STUDENTS' DEFINITIONS OF

LEXICALLY AMBIGUOUS MATHEMATICAL VOCABULARY

by

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Abstract

Fifty-five students in three high school geometry classes participated in a vocabulary survey asking them to write out, exemplify, and/or illustrate with drawings their definitions for fifteen mathematical vocabulary words: *acute, area, coordinate, diagonal, difference, exponent, factor, irrational, mean, multiple, prime, product, reduce, square,* and *variable*. All of these terms are characterized by lexical ambiguity, meaning that they have different meanings in different contexts.

The students' responses were analyzed qualitatively, driven by the following research questions. First, in light of past studies in which findings seem consistently to reveal that a large portion of the participants have inadequate comprehension and/or inability to articulate their understanding of "basic" mathematical vocabulary, what are students' ideas about the meanings of certain vocabulary words? What strands or themes of meaning attributed to the words are evident from students' responses? Do students' responses seem to indicate that lexical ambiguity causes confusion for them in their definitions? A variety of ideas and interpretations emerged from the students responses for each of the words. Some of the students' ideas conformed to conventional mathematical definitions of the terms, but many were also characterized by vagueness or confusion. Interference from the lexical ambiguity of some of the words did appear in the data, particularly with the terms *diagonal*, *irrational*, and *prime*. A secondary purpose of this research was driven by the question: what forms of written expression do the students use to communicate their meanings for the vocabulary words posed to them?

After a preliminary analysis of the data from the surveys through systematic theoretical sampling, nine students were selected to participate in follow-up interviews in which supplementary information was gathered. The interview method utilized stimulated recall, and the interviews were video-taped and transcribed.

Because students construct their own individual meanings for mathematical terminology, their ideas about specific words do reflect a spectrum of interpretations. Through focused discussion and articulation of meaning where lexical ambiguity exists, students can more confidently enter into mathematical communication and discourse.

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Chapter 1

Introduction

Foundational and complex mathematical concepts and constructs are often encapsulated in single words which have been coined or have evolved to communicate specific quantitative relationships and mathematical ideas and procedures. The teaching of these words and the ideas which they conventionally convey in mathematical discourse is, in my opinion, an important component of mathematics education. Opening the doors of opportunity for young people to enter into mathematical discussion as it relates to the world around them requires that the language and the words of that discussion and discourse be made familiar to them. Key words, the technical terms of mathematics that conventionally denote precise concepts and relationships, can instead seem to be a foreign language to learners of mathematics who are not familiar with the vocabulary. This lack of familiarity can be due to various factors, but if students are not exposed to mathematical vocabulary in school, their opportunities to construct mathematical meaning for mathematical terms will be severely limited. Occasions for students to construct mathematical meaning for vocabulary terms depend on teachers introducing and using vocabulary in the mathematics classroom, students paying attention to instruction concerning vocabulary, and commitment on students' parts to make sense of the vocabulary which they hear.

This paper reports a research project designed to elicit students' interpretations of vocabulary words which have both mathematical and non-mathematical uses. In the field of linguistics, such terms that have more than one use or meaning are called *lexically*

ambiguous. Lexical ambiguity in mathematics vocabulary has been shown in past research to cause confusion (Durkin & Shire, 1990). Non-mathematical uses of words, perhaps heard more commonly than mathematical uses, can interfere with the learning of precise mathematical meanings of terms. A primary purpose of the study reported in this paper is to examine the effects of interference from non-mathematical uses of terms on the students' perceptions of these terms in a mathematical context.

A second major purpose of this research project is to document a sampling of the variety of interpretations which students have regarding mathematical vocabulary. Students participating in this study wrote out, exemplified, or illustrated their notions of various meanings which were familiar to them for fifteen vocabulary words that commonly appear in middle and high school mathematics courses. From their responses, a wide spectrum of ideas pertaining to each term emerged. This range of perceptions and interpretations exemplifies the variety of ideas that students have concerning mathematical vocabulary. Some of the students' definitions conform to conventionally accepted mathematical meanings for the terms, but many do not. This paper describes in detail the various notions that the student participants in this study held for fifteen selected vocabulary terms.

Background

Various studies over the last two decades (Malone & Miller, 1993; Hardcastle & Orton, 1993; Monaghan, 1991; Durkin & Shire, 1990; Nicholson, 1989, 1977; Otterburn & Nicholson, 1976) have explored students' comprehension of mathematics vocabulary. Several of these which are particularly relevant to this study will be described in the next

chapter. These studies have, for the most part, been quantitative in nature, gathering data from large numbers of students and reporting the percentages of students able to provide acceptable or unacceptable definitions for selected mathematical terms. In undertaking this study, I seek to add a qualitative perspective to the discussion of students' comprehension of mathematics vocabulary. Through qualitative analysis, the question explored is no longer simply whether or not students comprehend mathematics vocabulary, but what they actually are interpreting common mathematics vocabulary to convey when they hear or read it.

Description of the present study

The starting point of this study was the overarching research question, "what do high school students think that the vocabulary terms posed to them mean?" More specifically, in this project, I endeavored to take the high school students' definitions of fifteen lexically ambiguous terms as written on survey forms and organize that information into strands of recurring themes. The students' responses were analyzed with the following questions in mind:

1)What distinct ideas are contained in each entry on the vocabulary surveys?

2) Which of these distinct ideas emerge as recurring themes in the conceptions held by the different students in this study?

3) What confusion or interference, if any, caused by non-mathematical or "everyday" meanings of these same words, is evident in the students' responses?

4) What modes of written communication do the students utilize in their attempts to express their conceptions of the terms? Do their definitions take the form of

memorized textbook definitions, or are their expressions formulated simply from their own mental images and recollections of how the terms are used in particular contexts?

5) Are there noticeable differences in the quality or correctness of responses from students in different classes based on the amount of emphasis their respective teachers claim to devote to mathematics vocabulary?

Outline

In the ensuing chapters, I relate background material, methodology, and the findings of the project. In the second chapter, the questions of this present research are situated through a review of literature in several contexts. I first briefly review background theory on students' learning and development as adolescents. I then discuss the interaction of mathematics and language concerns. Concluding chapter two, I describe and relate to the present study past research on students and mathematical vocabulary.

In the third chapter, I discuss the qualitative methodology followed in this study. The chapter details the design and administration of the data gathering instrument, and the organization and analysis of the written data from the students is elaborated. I also fully describe the planning and carrying out of follow-up interviews with selected participants for the purpose of gathering supplemental information to that gathered on the written survey forms.

In the fourth chapter, I describe in detail the range of responses given for each of the fifteen vocabulary words contained in the vocabulary survey. Included in each of these fifteen sections of this chapter are anecdotes from the follow-up interviews. Also,

comparisons are made between the responses given by the participants of this study to the responses of other student participants in similar research studies on students' comprehension of mathematical vocabulary.

The fifth chapter recounts the analysis of the data in regard to the ways that the students communicated their responses: by words, diagrams, or examples, or combinations of these three modes. Included in this chapter are detailed tables, one portraying the types of responses that were given for each word and the second displaying the response modes used by each student participant in this study for each of the fifteen words.

Finally, in the concluding chapter, I examine the question of whether or not teachers' self-reported emphasis on vocabulary in the classroom makes a notable difference in students' definitions of lexically ambiguous vocabulary. Additionally, I discuss the issue of everyday meanings of words interfering in the learning of conventional mathematical definitions for lexically ambiguous terms. The findings detailed in chapters four and five are summarized. In conclusion, questions are raised concerning the emphasis of mathematics-specific terminology in classroom discourse and possibilities for further research are posed.

Chapter 2

Literature Review

Key words in the discussion of this research

This research project explores students' responses and ways of expressing their responses on paper when asked what they think mathematical vocabulary terms mean. The words *meaning, definition*, *conception*, *perception, interpretation*, *understanding* and *comprehension* are used in my discussion of this research study. To communicate clearly in this paper, it is important for me to be as explicit as possible about my use of these words from the very start.

The first of these key terms, *meaning*, has a broad span of uses in the English language. Barnes (1990) argues that "meaning does not lie in words but in the cultural practices of those who use them" (p. 26). Wittgenstein (as cited in Bauersfeld, 1995) says "the meaning of a word is its use in the language" (p. 274). Athey (1983) distinguishes two aspects of the word *meaning*, calling one the "universal" aspect and the other the "individual" aspect: "The universal aspect includes those characteristics that are essential...and make it possible for us to communicate about something....The individual aspect includes those characteristics peculiar to a person by virtue of past experience" (p. 22). My own use of the word *meaning* in this paper applies to the "universal aspect" or those ideas which are conventionally associated with words, particularly in a mathematical context. To narrow in on one aspect from Webster's dictionary, I use *meaning* as "that which is...conveyed,

denoted...by...language; the sense, signification, or import of words" (p. 1115). I use this aspect of the word when I speak of "mathematical meanings."

Definitions give brief descriptions or explanations of the terms being defined and highlight the distinctive aspects of the ways the terms are typically used. Of *definitions*, Pimm (1993) says, "definitions, by definition, place limits around what is defined. They are often seen as...exclusive rather than inclusive. To define is to focus attention on the part at the expense of the whole" (p. 262). Borasi (1987,1989) characterizes mathematical definitions as worded expressions which provide essential information, establishing the necessary and sufficient conditions for the application of a term to a particular situation. My use of the term *definition* in this paper refers to the idea of focusing in, but I will use it in two distinct contexts. In the first context, the focus will be on particular conventional connotations, specifically when I refer to "the mathematical definition." In this case, the key issue is the satisfaction of the necessary and sufficient conditions associated with the mathematical use of a term. The second context will be in reference to an individual's ideas concerning a word, as in "the student's definition." This use of *definition* will pertain directly to the verbatim data provided by the students in written or spoken form in their answers to the question of what they think certain words mean.

This second sense of *definition* is a outward reflection of students' inward *conceptions, perceptions*, or *interpretations*. My use of these terms *conception, perception,* or *interpretation,* refers to students' ideas of words in their individual uniqueness. In these cases, I refer particularly to what the students themselves have put together in their minds as being communicated by a word. While I, as an outside observer, actually am only able to

guess about individual students' conceptions of terms (Pirie & Kieren, 1994), my use of this term is always in reference to a conjecture based on the clues that the students provide through their responses in written form or in informal interviews.

Understanding, also, is personal, and is not measurable. My own sense of *understanding* is that in the context of the mathematics classroom, or any other classroom, understanding is a dynamic sense-making of the ideas being discussed (Pirie & Kieren, 1994; Bauersfeld, 1995). A student's individual understanding of concepts is in a constant state of evolution, and continues to grow through varied experiences in a widening range of contexts. When I talk about students' understanding of terms in this paper, I am again making conjectures based upon the information provided by the students in their written or spoken responses. My use of *understanding* refers to the degree to which students connect the particular term in question to related topics and situate that term within a broader network of pertinent concepts and appropriate contexts. *Comprehension*, on the other hand, I use in relation to the students' demonstrated ability to recall and communicate the conventional, mathematical definitions that students would encounter in their mathematics textbooks or courses.

Having now indicated how these terms, *meaning, definition*, *conception*, *perception, interpretation*, *understanding* and *comprehension* are used throughout the discussion of this research project, I turn to discuss background literature which provides a context for the focus of the present study. First, some background theory on the development of adolescent thought, and then, more specifically, constructivist theory concerning mathematical learning is briefly described. The review of literature continues with the broad context of the

classroom and how language is used in the classroom, after which the discussion narrows to the language which is particular to mathematics. Finally, past research pertaining to mathematical vocabulary and students' comprehension of that vocabulary is considered in detail as it relates to the present study.

Developmental Theory and Mathematical Learning Theory

Many current theories of intellectual development in mathematics education literature trace their roots to the theories of Jean Piaget and Lev Vygotsky (Confrey, 1994). Both theorists focused on the process of concept development, but emphasized opposite ends of the spectrum as to the relationship between language and thought. Vygotsky held that language development was prerequisite to the development of thought, but Piaget maintained that thought preceded language development (Zepp, 1989). For the purposes of this research, however, I choose to highlight two points of agreement in their theories from developmental psychology: (1) adolescence is a time of mental transition in which the adolescent becomes capable of abstract thought, yet still relies heavily on concrete thought, and (2) the development of thought and expression through language go hand in hand throughout this transitional phase into formal adult thinking (Copeland, 1984; Vygotsky, 1931/1994).

Based upon these two key ideas from developmental psychology, two important points should be noted. First, the subjects of this present study, as adolescents, are in a developmentally transitional period when it comes to their ability to take in and understand new concepts. They are developing at their own unique paces. Some students remain in the

stage of "concrete thought" for much longer than others, and require more visual models, tangible experiences and concrete examples to be able to grasp concepts. Second, because thinking and language are both developing at distinct rates through adolescence, these same students' language skills may or may not be at a point at which they can articulate the conceptions which they have in their minds. Also, in endeavoring to communicate their ideas about particular concepts, they may rely on concrete models, diagrams, or examples to communicate their mental images of the concepts they are learning, because they are not able to describe their ideas solely with words. Thus, because of developmental variations, some students are better able to understand concepts and communicate concepts easily, while others struggle to do either very well.

Constructivism is a broad ideology with many proponents in the field of mathematics education. While there are various interpretations and divisions among those who claim to be "constructivists," a fundamental tenet of constructivism is that knowledge is constructed or built in a thoroughly unique manner in the mind of each individual learner (Bauersfeld, 1995; Cobb, et al, 1992; Confrey, 1994; Pirie & Kieren, 1994). Thus knowledge is considered not to be an objective entity to be taken in, but a subjective and unique creation in each individual's mind. Certainly there are ideas which are widely accepted conventions, but they are not pre-packaged, objective truths which a student simply absorbs. The building of knowledge requires the active taking in of impressions and forming of conceptions, ordering and structuring ideas based upon prior experiences and learning (Bauersfeld, 1995). While initially, the individual impressions upon which students' conceptions are formed may be sketchy and not in conformity with conventional interpretations of concepts, recurrent

experiences and discussion with others about these concepts provide more building tools with which the students modify and construct their knowledge and understanding.

The transitionality of the developmental phase of adolescence combined with the individual structuring and ordering of knowledge and understanding based on unique experiences, contribute to the wide variations in students' notions of the meanings of mathematical words as found in this study. Also contributing to the variations in students' interpretations of mathematical terminology is their exposure to that terminology and other language factors in the classroom. This is the topic to which I now proceed.

Language Use in the Classroom

Language plays an enormous role in education. As MacGregor (1993) notes, "Much of learning is possible only through the use of language, both private (to oneself) and public (communication with others). Language and concepts grow together" (p. 55). This echoes Halliday (1989), who says, "most of what we learn, we learn through language.... Language is so central to the whole of the educational process that its role was never even talked about, since no one could conceive of education without it" (p. 96). Nonetheless, as a crucial factor in the effectiveness of the education process, its role is no longer being taken for granted, but is being talked about extensively. Discussion concerning observations of classroom discourse and emphasis on language use (or lack thereof) in the classroom (Yackel, 1995; MacGregor, 1993; Barnes, 1990; Pimm, 1987; Gruenwald and Pollak, 1984) provides a broad context in which this particular study concerning mathematical vocabulary is situated. Classroom discourse and experiences provide students with much of the critical input by

which they form impressions and construct their academic knowledge. It is my belief that students' participation in classroom discourse directly influences their comprehension of and ability to use mathematical vocabulary. Additionally, taking part in the discussion of mathematical topics helps students develop their verbal articulation of what they are learning. The ways which students articulate their interpretations of mathematical vocabulary are also of interest in this study because it is only through their articulation, their putting thoughts into words, that their thinking can be observed.

The interaction of student language, teacher language and the distinctive register of each different academic discipline creates a dynamic mix that comprises the language of the classroom (Gruenwald and Pollak, 1984). The traditional classroom model had the teacher as instructor transferring ready-made "knowledge" to the students. This Platonic model is widely questioned if not fully rejected by many educators and researchers today (Confrey, 1994, Bauersfeld, 1995). A great impetus is now seen in reform movements to increase significantly the opportunities for active student participation and interaction in the classroom (NCTM, 1989). Concerning teacher-dominated classrooms, Gruenwald and Pollak (1984) warn that "an excessive amount of talking tends to create nonlisteners or at best, passive or confused listeners" (p. 50). While the call for reform continues, the question remains as to how much teachers' and students' roles in classroom discourse have actually changed over recent years.

Barnes (1990) notes that "an enormous amount of talk washes over pupils in lessons. Their problem must be to select from it those utterances which make explicit the criteria by which their performances will be judged" (p. 37). For instance, if mathematical vocabulary is

important in mathematical learning, students must be able to select out from all the classroom talk the specific terms and their mathematical meanings which they are to learn. The fact that students employ a selective attention, concentrating on the instruction at times, being distracted at other times, is obvious to most teachers. Teachers may direct students' attention to what they regard as the most important information to be learned by providing advanced organizers (outlining the content of the lesson from the beginning) and structuring classroom discourse through question-asking (Barnes, 1990). Even so, while these efforts may assist in classroom control and give indication of students' abilities to follow the agenda provided, "pupils may supply appropriate phrases without having grasped the thinking that justifies them" (Barnes, 1990, p. 39). This may create an "illusion of competence" by which students appear to comprehend more than they actually do, simply because they seem able to stay within the teacher's structure (Gregg, as cited in Yackel, 1995).

Teachers and students come into the classroom with widely varying frames of reference. Much of the disparity in what teachers believe they are communicating through their language use and what students actually can take in may be attributed to this fact (Barnes, 1990). In the mathematics classroom, teachers use mathematical terms which function to communicate specific concepts and properties. The conventional, mathematical meanings of these particular words that the teacher holds from years of experience with these concepts do not necessarily find resonance with the very limited experiences and individual conceptions for these same words held by the students. Bernard (1989) contends that "pupils who do not understand the vocabulary will not benefit from the teaching" (p. 11).

The written language used in textbooks may not necessarily shed much light for students on specific mathematical terms and concepts either. This may partially be due to the fact that the reading material seems irrelevant, far removed from the personal, immediate interests in the students' lives. The discussion of school topics is often, and particularly so in mathematics, factual and impersonal in nature. Just as students employ selective attention in listening, they do so in reading as well: "the evocation of meaning from the text requires a selecting out from the reservoir of thought and feeling, the acceptance of some elements into the center of attention and the relegation of others to the periphery of awareness" (Rosenblatt, 1983, p. 124).

Bauersfeld (1995) argues that "there is no transportation of information by language means...there is active (internal) construction of meaning only" (p. 275). Thus, differing frames of reference are inevitable, yet the existing gap between these various frames of reference can be lessened through purposeful classroom dialogue. Students should be able not only to comprehend the language used by their teacher or their textbooks, but to practice and utilize that language in relevant, interactive situations (Barnes, 1990; NCTM, 1989). Vocabulary terms are human creations which serve as social conventions for the widespread sharing of mathematical ideas. The mathematical meanings are "taken-as-shared," or assumed to be essentially the same, among those who use them (Cobb, 1992). Nonetheless, in the communication of the mathematics classroom, it often becomes evident to the participants that their individual perceptions of words are not actually shared. In these cases, deliberation and discussion among students and with the teacher can help the class to arrive at a meaning that is, once again, assumed to be shared.

As Bauersfeld (1995) points out, the process of interpreting and organizing all that we take in through our senses "is a historical, situated, and personal process, and by far it does not lead all people to the same reactions" (p. 278). For both teacher and student, therefore, the communication of that which is intended to be communicated must be jointly and actively worked toward through the creation of what he calls a "specific and differentiated language game, based on taken-as-shared experiences, activities, and objects" (p. 279). Active involvement and an abundance of interaction and sharing of ideas are essential to building a basis for mutual understanding in the classroom.

The Mathematics Register

Language is used in different ways, and these variations are of two major types: dialect and register. Dialect varies by membership in groups specified by region, social class, age, gender, or other common link within a larger language community (Halliday, 1989). Register is a variation of language that is determined by "the function it is being made to serve: what people are actually doing,...who the people are that are taking part,...and what exactly the language is achieving" (Halliday, 1989, p. 44). While dialects are probably better recognized by most people as being significant language differences within an encompassing common language, registers actually account for a greater range of language variation both in terms of structure and terminology (Biber, 1995). This paper does not actually address the subject of dialect; the mention is made here simply as a familiar phenomenon to which the idea of register can be compared. Although different dialects are hard to miss, it is possible

that distinct registers used in a school setting are not commonly recognized and accounted for, resulting in language that can place barriers in the learner's path (Barnes, 1990).

The mathematics register, used in communicating mathematical ideas, is filled with a multitude of distinct symbols, exact structural forms, and a technical vocabulary having precise mathematical definitions. The condensed symbolic code of mathematics (numerals, variables, operation signs, grouping signs, relational signs) and the rules which govern the combination and algebraic manipulation of these symbols is a topic given high priority in mathematical instruction. However, this aspect of the mathematics register is not the concern of this present study except that it may dominate the mathematics classroom to the exclusion of the expression of mathematics which is in words, the part of the mathematical register on which I wish to focus. The specific emphasis in this study is on terms and phrases which assume conventionally accepted mathematical meanings through consistent use in particular mathematical contexts but which also have other uses in other contexts. If perceptions concerning mathematics are that it is a topic having only to do with numbers and its condensed symbolic code, then introducing students to the vocabulary of mathematics and emphasizing ways of expressing mathematics with words may be passed off as being of secondary importance and given only cursory attention. Part of the intention of this paper is to encourage an appropriate focus on vocabulary in mathematics instruction, as the vocabulary is a critical part of the mathematics register.

Some of the vocabulary of the mathematics register are encountered only in mathematical contexts. Many of these words have Greek or Latin roots, and an emphasis on the study of these roots can be greatly helpful in learning such words (Milligan & Milligan,

1983; McIntosh, 1994). Examples of such roots are *dia, hemi, hyper, hypo, para, poly, equi, gonia, inter, trans, isos, bi, tri, quadri, penta, hexa, hepta, okta, nona, deci*...and the list goes on. On the other hand, Pimm (1987) declares that "in the case of mathematics, the register's most striking characteristic is the number of terms it contains which have been 'borrowed' from more everyday English....face, degree, relation, power,..." (p. 78). These are terms which Durkin and Shire (1990) call "lexically ambiguous" (p. 71). This is the type of vocabulary which is the primary focus of this study: words which are used to denote different meanings in different contexts.

Words which could be characterized as lexically ambiguous are ambiguous for different reasons. Because of this, in the field of linguistics, several types of lexical ambiguity have been classified. Among them are *homonymy*, *polysemy* and *homophony* (Durkin and Shire, 1990). *Homonymy* refers to words which are spelled the same ("same name") but have completely distinct meanings. *Polysemy* denotes words which have different, but related meanings. *Homophony* occurs when two words sound the same, but are spelled differently. Beyond these classified varieties of lexical ambiguity is the rather common problem of sloppy or careless word choice when a more precise term could be used (Durkin & Shire, 1990; Pimm, 1989).

The learning of subject-specific vocabulary is a fundamental part of learning any school subject (MacGregor, 1993). However, in mathematics, because of the tremendous number of terms in the register which have both mathematical meanings and uses outside of a mathematical context, it is possible that this ambiguity causes more difficulties and confusions for students than the vocabulary of other school subjects. One purpose of this

present study is to examine student responses to lexically ambiguous terms for evidence of such confusion.

The written expression of mathematical ideas also offers other challenges. Usually, the wording is densely packed without the benefit of many context clues. In narrative passages such as students may encounter in history texts or in literature, a main theme is noticeable and holds the reader's attention. However, in mathematics texts, typically "there is no narrative, no action, and no easily discernible topic or main idea" (MacGregor, 1993, p. 56). The message of mathematical text often boils down to specific prepositions that are used and word order that capture the precision of what is intended. Some examples of this include *dividing into* as opposed to *dividing by*, *increasing to 125%* as opposed to *increasing by 125%,* and the three entirely different implications captured in the phrases *5 is less than 7, 5 less than 7,* and *5 less 7.* In order for students to follow the descriptions of how to perform certain algorithms as they are written in text requires concentrated attention to the small details of the instruction and the examples provided. In the long run, avoidance of the densely packed wording, complicated explanations, or detailed instructions that are often encountered in mathematical texts will not help students gain the necessary interpretive skills to read mathematical texts on their own successfully. Students need experiences grappling with mathematical texts and learning to decipher the ways that worded statements and relationships expressed in verbal forms translate into condensed mathematical expressions and equations written in symbolic form and vice versa (Barnes, 1990; MacGregor, 1993).

Thus, the ambiguity in much of the terminology used to denote specific mathematical concepts and the lack of context clues in mathematical text contribute to the difficulty of

learning mathematics and its register. Piaget (as cited in Copeland, 1984) notes that the "so-called aptitude for mathematics may very well be a function of the student's comprehension of the language itself....Lack of comprehension of any single link in the chain of reasoning causes the student to be unable to understand what follows" (p. 360). A crucial question, then, is to what degree are the students actively constructing their own definitions for the mathematical words they encounter. A need to know the terms and the concepts they convey serves as the vital catalyst for the learning of concepts (Zepp, 1989; Pimm, 1987). If students do not experience this felt-need to know, learning will be hindered. A perceived lack of need on the part of students can create the situation in which they have no meaning to attach to words that are assumed by their teachers to have importance and regular use in their mathematics classrooms.

Lexically Ambiguous Mathematics Vocabulary and Previous Research Findings

This research study focuses on students' definitions of mathematical vocabulary which can be characterized as lexically ambiguous. My purpose is to explore what students think that certain lexically ambiguous terms mean and to classify their written responses into categories of recurring themes. In other words, through a qualitative analysis of the students' responses, my goal is to pull out the aspects of the concepts represented by these words that stand out to the students and which they, in turn, write about when asked the meanings of these words. By finding and classifying these recurring themes which emerge from the responses of the participants in this present study, I hope to provide a picture for classroom teachers which may be a reflection of their own students' understandings of these same terms.

This picture highlights the aspects of the words' conventional mathematical meanings that students are, indeed, constructing for themselves, and also may give indication of which facets of these words' conventional meanings need greater emphasis so that students can have increased opportunity to build onto their present conceptions of these terms.

Various past studies on lexical ambiguity and student knowledge of mathematics vocabulary are particularly relevant to this study (Malone & Miller, 1993; Hardcastle & Orton, 1993; Monaghan, 1991; Durkin & Shire, 1990; Nicholson, 1989, 1977; Otterburn & Nicholson, 1976). Each study also suggests a discouraging level of student competencies with mathematics vocabulary. Miller (1993) calls the results of the Malone and Miller study "quite disturbing" (p. 312). Hardcastle and Orton (1993) characterized their findings as "somewhat alarming" (p. 14). Roughly half of the vocabulary words which I chose to include in my study because of their lexical ambiguity were also terms which were posed to students in these other studies as commonly used mathematical vocabulary terms. The responses elicited for these same words in these other studies provide an interesting means of comparison across two decades and three continents, comparisons which I will make as I discuss each particular word in the fourth chapter.

Durkin and Shire (1990) discuss the interference of everyday connotations with mathematical meanings of terms, drawing from a variety of studies on mathematics vocabulary and lexically ambiguous terms. Their paper focuses mainly on raising the issue of lexical ambiguity for educators' awareness, and provides anecdotal examples predominantly from the primary school years. Monaghan (1991) conducted a study concerning four terms or phrases used in calculus synonymously: *limit, converges, tends to,* and *approaches*. Of

significance for the present study is the attention Monaghan gives to the potential interference of everyday connotations of these terms or phrases to the mathematical meaning and use. While the terms have "equivalent" mathematical meanings (p. 20), the students' responses to these four phrases gave indication that everyday connotations do create confusion.

Malone and Miller (1993) conducted their study with over 2000 students in grades eight through eleven in schools across Australia. The purpose of the study was "to investigate secondary students' ability to communicate their understanding of math vocabulary in writing" (p. 178). A list of 20 words for each year level was compiled with the input of participating teachers, words which were regularly used in their classrooms, but which were not newly introduced. The students were instructed to define the words, give examples, or use pictures as necessary to show their knowledge of the word meanings. All the various types of responses gathered were then coded as blank or acceptable or unacceptable as a demonstration of an understanding of each term. The criteria for this judgment between acceptable and unacceptable was unspecified in the research report. From the few detailed examples of responses that were given and the coding they received, Malone and Miller's coding of an acceptable response seemed rather lenient. If that is so, then the broad results offered in the form of percentages of acceptable, unacceptable, or blank responses for each word may be even more optimistic than is justified.

The Hardcastle and Orton (1993) report of their study provides some detailed examples of British students' responses to mathematical terms. The overall results of their study were also quantitative, in that results were tabulated as to the percentage of students that gave correct, blank, or confused responses for each of the 12 words included on their

research instrument. In their study, level eight students were asked to look at a figure and then answer questions which incorporated the key vocabulary words and required some comprehension of those terms for students to be able to give an answer. In this way, the students were provided a focused context in which to think about and respond to the twelve terms included in that study. This format may indicate an effort on the part of the researchers to diminish the effects of lexical ambiguity. However, while many of the terms they included are lexically ambiguous, there is no explicit mention of lexical ambiguity being a consideration in their choice or analysis of these words. The few detailed descriptions of the responses given by Hardcastle and Orton (1993) are similar to the qualitative analysis which I seek in the current research project.

Otterburn and Nicholson (1976) report on a research project undertaken by Otterburn in Northern Ireland, and Nicholson (1977, 1989) follows up with two other similar, but revised studies, also in Northern Ireland, on mathematics vocabulary. Otterburn tested nearly three hundred students on 36 mathematics terms, asking the students to (1) say whether or not they understood the term, (2) give the symbol if a word had one (for example, = for equals), (3) demonstrate the meaning of the word through diagram or numbers or symbols, and (4) describe the meaning of the term in words. This combination of responses was then judged to be correct, blank, or confused, and the findings were reported as percentages of responses which fell into each category for each word. Nicholson (1977) followed this with revisions in which the words were given a more focused context by posing tasks which the students were to perform. As the instructions for these tasks included the terms and demanded a comprehension of these terms, the critical question became whether the students could

operate with these terms as opposed to being able to articulate the meanings of these terms. Responses to these tasks were again categorized as being correct, confused, or blank. The results of his similar study in 1989 were also reported in terms of percentages of acceptable responses.

Each of these five studies on mathematics vocabulary paint a rather bleak picture of students' overall comprehension of vocabulary words which, in each case, were tested with the assumption that the students should know those words. In each study, the results were reported as percentages of students who demonstrated some acceptable level of understanding of these terms. Thus, while the combined picture drawn by the Malone and Miller (1993), Hardcastle and Orton (1993), Otterburn and Nicholson (1976), and Nicholson (1977, 1989) research studies may have been painted with broad strokes, those broad strokes do highlight the problem of a vast number of students working with an inadequate comprehension of very commonly used mathematics terms. This provides a backdrop and rationale for asking the question raised in this present study: if students are not understanding these words in conventional ways that their mathematics teachers would think they should be, how, then, are they interpreting them? What, actually, are students thinking that these words mean?

Chapter 3

Methodology

The purpose of this research project is to explore the range of students' interpretations of fifteen mathematics vocabulary words which are commonly taught prior to and in first year algebra and geometry courses. Past research studies (Malone & Miller, 1993; Hardcastle & Orton, 1993; Durkin & Shire, 1990; Nicholson, 1989, 1977; Otterburn & Nicholson, 1976) examining students' abilities to define mathematics vocabulary indicate that students have many varied and faulty notions about the meanings of vocabulary words, but few of these studies describe the students' notions in detail. Thus, this research is undertaken to supplement the findings of other researchers concerning students' comprehension of mathematics vocabulary, and is driven by several related questions: "What, actually, do students say that certain mathematics vocabulary words mean? Are there notable trends in how they express their definitions for these terms? How can the content that students express in their definitions of these terms be classified into distinct categories based on the different ideas contained in their responses?"

Development of the Vocabulary Survey

The format of the data gathering instrument is a vocabulary survey list designed to resemble the instrument used by Malone and Miller (1993). Because the focus of my study was on vocabulary words which are lexically ambiguous, an important characteristic of the data gathering instrument was that it would provide very little context for the students other

than that the terms listed were words that were used in their mathematics classes. An adaptation of the Malone and Miller design was deemed to be well-suited for an inquiry into students' actual thinking about words because of the open format and the freedom of expression that it allows the students. The instructions were for the students to write any and all meanings that they could think of for the fifteen terms listed. I also gave students the option of drawing pictures or diagrams or providing examples which might help them give a more complete picture of their conceptions of these words. Pictures or examples which students associated with the vocabulary terms were important aspects of their thinking about the words that I wanted to capture in the data.

My instrument included fifteen words, all lexically ambiguous. My written instructions were for students to "write down what these words mean to you as completely as you can. You may use pictures, diagrams, or examples or any other means you can think of to illustrate your meanings for the word." I also emphasized in my spoken instructions to the students for them to write down as many different meanings as they could recall for each word. I wanted to observe to what degree students recognize the lexical ambiguities that exist, and I hoped that student responses might reflect their thinking in some more authentic, unique way than a regurgitated textbook definition. Therefore, I purposefully sought to de-emphasize the idea of a standard, textbook definition by specifically asking for what the words signified to the students and requesting as many different meanings for the words as they knew.

The instrument contained fifteen lexically ambiguous terms to which the students were to respond. Potential words for this study were gathered from Malone and Miller

(1993), The Pre-algebra Lexicon by Hayden and Cuevas (1990), the index of a first year algebra text, and my own personal recollection of terms I used regularly in my algebra and geometry classrooms. The initial list of vocabulary words was then shortened to include only terms that had both mathematical and non-mathematical uses, or meanings. This pool of lexically ambiguous words was compiled before the sample of students was selected. Thus the textbooks used by the students who participated were not used in the process of compiling the original list of potential words. However, once the student sample was arranged, the list was further narrowed to fifteen words, based upon the cooperating teachers' and my judgments that all of the words should have been encountered by the students prior to their first year algebra courses. The words selected for the survey were *acute, area, coordinate, diagonal, difference, exponent, factor, irrational, mean, multiple, prime, product, reduce, square,* and *variable*.

These fifteen words were arranged on the vocabulary survey three or four words to a page, leaving ample room for the students to write or draw or exemplify all they wanted about each term. Because of the possibility of certain words jogging the students' memories about subsequent words on the test, I made three forms with different orderings of the fifteen words to reduce potential bias that might otherwise occur. The first form was arranged alphabetically, the second in reverse alphabetical order, and the third, a mixture beginning in the middle alphabetically and working forward and backward alternately.

Sample Selection

This research is not intended to produce generalizable inferences, but to showcase the variety of ways in which students can perceive words. This variety occurs even within a single classroom in which the teacher may believe that the students are hearing and responding to key vocabulary words in a uniform way. Therefore, convinced that a wide range of ideas and understandings about mathematical terminology does exist within single classrooms of students, and hoping to explore that diversity of conceptions, I sought answers for my research questions from students in two local high schools in northwest Washington state. As most of my own teaching experience is with the ninth and tenth grade age group and mathematics level in the classroom, I targeted this particular age group for my research. The practical selection of the student subjects was based upon their willingness to participate, the cooperation of the school administrations and two classroom mathematics teachers who offered instructional days for their classes to participate in this project.

Ninth and tenth graders, in general, are enrolled in mathematics courses which are requirements for graduation, as opposed to elective courses. Therefore, by the time they reach graduation, the vast majority of students have been exposed to the vocabulary words which are included in this study. Most students at the ninth and tenth grade level are enrolled in first year algebra or geometry courses in the Washington public school systems. Therefore, I found my student sample by first seeking out teachers in local school districts who teach these courses who would be interested in this research project and willing to allow me to have some of their class time for conducting the study. I found two such teachers in two different school districts and received the approval of their administrators to proceed

with the research. The sample size needed to be large enough to give some breadth to the results, yet small enough to be manageable for doing a qualitative analysis of the data. Out of the sections of geometry or algebra courses that these teachers taught, three geometry classes were selected to participate. This provided a final sample size of 55 ninth and tenth grade students, predominantly from working and middle class, Caucasian families, with a very small number of minority students. Both high schools serve their entire small-town population as well as an extended rural area outside of city limits. The first high school, which I will refer to as school A, had a student population of 478; the second, which I will call school B, had a population of 1223 at the time of administering the vocabulary surveys in the spring of 1996.

Administering the Vocabulary Survey

I administered the vocabulary survey in three different, intact classes, the first in school A with a male teacher who claimed to place little emphasis on vocabulary, and the other two classes in school B with a female teacher who says she emphasizes vocabulary in her instruction. It was a benefit to have two teachers participating who place different levels of emphasis on vocabulary instruction so that the responses of the students in their respective classes could be compared. This comparison is recorded in chapter six. All three classes were geometry courses, with ninth, tenth, and eleventh grade students. At the request of the cooperating teachers, the eleventh graders participated with the rest of their class in writing the survey, but their responses were not included in the analysis because of my specific targeting of ninth and tenth graders.

Because I wanted to be sure that the allotted 45 minute time limit would be sufficient and that the instructions on the vocabulary survey would be clear, I asked a student who was in a different section of the first teacher's geometry class if he would be willing to do a pilot run with the vocabulary survey. This student volunteered his time, and worked through the survey in a test run before I administered it in the classroom. His responses were thorough, and he took most of the allotted forty-five minute period to complete the survey. His responses yielded no surprises, and he reported no difficulties with understanding the instructions or the intent of the survey. Thus, no changes were made to the survey form based on the trial.

The class periods ran for fifty or fifty-five minutes in the two schools where I conducted these surveys. Leaving five to ten minutes at the beginning of the period for attendance matters and instructions concerning the filling out of the surveys, I allowed for up to forty-five minutes for the students to complete the forms. The allotted time was more than sufficient, as students' completion times ranged from ten to thirty-five minutes.

When the appointed class periods came, the teachers briefly took care of attendance and class business, and then introduced me to the students. I thanked the students for their participation, distributed the data gathering instruments, and read the written instructions to them. The three different orderings of the vocabulary survey forms were distributed evenly in each of the three classes that participated. When the instructions were read, I additionally stressed that I was not going to grade their responses as right or wrong, but rather wanted their honest replies to what they thought these words meant, both in a mathematical context and otherwise. I also emphasized that I wanted them to express on their forms as many

different aspects of the terms as they knew. The students were then asked to fill out the surveys and return the forms to me after they completed them. The students put their student numbers on their surveys to allow a degree of anonymity while still maintaining the possibility of being contacted through their teachers for follow-up interviews. When the students were finished, they had various other class assignments that they could do while the rest of the class completed the surveys. The teachers remained present throughout the administration of the surveys, and all the students were able to complete their surveys in a quiet classroom without any distractions other than students bringing their completed forms to me at the front of the room.

Data Organization and Initial Analysis Methods

The first step in the analysis process that I followed was to number the survey forms as they were turned into me. This numbering allowed for quick reference to the original raw data throughout the analysis process. Second, I organized my data in such a way as to expedite examining and analyzing the data in two different ways. One way in which I analyzed the data was by the students' written meanings for each particular word. Thus, I separated onto fifteen separate computer spreadsheet files the 55 responses that pertained to each word. The second way I analyzed the data was an examination of the modes by which the students communicated their responses: by definition, description, example, diagram or by leaving the entry blank. The use of the computer spreadsheet was critical to the organization and consolidation of my data. I listed the students' survey numbers down the first column and their responses across the columns of the spreadsheet. I entered each

student's response verbatim, but broken up and placed into the appropriate columns according to the mode by which the student communicated his or her response.

Because information entered into a computer spreadsheet application can be present in the computer's memory, but not actually show up on a print-out, I formatted the spreadsheet so that all the data would be organized and viewable on a print-out. Because the columns had to be limited to reasonable widths, the number of rows allotted to each individual student varied according to the length of their responses. This resulted in data for some of the words taking up to four pages for the entire printout, while the data for other words required only two. I then taped these two to four pages together to create the fifteen lists of the responses for each word. The paper printouts were necessary so that all the data for each word was consolidated into one viewable list. These printouts became the focus of further analysis, both in reference to the ways or modes by which the students responded and in reference to the content of their responses to the specific words. Nonetheless, the original raw data remained close at hand for clarification as necessary.

The initial analysis of the data in terms of the modes by which the students chose to express their meanings for the words was accomplished during this process of the organization and input of the data onto the spreadsheets. Seven general categories were used, requiring seven columns on the spreadsheet. The first four, *definition, diagram, example*, and *blank* were categories that I expected even before reading through the responses for the first time. The other three categories, *description, repetition*, and *confusion*, arose from the data as it seemed necessary to separate from the category of *definition* worded responses that seemed to provide less complete information.

In general, I categorized as a *definition* those responses that provided more explicit information about what students thought a word was, and that included key properties that were associated with the word, whether these were correct or incorrect by conventional standards. What I labeled as *descriptions* were worded responses that tended to be along the lines of describing the context in which the word is used or the way to operate mathematically with the concepts represented by the word. For example, for the word *coordinate*, one student wrote, "A coordinate is a point on a plane. One might say, 'find the coordinate (-2,3).' So you would go back 2 and up 3 from the center, and you'd find yourself there." The first sentence was placed under the *definition* category, whereas once this student goes on to explain how to find a particular point by its coordinates, the rest of the worded response was slotted under *description.* For the word *area*, one example of what I called a *definition* was "the space within the perimeter of an object," but other responses like "one side of something times the other side" or "length times width" were dubbed as *descriptions.* With the word *exponent*, the *descriptions* were the responses similar to "it is the little # [number] on the top right side of a #," while included in *definitions* were expressions such as "an exponent is a number that tells how many times a number should be multiplied."

Repetition was a category in which the students used a form of the word in an attempt to define it. For example, a common response for the word *irrational* was "not rational." While such a response may indicate that the students recognize the prefix *ir-* to mean "the opposite of," no indication is given as to their understanding of what *rational* means. With the term *factor*, several of the responses which were slotted under *repetition* were "factor a number to the lowest it can go," "factor down," or "factor something out."

Confusion specifically contained responses in which it seemed apparent that the student confused another mathematical vocabulary word for the actual word before them, giving a definition-type response that seemed clearly to be correct for another word, but not for the word in question. Several of the responses to *multiple* seemed to qualify for this category: "to times something," "timing two numbers," "all numbers that can go evenly into another number is the multiples of that number." Other examples which fit under *confusion* were "prime is that little number above another number," and the definition of a diagonal as "a line that meets a circle at exactly two points."

Blank was the label indicating that no response was given for a particular word. Replies such as "I don't know", or "nothing" were also included under this heading.

If there were drawings, I described them as completely as possible under the heading *diagram* on the spreadsheet. *Angle, coordinate, diagonal* and *square* were the terms which were illustrated by the students most frequently with the use of diagrams. This category, in particular, required keeping the original surveys close at hand for more in-depth analysis of the students' actual drawings.

The category for *examples* included a range of different cases. Under the category of *examples*, number operations were regularly used to illustrate the calculation of *area*, *difference* (e.g. "12 - 2 = 10, 10 is the difference"), *mean* (e.g. "5 + 7 + 4 + 4 = 20 / 4 = 5 would be the mean of this problem"), and *product* (e.g. "4 x 2 = 8, 8 would be the product"). Quite a number of students used symbolic notation for formulas in exemplifying area: "such as b x h, $^1/_2$ h ($b_1 + b_2$)" or "circle = πr^2." Some students also wrote sentences which used the terms in specific contexts: "you are acting *irrational*" or "the test was *multiple* choice."

This initial division into these seven categories provided the basis for continued examination of the different modes by which students expressed their ideas about the terms. This examination was particularly concerned with the question of whether students' surveys might reveal noticeable trends in the ways (by word, example, or drawing) that they chose to communicate their meanings. A more detailed explanation of that analysis process and observations concerning the response modes chosen and used by the students is found in chapter five.

Word Analysis

Once the organization of the data was complete and the spreadsheets were printed out, the analysis process turned to examining the actual content of the responses to the fifteen words. The content analysis centered around the search for emerging, repeated themes among the students' expressed meanings for the words. This second stage of analysis and categorization of themes followed a rigorous examination of the data and was instrumental in the selection of students to interview in the second round of data gathering.

I used the consolidated data on the spreadsheet print-outs for my analysis. Taking each of the fifteen words one at a time, I studied each response listed on the spreadsheet in a search for as many different facets of meaning for each word as were contained in the responses of the students. Each of these facets of meaning became a category heading which emerged from the data. Each distinct idea brought out in a response was listed under an appropriate category heading either emerging from that response or which had previously come out in another student's response. For example, with the word *factor*, the first response

listed is "there are two factors in a multiplication problem. All numbers can be factored down to prime numbers." Out of these two sentences, I pulled out three distinct ideas: *factors* are numbers in a multiplication problem; *factor* is spoken of as a process; and "*factored down*," the phrase indicating that *factor* is seen as a process, is a repetition of the term *factor* in the attempt to explain the term. The very next response on the list states, "To factor a problem is to simplify it. A factor can also be a variable to be taken into consideration. $(x + y)(x - y)$ factored $= x^2 - y^2$" From this response, the first idea, simplifying a problem, fit under the already created category of *factor* as a process. A second idea contained in this response created a new category, something to be considered. This category was later broadened to encompass responses that generally referred to any everyday connotation of the word, rather than a mathematical notion of the word. The third portion of the response also fit into the category of factor being a process, and the response also created another category: an example which was not correct.

As I proceeded down the responses to each word, this process was repeated, the number of categories grew, and the number of responses fitting under category headings also increased. In this way, many individual responses were broken down as fitting into several categories. The growing list of category headings were recorded on individual pieces of cardstock, and when a response either created or fit under a category heading, the code-number of that respondent was listed under the category heading. Once several category headings had emerged, the process became faster, as subsequent responses often matched several of the ideas seen in earlier responses. With each new category that seemed to emerge for a term, I would check back through all the previous responses I had already examined to

determine whether those responses might also have that distinctive idea in them. This process is called systematic *theoretical sampling* (Glaser & Strauss, 1967). After analyzing the responses to each word, I would have anywhere from 10 to 30 different categories that had emerged from the data and were represented by the responses of the students.

Sample selection for interviews

Looking at all the categories on the cards, I then asked the question, "what issues are raised here about which I need more information?" I selected categories that surprised me, that troubled me, or that represented responses that were internally inconsistent or seemed to indicate an incomplete notion of the word. The criteria for selection varied according to the word and the corresponding categories. As I prepared particular questions that I would want to include in interviews, I wrote down the respondents' numbers whose responses actually raised each question for me. For each word, I wrote three to six different interview questions and listed on a single piece of paper all the numbers of the respondents to whom I might pose these questions. After completing this process for all 15 words, I tabulated the number of times each respondent's number was listed. By interviewing the respondents whose numbers were listed most often, I would more likely glean the additional data that would provide a larger and clearer database. Of the students whose numbers had appeared most often, I selected three students from the first class, four from the second, and three from the third class to interview, and contacted their teachers to arrange a date to come and conduct the interviews. By selecting these ten students, the questions raised concerning the responses could be posed to at least one of the ten, and most of the questions could be posed to several

of them. This selection also allowed for interviews with students in each of the participating classes.

The questions for the interviews were particular to each student's responses on their vocabulary surveys and were designed for eliciting more information or clarification from the students about their conceptions of the words and about their written responses. The interview methodology utilized *stimulated recall* (Pirie, 1996), which is a process of showing the students what they had written previously on the vocabulary survey forms and prompting them to elaborate on what they had recorded. I used this approach so that the interviews would provide additional information, and not simply elicit repeat responses of what the students had previously written on the vocabulary surveys. The interview schedules were limited so as not to exceed fifteen minutes, and this amount of time proved sufficient. The interviews were videotaped and transcribed to provide supplemental data to that gathered by the surveys.

Conducting the Interviews

I was able to conduct the first seven interviews in two class periods on the same school day. All of these seven students were taught by the female teacher in the second participating school. The students were dismissed one at a time from class to be interviewed in a quiet corner of the library. A videocamera was positioned to record both our conversation and any writing on papers on the table at which we were seated. Because the interviews were informal and questions arose from their individual responses, the format was very conversational. I found the greatest difficulty was maintaining the stance of

researcher/inquirer/listener. I observed myself on several occasions shifting into the instructor mode in which it became very easy to guide the student in a certain structure of thinking regarding whatever particular word we were discussing at the time. While the caution not to ask leading questions was uppermost in my mind throughout the interviews, the videotapes of these interviews revealed the struggle I had in not leading to some degree, as the students did not freely volunteer a great deal of information. In addition, the dynamic of the students' perspective of me as something of a "teacher-figure" seemed to make them more reticent in their conversation with me as they seemed to grope for "right" answers to reply to my questions. While I was not seeking particular, set answers, these students appeared to struggle in their attempts to provide what they thought I was wanting to hear.

I arranged to interview the remaining three students from the first participating school on a later date. When that day came, one of the three selected students was absent. Thus, I interviewed only two from that class. I found that even though these two students were from a different class and school, their interviews did not seem to give a novel perspective or add significantly to the data I had gleaned from the first seven interviews. Having already transcribed the first interviews by this time, I was also significantly more cautious about asking leading questions. Because of this, it is possible that I ended up not probing as thoroughly as with the first seven, not wanting to lead them down a particular path. In addition, these two particular students were even more reticent about volunteering information than the previous teenagers with whom I had conversed.

The videotapes were transcribed in order to be better analyzed and to capture the responses of the students as further data. Again, the raw data of the videotapes were kept

close at hand for clarification of the transcriptions as necessary. I also found that doing my own transcription was very valuable in re-familiarizing myself with the conversations that I had with these students. Further analysis and discussion of the interviews is included in chapter four.

Chapter Four

Categorizations of Responses Given for the Fifteen Words

Through the process of systematic theoretical sampling (Glaser & Strauss, 1967), the students' definitions, examples, and diagrams were analyzed to create a list of the distinct ideas that were present in the responses. Some of the words elicited a variety of ideas, while others seemed to have thoroughly taken-as-shared ideas associated with them. In this chapter, I describe the categories that emerged for each word and discuss these in detail. Comparisons are also drawn between the responses gathered in this research study with data from other similar studies in which some of the same terms were posed to other students.

Supplementary information was also gathered from the follow-up interviews which I conducted with nine of the 55 students in the sample. In each section of this chapter, I outline the questions that I posed to students in the interviews and include anecdotal portions from the interviews which shed more light on what these particular students who were interviewed said concerning their ideas about the vocabulary terms. In some of the interviews, it seemed that the students responded at times with denial or forgetfulness concerning their written responses recorded on the surveys. This is likely attributable to the somewhat awkward dynamic and expectations which these students may have brought into the interviews that they were being asked for "right" answers rather than for straightforward, individual replies. In the portions transcribed from the interviews, I refer to myself as "M," and the student is denoted by his or her first initial.

The fifteen targeted vocabulary words are listed below in alphabetical order and the responses pertaining to each from the written surveys and follow-up interviews are described.

Acute

This particular word yielded the least varied results. While it has usage outside of mathematics, its mathematical meaning pertaining to angles was the prevalent sense of the word in this study. Thirty-eight out of the 55 respondents replied that *acute* meant an angle measure less than 90 degrees. Four others responded that it meant less than 90 degrees accompanied by a drawing of an angle. An additional three students said *acute* meant less than 90 degrees without specifying angle measurement. Likely they saw no need to elaborate that their response was referring to angle measurement as opposed to temperature. Only eight gave answers that were clearly incorrect or blank, yet even the incorrect responses were confusions with right angles or obtuse angles. A lone individual gave what I categorized as an everyday response ("small, severe, or sharp") with no specific reference to a mathematical context. Thus, it seemed clear that a large majority of the students associate the word with angles, and most have a solid notion of the concept of *acute*.

I pursued a question concerning the application of *acute* to refer to triangles in a follow-up interview with one student whose written response had been "being less than 90 degrees. acute angle. acute triangle." This student was one of very few who had written about *acute* in the context of triangles. However, when I asked her what she meant by an *acute triangle*, she replied that she thought she remembered some mention of *acute triangles* in class, but she could not define it within the context of triangles.

Two of the students that I interviewed had written responses to *acute* that had not referred to angle measurement. Nonetheless, both associated *acute* with angle measurement immediately in the interviews. One of these students was the individual who had given only "small, severe, or sharp" as his response. When I showed him what he had written and asked him about it, he said, "severe, sharp aren't math, but small, less than 90 degrees is probably what I should've put." As I probed further about his conception of *acute angles*, I showed him an angle in standard position (see Appendix B) on the coordinate plane which would have coterminal angle measures of approximately 315 degrees or -45 degrees.

> *M: If I had my grid like this and I had an angle like that, would you call that acute?*
> *S: I would, but I think there's another word for it, but I don't know it.*
> *M: Is there another way to think of this other than just this angle* (referring to the acute angle)
> *S: There's the minor angle* (referring to the acute angle) *there, and then there's the major* (referring to the 315 degree angle)

Another student that I interviewed concerning his response to *acute* had responded on the vocabulary survey with "is a triangle that is over 90 degrees" and had drawn two different *obtuse* angles, one labeled 120 degrees and the other not labeled. When I showed him his written response and asked him if he could elaborate on *acute triangles*, he stopped mid-sentence, saying that it wasn't a triangle he meant, but just an angle. I asked if he would identify the *acute angles* from a page of angles (Appendix B). Interestingly, he actually picked out the *acute* angles in contradiction to his previous written response that *acute* angles are "over 90 degrees." I then asked if he could pick out an angle that was 90 degrees, and without hesitation, he selected the *right angle*. In this case, it is possible that he had learned the mathematical meaning for *acute* in the time interval between writing the survey and speaking with me in the interview, or he had simply miswritten his earlier response.

Area

Students had difficulty expressing the meaning of *area* without repeating the word *area* in their response. The most common term used in the attempt to communicate their notion of the word was "space," which paired with various prepositions created some interesting word-pictures: "space in" was the most prevalent (20 respondents), "space on," "space taken up by," and even "space around." Some of these phrases actually seem better representations for the concepts of *perimeter* or *volume*. Hardcastle and Orton (1993) also found that many of the confused responses gathered in their data concerning *area* were confusions with the concepts of *perimeter* or *volume*.

Almost one-quarter of the students (13) seemed to have a process-oriented conception of *area*, as their responses consisted entirely of some *area formula*, such as "length times width," or some worked example finding the *area* of a rectangle, triangle, or circle. Their responses seem to answer the question with which they are likely more familiar, namely "how do you find the *area*?" rather than "what is *area*?" This observation coincides with the findings of Hardcastle and Orton (1993), who found that only 49% of the 76 twelve-year-old students participating in their study could demonstrate correct comprehension of the word. They explain that "[t]he main reason for the low correct percentage is that many pupils did not explain the word at all, they simply performed an area calculation. The topic...had clearly been interpreted only as a number crunching exercise" (p. 14). Miller (1993) reported that in the study undertaken by her and Malone in Australia, only one-third of the ninth grade participants could correctly define *area* using words only. This figure rose to 75% when diagrams or examples or *area formulas* were accepted. In the present study, 26 students,

almost one half of the sample, included an *area formula* or worked example of finding an *area* in addition to some other written explanation about the word *area.*

Ten students associated area with "two-dimensional" or "surface." Only three students mentioned square units in their responses. While the idea of measurement specifically came out in seven of the responses, the wording in a few of the responses seemed to indicate a confusion with either *perimeter* (e.g. "distance around") or *volume* (e.g. "amount of water in" or "amount of space inside of...").

Other previous studies and literature pertaining to students' conceptions of *area* report similar findings. Battista (1982) writes that we generally find *area* by measuring lengths and then substituting those values into *area formulas.* He observes regretfully that this indirect manner of measurement leads to students' developing vague ideas about what *area* actually is. "Asked to define it, many of these students respond that area is 'L x W.' Asked to compare the areas of two irregularly shaped regions...many students respond that the 'distances around' should be found (p. 362). Stone (1994) refers to a 1983 study by Woodward and Byrd who "found that almost two-thirds of the subjects in a study of eighth grade students believed that rectangles with the same perimeter occupy the same area. They concluded that the traditional formula-based approach is not effective in instilling conceptual understanding" (p. 590).

One student's particularly intriguing response in the present study was a diagram of an irregular shape with a grid of square units drawn over it. The squares that were completely within the perimeter of this shape had X's drawn in them, seemingly indicating the number of square units contained within the shape. Yet the written explanation accompanying this

drawing said, "the sum of a shape all the way around." While this student apparently had a good, concrete notion on which his or her conception was built, the written expression of that notion seems more appropriate for defining *perimeter* than *area*. This example highlights the disparity that may exist between students' understanding or mental image of a concept and their ability to put that into words. It also serves as another example confirming Battista's (1982) observation of how many students respond to *areas* of irregular shapes by trying to find a distance around as a means of measuring *area*.

I raised questions with several of the students I interviewed concerning their responses that were limited only to "length times width" or a very similar response. I asked one if "length times width" were the only way *area* can be found. He replied that "base times height" or "π - r - squared" also could be used. In the case of base times height, he stated the same formula again simply in different terms. When I probed for his ideas about "π - r-squared," he replied that it would be used for circles. He seemed to recognize that different shapes require different formulas for finding *area*, but our discussion did not extend beyond a formula-based notion of *area*.

When I read another student's response of "height times width" back to her, she was quick to insert, "or base times height." This was rather interesting to me as they are simply different words in the same *area formula*. During our conversation on *area*, though, as I tried to ascertain from her to what figure specifically height and width are in reference, she again asserted, "um, I don't know. I wouldn't use that. I'd use base times height." Of course, "height times width" had been her original written response, but this interaction points out not only the fixation on a process-oriented conception of *area*, but also her failure to

recognize "base times height" and "height times width" as being the same notion, perhaps because the formula for that process is memorized in a very particular way.

Another of the students that I interviewed had also written in her response, "one side of something times the other side. L x W." She was quick to note in our interview that for shapes other than squares or rectangles, other formulas would be required. I wondered how she might approach finding the *area* of other polygons. When I asked her how she might find the *area* of a hexagon, for example, she said, "um, find the length and then get the height. You'd have to find -- like the lengths of the sides, and then find the height and then probably times it." Such a response seems to point out the dependence on a formulaic approach toward finding *areas.* Without a solid understanding of what *area* is, students' abilities to reason through the finding of *areas* for polygonal regions other than the standard triangle, rectangle, or circle would certainly be hindered.

Coordinate

The vast majority of students surveyed for this study associated *coordinate* with graphing in some way. Thirty-five students specified the context for *coordinates* as being on a *coordinate system*, the x-y axes, a graph, a plane, or a grid. Twenty-three students gave examples of ordered pairs, with eight of these and an additional four students detailing that *coordinates* are numbers. Nine students also spoke of *coordinates* in terms of x and y. Nine other students described *coordinate* as being a location.

Not surprisingly, some students made no distinction between the numbers that are the *coordinates* and the points that they represent. Twenty-six students define *coordinate* as a

point, spot, or dot. While some of these particular students included an accurate diagram of a point correctly plotted, the combination of their responses indicates a lack of perceived need to distinguish the point from its *coordinates*.

Only one student gave no mathematical meaning at all, referring only to the everyday verb *to coordinate*. Several students recognized the word only in connection with the phrase *coordinate system*, and made no distinction between specific *coordinates* and the entire grid system. Out of the 55 surveyed, only a single student left this entry blank.

Two students that I interviewed had given written responses which had been worded, but were not accompanied by any drawing to help clarify their words. In both cases during the interviews, they were very clear in their spoken explanations about the *coordinate* plane and the numbers indicating the locations of points on the grid. Another student had responded in writing that coordinate "marks a location, used in graphing" and had drawn a diagram of a *coordinate* plane with two different points plotted and labeled "coordinates on a graph." When I asked her specifically what on her drawing were the *coordinates*, she realized that she wasn't sure whether the points themselves were *coordinates* or whether the "two over and then two up" would constitute her *coordinates*.

This same confusion over whether the point or the numbers made up the *coordinates* recurred in another interview. The response of another student on the survey instrument had been a drawing of a *coordinate* system with two points plotted (4, 0) and (4, 3) and to the side he had written, "the dots would be a coordinate."

> *M: So you're saying that this* (pointing at one of the dots) *would be a coordinate?*
> *S: or like in between -- yeah, those are, just the dots are coordinates.*
> *M: OK, so what are these numbers?*
> *S: Those are where the dots are.*
> *M: Where the dots are -- and what do you call them?*

S: Those are called coordinates.
M: You'd call these coordinates, too? So there's no distinction between whether you're talking about the numbers?
S: I think these (the numbers) *are the coordinates and these* (the points) *are like -- or it's the other way around. I don't know.*

For another student I interviewed, the response that had been given on the original survey was an odd drawing of a long rectangle divided into four parts across, and under the middle two sections were what looked like two more connected squares, except they had their two lower outside corners cut out of them. This unidentifiable figure had (8, 3) to the side. When I asked if he could explain his drawing, he was not sure what he had drawn either. While "(8, 3)" gave some indication that he had made some subconscious association of *coordinate* to ordered pairs, he seemed to make no conscious connection with that in our conversation. He said that he had been thinking about the *coordinate plane*, and during the interview drew intersecting axes, plotted and labeled (with ordered pairs) a couple of points. As our conversation on this topic progressed, it became more and more apparent that he could not separate out from the entire picture what specifically were the *coordinates*.

Diagonal

The term *diagonal* brought forth the most intriguing results in the study. The everyday connotations of slantedness for *diagonal* appear to have a tremendous interference on the mathematical definition of *diagonal* with the students I surveyed. Pimm (1987) noted that a significant number of students will point out any slanted line as a *diagonal*. His observation was confirmed in this present study.

Half of the respondents' answers contained key phrases such as "slanting," "crossways," "at an angle," "not straight up and down," or "between horizontal and vertical." Nine of these 27 students drew accompanying diagrams consisting solely of a slanted line segment in no relation to any other figure. Two additional students drew this same diagram of a lone slanted segment as the entirety of their response.

Out of the 55 participants, no more than 15 gave any written indication of the mathematical definition of a line segment which joins non-adjacent vertices of a polygon, though their written expression of that was not in every case precise. Several of these spoke only in terms of quadrilaterals, as opposed to any other polygons. Twenty-one students drew diagrams of quadrilaterals with segments joining opposite vertices. However, with the prevalence of worded responses that attribute to *diagonal* only the characteristic of being "slanted," it is possible that even with a mental picture of a vertex-to-opposite-vertex segment, some of these students' conceptions may still be more heavily weighted toward the idea of a slanted segment than the idea of a vertex-to-non-adjacent-vertex segment within a polygon. I would guess that the examples of *diagonals* to which students are most often exposed in the classroom are vertex-to-opposite-vertex segments in quadrilaterals oriented such that their sides are essentially parallel to the sides of the blackboard. Thus, the corresponding *diagonals* are, of course, *slanted.* If this is true, then students may perceive the non-horizontal and non-vertical notion of *diagonal* as the necessary condition for a segment to be a *diagonal* because that notion is reinforced by, and is a reinforcement of, the everyday connotations of the word. This thinking of slantedness as the essential

characteristic of *diagonal* seemed to be the case with several of the students that I was able to speak with in follow-up interviews.

In the interviews, students were shown pictures of various polygons with *diagonals* drawn in them (Appendix C). Some of the *diagonals* were slanted, and some were horizontal or vertical as compared to the edges of the page. The conception of *diagonal* as a slanted segment was very firmly set in several of the students' minds. Of the nine students I interviewed, seven had given original written responses that were categorized by the notion of slantedness. All seven held firmly to their idea even when questioned directly in efforts to bring to their recollection the geometric definition. I was quite surprised at the stability and the strength of the connotation of slantedness which limited their view of what could or could not be a *diagonal.*

Three of these seven students had also drawn a rectangular figure with a segment connecting opposite vertices on the original survey. In probing their ideas about *diagonals* and asking them to pick out all the *diagonals* they saw from the figures I showed them (Appendix C), not one of those three wavered from their idea that the non-horizontal, non-vertical orientation was the necessary criterion for a line segment to be a *diagonal.* One asked if the lines had to be inside the figures. I deflected the question back to her and she decided they did not. Another had originally responded that a *diagonal* "cuts something in half," but when asked particularly about the horizontal *diagonal* of a rhombus, which did, of course, bisect the rhombus, he decided that this particular segment was not a *diagonal* because it did not have a tilt to it. I asked the third student directly "does the diagonal have anything to do with this figure around it?" He replied "no, not really." He went on to

question whether or not the lines on the lined paper that the figures were drawn on would be considered horizontal. With that as his reference, he then selected segments out of the figures that were at a slant.

This particular student was one of two who were interviewed who realized that if the paper's orientation were changed, the segments that would be considered *diagonals* might also change. Pimm (1987) writes about students holding to the slanted notion because they have not come to a point of mental conflict in recognizing that slantedness is purely a matter of orientation. Yet these two examples show that even when students do realize that the notion of *diagonal* as slantedness is an unstable characteristic dependent on orientation, they still may experience no mental conflict at all and will continue to hold firmly to their notion of *diagonal* as describing a slanted line.

Difference

The everyday usage of the word *difference*, meaning "not the same" is surely more common and widespread than the mathematical sense of the work. Yet, "the difference of two numbers" as a phrase representing the result of subtraction is not as far removed from the everyday connotations as are some of the other lexically ambiguous terms that are included in this study. For that reason, I was surprised by the number of students who had not accommodated the specific mathematical context of subtraction for the use of *difference*. Nine students gave no mathematical meaning at all in their response. Their responses all followed the lines of "not the same." Another six left this entry blank. Because of the commonness of this word, I assume that they left it blank because they could not think of a

mathematical meaning, not because they could not think of any meaning. Only thirty students replied either in words or by including an example that *difference* had something to do with subtraction, with twelve of those specifying that it was the answer to a subtraction problem. "Amount between two numbers" or "how far apart they are" were other ways that students expressed their ideas about *difference*. Hardcastle and Orton (1993) reported that over half of the students in their research project correctly explained *difference* in reference to subtraction. Thus the results of the present study in terms of proportions of students recognizing the term *difference* in the context of subtraction are similar in that regard.

In interviewing one of the students whose written response had been "4 - 6. There's a difference between those numbers," I was curious as to what he meant in that he had placed a dash between four and six, but had repeated the word *difference* in his worded response without any reference to subtraction.

> *M: Difference. When you say that there's a difference between these numbers, um, what kind of a difference are you talking about?*
> *S: They're just not the same, they're different.*
> *M: They're different. OK, have you heard difference used in any other kind of sense in math?*
> *S: Not really. Not that I know of.*
> *M: Would there be another way that there's a difference between four and six? If you were told to find the difference between four and six, would you just say, "they look different?*
> *S: No. Like you'd divide them or something.*

Another student's original response had been "means $4 \cdot 4=16$, $4 + 4 = 8$." As I showed the student his response, he blurted, "that's just a difference -- like normal." When I asked whether he had heard difference used in math class used in some other way than "not the same," he shook his head.

Another student had written on his original response, "subtraction. Find the difference of 3 and 4 = 1. 3 - 4 = 1." In our conversation, as I probed for more elaboration on what he was thinking of as to the relationship between difference and subtraction, he finally expressed it as "what or how many are different between the numbers." This student also realized that his answer should have been written as a negative one. Thus, I feel confident that this student's conception of difference in relation to subtraction was not faulty, but his struggle was in articulating that relationship without the repetition of a form of the word, *difference*, a struggle that recurred in several of the interviews and five of the written responses.

Exponent

The non-mathematical sense of the word *exponent* is possibly not as familiar to high school students as the *exponents* they encounter in math class, and the meanings are far removed from each other. Nonetheless, two students did include the idea of a spokesman for some issue as a meaning for the word. Ten out of the 55 students left this entry blank. Twenty-three students described the *exponent* as the small number next to a big number or put down a simple example of a base with an *exponent* with a pointer labeling the small, raised number as the *exponent*. While these students clearly can name or show how an *exponent* appears, none of these gave any explanation of the function of the small, raised number beside the big number. Fourteen other students either gave an example which showed the function of the *exponent* or explained in words that an *exponent* indicates the number of times the base is to be multiplied. Two students associated *exponent* exclusively

with scientific notation in their written responses, though one of these students in a follow-up interview recognized that *exponents* did not have to apply solely to powers of ten. The term *power* appeared only in two responses.

At the level of high school geometry, a definition of an *exponent* as a countable number of times a base is to be multiplied generally suffices. However, Borasi (1989) points out that "while exponentiation is originally defined as 'repeated multiplication' when first introduced within the set of whole numbers, this meaning and definition have to be soon relinquished if we want to consider negative and fractional exponents as well" (p. 12).

One student replied that *exponent* means "the cube or square of a number." Yet when I interviewed him later, he did not even believe there was any connection between his response and *exponent*. I began our conversation by reading his written response back to him:

> *M: means the cube or square of a number. What would you do with...*
> *S: I don't have a clue. I just wrote something down.*
> *M: for exponent? You don't know what cube or square is?*
> *S: No, I know what that is, I just don't know that, if that's right or not.*
> *M: So exponent, you didn't actually know?*
> *S: No. I didn't know many of them really. I couldn't remember any of them.*

While the word *exponent* had brought to his mind *cubing* and *squaring* at the time he wrote the survey, this student was actually unaware of why he made that connection. Even in this situation of being asked to think about it again, he was too unsure or unfamiliar with the term *exponent* to see the association that probably came from some past exposure to the word.

Factor

The responses showed that most of the students who defined *factor* identified it with only one of the parts of speech that it can fill. Some students thought of *factor* in its noun sense, as a number that is multiplied. Others thought of it in its verb sense of finding the values which multiplied together yield a particular product. Yet only one student seemed to include both senses in his or her response, and even that student, in both senses, used the term *factor* in the attempt to explain *factor*. In all, seven students repeated the word *factor* in their definition or description. In all of these cases, the students were thinking of *factor* in its verb sense. This repetition and inability to find some alternative words to express themselves is perhaps reflective of the process of *factoring* being taught and named, but not connected to the essence of what is being done in *factoring*: i.e. finding the numbers or expressions that, when multiplied, form a particular product.

Ten additional students' responses also classified or described *factor* as a process. Four students gave examples of *factor* trees, two students correctly *factored* a trinomial, and four other students gave an example of the FOIL process or some other use of the distributive property of multiplication and called it *factoring*. The process of multiplying binomials together to produce trinomials and the reverse process of *factoring* the trinomials back into the binomial *factors* are usually taught closely together. These examples seem to indicate that confusion can arise as to which precise terminology is to be applied to which of these reverse processes.

Factor and *multiple* are two terms also often taught in close proximity in a context of finding greatest common factors and least common multiples. Confusion between these two

terms was also apparent in the survey. In fact, it was particularly because of the inclusion of both of these words on the vocabulary survey that I was concerned about the ordering of the words on the survey and designed the survey with three different orderings. Eleven students left the entry for *factor* blank. Of these, two had surveys in which the term *multiple* came before *factor* in the ordering, and their response to *multiple* was an appropriate response for *factor*, but not for *multiple*. However, since neither of these two was interviewed, it is not clear whether the ordering influenced their responses. Another ten students who did give responses to both entries confused the two terms in one manner or another. Most of these confusions were not simple mismatchings of definitions to the two terms. Instead, if any trend could be noted in this *factor/multiple* confusion, it would be that the students tended to describe a *multiple* as what should have been attributed to *factor* in its noun sense. Their response, then, to *factor* only described its verb sense. This is exemplified by one student who wrote for *factor*, "to factor something" and gave an example of a factor tree for one hundred and another for twenty-five. Then in his or her response to *multiple* wrote, "a number that can be multiplied by another number to equal [25]. 5 is a mutiple [sic] of 25. 5 x 5 = 25. 7 x 3 = 21. 7 and 3 are multiples of 21."

Factor was one of five terms in the Malone and Miller (1993) study which was included on the list of vocabulary terms for all four grade levels participating in their study, years eight, nine, ten, and eleven. Tabulations of the percentages of students able to define *factor* with words only were very low, 8.5%, 6%, 4%, and 7% for the four grade levels. However, when examples and diagrams were taken into consideration with the worded responses, the percentages of acceptable responses rose to 19%, 77%, 27%, and 33% for

grades eight, nine, ten, and eleven respectively. What is most striking from this data is the dramatic discrepancy between the percentage of responses deemed acceptable for the year nine students as compared to the other three groups. It appears as though for the ninth grade year, the term *factor* or examples of the process of *factoring* are especially familiar. This is likely due to the emphasis on the topic of *factoring* quadratics in first year algebra.

One boy I interviewed had written on the survey, "factor means to factor something out, to take it all out." When I asked if he could explain his response without using the word *factor*, he did not think he could. I then asked if he could give an example of *factor*. In response he made a *factor* tree on the paper, writing down eight, then below that, two and four, and below the four, two and two. He explained his process as "splitting it up to 2 x 4." His thinking about *factor* as a process, but not additionally as a divisor, remained unchanged through that topic in the interview. However, when we got to talking about the word *reduce*, he was explaining the process to *reduce* a fraction, and he said, "you'd find a common number, a common factor that goes into each one." At that point, I pointed out to him that he had used *factor* in a noun sense. While this student was easily able to find the common *factor* in order to *reduce* the fraction that had been presented to him, he did not seem to feel confident in his defining of the term in its noun sense at all. He struggled to express *factor* in its noun sense as "a number that multiplies into another number to -- I don't know."

A second student had written on the survey that *factor* was "some kind of number thing." In the interview, he had no idea for a meaning for *factor*. A third student originally had left the entry for *factor* blank, but while discussing the term *multiple* in the interview (prior to the topic of *factor*), he could not provide a meaning for *multiple*, except that he

thought that it had to do with multiplication. Yet, he was not confident of this conclusion and said, "I think that's a factor, though." In other words, he was skeptical that both *multiple* and *factor* could be related to multiplication. When we did progress in the interview to the topic of factor, our conversation went like this:

> *M: How about factor? Now you were thinking back with multiple...*
> *N: Yeah, it's when you multiply numbers. It's what you get when you multiply them together. It's like 3 x 9, the factor is 27.*
> *M: The factor is 27?*
> *N: I think. Yeah.*

We had already in this interview skipped over the term *product* because I had not had questions concerning his original response to that: "what you get when you multiply two or more things. Something that tends to make a result." Here, it seemed that he wanted to call *product* and *factor* essentially the same thing.

Another student had written on her original response, "used for multiplication (multiplication factors of 1 - 12)." As we discussed her response, it appeared that she was simply talking about the multiplication tables, or the multiplication facts for numbers up to twelve. She finally said, "I just remember using that word, that phrase all the time in middle school." I then wrote out a multiplication fact for her, 6 x 3 = 18 and asked her if she could identify *factors* in that sentence. She realized that 18 would not be, but could not recall what the proper word was for 18. She initially wanted to call 18 the *sum*, but thought that was not right either. She was content to say that "6 x 3" were the *factors*. This is another instance in which a particular phrase was etched in the student's memory, but she was not confident about articulating a meaning for that phrase.

Irrational

Eighteen students in the survey left this entry blank, more than for any other entry. Of the remaining two-thirds of the participants, nine responded with an uninformative "not rational." Unfortunately, none of these gave any indication as to what *rational* meant in any mathematical sense. Three of these nine added some other explanation, namely, "doesn't make sense," "not responding properly," or "not smart or thinking clearly." These three responses were representative of a total of 25 responses that clearly leaned to an everyday interpretation of the word. Of these 25 students whose meaning for the word leaned toward "not making sense," seven made attempts to fit their conception of the word into some mathematical context. In general, these seven students applied *irrational* to mathematics as a term indicating a problem that could not be solved or does not work out.

Twelve separate responses indicated that irrational conjured up some connection to numbers, but only two of these responses offered examples which were indeed *irrational* numbers, π and the square root of two. Another two of this same group of twelve connected *irrational* to decimal numbers. One of these two gave an example of 0.59476342 which may give an indication that there is a recognition of *irrational* numbers having no repeating pattern after the decimal point, but nothing more is written to indicate that this example would be a non-terminating decimal. The other of these two went on to say that the decimal "goes on and on," which is getting a bit closer, but the example that accompanied this response was that one-sixth equals 0.1666 with a bar over the last six. Of course, one-sixth is a *rational* number, not *irrational*, so while a response of a non-ending decimal approaches being correct, this student clearly has not quite distinguished a requisite characteristic of

irrational numbers. Of the fifty-five students, only one student gave a mathematical definition which was recited flawlessly: "a number that cannot be expressed as a ratio of two integers."

Four of the nine students that I interviewed said that they had no recollection at all of ever hearing the term *irrational* in a mathematical context. One had left the entry on the vocabulary survey blank, the second had written "out of line, i.e. you are acting irrational." Two others had originally written that *irrational* is the opposite of or not rational. All four of these students could only say that they had some recollection of hearing the term *rational* in math class before, but could not remember what rational meant.

Of the remaining five students who were interviewed, two associated square roots with either rational or *irrational*. One had not given a mathematical meaning to *irrational*, but rather had responded with "extreme" on the vocabulary survey. In our discussion, however, he still did not believe that he had ever heard *irrational* in the classroom, but thought that rational had something to do with square roots. The other student had written, "doesn't make sence [sic], stupid, irational [sic] numbers" and put as an example the square root of two. I asked this student if he could give more examples and what it was about the examples that made them *irrational*. He replied, "oh, there's a lot of them, mostly involving square roots or something like that -- π....They just kind of continue on forever in a pattern that can't be predicted."

The last two students to be interviewed both had interference from the everyday connotations of lacking sense or reason for *irrational*. Both attempted to apply this to a mathematics context. One said in his original written response that *irrational* "means the

number doesn't work in the problem" and gave an example. When asked if he could expound on that in the interview, he could explain his example, which had been rather confusing, but could not add anything substantive to what he had written. The other of the interviewees who also had attempted to apply irrational in its non-mathematical sense into a mathematical context simply said "you can't do it" on her written response, and in our discussion "it can't be done."

Mean

Expecting many of the responses to be along the lines of an average of a set of numbers, I was quite unprepared for the responses to *mean*, particularly from the students in school B. Apparently, these students had been studying *geometric means* with similar triangles in their geometry course shortly before the vocabulary survey for this research project was administered.

Nineteen of the respondents were in school A, and there was no trace of the idea of the *geometric mean* in their responses. Of these 19, ten responded that *mean* was an average. Six of these delineated the process for finding the *mean* of a group of numbers. Of the remaining nine students from the first school, five left the entry blank; one confused *mean* with the intersection of two sets of numbers; one said that *mean* referred to "rude people;" one replied that *mean* is "the definition of a term;" and the last said it "means the number higher."

In school B, the distinction caused by the influence of their most recent encounter with the term was significant. Fifteen students of 36 gave responses having to do with

"geometric mean," "means-extremes," or examples in which two fractions are set equal to each other, sometimes with the denominator of the first and the numerator of the second circled to indicate that those two terms are considered the *means* as opposed to the *extremes*. Only ten of these 36 referred to an average or an "in-between number" in their responses. Yet none of these elaborated what they meant by an average, explained the process or gave an example of finding a *mean*. Of the five students who gave more than one meaning to the word, only one referred both to geometric *mean* and average, relating them saying that geometric *mean* was similar to an average, but the only elaboration added was, "I don't remember exactly what it is."

The everyday, non-mathematical senses of the word that appeared in the responses from both schools' participants were nine occurrences of the sense of "not nice," and two occurrences of "dictionary definition." Thirteen students left this entry blank, the second highest number of blanks for any single entry in the survey. As with the term *difference*, the fact that the entry was left blank is much more likely an indication that the students could not remember a mathematical meaning rather than that they could not come up with any meaning at all.

In my interviews with the students from school B in which the *geometric mean* was the context of *mean* freshest on their minds, my questioning focused on ascertaining whether the idea of the *arithmetic mean* could be remembered by these students. Interestingly, of the seven interviewees who were from this school, three thought that there was indeed another kind of *mean* in math, but they could not remember it. Only with my going so far as to give an example of having several test scores did one of the students recall the process of adding

the terms and dividing by the number of terms. It is possible that while knowing the process, the precise term of what was being calculated was not connected to the process, or even that *average* had been used rather than *mean* in that situation. Also, of significance with this group of interviewees was that while the phrases "geometric mean" or "means - extremes" were recently etched into their memories, only one of the students was able to fully explain *geometric means*, and only three students were convincing in their explanations of *means - extremes*. While *geometric mean* was a catch-phrase listed on these seven students' written responses, most could explain no more than that it had something to do with triangles. The best some could do in defining *means-extremes* was that it had something to do with fractions. One student clarified that the prevalent use of that term by these students was due to the fact that they had been told not to use "cross-multiply," because it was a slang term, but rather to refer to the principle of the means equaling the extremes when they were working with proportions.

Multiple

Only two respondents left this entry blank, and a third responded that he could not remember. As was discussed previously, some confusion was evident between *multiple* and *factor*. In addition, it seemed in several cases that there was some confusion between *multiple* and *multiply*. This finding directly concurs with the report of Otterburn and Nicholson (1976) who observed that over a third of their sample group of 300 returned confused responses. They comment that "those who did muddle it thought it was a misprint or a synonym for multiply and the others thought it meant factor. Of the 103 muddled

responses, 83 muddled multiple with factor" (p. 19). Overall, many of the responses in the present study were not explicit enough to determine conclusively whether it was actually a *multiple* or a *factor* that was being described. The relatedness of these two words was clearly a cause of difficulty to the students, and contributed to a tendency on the part of the students to be less explicit rather than more explicit in their responses. Only seven worded responses and five additional responses which consisted solely of examples could be considered as correct mathematically. Seven students associated *multiple* with "a number in a multiplication problem," with most of these accompanied by an example that would suggest a direct confusion with a *factor*. An additional eight students seemed to have applied the meaning for *factor* to *multiple*. Nine students referred to *multiple* as some type of process, or even more specifically as the process of multiplication.

Many of the students used examples in their attempts to communicate their meanings for *multiple*. For those who gave only an example as a response, six were correct, four were incorrect, and nine were ambiguous, with not enough detail or elaboration to make a judgment as to the correctness. One of the very interesting observations I made was that for those who also gave worded responses, the examples sometimes did not appear to be consistent with the worded responses given. Twelve of the examples given were incorrect for *multiple*, but were consistent with the worded responses. Only four examples which came in combination with a worded response were correct, but one was inconsistent with what had been written. Another of these examples pertained to the everyday sense "many" rather than a mathematical sense, leaving only two examples consistent with the written elaboration.

One other example given in combination with a written response was too confused to classify.

Eight students gave only an everyday meaning for the word and another eight gave everyday meanings in addition to a mathematical meaning. A few other varied responses reflected other misunderstandings of the word: "a number squared," or "a number doubled."

The interview questions I posed to the students simply were in efforts to understand better the ways they were using *multiple* on their written responses. One student clearly confused *multiple* with *factor*, saying that two and four are *multiples* of eight. Another student figured that not much clarification was necessary: "just multiply two numbers together," he said. A third student had written "used as multiples of something" on her original written response, and just laughed when she saw how little information that provided. But in talking together, she was unable to be more specific than *multiple* having something to do with multiplication. A fourth student had written several multiplication facts and a factor tree for 50 as part of her original survey response, but in looking at that response, could only figure that she must have been confused just thinking of multiply. Another student confessed that she had confused *multiply* for *multiple* as well, and realized that the two words were different during the interview. However, her attempt to explain *multiple* was incoherent to me, "like a different thing, like not every one, like a multiple or like if you had three things that would mean not all of them, just like some...I don't know about that." When asked if she might be able to clarify with an example, she replied, "I can't think of anything right now."

Only one of the interviewees was explicitly clear in elaborating that four is a *multiple* of two because two times two would yield four, and by the same token, six would be a *multiple* of two because two times three equals six. Another student also clarified her written response in such a way that indicated a solid conception of *multiple*. Her original written response included several different ideas: "several, i.e. multiple choice, one number in an equation, a series, add in multiples of 4, i.e. 4 + 4 + 4 + 4 + 4 ect [sic]" Clearly the idea of "several" recognizes the everyday connotation of the word. When asked about "one number in an equation," she could not elaborate, and actually seemed unclear as to what she had been thinking when she wrote that. We then turned to her idea of a series, and she explained, "I said a series like multiples of four, or multiples of three, like three, six, nine, like that" and for five, "five, ten, fifteen, like that." While skip-counting may not be the most sophisticated way of looking at *multiples*, it is certainly a more complete notion than a vague association with multiplication.

In one interview, confusion of *multiple* with squaring a number was revealed. This student had written on his vocabulary survey, "5 x 5 = multiple." When I asked him about it, he said, "is it a number when you times it by the same number, or whatever?" To clarify what I thought I was hearing him say, I asked if three times three would be a *multiple* also. He said he thought so.

Prime

Forty-nine students correctly connected numbers to *prime*. Twenty-two students answered with commendable accuracy with clauses like "a prime number has no factors but

one and itself," or "a number that only one and itself will divide into evenly." Nevertheless, the question of whether these responses reflect memorized textbook definitions or real comprehension of the term remains open. An additional eleven responses contained elements of inaccuracy, although they implied some grasp of the mathematical sense of a *prime number*. For example, seven of these responses suggest that a *prime number* is not divisible. Although not entirely true, this response certainly indicates some notion of "primeness." Five students left this entry blank, and one other wrote "means exact or origin." In comparison, Otterburn and Nicholson (1976) reported that just over half of their respondents provided correct explanations of the notion of *prime*, with over a third of their sample group leaving that entry blank.

Twenty-one students provided examples of their notion of *prime*. Twelve of these were correct examples of *prime numbers* and consistent with worded responses also written. Seven of the examples given were incorrect, but were accompanied by intriguing written definitions such as "a weird number like nine," or "a number that is the result of two of the same numbers being multiplied together," indicating apparent confusion with a square number. One student wrote "prime is that little number above another number" indicating either a confusion with *exponent* or some diversion from the notation of placing an apostrophe by a letter such as calling X' "x prime." Also of note are three responses that took *prime* in its everyday sense and applied it to numbers, as in "the largest number," or "the most important number."

The interviews showed that students could give good examples and articulate their meaning for prime more coherently through speaking than through writing. One of the

interviewees continued to try to apply the everyday sense of the word to a mathematical context, saying that *prime* referred to "the main number" or in a set, "the highest number."

Product

The everyday connotation of *product* is something which is made through some process or manufacture. This idea is related to the mathematical use of product since the product is made through the process of multiplication. The responses from the students seem to imply a large degree of interference from the everyday connotation with students not narrowing in on the specific mathematical meaning. Six students replied that the *product* is the answer or the result in general to any mathematics problem. Six students applied *product* specifically to the answer of an addition exercise, with three of these also providing examples with arrows indicating that they thought of the answer as the *product*. One student applied it to subtraction. Only six students left this entry blank.

Though some interference is evident, a large number of students also have learned the specific mathematical use for *product*. Thirty-five students provided answers or examples which show they connect *product* multiplication. By what is written, several of these students' replies could be construed as referring to the process of multiplication, rather than as the result of the operation. For example, one student writes, "the multiplication of numbers." Whether or not this association with multiplication is ample evidence of an accurate notion of the specific mathematical definition of a product is debatable. A lack of more specific language may be attributable to adolescents appreciating the usefulness of vague answers which may be more easily given the benefit of the doubt. As Wilson (1988)

notes, "Students' vocabularies are not sufficient to express their ideas. Often their words are imprecise. There is some evidence that this is not merely neglect but 'fluid words' can serve a necessary function. Ambiguity can be useful" (p. 29). On the other hand, the student may have communicated precisely his/her conception of the word.

In Otterburn and Nicholson (1976) and in a follow-up study by Nicholson (1977), the term *product* consistently fell in the lowest quarter of the list of terms tested with respect to acceptable responses, with not more than 21% of the participants providing explanations which were deemed as acceptable for any of the three investigations reported. Most of the confusions reported from Otterburn's research and Nicholson's follow-up investigations centered around students attributing the term *product* to the one of the other three basic mathematical operations.

Two of the students that I interviewed had answered on the vocabulary survey that the *product* was an answer in general. My questions to them were particularly aimed at seeing whether they would narrow their responses to being the result specifically of multiplication. Neither of these students, however, wavered from their application of *product* to any mathematical result. Another interviewee who had written on the survey, "the showing of a number," declared in the interview that he had no idea what *product* was. Three of the interviewees clearly related *product* to multiplication.

Two of the interviewees responses on the written survey indicated that a *product* was the answer when two numbers are multiplied. To these two I asked whether the answer would still be called a *product* if more than two numbers were multiplied. The first student

answered without hesitation that it would. But the second was baffled and said she did not know whether *product* would still apply or not.

Reduce

The mathematical sense which I was associating with the term *reduce* when I included it on this survey was the process of dividing the numerator and denominator of a fraction by their greatest common factor to simplify the fraction into its lowest terms. The more everyday connotation, though, of lessening an amount or making something smaller also appears in mathematics word problems. Thus, it is not surprising that both senses of the word showed up almost equally in the students' definitions. Eleven students' responses spoke of *reduce* as lowering an amount or making less. Another fifteen students specifically used the words "make smaller," in reference to *reduce*. An additional eight students related *reduce* to the operation of subtraction, either in their written responses or through examples.

The sense of *reduce* used with fractions came out in the responses of 25 students who either mentioned fractions or showed examples of fractions and their *reduced* form. Three of the fifteen students who used the phrase "make smaller" spoke of making a fraction smaller, which technically is incorrect, though it is possible that this type of imprecise language is common in the classroom. Perhaps these three students exemplify best an interference between the more everyday connotation of *reduce* and the *reduce* that is applied to the simplification of fractions. Nine students used the phrase "simplify a fraction" in their answer. Eight others also referred to the idea of making simpler or easier to read or work with. Six of these eight gave examples that suggest that this simplification would be applied

to fractions. In all, 20 students supplied correct examples showing a clear understanding of the reduction of fractions. However, only two students gave any explicit description of how the process of *reducing* fractions is done, one in the words, "to simplify a fraction by dividing the numerator and the denominator by a common number," and the other by showing each step of the *reducing* process in an example.

It was quite interesting to note the other mathematical jargon or catch phrases employed to explain this word. Such phrases as "cut down, bring down, round down, take down, and break down" showed up in twelve students' responses. "Taking to lowest or smallest form" recurred eight times. This raises the question again of whether students use phrases because the phrases make sense to them, or only because they are phrases etched on their memories from mathematics class. Also of interest is the fact that *reduce* and *square* were the only words on the survey that no student left blank. Perhaps these two words have been encountered most often overall by the students to this point in their school lives.

Several of the responses that had been elicited on the vocabulary surveys spoke of "bringing down" or "lower terms" without any additional specific reference to fractions. Therefore, with these particular students, my interview questions concerning *reduce* focused mainly on clarifying with the students the context for the "bringing down" or the "lower terms." In none of these cases were the students unable to specify that they were referring to fractions, and all the interviewees with whom this clarification was needed were able to demonstrate correctly the *reduction* of fractions.

One student had placed *reduce* in the context of subtraction in her written response. When asked if she could recall any other mathematical context in which the word was used,

she applied *reduce* to the idea of shrinking or enlarging images, such as is done with photocopy machines or in transformational geometry.

Square

Mathematically, this word has two common uses: the first as the shape learned from preschool days, and the second as the multiplication of a number by itself. The former would probably also be considered as the more everyday sense of the word. Because of the very early exposure for most people to this shape called a *square*, a lack of familiarity with this sense of the word would have been quite surprising unless a particular student were not a native speaker of English. But as was already mentioned, no students left this entry blank. The real questions in my mind in including this word on the survey were first, whether students have isolated out the necessary and sufficient conditions of what separates a *square* from every other quadrilateral, and second, whether students would recall and note both senses in their answers as both are quite common in a course of geometry. While separating the concept of *square* from other quadrilaterals with some of the same characteristics is a somewhat sophisticated classification task, this classification is a common objective in high school geometry courses, and my pursuit of this question simply was to find to what degree that objective may have been accomplished with these high school geometry students.

Twenty-one students noted the sense of taking a number times itself in their answers for this entry. Seventeen of these also included the shape sense in their response. This leaves four students who only connected *square* to this process of multiplying a number by itself. Another two students referred to this process as "doubling a number," which is certainly not

correct, but may be more a reflection of inaccurate terminology than inaccurate thinking. One of these also did not include the shape sense of the word, leaving 50 of the 55 students who, predictably, did associate *square* with a four-sided shape. Interestingly, though, this shape like the face of a box was as explicit as some of these geometry students' responses were. Four students went no further in their explanation of a *square* than to draw an unlabelled four-sided figure with no other detail. With just a bit more information, another student drew a four-sided figure and said it was a "smaller rectangle." The range of detail given for the essential properties of a *square* went from this unlabelled figure drawing to very explicit detail of all the necessary features of a *square*. Of 37 students that noted that the shape was either four-sided, four-angled, or both, 30 included that these four sides were congruent, five observed that the opposite pairs of sides were parallel, and 21 also included the requirement of right angles. Another student indicated that the diagonals of this four-congruent-sided shape bisect each other at 90 degree angles in addition congruent sides, right angles, and parallel pairs of sides. Still another defined a square as "a four side [sic] figure that has 4 lines of symetry [sic]." Thus in all, only 23 students provided a definition of square that would separate it from every other polygon. While I would dare to say that, if pressed, these other 27 students who connected a square to a box-shape could give more detail than they did, again this lack of explicitness exemplifies the findings of Wilson (1988) that "students do not think of definitions as necessary and sufficient conditions" (p. 29).

Hardcastle and Orton (1993) noted that only nine percent of the participants in their study could specify the requisite features of a square, and most of the worded explanations were deemed acceptable only because of the inclusion of a diagram which clarified their

writing. Of the 76 students in their sample, only one suggested the second sense of square, "the multiplication of a number by itself" (p. 14). In Otterburn's research (Otterburn and Nicholson, 1976), 94% of the pupils explained *square* as a shape acceptably, while 65% explained *square* as the process of multiplying a number by itself correctly.

Of the nine students that I interviewed, four of their original written responses had enough information to distinguish a *square* from other quadrilaterals. One of these four was the student mentioned above that described the *square* as having diagonals that bisect each other at right angles in addition to the congruency of the four sides and angles and the parallelism of the pairs of sides. When I asked her, "how much would you have to know to know that that's a square?" she then had difficulty paring down her answer to the essential features. Of course, different combinations of these features could define a *square*. She came up with a couple of combinations, but in one, she still gave more than absolutely necessary, and in the second gave specifications that would work for any rectangle rather than just a *square*.

To the other five students who were interviewed, I asked whether what they had written was enough to describe a *square* and nothing else. Two very quickly recognized that what they had written denoted a rhombus and added that right angles would be required as well. Of the other three, one had written only that the figure has four sides. In the interview, she amended her response to have the four sides congruent, but then was content with that. The other two were also content that four sides of equal length were a sufficient description of a *square*, even though one of them observed that a "diamond" also could have four equal

sides. Even so, he could not recognize what adjustments would be required in his answer to distinguish a *square* from a diamond.

Five of the interviewed students were able to name the second common mathematical use of *square* when asked whether they could think of any other way that *square* is used. This lends support to my contention that many students know both senses, but may be content having written just one. Clarke (1993) observed in more extensive studies that "given open-ended tasks in mathematics, science, English, and social studies, students were similarly disinclined to give multiple answers or to attempt to frame a general solution" (p. 217).

One conversation with a student about the term *square* proved particularly intriguing. The discourse went like this:

M: Is there another way you've used square in math?
C: Um huh. Square something.
M: So when you square something you do...?
C: So like four squared. Do you want me to write it down?
M: Sure.
C: Four squared which would be 16. (writes $4^2 = 16$)
M: OK, what did you do to this?
C: Oh.
M: What were you doing?
C: Um, you're squaring it. Well, like four sides four times. You're squaring it four times, to the same number. (She writes $4 \cdot 4 \cdot 4 \cdot 4$) *Like two squared here,* (writes $2 \cdot 2$) *you're squaring it to the same number. Four times four times four times four or two squared would be two times two...to its own number.*
M: So you took four, four, four, and four...does that mean you'd multiply them all?
C: Uh huh.
M: So four times four is...
C: is, uh sixteen.
M: Times four is...
C: um...I'm not sure.
M: How about if you did threes? What would you be doing with threes?
C: Well, three squared would be nine.
M: How do you get nine from threes?

C: Three, three, and three. Three, six, nine.
M: Are you adding them?
C: yeah.
M: So you're saying you're adding, adding, adding (pointing to her fours on the paper).
C: yeah. Wait...four, eight... yeah. Is that right?

Certainly the fascinating observation here is that this tenth grade geometry student knows that three *squared* is nine or that four *squared* is sixteen, but when asked to specify her process, it appears she resorts to a very concrete, repeated addition model in order to *square* numbers. It is not clear whether she grasps the connection of repeated addition to multiplying the number by itself. While her first thought is that four *squared* involves multiplying four times four times four times four, when I ask her to work that out and she immediately arrives at her previous answer of sixteen after only multiplying once, mental conflict results. Her model says there should be two more fours involved, but through working it out aloud, she recognize that they must not be multiplied after all. Perhaps the fact that her first attempt did involve multiplication could indicate an association of *squaring* with multiplying, but she seems not to have fully moved into the multiplication model of *squaring* a number.

Finally, several other connotations of *square* did appear in the survey responses and interviews. Another everyday sense of *square* is applied to certain persons thought to be "uncool" or "boring." Six students mentioned this idea. A few other students suggested *square root* and *square units* as other mathematical uses for the word.

Variable

Kuchemann (1981), Kieren (1991), Schoenfeld and Arcavi (1988), Wagner (1983), and Rosnick (1980, 1981), among others, have written on the topic of the *variable* and the

confusion that can arise from applying this term indiscriminately to any letter used for various purposes in mathematics. It is indeed common, although incorrect, for all letters used in mathematics to be denoted as *variables*. Letters serve various functions in mathematical notation, and operating with these letters requires differing levels of thought. Kuchemann (1981) notes six ways in which students interpret the letters they encounter in mathematics contexts. As Clements (1991) observes, however, two of these uses or interpretations are most common. Kuchemann's fourth level is very common in algebra, in which the letter in an equation represents a specific unknown value and can be isolated to one side of the equation to find that value. Letters or "literal symbols" (Wilson, 1983) used in this way should be labeled as *unknowns* rather than as *variables*, since they represent a single value rather than various values. At the highest level, the letter is, indeed, a *variable*, and can represent a changeable and unspecified set of values in which a clear-cut relationship is recognized to exist between two such sets of values. Kuchemann's findings show that very few adolescent students (ages 11-16) have the capacity to operate with letters which are in fact, *variables*.

Schoenfeld & Arcavi (1988), Wilson (1983), Rosnick (1980), and Kuchemann (1981) all allude to the elusiveness of a clear-cut definition for a *variable*. One reason for this is the multitude of ways and contexts in which *variables* are used. Nonetheless, Kuchemann defines *variables* in this way, "letters are defined as variables when a second (or higher) - order relationship is established between them" (p. 111). He elaborates that "second-order relationships...give an indication of the degree to which one set of values varies as a result of

changes in another set" (p. 112). This definition can give a base of comparison for the responses given in the present study.

Rosnick (1980) undertook an in-depth review of 41 high school mathematics textbooks to examine the ways in which the concept of *variable* is presented. He observes that most texts spend less than a page on explaining the concept of *variable*, this "despite the fact that High School Algebra is predicated on the existence of variables" (p. 3). He concludes that his study "has revealed an astounding diversity in the approaches to presenting the concept of variable....No book fully gets across the idea of continuous variability without becoming...cumbersome. Furthermore, a majority of the texts are in some way ambiguous or confusing" (p. 11). With this background, I will turn now to the responses of the students in my study.

Twenty-two students said that a *variable* is a letter, and sixteen of these specified that the letter *represents* a number. An additional seven students reiterated a similar idea of any symbol that substitutes or fills in for a number. Thirteen students claimed that a variable *is* a number. Sixteen students included the idea of an unknown or missing number in their responses. Only five students recorded an answer that suggested the idea of changeability, which actually seems a bit odd in light of the root of the word, *vary*. Two students referred to their experience with the term in a computer context, and three students spoke of *variable* in a science context. Twelve students left the entry blank or replied that they could not remember the meaning of *variable*.

In the interviews, three of the students could not recall a meaning for *variable*. One had originally written, "a number over another one? fraction?" but was quick to admit in

talking with me that she did not know what *variable* was, and was relatively convinced that it was not a fraction. A second had left the entry blank. In our conversation, she was still unable to express a meaning for *variable* verbally, but felt she would be able to recognize a *variable* if she were to see one. When I showed her an equation $6x^4y + 17y^3z = 29xyz^6$, she quickly recognized, "oh, xy and yz, those are the variables, right?" However, beyond connecting letters to the term, it was not clear if she at all had any understanding of the function of variables. A third student had written a response that gave indication of interference, rather than help from the word root, *vary*. He had written "a variable is a number that can be changed I think." While this response might be seen as indicating a notion of something which can take on different values, the conversation we had actually made it plain that this student was quite confused about *variable* in a mathematical sense.

> *M: Can you be more specific? Can you...*
> *S: No, I'm not sure what variable is. I know it, I just can't think of it.*
> *M: Have you heard it in math?*
> *S: yeah.*
> *M: And you don't remember what exactly it refers to?*
> *S: ugh uh.*
> *M: So when you talk about a number that can be changed are you talking about a number like fifteen that could become eleven?*
> *S: Yeah, I guess. Kind of like that. Isn't it just basically a number in a problem and it could be any number. It's just in the problem and they say "find the variable" -- I don't know. I don't remember.*

It is very possible that when he begins suggesting a problem of "finding the variable" that he does have in his mind the idea of finding a particular unknown as is a very familiar problem at this level of high school mathematics. However, this idea is not consistent with a changeable value as suggested in his original response.

The responses of all the students on the surveys and through the interviews indicate that these students do not distinguish the different functions that letters play in the notation of mathematics. Until such distinctions are made and emphasized between letters that represent "unknown" but singular values in equations and those which represent changeable values within equations denoting stable relationships among various values, students will likely continue to regard all letters as "variables," whether they truly are vary-able or not.

Discussion

The range of ideas for the fifteen terms described above included both predictable and unpredictable notions as expressed by the students. While the spectrum of ideas gathered in this research study might not be precisely repeated in other groups of ninth and tenth grade students, the findings of this study do showcase the variety of ideas that can exist in students of two geometry teachers. Although many of the students in these small towns had similar mathematical experiences, and within this school year were under the same instruction, their conceptions of these mathematics terms did vary. This is not surprising when it is recognized that the individual construction of meaning for vocabulary, as well as related, broader concepts is a unique and very intricate process. It is intended that the different categorizations of students' responses to the words drawn out in this research project will help the classroom teacher who encounters notions such as the ones described here to recognize them as "meaning-making" opportunities.

In this chapter, I have described the results found by analyzing the actual content of the students' responses in relation to the fifteen mathematics terms. In the next chapter, I will

report the ways that the students chose to express their meanings, by putting their understandings of these vocabulary terms into words, by providing examples, by drawing diagrams, or by combinations of these modes.

Chapter Five

Worded Articulation, Exemplification and Illustration in Responses

Analysis of the modes by which the students responded

Students' responses were analyzed in a second way, exploring the modes by which the students communicated their ideas about these fifteen terms. Certainly many different conceptions of the meanings of the words were suggested as discussed in the last chapter. But all the content of the responses was put aside to analyze very simply whether any trends might be evident in the ways the students communicated their ideas. In other words, the responses were examined with these questions in mind: did the students support their written definitions with examples or diagrams? If so, was that supporting information a consistent characteristic in the responses of these particular students, or was such elaboration more a feature of the particular term being explained?

In attempting to answer these questions, the initial organization of the data as it was entered onto the spreadsheets was critical, because it was at this point that I determined the various columns or categories according to the response type of the students' definitions. Once the students' verbatim responses had been entered onto the computer spreadsheets, the distinct columns made the differences in mode visually apparent. While the categorizations on the spreadsheets distinguished between definitions, descriptions, confusions, and repetitions, all four of these categories, in the final analysis, were attempts by the students to articulate their interpretations of the vocabulary terms with words. Thus, for the examination of students' choices of communication mode on their vocabulary surveys, these four

categories were combined under one heading which was simply called *words*. Diagrams and specific attempts to exemplify the term were considered to be separate modes of communicating the ideas. Therefore, the categories by mode into which I separated the students' responses came to be eight in number: *blank, words-only, examples-only, diagram-only*, and the various combinations of these, *words and examples, words and diagrams, diagrams and examples*, and all three together, *words, examples and diagrams*. As a matter of clarification, though drawings of certain terms may have been considered to be *examples* of the concepts being conveyed, all drawings were categorized only as *diagrams*. Thus the distinction between use of examples or diagrams cannot be clearly made here. Nonetheless, the delineation between an example or a diagram is not of greater importance here, but rather the examination of whether students supplemented their worded responses.

Findings

A quick reference for the overall picture of the responses by mode is found in Table 1. This table indicates the break down of the ways that the students responded to each of the fifteen words. As can be seen from Table 1, the number of responses for each word equals 55, the number of participants. The total number of individual responses was 825, including the entries left blank. The approximate percentages of responses which are included in each category are shown as well. At just more than a third of all the responses, words-only responses predominated in this survey. The tendency noticed by Clarke (1993) for students to give only one response even for open-ended tasks may factor into this result. This may also be, in part, due to the written instructions on the survey: "Please write down what these

words mean to you as completely as you can. You may use pictures, diagrams, or examples or any other means you can think of to illustrate your meanings for the word." Thus, the instructions invited them to give examples and diagrams, but did not require that they do so. Nonetheless, by way of contrast, in the study conducted by Malone and Miller (1993) in which the instructions were quite similar, they reported that "very few students were using words only" (p.180).

Just over one quarter of the students supplemented their written responses with written or procedural examples. These were categorized under *words and examples* in Table 1. In some cases, however, the words and the examples were not in reference to the same sense of the word. For example, with the term *square*, a student may have given a worded answer describing the shape and then given $3^2 = 9$ as an example of the other common mathematical connotation. Because both modes were used in the response, such an entry was listed under the *words and examples* category even though the meaning being communicated was not the same for these two parts of the student's response.

Each student's survey was also individually analyzed by examining that student's responses as they had been entered on the fifteen spreadsheets and tabulating the number of *words-only, examples-only, diagram-only*, or combination responses of their fifteen entries. The information on how each student responded to each of the fifteen words is condensed into Table 2. Each of the fifteen words is denoted by a different letter as indicated in the accompanying legend. Table 2 shows which words were given worded responses, examples, diagrams, or were left blank. Additionally, the worded responses which gave the students more difficulty in terms of *repetitions* or *confusions* are noted by underlining the symbols of

the words in which *repetition* occurred in the students' definitions and by putting symbols in boldface for which the responses seemed to be *confusions* with definitions of other terms. Worded responses and examples are aligned together in such a way to ease the comparison between which worded responses were supported with algorithmic examples and which were not, or to show which words were given only examples without worded explanations. Also, more general observations can be made by aligning the symbols alphabetically across and leaving spaces where the response for the particular word was not categorized in that mode of response. Because only seven of the words were diagrammed in any way by the participants in this sample, those seven symbols are aligned alphabetically in the *diagram* column. The entries left *blank* were less frequent and less consistent among the participants. For this reason, the symbols are simply listed according to the alphabetic ordering of the words they represent without aligning the symbols in that column.

For all students, a mix of the eight categorized ways of responding did exist. Nonetheless, for some students, there did seem to be trends in the way they responded to the vocabulary terms. Trends or tendencies that are noted here are solely intended to be observations of these students responses on this particular vocabulary survey. They are not conclusive in regards to individual students' long-term tendencies. On this vocabulary survey, some exhibited a tendency to give *words-only* responses, while others were prone to give examples or diagrams with their words, and others were even wont to leave entries blank. Twenty-two students most often gave *words-only* responses. In Table 2, these students' survey numbers are starred (*). At least seven of their fifteen responses were *words-only* responses.

Seventeen students regularly included examples or diagrams with their written responses or tended to give examples or diagrams only. For all included in this category, the key criterion was that the majority of their responses were not *words-only*. These seventeen students' survey numbers are marked by a plus sign (+) in Table 2.

For twelve students the mix of categories was too evenly balanced to classify them as tending to any particular mode. Their survey numbers are marked with an equals sign (=) in Table 2. The frequencies between their preferred modes differed by no more than two responses.

Of the entire group, five students had from five to nine blanks. One of these was slotted into the *words-only* preferring group because aside from the five blanks, ten entries were *words-only*. The other four were classified as tending to skip entries, and their survey numbers are accompanied by a minus sign (-) in Table 2.

Because of the relatively few entries in which diagrams were used, or were even very feasible as a means of conveying meaning for these terms, I took a second look specifically at the seven terms in which students did use diagrams. In descending order of the number of students (noted in parentheses) who drew diagrams as part or all of their response, these seven vocabulary words were *diagonal* (41), *coordinate* (35), *square* (35), *acute* (31), *area* (23), *difference* (4), and *mean* (3). No individual student, however, had drawn a diagram for all seven of these "drawable" words. Only six of the 22 students who mostly responded in *words-only* mode (27 %) had drawn either four or five diagrams on their surveys. On the other hand, ten of the 17 students (59%) who had more often included examples with their worded responses also included four or five diagrams on their surveys. Of the twelve

students whose modes of response were more evenly mixed, seven students (58%) had drawn four, five, or six diagrams. For these students that drew four or more diagrams among their fifteen responses, their survey numbers in Table 2 are additionally marked with an exclamation point (!).

A trend noted here is that there are certain students who seem more prone to elaborate their answers with supplemental examples or drawings and other students who are not so inclined. Beyond that, however, because of the discrepancy of numbers of words for which examples were feasible as compared to terms that could be diagrammed in some way, and because diagrams could also be considered to be "drawn examples," it is difficult to conclude convincingly that a specific number of students lean to drawing illustrations as opposed to writing examples.

Significance

The significance of looking at the ways that students respond to open-ended questions in which they are to share their conceptions of certain mathematical topics lies in the fact that students have different learning styles. The modes by which students respond to open-ended tasks can be indicative of their learning styles and their ways of perceiving mathematical concepts, information which mathematics teachers can use to teach their students more effectively. While the observations from this one survey are simply given to raise awareness of different ways that students try to communicate their ideas about words, classroom teachers may be able to glean valuable insights into students' learning styles and preferences through ongoing, long-term observations of this nature.

Table 1

Distribution of Modes of Response

	WORDS ONLY	EXAM-PLES ONLY	DIA-GRAMS ONLY	WORDS & EXAM-PLES	WORDS & DIA-GRAMS	EXAM-PLES & DIA-GRAMS	WORDS, EXAM-PLES, & DIA-GRAMS	BLANK
ACUTE	22	0	0	1	21	0	9	2
AREA	19	5	0	7	8	4	11	1
COORDINATE	17	0	4	2	6	3	22	1
DIAGONAL	13	0	9	0	30	0	2	1
DIFFERENCE	18	4	0	23	0	1	3	6
EXPONENT	8	12	0	25	0	0	0	10
FACTOR	22	10	0	12	0	0	0	11
IRRATIONAL	29	1	0	7	0	0	0	18
MEAN	19	4	0	16	0	3	0	13
MULTIPLE	17	12	0	23	0	0	0	3
PRIME	30	2	0	18	0	0	0	5
PRODUCT	28	3	0	18	0	0	0	6
REDUCE	25	0	0	30	0	0	0	0
SQUARE	12	0	2	8	23	0	10	0
VARIABLE	17	2	0	24	0	0	0	12
TOTALS	297	55	15	214	88	11	57	89
PERCENTAGE	36%	7%	2%	26%	11%	1%	7%	11%

Table 2

Students' Responses by Mode

<	acute	R	reduce
A	area	S	square
C	coordinate	V	variable
/	diagonal		
D	difference	Underlined symbol - student repeated word in definition	
E	exponent	Boldfaced symbol - student confused definition of word	
F	factor		
I	irrational	*	gave mostly words-only responses
M	mean	+	gave many examples
X	multiple	=	even distribution of modes of response
p'	prime	-	left many entries blank
P	product	!	drew many diagrams

Number			Diagram	Blank
1 *	**Worded Responses**	<, A, C, /, D, E, F, I, M, X, p', P, R, S, V		
	Example	A, C, E, M, X		
2 +!	**Worded Responses**	<, A, C, /, D, E, F, I, X, P, R, S, V	<, A, C, /, S	M, p'
	Example	<, A, C, /, D, E, F, X, P, R, V		
3 +!	**Worded Responses**	<, A, C, /, D, E, F, I, M, X, p', P, R, S, V	<, A, C, /, S	
	Example	<, A, C, /, E, F, M, p', P, S		
4 *	**Worded Responses**	<, A, C, /, D, E, F, M, **X**, **p'**, P, R, S, V	<, S	I
	Example	D, M, X, p'		
5 *!	**Worded Responses**	<, A, C, /, D, E, F, M, X, p', R, S, V	<, A, C, S	I, P
	Example	C, V		
6 =	**Worded Responses**	<, A, C, /, D, F, X, P, R, S, V	<, /, S	E, I, M, p'
	Example	A, V		
7 =!	**Worded Responses**	A, C, /, D, **E**, I, M, X, **p'**, P, R, S	<, A, C, /, S	F, V
	Example	<, A, C, X, p', S		
8 =	**Worded Responses**	<, A, C, /, E, F, I, M, X, p', R, S	<, C, /	D, P, V
	Example	A, E, M, X, S		
9 +	**Worded Responses**	<, A, C, /, D, E, F, I, M, X, p', P, R, S, V	<, S	
	Example	D, F, I, M, X, p', P, R, V		
10 +!	**Worded Responses**	<, A, /, E, F, I, **M**, **X**, p', P, R, S, V	<, A, C, /, S	
	Example	D, E, F, I, X, R, V		
11 =	**Worded Responses**	<, A, C, /, D, E, F, I, **X**, P, R, S, V	<, /, S	M, p'
	Example	A, C, D, E, X, R		
12 *	**Worded Responses**	<, A, C, D, I, M, X, p', P, R, S, V	/, S	E, F
	Example			
13 -!	**Worded Responses**	A, C, /, I, X, p', P, R, S	A, C, /, S	<, D, E, F, M, V
	Example	C, X, P		
14 *	**Worded Responses**	<, A, **C**, /, D, E, F, I, M, X, p', P, R, S, V	<, C, /	
	Example	A, D, E, p', R		
15 +!	**Worded Responses**	<, A, C, /, D, E, F, I, M, **X**, p', P, R, S, V	<, A, C, D, S	
	Example	E, F, M, X, p', P, R, S, V		

Number			Diagram	Blank
16 *	**Worded Responses**	<, A, C, /, D, E, M, X, p', P, R, S	<	F, I, V
	Example	S		
17 -	**Worded Responses**	<, A, D, E, M, X, p', R, S	<, /	C, F, I, P, V
	Example	E, S		
18 -	**Worded Responses**	A, C, E, R, S, V	S	<,/,D,F,I,M,X,p',P
	Example	A, E, R, V		
19 *	**Worded Responses**	<, A, C, /, D, E, F, I, M, **X**, p', R, S, V	<, C, S	P
	Example	<, X, S		
20 =!	**Worded Responses**	<, A, D, F, I, M, X, p', P, R, S	<, A, C, /, S	V
	Example	A, D, E, p', R		
21 *!	**Worded Responses**	<, A, C, D, E, F, I, M, X, p', P, R, S, V	A, C, /, S	
	Example	A, R, V		
22 *	**Worded Responses**	<, A, C, /, D, E, F, X, p', P, R, S	<, /, S	I, M
	Example	<, E, V		
23 *!	**Worded Responses**	<, A, C, /, D, F, I, M, p', P, R, S, V	<, A, C, /, S	
	Example	D, E, X, p'		
24 *	**Worded Responses**	<, A, C, /, D, E, X, p', R, S		F, I, M, P, V
	Example			
25 +!	**Worded Responses**	<, A, D, X, P, R, S, V	<, A, C, /, S	F, I
	Example	D, E, M, X, p', P, R		
26 =!	**Worded Responses**	<, A, C, /, D, F, M, X, p', P, R, S	<, A, /, D, S	E, I, V
	Example	<, X, R		
27 +	**Worded Responses**	<, A, C, /, D, F, M, X, p', P, R, S	C, /, S	E, I, V
	Example	<, D, F, M, p', P, R		
28 +!	**Worded Responses**	<, A, C, D, E, F, I, p', P, R, S, V	<, A, C, /	
	Example	D, E, F, M, X, p', P, S, V		
29 *	**Worded Responses**	<, A, C, /, D, E, F, I, **X**, p', P, R, S, V		M
	Example	D, E, X, P		
30 +	**Worded Responses**	<, A, C, /, D, E, F, I, **X**, p', P, R, S, V	/	M
	Example	D, E, F, I, X, R, S, V		
31 *	**Worded Responses**	<, A, /, D, E, F, I, M, **X**, p', P, R, S, V	C, /, S	
	Example	E, F, X, p'		
32 +	**Worded Responses**	<, A, C, /, D, E, F, I, M, X, p', P, R, S, V	C, /	
	Example	C, D, E, I, M, X, p', P, R, S, V		
33 *!	**Worded Responses**	<, A, C, /, D, F, I, M, **X**, p', P, R, S, V,	<, A, C, /, D, M	E
	Example	D		
34 +	**Worded Responses**	<, A, C, /, D, E, F, X, p', P, R, S, V	A, C, /	I, M
	Example	D, E, F, X, p', P, R, S		
35 =!	**Worded Responses**	<, A, C, /, D, E, F, p', P, R, S	<, C, /, D, S	I, M, X, V
	Example	D, E, F, R		
36 *	**Worded Responses**	<, A, C, /, F, I, M, X, p', P, R, S, V	C, /, S	D, E
	Example	<, F, I, M, X, P, R, S, V		

Number			Diagram	Blank
37 -!	**Worded Responses**	<, /, I, p', R, S	<, C, /, S	A, D, F, M, V
	Example	E, X, P		
38 *	**Worded Responses**	<, A, C, /, D, E, I, M, p', P, R, S, V	C	
	Example	E, F, X, S, V		
39 =	**Worded Responses**	<, A, C, p', P, R, V	/, S	I
	Example	/, D, E, F, M, X, R		
40 =!	**Worded Responses**	<, A, C, /, D, I, **X**, p', P, R, S	<, A, C, /, M, S	E, V
	Example	F, M, X, p', P, R, S		
41 *	**Worded Responses**	<, A, C, /, D, E, F, I, M, X, p', P, R, S, V	A	
	Example	A, D, E, I, M, X, P, V		
44 *!	**Worded Responses**	<, A, C, /, D, E, F, I, M, X, p', P, R, S, V	A, C, /, S	
	Example	A, C, D, M, R		
45 +!	**Worded Responses**	<, A, C, /, D, F, I, M, p', **P**, R, S, V	<, A, C, /	
	Example	<, A, C, D, E, F, M, X, p', P, R, V		
46 *	**Worded Responses**	<, A, C, I, M, X, p', P, R, S, V	/, D	F
	Example	D, E, R		
47 *!	**Worded Responses**	<, A, C, /, D, E, I, X, p', P, R, S, V	A, C, /, D, M	F
	Example	A, D, M, R		
48 +	**Worded Responses**	<, A, C, /, D, E, F, M, **X**, p', P, R, S, V	C, /, S	I
	Example	D, E, M, X, P, R, S, V		
49 +!	**Worded Responses**	<, A, C, /, D, E, F, M, **X**, p', P, R, S, V	<, A, C, /, S	I
	Example	<, D, E, F, M, X, P, R, S, V		
50 *	**Worded Responses**	<, A, C, /, D, I, M, **X**, p', P, R, S, V,		E
	Example	F, X		
51 *	**Worded Responses**	<, A, C, D, F, I, M, X, p', P, R, S, V	/, S	
	Example	D, E, F, X, P, R, S, V		
52 +!	**Worded Responses**	<, A, C, /, D, F, I, M, X, p', P, R, S, V	<, A, C, /	
	Example	<, A, /, D, E, F, X, p', P, R, V		
53 +	**Worded Responses**	<, A, C, /, D, E, F, I, M, X, p', P, R, S, V	C, /, S	
	Example	D, E, F, X, p', P, R, V		
54 +!	**Worded Responses**	<, A, C, /, D, E, I, M, X, p', P, R, S, V	<, A, C, /, S	
	Example	<, A, D, E, F, I, M, X, p', P, R, V		
55 =	**Worded Responses**	<, A, C, /, D, **X**, p', P, R, S, V	C	I, M
	Example	E, F, X, p', R, V		

Chapter Six

Summary, Discussion, and Conclusions

Outline

In this concluding chapter, the findings which are detailed in chapters four and five are condensed and presented to highlight the prominent recurring themes of word meanings which emerged from the data. A description of a comparison analysis between the classes of the two participating teachers is given. Following this, the results and findings of this study in relation to lexical ambiguity are considered. In conclusion, implications for classroom practice are discussed, and topics for further research are suggested.

Summary of Findings

Chapters four and five of this paper describe in detail the responses of participating students to the fifteen words included in the study. My purpose here is to provide a brief synopsis of the strands of meaning which emerged from the data for each of the words as they were described in chapter four. Percentages are provided simply for a general picture of the prominence of each strand of meaning, but should not be regarded as the essence of this study's findings. In addition, my observations concerning the different modes by which the students communicated their conceptions of the vocabulary terms are condensed from chapter five and presented.

Acute. Eighty-four percent of the students in the study associated *acute* with an angle measuring less than 90 degrees. The remainder confused *acute* with *obtuse* or *right* angles or simply drew angles with no specification about the measurements of the angles.

Area. The dominant association with *area* was with "space in," "space on," or "space taken up by." One quarter of the students provided one or more area formulas (most commonly the formula for the area of a rectangle) as their definition for *area.*

Coordinate. *Coordinate* was connected to graphing in the majority of responses. In many cases, students did not distinguish between the ordered pair of numbers constituting the *coordinates* from the point itself.

Diagonal. The notion of a slanted line dominated the responses to *diagonal.* Only one quarter of the students gave evidence that their interpretations of *diagonal* included the notion of a segment which joins non-adjacent vertices of a polygon. However, the interviews indicated that some of these students would only consider a slanted segment joining opposite vertices of figures to be a *diagonal.*

Difference. Fifty-five percent of the participants associated *difference* with subtraction, but only 22% of the sample specified that *difference* denotes the answer of a subtraction exercise. Other students gave definitions of *difference* in its everyday sense of "not the same."

Exponent. Forty-two percent of the responses described or showed an example of how an exponent appears as "a little number that sits on the top right side of a number." A quarter of the students explained *exponent* in terms of its function of indicating that a specific number of multiplications are to be performed with the base. Two students connected *exponents* solely to scientific notation.

Factor. All the participants in the study who provided a response except one either thought of *factor* strictly as a noun, or strictly as a verb. Those who recorded a noun sense of *factor* said it was a number which is multiplied or, less specifically, a number in a multiplication problem. For the students who tried to communicate their interpretation of *factor* in its verb sense, very few could find alternate words to express "to factor something." Eight students confused the use of *multiple* for the noun sense of a *factor.*

Irrational. One third of the students left this entry blank. Most of the remaining students either said that *irrational* was "not rational," or tried to apply the everyday meaning of "not making sense" into a mathematical context. Only twelve students connected *irrational* to numbers, and only two of these twelve actually gave a correct example of an *irrational* number.

Mean. Twenty students associated *mean* with an average or an "in-between" number. Only students from school B communicated applications of *mean* to the *geometric mean* and the principle of *means equals extremes.* Apparently, geometric means had been recently studied in their geometry classes at the time the surveys for this study were administered. Thirteen students left this entry blank.

Multiple. The term *multiple* seemed to be the most susceptible to confusion with other mathematical terms. Confusions were evident between *multiple* and *multiply,* and also between *multiple* and *factor.* Many students responded to this entry with examples, but some of the examples were inconsistent with the written responses in that they were appropriate examples of *factors* of a particular number instead of *multiples* of that number.

Prime. Forty percent of the students were able to give an accurate written definition of *prime* similar to "a number that can only be divided by one and itself and still be a whole number." Many of the students provided examples of prime numbers in their responses. Three students apparently experienced interference from the everyday sense of the word as evidenced by attempts to apply that sense into a mathematical context.

Product. More than fifty-five percent of the students associated *product* with multiplication, with most of them correctly noting that the *product* is the answer of a multiplication. Thirteen students applied *product* to an answer or result of any mathematical process.

Reduce. Thirty-four students wrote about *reduce* in its more everyday, yet still mathematical, sense of making less or smaller or in reference to subtraction. The responses of twenty-five students applied *reduce* to the simplification of fractions.

Square. Two dominant mathematical meanings appeared most frequently in the responses of the students. The sense of the word which refers to the shape with four right angles and four sides of equal length was noted by 50 of the 55 participants. Their responses ranged in detail from scrawling a four-sided figure in the entry to describing the characteristics of congruent sides with opposite pairs being parallel, right angles, and perpendicular, congruent diagonals which bisect each other. Twenty-one students applied *square* to the process of multiplying a number by itself or to having an exponent of two. *Square root* and *square units* appeared in a few responses as other mathematical applications of the term *square*.

Variable. Thirteen students said that a *variable* is a number, while twenty-two stated that a *variable* is a letter. Sixteen of these 22 spoke of *variable* as representing numbers. Almost one third of the responses contained the idea of *variables* representing unknown values. Twelve students left this entry blank.

Response Modes

The second perspective with which the data was examined looked at whether the students' responses came in the form of words, examples, or diagrams. Slightly more than one third of the responses contained only words. One quarter of the responses were supplemented with examples. Only seven of the fifteen terms were suited to being illustrated with drawings in the students' responses. For five of these terms, *diagonal, coordinate, square, acute,* and *area,* drawings were made by more than 40% of the participants.

Effects of Vocabulary Emphasis

In chapter three, I mentioned that the two classroom teachers whose classes participated in this research differed from each other in the amount of emphasis they claimed to place on mathematical vocabulary in their classrooms. As a means of comparison, I conducted a final, overview analysis of the responses from the written survey forms to see if notable differences could be detected in the overall performance of the students from school A, whose teacher concedes that he does not emphasize vocabulary, and those from school B, whose teacher claims to give specific attention to mathematical vocabulary. Separating the responses of the students from schools A and B, I examined their worded responses, examples, and diagrams, and classified their entire entry for each term as correct or incorrect.

A classification of "correct" encompassed many partially correct responses with a broad span of detail and completeness in the responses. If the wording was vague, but a clear, correct example comprised a part of the response, the response was classified as "correct." Otherwise, blank, incomplete or vague responses were categorized as "incorrect."

I used more explicit criteria in my classifications of "correct" or "incorrect" for specific terms. For the term *area*, responses that seemed more appropriate for *perimeter* or *volume* were considered "incorrect." For the term *coordinate,* I counted as "correct" only those responses which clearly pointed to the numbers marking the location of a point, and did not confuse the *coordinates* with the point itself. Concerning the term *diagonal*, responses which did not explicitly say that the *diagonal* joins two opposite or non-adjacent vertices of a polygon, or did not contain a picture of a segment joining opposite vertices of a polygonal figure were classified as "incorrect." Thus, direct references to the everyday connotation of a slant were categorized as "incorrect" responses for *diagonal.* With the term *exponent*, "correct" responses included both a general recognition of what an *exponent* looks like and the function that the *exponent* performs. The term *reduce* has at least two distinct meanings in a mathematical context which appeared frequently in the students' responses. Responses which contained either sense of *reduce* were counted as "correct." The students also responded to the term *square* in two dominant senses. Either sense was accepted as a valid sense in which to define the term, but responses which did not specify the multiplication of a number by itself , an exponent of two, or did not distinguish *squares* from other quadrilaterals were not considered to be correct. For the term *variable*, I counted as

"incorrect" only those responses which did not refer to the use of letters or symbols to represent a replacement set, whether that replacement set contained many or only one value.

Because the classification of "incorrect" responses was a more clear-cut process, and the classification of "correct" responses included a broad range of partially right descriptions of the terms, the results of this comparison analysis are presented as a percentage of "incorrect" responses. For each word, the number of "incorrect" responses was divided by the number of students in the group, 19 students from school A and 36 from school B. The mean of these percentages was then calculated.

As a group, the students from school A responded incorrectly on 49.3% of their entries. The students from school B had 35.2% of their responses judged "incorrect" according to the criteria noted above. Nevertheless, this analysis and the results are solely meant to be descriptive of the two groups participating in this study. The tabulations of percentages correct or incorrect should not be taken as conclusive, as this comparison analysis was undertaken simply based upon two teachers' self-evaluations of their own teaching and the relative attention they direct toward vocabulary in their classrooms.

While some distinction may be discernible between the overall comprehension of the students from school B and those from school A, the percentages of incorrect responses are high for both groups. The imprecision or incompleteness of the responses overall is disheartening. The teacher in school B claims to emphasize vocabulary, but is the way which she focuses attention on vocabulary providing the optimum conditions for students to construct their understanding of the concepts denoted by the words? Unless the students are actively gathering impressions about mathematical terms in use around them so that they

have material to organize and structure concerning the ideas conveyed by the vocabulary words, the efforts of the teacher may prove ineffectual. The teacher in school A claims not to accentuate mathematical vocabulary in his instruction. If this is so, his students are left to make sense of terms without structured support in the classroom.

The results of this comparison analysis indicate that for students in both schools, terms are only loosely associated with ideas, but when pressed as to the nature of that association, a third and more of these students were unable to specify. Without observation and more information about these two teachers' classrooms than their own statements about vocabulary emphasis, no claims can be made concerning the effects of vocabulary emphasis with these particular students. The comparison analysis does not seem to show any dramatic differences in the performance levels of students from school A and school B. Both teachers might find it worthwhile to devote more attention to key vocabulary words, without which the learning of mathematics becomes an exercise for students in trying to comprehend in a "foreign" language.

Interference of Lexical Ambiguity

The survey of studies related by Durkin and Shire (1990) on the effects of lexical ambiguity in mathematics education focuses primarily on the problems which young children encounter. Durkin and Shire (1990) cite their own and other research (Mason et al, 1979, as cited in Durkin and Shire, 1990) which shows that students will far more often apply the everyday sense of a word into a mathematical context than a mathematical sense into an everyday context. As students progress in their school mathematics, they encounter more

new vocabulary words which already have everyday connotations associated with them. These previously-learned connotations can interfere with the accommodation of the mathematical meaning and can lead students to try to apply their everyday connotations into the mathematical context.

This present study and other studies (Otterburn & Nicholson, 1976; Monaghan, 1991; Malone & Miller, 1993; Hardcastle & Orton, 1993) show that difficulties with lexically ambiguous mathematical vocabulary is not limited to young children. Monaghan's (1991) study provides evidence that through the high school years, the use of lexically ambiguous terms to name new and more complex mathematical concepts continues to cause confusion and interference for students trying to grasp the precise mathematical ideas signified by the terms.

Durkin and Shire (1990) noticed that the older students in their study produced fewer errors with the same terms than did younger students. This may suggest that given sufficient time and experience with lexically ambiguous words, students will be more likely to sort through the different aspects of the terms. Whether to primary age students or calculus students, the problems attributable to lexical ambiguity may tend to diminish as the students have more focused experiences with the words and the concepts conventionally conveyed by the words (Durkin & Shire, 1990; Monaghan, 1991). Thus, more structured opportunities to work with, use, and articulate meanings for lexically ambiguous terms appropriately in a mathematical context are necessary in order for students to construct conventional mathematical meanings for mathematical terms. Teachers should not assume that with the

status quo maintained, students will simply "pick up" the mathematical meanings of the terms used regularly in mathematical discourse.

The most prominent evidence of interference from everyday senses of the fifteen terms in this study occurred with the word *diagonal.* Despite the fact that the mathematical definition of a diagonal as a segment joining non-adjacent vertices of a convex polygon is more prevalent in a high school geometry course than any other high school course, half of the respondents held to the everyday notion of a diagonal being a line oriented at a slant.

Several of the other terms may have also been affected by interference from everyday aspects of the words, but evidence of such interference was not as pronounced as with *diagonal.* Even when posed the term *difference* in a mathematical context, nine students apparently had made no accommodation for *difference* to refer specifically to the result of subtraction. A large majority of the students in my sample gave no evidence of comprehending the mathematical meaning of *irrational.* Thus, they attempted to fit the everyday sense of the word, which they did know, into a mathematical context. To a much smaller degree, this phenomenon also occurred with the term *prime*, but only with two or three students who supposed a prime number to be "the main number" or "the largest number." *Product* was perceived by many of the students as that which was "produced" by any mathematical operation. Likely, a student's definition of *product* as "the answer of the equation" in general and "what a factory makes" is more influenced by products on store shelves than by the limited application of *product* in mathematics class to the results of multiplication.

The examples of difficulties experienced by the students in this study which seem directly attributable to lexical ambiguity are too many to ignore. Teachers need to be aware of the possible confusion that different uses of mathematical vocabulary words can cause. Only then can they consciously point out to students the distinctions and the commonalties of mathematical and non-mathematical meanings. A crucial language skill for students is to distinguish the use of words in different contexts and registers. Mathematics teachers have a role to play in helping students develop this skill. By emphasizing vocabulary and language aspects of the mathematics register and devoting much time and energy to the explicit discussion and articulation of meaning by the students in the classroom, teachers can enable students to participate competently in mathematical discourse and provide students with valuable language skills which will be useful to them in many different contexts.

Reform Mandates and Mathematical Literacy

The National Council of Teachers of Mathematics (1989, 1991, 1995) in recent years has called for overhauling and upgrading the curriculum, teaching, and assessment standards in high school mathematics based upon major changes in our society and the transition from an industrial focus to an information focus. A key issue in this reform movement focuses on enabling and encouraging students to communicate the mathematical concepts they are learning and to become more mathematically literate.

Mathematical literacy requires more than a learning of mathematics vocabulary. As Bullock (1994) observes,

> knowing the formal definitions of a large number of words is no guarantee that one can actually say anything comprehensible. One must obtain practice in expressing complete thoughts with language. In mathematics, this means...applications....What

> matters is that words must be given a context if we are to be enriched by understanding their meanings (p. 740).

Countryman (1992) warns that pre-packaged "definitions alone rarely throw much light on the ideas they represent" (p. 55), citing instances in which students could repeat the "correct" words, but the working out of those ideas in application did not necessarily follow. Vocabulary terms have no educational significance in themselves. The concepts and constructs which are represented by them are of greater importance, and these concepts are not grasped simply through memorizing textbook definitions. Barnes (1990) insists

> the meanings...provide the only valid justification for teaching the words. The words without the concepts they represent...are strictly useless to pupils. As their grasp of a concept develops, the technical term becomes a useful centre about which can cluster relevant experiences and understanding; presented too soon the term may actually inhibit the process of clustering and abstraction" (p. 49).

Thus, an emphasis on mathematical vocabulary should not revolve simply around words, but must focus first on providing valid, relevant experiences with the mathematical concepts. Students' experiences with new concepts should allow for active involvement and provide connections to previous learning. Then as the students make sense of the new concepts and begin to apply the concepts, a need is created to talk about and to have names for the concepts. It is through learning both the concepts and the words that name them that access into mathematical discourse is opened and further understandings of the concepts grow.

The sketchiness and imprecision of many of the students' responses in this research project give some evidence that either they do not clearly understand the concepts communicated by the fifteen terms of this study or they are unable to express their understandings clearly. The results of this study, combined with teen-agers' difficulties with

written communication, create a picture of students only partially making sense of words which are fundamental in their mathematics courses.

Further research in the area of mathematical vocabulary is called for as it fits into the larger picture of connecting the vocabulary to the underlying mathematical concepts. If what Barnes (1990) says is true that terms can be introduced too soon and actually inhibit students' understanding, when is the optimal time and under what conditions should mathematical vocabulary words be taught? What teaching strategies and activities provide students with effective experiences with the mathematical concepts? Does an emphasis on mathematical vocabulary in the classroom make a significant impact on students' negotiation of meanings with lexically ambiguous terms. How can lexical ambiguities be used as teaching opportunities to stimulate mathematical discussion? Many other mathematical terms are characterized by lexical ambiguity, and students' interpretations of them could also be explored qualitatively. The terms *simplify, negate, cancel, distribute, roots, operation, function, domain* and *range* name just a few. Further research could also explore the relationship of mathematics vocabulary aptitude with performance in other areas of mathematics.

Vocabulary comprises a small part of the world of mathematics, but its importance and role must not be overlooked. The words of mathematics must not be separated from the concepts they represent, but neither can the concepts be separated from the words. The words provide the vehicles for concepts to be discussed, to be questioned, and to be understood in relation to other concepts. Specific names conventionally used make it possible for mathematics to be a social enterprise. Access to the mathematical enterprise,

discourse, and learning is effectively denied or granted based upon understanding the language by which mathematics is communicated. Only through appropriate attention to vocabulary and other language factors of mathematics can students confidently enter into the enterprise we call mathematics.

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Appendix A
Vocabulary Survey

Name (Give the code number assigned by your teacher) _____

Grade: _____

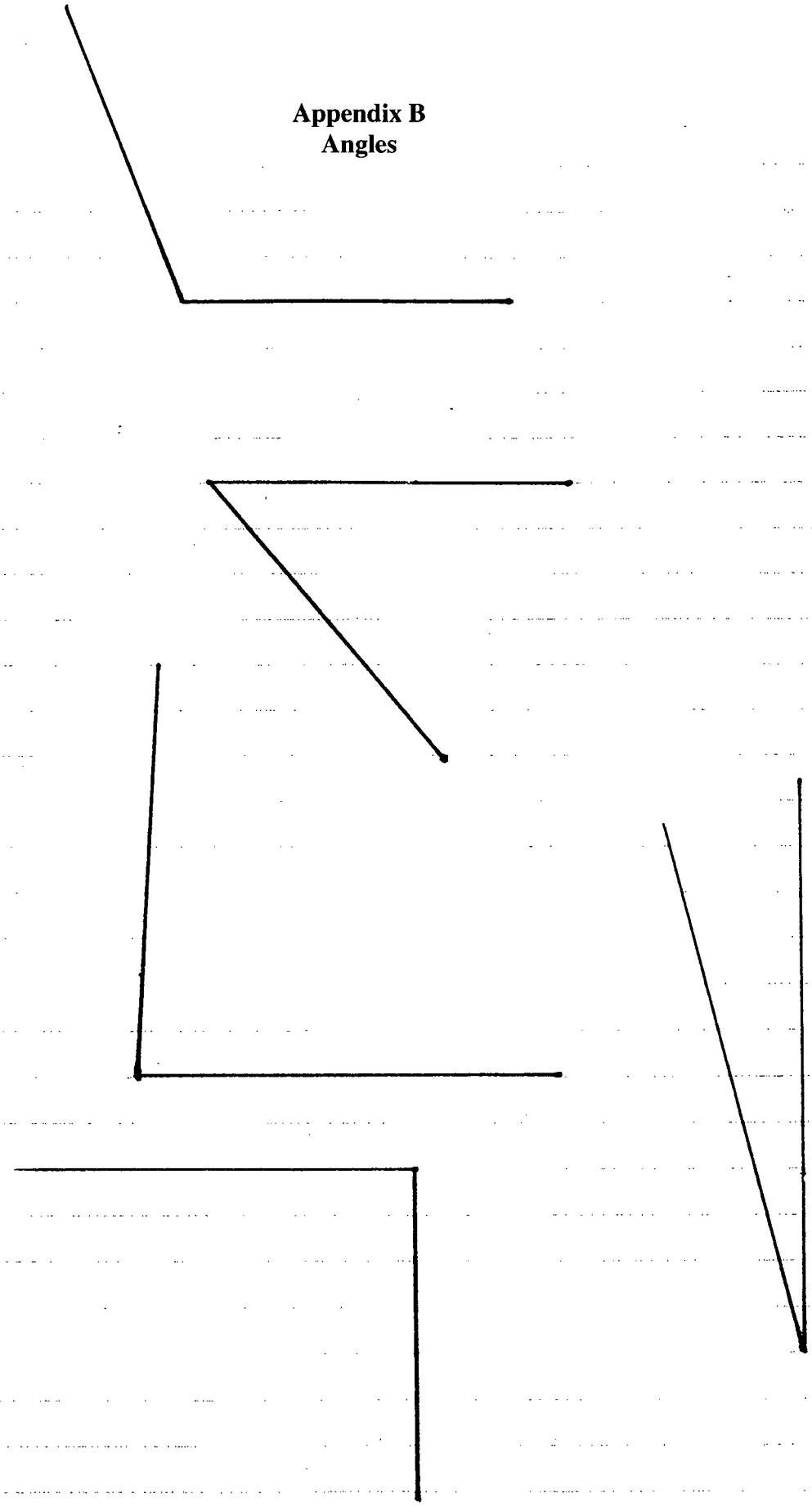
- What high school mathematics courses have you taken? (Check the ones that apply) Include the math course you are in now.

[ ] Prealgebra
[ ] First Year Algebra
[ ] Geometry
[ ] Second Year Algebra

- In your mathematics classrooms and textbooks, you hear and read many words that have special mathematical meanings. Some of these terms are listed below. **Please write down what these words mean to you as completely as you can. You may use pictures, diagrams, or examples or any other means you can think of to illustrate your meanings for the word.** If you need more room, feel free to use the back of the page or the extra paper that is provided.

1) acute
2) area
3) coordinate
4) diagonal
5) difference
6) exponent
7) factor
8) irrational
9) mean
10) multiple
11) prime
12) product
13) reduce
14) square
15) variable

Appendix B
Angles



Appendix C
Diagonals