticing such interventions in Karen's teaching, is of relevance to a wider audience than teachers, though. It also points to the value for researchers who engage in self-study of widening their methodologies to incorporate the study of others. I still believe that self-study is a valuable research mechanism, though. After all, not only did I overlook certain aspects of my own teaching until I had studied Karen's teaching, but, as the additional themes generated for the first time whilst studying the UK tapes show (Appendix 16), I also did not notice certain aspects of Karen's teaching until I noticed them in my own. This

reciprocal dependence of the two strands of my research is a strong justification for the particular methodology I have adopted. Although I recognise the value of supporting teachers' efforts to participate in self-study, and note that there is need for increased attention to be paid to teachers' own research findings (Cochran-Smith & Lytle, 1993), in adopting the unusual methodological approach of integrating self-study with the study of another teacher I claim to have demonstrated the potential for an alternative qualitative methodology to contribute to knowledge about teaching.

5.4 Implications and Contributions of the Study

5.4.1 Introduction

In this section I want to summarise the main contributions that I feel this study offers, and to present the main implications for research, teaching and learning that derive from it.

5.4.1 Implications for the Research Community

As I discussed in Section 5.3 above, the methodology of integrating self-study with the study of another (or several other) teacher(s) offers potential for supporting the study of teaching. It is not a methodology that I have been able to discern as a distinct approach in the literature, and yet I have found it to be extremely powerful. I therefore offer it as a significant contribution to the field of qualitative research methodology and I suggest that it warrants further investigation and exploration.

Linked to this methodology is the notion of the observant participator that I have developed. I claim that this orientation enables researchers (particularly those engaged in

self-study) to approach their study from a number of perspectives simultaneously, foregrounding one or other of the observer/participant perspectives at any time, but without losing the connection to the other. Thinking of my role as one of observant participator (rather than participant observer) allowed me to recognise the value of the insights I gained from my researcher-perspective, whilst not diminishing the insights gained from my teacher-perspective, and as such I offer the observant participator descriptor as a useful device, particularly in relation to the particular methodology I have adopted in this study.

In terms of the implications for other researchers who adopt the Dynamical Theory for the Growth of Mathematical Understanding as a theoretical tool of analysis, I have made a number of extensions to this theory. The first of these is the introduction of the term *image viewing* to describe instances where a student was observed to be passively "watching" an explanation by the teacher, and where the information being presented appeared to be new to the student. This phenomenon was previously unaccounted for by Pirie and Kieren.

The remaining developments (*incremental extending*, *incremental folding back*, *repeated folding back* and *jumps in understanding*) describe features on the students' mappings that had been noticed previously by Pirie and Kieren, but had not explicitly been labelled, or, as in the case of *jumps in understanding*, a similar but more encompassing definition had previously been generated which I have refined to draw attention to the significance of more expansive movements. Such identifying and naming provides a new source of

features for future researchers who adopt the Dynamical Theory to consider when mapping students' understanding, and particularly when establishing links between teaching activity and the growth of students' understanding. The providing of new language to describe a phenomenon is a significant advance because it *allows for* exploration. New descriptors enable us to think about and talk about phenomena that might otherwise be overlooked. Naming alerts us to the possibilities for deeper investigation, and therefore points to areas which might hold promise for future research.

5.4.2 Implications for Teaching and Learning

In terms of the implications of this study for teaching and learning, I claim to have made three main contributions. The first is that in identifying themes to describe the teachers' interventions in this study I have developed not only a complex, explicit and contextualised characterisation of the roles that teachers play in the classroom (Chazan and Ball, 1995) but also (through my development of the three teaching styles and their associated clusters of themes) a more precise means of describing the telling that teachers do.

Following on from this, my second contribution is that in drawing out the notion of a continuum of telling along which teachers interested in reforming their practice might move, I claim to have responded to concerns that researchers who exhort teachers to avoid telling often do not suggest anything that they might do instead (Chazan and Ball, 1995). I claim to have contributed a range of strategies (clustered in three teaching styles) through which teachers might move from a teaching style that celebrates telling (*showing*

and telling) through *leading* to a more subtle form of teaching (*shepherding*). My data has shown that such a progression is to be valued because there is evidence that the *shepherding* teaching style (whenever it manifested) promoted the growth of understanding, whilst there was conflicting evidence regarding both the *showing and telling* and *leading* styles. Sometimes these latter two styles were seen to contribute to the growth of understanding through extending or *folding back*, but at other times they did not appear to occasion any growth of understanding.

The third contribution that relates specifically to teaching and learning is my identification of three types of intervention that consistently contributed to the growth of students' mathematical understanding. As I discussed in Section 5.2.2 these are the themes of *inviting* and *rug-pulling*, and the teaching style, *shepherding*. In addition, I have identified and discussed (Section 5.2.2) an intervention theme which appears to inhibit the growth of understanding, that being *blocking*.

5.5 Concluding

In summary, I note that Chazan and Ball (1995) close their paper with the following appeal which I believe I have addressed through the findings I have presented in this chapter:

> We hope that the development of vocabularies for describing the teacher's role...will enhance opportunities for sustained, critical, and insightful discourse among researchers, teachers, and teacher educators (Chazan & Ball, 1995, p.23).

Although I am using this call to draw my writing to a close, it is my hope that the implications to be drawn from my study will now open such a discourse.

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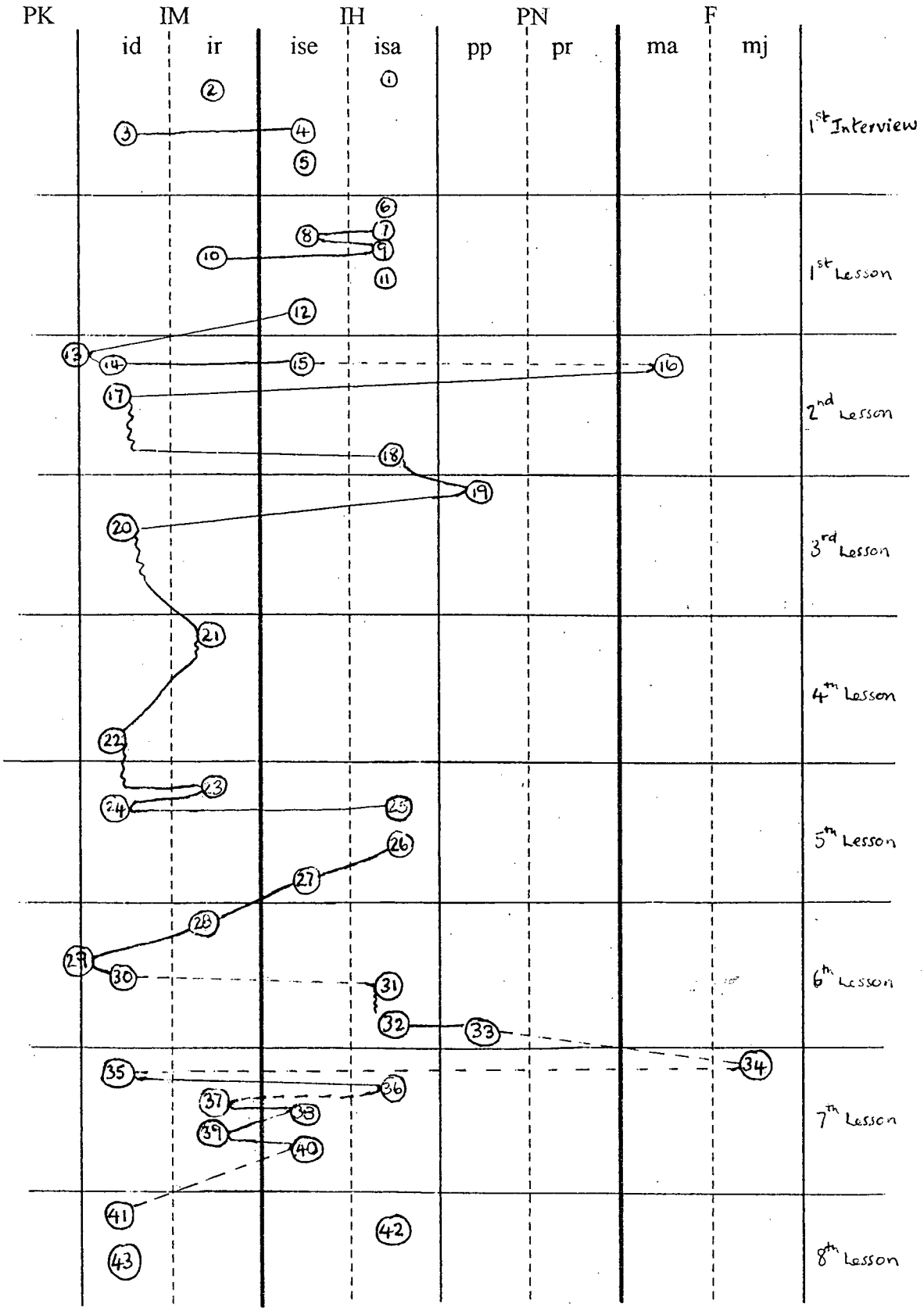
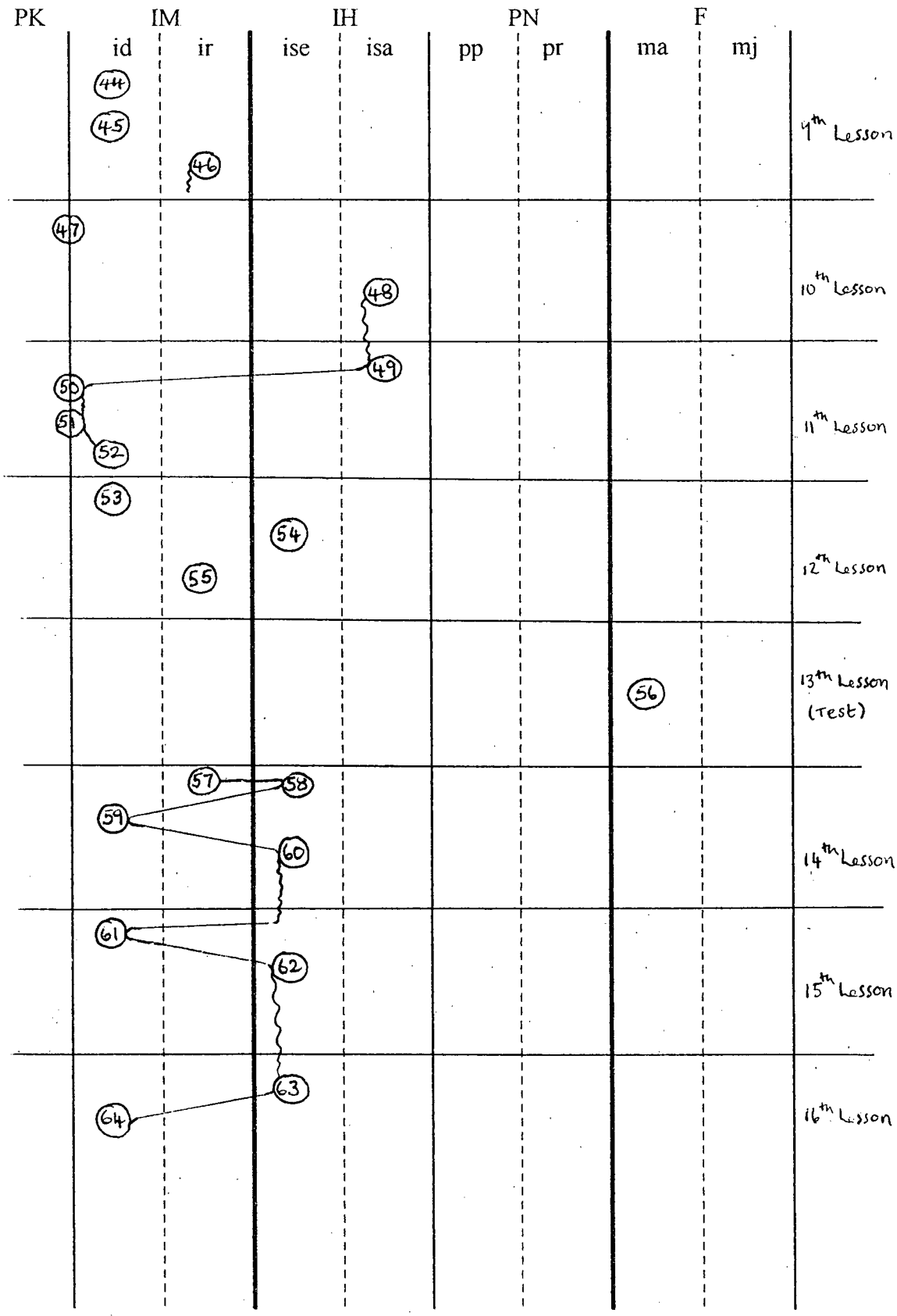
APPENDIX 1 - ABBY'S MAPPING DIAGRAM

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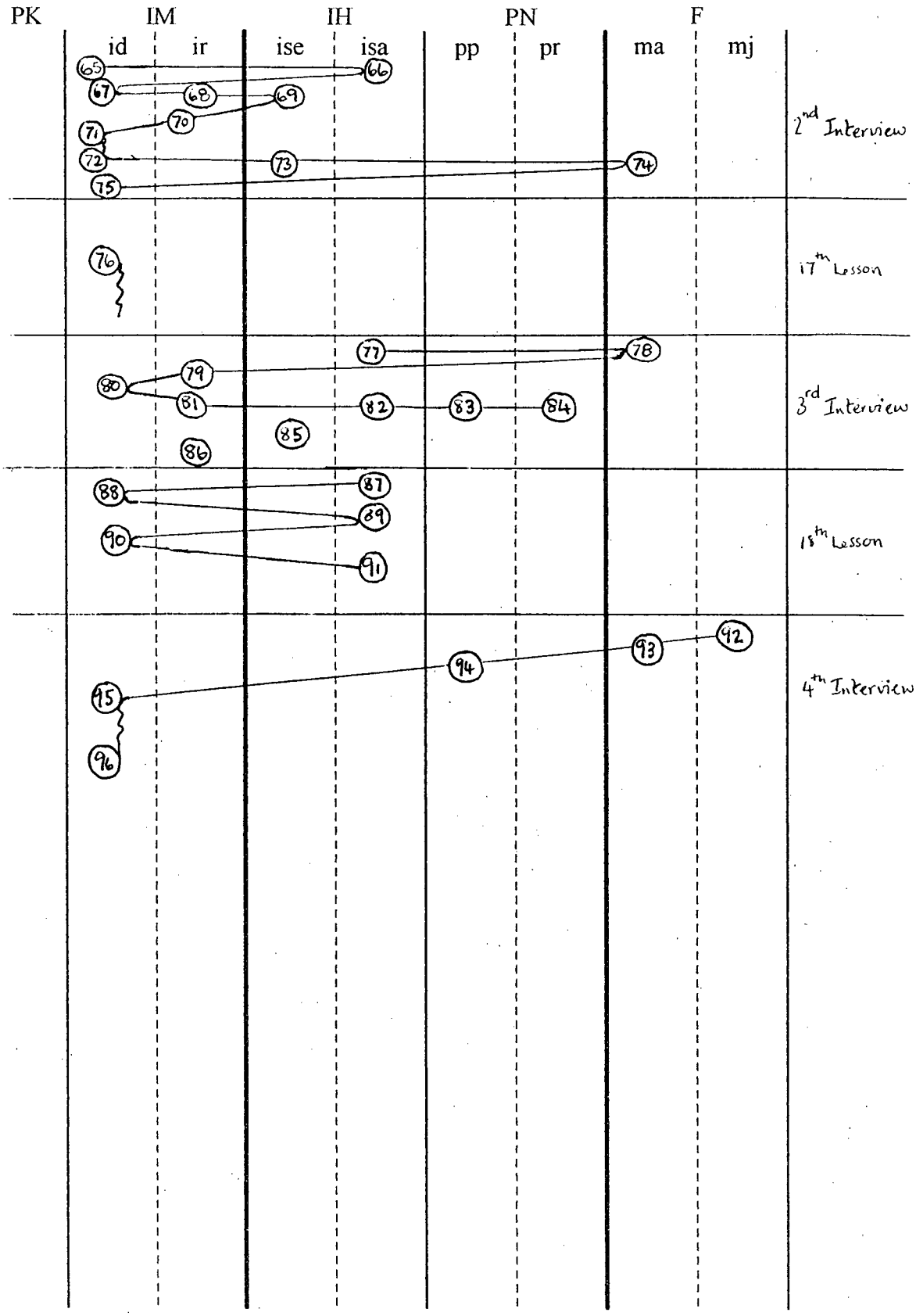
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APPENDIX 2

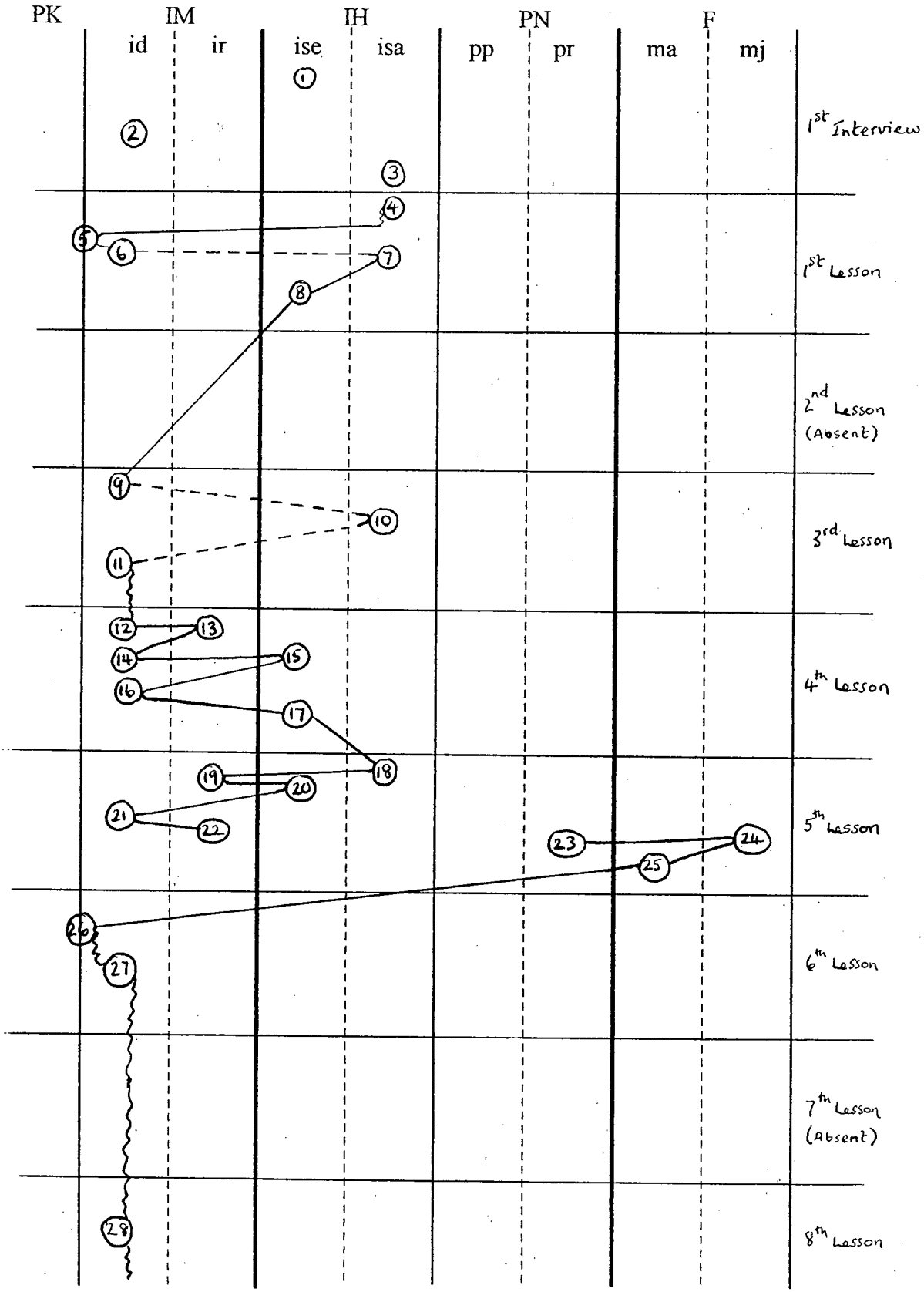
ABBY'S MAPPING POINTS

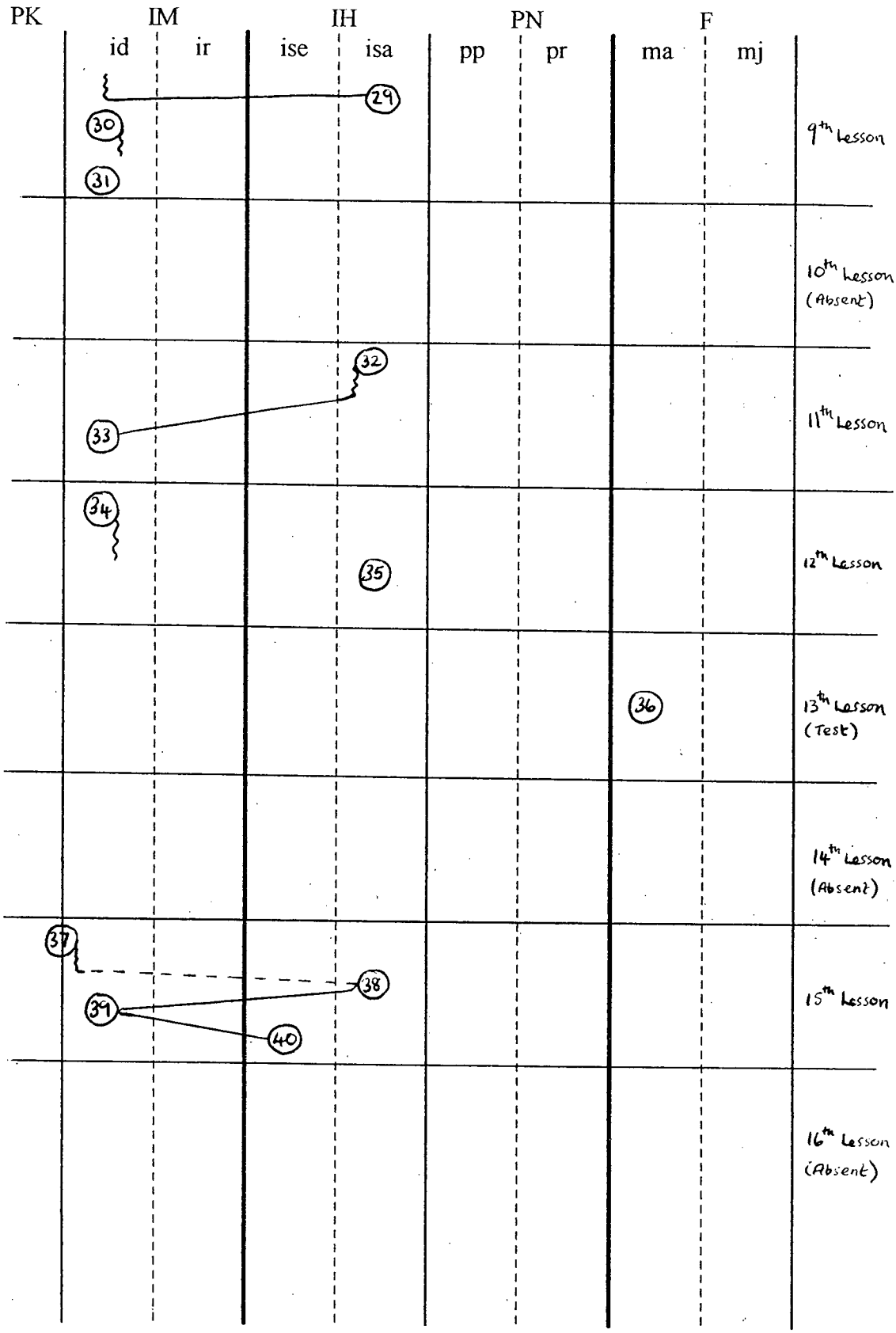
1	1st interview:	isa for the definition of the area concept, and for surface area of a cylinder, and area of 'deviant' rectangle
2	1st interview:	ir for volume of a cylinder
3	1st interview:	id for angle properties associated with parallel lines
4	1st interview:	ise for angle properties associated with parallel lines
5	1st interview:	ise for volume of combined solid
6	1st lesson:	isa for definition of the area concept
7	1st lesson:	isa for definition of the volume concept
8	1st lesson:	ise for properties of plane figures
9	1st lesson:	isa for area of a parallelogram
10	1st lesson:	ir for area of a trapezoid
11	1st lesson:	isa for volume
12	1st lesson:	ise for surface area of a right rectangular prism
13	2nd lesson:	iv for Pythagoras' theorem
14	2nd lesson:	id for Pythagoras' theorem
15	2nd lesson:	ise for Pythagoras' theorem
16	2nd lesson:	ma for surface area of a right rectangular prism
17	2nd lesson:	id for definition of a prism, and for surface area of a triangular prism drawn as a net
18	2nd lesson:	isa for Pythagoras' theorem
19	3rd lesson:	pp for surface area of a triangular prism (involving Pythagoras' theorem)
20	3rd lesson:	id for surface area of a combined solid
21	4th lesson:	ir for surface area of walls requiring paint
22	4th lesson:	id for surface area of a cylinder, continuing into 5th lesson for surface area of a half-cylinder
23	5th lesson:	ir for Pythagoras' theorem
24	5th lesson:	id for surface area of a combined solid
25	5th lesson:	isa for Pythagoras' theorem
26	5th lesson:	isa for volume
27	5th lesson:	ise for definition of a right rectangular prism
28	6th lesson:	ir for definition of a right prism
29	6th lesson:	iv for volume of a pyramid (water-pouring demonstration)
30	6th lesson:	id for volume of a cylinder and volume of a cone
31	6th lesson:	isa for volume of a combined solid
32	6th lesson:	isa for Pythagoras' theorem
33	6th lesson:	pp for dropping a perpendicular into a triangle to create a situation in which you can apply Pythagoras' theorem
34	7th lesson:	mj for volumes of a right rectangular prism and a cylinder
35	7th lesson:	id for volume of a triangular prism
36	7th lesson:	isa for volume of a triangular prism

37	7th lesson:	ir for volume of pyramids and cones
38	7th lesson:	ise for volumes of a cone and a pyramid
39	7th lesson:	ir for volume of a pyramid
40	7th lesson:	ise for volumes of a cone and a pyramid
41	8th lesson:	id for volume of a sphere
42	8th lesson:	isa for surface area of a cylinder
43	8th lesson:	id for ratio/comparison
44	9th lesson:	id for volume of a sphere
45	9th lesson:	id for cancelling algebraic fractions
46	9th lesson:	ir for visualising the net of a cone
47	10th lesson:	iv for surface area of a cone
48	10th lesson:	isa for definition of terms such as angle and line, and for properties of plane figures
49	11th lesson:	isa for further properties of plane figures (such as the sum of the angles in a triangle)
50	11th lesson:	iv for triangles on the surface of a sphere
51	11th lesson:	iv for properties of quadrilaterals
52	11th lesson:	id for acceptable justifications for angle properties associated with parallel lines
53	12th lesson:	id for surface area of triangular prism
54	12th lesson:	ise for volume of a triangular prism
55	12th lesson:	ir for acceptable justifications for angle properties associated with parallel lines
56	13th lesson (the test):	ma for surface area and volume
57	14th lesson:	ir for acceptable justifications for angle properties associated with parallel lines
58	14th lesson:	ise for several angle properties associated with parallel lines
59	14th lesson:	id for acceptable justifications for some angle properties associated with parallel lines
60	14th lesson:	ise for acceptable justifications for angle properties associated with parallel lines
61	15th lesson:	id for angle properties in situations involving circles
62	15th lesson:	ise for circle properties
63	16th lesson:	ise for angle properties associated with parallel lines
64	16th lesson:	id for angle properties associated with parallel lines, and circles
65	2nd interview:	id for the packaging four tennis balls problem
66	2nd interview:	isa for the packaging problem situation and for circumference of a sphere
67	2nd interview:	id for volume of a sphere
68	2nd interview:	ir for volume of a sphere, and for volume of a cylinder
69	2nd interview:	ise for space needed within the cylinder to fit four tennis balls
70	2nd interview:	ir for space available within the designs
71	2nd interview:	id for ratio of sphere to cylinder
72	2nd interview:	id for visualisation of right rectangular prism design

73	2nd interview:	ise for volume taken up by the tennis balls inside the right rectangular prism
74	2nd interview:	ma for volume of a right rectangular prism
75	2nd interview:	id for calculation of percentages of space taken up inside each package by the four tennis balls
76	17th lesson:	id for acceptable justifications within two-column proofs
77	3rd interview:	isa for visualisation of spheres within the third packaging design
78	3rd interview:	ma for volume of a right rectangular prism
79	3rd interview:	ir for ratio of spheres to prism for third packaging design
80	3rd interview:	id for the volume of a pyramid (fourth packaging design)
81	3rd interview:	ir for amount of wasted space within the packages
82	3rd interview:	isa for meaning of the percentages calculated earlier (amount of space taken up by the tennis balls)
83	3rd interview:	pp for wasted space
84	3rd interview:	pr for relationship between the wasted space and the volume of the package
85	3rd interview:	ise for surface area of a cylinder
86	3rd interview:	ir for the height of the pyramid (fourth packaging design)
87	18th lesson:	isa for definition of congruency
88	18th lesson:	id for congruency postulates
89	18th lesson:	isa for congruency postulates
90	18th lesson:	id for congruency postulates
91	18th lesson:	isa for congruency postulates
92	4th interview:	mj for angle properties associated with parallel lines
93	4th interview:	ma for angle properties within a circle
94	4th interview:	pp for angle properties within a circle
95	4th interview:	id for proof of angle properties within a circle
96	4th interview:	id for guided proof

APPENDIX 3 - TASHA'S MAPPING DIAGRAM

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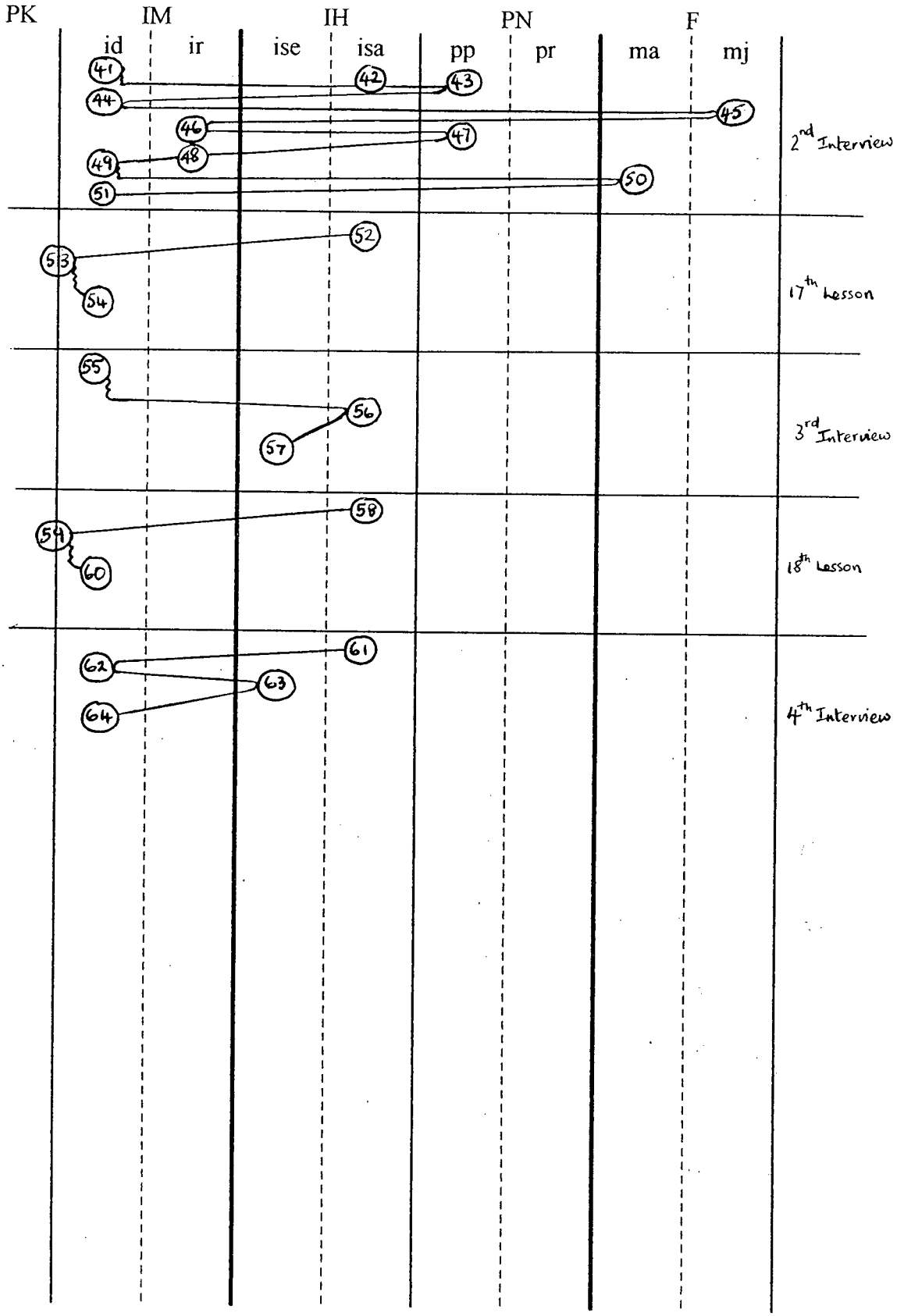
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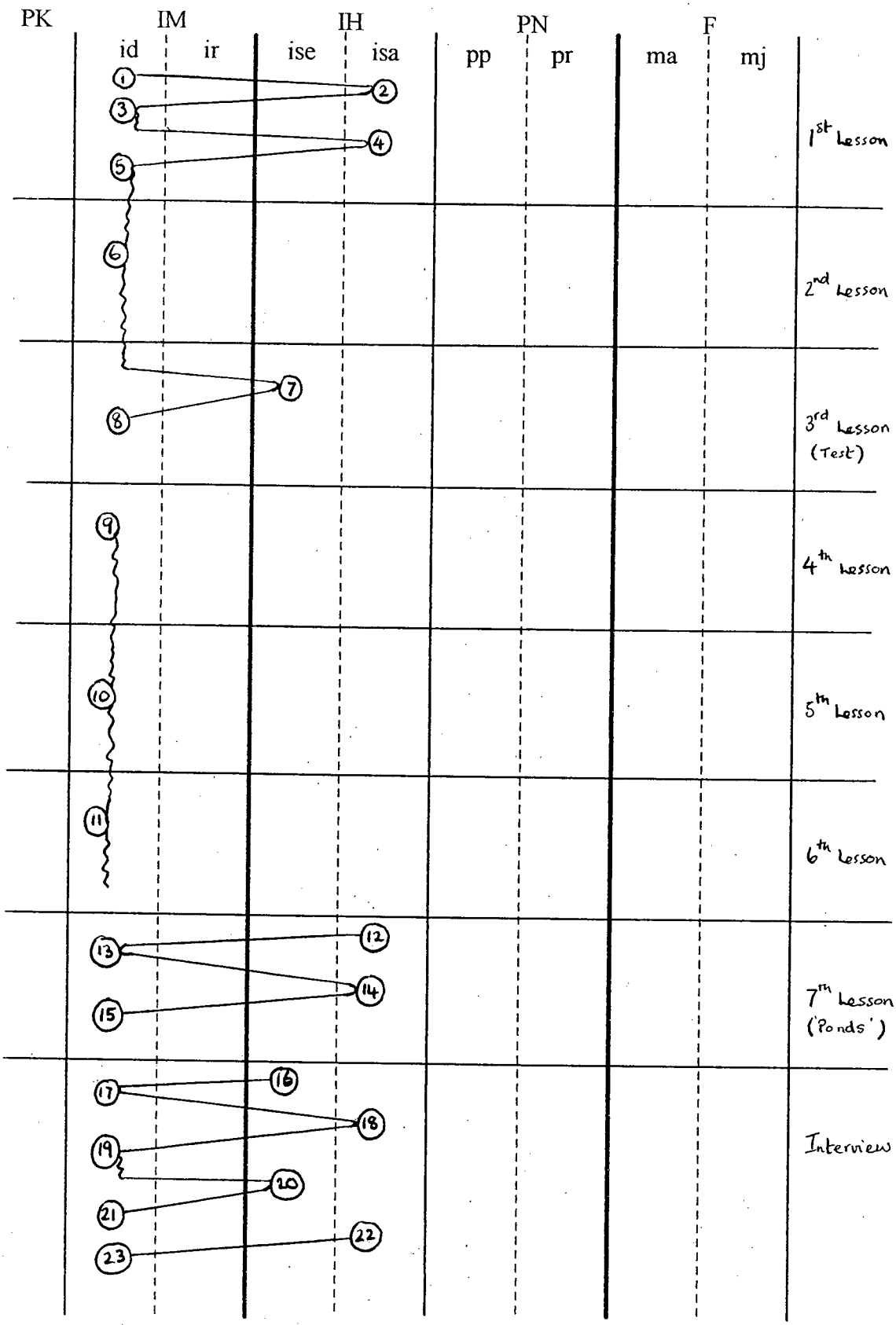
APPENDIX 4

TASHA'S MAPPING POINTS

1	1st interview:	ise for surface area of a cylinder, and area of 'deviant' rectangle
2	1st interview:	id for angle properties associated with parallel lines
3	1st interview:	isa for volume of a right rectangular prism
4	1stlesson:	isa for definitions for the concepts of area and volume, and for properties of circles and other plane figures
5	1st lesson:	iv for area of a trapezoid
6	1st lesson:	id for area of a trapezoid
7	1st lesson:	isa for area of combined plane figure
8	1st lesson:	ise for surface area of a right rectangular prism
9	3rd lesson:	id for Pythagoras' theorem
10	3rd lesson:	isa for area of rectangles and triangles
11	3rd lesson:	id for surface area of a square-based pyramid and for a combined solid
12	4th lesson:	id for surface area of walls requiring paint
13	4th lesson:	ir for surface area of walls requiring paint
14	4th lesson:	id for amount of paint required
15	4th lesson:	ise for surface area of a cylinder
16	4th lesson:	id for circumference of a circle
17	4th lesson:	ise for surface area of a cylinder
18	5th lesson:	isa for circumference of a circle and for Pythagoras' theorem
19	5th lesson:	ir for surface area of an octahedron and for surface area of a combined solid
20	5th lesson:	ise for surface area of a half-cylinder
21	5th lesson:	id for surface area of a triangular prism
22	5th lesson:	ir for surface area of a combined solid
23	5th lesson:	pr for volume of any prism
24	5th lesson:	mj for volume of any prism
25	5th lesson:	ma for volume of a right rectangular prism
26	6th lesson:	iv for volumes of a pyramid and a cone
27	6th lesson:	id for volumes of a cone, a cylinder, a half-cylinder and for Pythagoras' theorem
28	8th lesson:	id for surface area and volume of a sphere and of a cylinder
29	9th lesson:	isa for relationship between the radius and diameter of a circle
30	9th lesson:	id for cancelling algebraic fractions, and for area of a circle, and for volumes of a cone and a sphere
31	9th lesson:	id for visualisation for the net of a cone
32	11th lesson:	isa for angle terminology
33	11th lesson:	id for angle properties associated with parallel lines and for acceptable justifications for angle properties associated with parallel lines
34	12th lesson:	id for Pythagoras' theorem and for volume of a triangular prism

35	12th lesson:	isa for angle properties of triangles
36	13th lesson (the test):	ma for volume of cones, spheres, right rectangular prisms and for surface area of spheres and right rectangular prisms
37	15th lesson:	iv for acceptable justifications for angle properties associated with parallel lines
38	15th lesson:	isa for angle properties of triangles
39	15th lesson:	id for angle properties associated with parallel lines
40	15th lesson:	ise for angle properties associated with circles
41	2nd interview:	id for tennis balls packaging problem
42	2nd interview:	isa for circumference of a circle
43	2nd interview:	pp for amount of wasted space
44	2nd interview:	id for volume of a sphere
45	2nd interview:	mj for volume of a cylinder
46	2nd interview:	ir for volumes of a sphere and a cylinder
47	2nd interview:	pp for ratio of volume of sphere to cylinder
48	2nd interview:	ir for amount of wasted space
49	2nd interview:	id for ratio of sphere to cylinder and for visualisation of right rectangular prism
50	2nd interview:	ma for volume of a right rectangular prism
51	2nd interview:	id for assessment of which design wastes the least space
52	17th lesson:	isa for angle properties associated with parallel lines
53	17th lesson:	iv for guided proofs
54	17th lesson:	id for guided proofs and for acceptable justifications for angle properties associated with parallel lines
55	3rd interview:	id for visualisation of third packaging design, for percentages, and for area of the base of the fourth packaging design
56	3rd interview:	isa for comment linking the percentages and available space
57	3rd interview:	ise for surface area of a cylinder
58	18th lesson:	isa for the difference between congruency and similarity
59	18th lesson:	iv for congruency postulates
60	18th lesson:	id for congruency postulates
61	4th interview:	isa for vertically opposite angles
62	4th interview:	id for angle properties associated with parallel lines
63	4th interview:	ise for angle properties associated with parallel lines
64	4th interview:	id for angle properties within a circle and for guided proof involving congruency postulates

APPENDIX 5 - DONNY'S MAPPING DIAGRAM



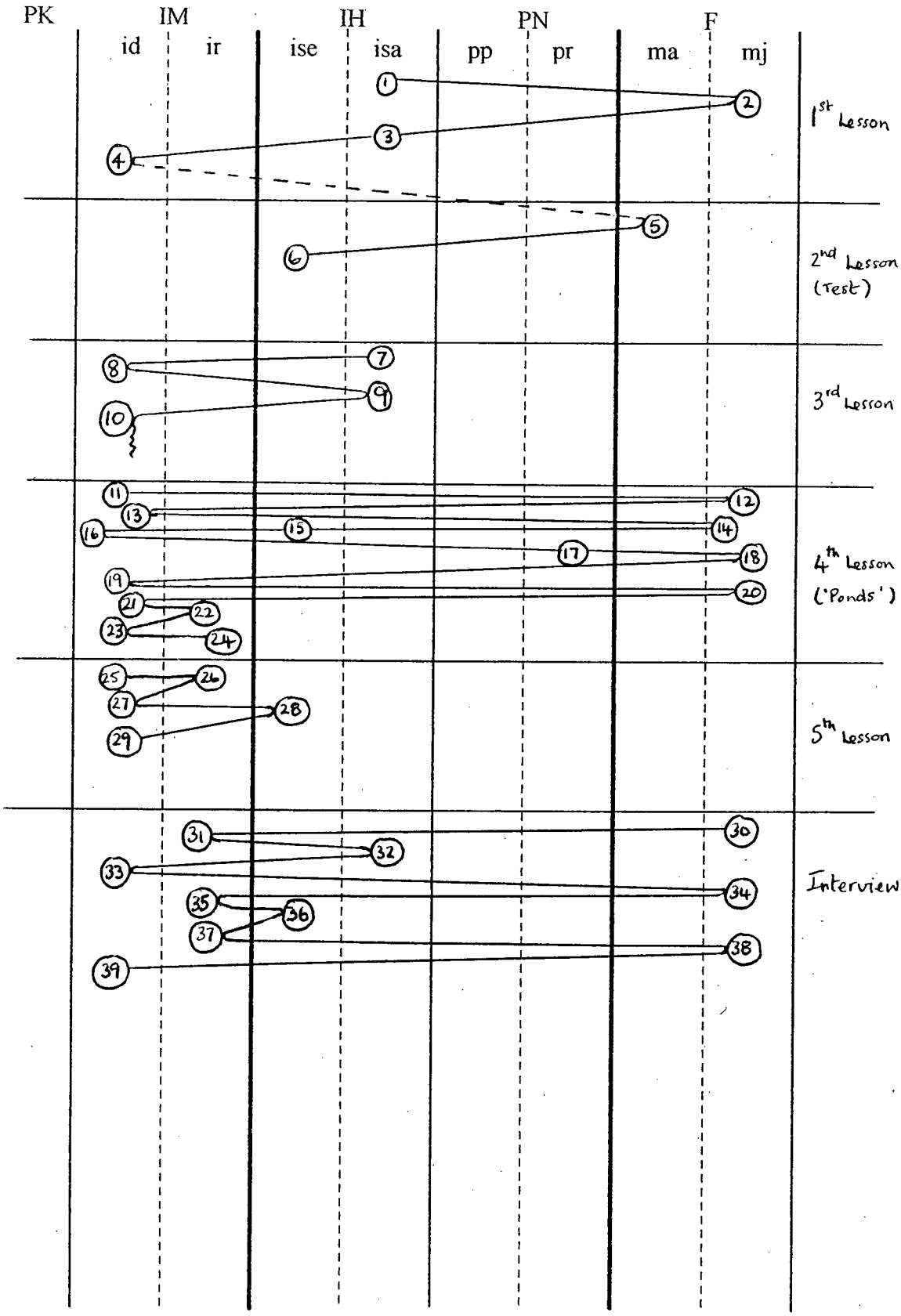
APPENDIX 6

DONNY'S MAPPING POINTS

1	1st lesson:	id for non-algebraic perimeter of a rectangle
2	1st lesson:	isa for non-algebraic perimeter of a rectangle given all the sides
3	1st lesson:	id for non-algebraic perimeter of rectangles with missing information
4	1stlesson:	isa for non-algebraic perimeter of a square given all the sides
5	1st lesson:	id for non-algebraic perimeters of squares and rectangles with missing information, and for 'deviant' rectangle with missing information
6	2nd lesson:	id for non-algebraic areas of a rectangles and squares (both with all sides given and with missing information), and for area of a 'deviant' rectangle
7	3rd lesson (test):	ise for non-algebraic perimeters of rectangles and squares (both with all sides given and with missing information)
8	3rd lesson (test):	id for non-algebraic areas of rectangles and squares (both with all sides given and with missing information)
9	4th lesson:	id for formulating and manipulating algebraic expressions, including collecting like terms, within the context of perimeter
10	5th lesson:	id for formulating and manipulating algebraic expressions, including collecting like terms, within the context of perimeter
11	6th lesson:	id formulating and manipulating algebraic expressions, including collecting like terms, within the context of perimeter
12	7th lesson:	isa for the sequence of diagrams in the 'Ponds' activity
13	7th lesson:	id for drawing the family of 'ponds', creating the table of results and finding a link between the 'pond' size and the number of paving stones needed to surround it
14	7th lesson:	isa for the 'add four' pattern in the table of results for the 'Ponds' activity
15	7th lesson:	id for the link between the 'pond' size and the number of paving stones needed to surround it in the 'Ponds' activity
16	Interview:	ise for non-algebraic perimeter of a rectangle given all the sides
17	Interview:	id for relationship of sides in a non-algebraic 'deviant' rectangle
18	Interview:	isa for the perimeter of a non-algebraic 'deviant' rectangle
19	Interview:	id for manipulating algebraic expressions, including collecting like terms, within the context of perimeter
20	Interview:	ise for a non-contextual algebraic question involving collecting like terms
21	Interview:	id for non-algebraic area of a rectangle with all sides given
22	Interview:	isa for generating values in the table of results and for predicting subsequent values, in a 'Ponds' activity involving rectangular ponds

23 Interview: id for creating a link between the 'pond' size and the number of paving stones needed to surround it

APPENDIX 7 - KAYLEIGH'S MAPPING DIAGRAM



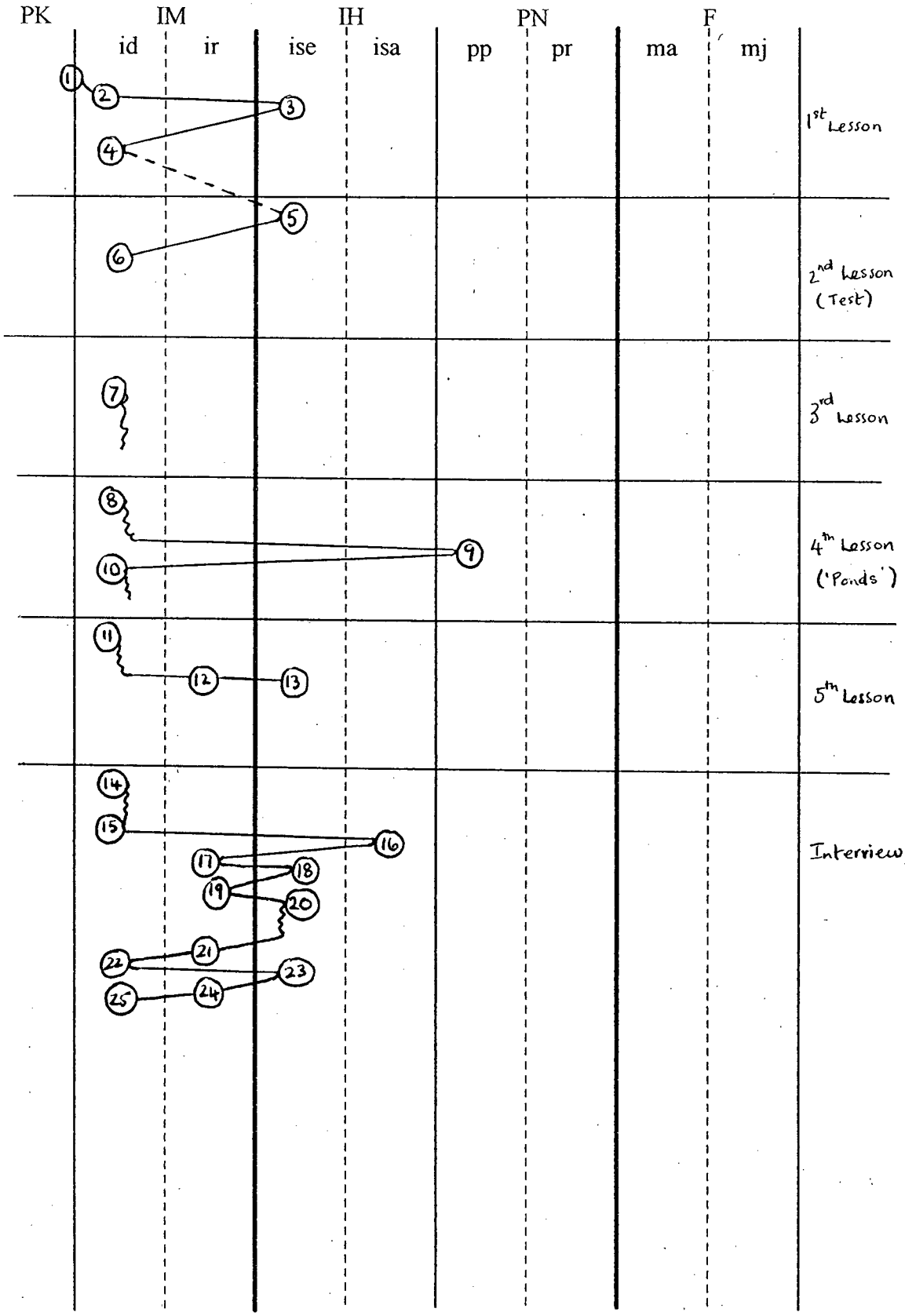
APPENDIX 8

KAYLEIGH'S MAPPING POINTS

1	1st lesson:	isa for calculating the missing lengths of a non-algebraic 'deviant' rectangle
2	1st lesson:	mj for non-algebraic perimeters of rectangles and squares
3	1st lesson:	isa for non-algebraic area of a triangle
4	1st lesson:	id for non-algebraic areas of combined shapes involving triangles
5	2nd lesson (test):	ma for non-algebraic and algebraic perimeters and areas of plane figures (both with all sides given and with missing information)
6	2nd lesson (test):	ise for formulating and manipulating algebraic expressions, including collecting like terms, within the context of perimeter and area, especially concerning 'deviant' rectangles
7	3rd lesson:	isa for formulating and manipulating algebraic expressions, including collecting like terms, within the context of perimeter
8	3rd lesson:	id for formulating and manipulating algebraic expressions, including collecting like terms, within the context of perimeter (missed out plus signs between the terms in some expressions for perimeter)
9	3rd lesson:	isa for formulating and manipulating algebraic expressions, including collecting like terms, within the context of perimeter
10	3rd lesson:	id for algebraic formulation for perimeter of a 'deviant' rectangle
11	4th lesson:	id for 'Ponds' activity
12	4th lesson:	mj for link between 'pond' size and number of paving stones needed to surround it for square ponds
13	4th lesson:	id for rectangular ponds
14	4th lesson:	mj for link between 'pond' size and number of paving stones needed to surround it for rectangular ponds of fixed width one
15	4th lesson:	ise for extended rectangular ponds (of fixed width three)
16	4th lesson:	id for rectangular ponds of any dimension
17	4th lesson:	pr for rectangular ponds of any dimension
18	4th lesson:	mj for rectangular ponds of any dimension
19	4th lesson:	id for the first L-shaped family of ponds
20	4th lesson:	mj for link between 'pond' size and number of paving stones needed to surround it for the first family of L-shaped ponds
21	4th lesson:	id for second variation of a family of L-shaped ponds
22	4th lesson:	ir for link between 'pond' size and number of paving stones needed to surround it for the second family of L-shaped ponds
23	4th lesson:	id for second family of L-shaped ponds (returns to drawing and counting)
24	4th lesson:	ir for link between 'pond' size and number of paving stones needed to surround it for the second family of L-shaped ponds
25	5th lesson:	id for formulating and manipulating algebraic expressions within the context of area

26	5th lesson:	ir for formulating and manipulating algebraic expressions within the context of area
27	5th lesson:	id for formulating and manipulating algebraic expressions within the context of area
28	5th lesson:	ise for formulating algebraic expressions within the context of area
29	5th lesson:	id for manipulating algebraic expressions within the context of area (ie. expressing in shortest form)
30	Interview:	mj for perimeter (non-algebraic), both with all sides given and with missing information
31	Interview:	ir for formulating and manipulating algebraic expressions within the context of perimeter
32	Interview:	isa for formulating and manipulating algebraic expressions within the context of perimeter
33	Interview:	id for formulating and manipulating algebraic expressions within the context of the perimeter of a 'deviant' rectangle with missing information
34	Interview:	mj for manipulating a non-contextual algebraic expression, including collecting like terms, and for formulating algebraic expressions within the context of area of rectangles
35	Interview:	ir for formulation of an algebraic expression within the context of area of a square
36	Interview:	ise for formulation of algebraic expressions within the context of area of a 'deviant' rectangle, and for a non-contextual question
37	Interview:	ir for formulating an manipulating algebraic expressions within the context of perimeter
38	Interview:	ma for collecting like terms (non-contextual)
39	Interview:	id for concept of an equation

APPENDIX 9 - CARRIE'S MAPPING DIAGRAM



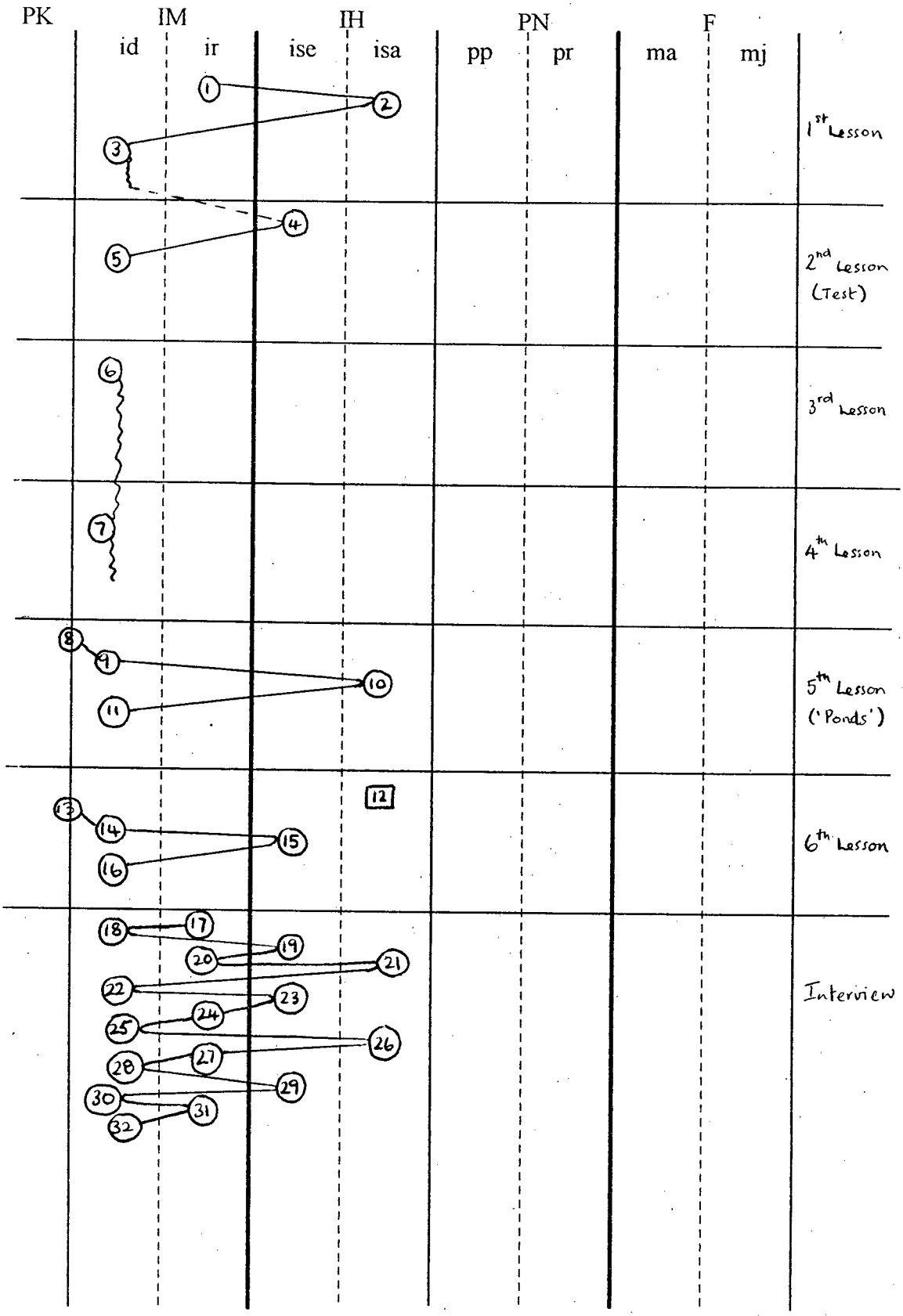
APPENDIX 10

CARRIE'S MAPPING POINTS

1	1st lesson:	iv for non-algebraic perimeters and areas of rectangles, and for perimeter of a 'deviant' rectangle with missing information
2	1st lesson:	id for non-algebraic perimeters and areas of rectangles and squares
3	1st lesson:	ise for non-algebraic perimeter
4	1st lesson:	id for non-algebraic areas of triangles and combined shapes involving triangles
5	2nd lesson (test):	ise for perimeter and area of rectangles and squares (both with all sides given and with missing information)
6	2nd lesson (test):	id for formulating and manipulating algebraic expressions, including collecting like terms, within the context of perimeter and area
7	3rd lesson:	id for formulating and manipulating algebraic expressions, including collecting like terms, within the context of perimeter
8	4th lesson:	id for 'Ponds' activity
9	4th lesson:	pp for link between 'pond' size and number of paving stones needed to surround it for rectangular ponds of fixed width three
10	4th lesson:	id for link between 'pond' size and number of paving stones needed to surround it for rectangular ponds of any dimension, and for families of L-shaped ponds
11	5th lesson:	id for formulating and manipulating algebraic expressions, including collecting like terms, within the context of area
12	5th lesson:	ir for formulating and manipulating algebraic expressions, including collecting like terms, within the context of area
13	5th lesson:	ise for formulating and manipulating algebraic expressions, including collecting like terms, within the context of area
14	Interview:	id for non-algebraic perimeter of a rectangle
15	Interview:	id for non-algebraic perimeter of a 'deviant' rectangle
16	Interview:	isa for relationship of sides in a non-algebraic 'deviant' rectangle (able to explain why she doesn't need to find the missing values
17	Interview:	ir for manipulating algebraic expressions, including collecting like terms, within the context of perimeter
18	Interview:	ise for formulating and manipulating algebraic expressions, including collecting like terms, within the context of perimeter
19	Interview:	ir for manipulating algebraic expressions, including collecting like terms, within the context of perimeter
20	Interview:	ise for formulating and manipulating algebraic expressions, including collecting like terms, within the context of perimeter
21	Interview:	ir for non-algebraic area of a rectangle
22	Interview:	id for formulating and manipulating algebraic expressions including collecting like terms, within the context of area

23	Interview:	ise for formulating algebraic expressions, including collecting like terms, within the context of area
24	Interview:	ir for formulating and manipulating algebraic expressions, including collecting like terms, within the context of area
25	Interview:	id for formulating and manipulating algebraic expressions, including collecting like terms, within the context of area of a 'deviant rectangle, and for a non-contextual question

APPENDIX 11 - KERRY'S MAPPING DIAGRAM



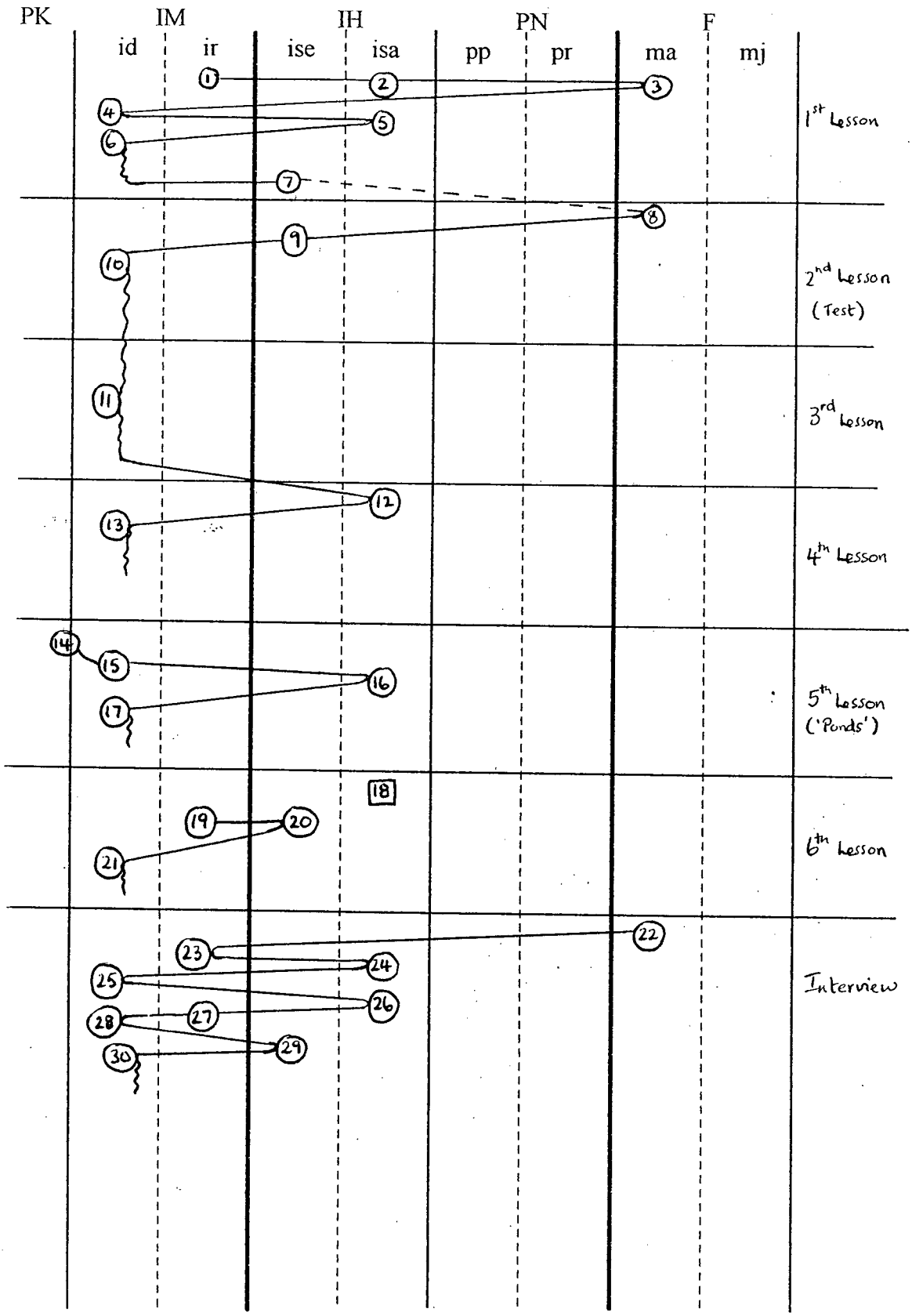
APPENDIX 12

KERRY'S MAPPING POINTS

1	1st lesson:	ir for non-algebraic perimeters of rectangles and 'deviant' rectangles
2	1st lesson:	isa for non-algebraic perimeters of rectangles with all sides given
3	1st lesson:	id for non-algebraic perimeters of squares (both with all sides given and with missing information), and 'deviant' rectangles, and for non-algebraic areas of rectangles (both with all sides given and with missing information), and areas of 'deviant' rectangles and other plane figures
4	2nd lesson:	ise for non-algebraic perimeters and areas of rectangles and squares with all sides given, and for non-algebraic perimeters of rectangles with missing information
5	2nd lesson (test):	id for formulating and manipulating algebraic expressions within the contexts of perimeter and area, and for non-algebraic perimeters of squares with missing information, and for non-algebraic areas of rectangles and squares
6	3rd lesson:	id for formulating and manipulating algebraic expressions, including collecting like terms, within the context of perimeter
7	4th lesson:	id for formulating and manipulating algebraic expressions, including collecting like terms, within the context of perimeter
8	5th lesson:	iv for 'Ponds' activity
9	5th lesson:	id for drawing the series of square 'ponds'
10	5th lesson:	isa for the pattern in the table of results created for the family of square 'ponds'
11	5th lesson:	id for the link between the 'pond' size and number of paving stones needed to surround it for the family of square ponds
12	6th lesson:	isa for formulating and manipulating algebraic expressions, including collecting like terms, within the context of perimeter
13	6th lesson:	iv for formulating and manipulating algebraic expressions, including collecting like terms, within the context of area
14	6th lesson:	id for formulating and manipulating algebraic expressions, including collecting like terms, within the context of area
15	6th lesson:	ise for formulating and manipulating algebraic expressions, including collecting like terms, within the context of area
16	6th lesson:	id for formulating and manipulating algebraic expressions, including collecting like terms, within the context of area
17	Interview:	ir for non-algebraic perimeter of a rectangle
18	Interview:	id for non-algebraic perimeter of a 'deviant' rectangle
19	Interview:	ise for formulating and manipulating algebraic expressions, including collecting like terms, within the context of perimeter

20	Interview:	ir for formulating and manipulating algebraic expressions, including collecting like terms, within the context of perimeter (figures other than rectangles)
21	Interview:	isa for formulating and manipulating algebraic expressions, including collecting like terms, within the context of perimeter (figures other than rectangles)
22	Interview:	id for formulating and manipulating algebraic expressions, including collecting like terms, within the context of perimeter of 'deviant' rectangles
23	Interview:	ise for formulating and manipulating algebraic expressions, including collecting like terms, within the context of perimeter
24	Interview:	ir for formulating and manipulating algebraic expressions, including collecting like terms, in a non-contextual question
25	Interview:	id for non-algebraic perimeter of a rectangle
26	Interview:	isa for formulating and manipulating algebraic expressions, including collecting like terms, within the context of area
27	Interview:	ir for manipulating algebraic expressions within the context of the area of a square
28	Interview:	id for formulating and manipulating algebraic expressions, including collecting like terms, within the context of area of a 'deviant' rectangle
29	Interview:	ise for manipulating algebraic expressions (i.e. expressing in shortest form) in a non-contextual question
30	Interview:	id for formulating and manipulating algebraic expressions, including collecting like terms, within the contexts of perimeter and area
31	Interview:	ir for manipulating algebraic expressions in a non-contextual question
32	Interview:	id for the concept of an equation

APPENDIX 13 - GRAHAM'S MAPPING DIAGRAM



APPENDIX 14

GRAHAM'S MAPPING POINTS

1	1st lesson:	ir for non-algebraic perimeters of rectangles and 'deviant' rectangles
2	1st lesson:	isa for non-algebraic perimeters of rectangles with all sides given and with missing information
3	1st lesson:	ma for non-algebraic perimeters of rectangles and squares (both with all sides given and with missing information), and 'deviant rectangles'
4	1st lesson:	id for non-algebraic area of a rectangle
5	1st lesson:	isa for non-algebraic areas of rectangles and squares (both with all sides given and with missing information)
6	1st lesson:	id for non-algebraic areas of 'deviant' rectangles, and triangles
7	1st lesson:	ise for non-algebraic area of a triangle
8	2nd lesson (test):	ma for non-algebraic perimeters of rectangles and squares (both with all sides given and with missing information)
9	2nd lesson (test):	ise for formulating algebraic expressions within the context of perimeter
10	2nd lesson (test):	id for manipulating algebraic expressions, including collecting like terms, within the context of perimeter, and for formulating and manipulating algebraic expressions, including collecting like terms, within the context of area
11	3rd lesson:	id for formulating and manipulating algebraic expressions, including collecting like terms, within the context of perimeter
12	4th lesson:	isa for formulating and manipulating algebraic expressions, including collecting like terms, within the context of perimeter
13	4th lesson:	id for formulating and manipulating algebraic expressions, including collecting like terms, within the context of perimeter
14	5th lesson:	iv for 'Ponds' activity
15	5th lesson:	id for drawing the series of square 'ponds'
16	5th lesson:	isa for the pattern in the table of results created for the family of square 'ponds'
17	5th lesson:	id for the link between the 'pond' size and number of paving stones needed to surround it for the family of square 'ponds'
18	6th lesson:	isa for formulating and manipulating algebraic expressions, including collecting like terms, within the context of perimeter of rectangles (both with all sides given and with missing information)
19	6th lesson:	ir for formulating and manipulating algebraic expressions, including collecting like terms, within the context of area
20	6th lesson:	ise for formulating and manipulating algebraic expressions, including collecting like terms, within the context of area of rectangles

21	6th lesson:	id for formulating and manipulating algebraic expressions, including collecting like terms, within the context of areas of squares and 'deviant' rectangles
22	Interview:	ma for non-algebraic perimeter of a rectangle
23	Interview:	ir for formulating and manipulating algebraic expressions, including collecting like terms, within the context of perimeter
24	Interview:	isa for formulating and manipulating algebraic expressions, including collecting like terms, within the context of perimeter
25	Interview:	id for formulating and manipulating algebraic expressions, including collecting like terms, within the context of perimeter of a 'deviant' rectangle
26	Interview:	isa for manipulating algebraic expressions, including collecting like terms, on a non-contextual question
27	Interview:	ir for non-algebraic area of a rectangle
28	Interview:	id for formulating and manipulating algebraic expressions, including collecting like terms, within the context of areas of rectangles with missing information, and 'deviant' rectangles
29	Interview:	ise for manipulating algebraic expressions (ie. expressing in shortest form) in a non-contextual question
30	Interview:	id for formulating and manipulating algebraic expressions, including collecting like terms, within the context of perimeter and area of 'deviant' rectangles, and for the concept of an equation

APPENDIX 15

VANCOUVER STRAND INTERVENTION THEMES

Checking	The teacher is checking for student understanding.
Showing and telling	An extended stream of interventions often involving the giving of new information but usually without checking that the students are following the explanation.
Leading	An extended stream of interventions aimed at directing the student towards a specific answer or position, often involving step-by-step explanations. Differs from *showing and telling* by its attempts to involve the students in the explanation through frequent questioning.
Reinforcing	Giving further emphasis to a significant point (often one already made by a student).
Inviting	Suggesting of a new and potentially fruitful avenue of exploration. More open-ended than *clue-giving* (see below).
Clue-giving	A deliberate attempt to point the student to the correct answer or preferred route.
Managing	Including disciplining, keeping students on task, giving instructions etc.
Enculturating	Inducting students into the language, symbolism and practices of the wider mathematics community.
Blocking	Preventing a student following a certain path (sometimes preventing a student from *folding back* to *Image Making* activities).
Modelling	The teacher explicitly models her own thought processes.

APPENDIX 16

UK INTERVENTION THEMES

Praising	Praising individual students, groups or the whole class.
Shepherding	An extended stream of interventions directing a student towards understanding through subtle nudging, coaxing, and prompting.
Rug-pulling	A deliberate shift of the student's attention to something that confuses and forces the student to reassess what he or she is doing. Often results in a return to *Image Making* activities.
Retreating	A deliberate strategy whereby the teacher leaves the student(s) to ponder on a problem.
Anticipating	Preventing students from falling into common pit-falls, trying to prevent mistakes before they happen, protecting students from error, or removing the challenging aspects of a task.

APPENDIX 17

INTERVENTION THEMES

Showing and telling	An extended stream of interventions often involving the giving of new information but usually without checking that the students are following the explanation.
Leading	An extended stream of interventions aimed at directing the student towards a specific answer or position, often involving step-by-step explanations. Differs from *showing and telling* by its attempts to involve the students in the explanation through frequent questioning.
Shepherding	An extended stream of interventions directing a student towards understanding through subtle nudging, coaxing, and prompting.
Checking	The teacher is checking for student understanding.
Reinforcing	Giving further emphasis to a significant point (often one already made by a student).
Inviting	Suggesting of a new and potentially fruitful avenue of exploration. More open-ended than *clue-giving* (see below).
Clue-giving	A deliberate attempt to point the student to the correct answer or preferred route.
Managing	Including disciplining, keeping students on task, giving instructions etc.
Enculturating	Inducting students into the language, symbolism and practices of the wider mathematics community.
Blocking	Preventing a student from following a certain path (sometimes preventing a student from *folding back* to *Image Making* activities).

Modelling	The teacher explicitly models her own thought processes.
Praising	Praising individual students, groups or the whole class.
Rug-pulling	A deliberate shift of the student's attention to something that confuses and forces the student to reassess what he or she is doing. Often results in a return to *Image Making* activities.
Retreating	A deliberate strategy whereby the teacher leaves the student(s) to ponder on a problem.
Anticipating	Preventing students from falling into common pit-falls, trying to prevent mistakes before they happen, protecting students from error, or removing the challenging aspects of a task.

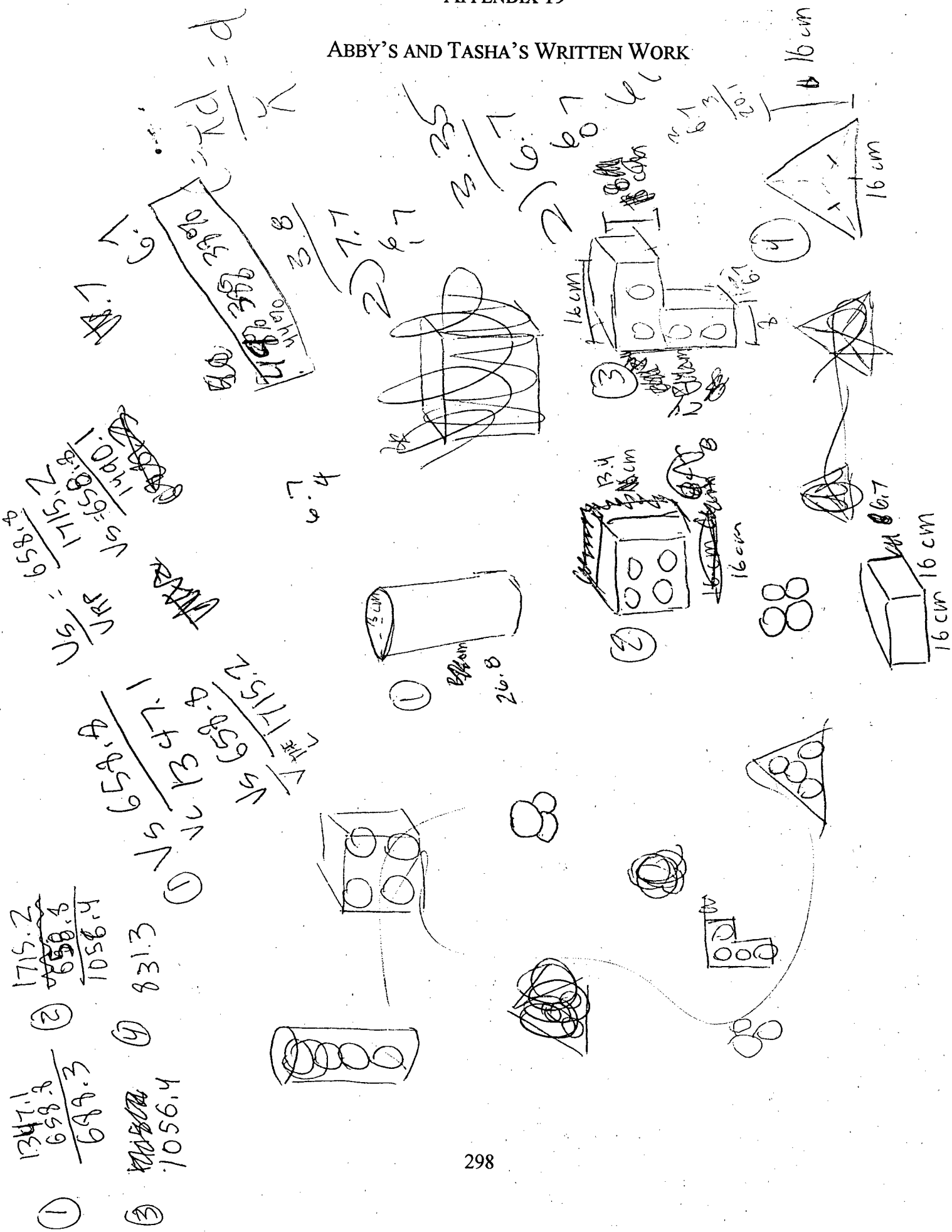
APPENDIX 18

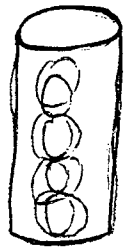
SHOWING AND TELLING, LEADING, AND SHEPHERDING CLUSTERS

Showing and Telling	**Leading**	**Shepherding**
Reinforcing	Clue-giving	Clue-giving
Enculturating	Blocking	Inviting
Modelling	Anticipating	Retreating

APPENDIX 19

ABBY'S AND TASHA'S WRITTEN WORK





5

3. $8 \times 16 \times 6.7 = 857.6\ cm^3$
$8 \times 6.7 \times 16 = 857.6\ cm^3$ $\Big\} = 1715.2\ cm^3$

$V = \frac{4}{3}\pi r^3$

$= \frac{4}{3} \times 3.4^3$

$= 164.7\ cm^3$
$\times\ 4$
$658.8\ cm^3$

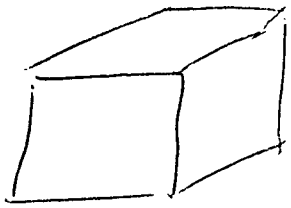
$\frac{\frac{4}{3}\pi r^3}{\pi r^2 h}$

$4r^3\ 2r$

$\frac{4}{3}$

$\frac{4}{3} \times \frac{1}{2} = \frac{4}{6} = \frac{2}{3}$

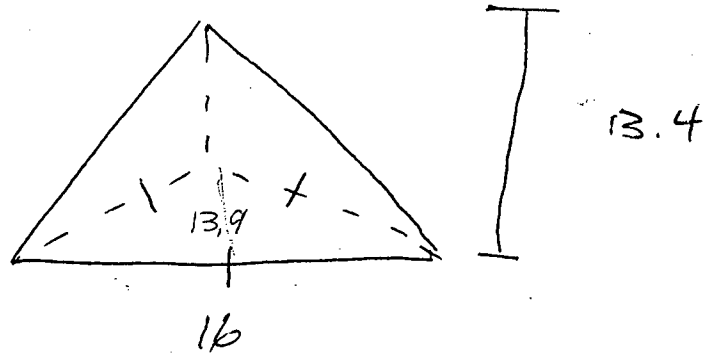
$\frac{3}{4}$



4 $V_S = 658.8\ cm^3$

$V_{RP} = 1715.2\ cm^3$

4.



$V = A.\ of\ b \times h$

$= \frac{1}{2}(13.9)(16)(13.4)$

$= 1490.08$

$a^2 + b^2 = c^2$

$b^2 = h^2 - a^2$

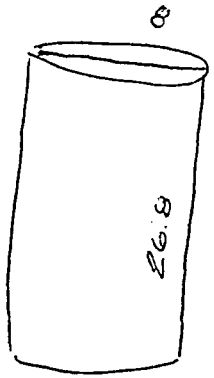
$= 16^2 - 8^2$

$= 192$

$= 13.9$

1.

Surface Area.



2 [1]

$C = \pi d \Rightarrow 25.1 \times 26.8 = 673.6\ cm^3 \Rightarrow$ [ ]

$\pi r^2 \Rightarrow \pi 8^2 \times 2 = 402.1$ 2

$SA = 1075.7\ cm^3$