CONCEPTIONS OF PROBABILITY HELD BY PRESERVICE TEACHERS OF SECONDARY SCHOOL MATHEMATICS

by

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Abstract

The purpose of this study was to explore the qualitatively different conceptions of probability held by two groups of preservice teachers of secondary school mathematics at the University of British Columbia. The study also explored the consistency of participants' conceptions of probability and their views on the utility of formal probability in solving everyday problems.

A set of written tasks, pair-problem-solving tasks, and interview tasks related to probability were given to the participants. A total of 40 preservice teachers participated in the written component, 16 of whom also participated in the pair-problem-solving and the individual interview components.

It was found that the preservice teachers held qualitatively different conceptions of probability. Their conceptions of probability were grouped into formal and non-formal. Formal conceptions of probability included the use of probability concepts such as independence and randomness, and the use of probability formulas, rules, and applications in solving problems. Non-formal conceptions of probability included participants' use of everyday experiences and heuristics as well as the use of science knowledge in solving probability problems. The participants' conceptions of probability varied widely among tasks depending on whether the tasks appeared to be taken from probability textbooks or from an everyday context. Many participants stated that knowledge of formal probability was not useful in solving everyday problems.

Two main conclusions were drawn from the results of this study. First, preservice teachers hold qualitatively different conceptions of probability that largely depend on contexts determined by tasks and settings. Second, students' understanding of probability may be influenced by their non-formal conceptions, and these should be used in teaching formal probability concepts.
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Mathematicians, statisticians, and mathematics educators consider probability and statistics to be valuable for people in understanding their everyday world (Borel, 1962; Dörfler & McLone, 1986; Good, 1983; Jacobsen, 1989; McGervey, 1986). Dewdney (1993) and Paulos (1989, 1994) have argued that people who lack a basic understanding of probability and statistics are often functionally innumerate. Probability and statistics are also considered important for advanced post-secondary studies; for example in the natural and social sciences. Dörfler and McLone (1986) claim that probability and statistics are not only important in many occupations but also relevant "for a general critical understanding of many facts in economics, politics, ecology and other fields" (p. 70). Falk and Konold (1992) argue that "probability will be as important in the 21st century as mastering elementary arithmetic is in the present century" (p. 151).

There is a growing movement to introduce elements of probability at both secondary and elementary school levels in North America (Mathematical Sciences Education Board, 1990; National Council of Teachers of Mathematics, 1980, 1989; Shulte, 1981). For example, the Curriculum and Evaluation Standards for School Mathematics published by the National Council of Teachers of Mathematics (NCTM, 1989) recommends that probability be taught at all grade levels in the United States. Despite this recommendation, probability does not seem to be widely taught in schools in North America (Romberg, 1992; 11t is necessary to mention, however, that they have not recommended replacing arithmetic with probability. The statement emphasizes the role of probability in students' everyday lives.

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1. It is necessary to mention, however, that they have not recommended replacing arithmetic with probability. The statement emphasizes the role of probability in students' everyday lives.

2. In North America, probability is underrepresented in school mathematics curriculum and instruction compared to other statistical topics such as graphs, averages, dispersion, and regression. This study focuses only on issues related to probability, although when researchers have discussed them together it is difficult to separate probability and statistics.
Shaughnessy, 1992). At present, it appears that very little instruction in probability is occurring at the classroom level (Garfield & Ahlgren, 1988; Shaughnessy, 1992).

One suggestion as to why teachers do not teach probability is that many of them do not have adequate preparation for its teaching (Fabbris, 1988; Jacobsen, 1989; Råde, 1985; Steinbring, 1989; Zuliani & Sanna, 1988). According to Steinbring (1989), most mathematics teachers think that mathematics is an objective and hierarchical discipline and so are not equipped to "handle the inexactitude and uncertainty" (p. 203) inherent in the study of probability. Researchers contend that the teaching of probability should be a part of teacher education programs in mathematics so that teachers will be prepared to teach it (Juraschek & Angle, 1981; Råde, 1985; Steinbring, 1989).

Based upon a literature survey, there is a lack of research on the teaching of probability in teacher education programs. There are no studies focusing on exploring teachers' knowledge about how to teach probability, nor are there studies that have explored inservice or preservice teachers' understanding of probability.

Bramald (1994) has provided some evidence that secondary school mathematics preservice teachers find it difficult to understand basic concepts of probability. However, his report is based solely on one snapshot of his classroom teaching and does not constitute systematic inquiry into preservice teachers' knowledge about probability. Preservice teachers' knowledge and understanding of probability have implications for the teaching of probability in schools in the future; it is therefore important to gain insights into preservice teachers' understanding of probability.
Conceptions of probability: What do they mean?

In studies related to students' understanding of particular phenomena, various terms such as concepts, conceptions, and conceptualizations have been used (Linder, 1989; Robertson, 1994a, b). According to Robertson, concepts are publicly held sets of rules or norms about the use of terms. Conceptions are an individual's interpretation of concepts, that is, the ways that individuals make sense of concepts within their individual and social frameworks. Conceptions include the understanding of concepts that may or may not be consistent with that of experts, but may be shared by some other individuals in a certain community. Conceptualizations, however, are "idiosyncratic personal understanding" (Robertson, 1994, p. 26) which are not necessarily shared with the expert community or another social community.

Conceptions include elements of both concepts and conceptualizations. They involve both mathematically correct and incorrect understandings of a topic held by a person. In mathematics education research, Thompson (1992) has identified several elements of conceptions such as "conscious or subconscious beliefs, concepts, meanings, rules, mental images, and preferences" (p. 132) held by students. Since these elements are used in mathematics education, as well as being closely related to the definition used by Robertson, Thompson's definition of conceptions is used in the present study.

In the light of Thompson's definition, conceptions of probability include students' formal knowledge of probabilistic facts, concepts, rules, and theorems as well as students' non-formal beliefs, meanings, intuitions, experiences, understandings, and images related to probability. Conceptions of probability, for example, include how students make sense of probability concepts (such as chance, randomness, and independence) that are usually accepted by the mathematical community. Conceptions of probability may also include students'
idiosyncratic views of everyday chance phenomena such as games, lotteries, accidents, weather, and so on. This study documents both the formal and non-formal conceptions of probability held by preservice teachers.

Why study preservice teachers' conceptions of probability?

Shulman (1986) argues that teachers require subject matter content knowledge, pedagogical content knowledge, and curricular knowledge. Subject matter content knowledge is the knowledge gained by understanding facts, concepts, procedures, and structures of the discipline. This knowledge is gained primarily by studying mathematics content courses in mathematics departments. Pedagogical content knowledge includes "the most regularly taught topics in one's subject area, the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations" (p. 9). The latter also includes "an understanding of what makes the learning of specific topics easy or difficult" (p. 9) for school students and the conceptions of the topic that the students hold. Pedagogical content knowledge is usually gained in mathematics methods courses.

In terms of Shulman's categories, preservice teachers' conceptions of probability are part of subject matter content knowledge and their understanding of students' conceptions of probability are part of pedagogical content knowledge. Both kinds of knowledge interact with each other in the teaching domain. Preservice teachers' conceptions of probability, their understanding of students' conceptions of probability, and the interaction between these two influence how preservice teachers view the teaching of probability. Gaining insights into their conceptions of probability can, therefore, be valuable for educators involved in the preparation of teachers.

Investigating students' conceptions is an important step for educators who believe in a constructivist view of learning. According to this view, students
subsequent mathematics learning and understanding are guided, filtered, and shaped by their existing conceptions of mathematics (Ausubel, Novak, & Hanesian, 1978; Confrey, 1991; Fischbein, 1987). Viewed from this perspective, preservice teachers' beliefs, meanings, rules, mental images, and preferences play a substantial role in their learning of subject matter content knowledge or pedagogical content knowledge about probability. Each preservice teacher constructs and reconstructs a wide range of mathematical understandings as his or her existing conceptions interact with the mathematics taught by a professor. Therefore, mathematics professors and mathematics education professors require insight into how preservice teachers conceptualize probability so that probability learning and the teaching of probability can be made more meaningful.

The importance of studying preservice teachers' conceptions of probability is also supported by researchers at the University of Wisconsin, such as Carpenter, Fennema, and their colleagues (Carpenter & Fennema, 1991; Fennema, Carpenter, Franke, & Carey, 1993; Fennema & Franke, 1992). Their innovative teacher training program is based on "cognitively guided instruction" which is founded on the argument that "instructional decisions should be based on careful analyses of students' knowledge and the goals of instruction" (Carpenter & Fennema, 1991, p. 11). To extrapolate this viewpoint, then, an understanding of how preservice teachers conceptualize probability is useful for mathematics professors in their teaching in the same way that the understanding of school students' conceptions is useful for school teachers. These researchers not only emphasize the importance of gaining insights into students' conceptions but also argue that teachers' conceptions affect how they view teacher education programs and how they teach. Thompson (1984, 1992) and Battista (1994) also support this viewpoint. Hence, preservice teachers' conceptions of probability can provide useful information for mathematics professors as they plan and
design strategies for teaching probability, and for mathematics education professors as they plan and design method courses to help teachers learn to teach probability in schools.

The purpose of the study and the research questions

Shaughnessy (1992) argues that the success of a reform movement in probability education relies on how well teachers are educated to teach this topic. If teachers do not have sufficient background knowledge and understanding of probability, they will not be able to teach it successfully. He therefore calls for research on teachers' conceptions of probability as an important step in investigating the teaching of probability in schools. He suggests that researchers "gather information from teachers at both the pre-service and in-service levels" (p. 489). Despite Shaughnessy's call to gather information about inservice and preservice teachers' understanding of probability, researchers have not given significant attention to this area. The purpose of the present study is to address Shaughnessy's concern and provide a picture of preservice teachers' understanding of probability. Specifically, this study seeks answers to the following questions:

1. What are some qualitatively different ways in which preservice teachers conceptualize probability?
   The aim of this question is to portray the varieties of reasoning used by preservice teachers to solve probability problems. Their reasonings portray how preservice teachers think about or understand probability.

2. In what ways do preservice teachers' conceptions of probability vary across tasks and settings?
   This question has two components. First, in what ways are preservice teachers' conceptions of probability consistent or inconsistent across
various tasks? Second, in what ways do their conceptions vary in individual and pair settings?

3. What are preservice teachers' views about the role of probability learned at the university level in solving everyday problems? This question addresses the issue of whether or not preservice teachers think that the probability courses they took at the university level were helpful in solving everyday problems. The responses obtained from this question will provide a picture of their conceptions of formal probability in coping with everyday problems.

**Theoretical framework of the study**

This study draws upon a constructive perspective that argues individuals construct knowledge based on their personal and social experiences (Cobb, 1989, 1994; Confrey, 1991). Cobb (1994) argues that "mathematical learning should be viewed as both a process of active individual construction and a process of acculturation into the mathematical practices of wider society " (p. 13). Davis, Maher, and Noddings (1990) provided a similar opinion stating that "each learner has a toolkit of conceptions and skills with which he or she must construct knowledge to solve problems presented by the environment" (p. 3). Consistent with the perspectives presented by these researchers, constructivism in this study is viewed in the following ways:

1. An individual constructs knowledge by attempting to make sense of his or her environment and by actively reorganizing his or her experiences (Cobb, Yackel, & Wood, 1991; Confrey, 1991; Resnick, 1983; Simon, 1994; von Glasersfeld, 1984, 1989).

3. Students bring prior experience to instruction, and these experiences should be utilized in teaching (Ausubel, Novak, & Hanesian, 1978; Carpenter & Fennema, 1991; Confrey, 1990a, b, 1991).

Having adopted a constructivist view of learning, this researcher was interested in exploring each individual's conceptions of probability during different phases of data collection and the analysis of the study. As a result, this study has elicited preservice teachers' qualitatively different conceptions of probability.

**Data sources and pilot study**

Data for the study were gathered through written tasks, pair-problem solving, and individual interviews with two separate groups of preservice secondary school mathematics teachers during the 1994 summer and fall mathematics teacher education programs at the University of British Columbia, Canada. The summer group was used as a pilot group for the study. Nine preservice teachers from the pilot group responded to a written task related to probability. Four of the nine participants, exhibiting a range of opinions and experiences, were then selected for the pair-problem solving and the individual interviews. Participants with different opinions and experiences were selected in order to increase interaction between them in the pair-problem solving activities and to explore qualitatively different conceptions of probability. All the pair-problem solving and interview sessions were audiotaped. The tapes were then transcribed and analyzed in conjunction with the written responses. The analysis of the pilot data demonstrated that the instruments and the data collection
procedures were robust and successful in depicting a variety of ways in which preservice teachers conceptualize probability.

The same procedures used for the pilot study were used for the main study. A total of 31 preservice teachers of secondary school mathematics participated in the written component in the fall of 1994. Twelve of the participants participated in the pair-problem solving and the individual interview tasks. As in the pilot study, all the pair-problem solving and interview sessions were audiotaped, and then transcribed and analyzed.

Analysis of participants' responses to written, pair-problem solving, and interview tasks indicated that they held different conceptions of probability. However, some participants in the pilot group held interesting conceptions of probability which differed from those of the main group. For example, a participant in the pilot group provided responses to tasks and settings that were markedly less consistent than those provided by the participants from the main group. Because of this, the pilot group's conceptions of probability were re-analyzed and included with the results of the main group. Since the data obtained from the pilot group were included with the data obtained from the main group, this study represents conceptions of probability held by 16 preservice teachers. A more detailed description of methodology and data sources can be found in Chapter 3.

Organization of the chapters

There are seven chapters in this dissertation. The first three chapters present the theoretical foundations and methodology for the study. The next three chapters present the results related to the three research questions. The final chapter draws the results together in relation to the literature and discusses the study's contribution to probability education both in terms of research and teaching.
CHAPTER 2
Review of Related Literature

The literature on probability can be organized around four themes: the need for teaching this topic in schools (Jacobsen, 1989; Pereira-Mendoza & Swift, 1981), the current status of probability in school mathematics curriculum (Ahlgren & Garfield, 1991; Kerkhofs, 1989; Scheaffer & Burrill, 1988), suggestions for teaching probability (Biehler, 1991; Gnanadesikan, Scheaffer, & Swift, 1987; Higgo, 1993; Holmes, 1988; Kerkhofs, 1989; Newman, Obremski, & Scheaffer, 1987; Schupp, 1989), and the difficulties that students face in understanding probability concepts (Garfield & Ahlgren, 1988; Green, 1983; Hope & Kelly, 1983; Piaget & Inhelder, 1951/1975; Shaughnessy, 1977, 1981, 1992). The present study is related to the fourth theme. Hence, in this chapter, the literature related to students' understanding of probability is reviewed. In the absence of research on preservice teachers' conceptions of probability, most of the review in this chapter is focused on school and college students' conceptions of probability.

Research on students' understanding of probability

Fischbein (1975) and Scholz (1991) provide a historical overview of research on students' learning of probability. Research paradigms discussed by Scholz can be summarized under two headings: behaviorist and cognitive.

Early behaviorist research on probability relied on stimulus-response theory where individuals' actions in uncertain situations were believed to be based on the reinforcement they received. This paradigm equated learning processes with a black box model where inputs and outputs observed in the experimental situation were used to assess people's understanding of probability. Individuals were "perceived as being governed by random processes" (Scholz, 1991, p. 215) rather than by their mental processes. The behavior of a research...
subject was analyzed "according to the standard interpretation of the situation or problem" (Scholz, 1991, p. 241). Individuals' cognitive acts were ignored because they were not observable. People's responses were assumed to be context and culture free. Researchers' judgments were either eliminated or minimized as far as possible. The behaviorist paradigm tried to control both subject and experimenter subjectivities. This paradigm perceived "subject-task relationship" as "ontological objectivism," which assumed that an individual's consciousness can be separated from outside reality (Scholz, 1991, p. 237).

In the cognitive research paradigm, the notion of ontological objectivism and the black box model of learning are rejected. The cognitive research paradigm assumes that students have their own ways of thinking and that their thinking can have a significant impact on their learning of mathematics. In probability, research conducted by Piaget and Inhelder (1951/1975), Fischbein and colleagues (1975, 1984, 1987, 1991), Green (1983, 1988, 1989), Kahneman and Tversky (1972), Tversky and Kahneman (1971, 1973, 1982a, b), Konold and colleagues (1989, 1991, 1993), Scholz (1991), and Shaughnessy (1977, 1981, 1992) can all be categorized as cognitive research although there is a large variation among the approaches taken. Green's studies are basically research surveys conducted with relatively large samples of students. Kahneman and Tversky's research relied on surveys but was conducted with smaller samples of students. Fischbein and his colleagues used both surveys and interviews. Konold and his colleagues conducted interview studies with relatively small samples of students. All these researchers reported students' understanding of probability in various ways.

There are basically two types of research on students' understanding of probability concepts: developmental research on precollege students such as that inspired by Piagetian studies and heuristic research on students, conducted with
both school and college students. Each of these is discussed in greater detail below.

**Developmental research on probability**

Piaget, Fischbein, and Green each conducted studies of children's understanding of probability from a developmental perspective. The research conducted by each of these individuals relating to children's understanding of probability concepts is discussed below.

**Piagetian Studies**

Piaget had a tremendous influence on research about children's thinking. He was one of the earliest to study children's conceptions of a wide range of mathematical topics such as number, space, time, motion, and chance. His research approach was a major breakthrough from the behaviorist research paradigm because of its attention to cognitive processes. Piaget posited that children's cognitive development occurs in stages, starting from the sensory motor stage then moving to the pre-operational stage, the concrete operational stage, and finally into the formal operational stage. Children who solved mathematical problems requiring logical reasoning (typically those at least 11 years of age) were considered by Piaget to be at a formal operational level.

This view, espoused by Piaget, is evident in many of his books including one co-authored with Bärbel Inhelder, *The Origin of the Idea of Chance in Children* (Piaget & Inhelder, 1951/1975). This was the first systematic, qualitative description of aspects of children's probabilistic thinking. The book describes tasks (such as throwing counters, drawing marbles, mixing beads, tossing coins, rolling dice, spinning spinners, and so on) that were used to explore students' conceptions of chance and randomness.
In Piagetian studies, researchers ask children for their ideas about natural events and listen carefully to what the children say (Piaget, 1929). Piaget was interested in exploring the children's notions of reality and causality (Solomon, 1994) with little or no influence from the researcher. The results of "clinical interviews" using these tasks with children of various ages indicated that pre-operational and concrete-operational children could not grasp formal probability concepts; and only by the age of 13 or 14 did children acquire basic probability concepts: for example probability as a ratio (Inhelder, 1976; Piaget & Inhelder, 1951/1975). Piaget and Inhelder found that children predicted certain outcomes on the basis of immediately preceding events and not on the basis of deductive analysis. Children believed that random events could be controlled by "smart" people and they could not distinguish between random and necessary outcomes. In Piaget and Inhelder's (1951/1975) view, the quantification of probability, or chance, and combinatorial thinking occurred only at the formal operational stage. Because of this, Lovell (1971) recommended that probability not be taught in school prior to grade six.

Some researchers have criticized Piaget's stage theory (e.g. Donaldson, 1978; Fischbein, 1975) and his structuralist view of learning (Ernest, 1991; O'Loughlin, 1992; Walkerdine, 1988). However, his theoretical premises that individuals construct knowledge by acting on their environment (Ellerton & Clements, 1992; Vergnaud, 1990) and that different individuals think in different ways (Confrey, 1990b; Vergnaud, 1990), as well as his use of clinical interviewing (Ginsburg, 1981, Ginsburg, Jacobs, & Lopez, 1993; Konold & Johnson, 1991) as a method for investigating children's thinking have become basic elements for many researchers interested in children's conceptions of mathematics and science. In addition, Piagetian studies of children's conceptions of different
topics have provided not only information on how children think but the foundations for constructivist views of learning.

**Fischbein**

Like Piaget, Fischbein was also interested in the development of probabilistic reasoning in children. His main area of interest was how intuition and logical structures can influence children's probabilistic thinking. He criticized Piaget and Inhelder for dealing only with logical and formal aspects of chance and probability and for ignoring "interpretive, explanatory, and problem-solving predispositions generated by the intuitive base" (Fischbein, 1975, p. 6). Fischbein (1987) defined intuition as "immediate, apparently self-evident cognitions" (p. 204). According to him, "Intuitions are always the product of personal experience, of the personal involvement of the individual in a certain practical or theoretical activity" (Fischbein, 1987, p. 213, emphasis in original). In his research, Fischbein concentrated on the relationship between intuitive and formal thinking in children.

Fischbein and his colleagues worked in the area of probability because many probability concepts are either intuitive or counter-intuitive for many students. Fischbein (1975) distinguished between the concepts of chance and intuition of chance. Re-analyzing Piaget and Inhelder's data relating to different experiments on children's conceptualizations of probability, Fischbein claimed that even pre-operational children before the ages of 6 or 7 did have some intuition about chance. As early as 1967, Fischbein, Pampu, and Minzat (1975) conducted experiments with children as young as 6 through 14 years of age to verify this claim. A total of 188 children were interviewed. Five inclined boards with different systems of progressively branching channels were constructed. Out of these five boards the first three were equiprobable and the last two were non-equiprobable. The problem posed to the children was, "I am going to drop a
marble in here (pointing to the top of the main channel)—where will it come out at the bottom?" (Fischbein et al., 1975, p. 162). After the subject's first response he or she was asked, "Will it always come out there, or can it come out in other places?" (p. 162). Another question followed, "If I drop the marble a lot of times, one after the other, will it come out at each place the same number of times or will it come out of some places more often than others?" (p. 162). Children were asked for their reasons for each answer.

Fischbein and his colleagues were surprised to learn that it was the youngest children who gave the greatest number of correct responses for the layouts with equiprobable routes. For non-equiprobable routes, correct responses increased with age. According to Fischbein (1975), for older children "chance was nothing but ambiguity and uncertainty" (p. 73). For younger children ambiguity and uncertainty did not affect the outcomes. Fischbein and his colleagues indicated that school experiences based on a deterministic view of mathematics were not helpful for developing appropriate formal concepts of probability. Their findings questioned the legitimacy of Piagetian stage theory that children's probabilistic conceptions develop only at the formal operational stage.³

In another study, however, Fischbein and Gazit (1984) reported that children's probabilistic thinking improved with age and instruction. When asked the probability that the sum of the digits on the roll of two dice is even, 44% of 55 grade seven children gave a correct answer of $\frac{18}{36}$ or $\frac{1}{2}$. Only 3% of the 70 grade 5 children and 16% of the 160 grade 6 children gave a correct answer to this task. All these children were tested after receiving instruction in probability. The

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³Piaget's stage theory can also be criticized on the basis of other research indicating that even adults believe that chance phenomenon can be controlled (e.g., Langer, 1982).
results indicate that children's performance substantially increased with age, although the problem was difficult for the majority of students at all grade levels.

In response to another question—whether a consecutive sequence of numbers such as 1, 2, 3, 4, 5, 6 or a random sequence of numbers is more likely to win a lottery—teaching seemed to have a significant influence at any grade levels. But the effect of age was not as substantial as in the first question. The question was found difficult by a majority of the students at all grade levels. More than 40% of the children thought that a random sequence of numbers would have a higher probability of winning the lottery. Although children gave mixed responses to different items, Fischbein and Gazit concluded that the teaching program in probability had a positive influence on children's performance.

In a more recent study, Fischbein, Nello, and Marino (1991) administered two sets of questionnaires with problems related to certain, likely, possible, and impossible events to 618 elementary (9- to 11-year old) and junior high school (11- to 14-year old) children in Pisa, Italy. The results of this study were not substantially different from the previous studies.

Although students' facility generally increased with age and instruction, there were certain items in which the facility rate decreased with age. For example, in the rolling of two dice children were asked, "Are you more likely to obtain 5 with one die and 6 with the other, or 6 on both dice? Or is the probability the same in both cases?" (p. 532). Of the elementary children, 73% responded that the probability of obtaining (5, 6) or (6, 6) would be the same. This error was made by approximately 90% of the junior high children. The responses were similar to those given in a coin tossing problem in which children were asked, "When tossing two coins which result is more likely: to get heads
with one coin and tails with the other, or to get heads with each of the two coins? Or is the probability the same for both results?" (p. 532).

The above tasks of recognizing equally likely cases were difficult, not only for Fischbein's subjects, but also for scholars throughout the history of probability. For example, even eminent mathematicians such as d'Alembert believed that the occurrence of 0, 1, or 2 heads in a throw of 2 coins would be equally likely (Borovcnik et al., 1991). D'Alembert failed to recognize the importance of order or permutation in the determination of equally likely cases. Similarly, another well known mathematician, Leibniz, reasoned that producing a sum of 11 and 12 on a roll of two dice was equally likely (Glickman, 1990). Like d'Alembert, Leibniz did not recognize the importance of the order of the events.

Although the recognition of equally likely cases in compound events (when two or more coins or dice are tossed together) is difficult, students' lack of facility in the studies conducted by Fischbein and colleagues may have resulted from the language used in the questions. In the present researcher's view, the language of the questions in their studies was not simple, and the children did not have an opportunity to ask questions. Similarly, students' responses were collected through questionnaires and the researchers did not have an opportunity to ask questions of, or discuss difficulties with, the children.

The studies conducted by Fischbein and his colleagues did not produce the same results as those of Piagetian studies. In Piagetian studies, development of the children's cognitive thinking was almost assured with an increase in age. In Fischbein studies, cognitive growth was not necessarily a product of age. It depended on the nature of the problem and the language used. Nevertheless, in their attempts to understand children's conceptions of mathematical structures, both Fischbein and Piaget emphasized the importance of understanding children's ways of thinking.
Green

Green's studies of elementary students aged 7–11 (1988) and of secondary students aged 11–16 years (1983) in England were large-scale surveys devoted specially to children's understanding of probability. Green's tasks for elementary children included a Piagetian raindrop problem, and a total of 1600 children participated in this task. In the Piagetian raindrop problem, about how 16 raindrops would land on the flat roof of a garden shed divided into 16 square sections of equal size, children's facility in answering ranged from 31% for 7– and 8–year olds to 37% for 10– and 11–year olds (Green, 1989). Although a significant proportion of young children demonstrated some understanding of randomness, their understanding did not increase substantially with age.

In an earlier study, Green (1983) administered a survey to 3000 secondary students that consisted of items on randomness, tree diagrams, a snowflake problem, spinners, and Piagetian marble tasks. The results for the snowflake problem, which is equivalent to the raindrop problem, were particularly striking. Green found that the percent of students choosing a random distribution of snowflakes declined with age. Students' facility for this item was approximately 25% for 11–year olds and 20% for 14–year olds. Although Green (1989) noted that students' responses on randomness varied on the snowflake, raindrop, and counter problems, he concluded that randomness was not well understood by students. Similarly, children's understanding of probability as a ratio was also weak. For example, on a task that asked students to decide whether or not there would be the same likelihood of picking a black marble from two separate boxes of marbles when one box contained three black marbles and one white marble and the other contained six black and two white marbles, nearly 50% responded

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4 A summary of results about children's understanding of randomness from both studies is reported in Green (1989).
that there would be a greater chance of picking a black marble from the box with six black and two white marbles.

Like Piaget, Green concluded that children did not understand probability as a ratio. In addition, like Fischbein, Green found that children had difficulty in understanding probabilistic language such as at least, certain, or impossible. Green's studies complemented those of Piaget and Fischbein by providing results from large samples of students.

In the United States, data from the National Assessment of Educational Progress provide some information about children’s knowledge of probability. Carpenter, Corbitt, Kepner, Lindquist, and Reys (1981) and Brown, Carpenter, Kouwer, and Swafford (1988) reported the results of probability items from two National Assessments. Both reports indicated that students seem to have some probabilistic intuition at an early age. However, the majority of the students did not show a grasp of simple probability concepts in terms of ratio. For example, only about half of grade 7 students and two-thirds of grade 11 students gave a correct answer in response to the question that asked the probability of picking a red object from a jar containing 2 red and 3 blue objects (Brown et al., 1988). In an item involving compound events that asked the chance of getting at least one tail when two fair coins are flipped, only 5% of grade 11 students selected the correct response of 3 in 4 (Brown et al., 1988). Brown et al. (1988) reported that 70% of the students who responded to the items selected "1 in 2". It appears that even grade 11 students did not go beyond an intuitive grasp of probability.

Probability was found to be difficult for students in British Columbia, as well. According to a large-scale study conducted by the Ministry of Education, only 45% of grade 10 students selected the correct alternative of $\frac{1}{4}$ when asked to calculate the probability of obtaining two heads when a person flips two quarters (Robitaille, 1990). As in the American case, 38% of students selected an incorrect
response of $\frac{1}{2}$ for this item. The probability of getting a total of 5 on the roll of
two dice, was found to be even more difficult for grade 10 students. Only 15% were able to select the correct response, $\frac{1}{5}$. A quarter of students stated that they did not know the answer, and another quarter gave a response of $\frac{5}{36}$ (Robitaille, 1990).

Nevertheless, the results from the National Assessments and the B.C. Mathematics Assessment show only how students perform on specific items, and do not provide enough information to make substantial claims about students' understanding of probability in general, for at least two reasons. First, there were only a few items devoted to probability. Second, the items were multiple-choice, and students' thought processes were not probed further.

Green's studies are more comprehensive than the U.S. National Assessments and the B.C. Mathematics Assessment because of the use of a wide variety of items related to students' understanding of probability. Nevertheless, the same criticisms apply to Green's studies. Most of the test items in his studies were multiple choice and children's ideas were not elicited.

The studies conducted by Piaget, Fischbein, and Green indicate that the development of probabilistic knowledge in children is not linear. Rather, it is complex and involves children's personal experiences as well as socio-linguistic factors.

**Heuristic research on probability**

Kahneman and Tversky have proposed a research paradigm called a *heuristic approach*. Heuristic research is designed to understand people's intuitive biases and cognitive representations of a particular task. Like the developmental research conducted by Piaget, Fischbein, and Green, Kahneman and Tversky's research emphasizes the need to understand people's ways of thinking (Scholz,
1991). On the basis of various studies, usually conducted with college students, Kahneman and Tversky argue that people estimate the likelihood of certain events by using different heuristics, which are often different from those proposed in accepted probability theory. They refer to these as judgment heuristics, which basically fall into two categories: representativeness heuristic (Kahneman & Tversky, 1972) and availability heuristic (Tversky & Kahneman, 1973). Subsequent studies conducted by Tversky and Kahneman, and others, have confirmed the use of judgment heuristics by children and adults of different ages and abilities (Agnoli, 1987; Kelly, 1986; Shaughnessy, 1977, 1981, 1992; Tversky & Kahneman, 1982a, b). Since these heuristics have the potential for informing our understanding of the ways in which people conceptualize probability, it would be useful to describe them further.

Representativeness heuristic

People applying a representativeness heuristic, base their arguments about how well an outcome represents the characteristics of its parent population and whether the outcome was generated by a random process. An event A is judged more likely than an event B if A appears more representative than B (Kahneman & Tversky, 1972). For example, in one study Kahneman and Tversky (1972) asked the following question to grade 10, 11, and 12 students in Israel:

All families of six children in a city were surveyed. In 72 families the exact order of births of boys and girls was GBGBBC\(^5\). What is your estimate of the number of families surveyed in which the exact order of births was BGBBBB? (p. 432)

If the judgment was made based on a formal view of mathematics, both sequences would be rated equally likely. But 75 of 92 subjects judged that the

\(^5\)G=girls, B=boys
number of families with the exact order of birth BGBBBB was much less than 72, the median estimate being 30. Similarly, in another item, the same subjects judged that the sequence BBBGGG was significantly less likely than GBBGBG. The sequence GBGBBG was regarded more likely than BBBGGG or BBBBBB for two reasons. First, unlike the sequence BBBBBB, it has an equal number of B's and G's. Second, B's and G's appear to be randomly distributed in GBGBBG, but not in BBBGGG and BBBBBB.

Similarly, in a study conducted by Shaughnessy (1981) in the United States, 50 out of 80 college undergraduate students prior to instruction in probability said that the sequence BGBGBG would be more likely than the sequence BBBBGB. Only 2 students selected the latter sequence and 18 students said that all the sequences had the same chance of occurring. Shaughnessy also asked students to provide written reasoning for their choices. They reasoned that the first sequence "fits more closely with the 50:50 expected ratio of boys to girls" (Shaughnessy, 1981, p. 91).

According to Kahneman and Tversky (1972), a representativeness heuristic is manifested in the so-called gambler's fallacy or the negative-recency effect. As early as 1951, Jarvik identified the gambler's fallacy which states that, in the case of two equally likely events A and B, subjects tend to predict outcome A after a long run of outcome B. The thinking behind this fallacy is that the occurrence of A will result in a more representative sequence than another occurrence of B. Hence, chance is viewed "as a self correcting process in which a deviation in one direction induces a deviation in the opposite direction to restore the equilibrium" (Tversky & Kahnemann, 1982a, p. 7).

A belief in representativeness also leads to the "conjunction fallacy" (Tversky & Kahneman, 1982b). Tversky and Kahneman asked the following
question to undergraduate and graduate students in Israel, Canada, and the United States:

Linda is 31 years old, single, outspoken, and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations.

Please rank the following statement by their probability, using 1 for the most probable and 8 for the least probable.6

(5.2) Linda is a teacher in elementary school.
(3.3) Linda works in a bookstore and takes Yoga classes.
(2.1) Linda is active in the feminist movement.
(3.1) Linda is a psychiatric social worker.
(5.4) Linda is a member of the League of Women Voters.
(6.2) Linda is a bank teller.
(6.4) Linda is an insurance salesperson.
(4.1) Linda is a bank teller and is active in the feminist movement.

(Tversky & Kahneman, 1982b, p. 92)

Since the description of Linda was constructed to be representative of an active feminist and not representative of a bank teller, Tversky and Kahneman expected that the mean rank of the compound last statement would fall between the mean ranks of its components. Interestingly, the result was as expected by Tversky and Kahneman. What is also interesting is that many subjects who had taken statistics and probability courses rated it as more likely that "Linda is a bank teller and is active in the feminist movement" than Linda is either a bank teller or an active feminist. The subjects, therefore, violated the conjunction rule of probability, namely \( P(A \text{ and } B) \leq P(B) \). Some researchers, however, argue that the conjunction fallacy can be easily corrected. For example, Agnoli (1987)

6The numbers in parentheses are the mean ranks assigned by the subjects.
claimed that even children of 11 to 13 years of age can be trained to avoid the conjunction fallacy by using Venn diagrams.

**Availability heuristic**

People applying an availability heuristic base their decisions mainly on their previous experiences of a particular instance or "by the ease with which instances or occurrences can be brought to mind" (Tversky & Kahneman, 1982a, p. 11). For example, if a person has been involved in a car accident, he or she is more likely to estimate a higher rate of car accidents than a person who has never been involved in an accident (Shaughnessy, 1981, 1992). Similarly, many people will say that it is more likely for an English word to begin with the letter k, for example "key", than to have k in the third position, for instance "like" (Kahneman & Tversky, 1972).

Similarly, Tversky and Kahneman's subjects estimated that more committees of 2 members can be made than committees of 8 members from a group of 10 people. In their research, the median estimate for committees with 2 members was 70 while the median estimate for committees with 8 members was 20. In fact, the number of committees that can be formed out of 10 people, with 2 or 8 members is equal. According to Tversky and Kahneman (1982a) more committees of few members comes more easily to mind than committees of many members, from the same group of people.

The vast majority of Shaughnessy's (1981, 1992) undergraduate students also said that more committees of 2 members could be made than committees of 8 members from a group of 10 people. Neither Kahneman and Tversky nor Shaughnessy reported whether their subjects had studied combinatorial logic prior to participating in their research. It would certainly be interesting to see if research subjects would still continue to say that more committees of 2 members
can be made than committees of 8 members from a group of 10 people after learning combinatorial principles.

**Critiques of heuristic studies**

Kahneman and Tversky's insights into how people think in making decisions have been very useful for mathematics educators. However, their studies lack the clarity that would have been provided by describing the research setting and subjects' characteristics in detail. Furthermore, they have been criticized for methodological and conceptual shortcomings. There is insufficient evidence, for example, to determine whether the research subjects comprehended the text, how they perceived the situation, and what meanings they attached to the concepts and tools used in the research (Scholz, 1991).

Borovcnik and Bentz (1991) reviewed students' responses to items used by researchers such as Green and Shaughnessy. Borovcnik and Bentz (1991) explained various forms of reasoning that students might have used in responding to items. Even when a student's answer was correct, it was difficult to know whether the student used a correct strategy or a naive heuristic to come up with the answer. They assert that students' incorrect answers might be due to linguistic and communication problems rather than lack of understanding of probabilistic concepts.

In much the same way, Nisbett, Krantz, Jepson, and Fong (1982) argue that, unlike Kahneman and Tversky's studies, many probability models or problems used in "social-psychological" studies do not have a single correct answer. Even if an error is determined, it is difficult to know without actually probing students' reasoning processes, whether the error is because of faulty reasoning, incorrect models, or incorrect prior beliefs, even when standard statistical problems are presented. Shaughnessy (1993) criticizes psychological research in general where "tasks are posed in a multiple-choice, forced-answer
format, where subjects' understanding of the task and their actual thinking processes are masked" (p. 72).

Scholz (1991) criticized the heuristic approach to research from a theoretical perspective for not considering any connections or mutual interdependencies among different heuristics. He proposed a "structure and process model of thinking" in which subjects' thought processes and their comprehension of texts and tools could be determined by tracing connections with what the subjects had said before. In addition, connections between different heuristics could be made by inferring the "complementarity of intuitive and analytic thinking" (p. 229). Scholz's proposal is based on an information-processing theory which metaphorically relates the human mind to a computer. However, Shaughnessy criticized Scholz's proposal by stating that it was solely cognitive. Scholz's model ignores students' subjective beliefs and the contextual situatedness of peoples' probabilistic thinking.

Further development of heuristic studies

The studies conducted by Konold and his colleagues considered that peoples' conceptions of probability were contextual and situated in the framework and the problem setting on the one hand, and in their personal history and belief systems on the other. Unlike Kahneman and Tversky's research questions about how people arrive at probabilistic judgments, Konold was interested in how people interpret probability problems.

Konold et al. (1993) claimed that when given probability tasks, high school and college students did not reason normatively; that is to say, the respondents did not reason according to accepted principles of probability. They conducted a number of studies of high school and college students, and found that students' conceptions of probability were inconsistent across various tasks. Many students who responded correctly to multiple-choice test items reasoned incorrectly in
written justifications and interview settings. Moreover, their studies indicated that students were not consistent even on the same multiple-choice test when items were phrased differently, for example changing "most likely" to "least likely."

According to Konold and his colleagues (1989, 1991, 1993), the respondents in their studies reasoned from an "outcome approach". That is, the respondents believed that their task was to determine what the next outcome would be in a single trial of an experiment rather than to estimate what was likely to occur if the experiments were repeated several times. Participants who reasoned from an outcome approach in Konold's studies were using a non-statistical conception of probability (Shaughnessy, 1992).

Based on a review of school and college students' probabilistic reasoning, Shaughnessy (1992) identified four types of conceptions of probability: non-statistical, naive-statistical, emergent-statistical, and pragmatic-statistical.

1. **Non-statistical.** Indicators: responses based on beliefs, deterministic models, causality, or single outcome expectations; no attention to or awareness of chance or random events.
2. **Naive-statistical.** Indicators: use of judgmental heuristics, such as representativeness, availability, anchoring, balancing; mostly experientially based and nonnormative responses; some understanding of chance and random events.
3. **Emergent-statistical.** Indicators: ability to apply normative models to simple problems; recognition that there is a difference between intuitive beliefs and a mathematized model; perhaps some training in probability and statistics; beginning to understand that there are multiple mathematical representations of chance, such as classical and frequentist.
4. **Pragmatic-statistical.** Indicators: an in-depth understanding of mathematical models of chance (i.e. frequentist, classical, Bayesian); ability to compare and contrast various models of chance; ability to select and apply a normative model when confronted with choices under uncertainty; considerable training in stochastics; recognition of the limitations of and assumptions of various models. (p. 485, emphasis in original).

Participants in developmental research such as those conducted by Piaget, Green, and Fischbein and subjects in the heuristic research reported by Kahneman and Tversky held either non-statistical, naive statistical, or emergent statistical conceptions of probability. None of these researchers reported research participants having pragmatic statistical conceptions of probability; this would indicate that the students lacked understanding of formal probability.

**Research on preservice teachers' conceptions of probability**

Most of the existing research in probability has focused on elementary, secondary, and first year college students' conceptions. Little attention has been given to preservice teachers' conceptions of probability. The only study of secondary school mathematics preservice teachers' understanding of probability was reported by Bramald (1994). He selected the following item from Green (1982) and assigned it to his class of preservice teachers.

Which of the following results is more likely?

1. Getting 7 or more boys out of the first 10 babies born in a new hospital
2. Getting 70 or more boys out of the first 100 babies born in a new hospital

(A) They are equally likely

(B) 7 out of 10 is more likely
(C) 70 out of 100 is more likely  
(D) No one can say.  

(Green, 1982 cited in Bramald, 1994, p. 86)  

Bramald (1994) was quite surprised to learn that the majority of his students, from a total of 31 in his class, thought that both events, 7 out of 10, and 70 out of 100 were equally likely. In further discussion, it was evident to Bramald that his students, like the school students, lacked a basic understanding of formal probability. Only three students chose the correct answer that 7 out of 10 is more likely than 70 out of 100. However, Bramald did not state nor speculate why students might have thought the way they did. The participants might not have thought about probability concepts, and might have simply used the concept of equivalent fraction. Since the fractions $\frac{7}{10}$ and $\frac{70}{100}$ are equivalent they may have decided that the statements were equally likely.  

Without further probing students' thinking processes, Bramald (1994) argued that preservice teachers are "pushed too quickly into the traditional route of manipulation of probabilities and then churning out 'right' answers, perhaps in the hope that understanding will follow later" (p. 86). His study provides some insights into how preservice teachers of secondary school mathematics with reasonably strong backgrounds in mathematics think about probability problems. However, his report is based on one snapshot of his classroom and does not provide any detail about students' conceptions of probability.  

**Diverse conceptions of probability: A historical account**  

Philosophers and mathematicians have held different conceptions of probability throughout history (Daston, 1988). At present, there are four distinct conceptions of probability that are derived from its historical tradition: classical,
frequentist, subjectivist, and structural (Borovcnik et al., 1991; Hawkins & Kapadia, 1984; Kapadia, 1988; McNeill & Freiberger, 1994; Steinbring, 1991b).

Classical probability was first defined by Pierri Laplace (1749–1827) as "the ratio of favourable cases to the total number of equally possible cases" (Hacking, 1975, p. 122). Classical probability is obtained by making an assumption of equally likely cases and is sometimes referred to as a priori, Laplacian, or theoretical probability. Although the credit for defining classical probability is given to Laplace, his definition was criticized because of the use of the word "possible," which has a dubious meaning. Contemporary mathematicians have refined this definition. For example, Kline (1967) expresses this as "if, of n equally likely outcomes, m are favorable to the happening of a certain event, the probability of the event happening is m/n...." (p. 524, emphasis in original).

Frequentist probability was "connected with the tendency, displayed by some chance devices, to produce stable relative frequencies" (Hacking, 1975, p. 1). Some people refer to this as a posteriori, experimental, or empirical probability. It is defined as the ratio of the observed frequency to the total number of trials in a random experiment over the long run. Mathematically, it involves the theory of limits and convergence.

Subjectivist probability has been common since medieval times although it received wide recognition only in the twentieth century. According to it "probabilities are evaluations of situations which are inherent in the subject's mind" (Borovcnik et al., 1991, p. 41). Daston (1988) defines it "as the intensity of beliefs" (p. 188). Also known as personal probability, it assumes that human beings are capable of estimating the probability of certain events and making adjustments when additional data are obtained. Mathematicians and statisticians call it Bayesian probability named after Thomas Bayes (1701–1761). Mathematically, it is associated with Bayes' theorem.
Structural probability, like other systems of formalist approaches to mathematics, is based on a system of axioms, definitions, and theorems (Borovcnik et al., 1991). The basic axioms of probability were first developed by Andrei Kolmogorov in 1933 (Daston, 1988). According to Kolmogorov, the basic axioms of probability are $P(E) \geq 0$ and $P(S) = 1$ where $E$ and $S$ represent an event and a sample space respectively (Kolmogorov, 1956). Based on these axioms and other terms, the addition rule, the multiplication rule, and other rules and theorems were developed and used to solve probability problems.

Mathematicians, logicians, and philosophers have debated which of the conceptions of probability is superior. Yet, these conceptions were closely associated with epistemological issues concerning how probability is learned. For example, philosophers or mathematicians who believed that knowledge can be derived deductively in a formal fashion, argued for classical and structural conceptions of probability. Those who believed in inductive reasoning supported the frequentist approach and those who believed in personal judgment supported the subjectivist approach to probability.

The different conceptions of probability that emerged have had significant influence on the teaching and learning of probability (Borovcnik et al., 1991; Hawkins & Kapadia, 1984; Kapadia, 1988; McNeill & Freiberger, 1994; Steinbring, 1991b). The subjectivist conception of probability seems to be compatible with a constructivist view of learning because of its emphasis on people's updated experience and knowledge in estimating probability. The subjectivists claim that they can only measure the degree of a person's belief, not an absolute truth. As in the constructivist view of learning, the subjective conception allows people to continually adjust their beliefs as they come to know more about the world. However, it is important to note that a constructivist view of learning does not support the argument that people's views and beliefs can be
updated or modified with a set of formulas as it is done in subjective probability by using Bayes' theorem.

**Formal and non-formal conceptions of probability: Another dimension in looking at students' conceptions of probability**

The historical debate about different kinds of probabilities (classical, frequentist, subjective, and structural) has influenced the research and teaching of probability. The conceptions that emerged throughout history can be used to categorize student' conceptions of probability. However, with the exception of subjective probability all these conceptions focus mostly on formal aspects of probability. Furthermore, a subjective approach to probability ultimately relies on Bayes' theorem (Moore, 1990), which is perplexing even for professional mathematicians and researchers (Falk, 1992; Shaughnessy, 1992).

By focusing merely on historical views to categorize students' conceptions of probability, there is a danger of losing important elements related to students' conceptions of probability, such as the representativeness heuristic (Kahneman & Tversky, 1972; Tversky & Kahneman, 1982 a, b), the availability heuristic (Tversky & Kahneman, 1973, 1982a), the outcome approach (Konold, 1989, 1991), and non-statistical, naive-statistical, emergent-statistical, and pragmatic-statistical conceptions of probability (Shaughnessy, 1992).

Shaughnessy's (1992) categories attempt to make a connection among various conceptions of probability. He categorized Konold's outcome approach as non-statistical conception and Kahneman and Tversky's different kinds of heuristics as naive-statistical. He used the label emergent-statistical to indicate some understanding of formal probability and pragmatic-statistical to indicate an in-depth understanding of formal probability. The difficulty with Shaughnessy's categories is the amount of overlap among the conceptions. It is difficult to determine whether a particular conception is non-statistical, naive-statistical,
emergent statistical, or pragmatic statistical. In addition, like Confrey (1991), the present researcher is not comfortable labeling a student's model as a misconception or naive conception from an expert point of view, because it fails to credit the student's perspective. Instead of labeling students' conceptions as misconceptions, naive conceptions, or pragmatic statistical conceptions, an alternative model of grouping students' conceptions is proposed for the purpose of this study.

This alternative model is to group students' conceptions of probability into formal and non-formal ones. Formal conceptions of probability will be defined as those in which students make explicit reference to and use of mathematical concepts and techniques. All other conceptions of probability documented in the study will be defined as non-formal conceptions.

Formal conceptions of probability include all the elements of pragmatic-statistical conceptions of probability discussed by Shaughnessy (1992). If someone uses probability concepts and techniques directly from their school or university courses they will be classified as formal conceptions. However, the use of a term such as chance, independence, or randomness is not enough to classify a student's conceptions as formal ones. It depends on how students have used those terms in solving problems. Steinbring (1991a, 1993), for example, provides examples of how meanings of probabilistic concepts differ even if the same words are used. Steinbring draws on a constructivist perspective and argues that the meanings of chance constituted by social interactions are different from the formal meaning of chance. Non-formal meanings of chance stem from everyday personal and social experiences, which are related to "luck, bad luck, fortune, etc." (Steinbring, 1993, p. 20). A student expressing a non-formal conception might regard a chance event as very rare, or nearly improbable, but one that can occur. In contrast, a formal conception of a chance event would be
associated with randomness and independence and does not necessarily imply a small probability of events (Steinbring, 1993).

Non-formal conceptions are developed by individuals from their experiences outside formal setting of university and schools. For the purpose of everyday problem solving they may be even more viable than formal conceptions of probability that are generally acquired through school and university education. For the purpose of this study, the outcome approach, the representativeness heuristic, and the availability heuristic will all be grouped as non-formal conceptions of probability because none of these are based on school or university knowledge of probability. Other subjective conceptions of probability, such as chance, luck, and weather forecasting will also be grouped as non-formal conceptions of probability.

For the purpose of this dissertation, formal and non-formal conceptions are not viewed as mutually exclusive. They are used for analytical purposes only and there are qualitatively different conceptions of probability within them.

**Overview of Chapter 2**

One of the major constraints in teaching probability in schools is students' difficulties in understanding probabilistic concepts as evident in various studies, such as those conducted by Piaget, Fischbein, Green, Kahneman and Tversky, Shaughnessy, and Konold. It is not only school and college students who face difficulties in understanding formal probability; school teachers may also be expected to face such difficulties. However, teachers' understanding of formal probability both at preservice and inservice levels has not been documented in the literature.

This researcher argues that preservice teachers' conceptions of probability, both formal and non-formal, are important for professors who have to plan and
teach probability. This study provides some of the qualitatively different conceptions of probability held by preservice teachers.
CHAPTER 3

Research Methodology and Procedures of the Study

The purpose of this chapter is twofold. The first is to provide a theoretical framework for the research methods used in the study. The second is to provide details of the research design: specifically, development of the instruments, data collection, data analysis, and data review procedures.

Theoretical framework for the methodology of the study

The methodology for this study derives from the constructivist perspective proposed by Guba and Lincoln (1989, 1994) and Schwandt (1994). Schwandt (1994) argues that constructivist research focuses on how "meanings are created, negotiated, sustained, and modified within a specific context of human actions" (p. 120). The importance of contexts and interactions as proposed by Schwandt is also emphasized by Guba and Lincoln who state that, "Individual constructions can be elicited and refined only through interaction between and among investigator and respondents." (Guba & Lincoln, 1994, p. 111, emphasis in original).

In order to strengthen continuous interaction between investigator and respondents through context-specific tasks, constructivist-oriented research demands a qualitative and interpretive approach. In order to achieve this in the present study, the researcher used three sources: written tasks, pair-problem-solving tasks, and semi-structured performance task interviews. Consistent with the constructivist and interpretivist approach to research, this study does not claim to have discovered a universal truth about preservice teachers' conceptions of probability. Rather, it provides an analysis of their thinking about probability within the context of the study.

7Context in this study is viewed as a product of task and setting.
Generalizability of the study

This study uses "naturalistic generalization" or "readers' generalization" (Lincoln & Guba, 1985; Stake, 1978, 1994) and analytic generalization (Firestone, 1993; Lather, 1991; Yin, 1989). Generalizability in the naturalistic sense is "the degree to which findings derived from one context, or under one set of conditions may be assumed to apply in other settings or under other conditions" (Shulman, 1981, p. 8). Riecken (1989) expressed the same view by saying that the transferability of knowledge claims made by one study "to other situations will depend on the degree of 'fit' between contexts" (p. 13). The results from a study may serve as a working hypothesis for other settings that are sufficiently congruent (Erickson, 1992; Lincoln & Guba, 1985). In this kind of generalizability, the reader compares his or her own experience with what is claimed in the research. It is the reader, rather than the writer, of the report who determines whether the knowledge claims pertaining to the study will be transferable to his or her context (Erickson, 1992; Shulman, 1981).

In analytic generalizations, researchers attempt to generalize their findings to a broader theory. The results from a study contribute to the theory because the researcher can find the weak points in the theoretical position in order to revise and extend the theory (Lather, 1991). The present study draws upon analytical generalization and argues for the role played by tasks and settings in the exploration of students' conceptions of probability.

Semi-structured interviews

In semi-structured interviews the initial tasks and questions are selected by the researcher but the phrasing of questions and their order may be altered to fit the characteristics of each participant and setting. The major characteristic of semi-structured interviewing is that it provides the interviewer with a substantial amount of freedom and autonomy for probing students' thinking (Denzin, 1989;
Fontana & Frey, 1994; Posner & Gertzog, 1982). Although an interviewer generally selects the topic, it is the interviewee's responses which determine the direction of the interview. The initial questions of the interview are quite general and the interviewer changes subsequent questions based on students' responses to the initial questions. Interviews are "guided by the prepared questions, but not bound by them" (Rowell, 1978, cited in Posner & Gertzog, 1982, p. 199). The interviewer listens carefully to what the student says, and observes what the student does. Based on what has been heard and observed, he or she "makes and tests hypotheses about the student's thinking" (Ginsburg et al., 1993, p. 242).

Curricular theorists such as Schwab (1989) and mathematics educators such as Davis (1971) have suggested that researchers use an in-depth, semi-structured interview method if the intention of the research is to explore deeper levels of students' understanding. According to Schwab (1989) even if tests are used to assess students' understanding of subject matter knowledge, they should be supplemented with interviews so that the students' modes of thought while acting on tests can be investigated.

Semi-structured interviews have been shown to be particularly effective in revealing students' understanding of mathematics and science content (Confrey & Lipton, 1985; Davis, 1971; Ginsburg, 1981; Ginsburg, Jacob, & Lopez, 1993; Linder, 1989; Lythcott & Duschl, 1990; Piaget, 1929; Posner & Gertzog, 1982). Since a constructivist assumes diversity of experiences among participants, he or she examines carefully "the student's use of examples, images, language, definitions, analogies etc. to create a mode which may well transform the interviewer's own understanding of the mathematical content in fundamental ways" (Confrey, 1991, p. 115, emphasis in original).

It is important that interview tasks be thought-provoking and interesting so that students will engage in the tasks (Confrey, 1991; Confrey & Lipton, 1985;
Denzin, 1989; Ginsburg, 1981). According to Confrey, the tasks in the interviews should require more than recalling facts, rules, and procedures. She emphasizes that a problem itself is not an important factor in probing students' conceptions; it is only a means of creating interactions between the interviewer and the student.

**Pair-problem solving**

In this study, pair-problem solving can be considered as a special type of group interviewing. In the pair-problem solving, participants work on problems with each other rather than being asked questions by the interviewer. The interviewer asks questions only if the conversation between the participants stops without any conclusion or if the interviewer is interested in some specific issues that arise in the course of the problem solving. The inclusion of pair-problem solving in the study can add all the major benefits of group interviewing. In particular, the construction of meanings by constant interaction and negotiation among individuals can be enhanced if the study includes both pair-problem solving and individual interviews.

The action of the interviewer or the respondent is very much shaped by the immediate action of the other in a specific sociocultural context (Clandinin & Connelly, 1994; Eisenhart, 1988; Erickson & Shultz, 1982; Gee, Michaels, & O'Conner, 1992; Hedges, 1985; Jones, 1985a, Mishler, 1986; Woods, 1992). Because of differing sociocultural contexts, results from the individual interviewing and the pair-problem solving can be very different. Hence, pair-problem solving contributes to the study by providing a different context for exploring the consistency of students' subject matter knowledge.

Pair-problem solving can be helpful in understanding an individual's thinking process because talking with other people "often helps people to analyze their own attitudes, ideas, beliefs and behavior more penetratingly and
more vividly than they could easily do if just alone with the interviewer" (Hedges, 1985, p. 73). Hence, pair-problem solving allows individuals to analyze their own thinking in terms of the context set by the pair.

Pair-problem solving also has the advantage of making the working situation less strange for interviewees and thus less stressful. The fact that the interviewer does not always need to ask questions can be considered as another advantage of pair-problem solving. The respondents ask each other different kinds of questions than those of the researcher and therefore could reveal new types of understandings (Denzin, 1989).

**Principles guiding research interviewing and pair-problem solving**

The following principles were applied in the study during the pair-problem solving and the individual interviews:

1. The researcher listened more than talked (adapted from Denzin, 1989; Erickson & Shultz, 1982; Wolcott, 1990) because as Denzin (1989) states, "understanding draws upon shared experiences. Persons can't share experiences if they don't listen to one another" (p. 109).

2. The researcher checked the meaning of words, phrases, or sentences of which he was unsure (adapted from Jones, 1985a). The researcher was always sensitive to contradictory statements and asked questions to gain insight into what the preservice teachers actually believed (adapted from Lythcott & Duschl, 1990; Posner & Gertzog, 1982).

3. The researcher was not satisfied with only the right answer. Instead he tested the strength and consistency of the preservice teachers' views by using repetition and countersuggestion (adapted from Ginsburg et al., 1993; Piaget, 1929; Posner & Gertzog, 1982). Follow-up questions were not
intended to elicit answers, but to explore, clarify, and extend preservice
teachers' tacit knowledge, doubts, and assumptions.

4. The researcher attempted to develop "rapport, trust and friendship,
sociability, inclusion" (Woods, 1992, p. 375) and showed interest in
preservice teachers' concerns, feelings, and cognitive orientations.

An illustration of these principles can be found in an extract from an
interview transcripts provided in Appendix D.

**Methods of data analysis**

The data analysis was based upon the constant comparative method
(Guba & Lincoln, 1989; Lincoln & Guba, 1985) and the interactive model
(Huberman & Miles, 1994). In addition, it drew upon the probability research
conducted by Konold and his colleagues (1989, 1991, 1993) to analytically
examine students' conceptions of probability.

The purpose of analysis is to make sense of data by identifying patterns,
categories, or themes that are meaningful for the interviewees, researchers, and
potentially useful to readers (Jones, 1985b; Eisenhart & Howe, 1992; Goetz &
suggested that data can be organized by predetermined categories but
researchers should still remain "sensitive to the unanticipated categories which
derive from the concepts of the research participants" (p. 58). She contends that
"categories do not just 'emerge' out of data as if they were objectively 'there'
waiting to be discovered" (p. 58). Different people, with different perspectives,
construct different categories to make sense of the data (Denzin, 1994; Holstein &
Gubrium, 1994; Jones, 1985b; Lincoln & Guba, 1990; Novak & Gowin, 1984;
Rosenau, 1992).
Jones (1985b) also suggests making connections to the concepts and theories one is working with to "confirm, elaborate, modify or reject them" (p. 59). Categories, constructs, and themes are constructed and reconstructed as new data are collected (Eisenhart, 1988; Spradley, 1979). Since the responses from the interviewees are "produced in the context of a developing sequence of interaction" (Hammersley & Atkinson, 1983, p. 193), it is important to interpret data in light of the interview contexts and settings.

After the collection of data and the construction of categories, the researcher should go to the participants to check whether they agree with the meanings constructed by the researcher (Guba & Lincoln, 1989, 1994; Lincoln & Guba, 1985). Guba and Lincoln (1989) call this process "member checking," whereas Hammersley and Atkinson (1983) call it "respondent validation" (p. 195). Lather (1991) supports this process of validation by arguing that the researcher should go back to respondents and discuss how the data are viewed by the researcher "both to return something to research participants and to check descriptive and interpretive/analytical validity" (p. 57).

Member checking allows the researcher to present his or her conjectures and hypotheses to respondents for comments, elaboration, corrections, and revisions (Guba & Lincoln, 1989). It provides an opportunity for the researcher to verify his or her constructions about the responses and also provides a chance for respondents to correct errors or provide additional information and clarification about their data. Member checking also provides additional information about whether respondents are consistent in their meanings after some period of time.

Member checking can be used to clarify ambiguities and confusion that may have occurred during data collection. This can take a form of stimulated recall interviews that provides an opportunity for both researcher and
participants to reflect on "uncomfortable moments" which occurred during the data collection period (Erickson and Shultz, 1982). In addition, it helps the researcher to understand what parts of the interviews or the pair-problem solving were thought salient and important by the research participants.

Research Method

Data in this study were collected through three types of instruments: a written task, a pair-problem solving task, and a semi-structured interview. Details of their development are provided.

Development of instruments

Each instrument's design was based on a review of literature and the researcher's previous experience in teaching probability. The tasks were finalized during field testing.

Construction of tasks

Several items used by previous researchers to examine students' understanding of probability (for example Kahneman & Tversky, 1972; Konold et al., 1993; Fischbein, 1975; Fischbein & Gazit, 1984) were reviewed, and their applicability for exploring preservice teachers' understanding of probability were discussed with colleagues and committee members. In addition to these items, several tasks designed by the researcher went through the same sort of testing and review.

A total of five tasks\textsuperscript{8} were selected for the written and the pair-problem-solving components and a total of nine tasks, which included two from these components, were selected for use in the interview. The written component included both multiple-choice and open-ended tasks. The written tasks required preservice teachers to justify their reasoning about their solutions. The interview

\textsuperscript{8}Tasks included written items as well as manipulatives, such as boards, marbles, dice, and thumbtacks. For the purpose of consistency everything will be called a task.
included tasks representing a variety of contexts such as coin tossing, car accidents, lottery winning, spinning a spinner, rolling thumbtacks, and dice throwing. These tasks were selected because they were deemed to be useful by the researcher in exploring preservice teachers' conceptions of probability and the results could be compared with the results obtained by previous researchers.

Field tests

A formal field test of the tasks assigned to the written, the pair-problem solving, and the interview components was carried out with two mathematics education graduate students at the University of British Columbia in May of 1994. Both graduate students had majored in mathematics and physics for their bachelor's degrees. Both had taken a probability course either at high school or university. The graduate students were chosen for the field test because they were readily available and had backgrounds in mathematics and probability similar to the preservice teachers who would participate in the study.

The field test of each component was conducted with these graduate students. The written, pair-problem solving, and individual interview sessions took approximately 35, 15, and 90 minutes respectively. Each pair-problem-solving session and interview was audiotaped, transcribed, and analyzed.

Both students found the tasks in the written component to be fairly clear and gave similar responses to each other. Thus the written component did not undergo major revision. In the pair discussion, however, both students agreed with each other and insufficient interaction took place. The pair-problem solving was changed to encourage more interaction. Some tasks in the interview were found to be lengthy and confusing for both students and these tasks were either revised or eliminated from the interview.
Criteria for selecting tasks for pair-problem solving and interview

Two criteria were used in selecting tasks for the pair-problem solving. First, the written and the pair-problem-solving components were not to be identical although some tasks could be repeated to explore students' responses in different social dynamics. Second, the tasks had to be interesting and challenging to the graduate students.

The tasks for the interview were developed based upon four criteria. First, the interview tasks had to be interesting enough to engage the interviewees in a sustained conversation to reflect on their understanding of probability. Second, the contexts of the interview had to vary from verbal mathematical problems to manipulatives and games. Third, the tasks could not be too hard or too easy since this might hinder interviewees' thinking processes. Fourth, the interview session had to last for a reasonable duration from both the interviewee's and interviewer's perspectives: about 45–60 minutes.

Description of tasks

The written component and the pair-problem-solving component each consisted of five tasks. Description of tasks is shown in Table 1.

The first task in the written component was called the birth sequence task, and asked participants to select the most likely sequence of births in a family. This task was similar to the head-tail sequence task, in the pair-problem-solving component, wherein participants were asked to determine the most likely sequence of alternatives that might result from flipping a fair coin six times. The second task in the written component, the rectangular spinner task, asked preservice teachers to decide the most likely section where the spinner would stop. The third task in the written component, the five-head task, required preservice teachers to decide the sixth outcome if the previous five outcomes
were all heads when a fair coin was tossed. The five-boy task in the pair-problem-solving component required the same reasoning. The fourth task in the written component or the third task in the pair-problem solving, the lottery and car accident task, required participants to decide whether they were more likely to win a jackpot in the B.C. lottery 6/49 or be killed in a car accident. The final task in the written component or the fourth task in the pair-problem solving, the circular spinner task, required preservice teachers to select two numbers out of eight so that they could have the best possible strategy to win the game. The final task in the pair-problem solving, the bank teller task, required participants to determine the number of tellers needed if one to six customers randomly arrive at the bank (see Appendix A and B for further details).

There were five tasks in the interview component, the head-tail sequence task, the lottery task, the board and marble task, the dice task, and the thumbtack task. The first task, the head-tail sequence task, was repeated from the pair-problem solving to explore changes in participants' conceptions from the pair to the individual setting.
The remaining four were new. In the second task, the lottery task, the preservice teachers were asked to resolve a conflict between two students who had opposing views about selecting numbers for a lottery 6/49. In the third task, the board and marble task, the participants were shown a wooden board with a system of channels progressively branched toward the base of the board and were asked where a marble dropped from the main channel, would exit at the bottom. In the fourth task, the dice task, the participants had to choose whether they wanted to be a player or a dealer in a dice game (sometimes called a "chuck-a-luck") in which three dice are rolled to determine whether the player or the dealer wins the bet. In the fifth task, the thumbtack task, the preservice teachers were asked whether a thumbtack will point up or point down when it is dropped or rolled. Finally, preservice teachers were asked to provide their opinions as to whether they found university or school knowledge of probability useful in solving everyday problems (See Appendix C for details of the interview protocol).

It is clear from the above description that some tasks that required the same sort of mathematical reasoning were included within and across settings in order to determine how preservice teachers' thinking would be similar or different across tasks and settings. The tasks and their relation to each other in written, pair-problem solving, and individual interview are shown in Table 2.

Preservice teachers required knowledge of randomness and independence to come to a mathematical solution in the birth sequence task and the five-head task in the written component. The same reasoning was required in the head-tail sequence task and the five-boy task in the pair-problem solving. Knowledge of randomness and independence were also required for the head-tail task, the lottery task, the board and marble task, and the dice task in the interview setting.
<table>
<thead>
<tr>
<th>Tasks</th>
<th>Written</th>
<th>Pair-problem solving</th>
<th>Interview</th>
</tr>
</thead>
<tbody>
<tr>
<td>Birth sequence task</td>
<td>R, I, O</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Rectangular spinner task</td>
<td>R, I, S</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Five-head task</td>
<td>R, I</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lottery and car accident task</td>
<td>C, Q</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Circular spinner task</td>
<td>E, A</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Head-tail sequence task</td>
<td>R, I, O</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Five-boy task</td>
<td>R, I</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lottery and car accident task</td>
<td>C, Q</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Circular spinner task</td>
<td>E, A</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Bank teller task</td>
<td>R, I, S</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Head-tail sequence task</td>
<td>R, I, O</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lottery task</td>
<td>R, I, C</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Board and marble task</td>
<td>R, I, P</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Dice task</td>
<td>R, I, O, M</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Thumbtack task</td>
<td>R, I, S</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Legend:
A = Addition rule
E = Recognition of equally likely cases
M = Multiplication rule
P = Physical symmetry
R = Randomness
C = Combinatoric
I = Independence
O = Recognition of order
Q = Probability as a ratio
S = Angles in sectors
Some tasks required an understanding of the addition rule and the multiplication rule of probability. The repeated tasks within and across different settings helped the investigator to probe further how each preservice teacher thought about probability in a different context.

**Data collection procedures**

After the development of the instruments, data were collected from two groups of preservice teachers of secondary school mathematics in the summer and fall of 1994 at the University of British Columbia. At the beginning of these classes, the researcher told preservice teachers the purpose of the research and distributed a student consent form. The data collection began as soon as the preservice teachers consented to participate in the study.

**Conducting written tasks**

All preservice teachers, nine from the summer group and 31 from the fall group, who wished to volunteer were asked to complete the written component of the study. The purpose of the written tasks was to identify preservice teachers who had diverse ways of thinking about probability. The participants were asked to provide some background information such as the number of courses in mathematics, statistics, and probability they had taken and what their major subject was. They were requested to provide accounts of their reasoning for the solutions of open-ended tasks. The written responses were analyzed for a range of opinions regarding participants’ beliefs, knowledge, intuitions, understandings, and experiences related to probability.

The responses provided by the preservice teachers in the written component were mostly consistent with the mathematics of probability. Out of 40 who participated in the written component, 32 said that all four options (BGBBBB, BBBGGG, GBGBBG, and BGBGBG) were equally likely in the birth sequence
Similarly, 34 out of 40 chose the mathematically correct response that the spinner would stop at section 3 because it had the greatest central angle. In the five-head task, most preservice teachers (34 out of 40) responded that the outcome heads or tails was equally likely to come up in the sixth trial even if there were all heads in the previous trials. The same proportion (34 out of 40) stated that it was more likely that they would be killed in a car accident rather than winning a jackpot in the lottery 6/49. Among all the written tasks, the circular spinner task evoked the widest variation. Out of 40 preservice teachers, 22 stated that they would choose any two numbers because all of them had the same probability of winning; eleven preservice teachers argued that they would choose numbers that were side by side. Five stated that they would choose numbers that were opposite each other.

Although participants' responses for all tasks were mainly formal in the written component, their justifications of the answers varied. These variations, both in answers and in justifications, were considered in the selection of participants for the pair-problem solving and interview components.

Selection of preservice teachers for pair-problem solving and interview

Three criteria were used to select preservice teachers for the pair-problem solving and individual interviews: that the preservice teachers had taken at least 10 mathematics courses in their undergraduate degrees, that they had taken some university or secondary school probability courses or had some background knowledge in probability, and that the responses they provided to the tasks in the written component differed from one another's.

A total of 16 preservice teachers, 4 from the summer group and 12 from the fall group, were selected for the pair-problem solving and the individual interviews (see Table 3 for their mathematics and probability backgrounds).
Table 3. Background information of preservice teachers selected for the pair-problem solving and the interview.

<table>
<thead>
<tr>
<th>Pseudonyms</th>
<th>Academic qualifications</th>
<th>No. of university mathematics courses taken</th>
<th>No. of university probability courses taken</th>
<th>Year the probability courses were taken</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alan</td>
<td>BSc Elect. Eng.</td>
<td>10</td>
<td>1</td>
<td>2nd, Grade 12</td>
</tr>
<tr>
<td>Anne</td>
<td>MA</td>
<td>10</td>
<td>2</td>
<td>2nd</td>
</tr>
<tr>
<td>Bill</td>
<td>BA</td>
<td>20</td>
<td>1</td>
<td>3rd</td>
</tr>
<tr>
<td>Feng</td>
<td>BASc Mech. Eng.</td>
<td>10</td>
<td>1</td>
<td>3rd</td>
</tr>
<tr>
<td>Gary</td>
<td>BSc Math</td>
<td>18</td>
<td>1</td>
<td>3rd</td>
</tr>
<tr>
<td>Gita</td>
<td>BSc Math/Biology</td>
<td>12</td>
<td>1</td>
<td>3rd, Grade 12</td>
</tr>
<tr>
<td>Ivan</td>
<td>BA Math/English</td>
<td>18</td>
<td>1</td>
<td>3rd</td>
</tr>
<tr>
<td>Jane</td>
<td>BSc Chemistry/Math</td>
<td>12</td>
<td>1</td>
<td>3rd (Part of a chemistry course)</td>
</tr>
<tr>
<td>John</td>
<td>BSc Math/Comp. science</td>
<td>16</td>
<td>3</td>
<td>2nd, 3rd</td>
</tr>
<tr>
<td>Lisa</td>
<td>BSc Stat</td>
<td>16</td>
<td>1</td>
<td>2nd</td>
</tr>
<tr>
<td>Mary</td>
<td>BA French/Math</td>
<td>14</td>
<td>3</td>
<td>2nd, 3rd</td>
</tr>
<tr>
<td>Mate</td>
<td>BSc Math</td>
<td>14</td>
<td>3</td>
<td>2nd, 3rd</td>
</tr>
<tr>
<td>Paul</td>
<td>BSc Math</td>
<td>17</td>
<td>2</td>
<td>3rd</td>
</tr>
<tr>
<td>Reid</td>
<td>BSc Physics/Math</td>
<td>10</td>
<td>2</td>
<td>3rd, Grade 12</td>
</tr>
<tr>
<td>Ruby</td>
<td>MSc Math</td>
<td>25</td>
<td>2</td>
<td>2nd, 6th</td>
</tr>
<tr>
<td>Ruth</td>
<td>BA Math</td>
<td>20</td>
<td>2</td>
<td>3rd, Grade 12</td>
</tr>
</tbody>
</table>
Eight pairs of preservice teachers, two from the summer group and six from the fall group, were grouped by the investigator. Each pair consisted of preservice teachers with diverse backgrounds and opinions so as to produce interactions that would help broaden the exploration of students' conceptualization of probability. For example, Gita and John were paired because Gita used mathematical reasoning in the written component whereas John used elements of the representativeness heuristic. In another pair, Ruby's reasoning was based on mathematical concepts while Ruth's reasoning moved between mathematical and everyday reasoning.

This kind of pairing was not always possible because there were more students who demonstrated mathematical reasoning than everyday reasoning. In such cases, pairs were made using other criteria, for example the number of mathematics and probability courses they had taken at university and their major subjects. The pairs are shown in Table 4.

**Procedures for pair-problem solving and interview**

The main purpose of the pair-problem-solving component was to gain insights into preservice teachers' probabilistic thinking in a different social setting from that of the individual setting of the written and the interview components. The main purpose of the individual interview was to probe in greater depth participants' beliefs as revealed in the written tasks and pair-problem solving, and to explore further their conceptions of probability, while remaining open to the possibility that new conceptions might emerge.

In the pair-problem solving, participants were asked to discuss possible solutions to tasks. That is, pairs discussed and attempted to solve tasks together. The investigator observed the pairs' work and made notes. Only in certain circumstances, for example if pairs stopped talking or reached an agreement very quickly, did the investigator provide prompts. The discussions were audiotaped.
<table>
<thead>
<tr>
<th>Pair No.</th>
<th>Participants and relevant selection information</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Gita (mathematical)</td>
</tr>
<tr>
<td></td>
<td>John (elements of representativeness heuristic)</td>
</tr>
<tr>
<td>2</td>
<td>Feng (mathematical, mechanical engineering graduate)</td>
</tr>
<tr>
<td></td>
<td>Reid (mathematical, physics graduate)</td>
</tr>
<tr>
<td>3</td>
<td>Alan (Elements of gambler's fallacy)</td>
</tr>
<tr>
<td></td>
<td>Ivan (mathematical and everyday reasonings)</td>
</tr>
<tr>
<td>4</td>
<td>Anne (conflict between mathematical and everyday reasonings)</td>
</tr>
<tr>
<td></td>
<td>Gary (mathematical)</td>
</tr>
<tr>
<td>5</td>
<td>Bill (mathematical and everyday reasonings)</td>
</tr>
<tr>
<td></td>
<td>Mate (mathematical)</td>
</tr>
<tr>
<td>6</td>
<td>Jane (elements of representativeness heuristic)</td>
</tr>
<tr>
<td></td>
<td>Paul (mathematical)</td>
</tr>
<tr>
<td>7</td>
<td>Lisa (elements of representativeness heuristic)</td>
</tr>
<tr>
<td></td>
<td>Mary (conflict between mathematical and everyday reasonings)</td>
</tr>
<tr>
<td>8</td>
<td>Ruby (mathematical)</td>
</tr>
<tr>
<td></td>
<td>Ruth (conflict between mathematical and everyday reasonings)</td>
</tr>
</tbody>
</table>

Each tape of the pair-problem solving was analyzed to make sense of preservice teachers' thinking about probability problems in a pair setting. Notes were made by the researcher during this analysis. Each interview was guided by these notes and the interview protocol (see Appendix C) prepared by the investigator. In the interview, respondents were asked to read tasks and provide their opinions. Subsequent questions were based on respondents' initial reactions to the tasks.
During the interview, each preservice teacher was allowed to ask questions related to the tasks. They were also allowed to perform experiments with marbles for the board task, roll dice to answer questions on the dice task, or roll thumbtacks to answer questions on the thumbtack task. They were also requested to think aloud so that their ideas could be audiotaped. The investigator also made observation notes immediately after each interview was concluded. These notes were added into the data transcripts.

Data transcripts

Before the tapes were transcribed, the researcher listened to them to get a sense of the pair-problem solving and interview sessions. It was noted that there were repetitions of words and sometimes sentences in the tapes. In some cases there were personal conversations with students unrelated to probability. Except for those repetitions and personal conversations, all other portions of each tape were transcribed. However, words such as "ya", "huhuh", "right", "you know", "well", "um" etc. were not transcribed unless they were meaningful in the contexts of the interview and problem solving. The researcher also added his notes on participants' interviews or the pair-problem solving to the transcriptions. In some cases notes were added to the transcripts directly from students' written work during the interview or the pair-problem solving.

Data analysis

Audiotapes from the pair-problem solving and the individual interviews were transcribed together with the researcher's summary of the important concepts and patterns that appeared (Hedges, 1985). A summary of each preservice teacher's responses to the pair-problem-solving tasks and the interview tasks was presented in words and sentences in large matrices. A similar summary of participants' responses to the written tasks was prepared.
These summaries were then used to categorize participants' responses into qualitatively different conception of probability within formal and non-formal conceptions.

Conceptions were called formal if there was a clear indication of explicit reference to and use of mathematical concepts. Conceptions were called non-formal if there was no explicit reference to and use of mathematical concepts.

Examples of preservice teachers' formal conceptions of probability include the following:

1. Use of a mathematical formula, for example, the probability of winning a jackpot in the lottery 6/49 is \( \frac{49!}{(49-6)! \times 6!} \).

2. Use of a mathematical technique, for example, the probability of getting any given sequence in the head-tail sequence task is \( \frac{1}{2^5} \), because the probability of getting each event is \( \frac{1}{2} \) and there are 6 events that are independent of each other.

3. Use of a mathematical idea, for example, the spinner in the rectangular spinner task is most likely to stop in number 3 because the section with number 3 has the largest central angle and the spinner is spun a large number of times.

4. Use of an experiment, for example, to determine whether the thumbtack is going to land point down or point up one should throw or roll thumbtack at least 30 times and calculate the relative frequencies. That is the ratios of points up and points down with the total number of trials.

5. Use of a mathematical reasoning, for example, "Any combination of six numbers is exactly as likely as any other combinations of six number in the 6/49."
The conceptions which could not be determined as formal were categorized as non-formal. For example:

1. The participants' references to their experiences of playing or observing games, observing customers at bank, predicting outcomes based on luck, and so on.

2. The use of representativeness heuristic, that the sequence THTHHT, in the head-tail sequence task, were most likely because the number of heads and tails were equal, and the events in the sequence were random.

3. The use of knowledge from disciplines other than mathematics.

In certain cases, however, when it was unclear as to whether their responses fell into either the formal or non-formal category, responses were categorized as jointly formal and non-formal. For example, in the head-tail sequence task, if preservice teachers recognized that the previous five outcomes did not influence the next outcome because the flip of a coin was independent and random, then they were categorized as a combination of formal and non-formal conceptions. This response might be from their learning of independence and randomness in school, university, or everyday experiences with coins.

In some cases, the preservice teachers responses to certain tasks were categorized as a conflict between formal and non-formal conceptions because the participants indicated that they were not sure as to which reasoning they should apply for solving the task. For example, in solving the lottery task, a response was categorized as a conflict between formal and non-formal, if a participant stated that intuitively a random sequence of numbers was more likely than a consecutive sequence of numbers to win the jackpot in the lottery 6/49, but mathematically any six numbers was equally likely to win it.
The categories were then interpreted in terms of the following questions:

What are the similar or different ways in which the preservice teachers conceptualize probability in the written, the pair-problem solving, and the interview settings? How do preservice teachers' responses vary from one task to another in the individual and pair settings? How does one participant modify or elaborate his or her conceptions of probability with respect to another participant's conceptions? What are participants' views about the role of academic probability in their everyday lives? Based on these questions the researcher wrote his interpretations of each preservice teacher's probabilistic thinking.

Data review and validation

Two methods were used for reviewing and validating the data in the study. First, an opportunity was provided for each preservice teacher to comment on his or her transcript. Second, an opportunity was provided for each participant to discuss the researcher's interpretation of his or her conceptions of probability.

Discussion of transcripts with preservice teachers

Transcripts for both the pair-problem solving and the individual interviews were given to the respective preservice teachers for their comments and elaboration. They were asked to clarify words, phrases, or statements that were not understood by the researcher. The participants were allowed to modify certain statements.

The preservice teachers returned the transcripts together with their comments within a period of one to two weeks. Some preservice teachers did not make any comments and agreed fully with what was written in the transcripts. Some others only made editorial comments. A few made substantial
comments and clarified their statements during the pair-problem solving and the individual interviews. The modifications made by the participants helped the researcher to determine their views of probability further.

**Discussion of researcher's interpretation with preservice teachers**

The researcher's interpretation of each participant's probabilistic thinking was distributed to that person for comments and criticisms. Everyone provided a response to the researcher's interpretations.

Some preservice teachers were happy with the researcher's interpretations, although they commented that they may not have come to the same conclusion as the researcher. Some others wanted some clarification about the researcher's interpretations of their thinking process. Only two preservice teachers disagreed with some of the researcher's interpretations of their probabilistic thinking. The reasoning they provided helped the researcher to probe their conceptions further. The final version of the analysis, interpretations, and conclusions of this study fully incorporates the preservice teachers' comments on the researcher's interpretations.
CHAPTER 4
Preservice Teachers' Conceptions of Probability

This chapter presents the analysis and interpretation of data for the first research question:

What are some qualitatively different ways in which preservice teachers conceptualize probability?

In order to answer this question, the qualitatively different conceptions of probability that the preservice teachers demonstrated during various tasks and settings of the study are discussed. The conceptions described in the study are derived from 16 preservice teachers' responses on the written tasks, the pair-problem-solving tasks, and the interview tasks.

Qualitatively different conceptions of probability

Based on the analysis of data obtained from the various tasks and settings, several themes are developed regarding preservice teachers' conceptions of probability. These themes are grouped into formal and non-formal conceptions of probability. The themes and the associated conceptions are presented as distinct for analytical purposes only.

Preservice teachers' formal and non-formal reasoning about all tasks in the written, pair-problem-solving, and interview components are shown in Table 5. Table 5 shows that the preservice teachers' responses varied based on tasks. All the participants, for example, used formal reasoning in the rectangular spinner task. Similarly, the majority of the participants reasoned formally in the birth sequence task and the lottery and car accident task in the written component, and the five-boy task and the circular spinner task in the pair-problem-solving component. In the dice task, however, the majority of them either reasoned non-formally or provided conflicting reasoning between formal and non-formal
Table 5. Preservice teachers’ formal and non-formal conceptions of probability.

<table>
<thead>
<tr>
<th>Written</th>
<th>Pair-problem solving</th>
<th>Interview</th>
</tr>
</thead>
<tbody>
<tr>
<td>BST</td>
<td>RST</td>
<td>FHT</td>
</tr>
<tr>
<td>Alan</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>Anne</td>
<td>F &lt;-&gt; N</td>
<td>F</td>
</tr>
<tr>
<td>Bill</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>Feng</td>
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<tr>
<td>Gary</td>
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</tr>
<tr>
<td>Gita</td>
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</tr>
<tr>
<td>Ivan</td>
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<td>F</td>
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<tr>
<td>Jane</td>
<td>N</td>
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</tr>
<tr>
<td>John</td>
<td>F &lt;-&gt; N</td>
<td>F</td>
</tr>
<tr>
<td>Lisa</td>
<td>N</td>
<td>F</td>
</tr>
<tr>
<td>Mary</td>
<td>F &lt;-&gt; N</td>
<td>F</td>
</tr>
<tr>
<td>Mate</td>
<td>F</td>
<td>F</td>
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<tr>
<td>Paul</td>
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<td>Reid</td>
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<td>Ruby</td>
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<tr>
<td>Ruth</td>
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</tbody>
</table>

Legend:
- BST = Birth-sequence task
- RST = Rectangular spinner task
- FHT = Five-head task
- LCT = Lottery and car accident task
- CST = Circular spinner task
- HTT = Head-tail sequence task
- FBT = Five-boy task
- BTT = Bank teller task
- LT = Lottery task
- BMT = Board and marble task
- DT = Dice task
- TT = Thumbtack task

- F = Formal thinking
- N = Non-formal thinking
- F N Formal and non-formal thinking without conflict
- F <-> N Formal and non-formal thinking with conflict
conceptions. Their responses were mixed with formal and non-formal reasoning in the five-head task during the written component, the lottery and car accident task and the bank teller task during the pair-problem-solving component, and the board and marble task, and the thumbtack task during the interview component. Unlike the dice task, the participants did not demonstrate a conflict between formal and non-formal thinking during these tasks.

It appears that the preservice teachers provided different types of responses to different tasks. If the task could be solved directly by mathematical rules or formulas, they applied formal reasoning. If the task appeared to be from an everyday context, and no mathematical rules or formulas came to their mind, they used non-formal reasoning. For example, they demonstrated a formal conception of independence and randomness on the rectangular and circular spinner tasks which were similar to textbook problems. But many of the same participants did not demonstrate a formal conception of independence and randomness on the dice task which was associated with an everyday context.

Qualitatively different conceptions of probability demonstrated by the participants within their non-formal and formal thinking are discussed below.

Non-formal conceptions of probability

Almost all the preservice teachers at some point interpreted probability based on their observations of how things worked in their everyday lives. Three conceptions—using everyday experiences, using the representativeness heuristic, and using science knowledge in solving probability problems—are discussed in this section.

Using everyday experiences in solving probability problems

Experiences gained by people in different facets of their everyday lives, such as playing or observing games in casinos, playing lotteries, observing the
Flow of people in banks, and making everyday decisions are considered as everyday experiences for the purpose of this analysis. The experience the preservice teachers gained in university mathematics or probability courses are not considered everyday experiences. The preservice teachers' use of their everyday experiences was particularly evident in the dice task during the interview component and the bank teller task during the pair-problem solving. Preservice teachers also used their everyday experiences in the lottery and car accident task, which was asked during both the written and the pair-problem-solving components.

According to the mathematics of probability, the dice task can be solved in three ways: (1) by calculating the expected value for a player, which comes to about a loss of 8 cents for every dollar; (2) by calculating the probability that a player will win, which is about 42%; and (3) by calculating the probability that a dealer will win, which is about 58%. If someone correctly uses his or her school or university knowledge of probability, he or she would come up with the solution that the dealer has the advantage.

Although the majority of the preservice teachers attempted to solve the dice task formally in the beginning, most of them abandoned their attempts and used everyday experiences to decide whether they would want to be the dealer or the player. All the respondents knew the probability of simple events, for example that the probability of getting a number in the roll of a die is \( \frac{1}{6} \).

However, most of them had difficulty in finding out the probability of compound events, such as the probability of getting the same number in two or three dice simultaneously. As a result, only three of 16 participants used appropriate mathematical strategies in solving the dice task; all the rest either attempted to solve it by using their everyday experiences or by using inappropriate mathematical strategies.
While attempting to solve the dice task, some preservice teachers used their personal experiences of playing or observing dice games and similar games in the real-world. For example, Anne stated that she would want to be the dealer because "most gambling casinos seem to win" and John stated that it is the dealers who eventually win money. Lisa said the same thing by stating "when you play cards with your friends the chances are a lot of times for the dealer to win." Ruby stated the following from her experience of observing casinos:

Well, I’ve not gambled but I’ve been to casinos just to watch people. I think it’s fun to watch them. They get really excited with games and they keep betting higher and higher amounts of money. Most of them couldn't pay any more because they lost it. Of course I could be unlucky but my chances are better if I’m the dealer.

Based on her experience of observing games in casinos, Ruby was pretty sure that the dealer was going to win. At her first sight of the task, she was not concerned with calculating the mathematical probability of the dealer or the player of winning.

Ruby was also drawing on the issue of luck when she stated "I could be unlucky but my chances are better if I’m the dealer." Her reasoning implied that luck plays a very important role in gambling. She thought that probability alone cannot be used to predict what is going to happen. The issue of luck, she was using in solving this problem, was drawn from everyday experiences and not from formal experiences received during university mathematics and probability courses.

Similarly, Mary and Ruth argued that winning $100 depends very much on luck. Mary thought that if she happened to bet a large amount, say $10, at the time when her number turns up then she would win. Moreover, if she were
lucky, the same number could turn up in two dice or even in three dice. It is important to note that Mary was not confusing chance with luck. She was differentiating between chance and luck by stating that "the chances are that one out of 6 times that I roll this dice it is going to be a 6. Luck is choosing the time it's going to be." If she were lucky, Mary thought, she could still win the money from the dealer even if the dealer had a higher chance of winning. Ruth was also concerned about luck. She stated that "if you win on the time when it's yours and you win like $20. And you lose on the other 5 times it's only $1; then you lose $5." Then she thought how lucky she would be to win this large an amount. Her solution to the problem ultimately relied on luck. Hence, Mary's and Ruth's final decisions to be the player were not based on the calculation of probability but on a feeling they developed from their everyday experiences.

Anne's and Lisa's view was that probability should be balanced with risk and control. Anne, for example responded to the dice task as follows:

The dealer has to risk 3 times in ... every roll of the three dice. He's risking three times the amount of money that the player bets. [Thinking] Yeah, the dealer takes greater risk. But ... it favors the dealer in terms of the other five numbers coming. So it's just a matter of evaluating whether that ends up putting a balance between player and dealer or whether or not it still favors the dealer. For me, in a situation like this, like I said, comes down to also other factors that don't have to do with necessarily probability, but just where I can see factors that I can control [laughs].

Anne's decision to be the player in the game was not because she calculated probability or the expected value mathematically but because she could control the amount of the bet as a player. Anne did not worry too much about the probability calculation because as a player she had the power of
controlling the bet. Like Anne, Lisa also talked about control. However, unlike Anne, Lisa stated that she wanted to be the dealer because "the possibility is controlled by the dealer." According to Lisa, the landing of the dice depends on how you throw the dice. It is notable that both Anne and Lisa showed their awareness that what they said was contradictory to the mathematics of probability. Nevertheless, they were quite happy in solving the problem with the feelings they had developed in their everyday experiences.

In the bank teller task, however, all pairs of preservice teachers attempted to use their mathematical knowledge as far as possible. Usually, they made a table or a diagram to look at the flow of customers arriving at the bank. Many preservice teachers looked at a worst possible scenario of six customers arriving per minute and proposed four tellers or five tellers to serve customers. However, at this point the reliance upon pure mathematical knowledge was suspended because they immediately questioned their solution in terms of cost efficiency. A bank manager in a real life situation would try to reduce the cost of banking as far as possible. Employing five tellers at the bank would not be a cost-effective solution.

When the preservice teachers attempted to think of a new strategy to solve the task they were distressed by the fact that 1 to 6 customers could arrive at the bank. There was no way to know how many customers would arrive at a particular minute. Although they stated that they could use the concept of randomness from their probability courses, they decided that the number of customers arriving at the bank is not random in a real life situation. Hence, they felt that using a simulation did not provide a useful solution to the problem. For example, Ivan and Alan had the following discussion about the task:

Ivan: Right, you don't know. But again if you look at other things ... to solve a math equation or to make this a math question the average
rate is three and a half. Overall of time it's three and a half.... For a particular second or for a particular hour you might have lots of people. I mean if you look at when banks are busy—in the afternoon and in the evening. And so if you really want to be more technical you want less tellers in the morning, less tellers two o'clock in the afternoon, more on lunch, and more on evening. More at three o'clock. Basically three and a half people coming in a minute is what you're working with.

Alan: These are interesting questions because it says the numbers of customers arriving varies between 1 and 6 customers per minute but it doesn't say what the distribution of those is over time.... We made a simplifying assumption which is that the numbers 1 and 6 are evenly distributed over time. There could be periods where you get six customers a minute or an hour. That information is not given to us.

Ivan and Alan used their everyday experience of observing customers at different times and decided that the distribution of customers arriving at the bank is not uniform throughout the day. They stated clearly that they decided to work with the average number of customers to come to a mathematical solution while knowing that the solution might not be useful in real life situations. Ivan and Alan gave a clear message that they make many simplifying assumptions in mathematics that may not be helpful in dealing with the complexities of real life situations. Anne and Gary had a very similar opinion on the problem.

Most pairs were not satisfied with their mathematical solution to the task especially because their everyday experiences differed from the mathematical solutions. Some participants argued that they would not go for a mathematical solution to the problem because it would not provide an appropriate
representation of a real life situation. However, preservice teachers were comfortable with building a mathematical model through observation of a real life situation in order to arrive at a more useful solution to the problem. For example,

In a real life situation, if I was the manager I'd ... observe it for a couple of days, record the observation of waiting time and decide according to that. I won't bother to make this so complicated. (Gita with John)

In the real world, it's going to be more complicated. The only way to assign the number of tellers is by observing what actually happens from day to day. Even if you decide once, you have to make changes from time to time. (Reid with Feng)

The above pairs argued that the solution to the problem becomes more meaningful when the decision is made by actually observing the number of arrivals at the bank for some days. They argued that the number of customers arriving at the bank at a particular minute is not random. Assuming the arrival to be random is risky for making good practical decisions. The preservice teachers were arguing that everyday experiences are more important than the mathematics of randomness in solving this task.

Similarly, the majority of the participants used their everyday experiences in the lottery and car accident task asked in the written and the pair-problem-solving components of the study. For example, Lisa stated that there is a lot of news on various media about car accidents and deaths. But there is hardly any news about the people who win the jackpot in the 6/49 lottery. Hence, for Lisa, the chance of being killed in a car accident was much higher than winning a jackpot in the 6/49 lottery.
However for some of the preservice teachers, like Feng, mathematics learned at the university helped to come to a solution to the problem. In the written component Feng stated that "the probability of winning a [jackpot] in 6/49 is about 1/14000000. I am sure the rate of a fatal car accident is much higher than that." She knew that the probability of winning the jackpot in the B.C. lottery 6/49 is approximately 1 in 14 million. Nevertheless, she used her everyday experience to decide that the ratio of being killed in a car accident in B.C. was much higher than 1:14000000.

Some of the preservice teachers; like Anne, Bill, Ivan, Jane, and Mary, disregarded probability calculations for the lottery and car accident task. They stated that it was more likely that they would be killed in a car accident because they did not buy lottery tickets but they did drive cars. For example, Jane stated the following when she was working with Paul:

Paul: First of all we have to find out the probability of being killed in a car accident.

Jane: I didn't have much problem with this question.... I don't play lottery 6/49. So the probability of me ever winning a jackpot without buying a ticket is not going to happen.

Jane was quite satisfied with her answer. But Paul argued that they should use mathematics to come to a decision rather than simply avoiding it. Paul stated that they could make assumptions to solve the problem, such as buying one lottery ticket per week. However, Jane indicated several problems such as:

Jane: You've to drive a car once a week too or walking.... To me the opportunity should be equal in both cases in order to be able to compare ... Again that would depend on how many people buy the
tickets and ... how many people are driving cars and how many streets you've to cross.... Actually you have to know more information.

Paul: Yeah. There're several interpretations for this question.... We have to make some assumptions such as the probability of one person being [killed] in a car accident. You have to assume that ... each individual in British Columbia assumes equal risk. Which is not true, so it's a wrong assumption. However, we have to start somewhere. The second you're born you have a continual probability of "whatever," throughout till the time you die.

Clearly, the task was not solvable by a simple mathematical model. Paul believed in making assumptions if the tasks were not well formulated. He attempted to convince Jane that since they were mathematics graduates they should be providing formal solutions even if the solutions were unrealistic. Gita and John went through the same sort of discussion. Gita stated that having a car accident depends on "whether you drive or not? What time of the day you drive? Probability of being killed if you are in an accident," and so on. John supported her and argued that they could not provide a formal solution to the problem that would be meaningful in the everyday world. Even after a solution was offered, the preservice teachers suspected that the solution would not actually reflect what would happen in the everyday world.

In some other tasks, for example, the circular spinner task, the board and marble task, and the thumbtack task, some of the preservice teachers used their everyday experiences and not the mathematics of randomness. Ruth, for example, had the view that probability could be controlled by people. In the circular spinner task in the written component of the study, Ruth stated that the
landing of the spinner depended on the amount of force applied to the spinner. She stated,

I would pick 4 and 5 because if the spinner is not spun hard enough, it might not go around to 6, 7, and 8. It is very likely that it would get past 1, 2, 3 even with a very light spin. (Ruth in the written component)

Ruth, however, made it clear that "if it was going to be spun hard, then any two pairs of numbers would have an equal chance of winning." Ruth's thinking was similar in the thumbtack task. She thought that the way in which the thumbtack lands depends on the height of the drop. She stated that the tack would be more likely to land point up if it was dropped from a larger height because the flat side of the tack was heavier.

Like Ruth, Ivan thought that results of experiments could be controlled by people. In the board and marble task he sent the marble in any number 1, 2, 3, or 4 he wanted, most of the times. He controlled the movement of his finger and stated that he was studying the bounces made by the marble. He argued that he would be more comfortable if the experiment was replaced with a computer simulation. Ivan argued that when experiments are manually conducted by human beings, the outcomes cannot be purely random. Gary also stated that the movement of the marble could be changed by changing the place where it was dropped from and the initial force applied to it. In all these examples, the preservice teachers were stating that there are many human factors involved in determining probability, and so simple mathematical solutions could not be useful for everyday purpose.

As discussed by the preservice teachers, the lottery and car accident task, the dice task, and the bank teller task could be solved mathematically. Many of the preservice teachers, however, chose to make a decision based on their
everyday experiences. Although these preservice teachers had strong backgrounds in mathematics and probability, they were quite satisfied with the answers they developed from their experiences. They were actually aware that the mathematics of probability differs from the non-formal reasoning that they had developed in their everyday experiences. Nevertheless, except in some tasks, their first instinct was drawn from everyday experiences rather than from formal knowledge of mathematics. They tried to use formal knowledge only when they were not satisfied with their non-formal solutions or when the researcher requested that they think formally.

The preservice teachers' everyday experiences were very important in the way they approached each task. In many cases, the preservice teachers' everyday experiences and their mathematical knowledge conflicted with each other. Some of them hypothesized that the researcher would value their formal rather than their non-formal knowledge of probability, so they attempted to develop a formal answer as far as was possible. Others stayed with their non-formal answers based on everyday experiences. In some tasks, the preservice teachers' everyday experiences were so strong that they looked at the tasks from their everyday perspective rather than from a formal mathematical perspective.

**Using a representativeness heuristic in solving probability problems**

According to Kahneman and Tversky (1972), the representativeness heuristic has two components: (1) the outcome in the sample should look like its parent population and (2) events in an outcome should appear random. Lisa's reasoning involved both characteristics. In the head-tail sequence task, in the interview setting, Lisa stated as follows:

Okay, first of all I thought that all four outcomes are possible and then I realize that ... it also deals with order. When you say for example (b) is
HHHTTT... the first one of course is half/half, either half head or tail. Then... if you want to produce a second head I think the probability of producing a head and tail would be different because since the first time is not a tail so I think increases the probability of having a tail instead of a head. That's why I'm thinking that the... four outcomes will not be equally likely. You have to look at the order to decide which one is most likely.... Here the first one is head and then tail then following four heads. I'm trying to think about ... a possible answer and also ... I try to use my common sense to decide. I think the first two are not most likely. (c) and (d) are probably likely in the sense they're more even like three tails and three heads but the order is different. It looks that (c) is more random than this one [d]. So ... I feel somehow that the random is more likely. If you want to produce an order that is most likely I would say probably (c).

Lisa argued that the most likely outcome should have approximately an equal number of heads and tails that were randomly distributed. Her argument followed the logic of the representativeness heuristic. Jane, another preservice teacher, used the first component of the representativeness heuristic in the head-tail sequence task in her interview setting as follows:

Jane: Basically my reasoning is—(a) and (b) ... seem to be a little bit less likely of occurring than (c) and (d). I don't see much difference between (c) and (d) because they're just sort of flipping back and forth between. My general understanding of flipping a fair coin will be that if you have a large enough sample, you're going to get basically about fifty percent of each type showing up. So in that sense ... there are two things that I'm thinking of; partly the samples because there is only a string of 6—kind of too small to be able to
tell. But given that I still think I know what the right answer should be ... which is (e).

Hari: Which of the following orderly sequences is least likely to result from flipping a fair coin six times? [shows the same sequences HTHHHH, HHHTTT, THTHHT, HTHTHT]

Jane: In my feeling ... (a) would be less likely.... The length of the string of the same thing occurring is probably less likely.

Jane stated that the right answer should be (e) wherein all sequences are equally likely especially because of the relatively small sample size. If the sample size was large enough, she would think that a sequence with nearly an equal number of heads and tails would be most likely. When asked which sequence was least likely Jane stated that a sequence with a long same-side run, such as HTHHHH, would be least likely. Jane stated that since the coin is fair a most likely sequence should have approximately an equal number of heads and tails. She was concerned with whether the outcome looked like its parent population, not with whether the event head or tail appears as randomly distributed. Hence, Jane was using the first component of the representativeness heuristic to come to her solution to the task.

Jane and Lisa also used the gambler's fallacy in the above task. The gambler's fallacy is discussed in this section because it satisfies the first characteristic of the representativeness heuristic. The gambler's fallacy states that, in the case of two equally likely events, A and B, subjects tend to predict outcome A after a long run of outcome B. The gambler's fallacy is closely related to the representativeness heuristic because people using this fallacy expect an equal number of heads and tails in an outcome. Jane said that since there were already five heads a tail was more likely to come because a fair coin was used. Otherwise, she suspected the coin might not have been a fair one. Lisa was also
using the gambler's fallacy when she said that the probability of getting a tail in the second trial increased because a head occurred in the first trial.

The most consistent user of the gambler's fallacy in the study was Alan. He used the gambler's fallacy in the dice task as follows:

Hari: Now it's time to decide. Now, you have the opportunity of becoming a dealer or a player?
Alan: I'll be the player.
Hari: Why is that?
Alan: Because I can look at the patterns and I can get some ideas of what might be happening.... The longer we play the game the more likely I think I am to win. [thinking] I think I'd still be the player but the reason for being the player is that I can look for cases where a certain number has not come up for a long time. Those cases are likely to come up. I'll start betting on them. That's the strategy that I would use.

Alan reasoned that he expected the probability of the occurrence of an event to be higher when it had not occurred for a long time. Alan not only said this but also applied the fallacy in choosing the numbers when the game was played. He was a player and the researcher was a dealer. Alan chose numbers which had not appeared on the previous rolls of the dice and argued that since those numbers had not occurred in the previous throw they had a slightly better chance of occurring next time. Alan used the gambler's fallacy in other tasks, such as the circular spinner task and the lottery task, as well.

Feng's response resembled the gambler's fallacy when she argued that she would keep doubling the bet and eventually win the money from the dealer. But her response was different from Alan's. She was concerned that the total amount
of money she had was only $100, which would not be sufficient to keep doubling
the bet. Her reasoning was somewhat similar to Gary's who stated,

You're stuck with a hundred dollars. I'm not really sure. I've no
reasoning as to why I say this. I think I rather be the player.... I'm
thinking more in terms of go on forever because there is only a hundred
dollars. I think if it went on forever it would be to my advantage.

But Gary knew that $100 would not be enough, even to double 10 times.
So Gary was not satisfied with the answer he provided as to why he wanted to be
the player. Finally, he remembered that he should have calculated the expected
values for the player to win. However, Gary stated that he forgot how to
calculate expected values and maintained his non-formal answer.

Using science knowledge in solving probability problems

Many preservice teachers studied physics in their undergraduate degrees.
As a result, their interpretation of some probability problems relied on their
physics knowledge. Their use of physics knowledge was particularly evident in
the case of the thumbtack task. To answer the question of whether it is more
likely that the thumbtack would land with point up or point down the preservice
teachers used terms such as stability, gravity, momentum, and so on. For
example, Alan's view is depicted in the following transcript:

Alan: I'd say point up is more likely.

Hari: Why is that?

Alan: Just because the weight of the head is going to be more likely to
cause it a fall like that.... This [point up] looks more stable than that
[point down]. So if we try [drops the thumbtack] it's more likely to
fall on the stable state.... I think the tack will more naturally go in
that position [point up].
Hari: Okay. But when you drop this it can also bounce. Right?

Alan: Right. [drops the tack]. But as it bounces it is going to naturally rotate the heavy end down. So I don't think it's random.

Bill's reasoning was very similar to that of Alan.

Bill: With the point up.

Hari: Why?

Bill: Because it's heavier at this end. It's going to fall that way. There is more weight at the bottom of the pin. So gravity is going to pull that towards bottom. Well, it might go down but the odds are. That's what my reasoning is. It's going to wind up like that [shows how it winds up].

Hari: So you don't consider the fact that it can bounce?

Bill: Even if it bounces ... it's just how much momentum it has when it goes to make a flip or something along the table. That's what I think. And you can work it out.

Similarly Lisa used her knowledge of the center of gravity to solve this task.

Lisa: Okay. I'll say this [point up] is more likely.

Hari: Why?

Lisa: The reason is—it's about physics. It looks that the center of gravity is based down here [center of the tack]. You can even feel it. So whenever you throw it I think it's more likely that it's going to turn up [point up] this way rather than this way [point down]. This is not balanced [stable].

Hari: Have you considered the bouncing factors when you drop it from here?
Lisa: No. But it may be. I didn't think about that. But when you say it I realized when it bounces—I think because the center of gravity is at the bottom. So even when it bounces, I think, provides the opportunity to land upside down. So, I think, this [point up] is more likely.

Similar to Alan, Bill, and Lisa, several of the other participants such as Feng, Ivan, and John, argued that the tack would land point up because the flat side of the tack is heavier and more stable than the pointed part. However, Reid mentioned that the way the tack lands depends on the curvature of the tack and that the bounce factors indicate that the tack should land point down. The above transcripts indicate that the participants did not consider the thumbtack task as a purely probability problem driven by a random process.

Likewise, some of the preservice teachers used their knowledge of genetics to solve the five-boy task in the pair-problem-solving component. They argued that the five-boy task may have had to do with a genetics factor rather than with a random phenomena. Anne, while working with Gary, stated that some couples may have only boys or girls because of hereditary factors. Reid also provided a similar statement when he was working with Feng. He first stated that the probability of having a boy or a girl for John and Tracy in the sixth trial should be equally likely. But a little later he stated,

But it's also a matter of genetics. If they have already five boys then actually the likelihood of having another boy is more because it is determined by male sperms. If the trend is already there for boys there is a tendency towards male children. (Reid, pair-problem solving)

Working with Ruth, Ruby also thought that genetics may have something to do with birth patterns in John and Tracy's family. She stated that because of
genetics factors some couples have girls after girls or boys after boys. She thought that if it was a genetics factor that was influencing the birth of babies in John and Tracy's family, then it was more likely that they would have a boy in the sixth trial as well. However, she argued that she needed more data about John and Tracy's family trees to come to this conclusion. Otherwise, she would simply assume that the chance of having a boy or a girl at a particular birth was 50% and so John and Tracy would have an equal probability of having a boy or a girl in the sixth trial. The discussion between Jane and Paul was similar to Ruth and Ruby. Like Ruby, Jane argued that the chance of having a boy or a girl may have actually been affected by male sperms and female fertility rather than affected by random processes.

The preservice teachers were using their knowledge of physics and genetics to solve some of the tasks posed in the study. When the task was related to throwing an object, they used their knowledge of physics, and when the task was related to the births of babies, they used their knowledge of genetics. In talking with the preservice teachers, it was at times difficult to determine whether they developed this knowledge from their science classes or from their everyday experiences. It appeared that such knowledge resulted from both science knowledge and everyday experiences. What was clear from the responses was that, for some preservice teachers, the use of probability knowledge, such as independence and randomness, was not applicable to this kind of problem.

Although many preservice teachers used non-formal conceptions of probability in some tasks, they also used formal conceptions of probability in other tasks. Table 5 shows that participants used formal mathematical reasoning in many tasks of the study.
Formal conceptions of probability

All the preservice teachers used their formal mathematics knowledge to solve particular probability tasks, such as the rectangular spinner task. The preservice teachers who used formal knowledge used such concepts as independence, randomness, and the addition and multiplication rules of probability. Two themes of conceptions: probability as independence and randomness and probability as formulas, are discussed.

Using independence and randomness in solving probability problems

The formal solutions to all tasks in all components of the study required an understanding of independence and randomness. Most of the preservice teachers demonstrated an understanding of independence and randomness in various tasks. Gita, for example, in the head-tail sequence task in the interview setting reasoned that

It's equally likely because the probability of having a head doesn't depend on ... a previous throw. The probability of getting a tail is not going to be greater if you had a head in the previous throw. If you had a head in the first time, the probability of getting a head in the second time is still one half. Doesn't matter if you've 10 heads in a row, the probability of getting the eleventh head is still one-half and the probability of getting the eleventh tail is still one-half.... What happened in the past doesn't influence the present throw.

Paul's reasoning was similar to Gita's in the lottery task. He had to advise students on whether a consecutive sequence or a random sequence of six numbers was more likely to win a jackpot in lottery 6/49.

Paul: Ill advise them that there is absolutely positively no difference in probability between 1, 2, 3, 4, 5, 6 or 1, 2, 3, 4, 5, 49 or 3, 5, 10, 36, 24
and whatever. Makes absolutely no difference. Any combination of six numbers is exactly as likely as any other combination of six numbers in the 6/49.

Hari: Do you think that the students will be convinced?
Paul: Convinced of that?
Hari: Yes. How would you convince the students?
Paul: One can think of this as there're 49 chips in a row. And the object is to pick six of those. They're arbitrarily numbered 1 to 49. Those numbers are strictly arbitrary. So they don't affect the likelihood of one chip being picked over another. Therefore these numbers are just as arbitrary. The probability of picking a 1 is exactly the same as picking a 47 or 48. Therefore picking 1, 2, 3, 4, 5, 6 is no more or less likely than picking any other six numbers.

Paul demonstrated a good understanding of independence and randomness in the above task. Later in the discussion Paul argued that no one can say for sure whether a sequence of numbers is random or not. Because whether or not a number is random depends on how it was selected. If the selection of numbers was random, Paul argued, the sequence of numbers is random even if they appeared consecutive. If the selection was not random, the sequence of numbers is not really random even if it appears random. Paul had a comprehensive understanding of random numbers.

But in the dice task several of the preservice teachers who attempted to solve the problem inappropriately used formal concepts of independence and randomness. Because the concept of independence was not properly utilized, Feng, Gary, and Mate thought the player had only 3 chances of winning out of 18 possible chances. The following transcript between the researcher and Mate illustrates this issue.
Hari: I think you have the feel for the game. Now it's time to decide whether you want to become the dealer or the player.

Mate: [Thinks for a while]. I'll probably choose to become the dealer.... When a person is putting down money on one number to occur ... out of 18 possible different outcomes here you have 3.... I'd probably think better off being the dealer in this case because the odds are in your favor. More or less, you've 15 out of 18 chances that a number other than ... one of the 1's, or whatever, will roll up here on dice.

Hari: Is it 15 out of 18 chances?

Mate: This is 1 in 6 chances, that is 1 in 6, that is 1 in 6. ... You have a 1 in 6 chances that a 1 will be up here on this dice, 1 in 6 chances that a 1 will be up there on that one, and 1 in 6 chances that a 1 will be up on that dice. So there is a 5 in 6 chance that a number other than 1 will appear on this dice. Same on this one. Same on that one. So the odds of losing are better than the odds of winning on any given dice.

However, Gary, Feng, and Mate were all concerned that their method of reasoning was perhaps mathematically incorrect. Feng thought that the probability for the player would not be that low. Gary thought that since the dice were independent of each other he had to look at each die individually. Later in the discussion Mate also thought that the dice were independent of each other and so he could not add probabilities. However, their knowledge of independence and randomness did not lead them to apply the multiplication rule of probability.
In the same way Bill, Reid, and Ivan got incorrect solutions because they used the addition rule of probability, when they should have used the multiplication rule.

My chance of getting 1 in each die is one in six. And there are three dice, so my chance of getting one in three of them is 3 in 6. That's fifty/fifty. Then my chance of getting 2 ones is, I guess, 1 in 36. Oh! hold on it is more than that [long pause, performs some mental calculations]. I don't remember how to do that. Let's see [Long thinking]. I don't know. Anyway, my odds are already fifty/fifty just by getting 1 to 1. Two 1's whatever the probability is on top of that one 1. And the probability of getting three 1's, even though that is very low (1 in 6³) that is also added to the total. So my odd of winning is better than fifty/fifty. (Reid)

There is a 1 in 6 chance you'll get your number here and 1 in 6 chance you'll get your number here and 1 in 6 chance you'll get your number here [shows different dice]. So the chance of me getting 1 is 3 in 6? Yes....

Actually I'm supposed to be the player. Because if I said 1, my chance of getting a 1 is 1 in 6, 1 in 6, 1 in 6. Right? So I should win half the times. ... But the chance of me getting two 1's should also be added to that. The chance of my getting two 1's is lower than my chance of getting one 1, obviously, or three 1's is lower than chance of getting one 1. But the sum of all those chances is higher than 50%.... It doesn't matter which number I pick. But my chances as a player winning are higher than that of the dealer winning. (Ivan)

Bill, Reid, and Ivan were all confused in their attempt to solve the task. All reasoned that the probability of obtaining a number on a die is \( \frac{1}{6} \). Since, there are 3 dice, the probability of obtaining a number on one of those 3 dice is \( \frac{3}{6} \).
or 50%. But the correct probability calculation places it at about 35% $\left[ 3 \left( \frac{1}{6} \times \frac{5}{6} \times \frac{5}{6} \right) = \frac{75}{216} \right]$. All these preservice teachers were aware that each die was independent of the other, and that the landing of a number on one die does not influence the landing of a number on the other dice. They also stated that they knew the multiplication rule of probability. However, when they were engaged in solving the problem the association between independence and the multiplication rule did not seem to occur to them.

Further, they did not appear to realize that the addition rule of probability holds in the case of mutually exclusive events, not in the case of independent events. The six faces of a die are mutually exclusive. However, one die is not mutually exclusive of the other. Rather, each one is independent of the other.

Some participants like Gita and Paul demonstrated an understanding and appropriate use of independence and randomness throughout the tasks and settings. Some others like, Bill, Feng, Gary, Ivan, Mate, and Reid thought about independence and randomness but could not correctly apply them in some probability calculations. These participants could appropriately use independence and randomness in some tasks, like the rectangular spinner task, but not in other tasks, like the dice task.

**Using formulas in solving probability problems**

Some preservice teachers attempted to solve all tasks in the written, the pair-problem solving, and interview components by using some mathematical formula. Reid, for example, simply applied a mathematical formula of calculating the probability to the birth sequence task. Since there were six births Reid stated that the probability of each sequence is $\frac{1}{2^6}$. When the number of trials was increased to 20 from 6, Reid was very quick to say $\frac{1}{2^{20}}$. Another participant, Paul, argued in the same way.
Reid and Paul not only applied mathematical rules and formulas as far as possible but also enjoyed making rules and formulas themselves. In the lottery and car accident task, for example, many of the preservice teachers used their everyday experiences and intuitions to come to a solution. But Reid and Paul came up with the following formulas to calculate the probability ($P_r$) of being killed in a car accident. According to Reid

$$P_r = \frac{\text{Number of people killed in a year} \times \text{Average human life span}}{\text{Total population}}$$

According to Paul

$$P_r = \frac{\text{Deaths in one year}}{\text{Total population}} \times \text{Expected years yet to live}$$

After writing the formulas, each of them stated that they could obtain data such as how many people were killed in a car accident in a particular year and what was the total population of B.C. in that year. Similarly, data about average human life span could be found.

The formulas provided by Reid and Paul are similar to each other except in some differences in language. Unlike some of the other preservice teachers who simply took the ratio of deaths to population without considering human life span, Reid and Paul thought that the ratio did not give a good estimate of being killed over a lifetime. When this matter was brought up in the interviews, they argued that they could be killed in a car accident during their lifetime so it was essential to find out their expected years to live and multiply this figure by the ratio.

Reid and Paul's methods are mathematically more complex because they also considered the role of human life expectancy in order to estimate the probability of being killed in a car accident. However, for them the probability of being killed in a car accident was simply plugging different numbers into the
formulas they provided. Because of their reliance on a mathematical procedure, they did not think of other complex issues inherent in the likelihood of being killed in a car accident (such as whether one drives and in which part of the country one lives) that other preservice teachers thought about. Their formulas could not provide justifications for issues such as: Is the probability of being killed in a car accident equal for all people who live in B.C.? According to Paul, not all people who live in B.C. have an equal probability of being killed in a car accident. He stated, however, that a mathematical solution to the task cannot be obtained without making the assumption that every one who lives in B.C. has an equal probability of being killed. In the pair setting when Paul was working with Jane, he stated that he had to make wrong assumptions if the task was not well formulated. He stated that although the probability of being killed in a car accident was not equal for everyone, he had to assume that the probability was equal for everyone in order to solve the problem.

Although many preservice teachers used their everyday experiences and intuitions during the dice task, three preservice teachers used university knowledge of probability to reach a solution. Gita was one of the three preservice teachers who used correct mathematical procedures to solve this problem. She argued that she wanted to be the dealer because the dealer had a greater probability of winning. On a piece of paper she wrote that \( \frac{1}{6} \times \frac{5}{6} \times \frac{5}{6} = \frac{25}{216} \); \( \frac{1}{6} \times \frac{1}{6} \times \frac{5}{6} = \frac{5}{216} \); \( \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} = \frac{1}{216} \). The total is \( \frac{31}{216} \). There are three different ways of obtaining these probabilities. So the probability for the player is \( 3 \times \frac{31}{216} = \frac{93}{216} \).

She concluded, "I think the probability of getting 1, in one way or the other, is 93 out of 216... 43% of the time you're going to win and the dealer is going to win the rest of the time. So I think I'd rather be the dealer." Gita's logic was based on formal mathematics. She was correct according to the formal mathematical principles of probability except for her failure to account for the fact that there is
only one way of getting the same number on all the dice. Gita’s error is not surprising because the recognition of order in such a task has been difficult for world famous mathematicians such as d’Alembert, Leibniz and so on (Borovcnik, Bentz, & Kapadia, 1991; Glickman, 1990). Nevertheless, she noticed her error when she was given her transcript to comment on, and provided the mathematically correct solution of \( \frac{91}{216} \).

John’s strategy was similar to that of Gita. At the beginning of the interview John wanted to be the dealer because usually it is the dealers who win the money. Later when the researcher asked him if he could come up with a mathematical solution he calculated the probabilities as follows:

Let’s see. It’s like 3 independent trials, right? So the probability of getting a number in one die is \( \frac{1}{6} \), for the three dice will be \( \frac{1}{6} \times \frac{5}{6} \times \frac{5}{6} \).

The probability of getting 3 wins is considerably lower, that would be \( \frac{1}{6^3} \).

The probability of 2 wins and 1 loss would be \( \frac{1}{6^2} \times \frac{5}{6} \). The probability of 1 win plus the probability of 2 wins plus the probability of 3 wins equals \( \frac{25}{6^3} + \frac{5}{6^3} + \frac{1}{6^3} \) equals \( \frac{31}{6^3} \). [laughs]

John felt very happy about his solution because his non-formal reasoning of becoming a dealer actually matched with his formal calculation. However, he did not look at the different ways that one can get \( \frac{1}{6} \times \frac{5}{6} \times \frac{5}{6} \) and so on. His solution was therefore incomplete. But the strategy he used was mathematically sound.

Another preservice teacher, Ruby who was totally satisfied with her non-formal reasoning in the first interview, decided to work on the problem further. When the researcher returned the interview transcripts for further comments and elaboration, Ruby looked at a probability text and realized that she should have
calculated the expected value if she wanted to come to a mathematically correct answer. When she returned the transcript she provided a mathematical solution as follows:

\[
E(\text{Player}) = \frac{-5^3 + 5^2 + 2 \times 5 + 3}{6^3} = \frac{-87}{6^3}
\]

Since the expected value for the player was negative, Ruby commented that she should be the dealer. She said in the follow-up interview that she was glad that her non-formal reasoning in the first interview was not incorrect mathematically. Ruby's strategy was correct. However, as did John, she did not recognize that there are three different ways of getting a number once and twice.

As stated above the inability to recognize the different ways one can get certain outcomes has been a problem throughout the history of probability. The same thing was true in the case of these preservice teachers who have training in mathematics and probability.

**Overview of Chapter 4**

In this chapter preservice teachers' qualitatively different conceptions of probability were discussed. These conceptions were grouped under formal and non-formal. It was argued that the preservice teachers used non-formal conceptions such as using everyday experiences, the representativeness heuristic, and knowledge from other disciplines in solving probability problems. Their formal conceptions of probability included knowledge of independence and randomness, and knowledge of mathematical formulas and procedures.

Examples of the participants' use of everyday experiences were prevalent in the lottery and car accident task in the pair-problem-solving component and in the dice task in the interview component. A few of the preservice teachers demonstrated other kinds of non-formal conceptions such as the representativeness heuristics and the gambler's fallacy.
Formal conceptions of probability were used by the majority of the participants in many tasks, such as the rectangular spinner task, the lottery and car accident task, and the circular spinner task in the written component, and the five-boy task and the circular spinner task in the pair-problem-solving component. For example, in the rectangular spinner task, all of the preservice teachers took a formal approach by looking at the greatest angle in different sections of the spinner. Similarly, they used a combinatoric formula to find out the probability of winning a jackpot in the B.C. 6/49 lottery.

Only a few preservice teachers demonstrated complex formal mathematical thinking infused with their knowledge of independence, randomness, formulas, axioms, and proofs in probability. Although the preservice teachers could explain what they meant by independence and randomness, they were confused when attempting to integrate these concepts with the addition or multiplication rule of probability in some tasks, such as the dice task.

It is interesting to note that the majority of the participants expressed their desire to use mathematics as far as possible to solve probability problems in the university setting. In the dice task, for example, they attempted to solve the problem mathematically. However, almost all the preservice teachers could not provide a mathematically correct solution to the problem because of their inability to relate independence and randomness to the addition rule and the multiplication rule of probability. On the other hand, the majority of the preservice teachers solved the bank teller task mathematically, but were unsatisfied with the applicability of the mathematical answer to a real life situation.
CHAPTER 5
Consistency Analysis of Preservice Teachers' Conceptions of Probability

This chapter presents outcomes of analysis of the data for the second research question:

In what ways do preservice teachers' conceptions of probability vary across tasks and settings?

The data for this question were generated from the preservice teachers' responses to the tasks presented in different settings\(^9\) of the study: the written, the pair-problem solving, and the interview. The outcomes of the analysis presented in this chapter are based mainly on the tasks that required an understanding of independence and randomness to solve them correctly.

The birth sequence task in the written component was mathematically identical with the head-tail sequence task presented in the pair-problem solving and the interview settings. Similarly, the five-head task in the written component was mathematically equivalent to the five-boy task in the pair-problem-solving component. The lottery task and the dice task in the interview were closely related to the head-tail sequence task and the five-head task in terms of independence and randomness.

The analysis focuses on how the preservice teachers' responses varied across similar tasks in the same setting and how the responses varied across settings on the same or similar tasks. For the purpose of the analysis, their responses were classified as consistent or inconsistent. If participants used formal reasoning correctly or incorrectly to solve at least half of the tasks in all the settings, they were classified as consistent thinkers. They were also deemed to be consistent thinkers if they used non-formal reasoning in a cluster of tasks

\(^9\)Settings and components are used interchangeably.
throughout all the settings. For example, if they used the gambler's fallacy in at least half of the tasks in which this fallacy was possible then they were labeled as consistent thinkers.

If they switched from non-formal reasoning to formal reasoning or vice versa and stated that they were confused as to which reasoning they should have been using to solve the task, they are referred to as inconsistent. The preservice teachers' inconsistent reasoning was often due to a conflict between their formal and non-formal reasoning.

Table 5 in Chapter 4 shows the use of formal and non-formal reasoning to the tasks posed in this study. As can be seen in Table 5, the participants' responses varied widely between formal and non-formal conceptions on the tasks posed within and across settings. In some tasks, almost all the preservice teachers used formal conceptions, and in other tasks they used their non-formal conceptions.

Although the preservice teachers' reasoning differed in different tasks, some of them, like Paul and Reid, were consistent in using their formal mathematical reasoning to solve all the tasks. Alan was consistent in using non-formal reasoning, particularly the gambler's fallacy, in solving a cluster of tasks in the study. However, other preservice teachers, like Anne, Jane, John, Mary, and Ruth demonstrated inconsistency in solving the tasks. Three preservice teachers' conceptions of probability are provided below to illustrate consistent and inconsistent cases of solving the tasks posed in this study. Paul is chosen to illustrate a case consistently using formal mathematical reasoning to solve the tasks. Alan is chosen to illustrate a case consistently using the gambler's fallacy while attempting to solve the tasks. John's case provides an illustration of how preservice teachers can change their conception depending on the tasks and settings. The cases of Paul, Alan, and John are summarized in Table 6.
Table 6. Examples of preservice teachers' consistent and inconsistent conceptions across tasks and settings.

<table>
<thead>
<tr>
<th>Components</th>
<th>Tasks</th>
<th>Paul</th>
<th>Alan</th>
<th>John</th>
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<tr>
<td>Written</td>
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<td>Pair-problem</td>
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<td>solving</td>
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<table>
<thead>
<tr>
<th>Consistent</th>
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<td>Formal</td>
<td>Non-formal</td>
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Legend:  
BST = Birth-sequence task  
FHT = Five-head task  
CST = Circular spinner task  
FBT = Five-boy task  
LT = Lottery task  
DT = Dice task  
RST = Rectangular spinner task  
LCT = Lottery and car accident task  
HTT = Head-tail sequence task  
BTT = Bank teller task  
BMT = Board and marble task  
TT = Thumbtack task  
F = Formal thinking  
N = Non-formal thinking  
F N = Formal and non-formal thinking without conflict  
F N = Formal and non-formal thinking with conflict  
G = Non-formal thinking (Gambler's fallacy)
Consistent formal: The case of Paul

Paul's reasoning consistently incorporated the mathematics of probability across tasks. In the head-tail sequence task in the written component, he reasoned that all four sequences were equally likely because each event, head or tail, is equally likely. Similarly in the five-head task, he chose that the outcome (head or tail) was equally likely because "the previous five outcomes have no bearing on sixth outcome."

In the pair-problem solving, Paul's reasoning was consistent in the head-tail sequence task and the five-boy task. For instance, Paul read the head-tail task aloud and answered as follows during the pair-problem solving:

These are all independent of one another and so there is no one more likely than the other one. It's more likely that you're going to have ... three heads and three tails than you're going to have say 6 tails or 6 heads. However it's because it's orderly, they're all equally likely. Because each one is independent of the other one. So on the first one it's 50/50 heads or tails, second one is 50/50, third one is 50/50, fourth one is 50/50. So they're all the same. Do you agree Jane?

Jane was largely a non-formal thinker. At this point she did not readily accept Paul's formal reasoning. Jane was thinking that the sequence THTHTHT or HTHTHTHT was more likely to occur than the sequence HTHHHH because the number of heads and tails in those sequences were equal. However, Paul argued that all sequences were equally likely in the following way:

Okay.... But (e) [all sequences are equally likely] is correct, right? I think what you're saying is that expectation of large sample you're going to have relatively close to 50/50 each way and that's true. That's a fact. However when you're given an orderly sequence then that eliminates this
because they each become independent of one another. Therefore any one is just as likely as any other one.

Finally, Jane agreed with Paul. In a discussion with Jane of the researcher's interpretation of her conceptions of probability, she stated that Paul's reasoning was highly mathematical. She thought that mathematical ideas were valued in university education and research and so she decided to agree with what Paul had said. Jane also stated that Paul was a very good mathematical thinker and she could trust him regarding his mathematical capability.

In the five-boy task, Paul argued that having five boys previously "has absolutely no bearing on the sixth". He argued that the present birth would be independent of the previous births "much like flipping a coin and getting heads and tails." He stated that the chance of getting five boys in a row is only 1 in 32. But it is the same for other outcomes as well.

In the interview setting, Paul recognized the head-tail sequence task immediately and argued that all sequences were equally likely. Later in the course of the interview, the researcher asked him about his written response to the question "which of the following orderly sequences is most likely to result from flipping a fair coin 100 times?" Four alternatives HTHHHH...(99 heads and 1 tail), HHH...TTT (50 heads and 50 tails), THTHHTHTHHT... (50 heads and 50 tails), and HTHTHT... (50 heads and 50 tails) were given in this problem. Paul argued that all sequences were equally likely because they were independent events. In the interview setting, he stated,

As long as it's orderly there is no difference at all. If it's not ordered then there is a difference.... If you flip a coin a hundred times ... your most likely solution would be 50 and 50. That would be your expected value. It's unlikely that you would get exactly 50/50. However, that's the most
likely possibility—50 heads and 50 tails. However, if you order them, then it eliminates that. Because on any given flip they're all independent. So on any given flip, it can be either heads or tails. It's a fifty/fifty chance. Therefore any order of heads and tails is ... equally likely. Doesn't matter how many flips you have. Doesn't matter at all. It's all equally likely.

When asked to clarify his statement that the "most likely solution would be 50 and 50," Paul talked about the expected value. He argued that while flipping a fair coin 100 times, the expected value would be approximately 50 heads and 50 tails. In general, Paul's reasoning was consistent with mathematical theory of probability. He did not change his conception even when the number of trials in the sequence was changed from 6 to 100. On top of that, Paul calculated the probability of each sequence to be $\frac{1}{2^6}$ when there were six events in the sequence. Paul had no difficulty extending this to $\frac{1}{2^{100}}$ if there were 100 events in the sequence in a particular order.

As in the head-tail sequence task, Paul reasoned mathematically in selecting a sequence of numbers for the lottery 6/49. He argued that "there is absolutely, positively, no difference in probability between 1, 2, 3, 4, 5, 6 or 1, 2, 3, 4, 5, 49 or 3, 5, 10, 36, 24 and whatever." For Paul "any combination of six numbers is exactly likely as any other combination of six numbers in the 6/49" because each number in the draw is independent of the previous one. He argued that the sequence of numbers 1, 2, 3, 4, 5, 6 is arbitrary just as is any other sequence of numbers. Paul's argument did not change even when the researcher provided a counter suggestion saying that no single consecutive sequence of numbers has ever won a jackpot in the lottery 6/49. Instead Paul argued consistently that "it makes no difference what six numbers the person picks as
long as they are all between 1 and 49 and there are no repeated numbers."
Hence, Paul's reasoning was consistent throughout the tasks in all the settings.

Although Paul correctly applied the concept of independence and randomness in almost all tasks, he had some difficulty solving the dice task. Like many of the other preservice teachers, his strategy did not consider the difference between mutually exclusive and independent events while solving the dice task. However, his solution was quite sophisticated compared to the other preservice teachers. First, he thought that it was a tie game. Later, he reasoned that the player had the advantage in the game. The following transcript helps to give a sense of his mathematical thinking:

If I assume that each number that shows up on the die is different, ... then it's clearly a game of fifty/fifty. I'll get it half the time. You'll get it half the time. There are three dice and six opportunities. I'll pick one of the three that wins. I win.... If I pick one of those three that doesn't show up, then I lose. So it's fifty/fifty. That's assuming that each number that shows up on a die is different. For example, ... if I pick a number randomly from 1 to 6 that's fifty/fifty chance of winning, fifty/fifty chance of losing because the numbers that are up in the dice are, let's call them 4, 5, and 6.... So I have got fifty/fifty chance. But that's making a false assumption that they're all different.... If it's a repeated number ... like 5, 6, and 6 to keep it kind of easy ... then my chance of winning are 1 in 3. And my expected winnings are going to be $2+$1 divided by 3 which gives the expectation of a dollar.... If they're all the same, there is a $\frac{1}{6}$ chance that I'm going to win and I'll win $3 + 0$ over 6. So it's 50 cents again.... I'm wrong here I think. Let's assume I'm right. Okay, now. The chance that
they're all different is \( \frac{1}{1} \times \frac{5}{6} \times \frac{4}{6} \). The chance that two are the same and one is different is [long pause] \( \frac{5}{6} \times \frac{1}{6} \times 3 \).

Later Paul calculated that if all the numbers were different then the probability would be \( 1 \times \frac{5}{6} \times \frac{4}{6} = \frac{20}{36} \); if two numbers were the same and one number was different then the probability would be \( \frac{5}{6} \times \frac{1}{6} \times 3 = \frac{15}{36} \); and if all three numbers were the same then the probability would be \( \frac{1}{6} \times \frac{1}{6} = \frac{1}{36} \). And finally the expected value was \( \frac{20}{36} \times (0.50) + \frac{15}{36} \times (1.0) + \frac{1}{36} \times (0.50) = \frac{51}{72} \). Since this was the probability for him to win as a player, he decided to be the player in the game.

Paul's thinking was formal because he applied the knowledge of expected value from his university probability courses. However, he did not come to a mathematically accepted solution to the problem because he did not treat each die individually. He reasoned that the probability of his number turning up in one of the three dice was 50%. In fact he was using the addition rule of probability. He had strong background knowledge of expected value and other theorems in probability, but he could not arrive at the correct mathematical solution because of an incorrect assumption.

With the exception of the dice task, Paul applied the concept of independence and randomness throughout the tasks in all the settings: written, pair-problem solving, and interview. He was also consistent in the discussion with the researcher which took place during the transcript checking and the researcher's interpretation of his conception. When the researcher questioned Paul concerning his solution to the dice task, Paul recognized that he had not looked at each die independently. He stated that he would obtain a different answer if he had to solve the problem again because he would look at the dice independently.
Paul valued a formal rather than a non-formal approach to solving problems. The only time he used non-formal reasoning was in the thumbtack task. However, the non-formal reasoning he was using in the thumbtack task was based on physics knowledge rather than everyday experience. He enjoyed working on the probability problems if they could be solved mathematically. He was actually frustrated by some of the tasks in which the assumptions were not clear. Paul stated that he enjoyed looking at mathematical problems in a theoretical way but he did not "enjoy haggling over whether a tack is going to fall one way or the other way." He clarified that he preferred "to be given some assumptions even if they're false" and let his "mind explore the possibilities beyond that."

As has been discussed, Paul consistently used formal reasoning to solve problems. Unlike many of the other preservice teachers, conflict between formal and non-formal reasoning was not evident in Paul's thinking.

**Consistent non-formal: The case of Alan**

Alan was one of the preservice teachers who was consistent in using non-formal reasoning, particularly the gambler's fallacy. In the five-head task in the written component, Alan stated that "over a larger and larger number of flips, there will be a tendency for the number of heads and tails to be equal. Therefore it is more probable that a tail will occur as the number of flips resulting in heads increases." Alan reasoned that since the occurrence of a head or a tail was equally likely and a head had occurred five times already it was more likely that a tail would occur in the sixth trial. Alan was using the gambler's fallacy in this task.

Alan also used the gambler's fallacy in the circular spinner task in the written component. He showed this fallacy in the following way:
It doesn't matter which two are picked initially. However, if a pattern develops that shows a certain number coming up often, I would avoid it next time. I would also look for numbers which never come up, and bet those on successive rounds.

Alan consistently used the gambler's fallacy in many tasks in the pair-problem solving and the interview settings. Alan's reasoning in the head-tail sequence task in the pair-problem-solving component was based on the gambler's fallacy. When he was working with Ivan in the pair-problem solving of the head-tail sequence task he reasoned as follows:

Ivan: It doesn't matter which order that it goes in. The probability is 50:50 for each trial.

Alan: One thing I was thinking when reading this question for the first time, I recognized this question before: ... is it not a lower probability, for example you will roll/flip all heads or all tails? [pause] That's the thing that's got me about this one because there're certain patterns in here [shows the sequence HTHHHH] which seem to me to be less likely.

Ivan: But are they? Is it not just chance, chance, chance?

Alan's argument indicates that the sequence HTHHHH was less likely because the number of heads and tails in this sequence were not evenly distributed. But Ivan did not agree with him, so Alan continued to argue as follows:

Another way to think about it is: ... If you think about how this sequence occurred—if you bet in the middle here ... you see here, we have HT, HT, HT, what would you bet the next two are going to be heads? You say well it is not very likely. So that's what worries me about this. Again, I'm not
thinking clearly about probability here.... Something intuitive about this says the possibility of having a sequence that has only one tail seems less likely than some even distribution.

But Ivan did not agree again. Ivan argued that "in terms of a grand scheme there should be an even distribution." Furthermore, Ivan argued that if the run of a string, of say three hundred tosses or more, are made then every single sequence HTHHHH, HHHTTT, THHTHT, or HTHTHT would be there. When Ivan increased the number of tosses to 300, Alan was more comfortable in agreeing that all sequences were equally likely. However, in the five-boy task, Alan repeated his gambler's fallacy as follows:

Alan: My thinking is that the nature tends to be in equilibrium. So if they already have five boys, I think it's more likely to have that the sixth child will be a girl.

Ivan: From the scientific point of view, it's completely random. It's absolutely completely random whether or not their sixth child will be a boy or a girl. So there is no more likelihood to be a girl or a boy. The actual chance is either XX or XY. I mean the first one is given by the whatever, mother, and determined when it's given by the father. Whether or not it carries Y chromosomes or X chromosomes and it's 50/50. It's completely random.

But Alan was not convinced by Ivan's argument. Alan went on to argue thus:

Alan: It seems to me that the more children they have the more likely it is that their population would resemble the total population.

Ivan: Yes. But that doesn't mean that the next one has to be a girl.

Alan: True, doesn't have to be, but it's more likely.
Ivan: No. It can be a boy and there are 300 more kids, then maybe eventually it will. It doesn't have to be though. Again, it could be a string of B's: boy, boy, boy, boy.... That's where they happensd to be a sequence in a grand scheme thing. Somewhere else, there is somebody having six girls.

Once again Alan stated that he would agree with Ivan if the number of births was large enough, say 300. If there were 300 births, then Alan would think that the ratio of boy to girl would be 50:50.

In the interview setting, Alan recognized the head-tail sequence task and maintained the same solution he negotiated with Ivan in the pair setting. Alan reasoned that even with 100 trials in the head-tail sequence task, all the sequences were equally likely. Alan was thus changing his conception based on the interaction he had had with his peer. However, when the task was changed, Alan did not maintain his new conception.

In selecting a sequence of numbers in the lottery 6/49, Alan stated that whether the sequence was consecutive or random it had the same chance of winning a jackpot "because the number coming up is random and there is nothing about the previous number coming up that has anything to do with the next number coming up." Later the researcher asked what Alan would say if the student was thinking that no string of consecutive numbers had ever won the jackpot and that it was therefore unlikely to win. Alan argued the opposite and showed his gambler's fallacy. He stated that "because consecutive numbers have never shown up they are slightly more likely to occur than [the random sequence of numbers]." He elaborated this argument as follows:

Alan: Let's say you just pulled out lotto 6/49 numbers to infinity. You'll eventually get all the outcomes over a long period of time. As you
continue to increase the number of times you had an outcome, you would eventually see an even number of all outcomes. ... In other words, let's say you've thrown ... the total number minus 1, then I'd say the chances are that you get the missing outcome in the last throw are fairly good. And the more chance you have to choose the numbers, the more likelihood is you're going to have an even distribution.

Hari: You said that here, too [shows his response in the written task].

Alan: Yes.

Hari: What is your conclusion? What would you tell to them?

Alan: I would be inclined to say that if the six consecutive numbers have never occurred then they might choose the six consecutive numbers. That means slightly more likely. But it's a very slight amount because the chances of occurring them at a given throw is pretty small.... You might be only slightly improving your chances.

Alan also demonstrated the gambler's fallacy in the board and marble task. He observed that the marble did not come to No. 1 when he dropped it several times. He stated that he was positive that the marble would start to come to No. 1 more frequently because it had not for a long time. He stated, "if I was now to bet, I would start betting on 1". After dropping the marble for 24 times he concluded that the board was reasonably fair even if the marble went to No. 1 only twice.

In the dice task, Alan decided to be the player in the game. When asked for his reasons, Alan provided an argument which also resembles the gambler's fallacy:
Hari: Now it's time to decide. Now, you've the opportunity of becoming a dealer or a player?
Alan: I'll be the player.
Hari: Why is that?
Alan: Because I can look at the patterns and I can get some ideas of what might be happening. ... The longer we play the game, the more likely I think I am to win. [thinking] I think I would still be the player but the reason for being the player is that I can look for cases where a certain number has not come up for a long time. Those cases are likely to come up. I'll start betting on them. That's the strategy that I would use.

Alan reasoned that the probability of the occurrence of an event would be higher when it had not occurred for a long time. Alan also applied the gambler's fallacy in choosing numbers during the playing of the game, when he was player and the researcher was dealer. He chose numbers which had not appeared on the previous rolls of the dice.

Although Alan was inconsistent in solving some tasks, like the lottery task in the interview setting, he was consistent in his use of the gambler's fallacy across many of the tasks in all three settings—the written, the pair-problem solving, and the interview. He temporarily withdrew this fallacy in some of the tasks, for example when he was working with Ivan in the circular spinner task, but he continued to use it in other tasks and settings. His fallacy was temporarily changed because of peer interaction with Ivan. After the pair-problem-solving component, Alan mentioned that he had a hard time convincing Ivan. Alan thought that Ivan's thinking might have been mathematically more sound than his, but he was not sure.
In a follow-up interview, Alan mentioned that he had developed this belief (gambler's fallacy) from his experience in playing black jack in his real life. He stated that he had used the same strategy in a gambling situation and had actually won money several times. Alan was concerned that his way of reasoning might not be consistent with the mathematical theory of probability. However, he also stated that during his school and university courses, he had never had an opportunity to use probability to analyze everyday problems; because of this he felt that he lacked the necessary skills for applying probability to everyday problems. Alan also confided that he had difficulty conceptualizing mathematical probability because he could not make mental models to solve probability problems, as he could when solving problems in number theory or geometry.

**Inconsistent: The case of John**

John's reasoning was inconsistent within some and across many tasks in this study. His response to the birth sequence task in the written component was influenced by the representativeness heuristic, even though he said that "having a boy is as equally likely as having a girl" and also "each birth is independent of any previous births." Despite his awareness of independence and randomness in the birth sequence task, John thought that the sequences BBBGGG, GBGBBG, and BGBGBG would be the most likely. This implies that the sequence BBBBBB would be the least likely. John chose these sequences as most likely because they contained equal numbers of boys and girls. He was, therefore, not correctly utilizing his formal knowledge of independence and randomness in the task.

Unlike his response to the birth sequence task, John's response to the five-head task was driven by his formal knowledge of independence and randomness. He stated that both outcomes (heads and tails) were equally likely. He provided the following reasons for his selection:
The sixth outcome is equally likely to be a head or a tail because the coin is fair. Also all trials (flipping of a coin) are independent from other trials. Just because you happen to flip 5 heads does not mean that your sixth flip will be swayed either to heads or to tails.

Hence, John was inconsistent in his treatment of the birth sequence task and the five-head task during the written component of the study. When he was working with Gita in the head-tail sequence task during the pair problem solving, she reminded him of the importance of order in the task. He was quickly convinced by her suggestion and agreed that all the sequences were likely. The following transcript depicts the pair-problem-solving situation:

Gita: I would say that they are all equally likely.... What do you think?
John: I don't think so because in sequence HTHHHH there is only one tail. The probability of getting a head or a tail is about 50% and so this [HTHHHH] would be more unlikely than the other sequences, for example, which have three heads and three tails.

Gita: Yes, but if it is the order (first you get the head and second you get the tail) then I don't think that the number of heads or tails in the total matters. I'm not sure if the question is asking that.

He agreed with Gita that all the sequences in the task were equally likely. John carried this change of conception to the individual interview setting where the head-tail sequence task was presented again. As he had concluded in the pair-problem solving with Gita, John stated that all the sequences were equally likely. Upon further probing, however, he was not highly confident about the equal likelihood of all these outcomes.
Hari: Do you mean that even with 5 heads and 1 tail in one sequence and 3 heads and 3 tails in the other sequence does not make any difference?

John: Um-. I still have some doubt, I guess, obviously about choosing that they are all equally likely. I still have to say that they are all equally likely, but I'm not quite sure if my answer is right. There are different ways you can look at it. I'm not sure which is the right way. The mathematical way is heads and tails are equally likely and each event is independent and you come to the conclusion that they are all equally likely. Or you could take a count of how many heads and tails you get. For (b), (c), and (d) [sequences HHHHTT, THTHHT, and HTHTHT] you got 3 heads and 3 tails. So those will have the same probability. And ... if you think that in 6 throws of a coin you should get 3 heads and 3 tails is a fallacy. You could get anything really.

The above excerpt shows that John was not sure whether all sequences are equally likely. Although John showed a change of conception in the pair-problem solving, he was not confident as to whether the question was concerned only with the independence of the flips or also with the total number of the flips. When the researcher asked whether the outcomes would be equally likely if the total number of trials was 20 and there were 19 heads and 1 tail in one sequence and 12 heads and 8 tails in the other sequence, John replied that he would change his answer. John thought that when the number of trials was increased up to 20, then the sequences with roughly half heads and half tails would be more likely than the sequences with an unbalanced number of heads and tails, for example, nineteen heads and one tail. He elaborated this view as follows:
Hari: Do you think that all the sequences would be equally likely, if the total number of trials was 20 and in (a) I had 19 heads and 1 tail and somewhere, in (c) for example, I had 12 heads and 8 tails or 10 heads and 10 tails? [shows the examples of sequences HTHHHH..., HTHHHHTTHHHH...]

John: Um. No, I think it changes as the number of trials increases. But the thing is it's a fallacy to assume that you get 10 heads and 10 tails in 20 trials.

Hari: I understand that. But what about comparing those 19 heads and 1 tail and 12 heads and 8 tails? Do you think that they're equally likely?

John: No. Intuitively I'd not think so. I would expect that 12 heads and 8 tails would be more likely. I'm trying to think how you can calculate the probability of these sequences mathematically, but I kind of forgot. [shows the head-tail sequence task] Intuitively speaking, I'd still think that these three sequences (b), (c), and (d) [sequences HHHHTT, THTHHT, and HTHTHT] are more likely than (a) [HTHHHH], that (a) is less likely. Um-, but that's intuitive. I'm thinking that any of these sequences are equally likely because the number of trials you've got is small, you only have six trials; and this whole thing that the trials being independent and the probability of heads and tails being equal, I just have to say they're equally likely.

Hari: Even in 20 trials?

John: If you extend this to 20 trials, I might change my answer. If you increase the number of trials, I would go more with my intuition and say those sequences have roughly half heads and half tails.
What I would try to do is somehow reflect that in the mathematics. But right now I can't quite figure out how to reflect.

As can be seen in the above transcripts, John stated that the probability of \( \frac{1}{2} \) when the researcher requested a mathematical solution. However, he did not extend this to the case when the number of trials was 20.

In the above two transcripts, John expressed his concerns about two issues: First, you cannot expect the same number of heads and the same number of tails in a sequence with a small number of trials. If that happens, one has to doubt its validity. His concern was clear when he stated that it was a fallacy to assume that you would get 3 heads and 3 tails in 6 trials or 10 heads and 10 tails in 20 trials. Although he thought this was a fallacy, he still asserted that the outcomes with an equal number of heads and tails would be more likely than the outcomes with an unequal number of heads and tails. The second issue is related to the number of trials. If the number of trials was small, John thought that the sequences, whether they appeared balanced or not would have the same probability of happening. However he thought that if the number of trials was large enough, the sequences where heads and tails were balanced were more likely to occur than the unbalanced sequences. This indicates that although John was aware of independence and randomness, he did not look at each event independently, especially if the number of trials was larger. His faith in the law of large numbers overshadowed his knowledge of independence and randomness.

John's struggle in the above transcript was to find a resolution between his non-formal and formal thinking. He indicated that he sometimes used non-formal reasoning and sometimes formal reasoning. John found it easy to make a
decision if his non-formal and formal thinking did not conflict with each other, but when they conflicted he lacked confidence in his decision.

John continued to show both non-formal and formal thinking in the lottery task. For this task, which involved choosing six numbers for a lottery 6/49, John gave the following reasoning:

I would probably talk to them about different ways of looking at them. You could look at it mathematically and say that the probability of getting a certain number is equally likely as getting any other number. So in a sense it doesn't matter which number you pick. You know you are still going to have the same chance of winning as picking any other numbers. Then again, you can look at the past history of the numbers drawn from 6/49. You can keep track of what the winning numbers are and then look for trends or patterns of the numbers that were chosen a lot. You can come up with certain numbers that seem to be chosen quite frequently.

Later, the researcher provided him with information from *Luck* magazine stating that no consecutive number sequence has ever won a jackpot. John stated that a consecutive sequence had a smaller chance of winning the jackpot than any other random sequence of numbers. The conversation was as follows:

Hari: Not a single consecutive number has won the jackpot so far. What do you think now?

John: That's actually crossed my mind. You know whenever I see the lottery winnings, I've never seen the consecutive numbers. Um-, I'm getting into ethics here. I would have to agree with the other student who thinks that the chance of getting a sequence of six consecutive numbers in the lottery is smaller than getting a random sequence of any six numbers. Then again, if I'm advising these
students I would only present the information and let them come to a decision.

Hari: What's your personal thinking?

John: A random sequence is more likely than choosing six consecutive numbers. That's my opinion.

Hari: What's the reasoning for that?

John: I'm just basing my decision on the fact that intuitively six consecutive numbers seem unlikely to me and by the information you've just provided it has never happened. My reasoning is purely based on intuition. Mathematically speaking, any six numbers is equally likely, right? There're mathematical ways of looking at things and there is your intuition. Sometimes you have got to separate the two or make compromises or whatever I don't know.

Once again, John was struggling with whether to use non-formal or formal reasoning. He made it clear that mathematically, any six numbers chosen for the lottery had an equal chance of winning. But his non-formal reasoning did not support that. However, his choice of non-formal or formal reasoning depended on the setting and the person whom he was talking.

**Overview of Chapter 5**

This chapter described both consistent and inconsistent reasoning that the preservice teachers used in solving the tasks posed in the study. In varying degrees, most tasks evoked conceptual dispersion that included non-formal and formal conceptions of probability. For example, the rectangular and circular spinner tasks evoked a much smaller conceptual dispersion than other tasks such as the dice task. This means that preservice teachers' responses varied on tasks
within and across settings. However, variation among responses to different tasks was not the same for all participants.

Some participants, like Paul, used consistent mathematical reasoning in most tasks. Others, like John, used inconsistent reasoning that was formal, non-formal, or a mixture of both. The participants were either confident of their formal mathematics and were able to solve problems using it, or they had forgotten elements of mathematics that they were taught and had to rely on their non-formal reasoning.

When the preservice teachers recognized inconsistencies in their explanations they admitted to them arguing that human beings are inconsistent in their thinking. The majority of them commented that their thinking differed dramatically when they solved problems in their everyday lives and when they solved problems in their university examinations. Some of the preservice teachers who changed their conceptions from non-formal to formal, stated that they tried to do so because the research was conducted in a university setting where formal mathematics is valued more than non-formal.

In the pair-problem solving, formal thinkers always dominated the non-formal thinkers. Paul dominated the discussion with Jane. Jane agreed with Paul because she thought that he was a highly formal thinker. Alan temporarily changed his gambler's fallacy to formal reasoning in the head-tail sequence task and the five-boy task. Similarly, John changed his conception based on a discussion with Gita who applied formal reasoning to solve the problems. Although, Jane, Alan, and John all temporarily suspended their non-formal reasoning when challenged by their peers, they continued to use non-formal reasoning in the interview setting, which followed the pair-problem solving.
CHAPTER 6
Preservice Teachers' Perceptions about the role of
Probability in their Everyday Problems

The third research question was presented in Chapter 1 as:

What are preservice teachers' views about the role of probability learned at the university level in solving everyday problems?

The data for this question were drawn from two types of questions asked of the preservice teachers. First, the preservice teachers were asked to provide their opinions as to whether they found university knowledge of probability useful in solving their everyday problems. Second, the preservice teachers were given a variety of tasks to solve as described in Table 2 on page 49. The preservice teachers' responses to different tasks and settings are thoroughly described in Chapters 4 and 5. It is not intended to repeat those in this chapter; the gist of the responses will be provided as well as a detailed description of their views of the role of probability in solving everyday problems. The participants' responses to the tasks summarized in this chapter provide some insight into how they use formal probability in solving everyday problems, in addition to how they view the role of formal probability.

Participants' views about the role of probability in solving everyday problems

The statements of participants in this study suggest that their knowledge of formal probability is not useful in solving everyday problems. However, two kinds of responses were obtained from the preservice teachers. About half of them thought that probability in university settings and probability in everyday life are two different phenomena. The rest argued that knowledge of university
probability strengthens reasoning skills, which in turn helps in solving everyday problems.

**Probability in university and probability in everyday life: Two different phenomena**

Some preservice teachers in the study commented that they never calculated probabilities while solving everyday problems. They responded that the university probability courses were not useful because they were too theoretical and hence had little or nothing to do with everyday problems. Gita, for example, expressed her feelings in the following way:

> You probably do [use probability in everyday problem-solving] but I don't think about using it.... You don't think, "I'm using probability," you think of it as using common sense. I don't play the lottery because the chance of winning the lottery is so low. You might not work out the probability but just intuitively you know that the probability is low. There are 49 numbers and I pick only six numbers. The chance of my winning number is very low. You know that. You don't need to sit down and figure out the probability.

Gita's comments indicate that she considered university probability to be purely formal—not closely related to her everyday life. Gita thought that her non-formal thinking was enough to decide that the probability of winning a lottery was very small, and so it would not be worthwhile to spend money on lotteries. The above transcript indicates that Gita's non-formal thinking is a product of her common sense and some formal knowledge of probability. However, in a further discussion Gita elaborated that if she did use formal knowledge of probability to solve problems then it would be what she learned in Grade 12, not what she learned in university probability courses.
Like Gita, many of the other preservice teachers such as Alan, Anne, Jane, John, Mate, and Ruth stated that the probability they learned in university courses was not helpful in solving everyday problems. They argued that they could not use formal probability in their everyday world because the probability courses at the university did not emphasize solving everyday problems. They thought that if their probability courses were intended to help them in solving everyday problems, they might have used it in their everyday lives. According to them, the purpose of the probability courses was to prepare them for other courses in mathematics, statistics, physics, chemistry, and engineering.

The preservice teachers also argued that they might have been able to use probability in their everyday problems if their understanding of probability was good. Unfortunately most of the preservice teachers admitted that they did not have a strong conceptual grasp of probability. According to these preservice teachers, the courses in probability were taught in a rote fashion with no emphasis on conceptual understanding. They rarely had time to clarify concepts through discussion. For example, Gita stated the following:

It was taught in a very formulaic way. You look at the problem. You identify it. You match it with a formula and solve it that way. I never understood the formula and I never understood how the formulas were derived. I could never remember them. So I don't think I've used them properly. I used them more maybe in terms of how to read problems. To look where the events are independent or dependent or things like that. More than trying to fit them into a formula. No, I don't think that course was very helpful.

Although Gita was highly critical of the way probability was taught at the university level, she was one of the few participants who used formal
mathematical reasoning in most of the tasks posed in this study. When this issue was raised with Gita in a follow-up interview, she mentioned that her Grade 12 probability unit was much more helpful in solving the tasks because she had several opportunities to experiment with coins, dice, and spinners in that unit. However, another preservice teacher, Alan, forgot what he had done in Grade 12. He did remember the probability course he took at university and stated that it did not provide any help in solving everyday problems. He mentioned that his professor did not communicate well and spent most of the time writing "lots and lots of equations that didn't make sense" to him. He spent his time in the course routinely following the formulas and applying them to problems in a rote manner without actually understanding why. Hence, for Alan, using probability formulas in everyday life was removed from his way of thinking. It is interesting to note that he did not use formal probability in solving everyday-oriented tasks posed in the study. Instead, he used non-formal reasoning to solve them.

Ruth did not like university probability because she actually found it counter-intuitive to her thinking. According to her many things in probability did not fit with her "feelings." Ruth's "feeling" is the same as non-formal reasoning. Ruth stated that in the everyday world human feelings play an important role in the solution of a problem, while in university courses feelings do not have much place. Ruth found that her feelings drawn from everyday experiences were incompatible with the probability theories she had studied at university. Because of this, she followed the formulas and knew their applications, but without much understanding. In a follow-up interview, Ruth mentioned that in everyday life she always relied on her feelings, while in university and similar settings, she tried to apply her formal knowledge of probability. Ruth further explained that formal probability at a university does
not have much place for people's everyday feelings. In much the same way, her everyday world did not make use of formal probability.

Anne thought that probability was an interesting subject. However, she did not believe that teaching probability without conceptual understanding would be helpful in everyday life. Anne described her probability class as follows:

Even though I found it a very fascinating subject ... He presented in such a theoretical [way].... It seems to tie with a lot of things, but he didn't make as many connections.... It was very much torture. Just learn the formula and, you know, we weren't necessarily understanding why.

It was evident that all the preservice teachers mentioned above saw applications of university probability as different from chance phenomena that people encounter in everyday lives. John, for example, saw probability as a "purely academic" matter that is unrelated to his "day-to-day life." Although he took risks in his day-to-day life and knew that those were related to probability, he actually found it difficult to explain exactly how school and university knowledge of probability helped him to make judgments in his everyday life. John mentioned that he would rather use intuition and personal experience to solve everyday problems involving probability. According to him, solutions to everyday probability problems do not require the use of the mathematics that he learned at university. Moreover, he found that formal probability was too theoretical to be applicable to everyday problems.

Many of the preservice teachers stated that they would try to use university knowledge for questions asked in a university setting or in other similar settings. However, in solving problems encountered in everyday life they would feel secure relying on everyday experiences and intuitions. Anne argued
that in everyday life she used “human probability” instead of university probability. Human probability is taken to mean the same as non-formal probability—that is thinking about chance phenomena in the everyday world without explicitly referring to formal probability. For example, how likely is it that her sister will call her before going to the grocery store? How likely is it that the bus is going to come within the next 10 minutes? For Anne, non-formal probability was different from university probability because the former is not based on formulas, procedures, axioms, or theorems. Instead, it is based on people's everyday experiences and intuitions.

**Knowledge of university probability enhances non-formal probability**

Some of the preservice teachers were more positive about their learning of probability at the university. They reasoned that although probability learned at university is not directly applicable to their everyday lives, it enhances the thinking skills and problem-solving abilities which can be applied to their everyday problems. Therefore, they argued that they use probability learned at the university in an indirect manner. For example, Lisa stated the following:

> I guess in a sense you use it but you don't ... realize. You can think what could be the percentage of something happening, how likely it'll happen. I think everyday you think about something, you figure out how to solve problems, how to make decisions. I think that part, probability part, is a part of thinking process. You just don't realize it.... I will not look at [the fact that]: there are so many combinations, which one would be most likely? I don't look that way because I'm not very comfortable in predicting outcomes.

Lisa thought that the probability she had learned at university was helping her to make decisions in an indirect manner. However, upon further
probing, she was not sure whether the assistance in making decisions had come from university probability courses or from her everyday experiences. Lisa's statement implies that her non-formal thinking was influenced by the probability course she had taken at university. That may be the reason why Lisa's responses to the probability tasks in this study were characterized by a conflict between her non-formal and formal thinking.

Reid had views similar to Lisa's regarding the use of probability in everyday life. The following transcript illustrates the ways in which Reid thought that probability was useful in solving everyday problems:

Reid: Even though I don't remember the exact formulas and so on, I still got through the thinking process in looking at complex problems. You're not just bringing your intuition, but you're bringing knowledge and experience.

Hari: Do you use your university or school knowledge of probability to solve your everyday problems?

Reid: Yeah. For instance, you drive into school. And you say, well I'm only going to be here for 2 minutes to pick up some books. So why don't I just park in the meter parking and not put any money into it. Because what's the probability that someone will come along and give me a ticket in 2 minutes? I can make some in-depth thinking. Let's say this guy comes every hour. The likelihood that he's going to come by in those 2 minutes is very small. Even if you aren't making the actual number calculations all the time, you're making some relative calculations in your mind. And you do this sort of thing all the time in real life.

Hari: Is that because of your university mathematics or you do that anyway?
Reid: Good question. That's difficult to say. But it certainly comes before university education. It's a sort of common sense.

Reid was one of the few consistent mathematical thinkers in the study. He stated that he would use intuition and formal probability to solve problems in everyday life. Moreover, his intuition was influenced by formal probability. Being a formal thinker, Reid thought that he would use probability in his everyday life. However, his example of parking a car at a meter did not actually require a university level understanding of probability. People could solve Reid's problem without knowing probability that is taught at universities. It is notable that even Reid, who was an advocate of university probability, was actually providing an example of the non-formal probability that Anne spoke of.

Mary and Bill also provided examples of non-formal probability. Mary's example of non-formal probability can be seen in the following transcript:

Hari: How often do you use probability in everyday life?
Mary: Actually I think I use probability every day.... Like the chances are if I go to the university to work on my project, nobody will bother me. Whereas if I stay at home and try to work on my project, my sister will be there. The chances are she's going to come in and bother me a lot and I'm not going to get much work done. So I better go to the university.

Hari: Do you need university probability to decide those sorts of things?
Mary: [pause] To decide those kinds of things, no I don't. But ... to me probability gives a kind of way of looking at things which is sort of different than [everyday intuitions].... In the courses I took, there was a lot of material done in a very short period of time. And my retention from those courses about probability is actually very low
[laughs]. I think if I had just taken a course in probability ... I think courses like that provide students the better opportunity to learn the probability to practice. I didn't get much practice in my university learning probability to effectively use it in everyday life. Although it gave me a better intuition. Like I've a feeling for it which I didn't have before.

Mary thought that her intuition may have been influenced by her university experience in probability. She expected that her non-formal conception of probability was a function of her everyday and university experiences. However, Mary's example of her sister interrupting her when she was studying is a good example of non-formal probability. Mary's reasoning stemmed completely from her everyday experiences and not from the kind of probability learned at university. Mary admitted that she did not use university probability to decide whether or not she should go to the university to work.

Mary was concerned that her probability course did not use everyday problems such as coin tossing and dice rolling. She had never had an opportunity to solve probability problems that may be encountered in everyday life. She thought that if she had had an opportunity to practice solving everyday problems in her university courses she would now be better at solving everyday problems. Bill, however, argued the opposite. He thought that the probability course he had taken at university was useful because it helped to eliminate his intuition. This is clear in the following transcript:

Hari: You've done some probability at your university. How often do you use probability in your everyday life?

Bill: Pretty often.

Hari: Pretty often? Can you give me an example?
Bill: [pause] No. [laughs] Okay, go to the bus this morning and it says that it'll come at 7:21 or 8:21. But based on previous experience I know that it doesn't usually come right at 8:21. It usually comes a couple of minutes later. So, even at 8:21 I'll still run to the bus stop and probably catch a bus. You know the odds are better that I would [catch the bus].

Hari: Interesting. Do you need university probability to make that decision?

Bill: No. [laughs] [pause] When do I use that? If I'm feeling really keen. If someone asked me a question and I really wanted to know the answer, I would probably look at my textbooks and use a formula. But my probability course did teach me not to rely on your intuition kind of things. There is a lot of other factors that influence, like the bus schedule doesn't mean that the bus is going to show up. Because there is traffic. Maybe it broke down. Maybe they have more buses on schedules because it's a busy time.... There are lots of other factors. Like we did a birthday problem beginning of stat class.... He [the instructor] made a bet with the class that at least two people in the class will have the same birthday. And the fact that you've a class of larger than 20 people ... the odds are better that two people in the class are going to have the same birthday.... Your intuition might tell you that there is only a 1 in 365 chance that you have the same birthday as [someone else].... You can't always rely on your intuition. This thing I got out of my stat class.

Bill argued that he no longer relied on his intuition after taking probability courses at the university. The above transcript indicates that Bill's probability
course was designed to show that students' intuitions were faulty. For example, the birthday problem was used by the instructor to argue that students' intuitions are often inferior to mathematical probability. Bill was convinced from his course that he should use mathematical reasoning in solving probability problems. It is interesting, however, that the example (waiting for the bus) he provided was an example of non-formal probability not an example of university probability.

There were a few other students who argued that university probability was useful in solving everyday problems. However, like Reid, Mary and Bill said they had a hard time providing a suitable example of a problem and exploring how they would use it.

It is notable that all the students mentioned above, except Bill and Reid, did not enjoy the learning of probability in university courses. Two other participants, Gary and Paul, enjoyed probability at the university level. They expressed themselves as follows:

Oh! It was fantastic. We had an incredible teacher. Very, very difficult. It went like: binomial theorem, multinomial theorem.... He made us actually prove this.... Really, really difficult course but I really learned a lot.... I wish I could have taken a little bit more after. There was so much so fast. A lot of it just went in and out. At least I know that there is a way to figuring out, like you are not remembering how to do expected values. I know there is a way to do that. I can always go and look at it.... I really enjoyed it. Really interesting! (Gary)

I enjoy taking up things in my own mind.... Taking a theoretical, and how does it apply. I enjoy that. But I don't enjoy haggling over whether a tack is going to fall one way or the other way. I mean I've got to be given some
assumptions even if they're false. Then let my mind explore the possibilities beyond that. (Paul)

Both Gary and Paul enjoyed probability, not because probability had a lot to do with their everyday lives, but because they could make hypotheses and compare them with theory. In general, Gary and Paul both enjoyed their probability courses at the university because the courses were mathematically sound. Their arguments showed that they were both deductive thinkers and drew conclusions based on axioms, assumptions, and hypotheses. Although they found university probability sound and useful from a formal perspective, they argued that the mathematics of probability was not useful in everyday life. According to Paul, in real life "you go on with your intuition." Thus almost all the participants believed that the probability they learned at the university was not useful for them in solving problems that they encounter in their everyday lives.

Preservice teachers' methods of solving everyday problems

Some tasks, such as the lottery and car accident task in the written and the pair-problem-solving components, the bank teller task in the pair-problem-solving component, and the dice task in the interview component, were related to the preservice teachers' everyday world. All the preservice teachers either drove cars or were familiar with car accidents. All had been to a bank at different times of the day and had observed a line up. All but one knew how to play the lottery 6/49. Some played card games, such as black jack, and some observed people playing card and dice games in a casino. The preservice teachers could have used their university knowledge of probability in solving these tasks. However, as has been discussed in Chapters 4 and 5 many of the
Preservice teachers did not use their university knowledge of probability to determine solutions for these tasks.

As discussed earlier many of the preservice teachers, Alan, Anne, John, and Ruth for example, mostly used non-formal reasoning in solving the probability tasks presented in the study. They basically developed non-formal reasoning from their past experiences in their everyday lives. However, to some extent their non-formal reasoning was framed by their school and university knowledge also.

In a task like the dice task, the majority of the preservice teachers attempted to solve the problem mathematically. However, they found it difficult to solve. They stated that the probability of getting a number in the roll of one die is $\frac{1}{6}$. So the dealer had $\frac{5}{6}$ chance of winning the bet. However, they were confused as to how they would calculate the probability when there were three dice. They could not figure out whether they had to apply the addition rule or the multiplication rule of probability. They were highly confused with the provision that the player could win double or triple the amount of money he or she bets if the same number occurs in two or three dice. Because of such confusion and conflict they could not solve the problem formally and they stayed with their non-formal reasoning.

Many of the preservice teachers could not do the probability calculation formally because their probability learning was not conceptual enough to apply to everyday problems. Moreover, they forgot the mathematical rules and theorems from their probability courses. Mary and Ruth, for example, believed in luck only when they failed to solve the problem formally. In the second interview both of them mentioned that they wanted to solve the academic problem formally because both of them felt that formal knowledge was more valued in university teaching and research. However, in their everyday lives
they believed in luck and feelings. Hence, for these preservice teachers, probability at university and probability in everyday life were two different phenomena with only a small overlap.

**Overview of Chapter 6**

Most of the preservice teachers in this study argued that the knowledge of probability they gained at university was not useful in solving problems that they might encounter in their everyday lives. Of those who thought that their probability knowledge would be useful in solving everyday problems, most could not give a suitable example of how university probability was useful. On the contrary, the examples they provided were more suitable to illustrating that the probability they learned at the university contributed very little to problems they had to solve in their everyday lives. Mary's example of her sister interrupting Mary in study, Reid's example of parking a car at a meter, and Bill's example of waiting for a bus are all related to chance factors. However, the solutions to these problems do not require university knowledge of probability.

Even the preservice teachers who argued for the benefits of university knowledge of probability realized that the benefits were not direct or immediately applicable. They argued that their knowledge of university probability helped to enhance their thinking skills which in turn made them better problem solvers. However, many participants did not come up with a mathematically acceptable solution to different tasks, such as the dice task, that were contextualized in a real-world situation. Many of them either went with their non-formal thinking or applied inappropriate mathematical procedures and strategies. When this issue was raised, they stated that probability learned at university was not readily applicable to simple everyday problems.
CHAPTER 7
Conclusions, Discussion, and Implications

The purpose of the study was to investigate preservice teachers' conceptions of probability (first question), the consistency of their conceptions (second question), and their views about the role of university probability courses in solving everyday problems (third question). A variety of tasks involving probability were given to 16 preservice teachers in three different settings: written, pair-problem-solving, and interview situations. The outcomes of the analyses and interpretation of the data with respect to each research question has led to a number of conclusions which are discussed in this chapter.

Conclusions

In relation to the first research question—What are some qualitatively different ways in which preservice teachers conceptualize probability?—non-formal and formal conceptions of probability with qualitatively different conceptions within each category were identified. In answer to the second research question—In what ways do preservice teachers' conceptions of probability vary across tasks and settings?—it was found that their conceptions varied depending on the types of tasks and settings as well as their personal experiences. Inquiry into the third research question—What are preservice teachers' views about the role of probability learned at the university level in solving everyday problems?—led to the conclusion that the preservice teachers viewed university probability as being not closely related to their day-to-day lives. The conclusions to each of the research questions are discussed in greater detail below.
Question one: What are some qualitatively different ways in which preservice teachers conceptualize probability?

The preservice teachers held qualitatively different conceptions of probability. These qualitatively different conceptions were grouped under non-formal and formal conceptions of probability. Non-formal conceptions of probability included the preservice teachers' use of everyday experiences and heuristics in solving problems. Formal conceptions included the preservice teachers' knowledge of independence, randomness, probability formulas, rules, and applications.

Both non-formal and formal conceptions identified in this study are unique compared to earlier studies reported in the field of probability research. Some of the non-formal conceptions such as the outcome approach discussed by Konold (1989, 1991) were not found to be used by the participants in this study. Rather than attempting to answer a particular problem based on one trial, all the participants in this study appeared to believe in the law of large numbers, although their conceptions of large numbers were different. The number of trials for a large number varied from 30 to a million. For participants who were strongly influenced by formal probability, a large number required at least a hundred trials. For others, it was a matter of context depending on tasks and experiments.

Non-formal conceptions of probability such as the representativeness heuristic and the gambler's fallacy were found to be used by some of the participants in this study. However, the conceptions as discussed by these participants seemed to differ from the ones reported in previous research in at least three respects.

First, not all the participants who used a representativeness heuristic consistently thought that the outcome in the sample should look like its parent
population, or that the events in an outcome should appear random. Instead, the participants who used this heuristic focused on the first characteristic that a sequence with nearly equal number of heads and tails is most likely to occur. In the head-tail sequence task for example, the sequences THTHHT and HTHTHT were thought to be equally likely because both of them had the same number of heads and tails. If the preservice teachers had used the representativeness heuristic in the sense that Kahneman and Tversky (1972) have described, the participants would have stated that the sequence THTHHT would be more likely than the sequence HTHTHT because the former sequence appears to be more random than the latter which has a systematic (alternate heads and tails) occurrence of heads and tails.

Unlike Kahneman and Tversky's studies, some of the participants who stated that the sequence with nearly equal number of heads and tails would be most likely to occur did not hold the same view when the number of trials was increased from 6 to 20 or to 100. Participants in this study thought that with a larger number of trials all the sequences would be equally likely. Hence, the representativeness heuristic was used only in the case of small samples. These statements provided by the participants indicate that the representativeness heuristic is far more complex than has been suggested by previous researchers.

Third, the participants used these heuristics because of everyday experiences of success in solving real life problems. For example, one participant who consistently used the gambler's fallacy got this idea from his experience of playing blackjack and other games in a casino. He often won money in real life by using this strategy. He made it clear that his reasoning might not be consistent with the mathematical theory of probability, but was fine for him in his everyday life.
One of the uses of formal probability discussed in the literature is that it helps people in making reasonable decisions in their everyday lives (Borel, 1962; Dörfler & McLone, 1986; Good, 1983; Jacobsen, 1989; McGervey, 1986). But this participant thought that his decision, while not making use of formal probability, was reasonable because he frequently won money. Thus for him the heuristics were not fallacies; instead they were successful strategies for solving everyday problems. Nevertheless, he stated that he would not use the gambler's fallacy in a university examination because it is not valued in that context. The reasoning provided by the participants indicated that they viewed the mathematics of probability or the heuristics in terms of contexts and situations.

The participants in this study also demonstrated conceptions similar to the availability heuristic. Most of the discussion provided in Chapter 4 regarding the use of everyday experiences in solving probability problems is related to this heuristic. Researchers argue that people use previous experiences without evaluating how appropriate such experiences are in solving problems (Tversky & Kahneman, 1973, 1982a; Shaughnessy, 1992). According to these researchers, the lack of evaluation may mislead people's attempts to solve probability problems. However, in this study, the participants' argued that they had to use their everyday experiences in solving some of the problems because formal probability was not always useful. The participants provided various examples of why they thought that mathematical solutions to the problems were not applicable to real life situations. For example, their suspension of a mathematical solution to the bank teller task because of its lack of applicability to the real life situation indicates that the participants in this study were thinking beyond a simple mathematical model of probability.

By the same token, the preservice teachers' argument that a person was more likely to be killed in a car accident than win a jackpot was based on a non-
formal conception that is not incompatible with the mathematics of probability. The same is true for preservice teachers' decision to be the dealer in the dice game because, "It is the dealer who usually wins money." Their solutions to the problems as well as the reasoning processes they used to come to the solutions were sensible when viewed in terms of everyday life situations. Hence, the availability heuristic is connected with a person's everyday life and so may be more useful than formal mathematics in thinking about problems related to their personal life.

In addition to reporting the use of these heuristics in different ways than described in earlier research, this study has added some new and qualitatively different conceptions of probability. One such conception is the participants' use of knowledge from other disciplines in solving probability problems. Some of the preservice teachers used their knowledge of physics and genetics to solve some of the problems. It appeared that the participants' thinking about probability was interwoven with knowledge from different disciplines.

In some of the tasks, the participants' use of formal probability concepts, such as independence and randomness, and probability formulas and rules, was reasonably easy to see. The participants' responses discussed in Chapters 4 and 5 indicate that they had a more complex understanding of formal probability than students in the earlier studies, such as those reported by Piaget and Inhelder (1951/1975), Fischbein and colleagues (1975, 1984, 1987, 1991), Green (1983, 1987, 1988), Kahneman and Tversky (1972), Tversky and Kahneman (1973, 1982a, b), Shaughnessy (1977, 1981, 1992), and Konold and colleagues (1989, 1991, 1993). The participants' mathematically correct responses were not surprising because they had better backgrounds in mathematics and probability than the school children and first year college students in earlier studies. However in this study, many of the participants who correctly used probability concepts, such as
independence, randomness, and the addition and the multiplication rules, could not use these concepts correctly on some tasks, such as the dice task. Their different responses to similar tasks in different settings indicate that their conceptions were situated in contexts generated by tasks, settings, and different social dynamics.

Question two: In what ways do preservice teachers' conceptions of probability vary across tasks and settings?

Preservice teachers' conceptions of probability varied across tasks. They used formal conceptions if a task appeared similar to a textbook-type problem and could be solved by a relatively straightforward application of a mathematical formula or a rule. They used non-formal conceptions if the tasks appeared similar to everyday experiences. On some tasks, for example in the lottery task and the dice task, their conceptions of probability were characterized by a conflict between the non-formal thinking they developed from their everyday experiences and the formal probability they learned at university.

Not only did the participants' responses vary across tasks, but their responses also varied across settings. In the pair-problem solving, the preservice teachers who used non-formal conceptions agreed with those who used formal conceptions. However, many of the participants who seemingly changed their conceptions toward the formal ones in the pair-problem solving went back to their original conceptions when probed individually in the interview setting.

It is also interesting to note that the participants' responses were usually formal in the written component, and more non-formal or a mixture of both in the pair-problem solving and the interview components. The participants' formal conceptions were challenged when different social dynamics were created in the pair-problem solving and the interview components. When their formal conceptions were challenged by the researcher in the interview, they began to
shift their conceptions to non-formal ones. In much the same way, they began to shift their non-formal conceptions to formal ones when challenged.

The role of the tasks was also an important determinant of how the participants viewed probability. On a textbook-like task formal thinkers always dominated the non-formal thinkers. When formal thinkers were able to apply a mathematics formula to solve a task, non-formal thinkers usually supported them because in their view formal mathematics would be valued in such tasks. Once the participants changed their conceptions from non-formal to formal in textbook-like tasks, they used this formal conception in solving other tasks, too. But that was not true for everyday-oriented tasks. On tasks set in an everyday context, participants' non-formal conceptions were difficult to change. They found that non-formal conceptions were more helpful than formal ones if the solutions had to be applicable to real life situations.

These findings, derived from the preservice teachers' responses generated from different tasks in various settings, provide a richer understanding of students' conceptions of probability. Research conducted by Piaget and Inhelder (1951/1975), Fischbein and colleagues (1975, 1984, 1987, 1991), Green (1983, 1987, 1988), Kahneman and Tversky (1972), Tversky and Kahneman (1973, 1982a, b), Shaughnessy (1981, 1992) all concentrated on students' understanding of probability by giving different tasks but not providing different settings. Those studies provide some insights into how students' reasoning varied in different tasks, but fail to show how their reasoning changes across different settings.

This study has reported not only qualitatively different conceptions of probability but has also provided examples of how and when participants' probabilistic reasoning varies. When the participants moved from individual written tasks to the pair-problem solving and then into the interview setting,
their responses varied based on social interactions between the respondents and also between the respondents and the researcher.

**Question three: What are preservice teachers' views about the role of probability learned at the university level in solving everyday problems?**

The participants generally viewed the teaching of university probability as a highly theoretical area which did not emphasize conceptual understanding. They viewed the lack of emphasis on conceptual understanding of formal probability as contributing to their inability to use it in solving everyday problems. That was one of the reasons why the preservice teachers relied on non-formal probability in solving problems that they encountered in their day-to-day lives. Despite an ever increasing appeal for a need to learn and teach mathematics with understanding (Hiebert & Carpenter, 1992; Skemp, 1976), conceptual understanding of mathematics does not seem to have been achieved for these participants at the university level. This seems to be a concern which needs to be addressed by mathematicians and mathematics educators who are concerned about students' conceptual understanding.

Some of the participants in the study thought that they were unable to use formal probability in solving everyday problems because their probability classes never emphasized its use in that context. They thought that if they had had opportunities to solve everyday problems in their probability classes they would have been able to use it in their everyday lives. According to them the use of everyday problems in probability classrooms would have provided them with better understanding of probability, which in turn would help them in solving everyday problems. Although reform documents such as *Everybody Counts* (National Research Council, 1989), *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989), and *Professional Standards for Teaching Mathematics* (NCTM, 1991) emphasize the role of everyday contexts in the teaching and
learning of mathematics, the importance of such contexts appear not to have been valued in the teaching of probability at the university level.

Some of the participants in the study thought that university probability enhanced their reasoning skills which in turn helped them solve everyday problems. However, the examples of problems that they gave did not require university knowledge of probability. Instead, the examples they provided illustrated how university probability was not required to solve everyday problems. This indicates that the participants' immediate thinking about everyday problems had very little to do with university probability.

By probing the participants' views about the role played by university probability in everyday problem-solving, this study has provided additional insights into students' conceptions of probability. Their examples of probability problems from their everyday contexts were useful for the researcher in probing their conceptions further.

Discussion of critical issues arising from the study

In Chapter 2, it was noted that the major thrust of research into students' conceptions of probability was concentrated at elementary and junior high school levels. A relatively small number of research projects were also carried out with first year college students. This study was unique in that it focused on preservice teachers who had strong backgrounds in mathematics. This study contributes to the literature by raising two different issues. First, it highlights the role of contexts in students' methods of solving problems and second, it provides insights into the relationship between an understanding of formal probability and its use in solving everyday problems.
Role of contexts in solving problems

This study has provided information regarding preservice teachers' understanding of probability by portraying how they think differently about problems in different tasks and settings. Their solutions to the problems depended on tasks and settings and also depended on their personal and institutional experiences.

Many of the participants used non-formal conceptions in solving everyday problems, but they began to shift toward formal ones when they were asked what answer they would give on a mathematics examination. They indicated that, for examination purposes, they would suppress their non-formal conceptions because only formal conceptions are valued in that context. This suggests that to be successful in an academic environment, they believed they should be able to suspend their non-formal conceptions.

Although the participants indicated their tendency to suspend their non-formal conceptions for the purpose of university examinations, some non-formal conceptions of probability do not conflict with formal conceptions, and the former may help in the learning of the latter. For everyday purposes like waiting for a bus and parking a car the calculation of formal probability may appear ridiculous, although a calculation can be done. In order to solve such problems a non-formal conception of probability appears to be enough. In more abstract problems, formal probability may be essential. Instead of devaluing students' non-formal conceptions, educators may serve their students better by making them aware that competing conceptions of probability and their meaningful use may depend on contexts.
Understanding probability and using it in everyday life

The participants in this study stated that their probability courses did not value their experiences and intuitions, and thus they were not confident in solving everyday probability problems from a formal perspective. This view expressed by the preservice teachers, that their probability learning was not meaningful because they never had an opportunity to reflect on their experiences, is consistent with the arguments presented by many constructivist educators (Ausubel, Novak, & Hanesian, 1978; Bishop, 1988; Confrey, 1990a, b; Fischbein, 1975, 1987). These educators argue that students come to class with their own prior experiences and they cannot develop a conceptual understanding of formal knowledge if their prior knowledge is not elicited and extended.

The basic premise of a constructivist view of learning is that students' prior experiences should be recognized, respected, and used in teaching. Otherwise, students will continue to show incoherence, inconsistency, and fragmented knowledge of subject matter as was evident in this study. Probability courses for preservice teachers therefore should aim at a strong conceptual emphasis by providing opportunities for them to connect formal knowledge to their everyday experiences in the context of social interaction in the classroom. As indicated earlier, the participants in this study argued that formal probability would have been more meaningful to them if the teaching had attempted to establish a link between everyday chance phenomena and formal probability. This view is consistent with the views argued by educators such as Gallimore and Tharp (1990) and Hennessy (1993). Stigler and Baranes (1988) have argued that students can be comfortable in solving school problems and applying them in the real-world only if school activities are similar to the practical activities performed in day-to-day life.
The arguments provided by the participants in this study and by other educators imply that if a link can be established between students' non-formal and formal conceptions, there is some hope that students can understand formal concepts and in turn be able to apply that knowledge to problems that they encounter in their everyday lives. However, as Walkerdine (1990, 1994) argues, educators should be aware that the everyday practice of mathematics is discursively different from school practice and the link between these two is not readily obvious. For example, the participants' examples of everyday chance phenomena, such as waiting for a bus, parking a car, and the rise and fall of stock prices do not follow a mathematical principle of randomness, so it is difficult to connect this to formal probability learned in school and university. Whether the use of preservice teachers' everyday experiences in the teaching of probability helps them solve their everyday problems is a question that was not directly addressed in this study and would need to be pursued.

It is important that mathematics programs for prospective teachers provide an opportunity for them to make sense of what it means to learn mathematics. This is different from the practice of simply learning definitions, rules, formulas, and routine applications that seems to be prevalent in the teaching of mathematics from school to university levels (Battista, 1994). A program of mathematics should emphasize problem-solving "in an environment that encourages students to explore, formulate and test conjectures, prove generalizations, and discuss and apply the results of their investigations" (NCTM, 1989, p. 128). Students' intuitions and everyday experiences do not always hinder such mathematics learning but they could be helpful in making such programs meaningful to students (NCTM, 1989).

The quotation from the NCTM Standards emphasizes a new vision of mathematics learning that enhances conceptual understanding. Many of the
participants argued that they could not use formal probability in everyday problem-solving because they did not understand it conceptually. There is not enough evidence from this study to suggest that an emphasis on conceptual understanding helps students in effectively solving everyday problems. However, it is certainly an interesting area for further study because many mathematics educators have argued that the learning of mathematics should emphasize conceptual understanding (Hiebert & Carpenter, 1992; Skemp, 1976) while others have stressed that one of the purposes of doing mathematics is to be able to solve everyday problems by using it (Dörlfer & McLone, 1986; Jacobsen, 1989).

Implications for teacher education

Educators who argue for the teaching of probability in schools and universities are interested in understanding how prospective and practicing teachers conceptualize probability (Shaughnessy, 1992). There is very little in the literature to inform educators about how preservice teachers with strong backgrounds in mathematics think about probability. This study provides a portrayal of two groupings of qualitatively different conceptions of probability that a group of preservice teachers held. The various conceptions demonstrated by the participants in this study can inform both mathematics professors and mathematics education professors about how people with reasonably strong backgrounds in mathematics solve probability problems.

The study provides evidence that university students, even after taking many mathematics courses, including a probability course, continue to use non-formal reasoning to make sense of problems related to probability. Insights gained into how probability is conceptualized by preservice teachers can function as points of discussion, both during the teaching of subject matter knowledge of probability in university mathematics or statistics departments and
the teaching of pedagogical content knowledge of probability in mathematics education departments. Some instructors may find that their own students may hold conceptions of probability similar to those identified in this study. Since the conceptions in this study are identified from the preservice teachers' reflection upon their everyday and mathematical experiences they are likely to be meaningful and relevant to other preservice teachers who have similar experiences. All this information will help instructors plan and teach courses in probability and probability education.

By knowing some of the different ways university students conceptualize probability, mathematics professors may respond to their students' difficulties in understanding formal probability by providing alternative explanations of the probability concepts to be taught. Mathematics education professors may be able to help preservice teachers by addressing the differing conceptions of probability held by other preservice teachers and also held by school students. Students and professor may wish to explore such questions as: To what extent do school and university students' conceptions differ and what do these differences pose for teaching? Furthermore, mathematics professors may be willing to discuss what types of probability conceptions their own group of students hold and whether or not these conceptions are mathematically correct. Since students' conceptions serve as a powerful force in their understanding of mathematics (Borasi, 1990), discussion of these conceptions may prove to be beneficial for preservice teachers' understanding of probability. The diversity of the preservice teachers' conceptions elicited in the study provides a rich context for examining the teaching of probability. For example, students in a probability classroom could be asked to evaluate the appropriateness of using everyday experiences or science knowledge in solving probability problems such as the lottery and car accident task and the thumbtack task posed in this study.
Furthermore, mathematics and mathematics education professors can obtain insights into their own students' conceptions by using varieties of tasks in different settings such as those presented in this study. The eliciting of conceptions held by their students may help professors to extend students' experience-based thinking to more formal mathematics.

Implications for curriculum and instruction
This study has implications for the field of curriculum and instruction. A question that arises from this study is the meaning of literacy in probability: What does it mean to be literate in probability? Are university mathematics graduates literate in probability? Can a person be considered literate if he or she knows how to solve textbook probability problems but cannot apply them in solving everyday problems? Can someone be called literate if he or she can solve everyday problems without using formal probability? The works of Dewdney (1993) and Paulos (1989, 1994) imply that being literate in probability means being able to apply formal probability to solve problems related to chance phenomena that people encounter in their everyday lives. If being literate in probability is being able to apply it in solving everyday chance phenomena by using formal probability, then schools and universities do not seem to help students to be literate in this regard, as expressed by the participants in this study.

However, if being literate means to be able to solve everyday problems without using formal probability, then many people without any knowledge of formal probability can still be considered literate in probability. There may be people without training in formal probability who are better at predicting phenomena in real-world situations, such as weather forecasting, stock prices, interest rates, etc. than the ones who have a good knowledge of formal probability.
This finding has an important implication regarding the nature of probability and its teaching at school and university levels. If people with strong backgrounds in mathematics and probability do not find their knowledge of mathematics useful in the solution of everyday problems, why would ordinary people and students with little mathematics background do so? Perhaps more fundamentally, is a formal conception of probability useful in an everyday context? If the purpose of teaching probability is to help students solve formulated textbook-like problems, then an emphasis on formal conceptions of probability is certainly necessary. However, if the purpose of teaching probability is to help students in solving problems that can be encountered in day-to-day life, then more than an emphasis on formal conceptions seems needed.

Based on the findings of this study, a question can also be raised about a conceptual change view of learning. In classroom teaching, students and teachers bring differing cultural background, and personal experiences (Bishop, 1988) which significantly affect classroom discourse. Because of this social interaction, students may demonstrate a change of conception in classrooms and examinations. But in everyday thinking, they may continue to adhere to their non-formal conceptions. The results of this study provide evidence that students never entirely dismiss their original conceptions. Even after studying several courses in university mathematics and a course in university probability, the participants in this study continued to utilize their everyday non-formal conceptions while solving problems related to probability.
Recommendations for further research

It could be expected that the preservice teachers might demonstrate different conceptions of probability if their probability courses had provided them with an opportunity to compare their non-formal and formal conceptions of probability. However, how students would develop their understanding of formal probability if their non-formal conceptions were explored and used during teaching has not been researched. A useful project for further study would involve eliciting students' conceptions of probability prior to instruction and exploring how those conceptions would change over time if their non-formal conceptions were used in teaching.

The participants' responses in this study were a function of tasks and settings. A second project could be to conduct a study of students' views about probability tasks and settings. It would be interesting to investigate why students' conceptions are non-formal during certain types of tasks and settings and formal in others. Exploring students' conceptions of tasks and settings can be beneficial for teachers and professors who teach probability because the results from such studies may help them in selecting interesting, thoughtful, and challenging tasks for their teaching to be used in various settings.

A third implication of this study could be to introduce an undergraduate mathematics course for future teachers that emphasizes eliciting their conceptions through interviews and journal writing, and using the information thus obtained in teaching. Important research questions can be: What do students think about mathematics learning and teaching prior to instruction and how do their views about mathematics evolve and change during and after instruction? What does it mean to learn mathematics that emphasizes conceptual understanding? In what ways do prospective teachers' understanding of mathematics influence their teaching?
References


# Appendix A - Written component

## Background Information

Circle either Yes or No in the following questions. If the questions are not followed by Yes or No, please write your responses.

<table>
<thead>
<tr>
<th>Student's ID:</th>
<th>Sex: M/F</th>
</tr>
</thead>
</table>

- Are you a preservice teacher? Yes / No
- Are you an inservice teacher? Yes / No
- How many university graduate or undergraduate mathematics courses have you taken?
- Your highest academic qualifications:

- Have you taken any statistics course? Yes / No
  - How many courses?
  - At what levels?

- Have you taken any probability course? Yes / No
  - How many courses?
  - At what levels?

- Have you had any teaching experience? Yes / No
  - How many years of teaching experience?
  - Have you ever taught probability? Yes / No
    - If Yes, how many years?
    - At what levels?
Written tasks

Below* are some multiple-choice and open-ended questions. Please answer each question and provide your reasoning. Not only is your final answer important but also the method you used (or would use) to solve each of these problems.

1. Birth sequence task

A couple plans to have six children. Which of the following orderly sequence of birth is most likely? (Boys = B, Girls = G).

(a) BGBBBB
(b) BBBGGG
(c) GBGBBG
(d) BGBGBG
(e) All four sequences are equally likely

Give reasons for your choice.

(Modified from Kahneman & Tversky, 1972 and Konold et al., 1993)

* In the actual research the tasks were provided on separate sheets of paper.
2. **Rectangular spinner task**

In the diagram above OA is a spinner. If it is spun a large number of times on which number is it most likely to stop.

(a) 1  
(b) 2  
(c) 3  
(d) 4  
(e) 5  
(f) The spinner will stop equal number of times in each case.

Give reasons for your choice.

3. **Five-head task**

A fair coin is flipped five times, each time landing with heads up. What is the most likely outcome if the coin is flipped a sixth time?

a) Another head is more likely than a tail  
b) A tail is more likely than another head  
c) The outcome (head or tail) is equally likely

Give reasons for your choice.

(Adapted from Konold et al., 1993)
4. **Lottery and car accident task**

Which one is **more likely**?

(a) You will win a jackpot in BC lottery 6/49  
(b) You will be killed in a car accident

Describe how you would solve this problem.

5. **Circular spinner task**

In the diagram below OA is a spinner. Four people are playing this game. Each person will pick two numbers from 1–8 and if the spinner stops at any of the numbers picked the person wins. If you are playing this game which two numbers will you pick to have a best chance of winning? Why?
Appendix B - Pair-problem solving

Below* are some multiple-choice and open-ended questions. Please answer each question and provide your reasoning. Not only is your final answer important but also the method you used (or would use) to solve each of these problems. Please speak aloud whatever you think so that your thinking can be recorded.

1. **Head-tail sequence task**
   Which of the following orderly sequences is most likely to result from flipping a fair coin six times? (Heads = H, Tails = T).
   
   a) HTHHHH  
   b) HHTTTT  
   c) THTHHT  
   d) HTHTHT  
   e) All four sequences are equally likely

   Give reasons for your choice.

   (Modified from Kahneman & Tversky, 1972 and Konold et al., 1993)

2. **Lottery and car accident task**
   Which one is more likely?
   
   (a) You win a jackpot in BC lottery 6/49  
   (b) You will be killed in a car accident

   Describe how you would solve this problem.

* In the actual research the tasks were provided on separate sheets of paper.
3. **Five-boy task**

John and Tracy live in the interior of British Columbia. They have five children, all boys. Both John and Tracy want to have a daughter. Since they already have five boys John and Tracy think that their sixth child would more likely to be a girl. Do you agree with John and Tracy? Why, or why not?

4. **Circular spinner task**

In the diagram below OA is a spinner. Four people are playing this game. Each person will pick two numbers from 1–8 and if the spinner stops at any of the numbers picked the person wins. If you are playing this game which two numbers will you pick to have a best chance of winning? Why?

![Circular spinner diagram](image)

5. **Bank teller task**

A local bank has two teller windows open to serve customers. The number of customers arriving at the bank varies between one and six customers per minute. Customers form a line, and the person at the front of the line goes to the first available teller. Each teller services one customer per minute. If you were the manager of the bank and you wish to make sure the average waiting time is not more than 3 minutes, would you increase or decrease the number of tellers?

(Adapted from Gnanadesikan, Scheaffer, & Swift, 1987, p. 27)
Appendix C - Interview protocol

1. **Head-tail sequence task**
   
   Which of the following orderly sequences is most likely to result from flipping a fair coin six times? (Heads = H, Tails = T).
   
   a) HTHHHH
   b) HHHHTT
   c) HTHTHT
   d) HTHTHT
   e) All four sequences are equally likely

   Why?

   (Modified from Kahneman & Tversky, 1972 and Konold et al., 1993)

2. **Lottery task**

   Two students in a probability class have opposing views about selecting numbers for a lottery 6/49. One student prefers to choose consecutive numbers like 1, 2, 3, 4, 5, 6. Other student thinks that chance of getting a sequence of six consecutive numbers in a lottery is smaller than getting a random sequence of any six numbers. Both students come to you for an advice. How would you advise them?

   (Adapted from Fischbein & Gazit, 1984)

3. **Board and marble task**

   A wooden board with different channels as given in the diagram below was shown to each preservice teachers.
When the preservice teachers observed the board and the channels the following question was asked: "If a marble is dropped from the top of the main channel where do you think the marble will come out at the bottom?"

(Adapted from Fischbein, 1975)

4. Dice task

Three dice with each face numbered 1 through 6 were shown to each preservice teacher. He or she was told that all the dice would be rolled. The person who rolled the dice was the dealer and the person who placed bets was the player. The rules of the game were the following:

"The player can choose any number 1 to 6. Suppose the player bets $1 for face 1. If face 1 turns up on any of the dice then he or she will be paid $1 by the dealer. If face 1 turns up on two dice then he or she will be paid double the amount he or she bets, $2 in this case. If face 1 turns up on three dice then he or she will be paid three times the amount he or she bets, $3 in this case."
The player can go on doubling the bet each time he or she loses. For example, if the player bet $1 for the first time and lost he or she can bet $2 for the second time, $4 for the third time, and so on. The player and the dealer each has a total of $100 to play."

When the preservice teacher read the rules the following questions were asked: If you have an opportunity of becoming a dealer or a player which would you choose? Why?

5. Thumbtack task

Each preservice teacher was shown a thumbtack and shown two positions of its landing on a wooden table, with the point down or with the point up as shown in the diagram below.

![Thumbtack Diagram]

When the preservice teachers observed the thumbtack the following question was asked: Which one (landing with the point down or point up) is more likely? Why?

(Adapted from Willoughby, 1977, p. 106)

6. Question about the relevance of probability to solve everyday problems

How often do you use probability in your everyday lives? Do you think that university or school knowledge of probability is useful in solving everyday problems?
The head-tail sequence task

Hari: What's your answer to this problem [shows the head-tail sequence task]?

John: (e).

Hari: Why did you say (e)?

John: First of all throwing a head or a tail is equally likely. And the fact that going from the first trial to the second trial to the third trial or whatever trial to the next trial they are independent. Therefore you calculate equal probability for each of those.

Hari: Back in the written component you said something different [shows his response to birth sequence task], Do you remember this?

John: Yes, I do [laughs]. I changed my mind.

Hari: Do you mean that even with 5 heads and 1 tail in one sequence and 3 heads and 3 tails in the other sequence does not make any difference?

John: Um-. I still have some doubt, I guess, obviously about choosing that they are all equally likely. I still have to say that they are all equally likely, but I'm not quite sure if my answer is right. There are different ways you can look at it. I'm not sure which is the right way. The mathematical way is heads and tails are equally likely and each event is independent and you come to the conclusion that they are all equally likely. Or you could take a count of how many heads and tails you get. For (b), (c), and (d) [sequences HHTTHT, THHTHT, and HHTHTHT] you got 3 heads and 3 tails. So those will have the same probability. And ... if you think that in 6 throws of a coin you should get 3 heads and 3 tails is a fallacy. You could get anything really.

Hari: Do you think that all the sequences would be equally likely, if the total number of trials was 20 and in (a) I had 19 heads and 1 tail and somewhere, in (c) for example, I had 12 heads and 8 tails or 10 heads and 10 tails? [shows the examples of sequences HTHHH...], HHTHTHHHTHTHTHH...}
John: Um. No, I think it changes as the number of trials increases. But the thing is it's a fallacy to assume that you get 10 heads and 10 tails in 20 trials.

Hari: I understand that. But what about comparing those 19 heads and 1 tail and 12 heads and 8 tails? Do you think that they're equally likely?

John: No. Intuitively I'd not think so. I would expect that 12 heads and 8 tails would be more likely. I'm trying to think how you can calculate the probability of these sequences mathematically, but I kind of forgot. [shows the head-tail sequence task] Intuitively speaking, I'd still think that these three sequences (b), (c), and (d) [sequences HHHTT, THTHHT, and HTHTHT] are more likely than (a) [HTHHHH], that (a) is less likely. Um-, but that's intuitive. I'm thinking that any of these sequences are equally likely because the number of trials you've got is small, you only have six trials; and this whole thing that the trials being independent and the probability of heads and tails being equal, I just have to say they're equally likely.

Hari: Even in 20 trials?

John: If you extend this to 20 trials, I might change my answer. If you increase the number of trials, I would go more with my intuition and say those sequences have roughly half heads and half tails. What I would try to do is somehow reflect that in the mathematics. But right now I can't quite figure out how to reflect.

Hari: Can you tell me the probability of getting this sequence [shows sequence a]?

John: 1/2^6

Hari: This sequence [shows the sequence b]?  

John: 1/2^6. They're all 1/2^6.

Hari: If you look at these sequences their orders are different from each other. Do you still think that they're equally likely?

John: Yes.

Hari: Which of the above sequences [shows a), b), c), and d)] is least likely?

John: You're asking me the same question in a different way.
The lottery task

John: [reads the question] One student wants to select consecutive numbers. Another student thinks that the chance of getting a sequence of six consecutive number is smaller than getting a random sequence of six numbers. So one thinks the consecutive is higher and the other thinks the random is higher. I'd say Um-

Hari: No, the first student doesn't say that it is higher. S/he just prefers. But the second student thinks that consecutive sequence has smaller chance than the random sequence.

John: I would probably talk to them about different ways of looking at them. You could look at it mathematically and say that the probability of getting a certain number is equally likely as getting any other number. So in a sense it doesn't matter which number you pick. You know you are still going to have the same chance of winning as picking any other numbers. Then again, you can look at the past history of the numbers drawn from 6/49. You can keep track of what the winning numbers are and then look for trends or patterns of the numbers that were chosen a lot. You can come up with certain numbers that seem to be chosen quite frequently.

Hari: That's interesting. "Luck" magazines publishes all the jackpot winning numbers so far. But there is not a single consecutive number which has won the jackpot so far.

John: I'd definitely look at that. That'll cast some doubt on picking consecutive six numbers.

Hari: Not a single consecutive number has won the jackpot so far. What do you think now?

John: That's actually crossed my mind. You know whenever I see the lottery winnings, I've never seen the consecutive numbers. Um-, I'm getting into ethics here. I would have to agree with the other student who thinks that the chance of getting a sequence of six consecutive numbers in the lottery is smaller than getting a random sequence of any six numbers. Then again, if I'm advising these students I would only present the information and let them come to a decision.

Hari: What's your personal thinking?
John: A random sequence is more likely than choosing six consecutive numbers. That's my opinion.

Hari: What's the reasoning for that?

John: I'm just basing my decision on the fact that intuitively six consecutive numbers seem unlikely to me and by the information you've just provided it has never happened. My reasoning is purely based on intuition. Mathematically speaking, any six numbers is equally likely, right? There're mathematical ways of looking at things and there is your intuition. Sometimes you have got to separate the two or make compromises or whatever I don't know.

The thumbtack task

Hari: [shows a thumbtack] If I throw, or roll this thumbtack on this table there are two possibilities of landing: with the point down or with the point up [shows the position on the table]. Which one (landing with the point down or point up) is more likely?

John: (b)

Hari: Why do you think so?

John: Because, as it's falling most of the weight is here [shows the tack part]; it's in the head. So the heavy part will go on the bottom.

Hari: What about bouncing factors?

John: Oh! that's a good question. Oh! dear, I've never thought of that one. [tries rolling the tack]. It [bouncing] can make anything. So I'll probably just do like many trials, records and just base on that. I'll not try to theorize it. [keeps on trying with the tack] I'll not try to analyze this bouncing behaviors, that's absolutely ridiculous! Look at that spinning! it's too complicated. I'll just keep track of the outcomes over many times.

Hari: What about the heights? Does throwing or dropping from different heights make the difference?

John: I'll just do whole bunch of trials with different heights and stratify them.

Hari: How many trials do you need?

John: Thousands, it's just lots!