Collective Clutter and Co-Emerging Complexity
Enactivism and Mathematical Paths of Understanding

by

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Abstract

This thesis reports on a qualitative study in which three fifth grade children were presented with six nonroutine mathematical problems involving six different 3-D pyramids, constructed out of multi-link cubes. The children were videotaped while they worked without any adult help as a cooperative group to solve the pyramid problems. During these sessions, the students produced various 3-D cube models, 2-D drawings, and written records of arithmetic calculations as their solutions to the six problems. Through the lens of enactivism, this study describes and interprets the co-evolutionary processes of the group’s path of mathematical understandings as it unfolded during the six videotaped sessions. The results revealed building, drawing, and numbering as modes of representation of this group’s problem solving work. An analysis of these three modes of representation explored the co-emergence of the children’s individual and collective understandings, as well as the interrelationships which existed between their spatial structuring and their use of numerical operations in solving the pyramid problems.

1 A multi-link cube is a 2 cm x 2 cm x 2 cm plastic cube that can be affixed to other cubes on any one of its six faces.
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A Beginning

Every story has a beginning. This story begins with me sitting in front of the computer—thinking. I have many ideas about mathematics education that I would like to share, but for some reason I am unable to type them onto the screen. My eyes wander down to the floor where recorded thoughts, several books, and many more journal articles have become the ground which surrounds me. I am not thinking about what to write at this moment. Rather, it is my mother’s voice that I am hearing...

“Jennifer, your room is always cluttered... you need to get rid of all of this... put these back in their place”. My head starts to feel heavy, my stomach turns, and I begin to feel overwhelmed. Just then, a memory of a friend who had come to visit me enters my mind and puts me at ease. This friend had come to my house, looked at my collections of clutter and exclaimed, “Wow! You come out of the door looking so well-put-together!” I am sure that you can identify with this beginning and have had similar experiences—that of my mother, my friend, myself, or all three. I can understand why my mother tells me this—it has something do with who she is, but even though I have attempted on many occasions to get rid of things, I find I cannot part with them because for some strange reason, I need them. So I try to put my things where my mother thinks they should belong, but these things always find their way back to the floor, on tables, and down the hallway, in happy clusters. I now know that clutter is a part of who I am and the way in which I choose to live in this world.

My eyes wander back to the computer screen, and it is in this spirit that I would like to continue my story. As an elementary teacher and a graduate student, I find myself constantly orienting and reorienting myself in the field of mathematics education—specifically in developing a deeper understanding of children’s mathematical growth. My inspiration for this venture came from a piece written by
Norwegian philosopher Arne Naess (1985) who composed a summary, *The Shallow and the Deep, Long-Range Ecology Movement*, in which he interrogated and synthesized the political, cultural, scientific, and philosophical issues that differentiate *shallow ecology* from *deep ecology*. He describes shallow ecology as a movement which privileges humans (especially those societies in developed countries) over all other forms of nature, and concerns itself with issues that are directed toward sustaining human health and affluence such as pollution and resource depletion. Conversely, Naess' definition of deep ecology is elucidated by Capra (1996) when he expresses deep ecological awareness as being a view that:

...does not separate humans—or anything else—from the natural environment. It sees the world not as a collection of isolated objects, but as a network of phenomena that are fundamentally interconnected and interdependent. Deep ecology recognizes the intrinsic value of all living beings and views humans as just one particular strand in the web of life" (1996, p. 7).

This movement also involves the asking of deeper questions from which a “perspective of relationships to one another, to future generations, and to the web of life of which we are part” (Capra, 1996, p. 8) is possible.

Naess' piece provides a place of opportunity for me to search for relationships which exist within mathematics education. Children cannot be regarded as isolated minds; as humans they are an integral part of a world of possibilities that affects and is affected by their growing mathematical understandings. Mathematics is a part of them and their world. In similar vein to Naess' argument, the current clutter that has accumulated in the realm of mathematics education begs for interrogation. In the next section, I explore constructivism and look for similarities and differences between current constructivist perspectives. Following this, I discuss the ways in which constructivism has informed our view of mathematical knowledge, children's learning
of mathematics, and the relationships that exist between the teacher, the students, and their educational environment. I then move into the areas of deep ecology, biology, phenomenology, and enactivism to examine and investigate the ways in which these alternative views can provide new and deeper spaces for children's learning of mathematics and my inquiry into the meaning of mathematical growth.
Polysemous Constructions of Constructivism

Constructivism, once a distinct epistemology has become a commonly used and increasingly misinterpreted educational term, an obscured learning theory, and grounds for every style of teaching--from the transmissive, to the transactive, to the transformative. Consequently, constructivism's loss of identity and claimed place in virtually every mathematics classroom has resulted in frustration and confusion among educators today (Noddings, 1990).

In any event, regardless of the different branches of constructivism--radical, social, or trivial, there are two common principles. The first tenet is the view that knowledge is actively built by the cognizing subject, and it is the individual's communication of his or her ideas and thoughts which facilitate the construction of personalized meanings. The second tenet is that an absolute view of the world cannot be 'found' because we can only know the world through our personal experiences. Therefore, to search and to acquire universal truths is not an issue. Reality and knowledge are regarded as viable as long as they fit with the individual's actions, thoughts, and experiences (Chi, 1978; Piaget, 1973; von Glasersfeld, 1989a, 1989b).

In the classroom, the role of the teacher is to establish an environment that promotes such viability and to offer learning challenges which confront and extend the individual's constructed knowledge. Constructivists conceive knowledge as being a continual non-linear process of self-organization\(^2\) (Nash, 1970; Piaget, 1973; von Glasersfeld, 1989a, 1989b).

Within the realm of constructivism lies social constructivism. It is this branch of constructivism which incorporates into its framework, the co-construction of knowledge and the negotiation of meanings that stem from social interactions. In addition to the

\(^2\)Self-organization is taken to mean the way in which an individual 'makes and re-makes sense of' existing schema based on his or her experiences. This organization relies on the successful assimilation and accommodation of the individual's experiential knowledge--"Assimilation is the process whereby changing elements in the environment become incorporated into the structure [schema] of the organism. At the same time, the organism must accommodate [unconsciously or deliberately] its functioning to the nature of what is being assimilated" (Nash, 1970 p. 360).
two other constructivist principles, the social dimension of experience is viewed as being equally important in the self organization of the individual. Social constructivism supports Vygotskian frameworks which identify social interactions as being essential in the process of meaning making and self-organization:

Any higher mental function [is] was external and social before it [is] was internal. It [is] was once a social relationship between...people. Any function appears twice or on two planes. It appears first between people as an intermental category, and then within the child as an intramental category (Vygotsky, 1960, p. 197-198).

**Constructivism in Mathematics Education**

Constructivism in the field of education has provided new vantage points which have brought about positive changes in the way educators perceive the teaching and learning of mathematics. This is made apparent by the extensive research that has been and is currently being conducted in mathematics education (e.g., Cobb, 1994a, 1994b; Cobb, Yackel, & Wood, 1992; R. B. Davis, 1992; R. B. Davis, Maher, & Noddings, 1990; Hiebert & Wearne, 1993; Lesh, Post, & Behr, 1978; Lo, Wheatley, & Smith, 1994; Phillips, 1995; Pimm, 1987; Wheatley, 1991, 1992). As well, the widespread acceptance of constructivism among mathematics educators today can largely be attributed to the National Council of Teachers of Mathematics' (NCTM, 1989) publication of *Curriculum and Evaluation Standards for School Mathematics* in which the Council advocated for the instruction of mathematics to be centred on the conceptual development of children's mathematical knowledge.

However, in other respects, constructivism remains shallow and problematic. Firstly, constructivism's scope is narrowed and restricted because it exists only as an epistemology founded in cognitive psychology. Its lack of philosophical perspectives makes constructivism unable to encompass and address diverse but influential issues.
that are connected to one's knowledge such as morals, ethics, and cultures (Bowers, 1995, 1997a, 1997b; Bowers & Flinders 1990; A. B. Davis, 1996; A. B. Davis, Sumara, & Kieren, 1996; Frankenstein, 1983, 1989). Secondly, constructivist views hold all learning and knowing as being ultimately directed back towards the individual's internalized knowledge. This separation further alienates the individual from the rest of his or her environment because the view assumes the individual as an independent entity. Thirdly, constructivism's inability to move or connect itself to such issues as the transcendence of subjectivities (A. B. Davis, 1996; van Manen, 1990) or the holistic nature of knowledge, has created the perception of knowledge as existing as an object (Wheatley, 1991, 1992). Subsequently, to conceive knowledge-as-action (Bateson, 1972, 1979), knowledge-as-interaction, or knowledge-as-experience (Abram, 1996; Capra, 1996; A. B. Davis & Sumara, 1997; Maturana & Varela, 1987) becomes problematic.

Being pulled in these three directions has consequently resulted in the fragmentation of constructivism and the dissection of it in terms of dichotomies. For example, the debate regarding whether it is the individual's knowledge which informs or is derived from social mediation continues to be a topic of ongoing tension among constructivists (Cobb, 1994b; A. B. Davis, 1996; Dewey, 1963; Edwards & Mercer, 1987; Minick, 1989; Vygotsky, 1960). This leaves me wondering—Is constructivism connected to a larger ontological framework? Are there any alternative views which can address the problematic issues of constructivism? If so, what implications do these perspectives pose in terms of students' learning of mathematics? It is here that I wish to leave constructivism, and turn the reader's attention to the areas of ecology, biology, phenomenology, and enactivism in order to address and respond to these questions.
Moving Beyond Constructivism and Deeper Into Ecology, Biology, Phenomenology, and Enactivism

In Fritjof Capra's book (1996), The Web of Life: A New Scientific Understanding of Living Systems, he communicates that because of Western society's rapid movement toward ecological crises and radical developments in science, it is necessary for us to shift from a mechanistic worldview of Descartes and Newton to a holistic and ecological view. This paradigmatic turn Capra suggests, demands the repositioning of our perceptions and our thinking—from the self-assertive which focuses on rational, analytical, reductionist, and linear ways of thinking to the integrative which implies intuitive, connected, holistic, and nonlinear ways of thinking. As well, he believes our values can no longer be anthropocentric beliefs which instil competition, quantity, and domination as being inherently desirable, but must be values which are ecocentric and emphasize cooperation, quality, and conservation. Deep ecology brings with it a world not viewed as being made up of separate entities, but rather a world which are an integrated whole—a phenomenological network of all living and social systems that are interconnected and interdependent. In this way, the world is not understood through the detachment and isolation of its natural and social entities but conceptualized as a highly complex unity in which all systems are interrelated and connected to each other.

For the reader to appreciate what Capra means by this new scientific understanding, our need to view the world in a holistic and ecologically sensitive manner, and eventually come to an understanding of how these arguments implicate mathematics education, an exploration of alternative views on issues regarding autonomy, evolution, and cognition is necessary and will now be discussed.

Re-Viewing Autonomy: Autopoiesis, Organization, and Structure

Humberto Maturana and Francisco Varela (1980, 1987) describe all living systems...
systems as being autopoietic or self-making. What differentiates one distinct autopoietic system such as a human being from another autopoietic system such as a horse is its organization-"the relations between the components that make the unity" (Maturana, 1987, p. 70). Moreover, what distinguishes one living system from another living system within the same category is the structure of that living system. Structure is defined by Maturana (1987) as "the components and the relations that make up a particular unity" (p. 71). Unlike the organization of a living system which remains constant or self bounded, self generating, and self-perpetuating (Capra, 1996; Flieschaker, 1990), its structure is individually unique and is continually changing through the process of structural coupling within its organizational boundaries. Maturana and Varela (1987) define structural coupling as occurring "...whenever there is a history of recurrent interactions leading to the structural congruence between two (or more) systems" (p. 75). These structural changes emerge from the living system's interactions with the environment, which is comprised of human and nonhuman entities.

Furthermore, perturbations that are recognized by an organism as environmental disturbances, trigger but do not determine the structural changes which will occur in the organism. Rather, it is the living system that perceives the environmental perturbations and determines the outcome of such perturbations, whether they be new pathways or new connections within the autopoietic unity. Therefore, it is not possible to anticipate future outcomes with any exactitude. Each change not only alters the inner structure of the living being, but at the same time, alters the way in which the living system and the environment will interact and respond to one another.

**Evolution: A Process of Mutual Specification and Codetermination**

defines an organism, its environment, and how each responds to the other are interdependent. Richard Lewontin’s (1983) description of this dynamic circularity makes the authors’ point clear:

The organism and the environment are not actually separately determined. The environment is not a structure imposed on living beings from the outside but is in fact a creation of those beings. The environment is not an autonomous process but a reflection of the biology of the species. Just as there is no organism without an environment, so there is no environment without an organism (in Varela, Thompson, & Rosch, 1996, p. 198).

Living systems and their environments do not exist independent of one another, but are “in relation to each other through mutual specification or codetermination” (Varela, Thompson, & Rosch, 1996, p. 198). This view of evolution contrasts with the notion of natural selection, which assumes survival of the fittest. Instead of evolution as being one of optimality, this alternative perspective asserts that the unfolding of life occurs by natural drift in which recurrent interactions of living systems with their environments result in a co-evolution. Co-evolution implies survival of the fit, one of satisfying--through creative, cooperative, reciprocal, diverse, and complex processes.

Cognition

Maturana and Varela’s theory of autopoiesis opens a space for cognition to be understood as the process of living--“To live is to know” (Maturana & Varela, 1987, p. 174). Capra (1996) supports this view when he forms the argument that although Western society holds the view of information existing as the basis of thinking, in reality, humans think with ideas and not information. He further substantiates this claim by connecting Theodore Roszak’s (1994) assertion that “information does not create ideas; ideas create information. Ideas are integrating patterns that derive not from information but from experience” (p. 70). Moreover, as humans, it is this continual bringing forth of a world, through our experiences which implicates our very being; that
is, our living bodies enable us "the very possibility of contact, not just with others but with oneself--the very possibility of reflection, of thought, of knowledge" (Abram, 1996, p. 45).

In reference to the work of Maurice Merleau-Ponty (1962) and the interconnectedness that exists between our perceptions, our thinking, and our actions, David Abram rejects the need or possibility of separating our minds from our bodies and our being from the world; "One's perceptions influence one's actions, and one's actions allow for perceptions" (p. 59). By weaving Merleau-Ponty's theory of the phenomenology of perception, that is, the concerted activity of all the body's senses as they function and flourish together, into what has already been discussed regarding cognition, 'knowing' now becomes even more complex. Cognition of living systems not only involves the process of mutual specification of autopoietic organisms and their environments, but as human beings, it is also our perceptions that become the very roots from which cognition emerges. To clarify the latter statement, a discussion regarding the nature of social systems is necessary.

**Third-Order Systems and Social Phenomena**

Arriving at an understanding which establishes who we are and how we know as taking place through interactive life processes cannot be fully achieved unless our understanding also incorporates and accounts for how we interact as collective unities. Just as organisms and their environments co-evolve as autopoietic entities, social systems function in similar ways and are critical in comprehending what Capra refers to as the web of life. It is interactive patterns and relationships within and among living and social systems which constitutes the world as a phenomenological network.

Maturana and Varela (1987) use biological phenomena and concepts from the theory of autopoiesis as analogies to understand the social dynamics which occur between members of a particular group—whether the group is a colony of ants, a pack of wolves, a pair of African parrots, or human beings. The authors' characterize social
interactions as third-order couplings\textsuperscript{5} which take place when:

organisms take part in recurrent interactions, these couplings will occur—with

definite complexity and stability, but as a natural result of the congruence of

their respective ontogenic drifts\textsuperscript{4} (p. 181).

It is this inter-activity between two or more autopoietic organisms of the same class that
gives rise to higher order entities and social phenomena.

Maturana and Varela explain that third-order unities and social phenomena can
only exist when the structures of the individual members function as a collective
network or, co-ontogenies. Through social interactions, perturbations occur, and the
reciprocal structural coupling or mutual specification of 'like' organisms' actions that
result from the members' responses to these perturbations is what enables a group to
function as a collective system. These events which take place as a result of the
group's social activity can be short lived such as the unique duets that are created by
African parrots during their mating rituals, or, they can continue from generation to
generation, and become what are known as cultural behaviors.

It is these coordinated ways of being that evolve from social coupling and define
what Maturana and Varela call communication. They are also the actions which make
up what is known as the linguistic domain. It is in this realm that communicative ways
of being are differentiated from instinctive ways of being. Maturana and Varela state
that because instinctive actions do not depend on the stability of a social system in
order to occur, but rather, rely on the genetic information of the organism, instinctive

\textsuperscript{5}Characterized by the authors as a process of "co-drifting" (p. 180) of organisms whereby "interactions
between organisms acquire in the course of their ontogeny a recurrent nature. This will necessarily result
in co-ontogenies with mutual involvement through their reciprocal structural coupling, each one
conserving its adaptation and organization" (Maturana & Varela, 1987, p. 180). Based on the assumption
that cellular systems are 'first-order unities', and that metacellular systems are 'second-order unities', the
authors distinguish this coupling process as being more complex than cellular or metacellular couplings,
and hence, being of 'third-order'. As well, the ways in which these third-order couplings occur and the
phenomena they give rise to vary, depending on the particular system.

\textsuperscript{4}Ontogenic drift is defined by Maturana and Varela (1987) as being "the history of structural change in a
unity without loss of organization in that unity. This ongoing structural change occurs in the unity from
moment to moment, either as a change triggered by interactions coming from the environment in which it
exists, or as a result of its internal dynamics" (p. 74).
actions are not included in the linguistic domain. As well, it is important to note that while the authors view linguistic actions as giving rise to language, they do not equate the two as being the same or consider language as being a prerequisite for identifying this domain.

**Cultural Behaviors and Cultural Phenomena**

As mentioned earlier, when social phenomena of third-order systems result in reciprocal couplings that become the patterns which maintain a collective unity from generation to generation, these patterns are understood as *cultural behaviors*. And in the same manner that reciprocal couplings give rise to social systems and social events, cultural behaviors are what enable cultural systems and cultural phenomena to exist. The process by which a collective unity achieves transgenerational stability over its social dynamics and evolves, is recognized as *cultural drift* (Maturana & Varela, 1987). Maturana and Varela state that cultural drift in social systems is as essential to the development of collective communities as natural drift is to environmental and living systems. Just as the evolution of an environment or an organism is understood as structural responses, historical transformation is understood as being a cultural system's structural changes that serve in maintaining or shaping its ontogenic course. In other words, historical transformation is deemed necessary for the continuity of cultural entities and takes place when a collective unity re-coordinates the collective actions of its members in response to perturbations which bring into question the group's shared beliefs, values, or actions. When a group's learned ways of knowing and doing become encoded in their natural or 'taken for granted' ways of being, the group's culture can be viewed as the *product* or the structure of the group's social stability. At the same time, the group's cultural actions can be conceived as functioning as a *producer* because it is these actions which give rise to future phenomena. Further still, the ways in which older generations interact with younger generations become the *process* that enables historical transformation.
or cultural drift to occur. It is this dynamic circularity of product, producer, and process that enables cultural systems to exist and evolve.

The fact remains that we are continuously immersed in this network of interactions, the results of which depend on history. Effective action leads to effective action: it is the cognitive circle that characterizes our becoming, as an expression of our manner of being autonomous living systems (Maturana & Varela, 1987, p. 241).

Therefore as humans, the world we know is not only created through our experiences as individual autopoietic beings, but is also shaped or specified through our existence and participation as members of cultural systems. For instance, communicative structural coupling can be observed in the way we are taught to read and write. As members of Western society, we learn to read and record textual information in a left to right, top to bottom directionality and accept the reason for acting in this manner as being ‘the way we do things’ in order to ‘make sense’ of what we are reading or writing. This example of social coupling is just one of the countless culturally learned ways of being that constitutes what it means for us to participate as members of a collective network. Moreover, as cultures develop their taken for granted ways of being, double binds are also created. Maturana and Varela explain:

By existing, we generate cognitive “blind spots” that can be cleared only through generating new blind spots in another domain. We do not see what we do not see, and what we do not see does not exist. Only when some interaction dislodges us—such as being suddenly relocated to a different cultural environment—and we reflect upon it, do we bring forth new constellations of relation that we explain by saying that we were not aware of, or that we took them for granted (1987, p. 242).

It is only when we are confronted with, say, Asian methods of written communication that we even question the need for a ‘reason’ for the way we read and write. A more
elaborate description of this ‘blinding effect’ can be found in the works of Gregory Bateson (1972) and Chet Bowers (1997a; 1997b) who have written extensively on this idea but use the metaphor, cultural map. The metaphor is used to explain how a group’s cultural forms of intelligence such as its beliefs, values, and its experiences that are based on continual responses to the interactive patterns of the group’s environment emerge as conceptual ‘blueprints’ or ‘maps’ (Bateson, 1972; Bowers, 1997b). A cultural map is defined as being the spoken, written, spatial, temporal, and symbolic language of a group which mediates and influences what a particular social unity will and will not recognize as important. For instance, in Western society, our cultural map influences us to recognize technological advancements as being important for human progress, whereas when looking at the Western Apache’s conceptual map, it is the belief that their cultural identity must be rooted in their regional landscape which is instilled (Basso, 1996). Cultural maps are the ways in which social systems are able to establish and reestablish values, beliefs, roles, and responsibilities among their members. Thus, these maps serve as cultures’ metaphorical ways of being, interacting, and bringing forth of a world.

The Emergence of an Enactivist View

It is from delving deeper into the areas of ecology, biology, phenomenology, and philosophy that the emergence of an enactivist view of cognition is possible and can be conceptualized:

....as embodied action within the context of evolution as natural [and cultural] drift provides a view of cognitive capacities as inextricably linked to the histories that are lived, much like paths that exist only as they are laid down in walking. Consequently, cognition is no longer seen as problem solving on the basis of representations; instead, cognition in its most encompassing sense consists in the enactment or bringing forth of a world by a viable history of structural coupling (Varela, Thompson, & Rosch, 1996, p. 205).
Furthermore, enactivism surfaces as an ontological framework which enables a more complex understanding of what it means ‘to know’. In order to address the questions which I posed earlier regarding constructivism’s place within a larger ontological framework and the impact that enactivism will have in the field of mathematics education as an alternative perspective, it is necessary to move the two viewpoints into the area of political-educational ideologies. This allows me to interrogate constructivism and enactivism under a common lens and make sense of the theoretical ‘clutter’ that once again surrounds me.
Locating Constructivism and Enactivism Within Political-Educational Ideologies

As the reader, you may wonder what purpose(s) political-educational ideologies serve in terms of developing an understanding of enactivism, constructivism, or mathematics education. However up until now, enactivism has been described as a philosophical orientation which embraces the complex interdependence of living and social systems, and to some extent it could be interpreted as a framework that implicates constructivism by extending and compensating for its philosophical limitations. But when constructivism and enactivism are located within political-educational frameworks, it becomes clear to me that the two perspectives cannot be situated in similar realms and therefore, cannot move in compatible directions. So it is through the conceptualization of how political ideologies serve as the metaphorical roots which underpin educational movements that I am able to appreciate how enactivism is distinct from constructivism. I elaborate on this distinction below.

The Influence of Liberal Ideologies on Education

Bowers (1995; 1997a; 1997b) asserts that on a superficial level, current educational movements appear to be dissimilar from one another, as seen in the case of critical theory and constructivism which are recognized as two distinct epistemologies. But when one examines the various educational positions at a deeper level, common roots of liberal ideologies are revealed. The author identifies three forms of liberalism: emancipatory, technocratic, and neo-romantic. Bowers describes emancipatory liberals as being individuals like John Dewey and Paulo Freire; those persons or groups who understand critical reflection of the individual to be the most important in bringing about human progress. Peter Berger, Brigitte Berger, and

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5 Ideology is a framework that guides how basic aspects of human experience are to be understood and acted out (Bowers, 1997b).
Hansfried Kneller are classified by Bowers as being technocratic liberals. He characterizes this group as being those who commoditize knowledge and emphasize such things as *efficiency, measurable outcomes, and products* of effective behavior *management*. Furthermore, Jean Jacques Rousseau and Carl Rodgers are defined by Bowers as being neo-romantic liberals. It is this group that constructivists are a part of because, neo-romantic liberals are those who believe that individuals create their own ideas, values, and understanding of relationships through direct experiences.

While Bowers (1997b) acknowledges that these forms of liberalism differ in the aspects of modernity which they amplify, he stresses that all three forms of liberalism hold four common assumptions which serve as the core conceptual and moral underpinnings present in Western society’s modern culture. These views which the three types of liberalism share are as follows:

i) equating change such as ideas, values, and technologies, as being expressions of social progress,  
ii) viewing the environment from an anthropocentric perspective; that is, the environment as being an external entity which is not connected to humans,  
iii) representing the individual as the basic social unit responsible for establishing authority of ideas and values, and  
iv) promoting the overturning of tradition in all areas of cultural life, based on the belief that ‘new’ implies the notion of being ‘better’.

The author perceives the first and fourth attitudes as resulting from these groups’:  
...view of time as a linear unfolding and [their focusing on] improvement in human possibilities makes it difficult to acknowledge that most of what is viewed as today’s innovations involve building upon what has come down to us over time” (1997b, p. 117).

Bower’s examination of liberalism as a cultural map establishes liberalism as an ideology that focuses on maintaining forms of intelligence which view life processes in
mechanistic terms such as *data-based* decision making, or *checks and balances*. This type of thinking he contends, gives rise to anthropocentric ways of understanding human relationships with nature by disconnecting the environment from us and commoditizing it as being natural *resources*, claiming the world as our world, and viewing science as the most powerful and legitimate source of knowledge. Moreover, Bowers insists that because these beliefs evolved long ago during the Industrial Revolution and continue to be the guiding metaphors which underpin current trends in education, they perpetuate ways of being which are environmentally and culturally destructive. Consequently, this cultural map puts out of focus any need for us to recognize or understand the ways in which traditions and moral structures result from the reciprocal relationships that exist between cultural practises and the natural world; that is, how our social relationships play critical roles in shaping our (cultural) knowledge.

For me, the underlying assumptions that Bowers identifies as belonging to liberalism are inherent in constructivism and are interestingly the same issues which were previously discussed as being critically problematic with constructivism--the reasons that led to the search for enactivist theory! Therefore if enactivism was to be connected to constructivism, this would mean situating enactivism within the realm of liberalism. In light of what has been discussed, I do not believe this to be a possibility because doing so would compromise everything that constitutes enactivism. In the next two sections I will define the ideology of cultural/bio-conservatism, explain why I feel it is necessary for enactivism to find a place within this framework, and what this might mean in terms of the responsibilities of mathematics educators if such a position is taken.

**From Clutter to Cohesion and Complexity: Cultural/Bio-Conservatism**

Bowers (1995; 1997b; 1997b) explains that what separates cultural/bio-conservatism from liberal ideologies is that it does not share the same assumptions
that are inherent in forms of liberalism. Firstly, cultural/bio-conservatism as an ideology emphasizes cultural drift as being part of all social systems and necessary in understanding the continuities that exist between the past, present, and future structures of knowledge and identities. He (1997b) interprets the concept of cultural drift as *tradition*, and defines tradition as being the ways in which cultural patterns are handed down from the past and reproduced in current contexts. It is through the processes of revision and renewal that cultural patterns in turn affect future relationships within the social system and the larger environment. Secondly, rather than only recognizing the authority of the individual, a cultural/bio-conservative also acknowledges the authority of collective communities in establishing ideas, values, and actions. These two tenets emerge from the underlying assumption of individuals as being nested in cultural systems, and cultural systems being nested in ecosystems. In brief, the following excerpt from his book, *The Culture of Denial: Why the Environmental Movement Needs a Strategy for Reforming Universities and Public Schools* provides an explanation of the guiding principles and the position that is taken on by cultural/bio-conservatives:

...to maintain a balance between critical reflection, and an understanding of the complexity of our embeddedness in traditions on the one hand, and the need, on the other hand, to assess and renew traditions on the twin bases of whether they contribute to an equitable and just community and have a minimal impact on the environment (Bowers, 1997b, p. 46).

Bowers asserts that this ideology puts forth the understanding that as individuals and members of Western society we are but one part of the world, a world similarly described by Capra (1996) as being a phenomenological network. Given this view, one living or cultural system is not privileged over another, but all are understood as being interconnected and therefore, interdependent. Bowers cautions the reader that before any critical reflection or assessment of cultural traditions are made, such
evaluations need to be conducted from a position that is respectful towards the larger environment. This implies the importance for cultural ideas, values, and actions to be those which promote the sustainability of, and minimize negative impact on other living and cultural systems in the world.

Distinguishing Enactivism from Constructivism: Situating Enactivism in Cultural/Bio-Conservatism

For me, the examination of constructivism and the exploration of its problematic issues in broader and deeper contexts enables enactivism to surface as a more encompassing, embodied view, which informs educators' perceptions about learning. The idea of learning embodying co-emergent processes illuminates ways of knowing, doing, and being as interrelationships that exist between the teacher, students, and the environment(s). In the field of mathematics education, enactivism supports the constructivist view that recognizes children's effective mathematical learning as taking place within a proscriptive environment. But where constructivism sees the child as being separate from the world, enactivism places the child with the environment as an autopoietic system and a member of increasingly larger systems such as groups, classes, schools, communities, and societies. Secondly, interaction is considered to be a process of co-evolution of knowings by which mathematical knowings shape individual and group ways of knowing, doing, and being. And thirdly, children's mathematical growth is understood as beginning and evolving from their embodied mathematical actions. This interpretation of enactivism is elucidated later in this thesis. From an enactivist perspective, cooperative, creative, co-evolutionary, diverse, and reciprocal become words which describe the nature of children's mathematical learning. I feel these qualities are consistent with principles of cultural/bio-conservatism, and define an ecologically sound environment in which children's mathematical understandings can grow. The insight that Bowers demonstrates in rectifying political-educational vocabulary in order to investigate how these ideologies
influence educational movements not only situates constructivism within a larger conceptual framework of liberalism, but at the same time, creates a deeper space for enactivism to be distinguished as a philosophy and a form of cultural/bio-conservatism that will bring forth ideals that divert from liberalism and embrace those which are ecocentric.

As enactivism emerges on the fringe of the field of mathematics education, I believe the task of situating enactivism within the realm of cultural/bio-conservatism now becomes a priority of great significance for educators taking on this perspective. For if the philosophy's roots of cultural/bio-conservatism are not revealed, and the ideological differences between enactivism and other forms of liberalism are not communicated, I predict that the clutter and fragmentation which now surrounds constructivism will be a similar consequence for enactivism and its educators. Establishing an enactivist position in mathematics education then not only involves the emphasis of mathematical settings that value the complex circularity which exists in children's active and formulated ways of knowing, but also requires educators to assume the responsibility for the critical assessment of what mathematical concepts, procedures, language, attitudes, and actions will be passed on as cultural traditions, based on the guiding principles of cultural/bio-conservatism (Bowers, 1990; 1995; 1997a; 1997b). Accomplishing these tasks requires the efforts of all educators adopting an enactivist perspective--professors, reformers, and classroom teachers. If we consider the impact that cultural-mathematical maps have on children in terms of the meanings that they will develop from these maps and the ways in which children will enact these understandings in their daily lives--as members of society, the issue of establishing enactivism in mathematics education becomes one of urgency and importance. It is only through the critical and conscientious efforts made by educators that enactivism can maintain its integrity, its momentum, and its potential to bring about positive reforms in the field of mathematics education.
Enactivism and Mathematical Knowing

To apply an enactivist view which is situated in the framework of cultural/bio-conservatism to mathematical knowing means that knowing does not exist merely as an inner mental state--as mental representations and information, but really begins with our perceptions which involve our subjectivities, bodily sensations, (en)actions, and emotions. Furthermore, because our perceptions are specified by cultural maps, they become embedded as embodied and co-evolving actions with the environment. Thus, "meaning does not reside in the piece of information, but rather in the context in which it has been placed" (Capra, 1996, p. 272). The development of chaos and complexity theories, as well as fuzzy logic make it possible for us to understand mathematical knowledge as not predetermined and static, but rather, that it too evolves from our actions and interactions with the world (A. B. Davis, 1996; Capra, 1996).

If we reposition our view of cognition from the inactive or mechanistic to the enactive or embodied, we then cannot conceive school mathematics to be value-free, inert, or an already-invented-only-to-be-discovered body of knowledge. Instead, mathematical knowings and the enactment of them co-emerge and are continually shaped through students' and teachers' individual and collective ways of being with the (educational) environment(s). Regardless of whether or not the mathematics is formalized, children's meaningful and mindful learning now becomes a co-evolution of understandings which are inextricably embedded in the contextual depths of their experiences.

The Collective and Co-Emergent Nature of Mathematical Knowledge

An intriguing aspect to mathematics that seems to distinguish it from the arts...is the extent to which mathematicians...collaborate in their work.... [In] mathematics the collaborative process goes much deeper to entwine the
authors in a process...by which they are able to produce a result that could not have been half-reached by one of them (Barrows in A. B. Davis, 1996, p. 75).

Collective mathematical knowings cannot be considered a thing that emerges from social interactions of re-negotiation and consensus between participants (A. B. Davis, 1995, 1996; A. B. Davis & Sumara, 1997), nor can they exist as a vehicle for developing personal mathematical understandings. Shared mathematical knowings are rather, co-emergent phenomena (A. B. Davis & Sumara, 1997) which are brought into being by the socio-mathematical inter-activity of collective unities. The creation of such mathematical ways of knowing, doing, and being serve to coordinate group actions and enable collective unities to function. Mathematical knowings co-exist as part of every learner, and establish groups of learners and educators as social systems.

If we believe mathematical knowings evolve not from information but from lived experiences, then children's mathematical growth must be understood from the perspective that views their inner and physical reintegration of mathematical meanings existing as complex interrelationships of structural and reciprocal coupling which are located in the spaces of mathematical inter-activity (A. B. Davis, 1996; Sfard, 1994). Moving with this, the only way educators have of assessing such mathematical knowings is through the inquiry and observations of learners' mathematical ways of doing and being as unified, embodied, and co-emerging phenomena (A. B. Davis, 1996; A. B. Davis & Sumara, 1997; Kieren, 1990; Pirie & Kieren, 1994a, 1994b; Sfard, 1994). I would now like to present the Pirie-Kieren model of dynamical theory regarding the growth of mathematical understanding. I shall connect it with the work of Anna Sfard in order to explain how I have arrived at conceptualizing spaces of mathematical inter-activity as being the sites where educators can delve more deeply into understanding students' mathematical ways of knowing, doing, and being.
Understanding Mathematical Growth: The Pirie-Kieren Model

The Pirie-Kieren model of mathematical growth (1989) (see Figure 1) not only reflects an enactivist perspective, it brings enactivism into the field of mathematics education and provides a lens which focuses specifically on the complex and unpredictable nature regarding mathematical knowings. Mathematical understandings are viewed as occurring through interrelated and fluid processes which reinforce enactivism's view of cognition as being fractal-like in nature (Kieren, 1990; Pirie & Kieren, 1989; Pirie & Kieren, 1994b) (see Figure 2). And because of this, the authors describe their model's structure as being neither hierarchical nor linear. Rather, the realms of mathematical knowings in this model exist as embedded, unbounded circles and are self-similar and compatible to one another. The model is consistent with Maturana and Varela's (1987) axiom of “all doing is knowing, and all knowing is doing” (p. 26) because it locates primitive doing as being the experiential roots for all other mathematical knowings.

Susan Pirie and Thomas Kieren's cognitive mappings, of individuals and groups of students, identify mathematical knowings as being simultaneously individual and collective, dynamic, occurring on many levels at once, and revealing qualities of transcendence and recursiveness. The authors believe learners to be autopoietic beings who determine what will be experienced as perturbations and specify the ways in which they structure their mathematical thinking. Pirie and Kieren also stress that mathematical abstractions cannot be developed through the acts of outside agents imposing these on the learner(s), but will only be achieved when structural changes within, between, and among learners occur. It is this part of their argument which emphasizes this notion of mathematical inter-activity as being critically important for mathematical learning to grow. Furthermore, Pirie and Kieren define mathematical existence as the embodiment of all verbal, physical, and written acts:
Figure 1. Model of a dynamical theory of the growth of mathematical understanding (Pirie & Kieren, 1994a).

Figure 2. Model illustrating primitive knowing as the source of all other mathematical knowledge (Kieren, 1990).
Mathematical understanding can be characterized as levelled but non-linear. It is a recursive phenomenon and recursion is seen to occur when thinking moves between levels of sophistication. Indeed each level of understanding is contained within succeeding levels. Any particular level is dependent on the forms and processes within and, further is constrained by those without (Pirie & Kieren, 1989, p. 8).

The authors make it clear that their model is not intended to be used to define or prescribe a particular sequence of static levels which constitute students’ mathematical learning but rather, a way of conceptualizing the learning of mathematics as unpredictable and complex phenomena. As well, Pirie and Kieren (1994b) do not distinguish mathematical growth as being monological pathways, or privilege one's fluency to use formal language and mathematical symbols as representing formal mathematical understandings.

In keeping with the latter statement, a study conducted by Anna Sfard (1994) examined how expert mathematicians come to know when they truly understand a mathematical concept. Interestingly, her subjects revealed that it was neither a feeling of proficiency nor the production of proofs that brought about their sense of true mathematical knowing, but rather sensations which were personal, intimate, and transformative. The mathematicians described true mathematical knowing as going beyond the skill of operating the mathematics to inhabiting the mathematics, becoming intimate, and understanding the mathematics:

...like a person whom you really know and understand, (the mathematical construct) will perform certain operations (behaviour) or will react in a certain way to your action. This intimacy is exactly what I had in mind: you know what is to happen without making any formal steps. Of course, as in the case of human relationships, you may sometimes be wrong! (p. 49).
Pirie and Kieren substantiate Sfard's findings by viewing the process of mathematical knowing as involving the need for the learner or collective unity to fold back to previous levels. Folding back is not a redoing of what has already been done, but moves the learner or group of learners back to inner levels of mathematical knowings where they will reintegrate understandings as a result of the perturbations experienced in previous outer levels before moving on. Thus, this model also reflects the notion of mathematical knowings existing simultaneously as a product, producer, and process, (A.B. Davis, 1996; Maturana & Varela, 1987). The authors believe that mathematical understandings are enriched by the act of folding back⁶ and it is this process which gives rise to the co-emergence of self-referencing, remembering, and reintegration of mathematical knowings through structural and reciprocal changes. It is in these ways that the Pirie-Kieren model provides me with a cohesive yet complex way to engage in my inquiry regarding the spaces of individual and collective mathematical inter-activity where children's mathematical growth occurs.

⁶That is, the learner or group of learners return to previous realms of mathematical knowings within the Pirie-Kieren model.
Spaces for Mathematical Possibilities

In light of everything that I have discussed, adopting an enactivist perspective in the field of mathematics education means that students are regarded as being autopoietic, and they, as well as their teachers, are co-dependent, co-determining agents (A. B. Davis, 1995; 1996) in the classroom. Students need to be allowed and enabled to develop their mathematical ideas in settings that encourage students’ useful action (Bateson, 1972; Bowers 1990, 1995, 1997a, 1997b; Varela, Thompson & Rosch, 1996). In mathematical environments, the teacher is provocatively and invocatively, but certainly not the source of all perturbations. Children as autopoietic beings and members of social systems determine whether or not they perceive interactions as perturbations (Kieren & Pirie, 1992), and therefore it is the learner or group of learners who will ultimately determine what will be learnt (Steffe & Tzur, 1994).

Mathematical learning as continual growth through lived experiences implies children’s constant interaction within themselves and with the environment. Enabling students to question, wonder, and create their mathematical knowings, rather than merely producing correct answers, will open new spaces for them to be mathematical—to participate as individuals and collective unities, letting them dwell, reflect, and venture in their learning of mathematics. But before such a feat can be attempted, possible settings for such mathematical learning, and the roles of educators and learners take on must be considered.

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7 Provocative teaching is described by Pirie and Kieren (1994b) as the teacher’s active intention “...of moving the student[s] outwards” (p. 188). That is, moving the learner(s) from one (nonlinear) level of knowing to different realms of knowing within the Pirie-Kieren model of mathematical understanding to promote mathematical growth.

8 Invocative teaching on the other hand, is described by Pirie and Kieren (1994b) as being the teacher’s conscious act of engaging the student(s) “...to fold back in order to enlarge or alter his [their] image” (p. 188), with respect to the authors’ model.
The Mathematical Environment

Establishing spaces of inter-activity where students can learn mathematics through individual and collective problem solving involves much more than exposing them to nonroutine mathematical problems⁹. Edward Silver (1994) and Robert Davis (1992) support the NCTM’s (National Council of Teachers of Mathematics) position that “the very essence of studying mathematics is itself an exercise in exploring, conjecturing, examining, and testing” (Silver, 1994, p. 19). The development of children’s understanding of what it means to be mathematical can only occur when learners are able to experience, act, and realize their abilities in solving mathematical problems through perturbations and then couple their understandings before, during, or after solving problems in many different situations (Brown & Walter, 1983; Gonzales, 1994; NCTM 1990; Silver, 1994; Walter & Brown, 1993). Viewing students as problem solvers means that educators’ attention must be given to the active participation of individuals and collective groups in solving mathematical problems, as well as the development and use of their metacognitive skills, self-questioning and self-regulatory techniques (R. B. Davis, 1992; Silver, 1994). A problem solving environment then is a prescriptive setting which can, if properly managed, encourage students’ use of invented methods for solving problems, their mathematical verbalizations and embodied (en)actions, as well as the theories or principles that are generated from such inquiries.

Children as Mathematical Problem Solvers

The development and effective use of students’ self-questioning (e.g., “What if...?” and “What if not...?”), self-regulatory techniques, and metacognitive skills are crucial to their success as problem solvers. James Hiebert (1989), David Pimm (1987), and Richard Lesh (1981) define successful problem solvers as individuals who are able to accurately assess situations as being problematic, ⁹ I define ‘nonroutine mathematical problems’ later in this thesis.
perceive opportunities to apply their problem solving capabilities, and devise effective strategies to resolve such dilemmas. As well, it is important for the individual to demonstrate perseverance (Taplin, 1995) and exercise control (Carraher, Carraher, & Schliemann, 1987; Hart, 1989; Hiebert, 1989; Schoenfeld, 1985; Taplin, 1995). In the context of mathematical problem solving, perseverance refers to student’s intuitive, experiential sense in knowing when to continue with and not give up too soon on a chosen strategy or action, but also, knowing when to abandon a particular strategy or action and search for a more effective or useful one. The second managerial strategy, control, refers to the way in which a student selects goals and subgoals, monitors, revises, and assesses their progress of a problem solving activity. Control also includes how a student makes use of and sense of given or found information in attempts to solve a problem.

The Teacher’s Role

By adopting the guiding principles of cultural/bio-conservatism and integrating these with the philosophy of enactivism, educators in mathematics education can now begin to define and direct the philosophy’s influence in terms of understanding and inquiring into the moral dimensions regarding the participation of teachers and students in the learning of mathematics. This implies the critical assessment and possible strategies for the re-shaping of mathematics education, its teaching practises, curricular content, and students’ learning environments. Ultimately, I believe it is these efforts made by educators with an enactivist perspective that will allow for children’s mindful and meaningful learning of mathematics.

Moving into the context of the mathematics classroom, the role of the teacher with an enactivist perspective cannot be only to dispense novel problems for the students to solve, but needs to also open up possible situations where students can

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10 As these issues regarding problem solving have only been investigated by the above mentioned authors within constructivist contexts, they represent the individual and not collective unities. This problem can be resolved if, when authors have referred to problem solvers as “the individual(s)” or “the student”, the reader alters these references by reading them as “the individual or collective group...."
mathematically move and act as individual and collective problem solving unities. Moreover, mathematical inquiries that promote children's use of effective actions which are satisfactory and not necessarily optimal means that the ways in which students investigate mathematical problems are open to their use of strategy, experimentation, and generalization (Gonzales, 1994; Lesh, 1981; Worth, 1992) but also to their interpretation and embodiment of mathematics itself.
Another Beginning

In order to take an enactivist position and bring about positive changes in mathematics education, we as educators must reposition ourselves in understanding and communicating children's mathematical ways of knowing, doing, and being as the unfolding of social-cultural phenomena which are rooted in their lived experiences. If "all doing is knowing, and all knowing is doing" (Maturana and Varela, 1987, p. 26), then we cannot assume that children will enact mathematical understandings in ways that are necessarily conscious, fixed, formulated, or removed from who they are or the contexts from which these actions co-evolve. The Pirie-Kieren model furthers our understanding that mathematical growth occurs through the complex interplay of experiences, understandings, and meanings which co-emerge and are rarely if ever a progression from the concrete to the abstract, but rather manifest into the dynamic and constantly changing structures of the learners. Because of this, we need to probe deeper into investigating the co-emergence of mathematical knowings which takes place in the inter-activity of individual and collective unities--the instances where sophisticated mathematical understandings are enacted.

Our role as teachers becomes both a phenomenological and hermeneutic endeavour in seeking to understand the meanings that embody students' mathematical experiences (A. B. Davis, 1996; van Manen, 1990). And to do this, learning spaces need to be created where children can develop their effective problem solving actions as individuals and collective groups. Only then, can mathematics be appreciated for its wholeness and for its qualities that cannot be found through the dissection and decontextualization of its parts. Learning settings which open up possibilities for problem solving allow children to inhabit mathematics in an intimate and ecologically sensitive way.
Creating a Space for Mathematical Possibilities

My Background and Area of Research

As well as being a full-time teacher of second and third grade students, I am also a graduate student in mathematics education at the University of British Columbia. I completed my undergraduate training at the University of Victoria with a concentration in mathematics. I then entered the teaching field as a primary constructivist educator committed to immersing my students in a mathematical environment that not only engaged them actively through hands-on and minds-on investigations, but also provided them with learning settings that emphasized their growth as mathematical problem solvers. Having reflected on these classroom experiences, I still find myself to be intrigued by the unpredictable revelations and queries that students make, but more specifically, the dynamic ways in which their mathematical understandings take shape as they interact as members of problem solving groups. It is this curiosity of mine that has become my current area of research.

The Emergence of My Research Question

Hoping to gain a deeper understanding and a way to interpret my students' learning of their mathematics, I continued to read and study various literature regarding constructivism and its application to mathematics education. As a result of my examination of constructivist perspectives and how they inform the field of mathematics education, I now realize that these views cannot provide adequate means for my understanding of the fluid and collective movement which occurs when students engage in mathematical conversations, or, the unexpected emergence of their exploration of mathematical concepts and theories that occur when they are working collaboratively to solve mathematical problems.

Rather, it is the philosophical lens of enactivism which provides me with a
coherent framework where complex issues regarding the development of collective group mathematical understandings, and children's individual and collective mathematical knowings can be explored, interpreted, and more deeply understood. It is enactivism's embracement of biological, ecological, phenomenological, and philosophical perspectives regarding cognition that enables this paradigm to present an alternative view of knowledge. That is, knowledge is not a separate thing which connects the subjective to the objective, the knower to the world, the individual to the collective, or the mental to the physical, but is that which renders all of these inseparable, co-existing and co-developing simultaneously. Furthermore, by adopting an enactivist view, verbal, written, and physical actions cannot be assumed to be consequences of constructed knowings but are considered as being observable qualities of embodied and co-evolving knowings. It is in these ways that an individual participates in creating mathematical understandings not only as an autopoietic unity but also as a member of collective unities of increasingly larger social systems such as a small group, class, school, community, society and so on. Thus, it is the spaces of inter-activity where the co-emergence of mathematical knowings can be found. The efforts made by mathematics educators such as Brent Davis, Susan Pirie, and Thomas Kieren not only identify the inherent complexity involved in children's learning of school mathematics, but also open up spaces for further conversations and investigations to be conducted regarding our understanding of students' mathematical knowings and mathematical understandings as being highly dynamic and co-emergent phenomena.

So, my story continues. Beginning from a place where I perceived an accumulation of clutter, to the unfolding of my experiences as a mathematics teacher and graduate student, I have reemerged into a new space with the resonating sense of cohesion and complexity. It is from this place that I began asking the following question:
How do children’s individual and collective mathematical knowings co-emerge to create a path of group understandings in a problem solving environment?

Definitions of Terms and Tacit Assumptions of my Research

To explain the meanings that were implied by posing this question and to clarify my purposes for such an investigation, I will once again make use of Varela, Thompson, and Rosch’s (1996) view of cognition as being “much like paths that exist only as they are laid down in walking” (p. 205) to guide this discussion. Similarly, I believe mathematical understandings to be paths which are created dynamically and unpredictably through the inter-activity of children’s individual and collective knowings. It is a path of mathematical group understandings, brought forth by students during their problem solving of mathematical challenges that I wished to investigate. More specifically, I wanted to inquire into the ways in which children work together as an autopoietic group, and how their personal and co-emergent mathematical knowings and understandings become the twists and turns which shape such a path.

In regards to the term--knowings, I applied an enactivist metaphor which assumes knowledge to be a network in which there exist no foundations. That is, knowledge is made up of perceptions--relationships and descriptions which are formed when concepts and models are interconnected. This network also includes the material universe which is comprised of interrelated events (Capra, 1996). Furthermore, it was because of this notion of knowledge as being constantly created, integrated, and recreated that I chose to use the term, ‘knowings’ instead of ‘knowledge’ to further distinguish mathematical knowings as being fluid and not disconnected or static ‘objects’. My conception of individual knowings was taken to mean any enactions of mathematical concepts or skills that are brought to a group’s problem solving activities by an individual and are not perceived by the collective unity.

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as being a part of their current network. As well, collective knowings were understood by me as being any enactions of mathematical concepts or skills that evolve during group interactions or are brought to the group by a student or students and become part of the group's present and/or future networks. Therefore, any new collective mathematical enactions which emerge while a group is problem solving can be traced back to previous instances regarding the children's workings. Finally, my use of the words problem solving environment implied a learning setting that would be consistent with what has been previously described in earlier sections as a proscriptive mathematical environment.

It is important for me to make it explicit to the reader, that this research question served as a general, foreshadowed area for my inquiry to take place and the definitions regarding its terms were generated from the literature and my personal reflections prior to conducting my research. Therefore, it was my intention to keep the question 'open ended' so that as my research progressed, specific ideas and questions could emerge and be addressed. This enabled me to develop meanings and significance for the question based on the events which unfolded during this case study.

**Significance of my Research**

My engagement in this type of exploratory research aimed to describe, analyze and provide interpretations that would inform and encourage further research on the part of mathematics educators into such areas as:

i) the mathematical paths of understandings which are created by collective unities,

ii) the development of a deeper understanding for the fluid movement of mathematical understandings; as they are brought forth and grow into
mathematical knowings when students work together to solve mathematical problems, and

iii) the elaboration of the interrelationships which exist and give rise to individual and collective mathematical identities; that is, children's mathematical ways of knowing, doing, and being.
The Research Project

The following is a brief overview regarding the ways in which I began to organize the research project, the methods I devised to collect the data, and the general techniques I anticipated using for the analysis prior to conducting my inquiry. As well, it is during this discussion here and in following sections of this thesis that I describe how it became necessary for my research methods to evolve with the project as it proceeded. It is these explanations which serve to further illuminate the study as being emergent in design because my actions as a researcher could not be entirely preplanned and, thus, needed to co-emerge with the events that were unfolding during the investigation.

A key consideration for me in terms of this research project was to create a proscriptive mathematical environment which would be in keeping with enactivist views—a setting that would allow me to examine in depth, the dynamic and unpredictable qualities of group mathematical understandings through the development of children's individual and collective knowings. In order to do this, I proceeded to design a qualitative case study\textsuperscript{12} which involved three fifth grade students. While being videotaped, these children would be required to work as a collective unity and problem solve their way through six mathematical problems\textsuperscript{13} without adult assistance.

Teacher-Researcher Intervention

As my aim was to study the group’s path of mathematical understandings and how it was being shaped by the children’s individual and collective mathematical knowings, I realized from the beginning stages of this project, that I could not be

\textsuperscript{12} Case study design is defined as the particular situation(s) where phenomena will be described in depth and analysis of the case study’s events will be specific to the context of the given study. In this study, the events will be three students problem solving six mathematical problems as a cooperative group without adult assistance.

\textsuperscript{13} These “mathematical problems” are also referred to as mathematical “challenges”, “activities”, or “(group) projects” in this thesis.
physically present or available to help them during their problem solving of the group projects. However, because I planned for six mathematical problems to be given as written activity sheets, I felt it was necessary for me to be present in terms of posing each of the six mathematical challenges to the group and to facilitate their reading and comprehension of the six activity sheets. My reason for participating in this manner was to ensure that the mathematical problems and their conditions were as clearly understood as possible by the group. By doing so, I allowed the students to concentrate on the mathematical tasks at hand and lessened the demands on them in terms of having to reach group consensus entirely by their own devices.

In this context, I defined reading as being the students’ ability to identify and verbalize the written text on the task sheet, and comprehension as being the group’s ability to agree on and verbalize in their own words, the meaning(s) of what they had read (Pimm, 1985). Although my physical presence and involvement in terms of participation with the group was kept to a minimum, I acknowledged that my role as a facilitator would have an effect on the group’s comprehension of the six mathematical activities. For example, through my participation in helping the students’ reading and comprehension of the mathematical problems, I became involved to a certain degree, in the shaping of their taken as shared meanings (Cobb, Yackel, & Wood, 1992) regarding the written text of the activity sheets. This however, proved to be useful in providing me with a ‘taken as known’ starting point for conceptualizing the children’s growth of mathematical understandings.

Upon the conclusion for each of the six activities’ introductions, I reminded the students that they were not to seek assistance from anyone outside of their group, and gave them a time limit to work on the group project. I then left the room and sat nearby but outside of the conference room so that the group could work on the projects by themselves.
The Participants

Three fifth grade students were the participants in this case study. At the time of this study, student A was a male of ten years, student R was a male of nine years, and student V was a female of nine years. I made the decision to limit the group size to three children in order to allow for maximum participation and interaction among the members of the group during their problem solving of the six mathematical activities.

I feel that it is important at this point for me to explain that my study did not include any interviews with the children or formal viewings of the videotaped sessions\(^\text{14}\) with the group in regards to the events that occurred during the six problem solving challenges. I made this decision based on the fact that because the main objective of my research was to examine the children's mathematical path as it was unfolding, conducting interviews with the children or watching the videotaped sessions with the group would have at best, only served as records of the students' reflections or recollections of past episodes regarding the problem solving sessions (Pirie, 1996). Therefore, I did not see the use of interviews or the group's viewing of the videos as enhancing my analysis or interpretation of the events in this particular case study. Moreover, I believed the use of such interviews and video-viewing-and-response procedures would have considerably altered the intentions and diminished the philosophical integrity of this research project.

So, in order for me to describe and generate interpretations that addressed my original research question, I could not conduct interviews or ask the participants to corroborate my findings by watching the videos. This meant that the degree of rigour that I was able to exercise in analyzing and interpreting the video data placed a heavy reliance on the group's responses to the six mathematical problems, their ability to communicate by 'thinking aloud', and cooperate as a collective unity during the

\(^{14}\text{In this thesis, a "session" is being used as a way to group the instance(s) when the three students met to solve a particular pyramid problem. There are six sessions in total--one for each of the six pyramid problems. Therefore, one or more meetings during which the children came together to solve a particular pyramid problem may constitute one session.}
sessions. In addition to this, my study demanded the absence of any adults and because the primary video data consisted mainly of unsupervised sessions, I deemed it necessary to select three students who were able to work within these constraints.

The three children selected to participate in the study had just entered the fifth grade and at one time had also been former students of mine. Students A and V were taught by me during their second and third grades of schooling, and student R joined V and A's class during the last four months of grade three. One reason why I chose these three particular students is that V, R, and A had also participated in a previous case study in which they worked as a cooperative group on a mathematical problem (Thom & Pirie, in progress) and were therefore familiar with each other and with working together in this type of mathematical setting. A second reason why I selected these students to be the participants was because of their verbal and expressive abilities; that is, all three children spoke English as their first language, and therefore analyzing native speakers facilitated my analysis of their speech. In addition to these considerations, students A, R, and V were considered by their last year's teachers to have met the provincially set expectations for the fourth grade curriculum standards in mathematics (Ministry of Education, Province of British Columbia, 1995) and I was not, in this particular study, interested in examining children with special needs.

**School Setting**

The public elementary school where the study took place is located on the Lower Mainland in the province of British Columbia. The school's population at the time was approximately 400 students and their principal described the socioeconomic status of the school community as being comprised of lower-middle to middle class families. Student A and student R were currently registered at this school. Student V who was a former student at this school, was attending classes at a nearby school.
Specific Research Setting

All of the problem solving sessions were videotaped and commenced shortly after the students had finished their school day at 3:00 PM. In a small conference room located within the school I set up a large table and three chairs to be the group's work area and set out a variety of materials which were left on a nearby table for the children's use. These materials were as follows:

- a clock for the students to monitor their working time
- a large pad of chart paper
- containers of multi-link cubes\(^{15}\)
- one felt pen
- a set of 20 cm x 27.5 cm plain paper
- a set of 20 cm x 27.5 cm coloured dot paper\(^{16}\)
- a calculator
- a pair of scissors
- a 31 cm ruler

I showed these materials to the students but explained that it was entirely up to the group as to whether or not and how they chose to use them during the problem solving sessions.

In the corner of the room there was an auto-focusing video camera set up on a tripod stand to record the six problem solving sessions. The camera focused on the students, the table where they did their work, and the immediate area around them --just in case they chose to record items on the chart paper which was mounted to the wall. As these students had been videotaped in the sessions of the previous study and appeared comfortable enough to laugh and engage in conversations with each other, I

\(^{15}\) A multi-link cube is a 2 cm x 2 cm x 2 cm plastic cube that can be affixed to other cubes on any one of its six faces.

\(^{16}\) The dot paper used in this study were sheets of paper which had a series of single dots or 'points' that were approximately 1 cm apart from one another. This paper is commonly used in mathematical activities for drawing 2-D or 3-D diagrams and makes it easier for one to draw lines or images with accurate proportions by simply 'connecting the dots' together.
did not anticipate that they would be greatly bothered by being videotaped in this study. In any event, I gave the children time before the commencing of the sessions to ‘explore’ the conference room as a precautionary measure to prevent distractions during the videotaping of these sessions.

**Collecting the Data: Rationale for Videotaping the Problem Solving Sessions**

The primary data generated from this case study consisted of a series of videotapes which recorded the group’s work during the six problem solving sessions. My first reason for videotaping these sessions was to eliminate the need for me to be physically present while the students were working. Secondly, the videotapes allowed me the opportunity to view and re-view my data in order to prepare the subsequent mathematical problems--an issue which I address later in this discussion. Thirdly, by videotaping the sessions, I had permanent records of the children’s *embodiment of their mathematics*--their verbal, written, and physical mathematical actions and interactions of their problem solving activities. And fourthly, by viewing the videos as many times as I felt necessary and examining them in different ways I was able to develop ‘depth’ in terms of my descriptions, analyses, and interpretations of these sessions (Pirie, 1996). Other forms of data that were collected and integrated into my analysis were the written records and 3-D cube models produced by the students during the six sessions. These artifacts were photocopied or digitally photographed and accompany my analyses and interpretations in this thesis.

**An Overview of the Case Study Events: The Six Nonroutine Mathematical Problems**

The events of the case study focused on the group’s problem solving of six *nonroutine* mathematical problems about pyramids. For each of the six challenges, students A, R, and V were given a maximum of two sessions of approximately 45

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[^7]: See the next section of this thesis for a definition and the criteria of nonroutine mathematical problems.
minutes per week to complete a given problem. On the first day the group was presented with an activity sheet which had a set of problems about rectangular pyramids. These tasks involved the group's guessing and then their identification of the number of cubes that would be needed to build particular levels in the rectangular pyramid. The second and third mathematical problems were investigations which followed the same format as the first activity sheet but were based on the group's problem solving of cross and corner pyramids. The fourth, fifth, and sixth mathematical problems also had similar investigations outlined on the activity sheets but these explorations were based on the three pyramids (i.e., line, half-line, and half-cross) which the group designed as part of their group projects. Each of the six pyramid problems were introduced on separate days and only after the group had completed the previous one. Samples of the six nonroutine mathematical problems and digital photographs of the six types of 3-D pyramids which were used in this study are provided and discussed later in this thesis.

With my minimal participation in the problem solving sessions, it was critical that the six mathematical problems be thoughtfully designed in order to encourage the unfolding of the students' individual knowings, collective knowings, and their group's path of mathematical understandings. As well, because all six of the mathematical activities were to be a group effort, I expected that the students would need to negotiate and generate taken as shared definitions regarding the meanings of words and concepts as they surfaced during these sessions (Cobb, Yackel, & Wood, 1992).

**Definition and Criteria for Nonroutine Mathematical Problems**

Although each of the six nonroutine problems involved a different pyramid each time (i.e., rectangular, cross, corner, line, half-line, and half-cross), all six projects were created using the same criteria which was developed in my previously mentioned case study for designing 'nonroutine' mathematical problems. These guidelines are specified below.
First, the mathematical challenges needed to be meaningful and interesting to the participants (Lesh, 1981; Papert, 1972; Saari, 1977). I chose the topic of 3-D pyramids as the theme for the group's investigations because this was not an area that had been formally addressed by the students' teachers in any of the children's previous mathematics lessons. The pyramid problems posed to the group needed to provoke the students' curiosities in a creative, cognitive, and embodied manner. I wanted each mathematical challenge to prompt the students to ask exploratory questions, encourage them to conjecture, investigate, and analyze. The problems needed to lend themselves to more than one method of solution and, solutions that would require the students' use of strategy, experimentation, and generalization. As well, I wanted each problem to facilitate the students' development and use of basic mathematical concepts and skills regarding their spatial-numerical sense.

Second, the mathematical problems needed to be flexible and open enough to allow for the children to translate their mathematical thinking from one mode of representation to another. In other words, the problems were designed to accommodate the group's possible representational movements among manipulative models, pictorial, formal written or verbal mathematical language using symbols, or informal mathematical language. In this context, informal mathematical language was defined as being the students' natural language, dramatizations, or descriptions of real world situations (Gonzales, 1994; Lesh, 1979).

**Designing Six Mathematical Problems**

I prepared the first and second mathematical problems prior to the beginning of the problem solving sessions. After I watched the two videotaped sessions, I identified relevant mathematical concepts by questioning the contexts and consequences of the students' problem solving activities in particular events in the video data. Consequently, I modified the third pyramid problem, and structured the fourth, fifth, and sixth projects around the group's knowings and understandings of mathematical
concepts which resulted from their activities as problem solvers during the previous session(s). For example, while I was watching the videotapes of the group's first and second sessions in which the children were problem solving for the rectangular and cross pyramids that involved each pyramid problem as increasing successively by one level each time, I wondered what type of mathematics would be revealed if the group was also asked to solve tasks that involved the increasing of a pyramid's levels by more than one level. This curiosity led me to extend the remaining four pyramid problems to include two more tasks. The first extension involved the group's guessing, testing, and justifying of the number of additional cubes which would be required to build a pyramid of thirteen levels. This made for an increase of five levels from the previous question and involved the group's problem solving for the number of cubes that would be needed to build the eighth level of a pyramid. The physical building of this level was feasible if they so wished. Further still, an increase of sixteen levels as a second, additional task required the students to talk about and explain, how many more cubes they thought would be needed to build the twenty-ninth level of a particular pyramid. Unlike the last problem, it would no longer be feasible to build to this level.

The fourth, fifth, and sixth remaining mathematical projects asked the group to create pyramids that were different from the rectangular, cross, and corner ones which I presented to them and to solve the set of tasks based on these pyramids. Thus, it was in these ways that the videos helped me to view what was going on and delve more critically into understanding the group's mathematical path by integrating mathematical themes which I perceived as emerging and creating appropriate problems which challenged and allowed for the group's presentation of subsequent mathematical phenomena to take place.

\(^{18}\text{Here, the use of the word } presentation \text{ is taken to mean the group's eliciting of mathematical understandings.}\)
Schedule for the Case Study

The problem solving sessions were conducted over a Monday through Friday, four week period. As mentioned earlier, all problem solving sessions commenced shortly after the students’ school day approximately at 3:00 PM. To ensure the group’s concentrated efforts and continued movement through each of the six pyramid problems, the children and their parents agreed to fit within one week as many sessions as needed by the students for any particular problem. Therefore, on the occasions when the group was unable to complete their work of a particular pyramid challenge within the first 45 minute period, a second session was scheduled as soon as possible in the same week so that they could finish their tasks. As a result, the group needed a maximum of two 45 minute working sessions within any one week period to finish a given project. Agreement regarding the scheduling of the sessions involved students V, R, and A, their parents, and myself.

Research Ethics

It is clear that I cannot assume Bowers’ ecological position and leave implicit the steps I took concerning the ethical considerations essential to my research. Therefore, I outline my procedures below. After I received formal written consent (see Appendix A) and a certificate of approval (see Appendix B) from the University of British Columbia Behavioural Research Ethics Board, I then sought out written consent from the school’s district principal (see Appendix C) and the school’s principal (see Appendix D) in order to conduct my research on the school’s premises. Having received approval from these authorities, I made initial contact with the participants by sending letters to students A, R, and V and their parents two months prior to conducting the case study. The letters explained the details of my study and asked for the children’s participation in the research project (see Appendix E). With the written consent from A, R, and V’s parents, I then ensured that the students wished to volunteer to participate in the case study. When I asked the children whether or not
they wanted to participate, all three children expressed that they were eager to begin the research project and enthusiastically consented to be participants.

To ensure the anonymity of the students I asked the children to choose fictitious names for themselves. I then used these names in all of the written documents regarding this research and coded the videos accordingly. I sought written permission from the students and their parents after they themselves had viewed the videos and have received approval to use the videotapes for the purposes of supporting the findings of this case study (see Appendix F). The videotapes are kept in a locked cabinet and will not be destroyed unless I am requested to do so by a student or a parent. Finally, I plan to share the results of my research project with the students and their parents, as well as provide a copy of my thesis to the district principal upon its completion.

Data Collected from the Problem Solving Sessions

The data collected from the group's six problem solving sessions included the videotape of the six problem solving sessions (total time of 268 minutes), the activity sheets completed by the group, the 3-D pyramids they constructed from multi-link cubes, and all other written records that the students produced while working on the group projects.

Analyzing and Interpreting the Video Data: Rationale for Methods

It was not until I completed gathering the video data of all six sessions and begun my formal analysis of them as an integrated collection that I was able to define the specific ways in which the research question would be addressed, analyzed, and conceptualized. This was due to the fact that my investigation was examining new and complex phenomena in the light of enactivist views. In order to describe the events which unfolded during the problem solving sessions, I made use of some grounded
theory techniques\textsuperscript{19} and adapted these for the analysis and interpretation of the video data.

The reasons for why I incorporated the use of theoretical sensitivity\textsuperscript{20} and open coding\textsuperscript{21} methods for my video analysis were directly related to the nature of my research question--\textit{How do children's individual and collective mathematical knowings co-emerge to create a path of group understandings in a problem solving environment?} My use of the words--\textit{co-emerge} and \textit{create} indicated that the inquiry involved the examination into how the group's mathematical processes and movements shaped the children's mathematical understandings. As well, the distinction between \textit{individual and collective knowings} implied that my analyses required the conceptual development of two types of mathematical knowings. Thus, my question was addressing complex phenomena, and so I chose to use grounded theory techniques because of the method's strengths as an emergent, creative, and rigorous approach to qualitative research. An emphasis on identifying processes\textsuperscript{22} and patterns of action and interaction among \textit{social units}, the forming of conceptually dense \textit{relationships} through varied questioning and analytical techniques, and the generation of interpretations which \textit{fit} (Denzin & Lincoln, 1994), were all qualities that were consistent with that of enactivism.

It was through my exploratory inquiry and the ways in which I responded by integrating and devising techniques that I was able to refine the meanings of the terms in my research question. How these methods were integrated to fit this particular case study will be outlined in later sections. What emerged from my original question and the posing of the six mathematical challenges was my inquiry into the children's

\textsuperscript{19}Grounded theory is a qualitative research method that uses a systematic set of procedures to develop an inductively derived theory about phenomena. See Denzin & Lincoln (Eds.) (1994), p. 236-247 and p. 273-285; also Strauss & Corbin (1990).

\textsuperscript{20}The activity of constantly questioning the meanings in the data by going back and forth through the data, as well as verifying ideas against existing or emergent theories.

\textsuperscript{21}The procedure of open coding is outlined in Strauss & Corbin (1990), p. 57-142. See also McMillan & Schumacher (1993), p. 479-515

\textsuperscript{22}Processes are changes in patterns of action or interaction.
problem solving actions, their three spatial-numerical organizations, and how these
two topics underpinned the co-emergence of the group's individual and collective
mathematical ways of knowing, doing, and being. This was achieved through my act of
continually moving between my analyses and making conjectures about what
mathematical events were unfolding during the problem solving sessions, as well as
reviewing relevant theoretical literature throughout my analysis. As grounded theory's
methods of analysis are based on discovery and thus taken to be emergent, it was not
until I was doing my formal analyses of the videos that the specific categories and
relationships regarding the group's path of mathematical understandings surfaced
(Pirie, 1997).

The first stage of analysis consisted of viewing the videotapes from beginning to
end of each session several times and then repeatedly watching a series of sessions
or episodes until patterns or important events that related to my research question
emerged. Although viewing the video data in this manner took considerable time to
complete, I could not see any other method that would allow me to look at all of my
data and not exclude any items that might be important to the final results of my
research. By continually viewing the data in different ways, the presentation of new
ideas and information were possible, and I was constantly weaving and reweaving
these until they formed an integrated whole—the children's mathematical path. For
instance, during my two analyses, I asked a variety of questions such as: What
appears to be going on in this episode?, Are any of these events connected to any
previous events? If so, what may have given rise to these events?, And, how do these
events impact on the subsequent mathematical presentations regarding the students'
problem solving actions, the students' individual and collective knowings, and the
group's mathematical understandings of the three spatial-numerical structures? This
method of open coding which involved me going back to the video data and checking
my conjectures against any collected artifacts or related literature allowed me to verify,

23Category is the result of the grouping of concepts that seem to pertain to the same phenomena.
refute, develop significant concepts\textsuperscript{24}, and create dimensions of depth within the categories to understand the relationships which co-existed and co-evolved among these categories.

Finally, it was only during the later stages of my first and second analyses that I made transcripts of specific episodes in the videotapes. These consisted of those events which best served as responses to my research question and illustrated the children's spatial structuring of the pyramids' rods and layers, and their mathematical ways of knowing and doing. In this process of transcribing, I recorded the students' physical actions, their verbalizations, and the intonations of their voices to ensure that these episodes would be portrayed as accurately as possible on paper (Clarke, 1998).

**Reporting the Results of the Case Study: A Preview to the Next Sections**

I mentioned earlier that the purposes of this research were to be descriptive and interpretive. The next sections include my observations, excerpts of the participants' verbal and physical actions, digital photographs, and diagrams which represent the students' verbal and physical mathematical (en)actions. Using them, I have attempted to express and illuminate the co-evolutionary process(es) of the group's path of mathematical understandings as it unfolded during the six videotaped sessions.

\textsuperscript{24} Concepts are defined as discrete happenings, events, and other instances of phenomena.
The Unfolding of a Path of Mathematical Understandings

The Process of my Analysis and Interpretation

Before conducting any focus-driven analysis, I viewed the entire series of videos three times over, each time from beginning to end. One of the foci for analysis that evolved from this watching of the video data was how the students were using building, drawing, and numbering as modes of representation in their group's work. On my first and second viewings of the six sessions, I began to see specific events as being interrelated and certain parts of the children's path regarding their mathematical understandings started to emerge. It was during this part of watching the videos that I noticed a fluid yet unpredictable quality in the ways the children were moving mathematically within each of the six sessions, and how the group's mathematical understandings were connecting one session to another through their activities as problem solvers. In other words, it was the children's mathematical movements as problem solvers that I observed to be continually evolving from session to session, but these were not occurring in any predictable pattern.

From there, I went back and segmented the video data into episodes, and categorized them as building, drawing, and numbering. This was a crude first cut, but I was interested in the children's mathematical ways of being. Thus, the group's path of mathematical understandings was starting to take shape as the problem solving sessions were becoming integrated for me as a whole. At this point I began to make detailed notes about the children's actions of building, drawing, and numbering in order to develop these movements as three categories. Within each of the categories, talking and watching were defined as integral features that could also be observed as distinct activities. Descriptions of the specific mathematical actions are provided in the explanations below. It is important to note that these descriptions in no way attempted
to imply that there was not fluidity in the children's mathematical problem solving, but in fact, provided a means by which to interpret the key events that were shaping the evolution of the group's mathematical path. As well, there was no expectation of sequentiality through the three categories in any particular order and they were assumed to be flexible and not disparate. The three categories of the group’s mathematical ways of being were identified as:

**Building:**

*Doing.* A student, pair of students, or the group of students physically using multi-link cubes to construct three dimensional pyramids or physically putting the pyramids into a particular arrangement or arrangements.

*Watching.* The student(s) physically looking at a student or students building a particular pyramid or pyramids, or arranging the pyramids.

*Talking.* The student(s) verbally discussing or participating through engaging in talk/interactions regarding the building, structure, or arrangement of a particular pyramid or pyramids.

**Drawing:**

*Doing.* A student, pair of students, or the group of students making two-dimensional diagrams of the pyramid(s) on chart paper or dot paper.

*Watching.* The student(s) physically looking at the person(s) making the diagram(s).

*Talking.* The student(s) verbally discussing or participating through engaging in conversation(s) about the drawing of a particular pyramid or pyramids.

**Numbering:**

*Doing.* A student, pair of students, or the entire group physically counting by pointing and keeping track with their finger(s), verbally counting aloud, or performing
calculations. This category includes both verbal calculations and a student, pair of students, or the entire group’s production of written records regarding the solution to a particular arithmetical problem. These written records could consist of explanations, informal mathematical language such as ‘groups of’ instead of ‘x,’ numbers, mathematical symbols, and/or student or group-invented symbols.

Watching. The student(s) physically looking at the person(s) performing the counting, calculating, or recording activity.

Talking. The student(s) verbally discussing the counting or calculating activity.

After defining the three categories, I was then able to create the second layer of my analysis by interpreting the students’ problem solving actions. I did this by going back into the video data and developing descriptions for the students’ particular mathematical ways of being; that is, the mathematical activities that the children were engaging in while trying to solve the pyramid problems. Although this proved to be the most time consuming part of my analysis--because I was viewing the video data in many ways and several times over, it was the only way in which I could achieve a sense of rigour and depth in understanding the mathematical movements of the children. This theoretical sensitivity required me to have an “awareness of the subtleties of meaning of data” (Strauss & Corbin, 1990, p. 41)

This meant that I needed to be attentive to the smallest of details regarding the children’s mathematical actions, and found myself going back and forth through the video data to understand how these details were significant in the mathematical ways that the children were knowing, doing, and being. For example, by watching the videos in this way I was able to understand how an enaction of understanding, such as the way in which a student pointed as she or he counted the cubes in a pyramid, evolved into arithmetic calculations that mirrored the counting strategy, or the expressive way in which a spoken word was delivered triggered the group to carry forth previous
experiences and reenact them in what they were presently doing.

Further still, it was necessary for me to determine whether or not the mathematical actions were being demonstrated by one student, two students, or the entire group. So again, I went back into the videos to create the third dimension of my analysis. In order to identify the origin of each action as part of the students' mathematical path, I needed to distinguish the group's mathematical movements as being either individual or collective knowings. Therefore, any enactions that were not perceived as familiar by a member or members of the group were taken to be an individual knowing of a particular student and belonging only to V, R, or A. And any enactions that were observed as being previous experiences in earlier episodes or previous sessions and were now being utilized by a different student, pair of students, or the entire group were understood as forms of collective knowings. These collective knowings were also located as being the enaction of one student, the inter-activity of two students', or the 'doing' of the whole group. Together, my analytic strategies enabled me to understand the ways in which the group was laying down their path of mathematical understandings and how the path was unfolding and co-evolving as the children moved through the six mathematical challenges.

Results: Following The Children as They Lay Down a Path of Mathematical Understandings

In the next six sections, I provide descriptions regarding the six pyramid problems, examples of the activity sheets that were presented to the group, detailed accounts involving the children's strategies in solving each of the mathematical problems, as well as transcripts of key episodes that illuminate described actions of the children and the group. Following this, I discuss and provide interpretations regarding the co-evolution of the group's numerical and spatial understandings which emanated from their individual and collective knowings.
Group Project #1: Rectangular Pyramids

I presented the children with their first problem (see Figure 3) and showed them three 3-D models of rectangular pyramids that were made out of multi-link cubes. These structures consisted of a one-level, a two-level, and a three-level pyramid (see Figure 4). Following the format that was outlined in the task sheet, the group was asked to identify the number of cubes in the first pyramid, and then asked to determine the number of additional cubes in the second and third pyramids. The students passed around the first two pyramids, and quickly responded that there was one cube in the first pyramid and there were nine more cubes in the second pyramid. When I asked how many more cubes were in the three-level pyramid, both students V and A used their fingers to point and count the bottom layer’s width as five cubes and its length as five cubes, and responded with “twenty-five”. The fact that both V and A performed identical actions at different times suggests that both children were multiplying its length and width together to get the total number of cubes in the third layer. Student R however, turned the pyramid upside down, pointed with his finger as he counted by ones, and then announced the total number of cubes in the bottom layer as also being twenty-five. The group and I then finished reading through the task sheet and upon the conclusion of working on the first three questions, I left the students alone to work on their project.

Student V picked up the two-level pyramid, turned it upside down, and by keeping track with her finger, proceeded to point and count the number of cubes in the base by ones to arrive at a total of nine cubes. She also noted that the length/width of the square base was three cubes. Holding the third pyramid, student V then counted its base width/length as being five cubes each. She made the observation that the third pyramid had increased by two cubes in its width/length. Having watched her do this, the group then made the conjecture that the next pyramid would have a base of 7 x 7 cubes, or forty-nine cubes. While V was satisfied with this conclusion, R and A
**Introduction To Rectangular Pyramids**
(demonstration and discussion with teacher)

How many cubes are in pyramid #1 (1 level high)?

How many more cubes are in pyramid #2 (2 levels high)?

How many more cubes are in pyramid #3 (3 levels high)?

**Group Project #1:**

Using any of the materials that are available, can your group:

• find out how many more cubes will be needed to build a rectangular pyramid that is 4 levels high?

• find out how many more cubes will be needed to build a rectangular pyramid that is 5 levels high?

• find out how many more cubes will be needed to build a rectangular pyramid that is 6 levels high?

• *without actually building, guess* how many more cubes your group will need to build a rectangular pyramid that is 7 levels high.

• find out by testing and explaining, why your group's guess was correct/incorrect. To build a rectangular pyramid that is 7 levels high, you need to use this many more cubes:

• *without actually building, guess* how many more cubes your group will need to build a rectangular pyramid that is 8 levels high.

• find out by testing and explaining, why your group's guess was correct/incorrect. To build a rectangular pyramid that is 8 levels high, you need to use this many more cubes:

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**Figure 3.** The activity sheet of mathematical challenges involving rectangular pyramids that was presented to the students.
Figure 4. Digital photographs of the one-level, two-level, and three-level 3-D cube pyramids presented to the group during the introduction of the first problem solving session.
insisted that they needed to build the four-level pyramid to find out whether or not the solution is correct.

V: It has nine right? [referring to the total number of cubes that she has just counted in the bottom layer of the two-level pyramid] Three. [sweeps her finger across the length/width of the bottom layer] One, two, three, four, five [counting the cubes in the length/width of the three-level pyramid's base]. That's adding two, so, five plus two equals? [smiling]...

R and A: Seven!
V: Seven times seven equals...
R: Forty-nine.
R: Let's get the building blocks. We have to build it.
A: We might need some more cubes [reaches for another container of cubes].
V: [reaches for a pen to record their solution of '49' when R interjects].
R: We need to find out first!
V: But isn't that already correct?
R: We don't know that.

When the children began constructing the fourth pyramid, it was clear from their building strategies that A, V, and R had different conceptions regarding the way in which the pyramid was increasing in size. V and A began to build a horizontal fourth layer by putting together seven rods of seven cubes each, while R attempted to attach cubes so as to cover the entire existing three-level pyramid. Student A looked at what he and V were building and then looked to what R was making, giggled, and asked R what he was doing. Student V tried to take apart R's model. Objecting to their disregard for his work, student R reassured them that if they let him continue, they would see that what he was doing was correct. It was only after student A continued to watch R build his pyramid that A realized R was creating a four-level pyramid by attaching an outer layer of cubes around the third pyramid. Student A then began helping student R to complete the pyramid. Student V still disagreed with what R was building, and insisted that they needed to create a bottom layer. She then tried to get student A to come back and help her finish the bottom layer but student A told her that he and student R had "already got it".

---

25 Point 10 italic type style font is used to indicate my explanatory comments.
26 Point 12 italic type style font is used to indicate the action(s) of the student(s) as seen on the videotape.
V and A: [start to build rods of seven cubes each at which time, A looks over to see what R is building and laughs]

A: What are you doing?

R: You'll see.

V: [tries to remove the cubes that R is adding onto the 3-level pyramid, but he objects]

R: No, no, no, no! Don't take it off. I'll show you what I'm doing.

A: [watches R build the pyramid] I know. You're building them around.

A: [starts to help R complete the pyramid].

R: R! It's supposed to be built under!

A: You'll see what I'm doing, ask A.

A: Oh, I see what he's doing!

R: Now you know what I'm doing?

A: Yes, I see... you need three more on the top [gives R three more cubes].

V: [looks at R and A's pyramid, and continues to build a 7 x 7 layer of cubes]

V: A?! [says impatiently]

A: I think we already got it.

V stopped building, watched the other two students complete the pyramid, and then said that if the model proved their solution of forty-nine as correct, the group would not need to make any more pyramids. At this time, students R and A realized that the pyramid only had a 6 x 6 cube base. V counted the fourth layer and said that “it’s supposed to have seven”. Tilting his head from side to side, student R turned the pyramid around in his hands with a puzzled expression on his face. V became impatient and told the group that they should just record forty-nine as their tested solution. R continued to look at the model and then added a row of six cubes onto one side of the bottom layer to make a base of 6 x 7 cubes. Still not satisfied with his model, R paused briefly, added a row of seven cubes onto the width of the base and gleefully exclaimed “I did it!” (see Figure 5). V checked the dimensions of the bottom layer by counting aloud by ones and pointing with her finger, and confirmed that the pyramid’s fourth layer was indeed 7 x 7 cubes. The group agreed that this matched their solution of “49”. Here, through the emergence of two different building methods--V and A’s constructing of a horizontal 7 x 7 cube layer that was to be added as the base of the pyramid, and R’s act of attaching of cubes as an outer layer to cover
the entire structure of the existing third pyramid--student A was able to understand and help student V to see the two ways of building as both correct but as two distinct visualizations of pattern growth in the rectangular pyramid.

![Image of student R's 3-D four-level rectangular pyramid made out of multi-link cubes.]

**Figure 5.** Digital photograph of student R's 3-D four-level rectangular pyramid made out of multi-link cubes.

Having successfully solved and tested the fourth pyramid, the group simply used the identified and "proved" (V's word) pattern of increasing 'squared odd numbers' (although they do not use this term) to verbally explain that the fifth level would have "nine times nine" or "eighty-one" cubes and the sixth level would have "eleven times eleven" or "one hundred twenty-one" cubes. When the children went on to solve for the seventh pyramid, the group recited "thirteen times thirteen equals" but then stopped. When none of the students knew the answer to this, student A recorded the following mathematical calculation on a piece of large chart paper:

\[
13 \\
\times 13
\]

The children looked at the calculation and spent a few minutes trying to remember the algorithm for multiplying two digit numbers together! When none of them could recall what to do, they made use of the calculator and entered ‘13 x 13 =’ to get ‘169'. In
solving for the eighth and final level, the students quickly said "fifteen times fifteen". Student V wrote this down in a vertical fashion as student A had done for the previous problem, but again, they could not remember the procedure to finish the calculation. After making a guess of two hundred seven, the students then used the calculator a second time to enter \(15 \times 15 =\), and got the correct solution of two hundred twenty-five cubes. The group finished the session by going back and double checking their arithmetic for all of their recorded answers with the calculator.

**Group Project #2: Cross Pyramids**

The next day the group went on to investigate *cross pyramids*. When I gave the children the second task sheet (see Figure 6) and asked them how many cubes were in the first cross pyramid (see Figure 7) student A responded by saying, "one". This was accepted by the other two children as being the correct answer. But when the second question was posed regarding how many more cubes were in the two-level cross pyramid and students R and A both said "five", student V did not respond. Student R took the second pyramid, turned it over, and began counting the total number of cubes in its bottom layer. He pointed with his finger and counted the layer as being one group of four cubes with one cube in the middle (see Figure 8).

\[
\begin{array}{ccc}
1 & & 2 \\
4 & 5 & 3 \\
\end{array}
\]

**Figure 8.** Diagram which illustrates student R's explanation to student V regarding his method of counting the number of additional cubes needed to build a two-level cross pyramid.

Student V agreed with this, and the group moved on to the third question. Again, student R turned the pyramid over so that the bottom layer was showing, and counted aloud as V and A watched him. Student A joined him in the counting of the cubes and
Cross Pyramids

Introduction To Cross Pyramids
(demonstration and discussion with teacher)

How many cubes are in pyramid #1 (1 level high)?

How many more cubes are in pyramid #2 (2 levels high)?

How many more cubes are in pyramid #3 (3 levels high)?

Group Project #2:

Using any of the materials that are available, can your group:

- find out how many more cubes will be needed to build a cross pyramid that is 4 levels high?
- find out how many more cubes will be needed to build a cross pyramid that is 5 levels high?
- find out how many more cubes will be needed to build a cross pyramid that is 6 levels high?
- *[without actually building, guess how many more cubes your group will need to build a cross pyramid that is 7 levels high]*
- find out by testing and explaining, why your group's guess was correct/incorrect.

To build a cross pyramid that is 7 levels high, you need to use this many more cubes:

- *[without actually building, guess how many more cubes your group will need to build a cross pyramid that is 8 levels high]*
- find out by testing and explaining, why your group's guess was correct/incorrect.

To build a cross pyramid that is 8 levels high, you need to use this many more cubes:

Figure 6. The activity sheet of mathematical challenges regarding cross pyramids that was presented to the group.
Figure 7. Digital photographs taken from above of the one-level, two-level, and three-level 3-D cube cross pyramids that were presented during the introduction of the second problem solving session.
they got a total of nine. Student V looked at the pyramid and then looked to the other two members of the group. Student R showed her the bottom layer of the pyramid, and pointing with his finger, repeated his counting method as two groups of four and one cube in the middle (see Figure 9).

R: One, two, three, four.
   Five, six, seven, eight.
   Nine.

![Diagram](image)

Figure 9. Diagram which illustrates student R's explanation to student V regarding his method of counting the number of additional cubes needed to build a three-level cross pyramid.

Student V nodded her head in agreement and the group recorded nine as the correct solution.

It was when the group went to work on the problem involving the four-level cross pyramid that they unknowingly embarked on a rather scattered mathematical expedition--a journey that had the students taking turns being the leader and more often than not, losing his or her followers! The next episode tells the story of how they started out as a group, lost sight of each other's thinking, persevered, and finally found a way together, to generate a method to solve for any unknown level of the cross pyramid.

Student V started out by leading the group through a comparison of the second and third pyramids. In her explanation, she verbalized the operation of multiplication but actually added the numbers together in order to calculate the cubes needed to build the bottom layer of the fourth pyramid. In any event, student A was able to follow
along and the two successfully solved the answer as thirteen.

V: There's one here [holding the two-level cross pyramid] [referring to the four single cubes that are attached around the middle cube]...
and two here [holding and looking at the three-level pyramid] [referring to the four rods of two cubes each that are attached around the middle cube].
So that equals three ['three' being the number of cubes in each of the four rods that would be attached to the middle cube, thus making up the fourth level]...
So three times three times three times three (sic).

A: Twelve [takes her use of 'times' as meaning '+'].
V: Exactly. Plus one equals thirteen.

Meanwhile, R had been busy building a 3-D cross pyramid that was three levels high.
He was about to attach more cubes around the pyramid to build a fourth outer layer
when V and A took it away from him and added on a bottom layer of thirteen cubes.

Having solved the four-level pyramid problem and thus ostensibly being able to
move on to the next problem, the children did not move on, but remained working on
the fourth pyramid. This time their conversation took a turn and confusion set in as to
what exactly 'three' and 'four' signified in terms of the cross pyramid\(^\text{27}\). Student A led
the group's conversation and explained the pyramid's base was arranged in three
radiating groups of four cubes each around the middle cube. When student V kept
giving him the answer of "thirteen", student A became rather frustrated and repeated
his explanation to her two more times.

A: Let's just do the times-table thing, so four times four times four [is also
now misusing the operation of multiplication]...
V: Is thirteen.
A: No, four times four times four 'cause the last one..., four times four times
four...
V: Is thirteen [this time, in a louder voice].
A: No, no, no, four times four times four because everything has four now [in
a frustrated tone].

Student V tried to explain the fourth level from a different visualization of the cross
pyramid's pattern growth; which was, being arranged into 'four rods of three cubes'
and not 'three groups of four cubes' that were attached around the middle cube.

\(^27\) The interpretation of this whole episode is totally dependent on the data that was being analyzed. This
data consisted of the videotapes (i.e., visual) as opposed to their transcripts which would not have
revealed either reason for the students' confusion or its resolution.
V: This one only has three here [holding the fourth pyramid and pointing to the base that has rods of three cubes each that are attached to the four sides of the middle cube]. So that's the one which is twelve [points to each of the four rods of three cubes with her finger]. Twelve [sweeping her finger around the twelve cubes], plus one [pointing to the middle cube] equals thirteen. Do you guys agree?

Students A and R looked at the pyramid but said nothing. This may have been because they were still visualizing the arrangement of cubes as being three groups of four cubes, and not V's visualization of four rods of three cubes. Student V then turned the three-level pyramid upside down to reveal its base, and tried again to explain her thinking to them. What she did differently this time was as she pointed to the four rods of three cubes each that were attached to the centre cube, she made it clear to the others that the 'four' signified how many rods there were around the middle cube, the 'three' referred to the number of cubes in each of the four rods, and the middle cube was the 'plus one' in the calculation.

V: There are four sides here [using her finger, points to each of the four rods] and one is the plus [points to the middle cube]. So if you add one here ['one' meaning, another level], that'd make three here, three here, three here, and three here. So four times three plus one equals thirteen.

R and A finally agreed that the mathematical calculation made sense and thirteen was the correct answer. The children then moved on to the problem regarding the fifth cross pyramid.

V: Four times four times four times four plus one equals seventeen.

Even though she was still using the term "times" but was actually adding the groups of four together, the other two students did not correct her on this and appeared to understand her actions of adding as making sense. And although they appeared to agree with the calculation, it is possible that R and A were still understanding the cube arrangement of the bottom layer as being made up of four groups of four cubes each that were attached and surrounding the centre cube, and that V was still seeing the
cubes as being arranged as four rods of four cubes each that were attached to the middle cube. Either way, the value of the groups or the rods was ‘four’, and both visualizations produced the same mathematical result of \(4 \times 4 + 1 = 17\).

For the sixth and seventh pyramids, V, R, and A corrected their use of mathematical operations, but got confused as to which numbers and what order the values should be placed within the calculations. As the group went back and forth through a series of oral calculations, the group lost track of their mathematical actions, and soon realized that none of them was certain as to what needed to be done.

V: Five times four equals twenty [is not using her ‘\(4 \times n\)’ method], plus one equals twenty-one [the number of additional cubes needed to build a six-level cross pyramid]. Do you guys agree? [records ‘21’ as their answer]

A: [moving on to the seven-level pyramid] It’d be six. That’s going to be five... twenty-five\([5 \times 5]\) plus one equals twenty-six.

V: Do you guys agree with that? ’cause we found out that five times... I don’t get it [looks at A and R with a confused expression on her face].

A: See this? [pointing to their written answer of ‘21’ for the six-level pyramid] This was four times five [reverses V’s previous of “five times four”], that’s how you got twenty, [ignores R’s interjections] but then you plus one, equals twenty-one. So then you got, so then, that’s five times five.

V: Hold on for a second.

R and A: [stop what they are doing and look at what student V is doing]

V: [holding the three-level pyramid] Four levels high was thirteen, so... four times three [is now using her ‘\(4 \times n\)’ method again]. This one was five levels high, so that would mean that there was... If this [referring to the fourth pyramid] was three [cubes in each of the four rods], that’d make it four... so, equals six [the recorded answer for the sixth pyramid].

A: No, this is...

V: So five times four [points to the recorded solution for the sixth pyramid on the task sheet]. So if this one’s seven levels high... so that’d make five times four right? [with a tone of uncertainty in her voice, she looks to A and R].

A: Five times five... [looks at V] is twenty-five.

Student V stared at the base of the third pyramid and did not agree with what student A had just said. This in turn ignited an active debate which resulted in the students’ three attempts to clarify the ‘fiveness’ and ‘fourness’ of the cross pyramid.

V: But there’s not five here [pointing to the number of rods of cubes radiating from the middle cube].

R: But one, two, three, four, five [using his finger to count the four cubes and the middle cube], then four [pointing to the second group of four cubes].
A: No, two, four, six, eight [counting the bottom layer as being four rods of two cubes each with his finger].

R: I know, but one, two, three, four [counting the number of rods of two cubes each with his finger], five [pointing to the middle cube], ... then you add two [more cubes] onto each end [to create the fifth pyramid].

Realizing that their conversation was getting them nowhere, the children finally turned their energy toward making diagrams on the dot paper. As V, R, and A drew and talked their way through the 2-D pictures that showed the bottom layers for each of the pyramids, they were able to locate a pattern of adding on another group of four cubes to the one middle cube for each successive level, and translated their pictures into mathematical calculations that expressed the two-level pyramid's base as being ‘1 \times 4 + 1 = 5’; the three-level base as ‘2 \times 4 + 1 = 9’, and the four-level base as ‘3 \times 4 + 1 = 13’, and so on. Having made sense of what they were calculating, the group moved on to successfully solve for the number of additional cubes needed to build the seven-level pyramid as ‘6 \times 4 + 1 = 25’ and the eighth-level pyramid as ‘7 \times 4 + 1 = 29’ (see Figure 10). V finished this session by double checking their answers with a calculator.

Figure 10. The group’s 2-D diagrams and written records that depict the bottom layer of each of the eight pyramids with the corresponding mathematical calculations.
Group Project #3: Corner Pyramids

In the second week, the students solved a series of problems involving corner pyramids (see Figure 11). As I presented the group with the first two pyramids (see Figure 12), the children quickly responded without performing any counting actions by stating that the first pyramid contained one cube and the second pyramid contained three more cubes. When the third pyramid was shown to the students (see Figure 12), V and A watched as student R turned the pyramid upside down, pointed with his finger, counted by ones, and concluded that there were six cubes in its base (see Figure 13). V and A agreed with six as being the correct answer.

![Diagram](image)

Figure 13. Diagram which illustrates student R’s counting strategy of the bottom layer in the three-level corner pyramid to determine the number of additional cubes required for this pyramid as opposed to one with two levels.

The students went on to make several attempts in order to solve the next problem. Student A suggested that there would be nine more cubes needed to build a four-level pyramid, but then quickly dismissed it as being incorrect because it was not consistent with the number pattern on their task sheet (which was “1”, “3”, and “6”). R followed after him by making a guess of “eight” and attempted to add imaginary cubes by pointing to the places where they would be attached along the outside edge of the existing base, but then stopped. Student V took the third pyramid and tried to add on imaginary cubes onto the front of the pyramid by counting and pointing.

V: One, two, [pointing with her finger to either side of the base of the third pyramid]...
three, four,[pointing to the two outermost ‘steps’ on the bottom layer of the third pyramid]...
five, six, [pointing to the two ‘steps’ in the middle level of the pyramid]...
seven? [pointing to the one ‘step’ on the top layer of the pyramid]
How many would be here?
Introduction To Corner Pyramids
(demonstration and discussion with teacher)

How many cubes are in pyramid #1 (1 level high)?

How many more cubes are in pyramid #2 (2 levels high)?

How many more cubes are in pyramid #3 (3 levels high)?

How many more cubes are needed to build pyramid #4 (4 levels high)?

Group Project #3:
Using any of the materials that are available, can your group:

• find out how many more cubes will be needed to build a corner pyramid that is 5 levels high?

• find out how many more cubes will be needed to build a corner pyramid that is 6 levels high?

• without actually building or calculating in any way, guess how many more cubes your group will need to build a corner pyramid that is 7 levels high. It doesn’t matter if your guess is wrong!

• find out by testing and explaining, why your group’s guess was close to the correct answer or very different from the correct answer.
To build a corner pyramid that is 7 levels high, you need to use this many more cubes:

• without actually building or calculating in any way, guess how many more cubes your group will need to build a corner pyramid that is 8 levels high. It doesn’t matter if your guess is wrong!

• find out by testing and explaining, why your group’s guess was close to the correct answer or very different from the correct answer.
To build a corner pyramid that is 8 levels high, you need to use this many more cubes:

• without actually building or calculating in any way, guess how many more cubes your group will need to build a corner pyramid that is 13 levels high. It doesn’t matter if your guess is wrong!

• find out by testing and explaining, why your group’s guess was close to the correct answer or very different from the correct answer.
To build a corner pyramid that is 13 levels high, you need to use this many more cubes:

• talk about how many more cubes you would need to build a corner pyramid that is 29 levels high. You may use any of the available materials.

Figure 11. The activity sheet of mathematical challenges involving corner pyramids that was presented to the students.
Figure 12. Digital photographs of the one-level, two-level, and three-level 3-D cube corner pyramids that were presented during the introduction of the third problem solving session.
V, R, and A: [look at each other and there is a brief moment of silence]
R: Get the blocks!

Immediately the group started to build. Students A and V made a corner pyramid by attaching four horizontal layers on top of one another. R built his own pyramid but used a different strategy. He began by putting together a vertical rod of four cubes, which was presumed to be his response to the notion of ‘four-high’, and then completed the pyramid by filling in the three remaining levels. The students commented that the two models were identical in structure and that both had ten more cubes than the previous pyramid. Students V and A both identified these ten cubes through oral counting and pointing, but located the cubes as existing on different sides of the pyramid and counted from one to ten in reverse of each other. What the two counting methods demonstrated were different but complementary ways of making sense of the same solution. In V’s counting strategy, she carried forth student R’s method of visualizing the cross pyramid’s outer layers as steps when she incorporated this strategy into her enumeration of the cubes that had been added to create the fourth level of the corner pyramid (see Figure 14), and R brought forth his previous use of the bottom layer in the cross pyramid as a way of locating the number of additional cubes in the fourth level of the corner pyramid (see Figure 15).
V: points as she counts the number of 'steps' in each of the horizontal levels on the pyramid by starting at the bottom level of the pyramid and moving her way up]
One, two, three, four,
five, six, seven,
eight, nine,
ten.

Figure 14. Digital photograph and student V's counting strategy of the fourth pyramid's 'steps' for determining the number of additional cubes needed to build this four-level structure.
R: [points as he counts the number of cubes on the bottom layer of the pyramid by turning it upside down, beginning in the corner and moving outwards in a bottom-to-top and left-to-right direction]
One,
two, three,
four, five, six,
seven, eight, nine, ten.

Figure 15. Digital photograph of the fourth pyramid's bottom layer and student R's counting strategy for determining the number of additional cubes needed to build this four-level structure.

Moving on, the students were quick to predict that they would need fifteen more cubes to build a five-level pyramid. It was when student V interrupted student A as he attempted to check the group's prediction by using the four-level pyramid to 'count on' imaginary cubes that she brought to the group's attention, the notion of outside layers
by arranging the pyramids in front of one another--largest to smallest.

V: The more cubes are the ones you can see. That's how we'll figure it out, 'cause you can't see these ones [puts the three-level pyramid behind the four-level pyramid].

Eleven, twelve, thirteen, fourteen, fifteen [using the fourth pyramid, points with her finger and counts 'onto' its base where the fifth layer of visible cubes would be located] [see Figure 16].

How would you add? [the arithmetic pattern]

Figure 16. Digital photograph of the 3-D fourth corner pyramid illustrating student V's strategy for finding out the number additional cubes needed to build the fifth pyramid.
Student V and A continued to look at the pyramids, while R was busy building the fifth pyramid.

V: Add three [points to the third pyramid’s bottom layer], then add four [points to the fourth pyramid’s bottom layer]. So this time you have to add five. So if this is ten [referring to the number of additional cubes needed to build the four-level pyramid], then it'd equal fifteen [using the fourth pyramid, points to where the fifth pyramid’s bottom layer would be].

At this point, student A reexplained this to student R, by using the fourth pyramid to demonstrate this ‘adding on’ process (see Figure 17 for visual representation of the episode below).

A: Add five [with the pyramid upright, sweeps his finger across the front of the bottom layer]. Then six, seven, eight, nine, ten, eleven, twelve, thirteen, fourteen, fifteen [pointing as he counts up on the remaining ten ‘steps’ in the pyramid].

Seeing this as a complicated way of counting up the total number of cubes, student R suggested that rather than counting the steps on the front of the pyramid, they should just count the number of cubes in the pyramid’s base.

R: Why don’t you just count [turns the pyramid over so that its bottom layer is exposed]?

A: No, that’s only four levels.

Obviously not clear on what R meant, student V joined in on the discussion and by doing so, ‘integrated’ the two methods into one strategy (see Figure 18 as a visual representation of the episode below).

V: [holding the fourth pyramid upside down and sweeping her finger over the cubes, she explains the number pattern] Add two, add three, add four... [looks at students A and R] ... has to be add five.

This time, students R and A nodded their heads in agreement and the group recorded fifteen as their solution.
Figure 17. Digital photograph of the fourth corner pyramid and student A's explanation to student R as to how he and V added on five more imaginary cubes to the bottom layer of the fourth pyramid to arrive at their solution of an additional fifteen cubes being needed to build a five-level pyramid.
"Add add add has to be two...three... four... add five."

Figure 18. Diagram which illustrates student V's use of the bottom layer of the fourth corner pyramid to identify the structure's pattern of growth and to explain to R and A how many additional cubes would be required to build the five-level corner pyramid.

Students V, R, and A then systematically worked through the remaining problems, using oral arithmetic and applying their strategy of adding six to their previous answer of fifteen to get twenty-one for the sixth pyramid, adding seven to twenty-one to solve for the seventh container, and adding eight onto twenty-eight to solve for the eighth pyramid. The group then decided to use the calculator to keep a running total as they entered '36 + 9, + 10, +11, +12, + 13 = ' to solve for the thirteenth-level pyramid. They successfully monitored the calculation and got '91'. For the twenty-ninth pyramid, the children entered ' + 13 + 14 + 15 + 16 + 17 + 18 + 19 + 20 + 21 + 22 + 23 + 24 + 25 + 26', then '+ 29 =', and stated the solution as being three hundred ninety-three more cubes. Although they were using the correct procedure to solve the problem, the group did not notice that they had already added thirteen to their previous calculation of seventy-eight to get ninety-one, and should have begun by adding fourteen to ninety-one. A second error was made when the group did not add '27' or '28' as the next two values after entering '+ 26', but rather, ended their calculation with '+29'—despite student R's questioning of V and A on their reasons for doing so!
the group started their calculation by adding fourteen to ninety-one and continued to monitor the additions until they reached ‘+ 29’, they would have reached the correct total of four hundred thirty-five for the twenty-ninth level.

**Revisiting and testing the corner pyramids.**

On the next day, I asked the group to define their meaning of *testing*, as it was unclear by their actions in the previous session, whether or not the group was in fact testing their solutions. After a few minutes of conversing with each other, students A, R, and V defined *testing* as being able to prove a solution correct or incorrect by “working the problem out in another way and using different materials”. So, the group continued with the corner pyramid tasks and set out to test their answers for the seventh, eighth, and thirteenth levels. The children decided to draw diagrams and build a series of 3-D models using the multi-link cubes.

On the large chart paper and on a piece of dot paper V and A made diagrams that illustrated how many more cubes would be needed in order to build the seventh, eighth, and thirteenth pyramids (see Figure 19). To do this, the children drew three separate diagrams that showed two dimensionally, what the bottom layers of the three pyramids would look like and then numbered each square as a way to test and prove that their previous solutions were correct. V and A also built a seven-level pyramid (see Figure 20) and student R built an eight-level pyramid (see Figure 21). The children exhibited two different building strategies here. Student R built the bottom layer of his pyramid first, then attached a vertical rod of seven cubes to the ‘corner’ of the base, and then filled the rest of the cubes in like steps, constantly turning the pyramid as he was building it (see Figure 22). Students V and A first constructed what looked like a ‘corner’ (see Figure 23) with the cubes. This structure became the skeleton or frame onto which they attached more cubes and completed the pyramid.
Figure 19. The group’s 2-D diagrams of the bottom layers of the seventh, eighth, and thirteenth pyramid which show the total number of additional cubes needed to build them. The seventh and eighth level was created by A, and the thirteenth level was created by V.
Figure 20. Digital photographs of the fourth and seventh 3-D corner pyramids. The fourth pyramid was a construction of V, R, and A, and the seventh pyramid was built by A and V.
Figure 21. Digital photograph of the eighth 3-D corner pyramid constructed by student R.

Figure 22. Digital photograph of student R's method for building the eighth corner pyramid with multi-link cubes.
Group Project #4: Line Pyramids

In the third week, I gave the children a fourth activity sheet (see Figure 24) and asked them to create a pyramid (and give the pyramid a name) that was different from the three previous ones. After talking about several different kinds of pyramids and what the structures would look like, the children decided to build "line" pyramids. Student R quickly built the first three 3-D pyramids (see Figure 25), and showed these to the other two students. V and R then watched as student A arranged the pyramids by placing the one-level pyramid in front, the two-level pyramid behind it, and the three-level model behind the second pyramid. This was understood as a similar but slightly different action from V's sorting of the corner pyramids from largest to smallest, because the pyramid's design allowed the models to be 'lined up' nicely against one another and revealed the pyramids as increasing in size by their 'outer layers'. R and A went on to construct the fourth, fifth, and sixth pyramids with the multi-link cubes (see Figure 25). While they were doing this, student V counted the number of cubes that resembled 'steps' in the outermost layer of the fifth pyramid as being nine, and
Now It's Your Turn!

Group Project #4:
At the beginning of the previous sessions, I built the first three pyramids for you, now it's your turn. Using any of the materials that are available, can your group:

• **build a pyramid** that is different from the other three pyramids (i.e., rectangular, cross, and corner)?

• **build higher** pyramids?

The pyramids your group built in the previous sessions were called rectangular, cross, and corner. Can your group:

• give your pyramid a **name**?

Part Two

• What can your group say about your pyramids?

  **without** actually building or calculating in any way,
  **guess** how many more cubes your group will need to build a pyramid that it is 7 levels high.
  *It doesn’t matter if your guess is wrong!*

  **find out by testing and explaining,** why your group’s guess was close to the correct answer or very different from the correct answer.
  To build a pyramid that is 7 levels high, you need to use this many more cubes:

  **without** actually building or calculating in any way,
  **guess** how many more cubes your group will need to build a pyramid that it is 8 levels high.
  *It doesn’t matter if your guess is wrong!*

  **find out by testing and explaining,** why your group’s guess was close to the correct answer or very different from the correct answer.
  To build a pyramid that is 8 levels high, you need to use this many more cubes:

  **without** actually building or calculating in any way,
  **guess** how many more cubes your group will need to build a pyramid that it is 13 levels high.
  *It doesn’t matter if your guess is wrong!*

  **find out by testing and explaining,** why your group’s guess was close to the correct answer or very different from the correct answer.
  To build a pyramid that is 13 levels high, you need to use this many more cubes:

  **talk about how many more cubes you would need to build a pyramid that is 29 levels high.** Use any of the available materials to show what you mean.

Figure 24. The activity sheet of mathematical challenges presented to the children which involved the group’s invention of a new pyramid.
Figure 25. Digital photographs of the group’s first six 3-D cube models of the line pyramid.
announced to the group that if they counted the number of steps from the bottom to the top along one side of any given line pyramid, they could just "double it and minus one" to find out the total number of additional cubes that would be needed to build that particular level. The other two students did not acknowledge V's suggestion, and continued to build their models. The children then made separate estimates regarding the number of additional cubes that would be needed to build the seventh pyramid. V made a guess of thirteen, A guessed fifteen, and R guessed sixteen cubes.

To find out exactly how many more cubes would be required to build a seven-level line pyramid, students A and R began building the seventh pyramid, but abandoned their work (and then completed it towards the end of the session) and turned their attention to the counting strategy that student V was developing based on her comparison of the fifth and sixth 3-D pyramids.

V: [counts aloud the number of 'steps' along the outside edge of the six-level pyramid pointing as she counts up one side then down the other side of the pyramid]

One, two, three, four, five, six, seven, eight, nine, ten, eleven.

[counts while she points to the number of 'steps' on the outside of the five-level pyramid]

One, two, three, four, five, six, seven, eight, nine.

So the difference between nine and eleven is...

V and A: Two.

V: So eleven plus two will equal thirteen!

Knowing that each successive level of the pyramid would increase by two cubes, the students laughed as they made their guess that the eighth pyramid would need fifteen more cubes. Student V demonstrated that fifteen was the correct answer by counting the number of steps there would be in the outermost layer if it was eight levels high. She used the sixth pyramid to 'add on' two more imaginary bottom layers by counting aloud. She did this by placing her two index fingers on either side of the sixth pyramid, tapped alternate index fingers on the table while counting "one, two, three, four", and then by moving up a step each time with her fingers toward the top of the pyramid, she announced "...thirteen, fourteen, fifteen" (see Figure 26).
After making guesses of an additional twenty (student A), twenty-five (student V), and thirty (student R) cubes being needed to build a thirteen-level pyramid, student V led the group through a set of incorrect calculations in which she confused the 'double it and minus one' rule with their second procedure of adding on two to the previous total.

V: Eight plus what equals thirteen?
R and A: Five [the difference of the number of levels between the eighth and thirteenth pyramid].
V: The difference is five, so five times two equals ten, minus one equals nine [using the 'double it and minus one' rule].
So... hold on... so... fifteen [number of cubes to build the eighth level] plus ten minus one [the number of cubes being added to make the thirteenth level] equals twenty-four.

Student V recorded twenty-four as the group's solution to the problem. The group continued to use the calculation and solved for the number of extra cubes needed to build an eight-level line pyramid.

Figure 26. Diagram which illustrates the oral counting strategy that student V used in determining the number of additional cubes needed to build an eight-level line pyramid.

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28 At the start of the third problem solving session involving corner pyramids, the students and I had a conversation regarding their definition of what 'guessing' meant. The group determined that 'guessing' in the context of this and future sessions meant "a wild guess" (A's words)--"you can't count" and "you can't calculate" (V's words), or even generate an estimate. This gives the reader some insight into what may appear to be 'wild' suggestions made by students V, R, and A and their apparent inability to predict accurately.
build a pyramid with twenty-nine levels as ‘16 (difference of number of levels between the thirteenth and twenty-ninth level) x 2 (both sides of pyramid) = 32 -1 (taking away the extra ‘top cube’) = 31’. Unfortunately, the correct answer should have been fifty-seven. What the students did not realize is that if you are adding on extra cubes to create another level in this pyramid, you do not need to subtract one in the calculation, but you do need to add on the previous total; subtracting one would only work if you first multiplied the height of the pyramid by two, and then subtracted one from the product. The group should have used either ‘(16 x 2) + 25 = 57’ or ‘(29 x 2) - 1 =57’ to solve for these two problems.

The next afternoon, the students returned and continued working on their line pyramids. R finally finished building the seven-level 3-D model (see Figure 27), and then placed it against student V’s 2-D diagram on the chart paper. V’s diagram showed the front view of the seventh pyramid with squares drawn to represent the multi-link cubes. By putting the pyramid on top of the diagram, student R explained that this demonstrated how the 3-D structure and 2-D picture were similar representations. Then, in order to make the diagram for the eighth pyramid, student V simply extended the image of the seventh pyramid by drawing on more squares to create another outer layer of steps and marked the squares of the seventh pyramid with a cross (see Figure 28). By marking the seven-level pyramid, it was clear how many more cubes would be needed to build the eighth layer, and as well, the image showed the structure of the eighth pyramid emerging from the seventh pyramid. As the children continued to work, R and A joined in counting aloud by ones with student V as she recorded numbers for each of the squares in the outer-eighth-level of the diagram to illustrate that there were fifteen ‘cubes’. It was here that student V used a different counting strategy in her numbering of the squares from that which she used when she mentally visualized the outer layers of the pyramid.
Figure 27. Digital photograph of the seven-level 3-D line pyramid constructed by student R.

Figure 28. Student V’s 2-D diagram which shows the front view of the seventh and eighth line pyramids.

R completed his 3-D model of the eighth pyramid (see Figure 29) and the three students checked his work by counting aloud and finding that the model had fifteen ‘steps’ in its outermost layer. V then finished her third drawing which showed the thirteen-level pyramid as a separate image and had the seven-level pyramid shaded in (see Figure 30). R began the construction of the thirteenth-level pyramid by making
Figure 29. Digital photograph of student R's 3-D cube model of the eighth line pyramid.

Figure 30. Student V's 2-D diagram which shows the front view of the thirteenth line pyramid.

a rod of thirteen cubes for its base. He then attempted to attach the rest of the pyramid’s cubes as vertical columns to the base. Meanwhile, student A had drawn a 2-D front view of the thirteenth pyramid and started to shade in the seven-level pyramid when he became frustrated because the scale of his drawing was too small. After several attempts to redraw the diagram, student A abandoned his work and went
to help R with his building. Soon, another problem occurred. Because R had only used thirteen (and not twenty-five) cubes to make the pyramid's base, the two students ran out of cubes to which to attach the vertical columns. Student A paused, added on twelve more cubes to the base, and he and R completed the pyramid (see Figure 31).

![Figure 31. Digital photograph of student R and A’s 3-D cube model of the thirteen-level line pyramid.](image)

Meanwhile, student V was encountering another dilemma—a discrepancy between the 2-D image of the thirteenth pyramid and the group's numerical answer for the thirteenth level. Facing the chart paper, student V counted and recounted the number of squares in the outer layer of the thirteen-level pyramid diagram, each time ending with a total of twenty-five and not twenty-four (i.e., the group's original answer for this problem). After making several attempts by herself and with the other two students to try and make the answer come out as twenty-four, the group realized the correct solution had to be twenty-five. The session ended with the children revisiting the last problem on the task sheet and working together to create a new set of calculations to solve for the twenty-ninth level. Students A and R monitored student V as she worked through a series of written calculations that was organized into a chart or table format, and this time, the group came up with the correct solution of fifty-seven (see Figure 32).
V: Thirteen plus sixteen [the number of layers to be added on] equals twenty-nine. Twenty-five [the number of cubes needed to build the thirteenth-level] plus... twenty-five plus... [not sure what value is needed to add on to the previous total] One, two, three, four, five, ... sixteen groups of two [counting the number of '+2's that have been written down on the paper plus the twos for the twenty-fifth, twenty-sixth, twenty-seventh, twenty-eighth, and twenty-ninth pyramid levels] [the '+2's represent the addition of two cubes as being added on to the pyramid for each of the sixteen levels]. Twenty-five plus thirty-two ["thirty-two" being the sixteen groups of two, $16 \times 2 = 32$] equals fifty-seven.

Figure 32. Student V's table of written calculations for the group's solution for the twenty-ninth pyramid problem.

**Group Project #5: Half-Line Pyramids**

For their fifth project, I gave the students the same activity sheet as in the last session but this time, I asked them to create a pyramid that was more complicated than the line pyramid. The children engaged in a conversation and came up with two possible structures—a "half-cross" pyramid and a "half-line" pyramid. They made the decision to build half-line pyramids and quickly organized themselves in terms of who was going to build what pyramid(s) (i.e., the first, second, third, fourth, fifth, and sixth). All three children initially used the same building strategy. They began by attaching cubes together to form a vertical rod that would be the highest column in each of their pyramids. Thus, the number of cubes in these columns also identified the number of levels in each of their pyramids. Students V and A attached the rest of the cubes that

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29 From V, R, and A's talk, 'more complicated' for them, would be a pyramid that was not symmetrical in structure.
were needed to complete their 3-D models. During this part of the session, student R went on to experiment with other methods of building such as stacking and attaching horizontal rows of cubes one on top of another, as well as attaching vertical rods side by side to build his pyramids. When the group had completed the first six models of half-line pyramids (see Figure 33), V arranged the pyramids in the same manner as student A had done with the line pyramids—from smallest to largest, and then the two students proceeded to make observations about the 3-D structures.

V: If three had four more cubes on the outer edge of the pyramid than two, and this one has five more than [points to the four-level pyramid],

A: This one has six more [places the sixth pyramid behind the fifth pyramid].

V checked A's claim by counting the number of 'visible' cubes located in the outermost layers of all six pyramids, and confirmed the number pattern as being accurate.

V: So if the difference was two for two levels, and then... just keep adding one.

A: You do?

V: Yeah, because just look...

R: [turns and watches V and A]

V: If two has two more than one [points to the two 'more' cubes in second pyramid's outer layer and then points to the one cube in the first pyramid's outer layer],

three has three more than two [pointing to the third pyramid and then to the second pyramid],

four has four more than three [pointing to the fourth pyramid and then to the third pyramid],

four is one less than five [points to the fourth pyramid and then to the fifth pyramid],

five is one less than six [points to the fifth pyramid and then points to the sixth pyramid]...

The group agreed that based on their conclusions, the level of a pyramid also indicated the number of additional cubes that would be needed to build it. They then moved quickly through the remaining tasks and recorded their guesses for the seventh, eighth, and thirteenth levels as being seven, eight, and thirteen more cubes.
Figure 33. Digital photographs of the group's first, second, third, fourth, fifth and sixth 3-D cube models of half-line pyramids.
V, R, and A tested and found their guesses to be correct after building the seventh and eighth pyramids (see Figure 34) and drawing a series of two-dimensional diagrams for all three structures. The separate diagrams that V and A drew both showed the front views of the three pyramids, but the ways in which they drew them, produced different representations. Student A drew his diagrams on a piece of dot paper that illustrated the sixth, seventh, and eighth pyramid as one image and coded the squares to identify the three pyramids (e.g., "*" for the sixth pyramid, "•" for the seventh pyramid, and "▱" for the eighth pyramid). He did not make a diagram for the ninth pyramid, and went on to draw the thirteenth pyramid as a separate image on the same piece of dot paper (see Figure 35). Looking over at A's work, V complimented him on his system of coding, and she added these symbols to her pyramids. Here, V drew the sixth pyramid first, then extended the diagram to show the seventh pyramid by drawing on a seventh outer layer (see Figure 36). In another space on the same piece of chart paper, she then made a second diagram which showed the eighth pyramid in the centre, and the twelfth and thirteenth pyramids extending downwards from it.

Figure 34. Digital photographs of the group's 3-D cube models of the seventh and eighth half-line pyramids.
Figure 35. Student A’s 2-D ‘front-view’ diagrams of the sixth, seventh, eighth, and thirteenth half-line pyramids.

Figure 36. Student V’s 2-D ‘front-view’ diagrams of the sixth, seventh, eighth, twelfth, and thirteenth half-line pyramids.
The group then concluded that because their diagrams and models matched the answers that they had recorded on the activity sheet, their solutions had been “proved” (R’s word) correct. At the end of this session, student A completed the 3-D model of the thirteenth pyramid (see Figure 37) and the group provided the following explanation for why they believed the solution to the last problem regarding the twenty-ninth pyramid would be an additional twenty-nine cubes.

V: We all guessed...

V, R, and A: Twenty-nine!

V: Because with number two [pyramid] there were two [cubes] more, with number three there were three more, with number four there were four more, five-five, six-six, ...

R: And so on!

V: Seven-seven, eight-eight, nine-nine, ten-ten, and on and on.

Figure 37. Digital photograph of student A’s 3-D cube model of a thirteen-level half-line pyramid.
Group Project #6: Half-Cross Pyramids

For the sixth and final challenge, the children were given the same activity sheet as in the previous two sessions, only this time, they were asked to base their investigations on a pyramid that they had mentioned earlier but never built—the half-cross pyramid. Once V, R, and A decided who was going to build which pyramid(s), they used the multi-link cubes to make the first six models (see Figures 38 and 39). All three students used the same method. They first connected the cubes together as a vertical rod for the pyramid's height, then attached three horizontal rods for its base, and completed each of the pyramids by attaching the rest of the cubes onto the frames.

Student R then wanted to build the seventh pyramid but was told by the others that he needed to first make a guess. So, he predicted that the seven-level pyramid would require thirteen more cubes than the six-level pyramid, and proceeded to build the 3-D model in a horizontal manner, starting from the bottom and moving towards the top. V and A however, took more time to study the 3-D structures before making their guess.

V: With one there's one cube [A watches V, R continues to build but is also watching V].
   On the two-level one there's four extra cubes [points to the four cubes that make up the base].
   On three-level one ... one, two, three, four, five, six, seven [points and counts the number of steps on the outside of the third pyramid].
   So we're guessing ten right? [getting the four-level pyramid, she begins counting its total number of steps].
   One, two, three, four, five, six, seven, eight, nine, ten.
A: So you always add three.
R: [with his finger, counts the steps in the fourth pyramid again and also gets ten].
V: Yeah.
R: Add three [goes back to building the seventh pyramid].
V and A: Nineteen [10 + (3 levels of 3 cubes each)].
Figure 38. Digital photographs of the first, second, and third 3-D cube models of the half-cross pyramids that were built by students A, R, and V.
Figure 39. Digital photographs of the fourth, fifth, and sixth 3-D cube models of the half-cross pyramids that were built by students A, R, and V.
Just then, R stopped building. In front of him was a 3-D model that resembled a six-level half-cross pyramid. To find out what the actual solution for the seventh level was, he began attaching one cube at a time around the outside of this pyramid, and kept track of each addition by counting by ones. V chose to use the dot paper and drew the 2-D bottom view of the seventh pyramid. Student A also decided to draw a 2-D diagram of the seventh pyramid's bottom layer on the chart paper. A and V counted up the number of squares in their pictures, and both arrived at nineteen (see Figure 40).

Without waiting for student R to finish building, V took the sixth pyramid and student A watched her as she counted the number of steps on the outside of the pyramid by placing her finger on the top of the 3-D model and moving down one step each time toward the bottom. She counted the total number of steps as three sets of stairs—from one to six, seven to eleven, and twelve to sixteen. V stopped, announced “plus three”, and she and A both said “nineteen”. Meanwhile, R had been busy adding on cubes to create the seventh outer layer of the pyramid and keeping track of his additions by counting aloud. As he attached the last cube to the pyramid, he completed building the seventh layer of the pyramid (see Figure 41), and concluded that his actions had resulted in a total addition of nineteen cubes. R double checked and confirmed his
work as being correct by finger-counting the number of steps on the outside of the pyramid as "nineteen" in a similar fashion to that of V on the sixth pyramid (see Figure 42). Student V performed another check but this time counted the number of cubes in the bottom layer of the 3-D model and also got nineteen cubes (See Figure 40).

Based on what they knew regarding the pattern of ‘+3’, V predicted that the next pyramid would require twenty-two more cubes, and R and A made a guess of twenty-one cubes. Student V confessed that they were not really guessing at this point, because they already knew what the answer would be! Students V and A each made separate diagrams for the eighth pyramid, used the same method as in their drawing of the seventh pyramid, and labelled the squares to represent the twenty-two cubes that would make up the bottom layer. R began to build a 3-D model with the cubes but then decided not to continue. Instead, he reached for the seven-level model and applied his previous double-checking strategy to solve this problem. By pointing with his finger as he added on the ‘imaginary’ outer layer, and counted by ones after each addition, he was able to arrive at twenty-two cubes. For the thirteenth level, A made a
Figure 42. Digital photograph and R's counting strategy for double checking the number of cubes in the seventh outer-layer of the 3-D cube model of the cross pyramid as being nineteen cubes.

guess of forty, R guessed thirty-two, and V guessed thirty-eight as being the number of additional cubes that would be needed to build this pyramid. Instead of building or drawing diagrams, the group worked quickly through a set of oral and written calculations, initiated by student V.

V: Eight [level pyramid] was twenty-two, and what's the difference between eight and thirteen? Equals five, right? Okay, so five [levels] times three [cubes each for every level] equals fifteen. So fifteen and twenty-two equals, hold on, wait...
V recorded the following calculation onto a piece of paper, and the other two students watched and monitored V as she produced the correct answer of thirty-seven cubes.

15 [from 5x3, '5' being the number of levels that are added in order to build the thirteenth pyramid and then multiplied by three cubes for each level to get fifteen]

+22 [number of additional cubes required to build the eighth level]

37 [total number of additional number of cubes needed for the thirteenth level]

They continued to use this method and solved for the last problem--how many more cubes would be needed to build a pyramid that was twenty-nine levels high.

V: What’s the difference between thirteen and twenty-nine?
A: Sixteen.
V: [double checks this by recording it onto paper and vertically subtracting thirteen from twenty-nine].

Sixteen time three equals...

V recorded ‘16 x 3’ in a vertical fashion on the paper and then multiplied it by using the correct algorithm to get forty-eight. R and A watched her as she performed the arithmetic and upon completion, both nodded their heads in agreement. The group recorded forty-eight as their final solution and student R excused himself from the room for a few minutes. While he was gone, V and A started talking. Their conversation led them into reviewing their answer for the twenty-ninth pyramid, at which time student V began to question whether or not they had solved the problem correctly. The session finished with V and A working through this uncertainty.

A: It’s sixteen more.
V: [counting with her fingers from fourteen to twenty-nine] Sixteen.

V and A: So sixteen times three equals [V records this onto paper]

V: How do we do this right?

A: Okay, six times three equals... [using the algorithm for multiplication]

V and A: Eighteen...

V: Carry over one... [trading the group of ten ones in for a group of ten]

V and A: Three... four... [multiplying the digits in the tens column, then adding the other ten to get four tens]

A: Forty-eight.

R: [leaves the room].

A: Now...
V: I don't know if it's correct or incorrect [referring to their solution of 48]...
There's only like eleven difference between thirteen [levels] and twenty-nine [levels].

A: Is it?
V: Yeah.
A: No, it's sixteen.
V: No, I mean look, between thirty-seven and forty-eight.

V specified that she was looking at the difference in their answers for the thirteenth and twenty-ninth pyramids, not the difference of levels between the two pyramids.

V: It's only like... it's only eleven difference.
A: [starts counting on his fingers by ones from thirty-seven to forty-eight]
Thirty-eight, thirty-nine, ... forty-eight. [shrugs his shoulders and looks at student V]
Eleven difference.

V: And between thirteen and twenty-nine, there's sixteen difference [the difference of levels between the two pyramids].
It seems kind of odd [says quietly].

A: Use the calculator.

V: [looks down at the recorded answers on the activity sheet]
Oh! We forgot to add! [pointing to the answer of '37' on the sheet]
Forty-eight to thirty-seven.

48 +37
85

A: So... [records '48 + 37' vertically on paper]
fourty-eight plus thirty-seven equals...

R: [enters the room] What are you guys doing?[joins the group and watches as student V and A finish the calculation]

48 +37 85

V: Five, carry over one equals eight [talks as student A is doing the written calculation of adding up the numbers in the ones' column, regrouping and trading it in for a ten, and then adds the group of tens in the tens' column to get eight].

A: Equals eighty-five!
V: The answer is eighty-five. Now we're sure.
Looking at The Group's Path of Mathematical Understandings

The Interrelatedness of Spatial and Numerical Understandings

Spatial structuring is explained by Battista and Clements (1996) as being "the mental act of constructing an organization or form for an object or set of objects" (p. 282). This spatial understanding is believed to develop from the individual's physical and metacognitive experiences (Battista, 1994; Cobb, Yackel, & Wood, 1992; Hirstein, 1981; Piaget & Inhelder, 1967; von Glasersfeld, 1982, 1991). For example, when children engage in spatial tasks that require their use of concrete objects to count, arrange, or build, these actions in turn, develop mental capacities necessary for the formation of conceptual relationships and their ability to visualize 3-D objects. The development of this ability to visualize was very clear in the workings of V, R, and A. Battista and Clements (1996) assert that it is this type of conceptualization which enables children to calculate 3-D structures such as rectangular cube arrays, without having to physically count the individual units.

I contend, however, that counting should never be discarded or prohibited as a means of testing or checking—nor should it serve as an indication that children are at a less sophisticated level of mathematical understanding. V for example, moved between generalized rules and physical counting with fluidity and this was key for reflection, reintegration, and the co-evolution of new mathematical understandings with previous knowings. Thus, V's physical counting actions were combined with a flexibility to move between different modes or mathematical actions and revealed mathematical sophistication and not its converse. Battista and Clements do see the ways in which children make sense of their counting actions as being indicative of their level of spatial structuring, but I feel that simplicity of mathematical action is not a clear indicator of what these authors consider to be low level spatial structuring. R's building and counting may have appeared to be at first sight a low level mathematical
knowing, doing, and way of being, but, when probing deeper into how he was building, very sophisticated spatial-numerical structuring can be observed as taking place. This contention is further supported in the following sections.

**The Co-Emergence of Spatial-Mathematical Understandings with the Inter-Activity of Individual and Collective Knowings**

As a result of viewing the videos and exercising a sensitivity towards the subtleties in the children's mathematical ways of knowing, doing, and being, new ways of regarding the data began to emerge for me. These groupings were related to the three ways in which the students displayed their spatial structuring. By applying the method of "open coding" (Strauss & Corbin, 1990) I was able to identify the three conceptualizations as being the group's perceptions of the six pyramids being organized into: (i) *rods*, (ii) *horizontal layers*, and/or (iii) *outer layers*. These cut across the original categorizations of building, drawing, and numbering, and the manner in which these conceptual structures underpinned the group's mathematical path could not be understood by piecing together episodes and relating them to any (one) particular pyramid(s) or by sorting them into a sequence of events\(^3\). Rather, in order for me to develop an understanding for the interrelationships which were inextricably embedded in the children's individual and collective mathematical actions, I needed to examine the ways in which the students' conceptualizations of rods and layers emerged and reemerged throughout the course of the six sessions. By doing so, I was able to locate the group's spatial-mathematical organizations as co-evolving with the inter-activity of the children. In the next three sections, I explain how this co-emergence of V, R, and A's understandings and their reenactments of the three spatial structures became the group's mathematical ways of knowing, doing, and being through the six problem solving sessions.

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\(^{3}\) As a result of this, I did not go on to do any "axial" or "sequential" coding (Strauss & Corbin, 1990).
The Children's Structuring of the 3-D Pyramids as Rods

The strategy of building the 3-D pyramids by first connecting cubes into rods and then using this structure to attach the remaining cubes, was initiated by student R during his construction of the third corner pyramid. By standing the rod of three cubes up, he had efficiently defined the number of levels that needed to be built and was able to fill in horizontal layers without having to monitor the height of the pyramid. R's method became more elaborate when students V and A worked together to build the seventh corner pyramid. Here, the two students attached two horizontal rods of six cubes each onto two adjacent sides of the vertical rod's bottom cube. This structure looked like a corner and became the 'skeleton' on which V and A could quickly attach the rest of the pyramid's cubes. Their addition of the two horizontal rods is seen to be connected to V and A's previous work in which they had drawn 2-D diagrams of the seventh, eighth, and thirteenth corner pyramids' bottom layers, depicting the seventh pyramid as having two straight edges of six cubes each and being attached to two adjacent sides of the 'corner' cube. Moreover, it was while V and A were doing this that student R had also made his building actions more elaborate by attaching a vertical rod of seven cubes onto the corner cube of a base that was a 3-D version of V and A's 2-D diagram of the eighth corner pyramid's bottom layer.

R's first building strategy reemerged in its original form when all three students went to work to put together the first six 3-D models of the half-line pyramids. Only taking time to delegate which of the six pyramids each student was responsible for building, the children then proceeded to define their particular pyramids by building the highest vertical rod first and filling in the rest of the cubes in order to complete each of their 3-D structures. The group also integrated the 'skeleton' procedure a second time to build their 3-D models of the half-cross pyramids. To do this, they built the frame of the pyramid by making a vertical rod of cubes with the same number of cubes as the number of levels in the pyramid, and then attached to its bottom cube, three
horizontal rods that were one cube less than the first rod's height. From here, the children were able to complete each of their pyramids by simply adding on the rest of the cubes.

Curiously, the students never engaged in conversations regarding how or why they were building the cube pyramids in these ways. Yet somehow they intuitively knew when to make use of a previous strategy, and when they needed to reintegrate a particular method of building in order to make it fit the task at hand. The absence of such explanations not only supports the view that no discussions were deemed necessary by A, R, and V, but also that the group's mathematical knowings had resulted from a co-emergence which took place within their individual, pair, and group actions. The next two interpretations illuminate the children's enactions of the pyramids' horizontal and outer layers as also being dynamically complex, but at the same time, reveal how the group's individual and collective mathematical ways of knowing, doing and being enabled their devising of solutions for all six of the pyramid problems.

**The Children's Structuring of the 3-D Pyramids as Horizontal Layers**

The ways in which the children made use of their understanding of the 3-D cube pyramids as being organized into horizontal layers or levels is seen to have emanated primarily from their attention to the bottom layers or bases of the pyramids. What came out of the group's investigations were drawings, counting patterns, ways of building and arranging 3-D models, as well as numerical representations—all of which were then used by the group to generate strategies for solving the mathematical tasks.

For example, the co-evolution that took place regarding the group's spatial structuring of horizontal layers and their understanding of squared numbers can be found by going back to the first problem solving session and examining how V and A's study of the rectangular pyramid's base gave rise to this notion of squared numbers. It was through their separate but identical actions of finger pointing and counting of the
horizontal layer's width as five cubes and its length as five cubes, silently multiplying ‘5 x 5’, and quickly saying “twenty-five” that V and A identified the bottom level of the third pyramid as being a 5 x 5 cube array. Their observation emerged into an anticipated pattern of consecutive odd squared numbers when V found the second rectangular pyramid’s base to be a 3 x 3 cube array, compared it to the third pyramid’s base which was a 5 x 5 cube array, and predicted that because the bottom layers were increasing by two cubes widthwise and by two cubes lengthwise, the next level would be a 7 x 7 cube array. From here the pattern was employed as the spatial structure which V and A enacted through their building of the rectangular pyramid’s fourth layer by constructing seven rods of seven cubes each. And when the students discovered that R’s 3-D model of the fourth rectangular pyramid had a bottom layer which was 7 x 7 cubes, this confirmed the pattern as being accurate and V, R, and A’s understandings consolidated into a spatial-numerical knowing for the group. This new, collective knowing was embraced by the students with exuberance as they chanted in unison, “nine times nine”, “eleven times eleven”, and “thirteen times thirteen” and solved for the last three rectangular pyramids.

As the group laid down a path of understandings which included mathematical moments of shared eloquence, they also found themselves from time to time, moving in opposite directions—even when all three students were focusing on the bottom layer! The quandaries of the cross pyramid that the group encountered because of the emergence of two complimentary visualizations for the base of the pyramid, proved to be critical tests of the three children’s flexibility in making collective-mathematical sense out of the situations as they unfolded. Similarly, my job as the interpreter was equally confounding! For instance, in the beginning of this session, student R’s understanding of the cubes as being arranged in groups of four cubes surrounding the middle cube appeared to be making sense to student A when he joined in R’s activity of counting the cubes during the first three tasks. Likewise, it seemed to me that V was
also using this spatial organization as she watched R’s counting of the cubes on two separate occasions and nodded her head both times in agreement. But when the group moved on to the problem involving the fourth pyramid, it was clear from student V’s numerical representation of ‘4 x 3 + 1 = 13’ and her finger-pointing to the four rods of three cubes each, that she was not using the same spatial structure as R and A but rather, an opposite or complimentary structure of **four rods of cubes attached around the middle cube**.

From the following episodes in which R and A agreed with student V’s mathematical expressions for the fourth and fifth pyramids I began to wonder if the two students were now using the same spatial structure as student V. Only after I had repeatedly watched and connected the two episodes with the events that took place before and after them, did I realize that this was not the case. One reason for my conclusion was that even though A and R acknowledged V’s calculation as being a valid solution for the task, it could only be seen as their mathematical coordination of V’s knowing, and did not necessarily signify that the two students were working with the same conceptualization. Secondly, regardless of whether or not the children were visualizing the fifth cross pyramid’s bottom layer as an arrangement of ‘four rods of four cubes attached to the middle cube’ or ‘four groups of four cubes surrounding the middle cube’, the mathematical result of ‘4 x 4 + 1 = 17’ fit both views and therefore the group’s collective structuring remained uncertain to me. Thirdly, because the children’s verbal attempts to clarify what was ‘five’ and what was ‘four’ about the cross pyramid were not successful and resulted in the surfacing of three different views, it was clear to me that the group at this point, had only achieved mathematical coordination of their perceptions and not the coupling of one spatial image.

These variations in the students’ spatial-mathematical knowings triggered perturbations within the group which necessitated the group’s creation of a collective understanding of this problem. Thus, from a tangled clutter of spatial knowings to the
group's drawing of 2-D diagrams for all of the bottom layers of the cross pyramids, the students began using R and A's spatial structure, defining the cube arrangements as being the increasing number of groups of four cubes around a middle cube, and systematically solving the mathematical problems as a collective unity again. The written mathematical equations which were initiated by student V, revealed her spatial understanding as co-emerging with R and A's visualized pattern of the pyramid's growth. Moreover, when looking at the mathematical calculations the group had generated from the drawings (e.g., ‘1 x 4 + 1 = 5’ for the two-level pyramid, ‘2 x 4 + 1 = 9’ for the three-level pyramid, and ‘3 x 4 + 1 = 13’ for the four-level pyramid) and combining these with the fact that the children were soon verbalizing the calculations before they had actually drawn the images, I infer that the group was ready to move toward a formula which would let them solve for any given level of a cross pyramid. An example of such a formula would be: Let \( n = \text{the number of levels in a cross pyramid} \) and \( x = \text{the number of additional cubes needed to build the nth level} \), then \( 4(n - 1) + 1 = x \). The students did not perceive the necessity to do this, however, and because there was no teacher present, they were not provoked to do so.

The mathematical growth attained by the group regarding their ability to specify and use one spatial structure in the cross pyramids and to represent these images as mathematical calculations, proved to be important understandings which were carried forth by the children as they systematically worked through the corner and half-cross pyramids. These understandings were demonstrated in their mathematical ways of knowing and doing. Student R’s use of the 3-D three-level corner pyramid and turning it upside down in order to either enumerate the total number of cubes in its base or count on the additional cubes in a given layer coalesced into a collective understanding when student V combined R’s suggestion with her and A’s numerical pattern of ‘add two, add three, add four, then add five’. Working with this understanding, the group was able to solve for the rest of the corner pyramid problems
together, by using the calculator to keep a running total of their oral calculations, and by adding on the next number to the previous subtotal. As well, by making 2-D drawings of the bottom layers for the seventh, eighth, and thirteenth corner pyramids, and again for the seventh and eighth half-cross pyramids, the group was able to create visual representations for the number of cubes that would be needed for the particular pyramids.

**The Children's Structuring of the 3-D Pyramids as Outer Layers**

Although the group's spatial structuring in the first problem solving session focused mainly on their understanding of horizontal layers--one cube array stacked on top of another, it was during this same session that their conceptualizing of *outer layers* also emerged. Had R not persevered with his alternative method of building a separate fourth pyramid and had A not continually watched student R as he attached the cubes, the group may not have come to the realization that the rectangular pyramid's pattern of growth could also be achieved by attaching cubes "around" (A's word) the 3-D model and creating an outer layer. Subsequently, the group’s visualization of outer layers established a third form of spatial knowing and their constant re-enactments of it gave rise to several numbering, building, and drawing strategies.

The reemergence and reintegration of this spatial structure began with R's attempt to find out how many more cubes would be needed to build the third corner pyramid by the counting on of *imaginary* cubes around the two-level pyramid. Student V made a similar enaction when she tried counting on imaginary cubes as a series of *steps* to the front of the same corner pyramid. These attempts proved to be unsuccessful. It was not until three sessions later that the group's oral counting of the outside steps of the 3-D half-cross pyramids became their collective action. This time, the children’s knowing of outer layers proved effective in identifying the pattern as being the successive addition of three cubes. Moreover, R's progression from
physically constructing the seventh pyramid and keeping count as he attached nineteen cubes onto the outside of the sixth pyramid to counting on an eighth outer layer of twenty-two imaginary cubes illustrates how the group’s understanding gave direction to the development of his individual knowing. Together, these examples highlight that the constant back and forth movement which was present in the children’s building and counting actions was necessary for the co-evolution of group understandings and their subsequent use.

My act of delving even deeper into the complexity of the group's mathematical path brought into focus, the ways in which V and A’s actions of physically arranging the 3-D models allowed the group to elaborate on their structuring of outer layers and generate arithmetic patterns from these understandings. For instance, Student V’s placement of the 3-D three-level corner pyramid behind the 3-D four-level corner pyramid specifically defined the outer layers as “the ones that you can see” (V’s words). From here, student A’s sorting of the 3-D line pyramids from smallest to largest clearly demonstrated for the group that each increase in the pyramid’s levels could be conceptualized as the visible ‘step-like’ outer layers. This form of pattern visualization gave rise to V’s ‘doubling the pyramid’s height and subtracting one’ calculation—a method which determined the total number of additional cubes necessary for building any given level of the pyramid. Although this was the most efficient method devised for solving the line pyramids problems, it was not the procedure that the group ended up using. Rather, it was because the children had collectively made sense of the line pyramid’s pattern of growth as being an increase of two cubes, that this particular knowing was mirrored in their subsequent application of ‘+2’ as an oral counting strategy. Following this, it was the group’s 2-D front view drawings of the pyramids which enabled V, R, and A to visualize the pattern of growth, identify their error in counting of the thirteenth pyramid, and gain a collective understanding for why twenty-
five cubes and not twenty-four cubes would be required. This spatial-numerical structuring in turn helped the children to keep track of and make certain that V’s written calculations of the pyramid’s twenty-ninth level fit with what the students now knew regarding the line pyramid’s pattern of growth in its outer layers.

The spatial-numerical understandings that were laid down as a result of the group’s problem solving of the line pyramids had now become mathematical ways of knowing for the children—knowings which enabled their collective and fluid execution of the half-line pyramid problems. Here, the group’s method of sorting the 3-D models by size not only revealed to V, R, and A, the pyramid’s pattern of growth as an increase of one cube in its outer layer, but also gave rise to their understanding that the number of levels in any given line pyramid indicated the number of cubes that would be needed to build its outermost level. This became a second spatial-numerical knowing which eliminated the need for the children to guess the number of cubes which would be needed to build the seventh, eighth, and thirteenth levels for they already knew what the results would be. Furthermore, the group knowing also gave the children a clear and simple way to explain why the twenty-ninth level would require twenty-nine cubes.

Finally, it was the group’s interactions during the last session which confirmed for me that R’s previous individual knowing of outer layers had undoubtedly evolved into sophisticated and collective ways of mathematical knowing, doing, and being for the entire group. Through a series of choreographed mathematical movements by A, R, and V, they brought forth the oral and written calculation previously used in solving for the line pyramids by correctly changing the values in the calculations to fit with the pattern of growth of the half-cross pyramid and solved the seventh, eighth, and thirteenth pyramid problems. When the group moved on to the twenty-ninth pyramid and neglected to add the previous total of thirty-seven to what they had already calculated, it was the spatial-numerical knowings which the group had discovered and
found to be 'true' regarding the patterns of growth in the outer layers which caused V to be suspicious of the answer.

Student V’s descriptions of the solution as being “odd” and having “only a difference of eleven” from the previous total number of cubes indicated that this solution did not fit with what the group knew about the line pyramids: that somehow, the answer should have been a larger value. It was because of this perturbation with her spatial-numerical knowings that V reviewed the group’s recorded answers and was able to spot their error in calculating. This in turn prompted the children to work together through a set of written calculations and correctly answer the problem. Student V’s response, “now we’re sure” brought effective closure to the group’s mathematical journey and punctuated nicely, the co-emergence of V, R, and A’s spatial-numerical understandings.
By tracking V, R, and A’s mathematical footprints as they created a path of group spatial-numerical understandings, I was able to analyze and develop interpretations which shadowed the co-emergence of their mathematics. These emanated from the children’s problem solving ways of doing and their individual and collective knowings of the six pyramids. Working as a collective unity, the children embarked on a problem solving journey that proved to be unpredictable and dynamic in its co-evolution. Furthermore, V, R, and A’s mathematical path displayed their individual, pair, and group actions of building, drawing, and numbering--their forms of mathematical doing --as being inextricably connected to the growth of their spatial-numerical understandings. It is the development of these complex interrelationships that enabled the group’s problem solving actions and their understandings of the pyramids’ rods and layers to coalesce into such fluid individual and collective mathematical knowings. The students’ collaborative, very active, and dynamic mathematical ways of knowing, doing, and being serve to challenge views that assume mathematical sophistication as being the manipulation of purely symbolic, mental representations. Instead, V, R, and A’s activities as problem solvers support the conceptualization of mathematical eloquence as being the ability to move flexibly back and forth through experiential and representational realms of mathematical knowings, constantly integrating and reintegrating individual and collective understandings in order to respond effectively in problem solving situations.

Having explored the spaces within V, R, and A’s mathematical inter-activity, I became curious as to what kind of perspective I might gain by moving further away and re-viewing the group’s mathematical path as a whole, from above. From this vantage point I was able to take in the children’s mathematical journey as an integrated conceptual map. This map allowed me to see the group’s spatial
discoveries of the pyramids' rods and horizontal and outer layers as constituting the mathematical landmarks in their path of understandings. The patterns which linked these landmarks together and formed the mathematical path were the children's integration and carrying forth of the three spatial structures (rods, horizontal layers, and outer layers) through their problem solving enactions of building, drawing, and numbering.

The group's increased flexibility as a problem solving unity and the sophistication that was demonstrated in their mathematical knowing and doing are viewed as resulting from the perturbations or features of the mathematical landscape which the group encountered as their journey unfolded. These perturbations prompted V, R, and A to reflect and revise how they were knowing and solving the mathematical dilemmas. In doing so, these acts gave rise to the twists and turns in the group's mathematical path--as they moved through the landscape of pyramid problems. Thus, the students' quandaries can be taken to be the contours of this map. For example, when student R persisted in adding cubes around the rectangular pyramid, his action provoked the other two students' to shift their understandings of the pyramid's organization and realize 'outer layers' as being another spatial structure. Similarly to the twists and turns which occur in a road because of geographical obstacles, this group's mathematical perturbations necessitated critical structural and reciprocal couplings which then guided the children's problem solving ways of knowing, doing and being in the subsequent sessions. It is such couplings that codetermined the group's mathematical understandings and marked the children's path with locations of new and more sophisticated understandings in terms of the pyramids' structures and patterns of growth. Together, it was the emergence and reintegration of mathematical events and interrelationships which the children established that gave the group's mathematical path its eventual shape.
V, R, and A's path of mathematical understandings taken from a bird's eye view, reveals that the children's individual and collective mathematical ways of *knowing* of the pyramids' spatial-numerical structures, their *doing*, represented by their building, drawing, and numbering activities, and their *being*, as a problem solving unity, was inseparable and co-evolved, simultaneously and dynamically. Therefore, the collective path which the students moved along clearly did not occur in a linear fashion. It could not have been prescribed or anticipated. It was rather, the group's actions of carrying forth and continually making sense and use of their spatial-numerical experiences which provided opportunities for landmarks of increased mathematical understandings to unfold as fluid, highly complex phenomena.
The path of spatial-numerical understandings which the group laid down while on their mathematical journey around the six pyramids proved to be a co-evolution that existed within and was *dynamically shaped by* the complexity of V, R, and A's individual and collective mathematical acts of *knowing, doing, and being*. In keeping with this, my journey of inquiry into the group’s problem solving of the six pyramids, their development of spatial-numerical structures, and their mathematical path as a conceptual map, has also resulted in a path of co-emergent understandings for me as a teacher-researcher of mathematics education and enactivism.

Moving from the specific context of my research project and connecting it to the broader realm of research in mathematics education and enactivism has created even deeper spaces for me to explore children's mathematical *knowing, doing, and being*. My activities as the researcher in this case study required me to develop a sensitivity and flexibility for all aspects of the project. As a result, my movements as a researcher evolved from the different ways I worked with the video data. For example, by viewing the sessions immediately after each one had concluded, I found myself redesigning the nonroutine problems so that the challenges would accommodate the mathematics which were being demonstrated by the children. Moreover, because the methods I used to analyze, interpret, and represent my findings from this case study could not be anticipated prior to conducting the research, I had to adapt existing techniques and devise new ways of conceptualizing my study as I moved through the different stages of my research. As frustrating as this was at times, in order to begin to probe into the complexity of children's mathematical understandings through the lens of enactivism, the experimentation with research strategies and my reintegration of them to fit with specific (video) data was a critical part of this inquiry.
I also made several attempts to create a visual image that would represent the students' mathematical path as a map. I first tried to graph the group's path as a two-dimensional image and then went on to experiment with ways to express the different dimensions of the children's mathematical understandings (their individual knowings, collective knowings, and the mathematical landmarks of their path) by layering and superimposing overhead transparencies one on top of another, all of which proved to be unsuccessful. More important than the disappointments I experienced in doing this, I have realized that just as I could not discard the 'things’ in my house because my mother had told me to, or abandon my curiosities regarding the complexity in children's mathematical growth because constructivist perspectives could not address them, these maps could not be reduced to two-dimensional images to fit a traditional representational format without losing the meanings inherent in them. Instead, I see these visual conceptualizations as needing to be expressed as three-dimensional images in motion, and because of this, they continue to be a perturbation of mine. I console myself with the knowing that this predicament serves as yet another important opportunity and beginning for me as a researcher.

As enactivism itself, is not well understood or defined, the philosophy too is co-evolving with research that seeks to use it as a theoretical framework from which to question enactivist claims to illuminate our understandings. I offer my research as a superficial stone in this road of co-evolution. In any event, the conclusion of my small exploratory journey has opened up vistas of new territory; new beginnings which provoke my curiosities as a teacher-researcher even further in terms of my desire to study the growth of children's spatial-numerical flexibility as it occurs through their problem solving of mathematics and their interactivity as collective unities. As well, I see the need for critical investigations into the issues which were raised in my literature review regarding the future of enactivism within mathematics education. This would necessitate moving deeper into interrogating the ideological structures of
enactivism and looking in detail at what this might entail if the philosophy was situated within the framework of cultural/bio-conservatism. I believe doing so would open up new spaces where the discourse of such topics regarding mathematics as a cultural map, or the moral implications for teachers and students in adopting an ecological framework within mathematics education could take place. All such discourse would of course implicate our understanding of children's mathematical knowings as embodied and co-emergent phenomena.

Once again I return to the place where this story began--sitting in front of the computer, *still thinking*. However, now when I look down at the floor where recorded thoughts, several books, and even more journal articles have become the ground which surrounds me, the clutter no longer overwhelsms me. This clutter represents new meanings of significance for me. The complex understandings that I have gained from my experiences as a teacher and a graduate student could not have surfaced if I had not sought out constructivism's place within the larger framework of liberalism and had not moved deeper into ecology, biology, phenomenology, and enactivism. In other words, it was because of this clutter that I was able to cluster critical issues regarding constructivism and enactivism in ways which made sense to *me*. Consequently, enactivism emerged as the coherent framework and theoretical lens through which I could look and with which I could orient and reorient myself during this journey.

Now as I continue to look into and move within the realm of enactivism, I am carrying forth the belief that clutter *is* a part of *who* I am, and it is this appreciation for complexity in which I choose to live in this world and from which I strive to understand the field of mathematics education. Trying to take in the complexity of everything all at once can be overwhelming! But by my analysis of R, V, and A's working together, I have tried to illustrate the intricacies and yet the wholeness of co-evolving understandings. If educators and researchers who take an enactivist perspective, work co-emergently as collective unities, our efforts to examine specific issues while being
mindful of how they are connected and exist as integral parts within larger systems can foster the co-development and interrogation of such complexity within mathematics education and the philosophy itself. Doing so will undoubtedly open up new and creative spaces of possibilities for the meaningful and mindful development of our deeper understandings in children's mathematical growth and the evolution of enactivism.
References


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Appendix A

(Letter of consent from the University of British Columbia Behavioural Research Ethics Board)

THE UNIVERSITY OF BRITISH COLUMBIA

Office of Research Services

Phone: [Redacted]
Fax: [Redacted]

Our File: [Redacted]

June 14, 1998

Dr. S.E.B. Pirie

Dear Dr. Pirie:

Re: Your proposed study: Seeking a Deeper Understanding of Children's Mathematical Growth Through Problem Solving

The University of British Columbia Behavioural Research Ethics Board has reviewed the protocol for your proposed research project. The Committee found the procedures to be ethically acceptable and a Certificate of Approval will be issued upon the Committee's receipt of written agency approval from the School Board.

If you have any questions, please call me at [Redacted].

Sincerely,

Assistant Director, Ethical Review

Copy: Thom, Jennifer

31 After the completion of this study and through editorial revisions, the title which appears here and on all other consent forms within these appendixes, was subsequently changed to the title of this thesis.
# Appendix B

(Certificate of Approval from the University of British Columbia Behavioural Research Ethics Board)

<table>
<thead>
<tr>
<th>Principal Investigator</th>
<th>Department</th>
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<tr>
<td>Pirie, S.E.B.</td>
<td>Curriculum Studies</td>
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**Institutions Where Research Will Be Carried Out**

Public Schools

**Co-Investigators**

Thom, J.

**Sponsoring Agencies**

Seeking a Deeper Understanding of Children's Mathematical Growth Through Problem Solving

**Approval Date**

AUG 27 1998

**Term (Years)**

3

**Amendment**

- 

**Amendment Approved**

- 

**Certification**

The protocol describing the above-named project has been reviewed by the Committee and the experimental procedures were found to be acceptable on ethical grounds for research involving human subjects.

Approval of the Behavioural Research Ethics Board by one of:

- Associate Chair
- Associate Chair
- Director, Research Services

This Certificate of Approval is valid for the above term provided there is no change in the experimental procedures.
June 30, 1998

Jennifer Thorn

Dear Ms. Thorn,

Thank you for submitting your request to conduct a Research Activity in [School District]. This letter is to inform you that your project has been approved.

Please Note:

The anonymity of schools, teachers and parents who cooperate in your research must be guaranteed as required by Provincial Guidelines.

Your cooperation with the school(s) that you are involved with is of prime importance. Please use this letter as an indication of District Review when introducing yourself to the School Principal.

You are required to produce a copy of your final report as well as highlights of the study so the findings can be shared with the entire district.

All the best with your Research and I trust it will result in a positive Educational Experience for all participants.

Yours truly,

[Name]

District Principal

cc: [Name]
Appendix D

(Sample letter of consent addressed to the school's principal/administrator)

THE UNIVERSITY OF BRITISH COLUMBIA

ADMINISTRATOR INFORMED CONSENT FORM
SEEKING A DEEPER UNDERSTANDING OF CHILDREN'S MATHEMATICAL GROWTH THROUGH PROBLEM SOLVING

Principal Investigator: Dr. Susan E. B. Pirie, Professor and Faculty Advisor, Faculty of Education, University of British Columbia. Telephone: 

Co-Investigator: Jennifer S. Thom, Graduate Student, Faculty of Education, Department of Curriculum Studies, University of British Columbia. Telephone: 

The following research study is for my graduate thesis (Master of Arts Degree in Mathematics Education).

Purpose: I am interested in studying how children's individual knowledge and the knowledge that comes from them working in a group contributes to their understanding of mathematics when given non-routine problems to solve. Non-routine problems are those where which the students do not know, straight away, what mathematics they need to use to solve the problem and so they have to pool their ideas as a group and work together towards a solution. The focus of this study is not on what the students do as individuals, but rather, how they problem solve together and as members of a group to establish mathematical understandings.

Study Procedures: I have collaborated with Dr. Pirie from the University of British Columbia for the past year on a case study regarding students' problem posing abilities, the problem solving strategies they employ when given mathematical problems, and the mathematical understandings that emerge from their workings. I now wish to collect data on how students' individual and collective knowledge shapes their group's mathematical understandings in a problem solving environment.

I hope to observe and video-tape a small group of students as they work through a series of mathematical problems. I shall participate in the introduction of each of the mathematical problems to the group of students and then take video recordings of these students as they work on the problems. These video tapes will be used as the basis for my analysis.

I am very happy to show participants the videos and the only other people at this stage to see the videos will be my supervisor at the University of British Columbia. No one else at your school will see the video tapes. The students will, of course, be free to opt out of the project at any time and such withdrawal will not affect their grades or relationship with the school in any way.

Confidentiality: Any information resulting from this research will be kept strictly confidential. All documents will be identified only by code number and will be kept in a locked filing cabinet. Any report that is written concerning this research will, by giving false names, preserve the complete anonymity of all participants. At the end of the project I shall ask the students and their parents/guardians whether they are willing to let me use pieces of the video tapes as illustrations at conferences, but they do not have to make that decision until they have seen the video tapes. The students and their parents/guardians will also be offered the opportunity to have feedback on the findings of the study.

Page 1 of 1
Contact: If you have any questions or desire further information with respect to this study, you may contact Dr. Susan Pini at or Jennifer Thom at .

If you have any concerns about the treatment or rights regarding your school's participation in the research study, you may contact the Director of Research Services at the University of British Columbia.

Consent: I understand that my school's participation in this study is entirely voluntary and that the school may refuse to participate or withdraw from the study at any time.

Please check the box indicating your decision:

☐ I CONSENT to this school participating in the video-taped sessions as described in this form.

☐ I DO NOT CONSENT to this school participating in the study described in this form.

☐ I acknowledge that I have received a copy of this consent form for my own records.

Name of Administrator (please print) __________________________ Date ________________

Name of school (please print) _________________________________

Signature of Administrator _________________________________

Name of witness (please print) __________________________ Date ________________

Signature of witness ________________________________

Page 2 of 2
THE UNIVERSITY OF BRITISH COLUMBIA

PARENTAL INFORMED CONSENT FORM
SEEKING A DEEPER UNDERSTANDING OF CHILDREN'S MATHEMATICAL GROWTH THROUGH PROBLEM SOLVING

Principal Investigator: Dr. Susan E. B. Pirie, Professor and Faculty Advisor, Faculty of Education, University of British Columbia. Telephone:...

Co-Investigator: Jennifer S. Thom, Graduate Student, Faculty of Education, Department of Curriculum Studies, University of British Columbia. Telephone:... The following research study is for my graduate thesis (Master of Arts Degree in Mathematics Education).

Purpose: I am interested in studying how children's individual knowledge and the knowledge that comes from them working in a group contributes to their understanding of mathematics when given non-routine problems to solve. Non-routine problems are those where the students do not know, straight away, what mathematics they need to use to solve the problem and so they have to pool their ideas as a group and work together towards a solution. The focus of this study is not on what the students do as individuals, but rather, how they problem solve together and as members of a group to establish mathematical understandings.

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I hope to observe and video-tape a small group of students as they work through a series of mathematical problems. I shall participate in the introduction of each of the mathematical problems to the group of students and then take video recordings of these students as they work on the problems. These video tapes will be used as the basis for my analysis.

I am very happy to show participants the videos and the only other people at this stage to see the videos will be my supervisor at the University of British Columbia. No one else at your child's school will see the video tapes. Your child will, of course, be free to opt out of the project at any time and such withdrawal will not affect his/her grades or relationship with the school in any way.

Confidentiality: Any information resulting from this research will be kept strictly confidential. All documents will be identified only by code number and will be kept in a locked filing cabinet. Any report that is written concerning this research will, by giving false names, preserve the complete anonymity of all participants. At the end of the project I shall ask you whether you are willing to let me use pieces of the video tapes as illustrations at conferences, but you do not have to make that decision until you have seen the video tapes. You will also be offered the opportunity to have feedback on the findings of the study.

Page 1 of 1
make that decision until you have seen the video tapes. You will also be offered the opportunity to have feedback on the findings of the study.

Contact: If you have any questions or desire further information with respect to this study, you may contact Dr. Susan Pirie at [contact information] or Jennifer Thom at [contact information]. If you have any concerns about your child's treatment or rights as a research subject you may contact the Director of Research Services at the University of British Columbia, [contact information].

Consent: I understand that my child's participation in this study is entirely voluntary and that I may refuse to allow him/her to participate, or may withdraw him/her from the study at any time without jeopardy to his/her class standing, grades, or relationship with the school.

Please check the box indicating your decision:

☐ I CONSENT to my child's participation in the video-taped sessions as described in this form.

☐ I DO NOT CONSENT to my child's participation in the study described in this form.

☐ I acknowledge that I have received a copy of this consent form for my own records.

Name of parent/guardian (please print) ___________________________ Date ___________________________

Signature of parent/guardian ___________________________

Name of child (please print) ___________________________

Name of witness (please print) ___________________________ Date ___________________________

Signature of witness ___________________________