PERCEPTIVIST MATHEMATICS

EDUCATION

BY

BRIAN DWIGHT COPELAND
B.Ed, M.A., The University of British Columbia,
1982, 1985.

THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF MASTER OF ARTS.

IN

THE FACULTY OF GRADUATE STUDIES

(DEPARTMENT OF EDUCATION)

We accept this thesis as conforming to the required standards

THE UNIVERSITY OF BRITISH COLUMBIA
August 1989

In presenting this thesis in partial fulfilment of the requirements for an advanced degree at the University of British Columbia, I agree that the Library shall make it freely available for reference and study. I further agree that permission for extensive copying of this thesis for scholarly purposes may be granted by the head of my department or by his or her representatives. It is understood that copying or publication of this thesis for financial gain shall not be allowed without my written permission.

Department of Mathematics Education

The University of British Columbia
Vancouver, Canada

Date October 2, 1989

ABSTRACT

The aim of this paper is to outline and apply a perceptivist philosophy of education and a constructivitist theory of knowledge acquisition to a problem in secondary mathematics education. A working summary of perceptivism and constructivism is provided and a program and lesson materials are discussed within the context of perceptivist-constructivist ideas.

The main thesis of this paper is that the way to translate perceptivist-constructivist ideas into practice in mathematics is to emphasize activities that lead to actual perceptions. The traditional problem with this is that often the computational abilities needed to deal with reality are too much for most students to deal with. The information age innovation that makes a utilitarian mathematics education more possible now, where it was not possible previously, is the development of the personal computer. The computer can act as an information processing "step up transformer" to boost students past computation to real, perceptual mathematics.

The practical part of the paper consists of lessons aimed at a partial realization of perceptivism-constructivism in the classroom. The lessons concern concepts and skills from the traditional secondary mathematics curriculum areas; arithmetic, algebra, elementary function theory and calculus. The paper concludes with a report on field tests of the materials in the secondary classrooms of the author.

TABLE OF CONTENTS

Page

ABSTRACT ii

LIST OF TABLES vi

LIST OF FIGURES vii

CHAPTER I: A PROBLEM IN MATHEMATICS EDUCATION . . . 1
The need for an emphasis on applications . . . 6
Mathematics must become more meaningful . . . 8
The computer - a resource to be utilized . . . 10
Philosophy can help show the way 13

CHAPTER II: A BRIEF DESCRIPTION OF A PART OF PERCEPTIVISM 15
Orientation to the world 15
Orientation to being educated 20
Education rooted in perception 26
Ways of seeing 29
Perceptivism and mathematics education 34

CHAPTER III: A BRIEF DESCRIPTION OF CONSTRUCTIVISM 37
Knowledge is actively constructed 37
All experience is theory laden 40
The role of alternate conceptions 42

CHAPTER IV: AN INTERPRETATION OF PERCEPTIVISM . . . 46
The substantive role of the computer 47
Trajectories - using the computer to do mathematics 50
Extension of the concept - the trajectory envelope 53
Perceptivist-constructivist modelling 60
Computer assisted exploration of the exponential function 66
Functions as models of reality 72

CHAPTER V: AN INTERPRETATION OF CONSTRUCTIVISM . . 93
The active construction of meaning 93
Appealing to existing knowledge 100
Alternate conceptions and conceptual conflict 103
Summary 105

CHAPTER VI: PERCEPTIVIST-CONSTRUCTIVIST CALCULUS ON THE COMPUTER 108
Rationale 108
Numerical differentiation 112
Numerical integration 119

CHAPTER VII: REPORT ON A FIELD TEST OF THE CALCULUS UNIT 132
Purpose of the field test 132
Instructional sequence 134
Field notes 136
Questionnaire results 140

CHAPTER VIII: CONCLUSION AND RECOMMENDATIONS . . . 148
Conclusions 149
Suggestions for further research 153
Recommendations 156

BIBLIOGRAPHY 161

APPENDIX 171

LIST OF TABLES

Page

Table 1. Brauner's outline of the (nine) modes of perception. (Brauner, 1986, p.13) . . . 25

Table 2. Decay of carbon-14 86

Table 3. World population figures for the inhibited growth model 88

Table 4. Disease spread in a city using the inhibited growth model 90

Table 5. Student Questionnaire - Computer Calculus Unit 143

Table 6. Analysis of mean response to each question 144

Table 7. Analysis of responses by individual students. 147

Table 8. Computer Math: Possible program of activities 160

LIST OF FIGURES

Page

Figure 1. Orientation to the World 19

Figure 2. Hofstadter's depiction of Complementarity: seeing in different ways . . . 32

Figure 3. Computer used to amplify power of concepts 49

Figure 4. Trajectories 51

Figure 5. Lob vs. Clothesline Trajectory 53

Figure 6. The envelope of trajectories 54

Figure 7. Hits Inside the Envelope 55

Figure 8. Hitting Specific Locations Inside the Envelope 57

Figure 9. Perceptivist-Constructivist modelling; distance fallen as a function of time elapsed since dropped 61

Figure 10. "Periodic" Variation of Chemical Properties 63

Figure 11. "Periodic" Variation of Cultural Development 64

Figure 12. "Periodic" Variation in Heartbeat . . . 65

Figure 13. "Periodic" Variation in Sunspot Activity 65

Figure 14. Exponential Function and Population Growth (Smaller growth rate corresponds to lower curve) 68

Figure 15. Exponential Function and Inflation (smaller inflation rate corresponds to lower curve) 70

Figure 16. Exponential Function and Compound Interest 71

Figure 17. Bouncing ball, rapid decay 75

Figure 18. Bouncing ball, slow decay 75

Figure 19. Water depth variation in tidal cycles . 77

Figure 20. Average value of a function on the computer 77

Figure 21. Average tidal depth 78

Figure 22. Terminal Velocity in Motion with Resistance 80

Figure 23. Position as Function of Time with Terminal Velocity 81

Figure 24. Damped Oscillatory Motion 83

Figure 25. World population projections 84

Figure 26. C-14 decay 87

Figure 27. World population, inhibited growth model 88

Figure 28. Inhibited growth model of disease spread 90

Figure 29. Percentage buying a product, inhibited growth 90

Figure 30. Computer solution of degree one equations 97

Figure 31. Computer solution of degree two equations 97

Figure 32. Computer solution of degree three equations 98

Figure 33. Computer solution of degree four equations 98

Figure 34. Computer solution of degree five equations 99

Figure 35. Computer solution of degree six equations 99

Figure 36. Graphing a face 103

Figure 37. Slope by secant approximation 113

Figure 38. Time vs. patient temperature 115

Figure 39. Time vs. city's population 115

Figure 40. Dollars spent on advertising vs. T.V. sets sold 116

Figure 41. Time vs. fraction of doctors accepting a medicine 117

Figure 42. Width of rectangle vs. area of rectangle formed 118

Figure 43. Time of year vs. sales 119

Figure 44. $y = x^2$ 120

Figure 45. Month vs. sales in that month 122

Figure 46. Time vs. velocity 123

Figure 47. Time vs. Rate of Energy Use (4 Hrs) . . 124

Figure 48. Time vs. Rate of Energy Use (24 hrs) . . 124

Figure 49. Area between two curves 126

Figure 50. Average value of a function 127

Figure 51. Volume of revolution 128

CHAPTER I

A PROBLEM IN MATHEMATICS EDUCATION

Mathematics education in North America is in trouble. Compared to other leading industrial societies mathematics achievement in North America is lagging behind (McKnight, 1987). Since Dewey there has been no guiding theory in education as a whole or mathematics education in particular. Without a guiding theory we are not likely to do better. One of the main purposes of this thesis is to provide arguments to establish that a guiding philosophy of education is now available -- Perceptivism-constructivism. A second major purpose is to establish by way of argument and demonstration that the philosophy provides suggestions about ways in which mathematics education might be improved. In particular it can be used to generate suggestions and provide a rationale for ways in which the computer can be used to improve mathematics education. This thesis presents various computer integrated mathematics lessons and a computer integrated calculus unit. A field test of the calculus unit is reported and discussed.

This chapter is concerned with a group of problems centered around declining achievement in mathematics and lack of effort in using the computer as a resource to remedy the problem. The problem is twofold. First, mathematics performance kindergarten to grade 12, beyond mere computation, is especially poor. Second, without a guiding philosophy it is unlikely that the computer can be exploited to improve achievement and increase meaningful mathematics. The argument in this chapter is developed in five stages. (a) The details of poor performance in mathematics by secondary students is discussed. (b) A stress on applications is one suggestion in the literature to improve performance. (c) A second suggestion in the literature is to emphasize the meaningfulness of the technical details. (d) The computer is able to assist the development of applications and meaningfulness by relieving the student of the burden of computation. (e) It is then argued that philosophy can suggest ways in which the freedom the computer gives from the burden of computation can be exploited to help improve mathematics performance.

Second rate skill levels

High school graduates are not equipped to deal with the mathematical complexities of modern technological society. Henry Pollak (1987), a noted industrial mathematician, summarized what skills industry expects entry level employees to have. These include the ability to set up problem mathematically, the ability to apply a fluid repertoire of mathematical techniques to solve problems, the ability to abstract mathematical formulations from real world situations, the ability to work cooperatively on problem solving, the ability to see mathematical concepts and principles in common and complex situations. Students also needed an attitude disposed to toleration of open ended, not well formulated problems and a belief in the utility of mathematical techniques to render problems into a well formulated form. The present regimen of low level calculation does not provide such skills and abilities or in any way adequately prepare students for the work place or for further education. Commenting on this problem, Usiskin (1985) remarks:

> The biggest problem in secondary school mathematics today is recognized by all, regardless of feelings toward new math or back to basics. It is that a large number - perhaps a majority - of high school graduates lack the mathematical know-how to cope effectively in society, qualify for the jobs they would like, or qualify for the training programs (including those in college) leading to the jobs they would like.
>
> (Usiskin, 1985, p.8)

College and university officials note the poor level of college entering mathematical competence. Leitzel and Osborne summarize in these terms.

> About 45% of the graduating seniors in the United States enter some form of post-secondary education, and about 45% of this group lack the mathematical skills and understandings needed for success in post-secondary school mathematics.
> (Leitzel and Osborne, 1985, p.150)

A survey of undergraduate programs in mathematics and computer science documented that 15% of current enrolment in mathematics courses at public two year colleges is for remedial mathematics. Another 37% of the enrollment is for precalculus courses that should presumably have been completed in high school (Albers, Anderson and Loftsgaarden, 1987).

On a national level (U.S.A.), the National Assessment of Educational Progress in mathematics clearly showed that low level computational skills were at historically reasonable levels but that a majority of students do not understand basic concepts nor could they apply skills in problem-solving situations (Carpenter, Brown, Kouba, Linquist, Silver and Swafford, 1987). Internationally, compared with other industrial nations, particularly those in the Orient, average North American students are not competitive (McKnight, Crosswhite, Dossey, Kifer, Swafford, Travers and Cooney, 1987). McKnight notes how our "underachieving curriculum" expects less of our students, has them spend less time studying mathematics and has fewer students enrolled in advanced mathematics than students in other countries. The International Assessment of Mathematics and Science found a similar lack of mathematical ability compared to leading jurisdictions.

> Forty percent of Korea's 13-year-old students understand measurement and geometry concepts and are successful at solving even more complex problems. Less than 10 percent of those from Ontario (French) and the United States have the same level skills
> (Lapointe, Mead and Phillips, 1989, p.10).

There is reason to believe that of all the students we "pass" in school, many do not get a good mathematics education. The New Jersey College Basic Skills Placement Test is given to virtually all students entering New Jersey public colleges and some private institutions. After analyzing the 1984 results the Council reported:

> Of those students who completed a traditional college preparatory program in mathematics (Algebra 1, Geometry and Algebra 2), less

> than 4 percent were proficient in elementary algebra. Even more startling only 36% of these students were proficient in computation. Among students who completed only one year of algebra (N=2,030), only one student was proficient.
> (Morante, Faskow and Menditto 1984, p.30)

Voelker (1982), in a similar study of secondary school science, stated that "90 percent of all high school graduates fail" to meet criteria for scientific literacy. Passing courses does not seem to equate to understanding what went on.

North American high school seniors have this year (1989) fallen to 12th and last place in mathematical achievement among industrialized nations according to the U.S. Department of Education (1989). Other studies indicate similar deficiencies among biology and chemistry students. North American industrial performance has already fallen behind Japan's, a fact that may not be entirely coincidental with our problems in education.

The need for an emphasis on applications

The need to move mathematics education toward a more relevant, applied format has been recognized for a long time. Carson, in 1913, wrote:

> Among the many changes in mathematical education during the last twenty years ...

> one element at least appears throughout; a desire to relate the subject to reality, to exhibit it as a living body of thought which can and does influence human life at a multitude of points.
>
> (Carson, 1913, p.35)

The 1989 National Council of Teachers of Mathematics (NCTM) Standards Document emphasized that mathematics education need not get bogged down in an endless attempt to refine computation.

> It is a common assumption that mathematics computations are necessary before one can study algebra or geometry or investigate applied problems. This assumption is not warranted. Too many students are refused an opportunity to learn the mathematics that would make it possible for them to be productive members of society because they are not proficient at skills which are now done best on a calculator or computer.
>
> (NCTM, 1989, p.30)

Similarly, Ralston argues that there is no reason to spend time refining computational skills if the skills are not required. Rather, time and effort should be shifted to making mathematics an intellectually integrating and expanding activity.

> The focus of all mathematics teaching must soon become teaching students to understand mathematics rather than teaching them to manipulate symbols; that's what the computers are for.
>
> (Ralston, 1985, p.39)

Fey and Good comment in a similar vein that we must aim higher up on the cognitive scale. Rather than focussing predominantly on low level computational skills, leaving the intellectual meaningfulness of the technicalities to chance, mathematics education must evolve toward becoming a means of understanding and communicating about the real world.

> It should be possible for every teacher to place greater stress on situations in which mathematics is used to model the structure of real-life situations. There is a growing supply of resource material for this purpose ... and its use will force attention to important themes in the curriculum of the future.
>
> (Fey and Good, 1985, p.52)

Mathematics must become more meaningful

Preoccupation with calculation, at the expense of the broader intellectual context of mathematics, forces students to cover the material whether they understand it or not. It does not seem unreasonable to think that students would quickly pick up on how to play the game. A cynical, mechanical approach to learning would seem natural. Nigel Ford claims:

> There is increasing support for the idea that the way students think, believe and value is by no means synonymous with exposure to, and performance of, exercises in the comprehension, recall and manipulation of

> information and ideas. Most course-based activities certainly do not preclude the possibility that in terms of a student's personal acceptance and valuing of, and commitment to information and ideas, he is doing no more than "going through the motions" with shallow and short-term, if any, effects.
>
> (Ford, 1979, p.215)

Roy Forbes notes that the National Assessment of Educational Progress (NAEP) 1979 results indicate that "back to basics" mathematics emphasis in the years preceeding 1979 maintained low level skills at the expense of understanding.

> During a period when the public has placed great emphasis on the "basics," assessment data show that mathematics achievement has declined, especially in problem-solving and understanding of concepts.
>
> (NAEP, 1979, p.7)

Present emphasis on reform includes bringing mathematics education back from a curriculum driven by emphasis on extensive computation toward more meaningful mathematics for students. The NCTM Standards Document lists at least four important goals relating to bringing mathematics back to the student.

- Mathematics should be studied as an integrated whole so that students understand it as a dynamic discipline and an integral part of our culture.

> • Mathematics should help build students' abilities to reason logically.
>
> • Mathematics should be taught in a natural context.
>
> • Students should be encouraged to create, invent and participate.
>
> (NCTM, 1989, p.35)

Only by meaningfully involving the student, is there a possibility of discouraging just "going through the motions" and of encouraging actual learning. Only in this way can a "nation at risk" revitalize its "underachieving curriculum", especially in mathematics.

The computer - a resource to be utilized

The computer seems to have an enormous potential in mathematics education. Walker (1984) makes the claim that "the potential of computers for improving education is greater than that of any prior invention, including books and writing." But the computer is just a piece of technology and its potential may not be realized. Educational television held tremendous potential but has had little effect. The technology will not develop its own use as de Cecco remarks:

> All these facilities and equipment [computer, C.A.I.], I must remind you, are much more sophisticated than any theory of teaching we presently have. The temptation in a technological society is to allow our fantastic machines to determine our research problems and our educational practice. It is far more important that we subordinate these

> machines to the theoretical and practical instructional problems which, undoubtedly, the machines can help us solve.
> (de Cecco, 1968, p.418)

By relieving the learner of a considerable amount of computational burden the computer expands the potential for understanding. The rest of society seems to have discovered this fact, with the apparent exception of the education community. The NCTM Standards Document says:

> Most current mathematics programs fail to reflect the impact of the technological revolution affecting our society. The availability of low-cost calculators, computers and related new technology have already dramatically changed the nature of business, industry, government, sciences and social sciences. Unfortunately, most students are not educated to participate in this new society.
> (NCTM, 1989, p.25)

Released from the burden of computation, the student would be able to pursue the meaning of mathematics, its personal relevance, applicability and general intellectual usefulness, as the medium of quantitative literacy. Usiskin claims that history provides many examples of how technology and techniques liberate.

> The history of mathematics is filled with the development of techniques that take difficult problems and make them automatic. Usually these automatic procedures provide a benefit; with tedious calculations out of the way,

> understanding the ideas and purposes of the mathematics itself becomes easier.
> (Usiskin, 1985, p.16)

The freedom from calculation given by the computer could be used to explore applications of mathematics. Computational skills are not needed to the degree they were in the pre-information age culture, before computers. Technology will force a reworking of the curriculum in which the importance of low level skills will have to be reassessed. Use of computers may turn the curriculum on its head.

> For nearly every function of interest, the computer utilities make all ... questions accessible in some intellectually honest and mathematically powerful form to students who have not followed the conventional regimen of skill development. They open a fast track to the polynomial, trigonometric, exponential and algebraic functions that model interesting phenomena in the physical, biological, economic and social worlds. Computing offers an opportunity to turn the secondary school mathematics curriculum on its head. Instead of meeting applications as a reward for years of preparation, students can now begin with the most natural and motivating aspect of mathematics -- its applications.
> (Fey and Good, 1985, p.48)

What is needed, to indicate what to do with the freedom given by the computer, is a philosophy of education that shows where to look for the meaning of mathematics. In education, this is at hand in the form of Perceptivism-Constructivism.

Philosophy can help show the way

Because education has been without a philosophy for so long we have been buffeted from one fad to the next without any rudder to steady the course. Brauner describes the situation in these terms:

> Ever since the public school stopped trying to implement the program derived from the Pragmatism of John Dewey ... they have been without a working philosophy of education. That thirty-year intellectual drought has led to such a shortage of working theories in the area of teaching practises that policy considerations provide what little basis there is for justifying the current curriculum and its methods. Indeed, the grounds for public school practise have been so parched for so long that a whole generation of teachers has grown to retirement age without ever knowing just how it is that a working philosophy of education can inform classroom practises ...
>
> (Brauner, 1986, p.1)

This not to say that a philosophy of education is a blueprint. It is not as if there is some formal way to logically deduce specific educational practises from metaphysical, epistemological or axiological premises. What is being considered here is a looser notion, that of conditions for rational action. A reflective practitioner needs good reasons for what he does. This looser notion means that other theoretical views may also be consistent with certain practises and that many existing practises may be sound. But what is needed is a consistent philosophy to provide a systematic theoretical foundation on which to build innovation in practise. Meaningful, applied mathematics, facilitated by the computer, for example, would be a part of most new programs. But the grounds for and structure of the program might be very different depending on the reasons behind the program.

In chapter II, the writer will describe two parts of Perceptivism that bear on the problem of how to use the computer to improve the meaningfulness and applicability of secondary mathematics. The two parts of the theory concern an orientation to the world and an orientation to what it means to be educated. This theoretical foundation will then be used to generate suggestions as to what to do with the freedom from calculation that the mathematical use of the computer gives.

CHAPTER II

A BRIEF DESCRIPTION OF A PART OF PERCEPTIVISM

As a theory of education, Perceptivism has three major parts; an orientation to the world, an orientation to being educated and a moral and social theory. Chapter II describes the first two parts insofar as they might bear on the mathematical uses of the computer.

Orientation to the world

Perceptivism is being developed at the University of British Columbia by Dr. C.J. Brauner. Brauner's philosophy rests on a certain orientation to the world. According to Brauner (1985) anything that exists gives off impulses. These impulses contain information or data. Constellations or packages of these impulses, that impinge on an organism, can be received as signals. Any and all signals received generate impressions. Some impressions are left unnoticed, others are attended to. Private subjective packaging of impressions can give rise to notions, individualistic ways of "seeing things". Such formulae for packaging impressions, if they ever achieve widespread public usage can become concepts. A concept for Brauner,

> is a public formula for focusing attention on a limited number of characteristics of the entity under consideration.
>
> (Brauner, 1986, p.17)

By using publicly available packaging formulae language users can communicate and "see things" in similar ways. In fact the concepts shape the perception.

> The concepts of ordinary language are the instruments by which aspects of experience both real and imagined are singled out and imbued with significance. In a mature language setting concepts generate perception.
>
> (Brauner, 1986, p.16)

In Brauner's view, human perception is a process of a person selectively attending to the impulses received. Concepts, being publicly available and usually widely tested, are usually beneficial ways of packaging impressions. Without the stability of concepts the world would be chaotic. This author has obtained Brauner's agreement that the latter's views in this regard are similar to Cassirer's.

> The construction of our perceptive world begins with such acts of dividing up the ever-flowing series of sensuous phenomena. In the midst of this steady flux of phenomena there are retained certain determinate (perceptive) units which, from now on, serve as fixed centers of orientation. The particular phenomenon could not have any

> characteristic meaning except if thus referred to those centers. All further progress of objective knowledge, all clarification and determination of our perceptive world depends upon this ever progressing development.
> (Cassirer, 1923, p.165)

The fixed centers of orientation are concepts and conceptual systems. These fixed centers make perception possible. Brauner claims that

> Concepts, notions and dispositions are the prime instruments for sensitizing people to particular constellations of impressions in their physical, social, mental or moral surroundings.
> (Brauner, 1986, p.11)

It is only through the development and use of the best conceptual repertoire available that an individual can be meaningfully called educated (as opposed to schooled). The theoretical structures used by the individual determine what he sees and how he understands. Brauner is in agreement with Cassirer.

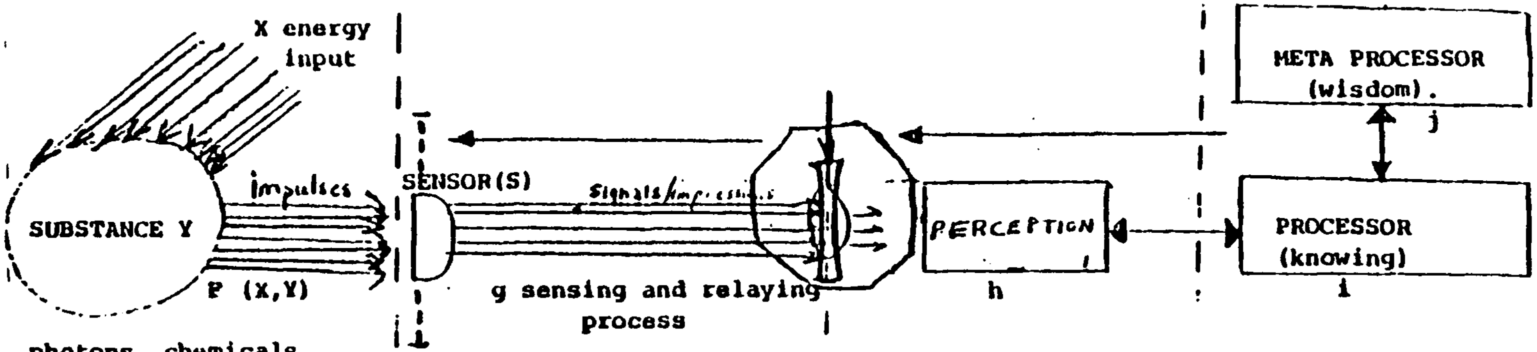
> There is no factuality ... as an absolute ... immutable datum; but what we call a fact is always theoretically oriented in some way, seen in regard to some ... context and implicitly determined thereby. Theoretical elements do not somehow become added to a "merely factual," but they enter into the definition of the factual itself.
> (Cassirer, 1923, p.475)

The world becomes "knowable" only through the mediation of sensory and conceptual equipment. The uneducated are literally blind to certain features of the world because they lack certain basic modes of perception. Brauner seems to be in substantial agreement with J.M. Jauch.

> When we try to understand nature, we should look at the phenomena as if they were <u>messages</u> to be understood. Except that each message appears to be random until we establish a code to read it. This code takes the form of an abstraction, that is, we choose to ignore certain things as irrelevant and we thus partially select the content of the message by a free choice. These irrelevant signals form the "background noise" which will limit the accuracy of our message.
>
> But since the code is not absolute there may be several messages in the same raw material of the data, so changing the code will result in a message of equally deep significance in something that was merely noise before and conversely: in a new code a former message may be devoid of meaning.
>
> Thus a code presupposes a free choice among different, complementary aspects, each of which has equal claim to reality, if I may use this dubious word.
>
> Some of these aspects may be completely unknown to us now but they may reveal themselves to an observer with a different system of abstractions.
>
> (Jauch, 1973, p.237)

In Figure 1, the writer offers a visual summary of Perceptivism's orientation to the world. The writer's view of mathematics, a version of mathematical realism,

Figure 1. Orientation to the World



photons, chemicals
pressures, heat, sound
etc. interface with
organism

-There is a tendency to perceive
(a)What has been perceived in the past
(b)What fits into a subject-field pattern
(c)What is in agreement with others
(d)What is important to the subject
-Errors seem to be unavoidable in the process
-Only parts of reality are relayed by "g"

f: perception of reality depends on the interaction subject, object , energy and

g: selected physical and logical elements of the input are coded and relayed on.

h: perception

i: understanding takes place, input compared to stored patterns and models-response signal possible

j: analysis, evaluation, speculation, synthesis-reflection on accumulated thought and experience implicit in the figure is elaborated in the Appendix. In Figure 1, substance Y gives off impulses which are received as signals. Signals can be packaged as impressions. The central octagon consists of private dispositions (impressions, notions) and of public formulae (concepts, theories). Concepts can build together to form a conceptual system.

Orientation to being educated

A conceptual system generates a mode of perception by setting out conditions of choice and grouping and so we partially select the content of the message. Irrelevant signals form just "background noise". Impressions of events are formed and perceptions generated in the interaction of the system of interpretation and the signals. Individuals develop dispositions to perceive in different ways and so come to have private notions of the way the world is. If these notions are recognizable as part of the publicly verifiable conceptual repertoire the individual has certain concepts. Perception presupposes a relatively free, but culturally delimited, choice among different complementary modes of perception. Perceivers have the capacity to see things in a number of fundamentally different ways.

In order to be considered educated, a person must be equipped with the conceptual world-builders of the culture. Brauner claims:

> Perceptivism is built around the idea that the educated person has a greater command of more worthwhile ways of perceiving things than someone who has not had the opportunity to get an education. Hence the purpose of education is to equip the learner with the capacity to perceive things in the different ways that the most worthwhile approaches to understanding provide.
>
> (Brauner, 1988, p.2)

Brauner has identified nine basic modes of perception. (a) The <u>first</u> basic mode of perception is standard perception. By using the most widely accepted and justified concepts, principles and theories of ordinary language standard perception is generated, for example, recognizing a table, seeing politeness in others. (b) The <u>second</u> mode is mythic perception. By drawing on religion, folklore and mythology, the images and elements depicted generate mythic perception, for example, Wayne Gretzky as hero, Mick Jagger as adolescent sex symbol, Rambo as law and order man. (c) The <u>third</u> mode is theoretic perception. The areas of the physical sciences such as physics, chemistry and biology generate theories of such power that those who understand them achieve theoretic perception. For example, water boiling is the escape of high kinetic energy particles, sunset is the Earth rotating around to obscure your view of the Sun. (d) The fourth mode is thematic perception. Fields such as philosophy, history and classical studies, often grouped as the humanities, develop their accounts through major themes. The themes are unverifiable, unfalsifiable but yet not totally arbitrary points of view that then serve as a basis for thematic perception, for example, the Marxist theory of surplus value which makes it possible to see business as exploitation of the worker, Veblen's theme of conspicuous consumption which shows up Hawaiian suntans in B.C.'s winter as ostentation. (e) The fifth mode is thesistic perception. Disciplines such as psychology, sociology, geography, law, political science, economics, education, sometimes referred to collectively as the social sciences, shape perception through the use of quasi-quantitative, partially verifiable theses. For example, Freud's theory of the unconscious makes dreams viewable as the expression of repressed thoughts. Keynes theory of economic equilibrium uses the analogy of an engine to be tuned, sometimes giving more fuel to increase the R.P.M. and sometimes less to decrease the R.P.M., in order to explain the business cycle. (f) The sixth mode is relational perception. Disciplines such as logic and mathematics provide the conceptual machinery needed to allow formal understanding and the perception of abstract form, for example, seeing change as functional variation, recognizing a reductio ad absurdum in conversation. (g) The seventh mode is primary perception. The discursive arts such as literature, poetry and theatre, use words to create the most far reaching portraits of human experience, for example, Paul Scott's conception of imperialism in retreat in The Raj Quartet, Melville's depiction of New England Puritanism in its relation to life in Moby Dick. (h) The eighth mode is primal perception. The nondiscursive arts such as dance, music, painting and sculpture provide the forms which allow the renewed, primal perception of appearance, for example, a Van Gogh starry sky, a Picasso face. (i) The ninth mode is operational perception. Through the use of technical concepts it becomes possible to see how mechanisms work, how to make new ones, how to fix old ones, for example, how to fix a carburetor and how to repair a flat tire.

Table 1 is Brauner's summary of the nine modes of perception. For each mode of perception the sources and types of concepts used are given. These concepts generate a certain type of perception. Brauner includes an example of each type of perception. Mathematics is part of relational perception. The relational propositions to which Brauner refers are the axioms, theorems and proofs of the various branches of mathematics (arithmetic, algebra, geometry, analysis, probability and statistics). An example of relational perception might be to see a mathematical function in a set of collected, raw data or to see an algebraic equation in a problem expressed in ordinary language. Acquiring the mathematical mode of perception opens up for the learner a new way of seeing the world and promotes a genuine understanding of a traditionally obscure discipline. Morris Kline's remarks here are most relevant.

> The meaning and purpose of almost all of mathematics do not lie in the series of logically related collections of symbols but in what these collections have to tell us about our world.
>
> (Kline, 1953, p.15)

Table 1
Brauner's outline of the (nine) modes of perception. (Brauner, 1986, p.13)

	Conventional Abstraction	Extraordinary Abstraction	Secondary Abstraction				Formal Abstraction		Primary Abstraction
SOURCES	Ordinary Language	Religion, Mythology, Folk Lore	The Technologies	The Sciences	Social Sciences	Humanities	Logic, Music Mathematics, Grammar	Literary Arts	The Arts (non-discursive)
	Employs common-sense concepts	Develop myths	Develop technical concepts	Develop genuine theories and paradigms	Develop a mix of thematic and theoretic constructs	Develop themes	Develop relational propositions	Develop megacepts in representing life	Develop unique representations
	Based on established concepts	Based on heroic figures or concepts such as Jesus or resurrection	Based on mechanisms	Based on technical concepts	Based on quasi-technical concepts	Based on specialized concepts abstracted from common sense, in the main	Based on relational concepts	Based on megacepts	Based on notions
KINDS	Standard Perception	Mythic Perception	Operational Perception	Theoretical Perception	Thesistic Perception	Thematic Perception	Relational Perception	Primary Perception	Primal Perception
EXAMPLES	To see a table or to recognize love in someone's conduct	To see Santa Claus as the bringer of Christmas gifts	To see how a car works or how to fix a tire	To see the horizon come up to cover the sun at "sunset"	To see a Hawaiian sun tan as conspicuous consumption	To identify the lion imagery in the Iliad as a symbol of bestiality in war	To recognize a contradiction or an irrational statement	To see a Carman, a Maurice Conch's or a "MASH" war.	To see "a Picasso face" or a van Gogh starry sky.

Education rooted in perception

The central idea of concepts generating perception creates an image of the learner not as a blank sheet to be trained in stimulus response patterns but as a rational, emerging perceiver, actively engaged in managing various conceptual systems. The higher level cognitive tasks undertaken in education take place in a conceptual ecosystem with various possible perceptions competing for acceptance. A conceptual survival of the fittest determines which perceptions survive and bear fruit and which wither on the vine. Learning is not just a matter of making the correct response but of balancing and selecting from possible conceptual alternatives. Perceptivism aims at developing rationality, a preparedness to respond to novelty with open, but not empty, minds. The world is always novel. The student must be prepared to use existing knowledge to respond in informed ways to new problems of life. The goal of education then is not simple mastery of facts or concepts but facility in applying knowledge to understand the world. Brauner states:

> Instead of teaching it [curriculum] for concept mastery alone and thereby limiting it to the top twenty percent in academic ability, an important change has been

> introduced. Perceptivism initiates the search for content on the basis of the perception that is desired and then insists that the concepts needed to produce it are taught in such a way that the learner actually achieves the sought for perception.
> (Brauner, 1987, p.27)

Brauner claims that the majority of secondary school students find concept mastery too abstract. In mathematics especially then, the meaningfulness and applicability of the learning will be minimal because the best the students can do is to perform the rituals by rote, without really understanding. By basing curriculum and instruction on perception the problem can be attenuated.

> The way to make teaching less abstract, while preserving the integrity of the concepts involved, is to make sure that the student has the actual perception that the concepts fosters. Indeed having the relevant perception is so important if the range and effectiveness of academic teaching are to be extended to the entire student body, Perceptivism would reconstruct the entire curriculum around preferred perceptions.
> (Brauner, 1987, pp.12-13)

Perceptivism tries to balance the need to respect the individual interests and understandings of the student with the need to expand understanding by developing the publicly available modes of perception. Relying on the spontaneous interests and abilities of the student is blind, it leaves the student uneducated, unempowered. Focusing on concept mastery without respecting the interests and abilities of the learner is empty because the concepts wither on the vine, when unused and unrelated to the experience of the learner. What has previously made the balance so difficult to achieve in mathematics education was the assumed lengthy involvement in technical calculation, with many students never emerging from the computational jungle. The computer makes the balance advocated by Perceptivism more attainable.

One of the motivations for writing this thesis is to provide an antidote to the current educational trend toward narrow, pedantic, academic specialization. In education this results in a fragmented approach to knowledge. Perceptivism is helpful in addressing the following issues:

(i) information organization, by discussing modes of perception that interact to interpret experience, and so to shape action (Table 1).

(ii) knowledge synthesis, by discussing relationships of bodies of knowledge, that knowledge be not just a grab bag of isolated facts but a relevant system of information that informs life.

(iii) application of knowledge, by stressing the organization of knowledge for use i.e. for perception. This brings knowledge out of passive retention and into use (Figure 1).

(iv) using knowledge to bring about an informed, rational view of self and world. This rationality is based not on intensive, exclusive absorption with one aspect of human knowledge but an extensive, inclusive repertoire of perceptual possibilities and their interaction.

Ways of seeing

According to Perceptivism, perception is mediated by (nine) basic conceptual schemes. The schemes make possible different ways of seeing the world. The schemes form a conceptual ecosystem with all the analogous interactions of a biological ecosystem. Higher learning and the construction of meaning take place in a dynamic perceptual environment, characterized by conceptual conflict, competition and completion.

What does it mean to perceive the world in different ways? Is this possible? Perceptivism claims it is. The following examples are offered to render the suggestion plausible.

1. What is a sunset? Common sense has it that the Sun drops in the sky and disappears below the horizon. But science tells us that a sunset is the Earth, with the observer attached, rotating around to obscure the observer's view of the Sun. This is the same event perceived in different ways, the latter being a nonstandard perception.
2. The moon, is it a perfect spherical smooth heavenly globe or a planetary rock-like body? The use of the telescope shows the moon to be mountainous and cratered and not a perfect heavenly globe as it might be more romantically perceived using the naked eye. The controversy surrounding Galileo's initial observations using the telescope attest to the social turmoil new perceptions can initiate.
3. Medieval art depicted man without physiological detail. Da Vinci and Michelangelo painted with attention to such detail. Whether the human body is looked at only on the surface or also at the level of "the bone beneath the skin" generates the differences in the human body as perceived in Giotto's Lamentation over Christ as opposed to Michelangelo's Creation of Man.

4. What is a table? It appears initially to be a stable, impenetrable object. But zoom in on the surface of the table with an electron microscope and it is mostly empty space with some atoms spread out at regular intervals. The electron microscope makes possible the achievement of a nonstandard perception of the table.
5. Rather than looking at the objects in the sky at night, look at the light in the sky. See it reflecting, blurring, diffusing. See the darkness next to the light. The van Gogh "starry sky" is perceivable. This is a different perception of a commonly observed phenomena. Each of the examples one to five show how to pay attention to and package signals and impressions in different ways using concepts. This theoretical principle is a cornerstone of Perceptivism's view of being educated.

Let the writer summarize with an analogy. This analogy is with reference to the Hofstadter (1979) diagram (Figure 2). We cannot perceive reality directly, unmediated by selection and processing. We can only perceive projections of reality, each of which contain only partial information. It is as if we can

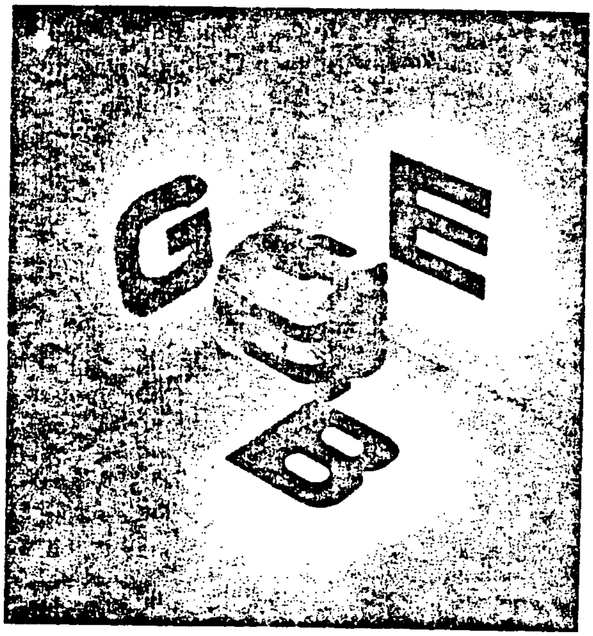


Figure 2. Hofstadter's depiction of Complementarity: seeing in different ways (Hofstadter, 1979, p.1)

only see the walls on Hofstadter's photograph, while reality remains the directly unperceivable generator of the projections. Only by paying attention to all the information contained in the various projections, is it possible to get an adequate conception of reality. Herbert Feigl's comment should be taken seriously.

> There are not two different sorts of reality, but there are two ways of providing a conceptual framework for its description. In fact, at least so it seems to me, there are a great many "perspectives" or frames - the extremes being the purely egocentric as the "lower limit" and the completely physical account as the "upper limit". In between are the many halfway (or part way) houses of the possible manifest images.
> (Feigl, 1967, p. 145)

There is nothing magical about seven, eight or nine modes of perception. Progress depends, in part, on the development of more conceptual schemes. Indeed when the writer began working with Brauner in September 1987, the latter wrote about seven modes of perception. He has added explicit mention of two additional modes, the primal and the operational. The primal concerns nondiscursive art such as painting, sculpture and music. The operational was first brought up in discussion in January 1988. It reflects a mode of perception, important always, but especially since the

Industrial Revolution, reflecting a knowledge of the mechanisms by which technology works. It depends on an operational abstraction, from experience, of the working principles of a piece of technology, social or physical (e.g. how a clock works, how to fix a bicycle, what to look for when your car doesn't work). But the important message in all of this is not the number of modes but the principle of the plasticity of human understanding behind the specifics.

Perceptivism and mathematics education

In a compact form, Chapter II was written to summarize Perceptivism in as an applicable form as possible. Students can be equipped with the modes of perception which will allow them to see the world the way an educated person does. The trouble, traditionally, with the realization of this program in mathematics has been computational complexity. The personal computer now lessens this problem and so opens up the possibility of more relevant, meaningful mathematics for more students.

It is only through the development and use of the best conceptual repertoire possible that an individual can be meaningfully called educated (as opposed to schooled). The theoretical structures used by the individual determine what is seen and how it is seen. A mathematics education that stops at mere fact or concept mastery and does not reach the level of perception is just schooling. A mathematics education preoccupied with its own liturgy that fails to locate itself within the conceptual ecology of its time is doomed to isolation and irrelevance. A mathematics education that wants to strive towards the goals of a perceptivist education needs the computer to overcome the computational difficulties inherent in the required applied, conceptual, interdisciplinary approach to mathematics Perceptivism implies. It is only within this type of information age framework that we can respond to Steen's challenge.

> The paradox of our times is that as mathematics becomes increasingly powerful, only the powerful seem to benefit from it. The ability to think mathematically--broadly interpreted--is absolutely crucial to advancement in virtually every career .... Confidence in dealing with data, skepticism in analyzing arguments, persistence in penetrating complex problems, and literacy, in communicating about technical matters -- these are the enabling arts offered by the new mathematical sciences. Whatever else the MAA* may do in the remaining years of this century, it must work to ensure that these

---

*The Mathematical Association of America

> new liberating, mathematical arts are made available to all students.
> (Steen, 1987, p. 6)

Mathematics must do better because a mathematical education is no longer a luxury. In an increasingly technological, competitive world, quantitative, mathematical literacy is fast becoming essential. And the luxury of catering to the abilities of ten to twenty percent of the population is gone. Modern social conditions demand the retention of most of the population in school mathematics for twelve years. The technological progress in society at large, in the form especially of calculators and computers is providing both problems and hints of solutions. But without an overall working philosophy we will just drift like a ship without a rudder somehow thinking the computer will, of itself, solve our problems in mathematics education.

CHAPTER III

A BRIEF DESCRIPTION OF CONSTRUCTIVISM

Constructivism is a theory of knowledge acquisition. It would appear to be compatible with the philosophy of education developed by Brauner (1988) called Perceptivism. Both of these perspectives provide a coherent framework for this thesis. This chapter is concerned with three central claims of most versions of Constructivism. The first claim is that all knowledge is actively constructed rather than passively received. The second claim is that all experience is theory laden so that past experience and notions, in Brauner's (1988) sense, are important to meaningful learning. The third claim is that alternate conceptions of things are endemic in the construction of meaning. An integral part of higher learning then is sorting out the tangle of conceptions that compete for attention.

Knowledge is actively constructed

It may be tempting to think of the learner as something of a blank sheet ready to be written on. If a clear presentation of new ideas is given, with practise and reward to follow, learning seems probable. Somehow the rational activity of the learner is left out. Simplified behaviorism seems to imply that a stimulus-response model works and that given the right empirical conditions learning is inevitable. Kenneth Strike argues:

> Behaviourists have interpreted traditional empiricism in such a way that epistemology is seen as unrelated to learning. Learning in turn is not seen as a rational activity. It is something that happens to people under proper empirical conditions.
> (Strike, 1982, p.51)

But the working out of understanding by the learner is critical in Constructivism. The constructivist principle is implicit in Dewey for example.

> No thought, no idea, can possibly be conveyed as an idea from one person to another. When it is told, it is to be one to whom it is told, another fact, not an idea. ... Only by wrestling with the conditions of the problem at first hand, seeking and finding his own way out, does he think.
> (Dewey, 1974, p.98)

Piaget was also a constructivist. He thought that most contemporary teaching of mathematics and science was inappropriate insofar as it rested on the simple transmission of knowledge rather than the development of intellectual independence.

> To understand is to discover ... the goal of intellectual education is not to know how to

> repeat or retain ready-made truths. It is in learning to master the truth by oneself at the risk of losing a lot of time and of going through all the roundabout ways that are inherent in real activity.
>
> (Piaget, 1973, p.218)

This type of a simplified transmission account of learning seems to be implicit in much of what goes on in mathematics classrooms. The NCTM Standards Document for example, claims that

> In most classrooms, the conception of learning is that students are passive absorbers of information, storing it in easily retrievable fragments as a result of repeated practise and reinforcement. Research findings from psychology indicate that learning does not occur by passive absorption (Resnick, 1986). Instead, individuals approach each new task with prior knowledge, assimilate new information and construct their own new meanings.
>
> (NCTM, 1989, p.28)

Similarly, Howson argues that many secondary classrooms operate on the basis of simple transmission.

> One is likely to find a stereotyped form of teaching in the bulk of secondary schools classrooms, a form which relies heavily on the textbook and the traditional pattern of exposition-examples-exercises. Apparatus is rarely used, and class teaching is still the norm.
>
> (Howson, 1985, p.75)

As a result much of what is learned is simply forgotten because it is not believed, but simply memorized and regurgitated. This cannot be regarded as a fault of the students - we must accept this as a basic flaw in course design and delivery.

What constructivists advocate is a central concern with the learner as constructor of meaning. Basic abilities to handle, process and apply information to construct meaning are more important than isolated specific performances. Lochhead writes:

> What I see as critical to the new cognitive science is the recognition that knowledge is not an entity which can be simply transferred from those who have to those who don't ... Knowledge is something which each individual learner must construct for and by himself. This view of knowledge as an individual construction ... is usually referred to as constructivism.
>
> (Lochhead, 1985, p.14)

<u>All experience is theory laden</u>

All experience is theory laden so that secondary students come to class with a repertoire of experiences and concepts. Many of these may be private, subjective, undeveloped notions in Brauner's (1988) sense. For them their subjective constructions of reality function as viable models. The learner can describe, understand and adapt to the world only through these existing conceptions. Because they are capable of producing concepts, relations and routines, learners can actively search for new regularities and meanings with which to model reality. But the descriptive means and models used in these subjective constructions are not totally arbitrary or unlimited because of the unifying bonds of culture and language the learner shares with others. New knowledge is constituted and arises within a framework in which there are three systems of constraints. First, there are the subjective structures of personal, intuitive knowledge. Second, there is the common web of public culture and language. Third, there are the constraining limits of objective reality. Learning then becomes a renegotiation of the terms of reference for constructing meaning, within the limits of these constraints. The parallels with Brauner's Perceptivism are obvious. Education becomes a process of creating in the young an appreciation of the fact

> ... that many worlds are possible, that meaning and reality are created and not discovered, that negotiation is the art of constructing new meanings by which individuals can regulate their relations with each other. It will not, I think, be an image of human development that locates all of the sources of change inside the individual, the sole child. For if we have

> learned anything from the dark passage in history through which we are now moving it is that man, surely, is not 'in island, entire of itself' but a part of the culture that he inherits and then recreates. The power to recreate reality, to reinvent culture, we will come to recognize, is where a theory of development must begin this discussion of mind.
>
> (Bruner, 1986, p.149)

The role of alternate conceptions

Within the constructivist paradigm education can be viewed as a process designed to transform a novice to become more of an expert in a particular knowledge domain. Referring back to Brauner's orientation to the world the learner must be able to sort out the noise from the message. Which signals are to be attended to, how are the signals and impressions to be packaged to construct meaning? Available are a competing swarm of notions and concepts. Because the expert is in possession of reliable theories he can often select and use the most effective conceptual scheme in the situation and can often understand a problematic situation quickly. The novice may fumble and fail, not knowing what to pay attention to. Simple, informal understandings that result from perceptions generated by relevant concepts are at the core of an expert's experience of mathematics and science. In contrast, novices are likely to use rote algorithms without the guidance of the perceptions generated by more powerful concepts and theories. When information can be managed with simple, informal, intuitive understandings this could be called expert processing because there is likely to be quick, intuitive access to relevant formal procedures. When random rote procedures are all that's available, with no way to sort out the relevant from irrelevant conceptions, this could be called remedial processing because access to powerful, formal procedures is chaotic. Education's efforts must focus on expert processing.

Constructivism focusses attention on the role of education in creating expert processing, as outlined in the previous paragraph. Learners must have a sense that reliable knowledge of significant worth has been passed on to them. They must have a sense of ownership of various ways of looking at the world. They must have a sense that an educated response to life matters. They can do this only if they are willing and able to think and act independently, choosing from a repertoire of modes of perceptions, according to what is sensible and reasonable based on the concepts and notions they currently hold. This would represent a rational response with an open but not empty mind. A child accustomed to rote acceptance of rules and procedures of faith will have allowed memory to take priority over reasoning power and so would at best be capable of only an uneducated response based on some private notions.

Constructivism holds that inquiry and learning occur against a background of the learner's current concepts. Current ideas will be interacting with new, often incompatible, ideas. Learning involves juggling various plausible alternate ways of looking at things. Becoming educated involves an ongoing rational reorganization and expansion of the learner's active conceptual repertoire. This is a process of accommodation that requires the active participation of the learner, that needs to take account of past experience and that needs to force alternate conceptions to compete for survival. Posner, Strike, Hewson & Gertzog (1982) write about the processes of accommodating and assimilating rival views of things and the struggle to construct meaningful, applicable understanding that this involves.

> Accommodation may ... have to wait until some unfruitful attempts at assimilation are worked through. It rarely seems characterized by either a flash of insight, in which old ideas fall away to be replaced

> by new visions, or as a steady logical progression from one commitment to another. Rather, it involves much fumbling about, many false starts and mistakes, and frequent reversals of direction.
> (Posner, Strike, Hewson & Gertzog, 1982, p.223)

In chapters II and III the writer has attempted to give a brief description of Perceptivism and Construction as theories. These theories will be used later to shed light on the central problem of this paper -- can the computer be used to improve the secondary school mathematics curriculum and instruction? Chapters IV and V to follow will attempt to interpret Perceptivism and Constructivism, on the basis of the descriptions given, in such a way as to make them relevant to the thesis problem, the educational role of the computer in improving mathematics education.

CHAPTER IV

AN INTERPRETATION OF PERCEPTIVISM

The specific purpose of this chapter is to relate a certain theoretical perspective in education to a particular problem in mathematics education. The theoretical perspective is called Perceptivism-Constructivism. It has been described in Chapters II and III. The problem in mathematics education is that a large percentage of the student population is getting an inadequate mathematics education. In particular, meaningful mathematics is lacking. Too many students seem to be just "going through the motions" getting schooled but not educated. Realistic applications are more important than ever in an advanced technological society yet are poorly understood by students (IAEP, 1988, pp.10-35). Against this background the enormous potential of the computer seems to be wasted because the computer is not being effectively utilized. These problems have been detailed in Chapter I. Chapter IV and V will interpret Perceptivism and Constructivism as they may bear on practise. Chapter VI will detail a calculus unit designed by the author, using perceptivist- constructivist principles, and delivered at Templeton Secondary in Vancouver. Chapter VII will report on field tests of the calculus unit and interpret the results. Chapter VIII will report conclusions and make recommendations.

The substantive role of the computer

The computer has so far been treated as an anomaly in mathematics education because there has been no guiding philosophy or theory of knowledge acquisition to inform its effective use. There is the temptation to think the technology will solve problems in education by itself. Perceptivism-Constructivism is the information age philosophy of mathematics education needed to advance mathematics education in the computer age. It provides the opportunity, to use an industrial analogy, to free students from digging ditches by hand so they may be freed to construct sky scrapers. By taking over the manipulative tasks, the computer frees students to become actively engaged in the understanding, application and interpretation of the major mathematical concepts that shape quantitative human perception in the information age. Fey and Good argue:

> Computing offers an opportunity to turn the secondary school curriculum on its head. Instead of meeting applications as a reward for years of preparation, students can begin

> with the most natural and motivating aspect of mathematics -- its applications.
>
> (Fey & Good, 1985, p.49)

Years of preoccupation with low level calculation may not be needed. The computer can take care of much of this. Ralston claims:

> No sound argument can be adduced to support a thesis that claims that high school students must be very skillful at polynomial algebra, trigonometric identities, the solution of linear or quadratic systems of equations or any of the myriad manipulative tasks that are part of the current high school mathematics curriculum.
>
> (Ralston, 1985, p.37)

What I am suggesting is that we "plug the computer" into the conceptual processing unit in Brauner's model (see Figures 1 and 3). Science has been able, as shown in Figure 3, to change the way we see the world because, in part, it has extended the range of impulses we can detect (e.g. via telescopes, radar, electron microscopes). In an analogous manner the computer can extend the possibilities for information processing through concept utilization. The range of useful mathematical procedures made available to all through the computer can change the way we see the world. Quantitative, mathematical literacy can become an active mode of perception as

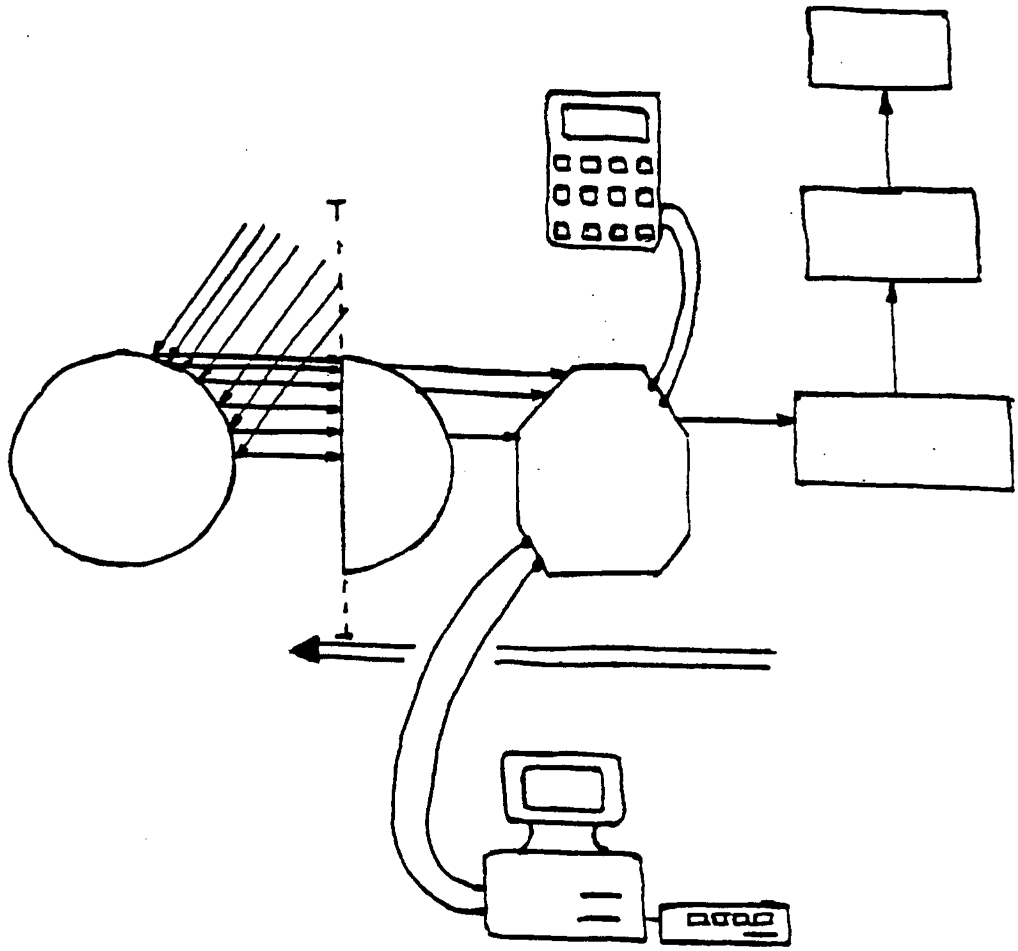


Figure 3. Computer used to amplify power of concepts.

previously laborious, unmanageable calculation is automated.

With the computer, two principles of Perceptivism seem more realizable. First, it should be possible for instruction to be aimed at actual perception and use. Mathematics should not start and end with itself but must contribute immediately to our understanding of the world and how we perceive it. With the computer, application need not be delayed. Second, mathematics can now be structured to involve the synthesis of student knowledge rather than its specialization, fragmentation and isolation. Students can understand the world using the dynamic interactions of conceptual systems managed in part by computers. Students can learn about the applicability, limits and interconnections of various ways of perceiving the world (refer to Figure 3).

Trajectories - using the computer to do mathematics

Consider the following illustration of these two principles in action in a computer learning activity for students (refer to Figure 4). Trajectories provide an opportunity to display the use of quadratic and trigonometric functions in modelling real world phenomena. Actual cases could be demonstrated or

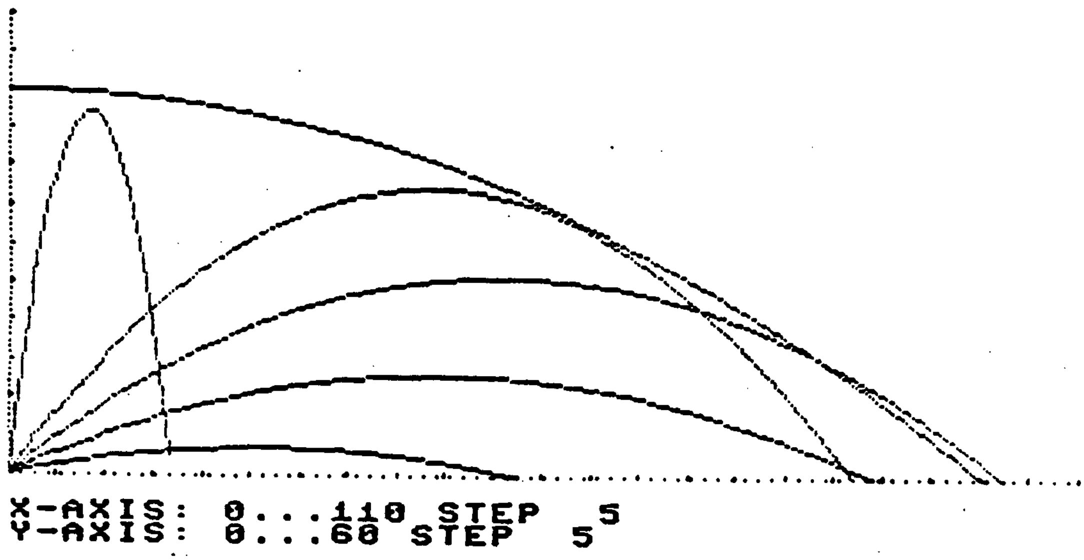


Figure 4. Trajectories

appealed to; cannon balls, footballs, rocks, arrows, shot puts, streams of water. In the two dimensional case the function which models the trajectory is given below, where g is the gravitational constant, V the velocity of the projectile and θ the projection angle (Kraushaar, 1982, pp.117-120).

$$y = x \tan(\theta) - \frac{gx^2 \, (1 + \tan^2(\theta))}{2V^2}$$

Get the students to plot some trajectories, as below. The functions listed below yield the trajectories in Figure 4.

Functions:
$y = x*.27-((9.8x^2)/2000)(1+.073)$
$y = x*.58-((9.8x^2)/2000)(1+.340)$
$y = x*1.0-((9.8x^2)/2000)(1+1.00)$
$y = x*1.7-((9.8x^2)/2000)(1+3.00)$
$y = x*11-((9.8x^2)/2000)(1+130.)$
$y = (10000-x^2)/200$

The following questions illustrate how students can be encouraged to reflect on their computer activity.

i) Which angle causes the projectile to go furthest? ... shortest distance? ... Why?

ii) Which angle causes the projectile to go half the maximum distance?

iii) What happens to the projectile if $\theta = 90$? ... $\theta = 0$? ... explain

iv) Explain what happens if g changes, for example, from 9.8 m/sec^2 on Earth to 1.6 m/sec^2 on the Moon to 0 m/sec^2 in outer space.

v) Explain and document what happens as v increases, θ and g constant. Give examples from everyday experience, for example, playing football, squirting friends with a hose.

Notice in Figure 5 how points on the x-axis can be hit with two trajectories. Get students to explore how the angles for the two trajectories are related. Have them develop a theory of the "lob". Ask them to apply

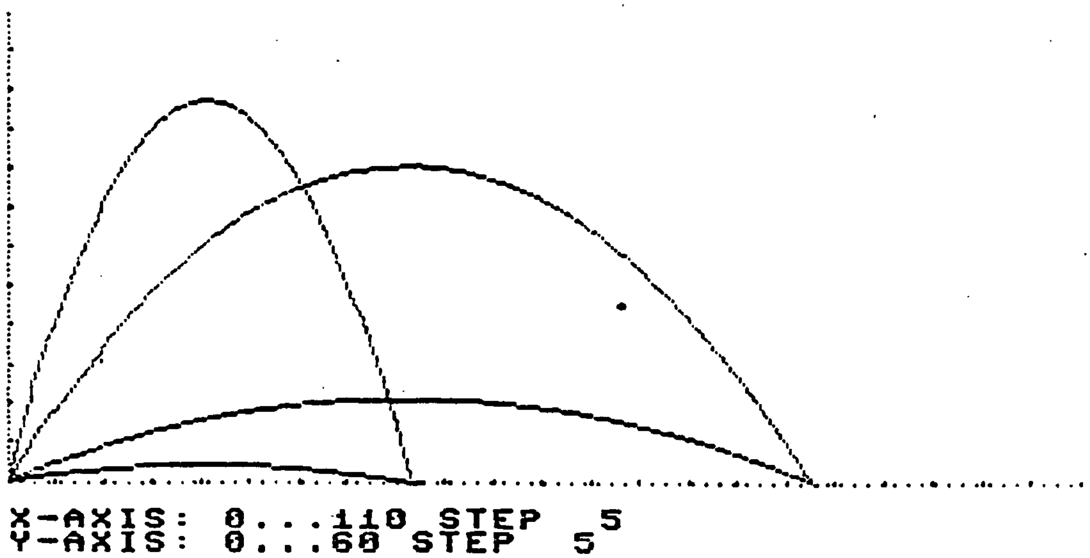


Figure 5. Lob vs. Clothesline Trajectory

the theory to ordinary experience, for example, throwing passes in football against man-to-man as opposed to zone defense, squirting a friend with water while he's standing behind a wall. Figure 5 uses these functions:

$$y = x*4.70-((9.8x^2)/2000)(1+22.11)$$
$$y = x*0.21-((9.8x^2)/2000)(1+00.05)$$
$$y = x*1.96-((9.8x^2)/2000)(1+03.85)$$
$$y = x*0.51-((9.8x^2)/2000)(1+00.26)$$

Extension of the concept - the trajectory envelope

By appealing to the idea of targets within range and targets out of range for a certain v, students can develop and explore the idea of a trajectory envelope.

For given v_o and g the equation of the envelope is;

$$y = \frac{\left(\frac{V_o^2}{g}\right)^2 - x^2}{2\left(\frac{V_o^2}{g}\right)}$$

Function:
$y = (10000-x^2)/200$

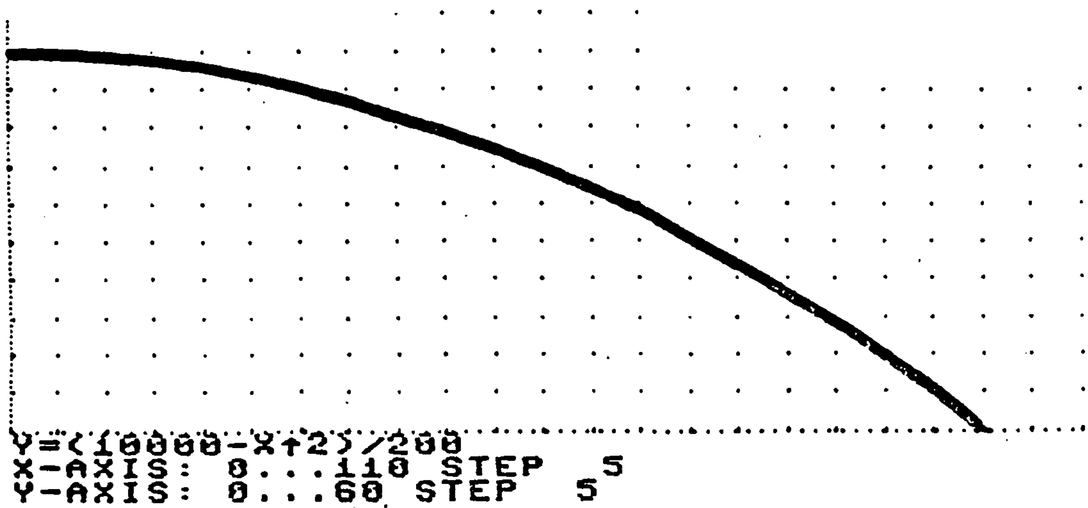


Figure 6. The envelope of trajectories

Figure 6 shows the envelope for a trajectory with $V_o^2=1000$ and $g=10$. The envelope curve divides the target space into three regions;

i) points outside the curve which cannot be hit with the given v,

ii) points on the curve which can be hit at only one angle,

iii) points inside the curve which can be hit with two angles.

Let students explore, document and discuss the possibilities. The computer takes care of the computational chores leaving the tasks of interpretation, analysis and synthesis up to the student. Computer managed calculation facilitates concept development (refer to Figure 3).

The business of hitting a target in the target space, with a given V, involves finding the angle θ. This can be done using the formula,

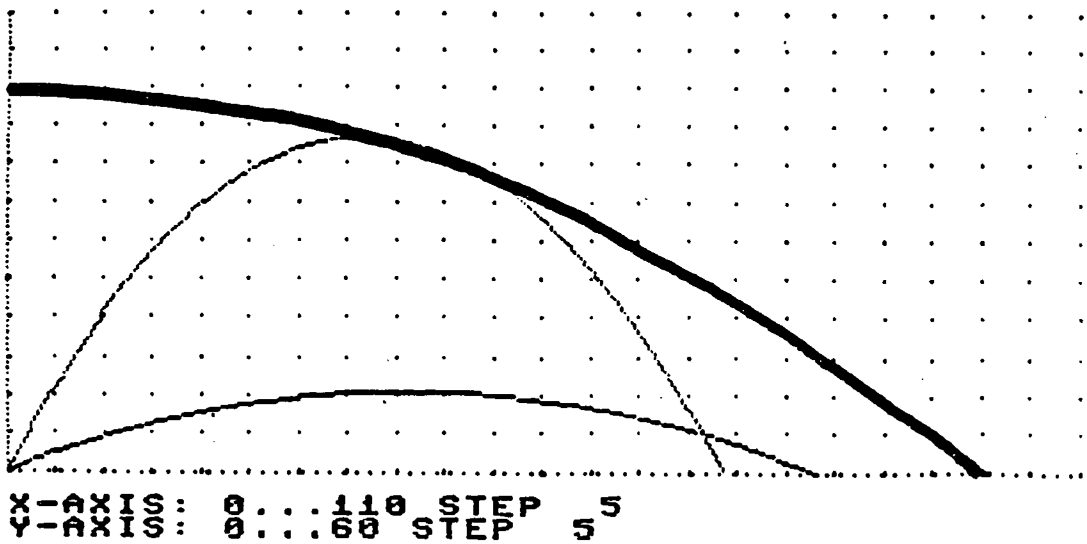


Figure 7. Hits Inside the Envelope

$$\theta = \tan^{-1}\left[\frac{1_m \pm\sqrt{1_m^2 - X^2 - 21_m Y}}{X}\right]$$

where 1_m = $(Vo^2)/g$, the range factor for the projectile.

Figure 7 show a point being hit with two different trajectories (see Kraushaar, 1982, pp.117-120). Have students hit some points with teacher chosen coordinates. They will find, gradually of course, that for some (x,y) the discriminant will be negative so there will be no angle (i.e. no solution), for some (x,y) the discriminant will be zero so there is just one solution (i.e. one angle) and for some (x,y) the discriminant will be positive so there will be two solutions (i.e. two angles). For example, Figure 8 shows two points inside the trajectory envelope each of which can be hit with two different trajectories.

Functions:
$y = x*1.00-((9.8x^2)/2000)(1+01.00)$
$y = x*1.43-((9.8x^2)/2000)(1+02.04)$
$y = x*3.27-((9.8x^2)/2000)(1+10.69)$
$y = (10000-x^2)/200$

For another example, assume that the point (55,30) is to be hit. Since it is inside the envelope, expect two angles. Substituting them,

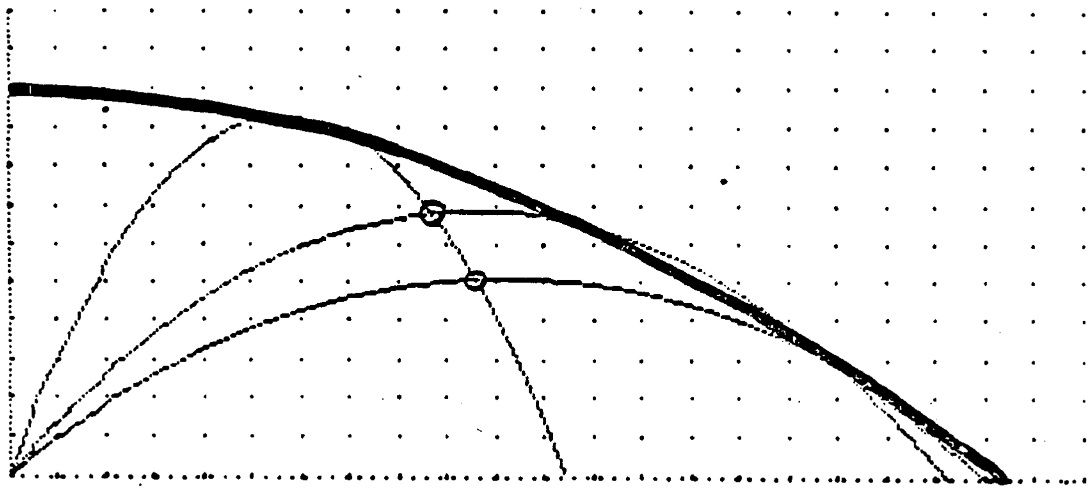


Figure 8. Hitting Specific Locations Inside the Envelope

$$\theta = \tan^{-1}\left[\frac{100 \pm \sqrt{100^2\ (55)^2 - 2\ (100)(30)}}{55}\right]$$

$\theta = 68°13'$

$\theta = 50°24'$

If a point on the x-axis is chosen two angles are obtained. Consider (50,0),

$$\theta = \tan^{-1}\left[\frac{100 \pm \sqrt{(100)^2 - 50^2 - 2(100)(0)}}{55}\right]$$

$\theta = 75°20'$

$\theta = 14°40'$

If (47,40) on the envelope is chosen, the discriminant becomes zero so only one angle is obtained

$$\theta = \tan^{-1}\left[\frac{100 \pm \sqrt{(100)^2 - (47)^2 - 2(100)40}}{47}\right]$$

$$\theta = 65°4'$$

If (60,60) outside the envelope is chosen, no angle is obtained because the discriminant is less than zero.

$$\theta = \tan^{-1}\left[\frac{100 \pm \sqrt{(100)^2 - (60)^2 - 2(100)60}}{60}\right]$$

$$= \tan^{-1}\left[\frac{100 \pm \sqrt{-5432.89}}{60}\right]$$

The trajectory exercise illustrates three important points about the perceptivist use of the computer in mathematics education. First, by taking care of elaborate and extensive calculation the computer allows perception of trajectories. Second, actual views of objects moving, stop motion photographs and computer simulations can be compared so the student can actually believe what is happening. Third, interaction of conceptual systems is encouraged. Ordinary language descriptions of football passes, squirting with water and cannon shots can be compared with concepts from physics such as velocities, acceleration due to gravity, angle of launch, range, height and these in turn can be related to quadratic functions, trigonometric functions, coordinates, curves from mathematics. This would promote conceptual discrimination and help assimilation and accommodation rather than leaving ideas in isolation (to wither and die from lack of use). Brauner argues:

> Whether it is in the sciences, the humanities, the social sciences or the logical sciences, secondary abstractions aim to generate an uncommon mode of perception that is sometimes at odds with ordinary perception. As a result ... education has a strong responsibility to provide the kind of help that makes the unique mode of perception aimed at a genuine and likely outcome.
> (Brauner, 1988, p.24)

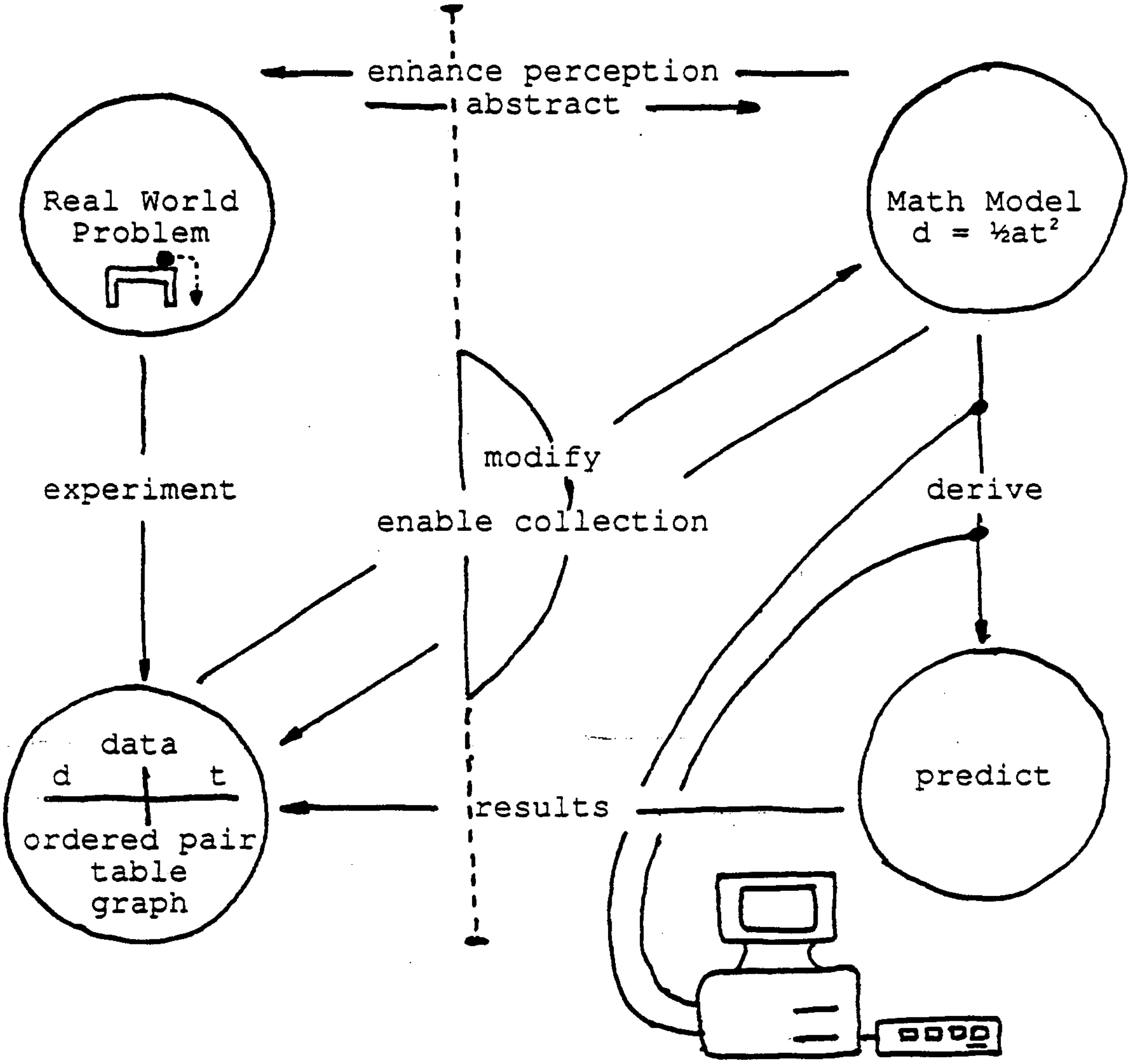
The trajectory project offers such an opportunity to learn new ideas in the context of a "genuine and likely outcome."

The author wishes to provide the reader with more examples of a mathematics education rooted in perception. What the reader should see operating throughout is the learning resulting in actual perception, the computer plugged into the conceptual system to facilitate information processing (Figure 3) and the mathematics operating across some of the nine modes of perception (Table 1) in an integrating fashion.

Perceptivist-constructivist modelling

The acquisition of the concept of a function, and the ability to use it, is one of the central tasks in secondary mathematics. It is the basis of real world modelling because it bridges the gap between reality and experience. It does this because the concept exists at five levels; as a set of ordered pairs, as a table of values (data), as a graph, as a rule (equation) and as a sentence in English. The main goal of present grade eleven and twelve secondary mathematics is to build a working knowledge of linear, quadratic polynomial, exponential and logarithmic and trigonometric functions.

Perceptionist-constructivist modelling involves four main components; a real world problem, data, a mathematical model and predictions (See Figure 9). Reality gives off impulses that may be selectively attended to with the use of concepts. In the case of mathematics these concepts would be those such as number, operation, variable, equation, curve, function, slope, integral. These concepts shape our perception of the world. Data can be collected from experiences and organized in tables and graphs. A model can then be constructed and often a rule devised. The (model)



function: ordered pairs
table of values
graph
rule
English sentence, leading to perception.

Figure 9. Perceptivist-Constructivist modelling; distance fallen as a function of time elapsed since dropped

rule will in turn influence the way we see the world. Predictions can be made with the model and tested against real data. The interaction between reality and model is consistently there. Reality, through data, puts constraints in our model. Our models and concepts shape the way we perceive the world: For example, the periodic variation of (a) properties of chemical elements (Figure 10), (b) rise and fall of civilizations (Figure 11) (c) impulses from heartbeat (Figure 12) and (d) occurrence of sunspots (Figure 13). The concept of periodicity is critical to seeing what's occurring in these various diverse phenomena.

Traditional mathematics education often gets trapped in the realm of the model, forgetting the other three realms and their interaction. What results is an intellectually sterile mathematics. The Perceptivist-constructivist alternative is to proceed quickly to applications so that the full, accurate picture can begin to take shape in the student. What makes this possible now, whereas it was just a pipe dream in the past, is the computing power of our information age personal computers, as Fey and Good argue.

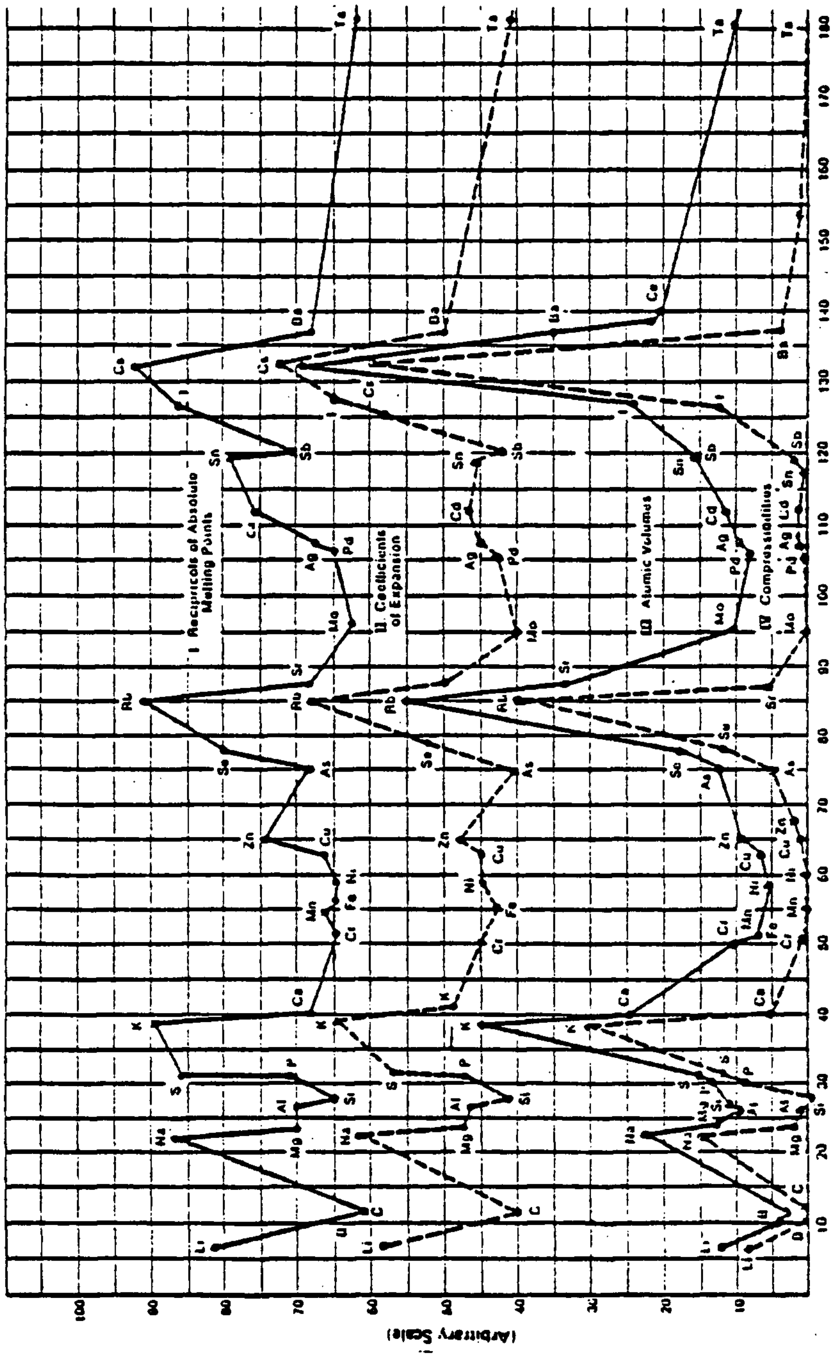


Figure 10. "Periodic" Variation of Chemical Properties (Considine, 1976, p.1731)

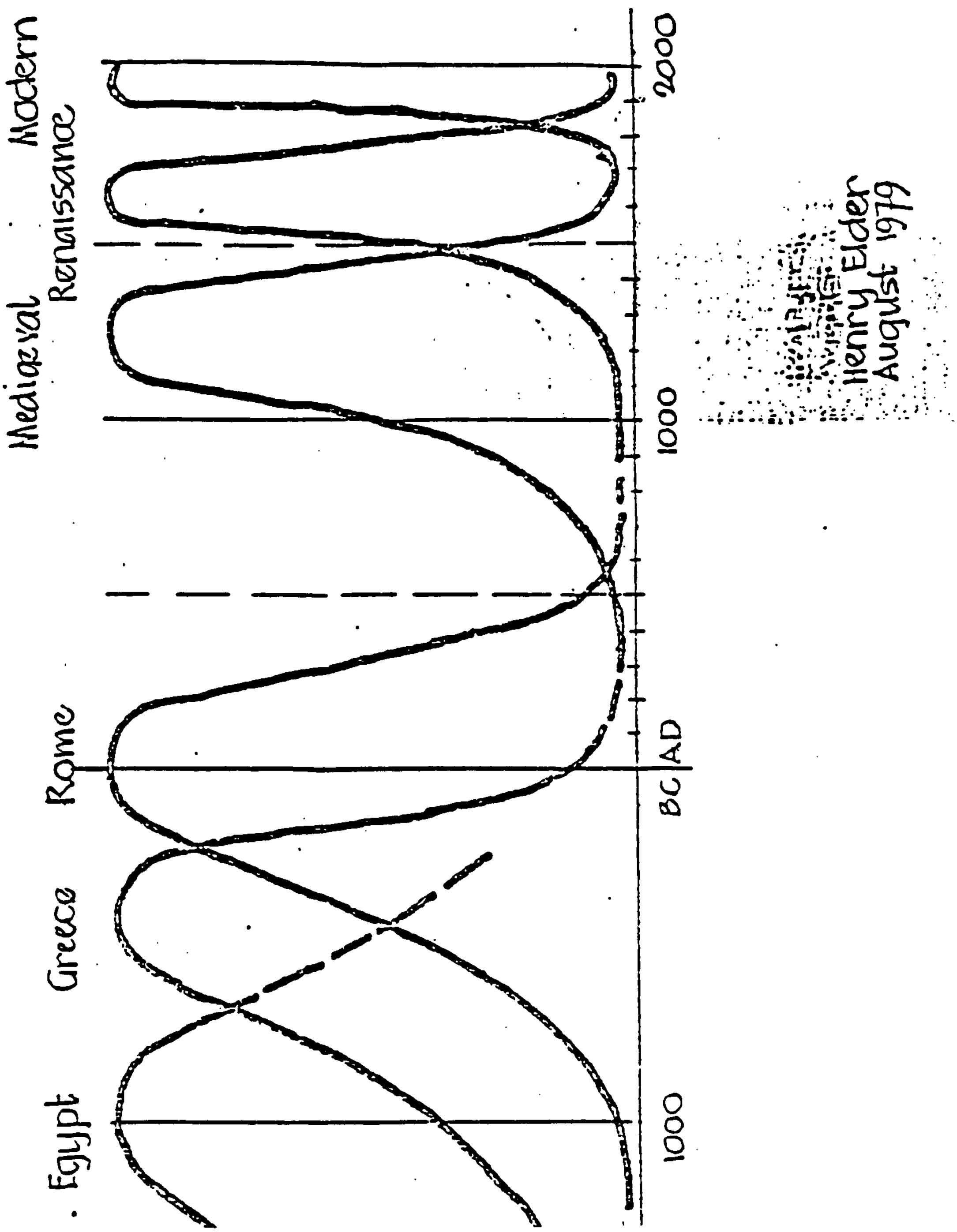


Figure 11. "Periodic" Variation of Cultural Development

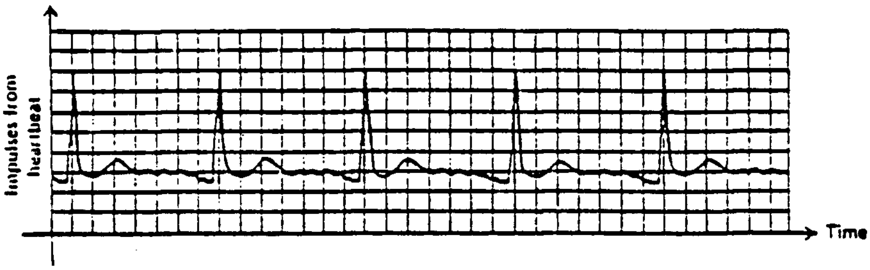


Figure 12. "Periodic" Variation in Heartbeat (Considine, 1976, p.1242)

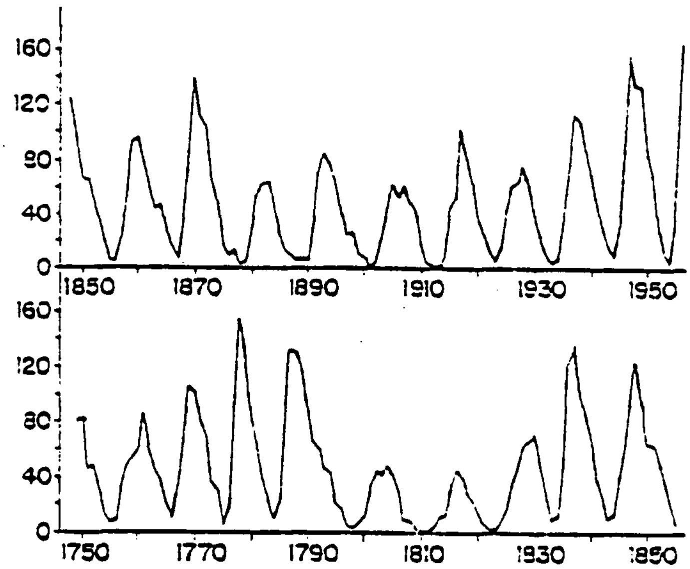


Figure 13. "Periodic" Variation in Sunspot Activity (Considine, 1976, p.2123)

> For a given function f(x), find--
> 1. f(x) for x = a;
> 2. x so that f(x) = a;
> 3. x so that maximum or minimum values of f(x) occur;
> 4. the rate of change in f near x = a;
> 5. the average value of f over the interval (a, b)
>
> For nearly every function of interest, computer utilities make all five questions accessible in some intellectually honest and mathematically powerful form to students who have not followed the conventional regimen of skill development.
>
> (Fey & Good, 1985, p. 48)

Computer assisted exploration of the exponential function

The exponential function, and its inverse the logarithmic function, can be discussed with regard to various applications in business, ecology, medicine and science. The functions model such phenomena as compound interest, forgetting, radioactive decay, drug absorption, probability, population growth and inflation. Graphs, tables of values and solutions to equations are available at the push of a button. The student's task shifts to the mastery and application of the concepts that weld together the experience and the model. The perceptivist notion of the educated person becomes approachable as mathematics shifts from number crunching to the informed application of quantitative reasoning to real world problems.

Following are three examples of the use of the exponential function in situations students are likely to be able to perceive as relevant to their lives: (a) population growth, (b) inflation, and (c) the compounding growth of long term investments. The first set of graphs (Figure 14), shows the effect of 100 years of population growth starting at our present Earth population of 5 billion and projecting 1, 2, 3 and 4 percent annual population growth rates. Carrying capacity is arbitrarily introduced as 30 billion. Useful discussions and exercises could be developed around issues such as:

1. What does the growth rate figure mean? How might it be determined? How is Canada doing in this regard?
2. What happens if we exceed the carrying capacity of the Earth? What would the graph look like? What would this mean in human terms?
3. What would a successful solution to the "population problem" look like graphically? Graphically speaking what is the "population problem"? Do you think there is a problem or is it all just so much worrying about nothing?

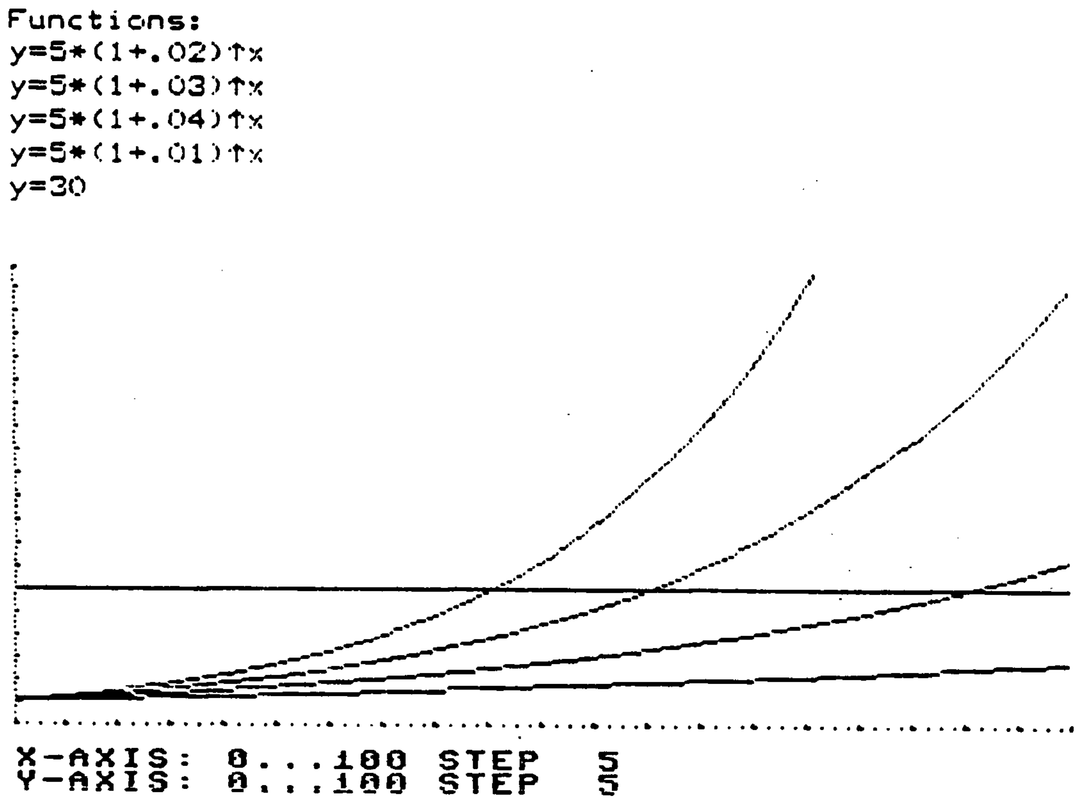


Figure 14. Exponential Function and Population Growth (Smaller growth rate corresponds to lower curve)

Perceptivism, by way of its orientation to the world and to being educated, is suggesting what to do. Get beyond mere fact or concept mastery to actual perception of the world. Constructivism is showing how to do it. Exploit the computer to allow active construction by the learner. The computer is making the information processing practicable. The student uses the calculating power of the computer to get beyond the technical details in order to attain concept mastery, understanding through use and actual perception.

The second group of graphs, (Figure 15), shows the effect of inflation on a $2.50 hamburger over the next 100 years at 3, 5, 10 and 15 percent annual inflation. A future reference point of $50 for the burger is included. Discussions and questions could focus on the effects of inflation on student's standard of living over the next 50 years of their lives (their income producing years). Financial planning strategies could be discussed and even attempted by students. A lot of the algebra and contrived word problems in senior mathematics classes are useful for future scientists and mathematicians and to act as a filter devices for higher education institutions but serve simply to mystify and alienate the vast majority of students. Computer assisted mathematics offers an opportunity to make a useful and relevant mathematics education available to considerably more students than at present.

The third set of graphs, (Figure 16), illustrates compound amount of an investment. Emergency, short

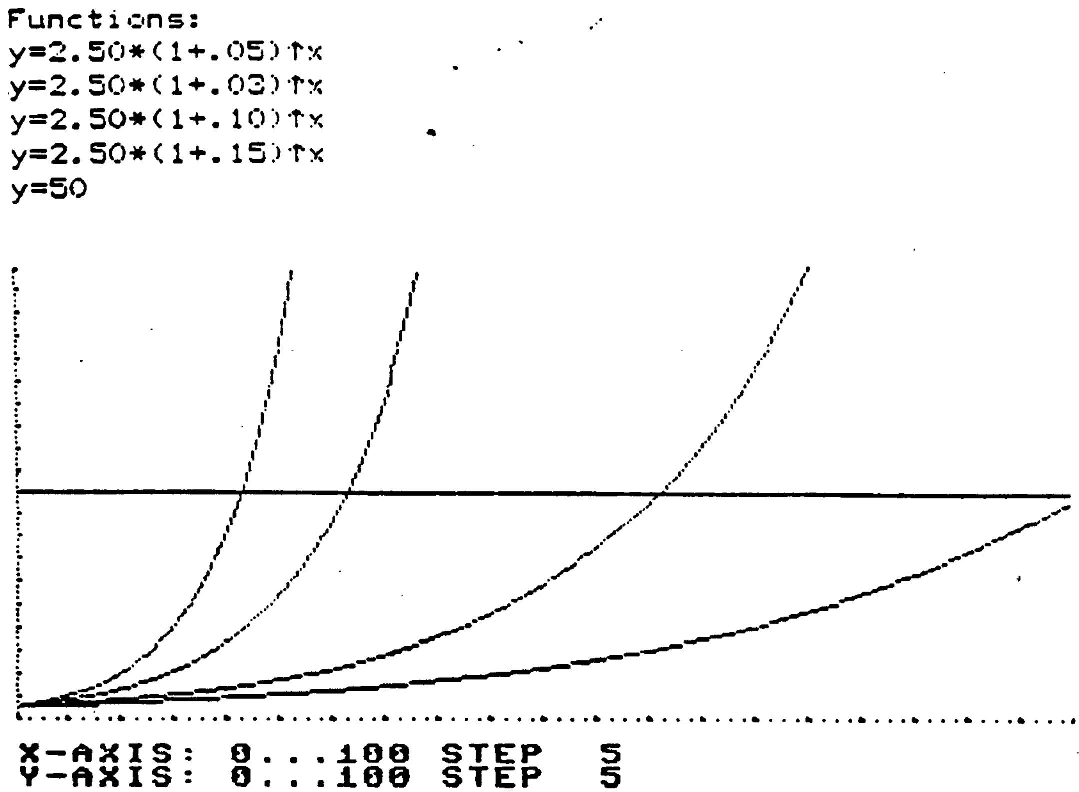


Figure 15. Exponential Function and Inflation (smaller inflation rate corresponds to lower curve)

term and long term savings could be discussed.

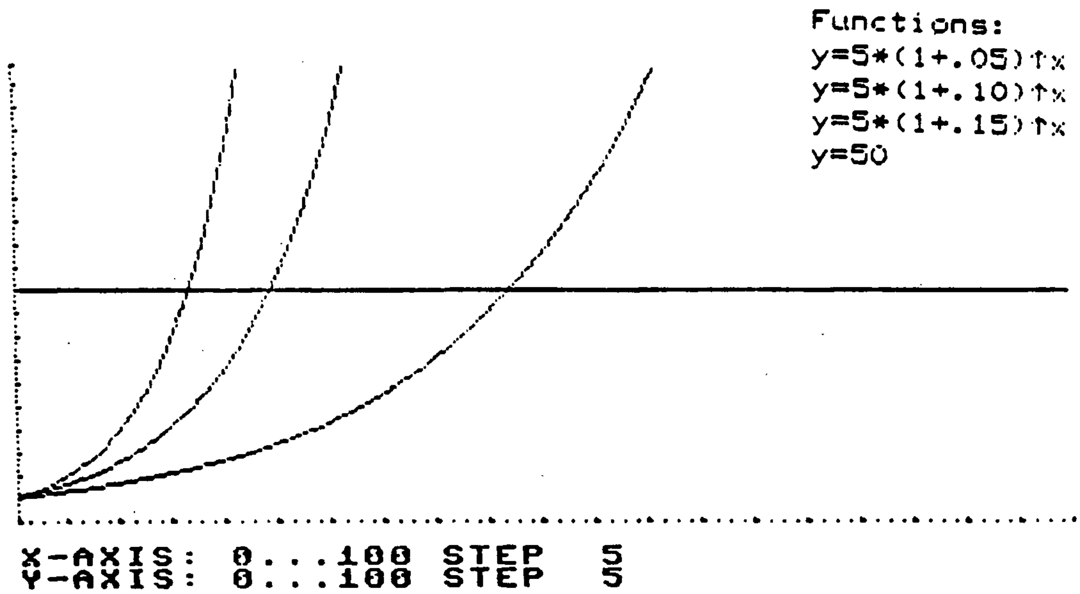


Figure 16. Exponential Function and Compound Interest

Interpolation could be used to find the value of the investment at various times. The writer's experience in the classroom indicates that even students of limited ability can make progress with the ideas here if the computer relieves them of the burden of the actual computations. Conceptual development in mathematics can then occur and so it becomes possible for the student to develop an informed, educated, quantitative understanding of their experiences of the world. By permitting mathematics to function simply as a filter mechanism we fail to educate by simple default. In failing to exploit the educational potential of the computer in mathematics education we are leaving uneducated the 70% filtered out.

Functions as models of reality

If mathematics education is to achieve a measure of intellectual respectability within a perceptivist philosophy of education it must show it is more than some esoteric tautology. It can do this by showing the student now that it can help us to see the world in new and useful ways. The computer seems to be an indispensable tool needed to overcome the computational difficulties encountered in applying mathematical functions to model real world phenomena (see Figure 3). This is not to say there is no place for the transmission of information. But students must quickly be given the responsibility of constructing meaning from the information, explaining by relating it to previous knowledge, orienting new knowledge with respect to old and integrating competing ways of looking at things so that they can achieve the desired perceptions. The essential facilitator in this process is the computer. The NCTM standards documents argues:

> Computer technology provides tools, especially spreadsheets and graphing utilities, that make the study of function concepts and their applications accessible to all students in grades 9-12. This technology makes it possible for students to observe the behaviour of many types of functions, including direct and inverse variation, general polynomial, radical, step, exponential, logarithmic, and sinusoidal. All students should use a graphing utility to investigate how the graph of y = af (bx + c) + d is related to the graph of y = f(x) for various changes of the parameters a, b, c, and d.
>
> (NCTM, 1989, p. 155)

The Bouncing Ball

"Bouncing ball" details the development of a model of bouncing ball released from above the ground and allowed to bounce. A combination of exponential, absolute value and cosine functions is found to model the damped motions of the bouncing ball reasonably well. Opportunities to allow for different elasticity for the ball, different gravity and variable damping effect can be explored by students. The performance of an actual ball (e.g. lacrosse) could then be modelled and compared with the performance of a different ball (e.g. tennis).

The bouncing ball needs three functions to model it:

1. the sine or cosine to get up and down,

2. absolute value to do away with negative values,
3. exponential to get the damping effect.

After playing with exploratory exercises such as those following (Figures 17 and 18), students would be in a position to model the behavior of specific bouncing balls, with different frequencies, amplitudes and decay rates. Questions about comparisons with what would happen on the moon and the possibility of a ball bounce that didn't decay but bounced higher and higher each bounce, could be set. Preformal experimentation hopefully will encourage student independence and confidence.

Tide Equation

"Tide equation" explores the ability to model tides. The sine function seems adequate. It is useful to note, in this case, how a graphing program can help provide five of the basic features of a function: value for a given value of x, value of x to produce a given y, slope at a point, minimum or maximum values on an interval and average value on an interval.

Water depth in a tidal region can be modelled with the sine function. Amplitude and frequency can be modelled and discussed in a way that connects with existing knowledge. The average value of a continuous

Function:
y=70*(2.71↑(-.05*x))*abs(cos(.25*x))

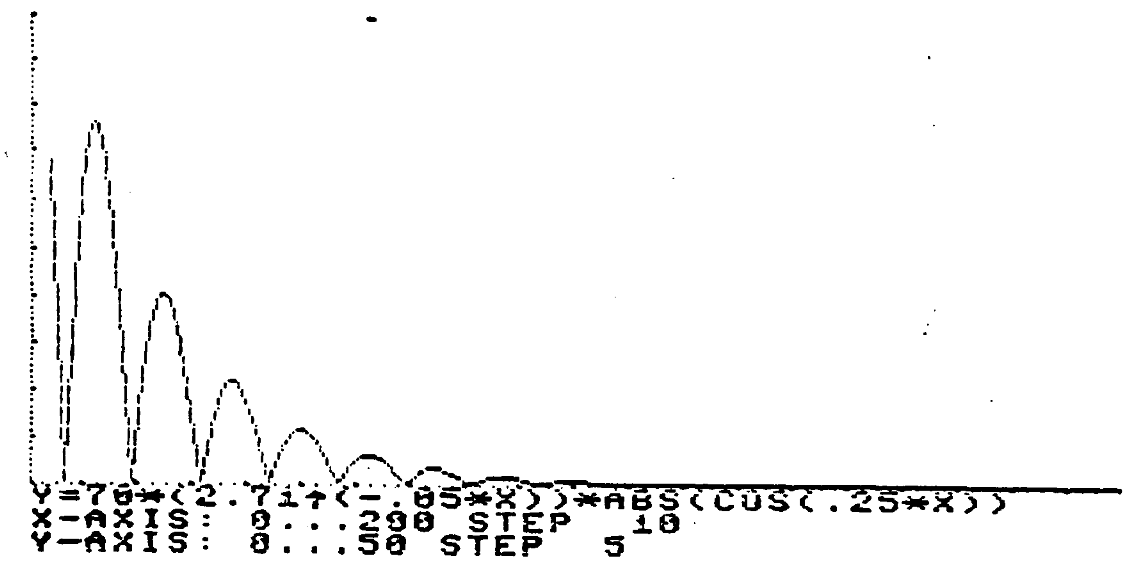


Figure 17. Bouncing ball, rapid decay

Function:
y=70*(2.71↑(-.02*x))*abs(sin(.50*x))

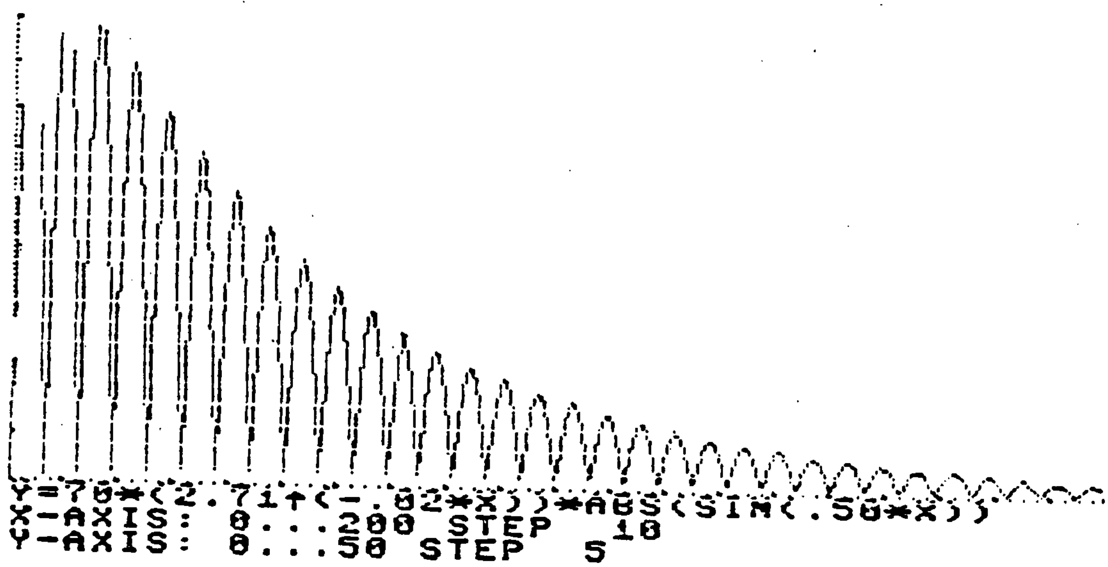


Figure 18. Bouncing ball, slow decay

function can be meaningfully investigated here, that is, average water depth over a tidal cycle. This average value can be calculated as follows;

$$\frac{\int_a^b f(X) \quad dx}{(-b-a)}$$

the definite integral being done on the machine.

Variations on the basic exercise might include changing the model to reflect deeper water and seasonal changes in tidal depth. A discussion of actual tides will unpack much of the information so compactly stored in the mathematical model.

Motion with Resistance

"Motion with Resistance" gives a full formal account of how falling objects behave while accelerating towards earth while experiencing air resistance. What is of value here is the exploration possible on the computer given the expressions for velocity and distance. The concepts of terminal velocity, average drag coefficient and acceleration due to gravity can be investigated numerically. The data can be used to explain the difference in behavior between a lead ball and a feather falling to Earth. This represents the kind of informed, quantitative

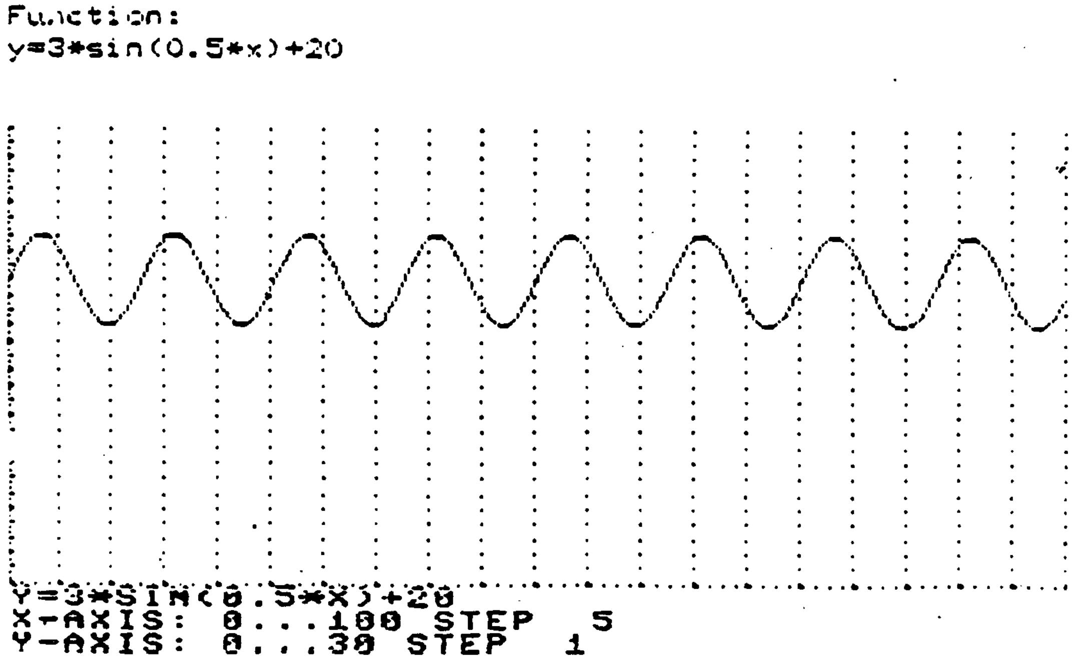


Figure 19. Water depth variation in tidal cycles

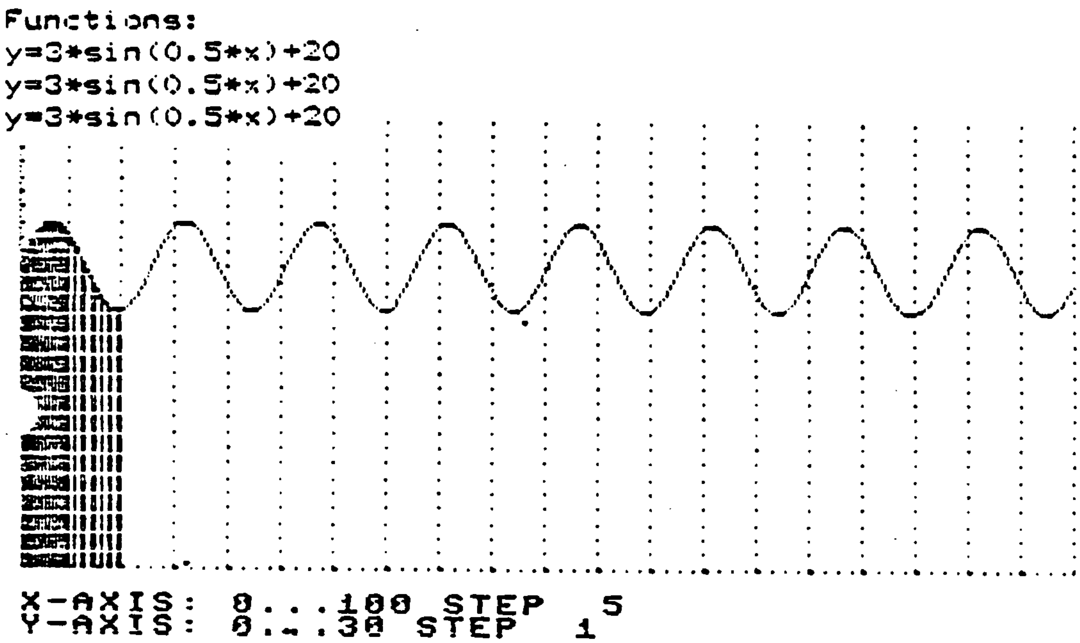


Figure 20. Average value of a function on the computer

discussion of aspects of the world that mathematical competency should enable.

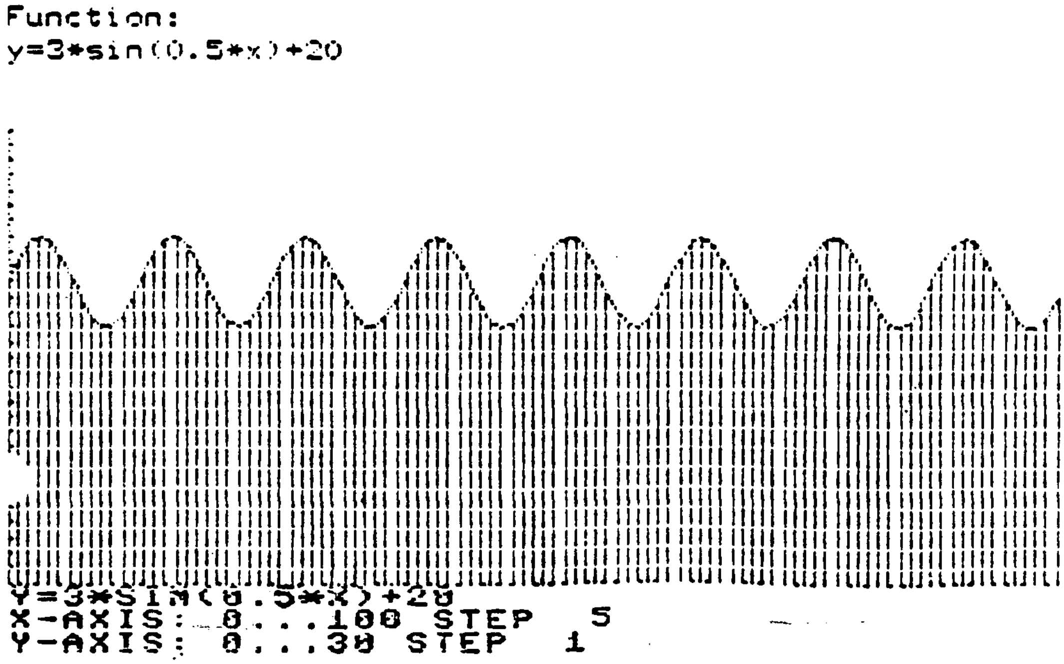


Figure 21. Average tidal depth

If an object is dropped from a height h above the Earth, air resistance ignored, the velocity (v) and distance (d) are given by

$v = at = 32t$

$d = .5at^2 = 16t^2$, where a is acceleration due to gravity

Velocity increases indefinitely with respect to time and supposedly all objects fall at the same rate since there is no mention of mass. With air resistance the situation is different (see equations 16 and 17 below). Some devices even depend on the effects of air

The computer facilitates the investigation of this type of motion under the constraint of resistance, in which

$v = (v_o + v_\tau)*exp(-pt)-v_\tau$,

where v_τ is terminal velocity,

p is drag coefficient,

v_o is original velocity,

t is time, and exp(x) is e^x

For simplicity assume $v_o = 0$ and use $v_\tau = g/p$ to get

g is acceleration due to gravity.

$$v = g/p * exp(-pt) - g/p \qquad (16)$$

Also the height of the object above ground is given by;

$y = y_o - yt + 1/p * (v_o + v_\tau) * (1-exp(-pt))$

where y is height above ground.

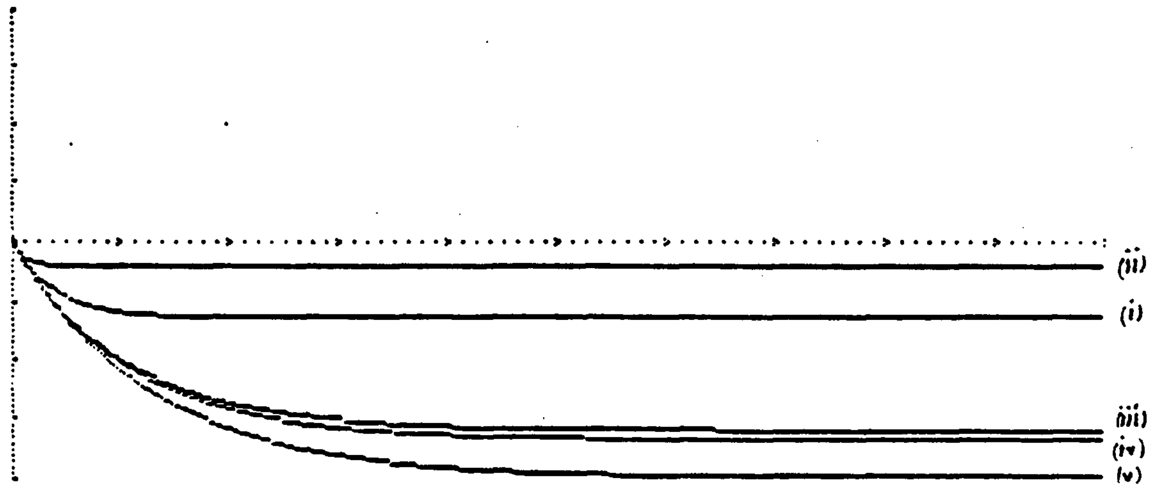
For simplicity use $v_\tau = g/p$ and assume $v_o = 0$.

$$y = y_o - g/p * t + 1/p * (g/p * (1-exp(-pt)) \qquad (17)$$

Notice below that the velocity does not increase indefinitely but rather a terminal velocity is attained. Experimentation should then proceed to determine what physical and corresponding mathematical factors will influence terminal velocity (Figure 22).

The height of objects in free fall can be studied using equation 17. Obviously feathers behave differently from cannon balls. Below, different free falls are plotted and the student can explore the

```
Functions:
i) y=(32/0.5)*(2.71↑(-0.5*x))-32/0.5
ii) y=(32/1.5)*(2.71↑(-1.5*x))-32/1.5
iii) y=(32/.20)*(2.71↑(-.20*x))-32/.20
iv) y=(32/.19)*(2.71↑(-.19*x))-32/.19
v) y=(32/.16)*(2.71↑(-.16*x))-32/.16
```



X-AXIS: 0...50 STEP 5
Y-AXIS: -200...200 STEP 50

Figure 22. Terminal Velocity in Motion with Resistance

falls are plotted and the student can explore the different possibilities.

1. Person with parachute p = 1.5

   v_τ = 21 ft/sec.

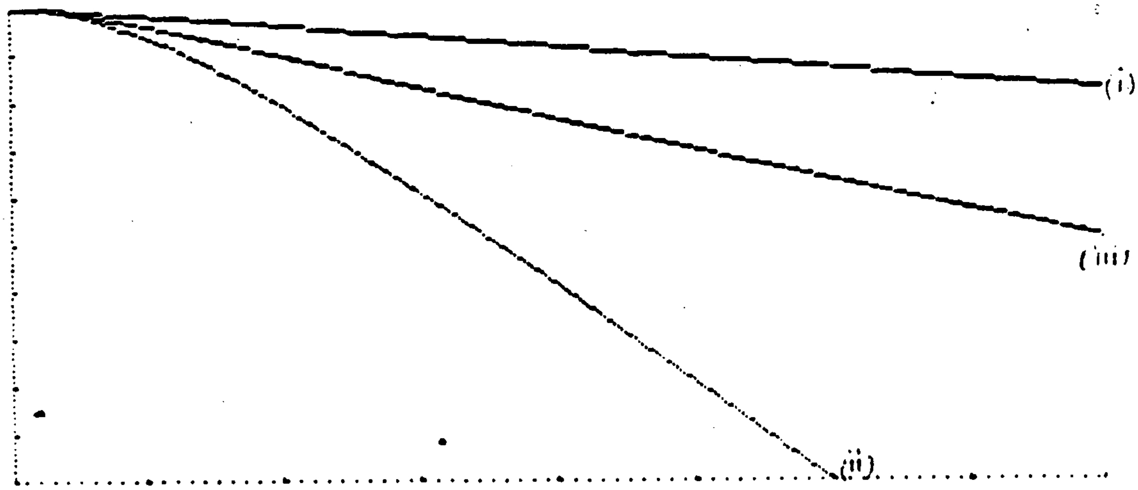
2. Person with no parachute p = .15

   v_τ = 213 ft/sec.

3. Person unbuttoned overcoat p=.5

   v_τ = 64 ft/sec. (Figure 23)

```
Functions:
i) y=5000-21.3*x+1/1.5*(21.3)*(1-2.71↑(-1.5*x))
ii) y=5000-213.*x+1/.15*(213.)*(1-2.71↑(-.15*x))
iii) y=5000-64.0*x+1/0.5*(64.0)*(1-2.71↑(-.50*x))
```




```
X-AXIS: 0...40 STEP 5
Y-AXIS: 0...5000 STEP 500
```

Figure 23. Position as Function of Time with Terminal Velocity

Damped Oscillatory Motion

"Damped Oscillatory Motion" attempts to model the gradually reducing back and forth motion of a pendulum or vibrating spring. A sine function will capture the periodic behavior and adjustments can be made for amplitude and period. Multiplying by an appropriate exponential function will then cause the amplitude to gradually decrease over time, duplicating the behavior of the physical system. Actual pendulum or spring systems can be studied and modelled using appropriate constants. Once again, informed, quantitative discussion of a real world situation is the vehicle for constructing meaningful understanding.

Vibrating springs and pendulums gradually fading out are both examples of decaying periodic behavior. Modelling periodic phenomena with decreasing amplitudes can be accomplished using a trigonometric function t and an exponential function exp(-bx). The equation is of the form A* t(X)*exp(-bx). Students can study the examples provided and attempt to model an actual spring or pendulum in oscillation, by adjusting parameters in the general function. See Figure 24. At this point in their development students should also be able to research applications of this type of functional variation. They should also be able to generate reverse damping and suggest applications, for example destructive vibration, amplification and sound theory.

Uninhibited Growth Model

"Growth Models" provides outlines for three different models of growth: (a) uninhibited exponential growth (and decay), (b) inhibited exponential growth to a limit, (c) inhibited exponential growth to limit where the rate is proportional only to the remaining

Functions:
i) y=2*sin(2*x-3.14) *(2.71↑(-x/5))
ii) y=4*sin(2*x-3.14) *(2.71↑(-x/3))
iii) y=8*sin(4*x-3.14) *(2.71↑(-x/3))
iv) y=9*sin(1*x-3.14) *(2.71↑(-x/5))

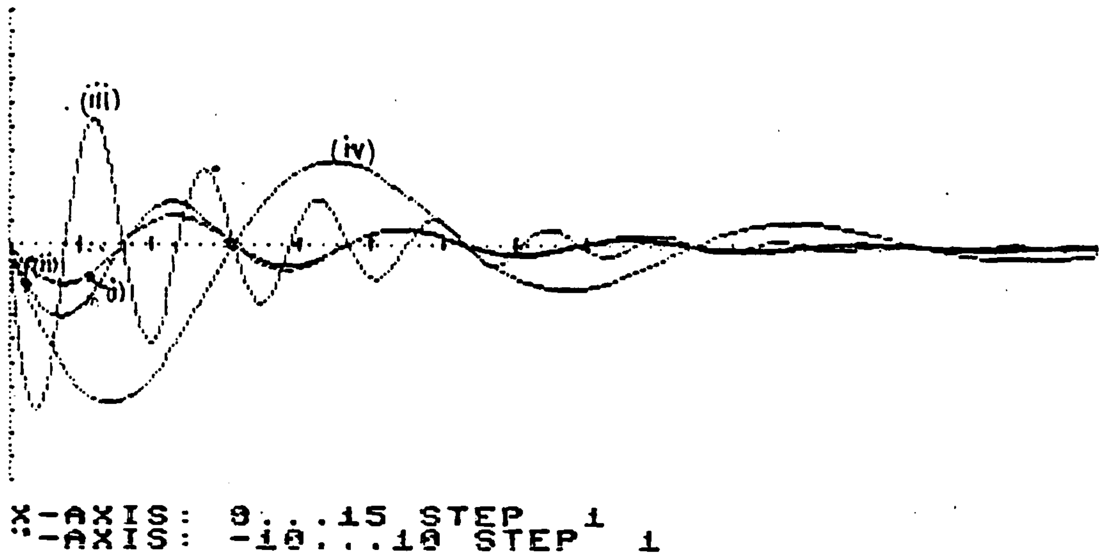


Figure 24. Damped Oscillatory Motion

room for growth (unlike the previous case where the rate also depends on the population size remaining). Application exercises in business, ecology, medicine, population theory, economics, physics, sociology and psychology give the student a chance to see and explore the diverse applications of the three models.

If a population reproduces itself at a rate proportional to its size, a normal sort of situation, then the population P is given by

$P(t) = P_0 \exp(kt)$ where k is the percent of growth per year as a decimal fraction.

Figure 25 below is an investigation of future world population under the dual assumptions of uninhibited growth and percentage increases ranging from .5% - 3%.

Functions:

i) y=5*2.71↑(.010*x)
ii) y=5*2.71↑(.015*x)
iii) y=5*2.71↑(.020*x)
iv) y=5*2.71↑(.030*x)
v) y=5*2.71↑(.005*x)
vi) y=5*2.71↑(.000*x)

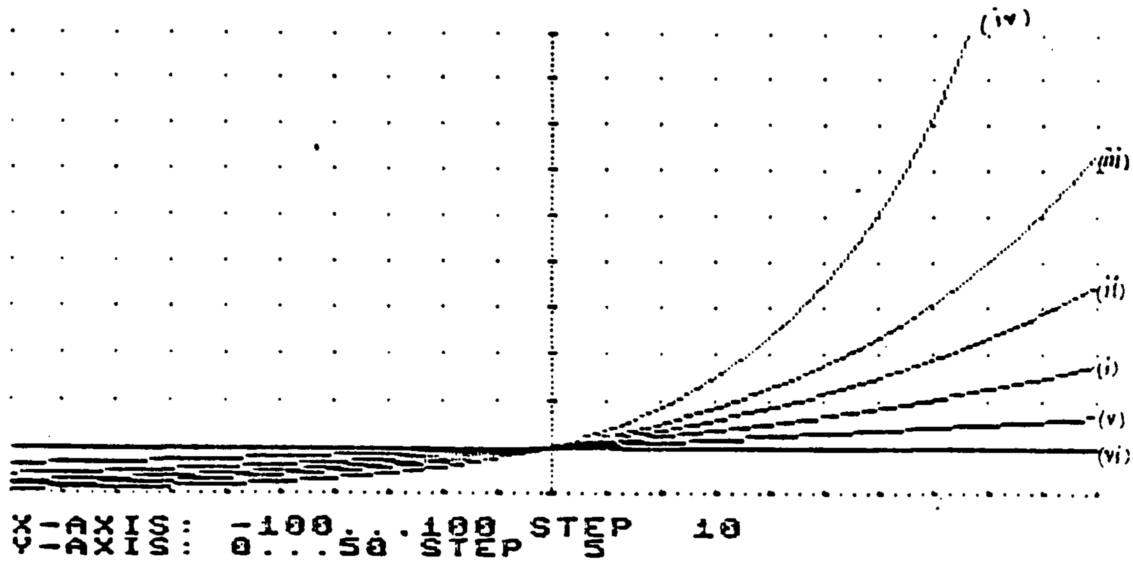


Figure 25. World population projections

Exponential Decay

If a population declines at a rate proportional to the present population then,

$$P(t) = P_0 \exp(-kt)$$

where k is the percent of yearly decline as a decimal.

The constant k for carbon-14 is .00012. The table of values in Table 2 and graph in Figure 26 can be used to show, for example, that the half life of the isotope is something like 5750 years and that if only 70% of an object's C-14 remains the object is about 3000 years old.

Inhibited Growth Model

If a population is constrained by certain limits to growth such $\frac{dP}{dt} = kP(L-P)$ where L is the limit and k the growth rate then the population is given by

$$P(t) = \frac{P_0 * L}{P_0 + \exp(-LKt) * (L-P_0)}$$

The inflection point of the curve where the rate of change reaches a maximum and then declines is P = .5L.

The graph in Figure 27 and table of values in Table 3 depict world population under the constraint of L = 30 billion.

Table 2
Decay of carbon-14

Age of object	Percent C-14 remaining
x	100*(2.7↑(-.00012*x))
0.	100.00
1000.	88.72
2000.	78.72
3000.	69.84
4000.	69.84
5000.	54.98
6000.	48.78
7000.	43.28
8000.	38.40
9000.	34.07
10000.	30.23
11000.	26.82
12000.	23.80
13000.	21.11
14000.	18.73
15000.	16.62
16000.	14.75
17000.	13.08
18000.	11.61
19000.	10.30
20000.	9.14

If the issue could be carried on using the model as a quantitative basis. Table 4 and Figure 28 depict an analogous growth situation in which disease gradually spreads to infect the entire population of a town of 2,000 people. The physical basis for the mathematical similarity could be discussed with students.

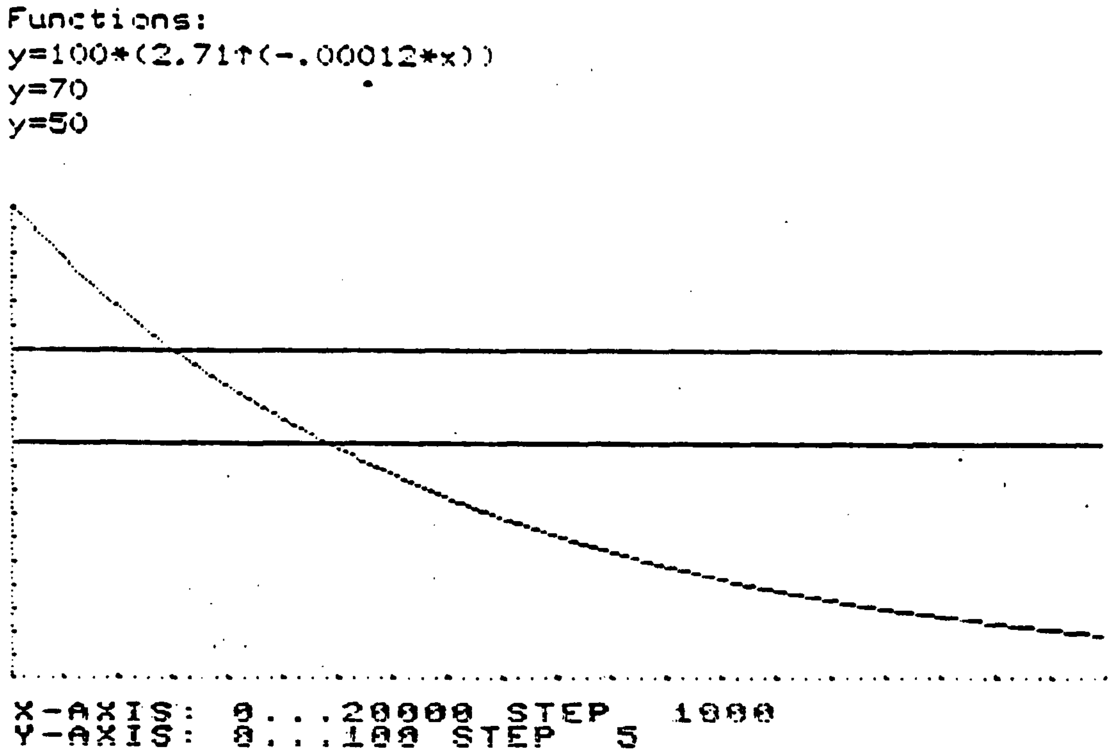


Figure 26. C-14 decay

Inhibited Growth (L-P)

A third model of growth assumes $\frac{dP}{dt} = k(L-P)$ where L-P is the available room for growth and P=0 when t=0. The population at any time is given by

$$P(t) = L(1-\exp(-kt))$$

Functions:

i) y=(5*30)/(5+2.71↑(-30*.01*x)*(30-5))
ii) y=(5*30)/(5+2.71↑(-30*.001*x)*(30-5))
iii) y=(5*30)/(5+2.71↑(-30*.005*x)*(30-5))
iv) y=(5*30)/(5+2.71↑(-30*.020*x)*(30-5))

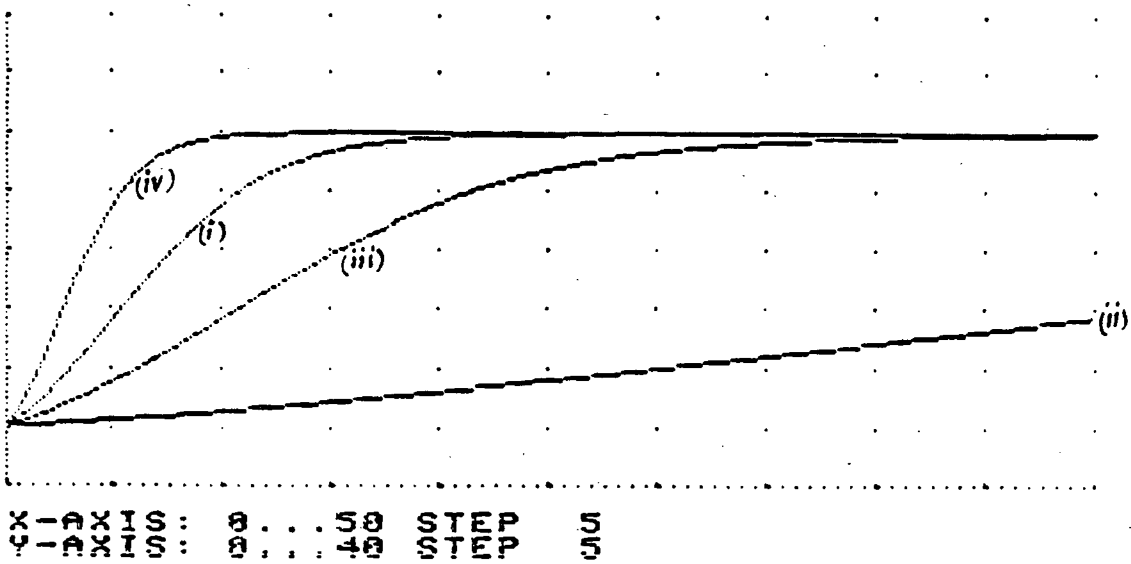


Figure 27. World population, inhibited growth model

Table 3
World population figures for the inhibited growth model (2% annual growth rate)

Years from present	World population in billions (5*30)/(5+2.7↑(-30*.020*x)*(30-5)
0.0	5.00
2.5	14.15
5.0	23.98
7.5	28.40
10.0	29.63
12.5	29.93
15.0	29.98
17.5	30.00

Table 4
Disease spread in a city using the inhibited growth model

Time in weeks	Number of people infected (10*2000)/(10+2.7↑(-2000*.003*x)*(2000-10)
0.00	10.00
0.10	18.11
0.20	32.70
0.30	58.69
0.40	104.25
0.50	181.25
0.60	307.81
0.70	497.19
0.80	751.35
0.90	1045.08
1.00	1331.22
1.10	1567.13
1.20	1736.31
1.30	1845.87
1.40	1912.21
1.50	1950.76
1.60	1972.62
1.70	1984.85
1.80	1991.64
1.90	1995.40
2.00	1997.47

Assume the limit L, percentage of the population that will buy a product is 100% i.e. L=1. Suppose it has been determined that the percent acceptance increase rate k = .07 so that the population buying the product is given by

$$P(t) = 1(1-\exp(-.07t))$$

The graph in Figure 29 shows the number of times to advertise to get any given degree of acceptance.

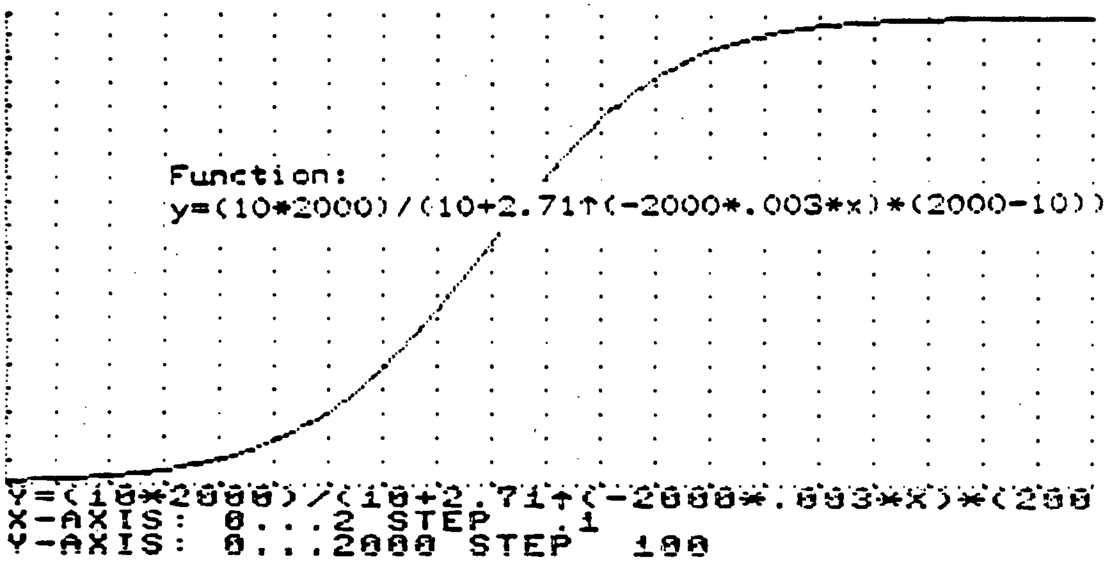


Figure 28. Inhibited growth model of disease spread

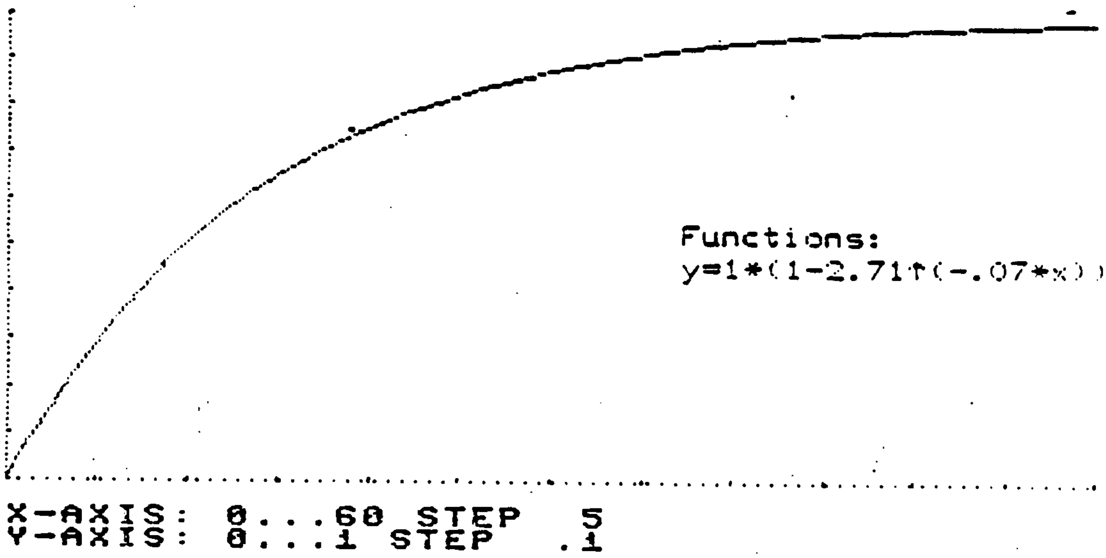


Figure 29. Percentage buying a product, inhibited growth

As the examples above illustrate, the computer can make the meaningful application of mathematics a reality for more students than ever before. The evidence suggests that mathematics educators have no program in place to take advantage of the powers of the computer to help enable students to master the fundamentals of quantitative literacy. This failure remains a chronic problem in mathematics education. Computer integrated mathematics programs must be made available that learners can explore domains of learning before describing them. Learners need to study concrete properties and relations before formulating abstract relations. Computer aided perception for many may be critical to attaining basic literacy.

The computer facilitates the realization of Perceptivist-Constructivist mathematics education. The computer, because of its information processing power, makes Perceptivist-Constructivist mathematics a realistic possibility rather than just an academic fantasy. Perceptivist-Constructivist theory in turn makes the computer the technological basis for concept development, concept application and thus meaningful learning of mathematics. The informed use of the computer can help get us beyond computation for the few and moving toward meaningful mathematics for all. The examples provided across Brauner's modes of perception have included examples from physics, business, population ecology, meteorology, economics and medicine. All the examples can be discussed in ordinary language as well as technical language. If we are guided by an information age philosophy such as Perceptivism, the range of examples can be expanded to build a curriculum rooted in perception. In this way mathematics education can move toward mathematics for all because it is mathematics about something to which we can all relate -- the world we share. It is mathematics using something to which we all have access, our ordinary language.

CHAPTER V

AN INTERPRETATION OF CONSTRUCTIVISM

The present chapter is an attempt to interpret three basic principles of Constructivism as outlined in Chapter III. The first principle indicated that instruction should actively involve the learner in the construction of meaning. By relieving the burden of computation the computer is able to encourage this. The second principle indicated instruction should draw on and relate to existing experience in a way that relates to existing knowledge. The third principle indicated that instruction should allow for the accommodation and assimilation of provisional or alternate conceptions. With the computer taking care of the tedious, the student can explore many conceptions of things that might be competing for his belief, rather than simply relying on the authority of the teacher. Understanding for use means more than just recalling and repeating, it involves actually believing.

The active construction of meaning

Constructivists make a distinction between information and knowledge. When the purpose of instruction is to transmit information, explanations are all that is required. Knowledge, the gaining of expertise in information processing and application, cannot be simply transmitted. It must be constructed with the active participation of the learner. Knowledge is something the learner can and must construct by himself. Discovery, re-invention, or active reconstruction is necessary.

Discovery, re-invention or active reconstruction in mathematics is usually difficult, at least in part, because of technical, computational difficulties. The computer can help with this; witness the trajectory example. There is also the problem of an abstract, formal approach to mathematics. Rather than starting with intuitive ideas and basic familiarity, proceeding to exploration and discovery and finally developing a polished, refined, axiomatized theory; mathematics teaching often dumps the finished product on the student. The extreme abstraction and compactness of the theory make it incomprehensible. The finished product is the goal but premature formalism is not the means. Eric McPherson notes with regard to geometry that it is a curriculum strand long dominated by the "pointlessness of premature and misdirected formalism" and that the traditional curriculum was often "too much of a preparation for a remote future that for many students never came" (McPherson, 1985, p. 67).

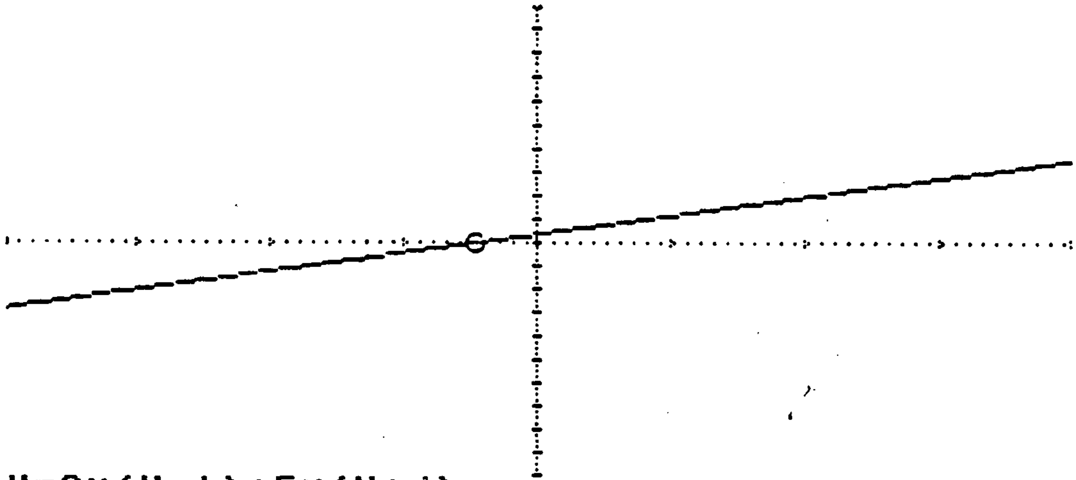
Howson echos a similar concern with an elitist, abstract curricula which seem to eliminate the possibility of meaningful mathematics for so many.

> Each stage is seen as a preparation for the next: primary prepares for lower secondary, lower secondary for upper secondary, which in its turn prepares for tertiary education. Yet by this stage only a minute proportion of the age cohort may remain within the formal education system; has all this "preparation" been just a waste of time and effort for the great mass of the students who are no longer there?
>
> (Howson, 1986, p. 29)

The computer can make reconstruction a real alternative to dumping the finished product on students. Consider this exercise, given by the author to grade twelve students at Templeton Secondary in Vancouver. The exercise is to reconstruct the fundamental theorem of algebra from a graphing perspective. The concept of a degree n polynomial equation having solutions where the graph of its associated function crosses the x axis can be easily investigated using graphing software and suitably selected equations with real variables. For degree one a straight line can only cross the x axis once. Different possibilities exist however and students should be asked to discover these (Figure 30). Second degree equations are represented by parabolas (Figure 31). One possibility is that it crosses twice. What are the other possibilities? Students can work backwards making up equations using the factor theorem (e.g. $0 = (x-3)\ (x+5) \Rightarrow x^2 + 2x - 15 = 0$). What solution possibilities are there for degree two equations, including the possibility of multiple and complex roots? Possibilities for degree three (Figure 32), degree four (Figure 33), degree five (Figure 34) and degree six (Figure 35) are suggested and the student must discover the other solution possibilities. Discovery, re-invention or reconstruction is the student's job. What makes this possible now, where it was impractical in the past, is the computational power of the computer. It is easy for students to explore a wide range of possibilities and satisfy themselves as to their conclusions. The NCTM Standards document claims:

> Calculators and computers with appropriate software transform the mathematics classroom into a laboratory much like the environment in many science classes, where students use technology to investigate, conjecture, and verify their findings. In this setting, the teacher encourages experimentation and provides opportunities for students to summarize ideas and establish connections with previously studied topics.
>
> (NCTM, 1988, p. 128)

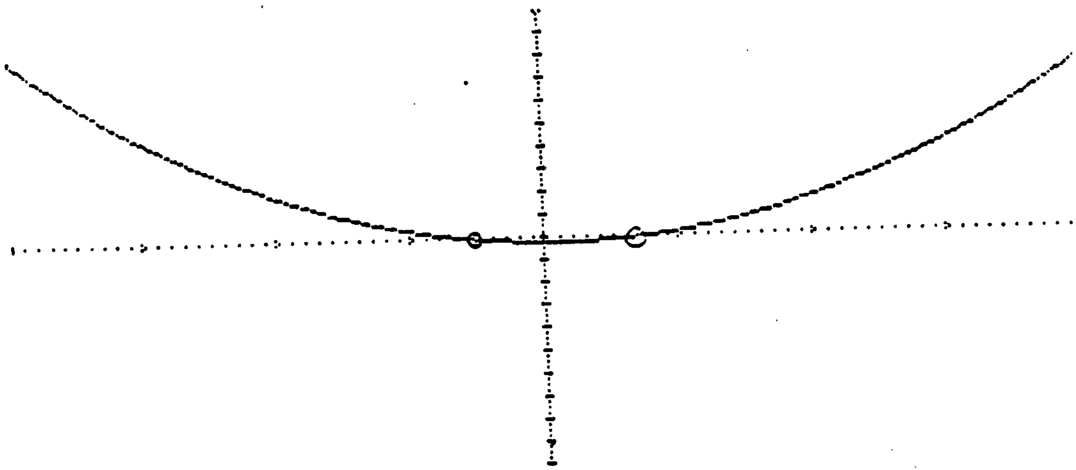
Function:
y=3*(x-1)+5*(x+4)



Y=3*(X-1)+5*(X+4)
X-AXIS: -20...20 STEP 5
Y-AXIS: -500...500 STEP 50

Figure 30. Computer solution of degree one equations

Function:
y=x↑2-x-12



Y=X↑2-X-12
X-AXIS: -20...20 STEP 5
Y-AXIS: -500...500 STEP 50

Figure 31. Computer solution of degree two equations

Function:
y=x↑3-3*x↑2-10*x+24

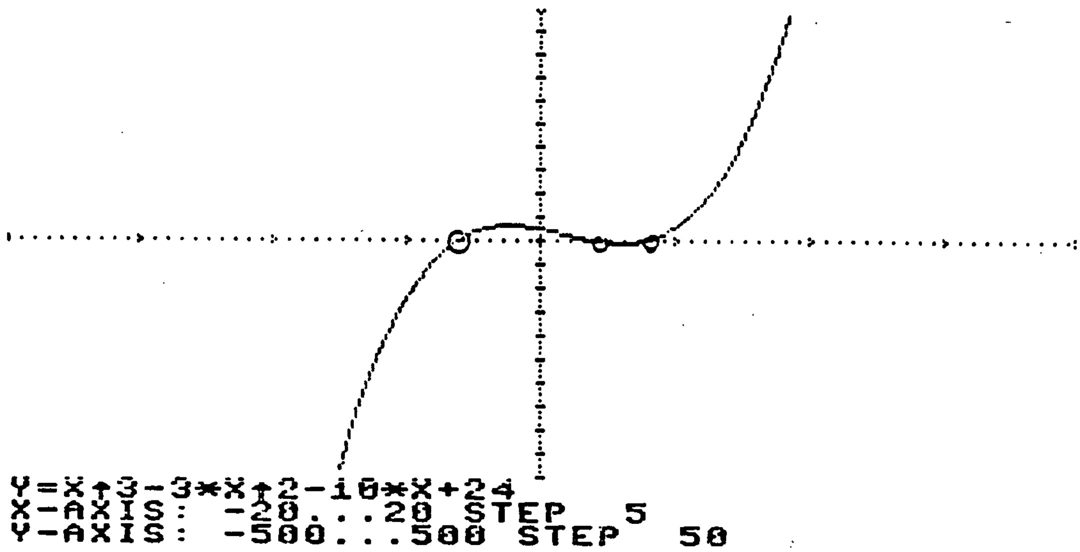


Figure 32. Computer solution of degree three equations

Function:
y=x↑4-4*x↑3-7*x↑2+34*x-24

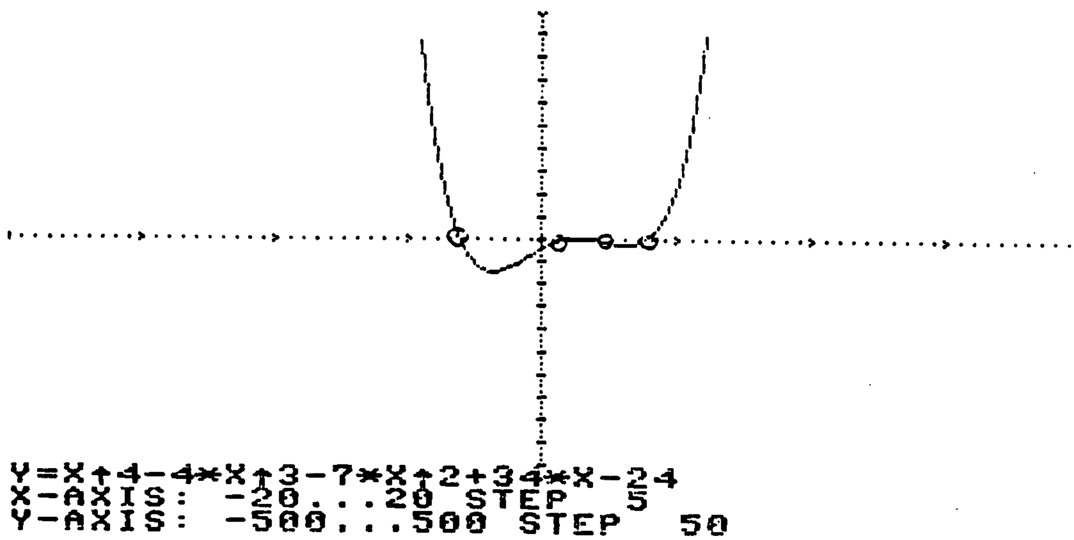


Figure 33. Computer solution of degree four equations

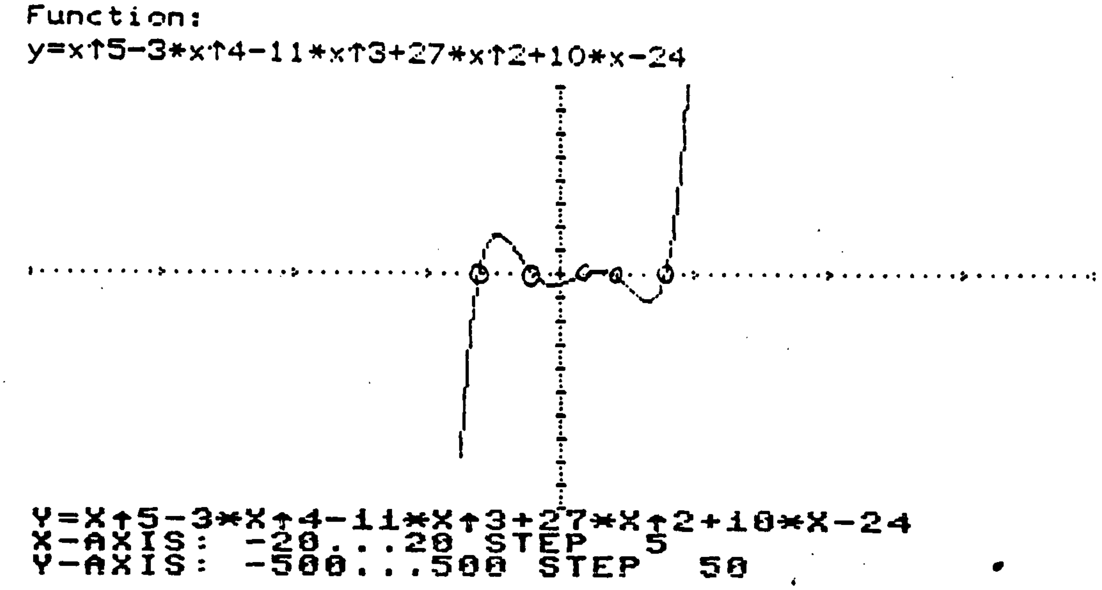


Figure 34. Computer solution of degree five equations

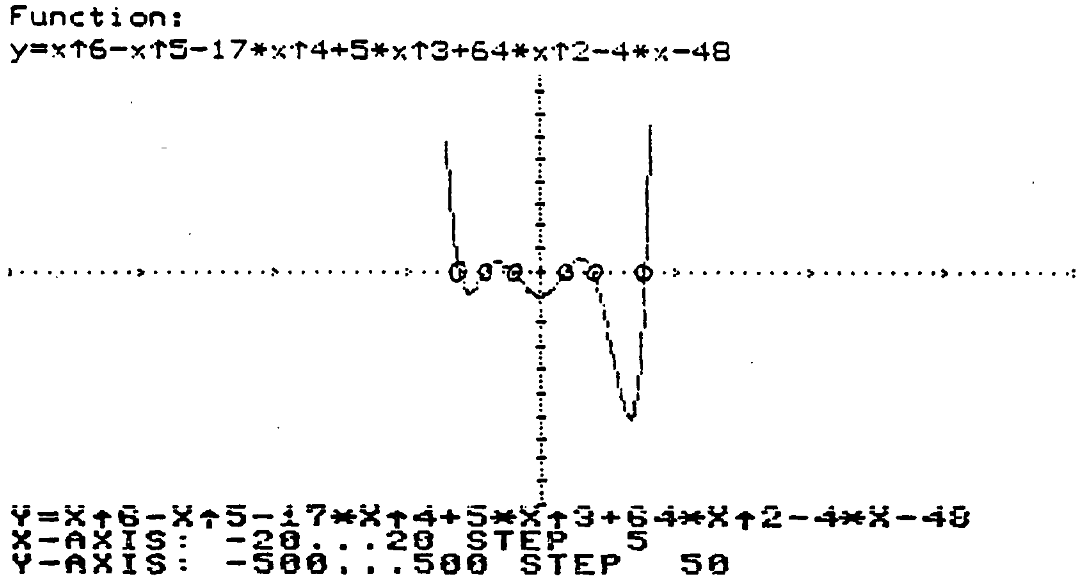


Figure 35. Computer solution of degree six equations

Appealing to existing knowledge

Kilpatrick (1987) argues that the constructivist must take account of existing experience and knowledge. Teaching, using procedures that aim at generating understanding, becomes sharply distinguished from the transmission of information, using procedures that aim at repetitive behaviour, in that assimilation and accommodation of competing conceptions must be facilitated. Processes inferred as inside the student's head become more interesting than overt behaviour. Linguistic communication becomes a process for guiding student learning rather than simply transferring knowledge. Taking into account what students think now and how they might be encouraged to change is critical. Student deviations from the teacher's expectations become means for understanding their efforts to understand. Less emphasis is put on teacher presentations of information more emphasis on teaching interviews as attempts to infer cognitive structures and modify them.

The computer can facilitate this type of process by shifting the focus of classroom activity from information transmission to knowledge construction. The burden of construction is placed with the student, with the help of the computer, so that the teacher then becomes a construction consultant. Much of class time for the teacher can be spent watching, diagnosing and suggesting learning strategies. The public nature of computer monitor encourages such consultation both between students and teacher and between students themselves.

The computer can relate new ideas to existing knowledge by facilitating the study of practical examples to which the student can already relate. The trajectory simulation, motion with resistance, and damped oscillatory motion are all examples of problems which can be discussed and are probably familiar to the student within the context of ordinary language, in Brauner's sense. This is likely to "provide a handle" and to help relate the new mathematical understanding to existing knowledge. This seems to deliver on Brauner's claim that ordinary language, and the standard perception it generates, is pivotal in assuring knowledge at the level of perception, and not just knowledge that consists of factual recall or concept mastery. Brauner argues:

> The key to the conversion of imagery back into linguistic form for use in thinking and communication is ordinary language .... To learn effectively they [students] need two things that are often not provided:

> 1. To be shown where new concepts stand in relation to familiar concepts, and categories of experience.
> 2. To be encouraged to form trustworthy ordinary language accounts of the key concepts and the categories of experience to which they belong.
>
> (Brauner, 1986, p. 19)

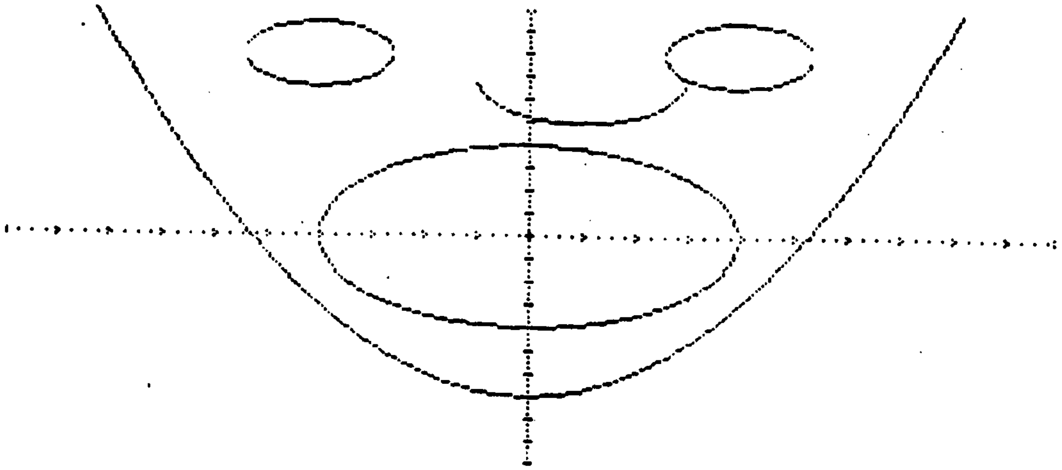
The computational power of the computer can also help to make the unfamiliar and imposing relate to existing experience and so make it less esoteric. A simple example is the use of quadratic functions and relations to make a face. Various properties of the algebraic representation of the function or relation can be mirrored in noses, eyes, ears etc. Mathematical variations can be examined in a more familiar context, that of moving, stretching and expanding (fig. 36). Problems could be set such as making a circular face, a parabolic face or an ellipical face.

The networking of ideas and experiences, facilitated by computer assisted applications, is likely to increase the meaningfulness of mathematics. The NCTM claims:

> Instruction that focuses on networks of mathematical ideas rather than solely on the nodes of the networks in isolation will serve to instill in students an understanding of, and appreciation for, both the power and beauty of mathematics. Developing mathematics as an integrated whole also serves to increase the potential for retention and transfer of mathematical ideas.

> Connecting mathematics with other disciplines and with daily affairs underscores the utility of the subject.
>
> (NCTM, 1989, p. 149)

```
Functions:
y=.25*x↑2-7
y=sqr(16-x↑2)
y=-sqr(16-x↑2)
y=-sqr(4-(x-1)↑2)+7
y=-sqr(2-(x-4)↑2)+8
y=sqr(2-(x-4)↑2)+8
y=sqr(2-(x+4)↑2)+8
y=-sqr(2-(x+4)↑2)+8
```



```
X-AXIS: -10...10 STEP 1
Y-AXIS: -10...10 STEP 1
```

Figure 36. Graphing a face

Alternate conceptions and conceptual conflict

Posner, Strike, Hewson and Gertzog (1982) have explained how for the constructivist, learning is conceptual change. Through active construction and accommodation with pre-existing knowledge, the learner changes the way he or she conceptualizes and so the way he or she perceives the world. Rather than keeping stimuli separate and simple, cognitive conflict should be created. Allowing activities to develop even if "errors" occur allows student's conceptions to compete for belief. The student will only learn if he believes the new conceptions because they work better. It is important for teachers to react constructively to "errors" and that they reward conceptual change and not just rote learning. The constructivist social contract in the classroom is not one that dwells on students doing what the teacher says but rather one that dwells on the student's obligation to construct meaning and the teacher's obligation to be a helpful consultant on the project. Constructivist teaching then depends on a trusting relationship. A teacher must trust students to solve problems and construct meaning. Students must trust teachers to respect their honest efforts. Given the often opposing tendencies of spontaneous student interest and the demands of the academic disciplines this trusting relationship will often be a difficult situation to create and so pose a constant challenge to constructivist teachers.

Because the computer can make calculations so rapidly students can be encouraged to investigate whatever possibilities they find plausible. This makes the program of conceptual change much more possible. Students don't need to rely on teacher authority. The example of the fundamental theorem of algebra problem is a case in point. Students can investigate for themselves and decide what to believe and why to believe it. There is no reason for the teacher to try to simply transmit information. There is every reason to aim at the construction of understanding through conceptual change. This is precisely the benefit of "plugging the computer" into the information processing system (Figure 3). The boost in processing ability empowers the student to assume responsibility for the meaningful construction of understanding.

Summary

The integrated use of the computer in mathematics education can help give more students more quantitative literacy than is presently the case. The information age understanding we need to move beyond computation and concept mastery to actual perception is the combined Perceptivist-Constructivist theory. The parts of the theory which are relevant to the problem addressed in this thesis center on five basic principles. First, instruction should be aimed at actual perception and use. Mathematics should not start and end with itself but must contribute immediately to our understanding of the world and how we perceive it. With the computer this application and use need not be delayed. Second, instruction should involve the synthesis of and interaction of all the student's knowledge on an ongoing basis, rather than its specialization, fragmentation and isolation. Only in this way can ideas be forced to compete for belief. Knowledge and understanding that is not believed is quickly forgotten. Third, instruction should actively involve the learner in the construction of meaning. Knowledge is something that learners must construct for and by themselves. There is no alternative. Discovery, re-invention or active reconstruction is necessary. The computational power of the computer makes this active involvement, in an abstract field such as mathematics, less of a scholarly dream and more of a realistic goal. Fourth, instruction should draw on and relate to existing experience in a way that taps existing knowledge and understanding. Meaningful, applied problems which relate to the experience of the student can be studied immediately with the computer because it can "crunch the numbers". Fifth, instruction should allow for the evolution and development of alternate and provisional conceptions.

With the computer taking care of the tedious, the student can explore many conceptions of things that might be competing for his belief, rather than simply relying on the authority of the teacher. Understanding for use means more than just recalling and repeating. It involves actually believing, on a non-inferential, perceptual basis.

These five principles will be operationalized in Chapter VI, a Perceptivist-Constructivist calculus unit. Chapter VII will report on field tests of the unit. Chapter VIII draws conclusions and makes recommendations.

CHAPTER VI

PERCEPTIVIST-CONSTRUCTIVIST CALCULUS ON THE COMPUTER

The computer makes the exploration and use of a numerical approach to calculus possible. "What you can do with it" becomes answerable right away. The theoretical derivation of algorithms, and associated manipulative techniques, can be left for college (i.e. later). The central ideas of calculus--limit, area under a curve, rate of change, and slope of a tangent line, can be tied to an existing notion of function. The emphasis of the program is providing students with a firm conceptual grasp of the uses of calculus rather than with the associated manipulative techniques.

Rationale

Mathgrapher, written by Steven Losse and published by Human Relations Media, and other computer graphing packages, have derivative, slope at a point, and area under a curve (definite integral) options. The slope is calculated using a secant approximation. The area under a curve is calculated from trapezoidal approximation. Problems from business, physics, psychology, sociology, economics, medicine and ecology can be shown to have solutions that involve definite integral or derivatives at a point. In this way we can attempt to build understanding rather just rote recall and so approach the NCTM standard addressed to calculus.

> This standard does <u>not</u> advocate the <u>formal</u> study of calculus in high school for all students or even for all college-intending students. Rather, it calls for opportunities for students to systematically, but informally, investigate the central ideas of calculus--limit, area under a curve, rate of change, and slope of a tangent line--that contribute to deepening of their understanding of function and its utility in representing and answering questions about real-world phenomena.
>
> The instructional approach to these concepts for all students, including college-intending students, should be highly exploratory and based on geometric and numerical experiences that capitalize on computer and calculator-based graphing utilities. The instructional activities should be aimed at providing students with <u>firm conceptual underpinnings of calculus, rather than with the associated manipulative techniques</u>. [emphasis added]
>
> (NCTM, 1989, p.180)

M. K. Heid remarks in a similar vein how symbolic manipulation programs at the college level offer the hope of calculus beyond mere computation.

> If students were allowed free use of symbolic manipulation programs such as muMath, calculus teachers could concentrate on development of concepts and as a result construct exams which test more than technique. The entire calculus curriculum would need to be rethought.
>
> (Heid, 1983, p.218)

Numerical integration proceeds on Mathgrapher by trapezoidal approximation. Infinite processes can be introduced by gradually increasing the number of trapezoids and seeing if the area approaches a limit (i.e. gets closer and closer to some specific value). Exercises follow that point out the geometric meaning of definite integral as area under a curve.

If the derivative of some function (F(x)) is denoted F′(x) then the area under the curve from a to b is the total amount of F from a to b. For example, if the velocity of an object, which is the derivative of distance, is given as $v(t) = 5t^2$ then total distance travelled by the object, from time a to b with the changing velocity, is $\int_a^b 5t^2dt$. The fundamental theorem of calculus is not even mentioned. Rather problems from business, physics, psychology, sociology, ecology and medicine <u>show</u> how computer graphs and definite integral give values of quantities on intervals, as the functions are changing value. What comes first is the actual perceptions and applications, with enough familiarity with symbolism to make the work of the computer meaningful. What will come later, when needed, is the justification, verification and formalization.

Notice how the above approach recapitulates the development of theory in mathematics and science. Initial insights, hunches and ideas are explored, refined, some abandoned, some generalized to gradually give shape to a finished product. The Perceptivist-constructivist approach to mathematics gives the student the opportunity, likewise, to work gradually toward a finished product. The Perceptivist-constructionist claim would be that such a process of "reinventing" the knowledge would be more likely to produce meaningful learning than giving the student the polished, refined, finished product outright and asking them to make sense of it.

Numerical differentiation proceeds on Mathgrapher by secant approximation. Infinite processes, as with numerical integration, can be introduced by gradually decreasing the size of the interval on which the secant is formed and seeing if the slope of the segment so far approaches a limit (i.e. gets closer and closer to some specific value). Mathgrapher does not have a visual process to depict numerical differentiation. Other packages could be used, or some could be done by hand, to display the geometric meaning of slope at a point ("definite derivative") as tangent.

The meaning of the mathematics is once again developed by using it. Problems from a wide range of disciplines show how rate of change is a useful concept. Minimum-maximum theory can be investigated simply by graphing, generating a table of values on an interval and looking for critical points. Finding the slope leads to the idea of slope 0 at critical points. By rooting the initial calculus experience in intuitively plausible activities, easily connected to previous knowledge and experience, the student can achieve a meaningful integration and organization of pivotal ideas. They are then more likely to understand, retain and apply the ideas rather than just memorize and then forget them.

Numerical differentiation

Calculus is concerned with finding areas under curves and slopes of curves. The former is called integral calculus, the latter differential calculus.

Mathgrapher finds slope at a point by secant approximation. Choose a point x and create a gap on each side, $x + \Delta x$ and $x - \Delta x$. The computer estimates the slope to be;

$$\frac{f(x + \Delta x) - f(x - \Delta x)}{2\,\Delta x}$$

Narrow the gap to get a better approximation. For the curve $y = x^2$ find the slope at different points (see Figure 37).

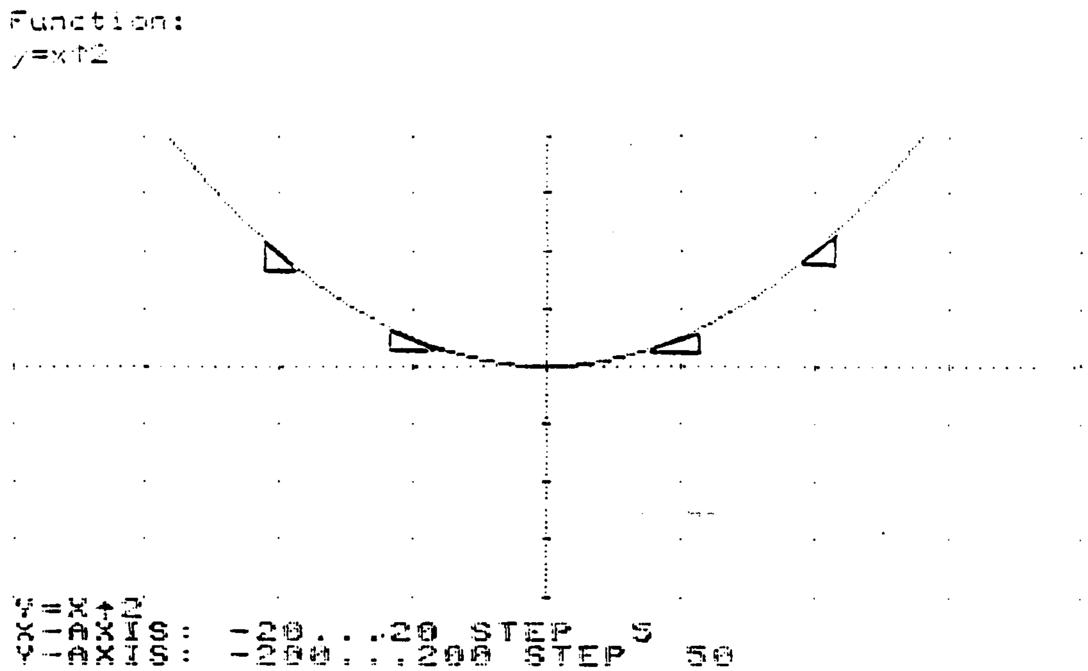


Figure 37. Slope by secant approximation

Is a steady answer approached as the secant x spread gets smaller and smaller? Numerical differentiation is based on 3 ideas

1. The slope of the tangent at P and the slope of the secant AB are approximately equal.
2. The smaller the value chosen for Δx, the secant x spread, the closer the slope of the

secant AB will be to the slope of the tangent at P.

3. If the Δx is small enough, the slope of AB will be close enough to the slope at P. The conventional notation for the problem above is $\frac{d(x^2)}{dx}$ = slope (some number at x = 5).

   This is called the slope at x = 5 (in a way the "definite derivative") and is a number. If a general expression for the slope of the curve at any point x is desired, then $\frac{d\,(x^2)}{dx}$, is called the derivative and is a function that gives the slope at any point x (as a matter of fact the derivative is 2x).

Doing differential calculus involves finding rates of change. Consider an example from medicine. A patient's temperature as a function of time is given by

$$y = -.1x^2 + 1.2x + 98.6.$$

Then at t = 3 is .6°/day, at 7 days, -.2°/day, at 10 days -.8°/day, as illustrated in Figure 38.

Calculus can be used in geography. A city's population starts from 100,000 and grows to an amount $P = 100{,}000 + 2000*t^2$. At time t = 5 years dP/dt is

20,000 people/year and at 10 years it is 40,000 people/year. This is illustrated in Figure 39.

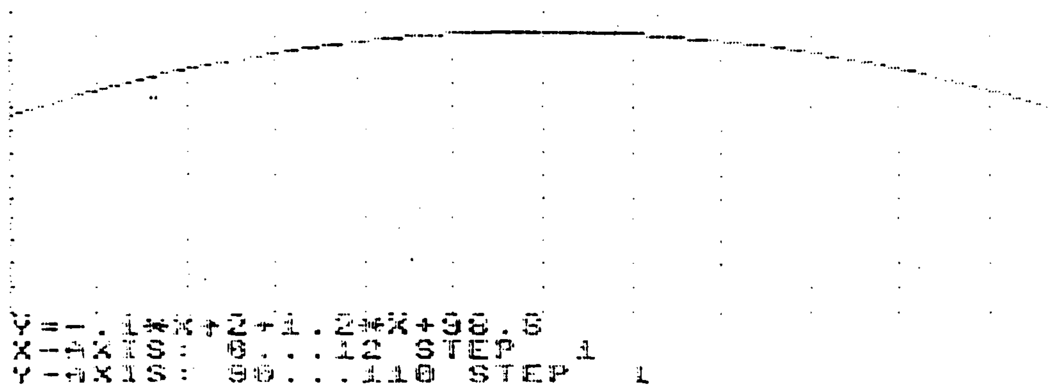


Figure 38. Time vs. patient temperature

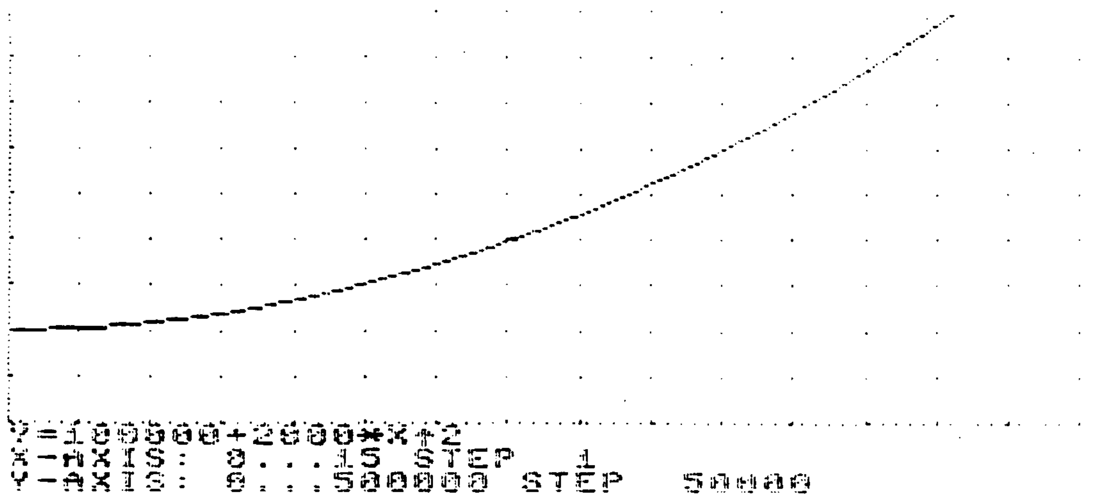


Figure 39. Time vs. city's population

Here is an example from business. A firm estimates it will sell n T.V. sets after spending x on advertising x in thousands of dollars. At $x = 100$ dn/dx is 100 T.V.'s/1000 advert, at $x = 200$ dN/dx is - 100 T.V.'s/1000, at $x = 300$ dN/dx is -300 T.V.'s / 1000. At some point advertising is a waste of money (see Figure 40).

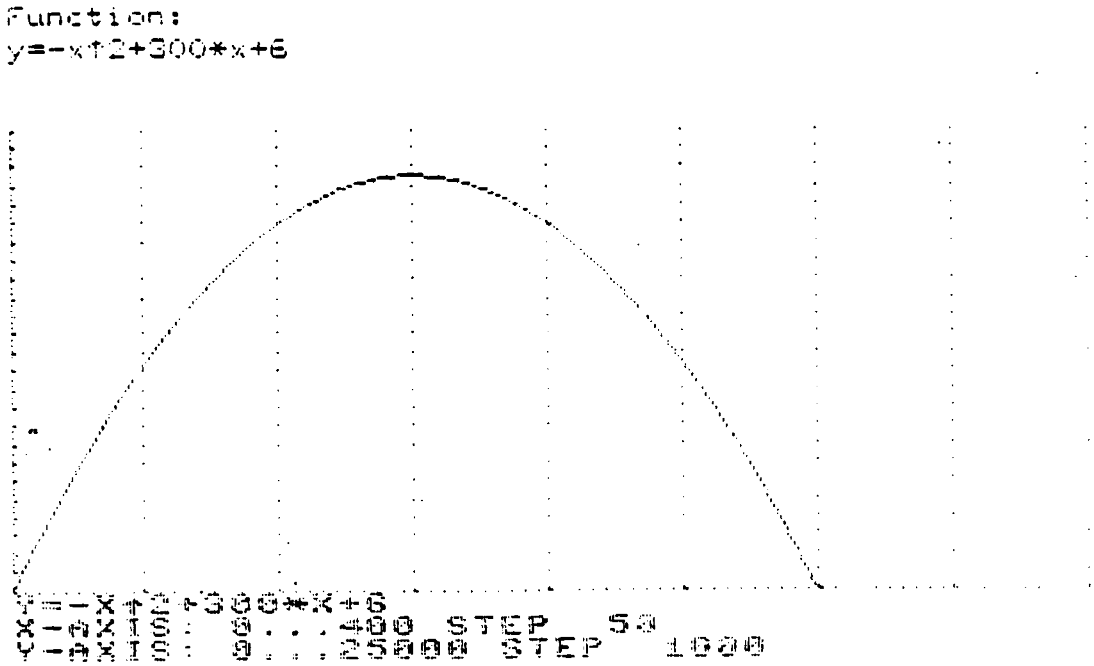


Figure 40. Dollars spent on advertising vs. T.V. sets sold

Consider another example from medicine. The percentage (expressed as a decimal fraction) of doctors who accept a new medicine is given in Figure 41. At 6 months dP/dx is 6% per month, at 12 months it is 1.8%

per month and at 18 months it .55% per month. The rate of change slows as the limiting value is approached.

Function:
y=1-exp(-0.2*x)

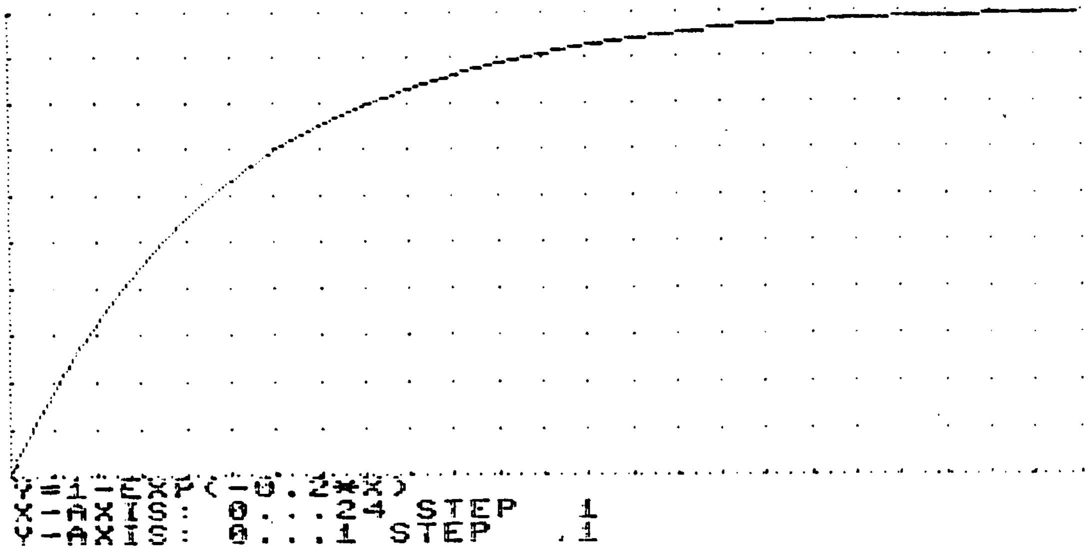


Figure 41. Time vs. fraction of doctors accepting a medicine

Geometry provides this example. Consider a 40 cm. long piece of string being used to form a rectangle of width x and length 20 - x so that area $20x - x^2$. Notice the maximum area is attained at x = 10. The slope there is 0. Optimization theory could be introduced and explored (see Figure 42).

Another example from business is the following. An up and down company finds that its sales during the X^{th} month of the year is given by the function y = 40,000 (sin(x) + cos(x)). Using the graph it can be found that

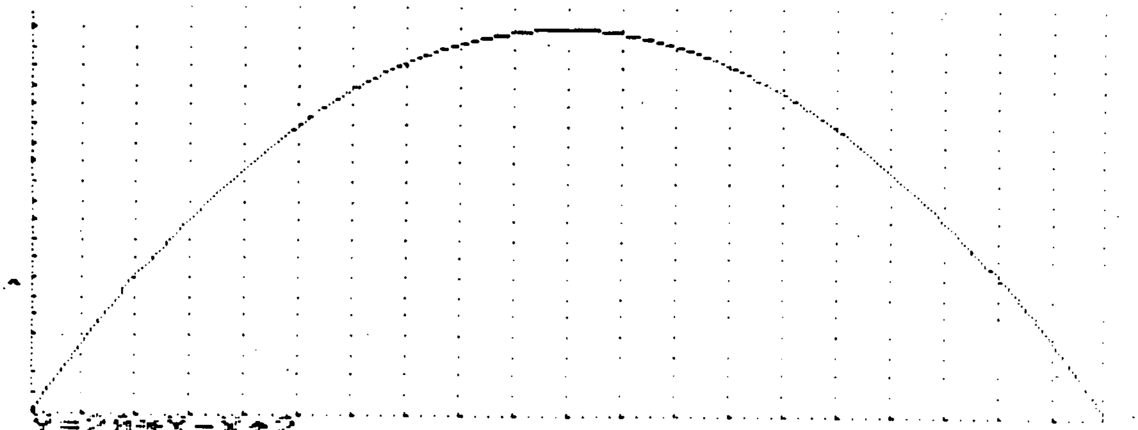


Figure 42. Width of rectangle vs. area of rectangle formed

S′ at x = 2 mo is -\$52930/mo

at x = 7 mo is \$3870/mo

at x = 8 mo is -\$45318/mo

at x = 12 mo is \$55125 /mo

The significance of positive, negative and zero slope can be discussed with reference to the calculus (see Figure 43).

The computer creates the possibility of having calculus come alive immediately with meaningful applications to which the student can relate. Calculus is then less a mystery and more a useful way of seeing the world.

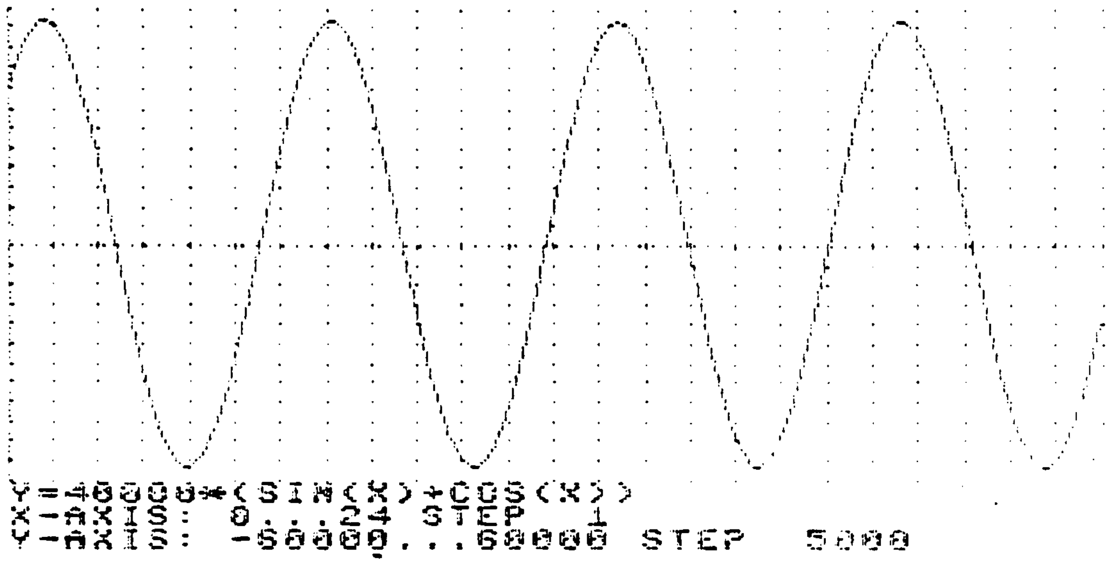


Figure 43. Time of year versus sales

Numerical integration

Consider the curve $y = x^2$ on the interval $0 \leq x \leq 10$. Use the integral option on the Mathgrapher program to find trapezoidal approximations to the area. Fill in the chart below and compare it with versions of Figure 44 generated by the computer.

No. Trapezoids	Area
2	______
5	______
10	______
15	______
20	______
25	______
30	______
100	______

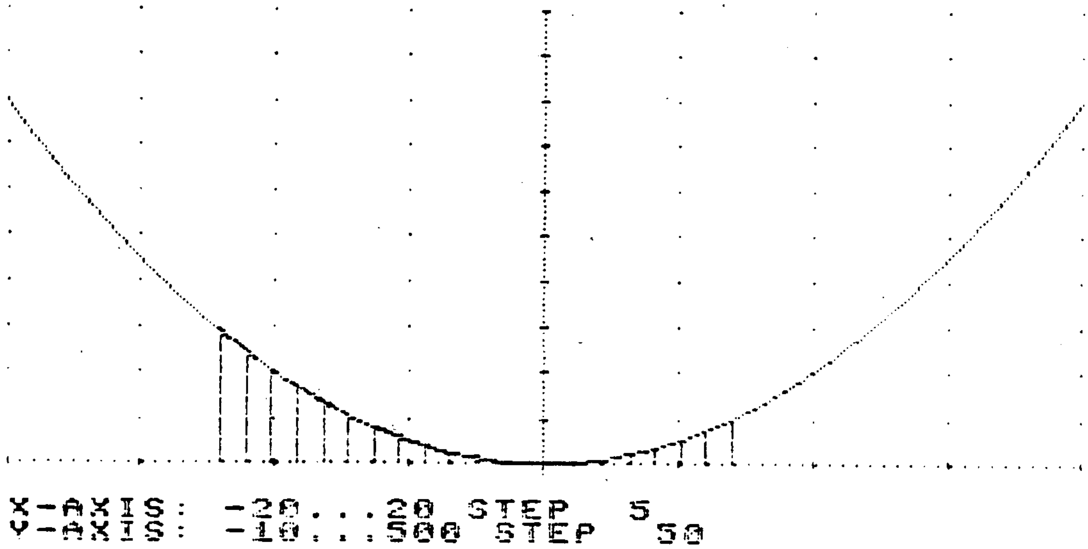


Figure 44. $y = x^2$

Is a steady answer approached as the number of trapezoids increases? Numerical integration is based on 3 ideas;

1. The sum of the areas of approximating trapezoids is approximately equal to the area under the curve.
2. The finer we make the divisions the closer we get to the actual area.
3. If we use a sufficiently fine division we will get as close to the actual area as needed to the for the application at hand.

The conventional notation for the problem above is $\int_0^{10} x^2\,dx$. This is called the definite integral from 0 to 10. It is a number, the area under the curve. If a general expression for an integral is desired, (i.e. $x^2\,dx$) this is called the indefinite integral and is a function that is a general expression for the area (as a matter of fact it is $Y = \frac{x^3}{3} + C$)

The area under the graph $y = x^2$ is found below with trapezoidal approximation. More trapezoids can be used to improve the approximation by forcing it to a limit.

$$\int x^2 {}^{7}_{-12} dx = 693.2.$$

Business can use ideas from integral calculus. A northern climate ski company determines its sales (S) of skis in x^{th} month is given by the function below, where S is in thousands of dollars for that month. The total sales for the year is given by,

$$\int_0^{12} 7*(1-\cos(\pi/6*x)) = \$84{,}000.$$ See Figure 45.

Consider an example of the application of calculus in physics. A particle moves in such a way that it keeps going faster and faster according to the equation $v(x) = 3x^2 + 2x$. The distance the particle goes

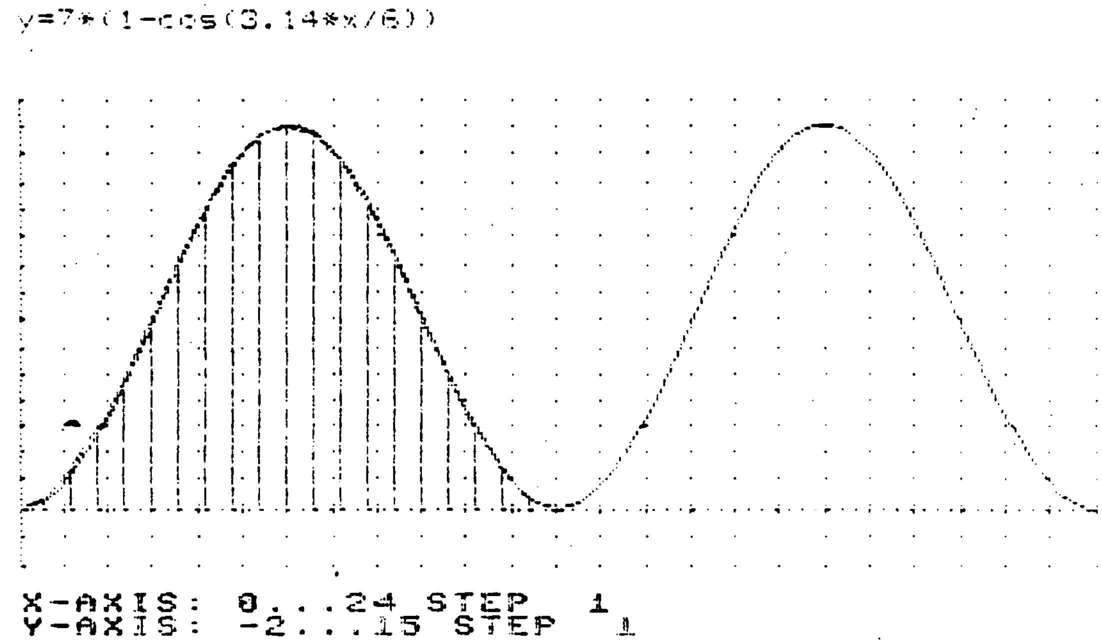


Figure 45. Month vs. sales in that month

between time 1 and time 2 is the definite integral. For example from $time_1 = 1$ to $time_2 = 5$ the particle goes,

$$\int_{x=1}^{5} 3*x^2 + 2*x \, dx = 148.$$ See Figure 46.

This procedure generalized allows for the computation of quantities determined by variables that are undergoing change through the course of the calculation--for example, distance as velocity changes, work as force changes, power as current changes.

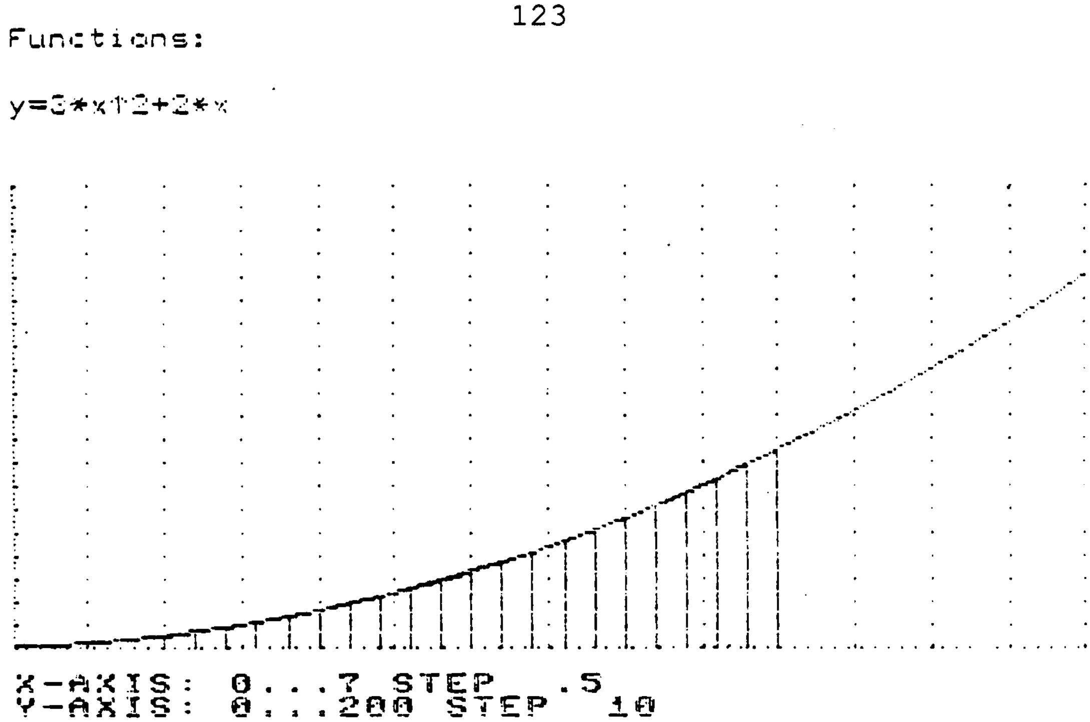


Figure 46. Time versus velocity

Another example of calculation of quantities in flux concerns the rate of use of electricity. A family uses electrical energy according to the equation below, where k is in kilowatt-hours, and x is time. During the first four hours the family uses;

$$\int_0^4 10*x*\exp(-x)\,dx = 9.1 \text{ kwh.}$$ See Figure 47.

For a whole day the family uses;

$$\int_0^{24} 10*x*\exp(-x)\,dx = 9.98 \text{ kwh.}$$ See Figure 48.

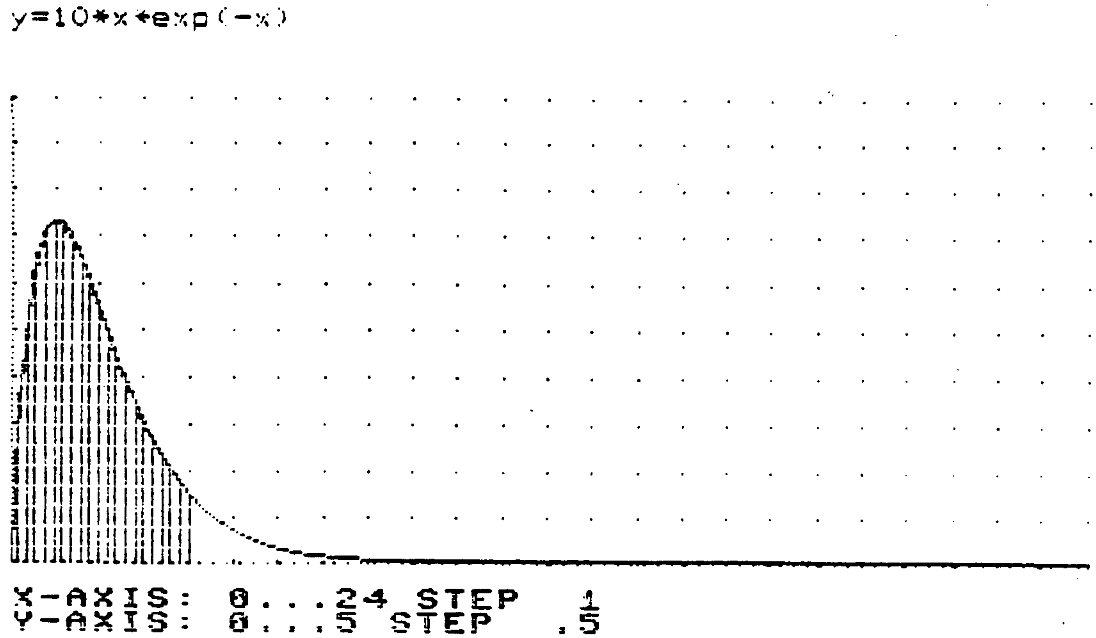


Figure 47. Time versus Rate of Energy Use: 4 hours

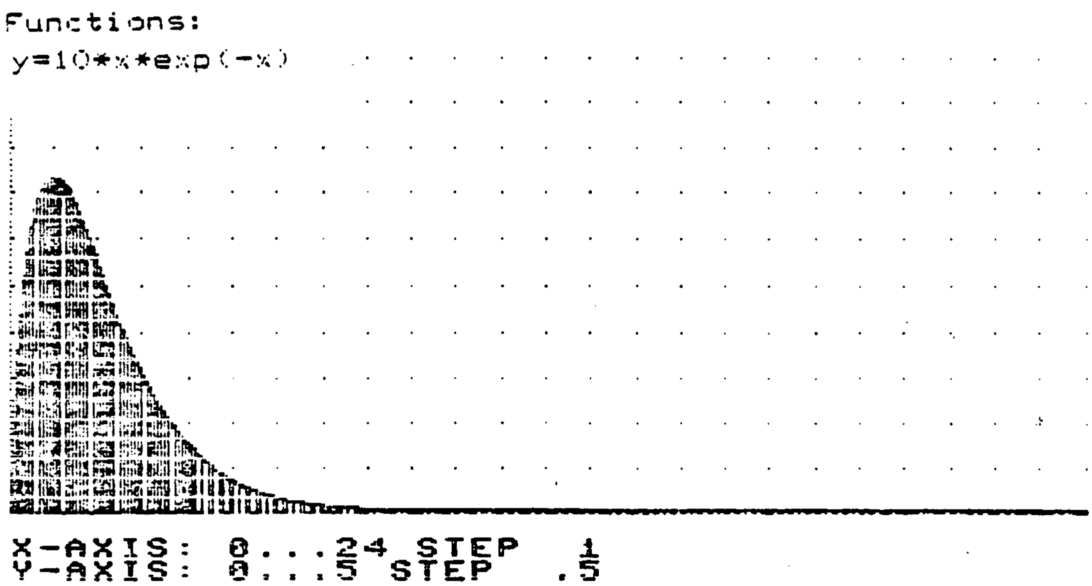


Figure 48. Time versus Rate of Energy Use: 24 hours

A whole web of questions can be addressed to connect the calculation to reality.

1. why the use distribution as it is?
2. what implications are there for the power company?
3. is the model realistic? .....

Consider an application of the integral from geometry, the area between the curves. With regard to the geometric idea of area between two curves, consider the area between the two curves below. The area from x = -2 to x = 3 is given by;

$$\int_{-2}^{3} (x^2+3)-x^2 = 15$$

The procedure is general and the computer takes care of the computation (see figure 49).

The average value A of a continuous function f is given by $\dfrac{\int_a^b f(x)\,dx}{(b-a)}$. This corresponds to some value of the function such that the rectangular area A*(b-a) is the area under the curve from a to b. For example, the following situation could occur in the field of medicine. The amount A of drug in a patient's bloodstream at time x after a dosage is administered is given by the function $A = 3e^{-.86x}$, where A is in milligrams per millilitre. The amount of drug in the

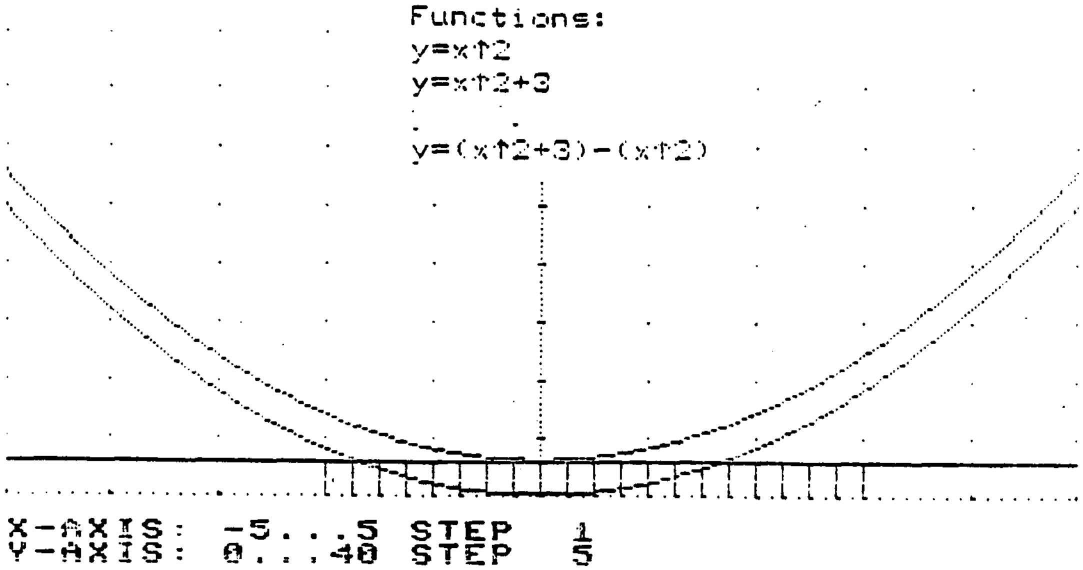


Figure 49. Area between two curves

body is going down as the body metabolizes the drug. The average amount of the drug in the body over the interval [a,b] is given by

$$\int_a^b \frac{3*\exp(-.86*x)\ dx}{(b-a)}$$

Over the first two hours the average is

$$\frac{\int_0^2 3*\exp(-.86*x)\ dx}{2} = 2.87/2 = 1.435 \text{ ml (Figure 50)}$$

Consider another example for geometry. If a plane figure is rotated about the x axis in space, a solid is formed. The volume of the solid of revolution is given by

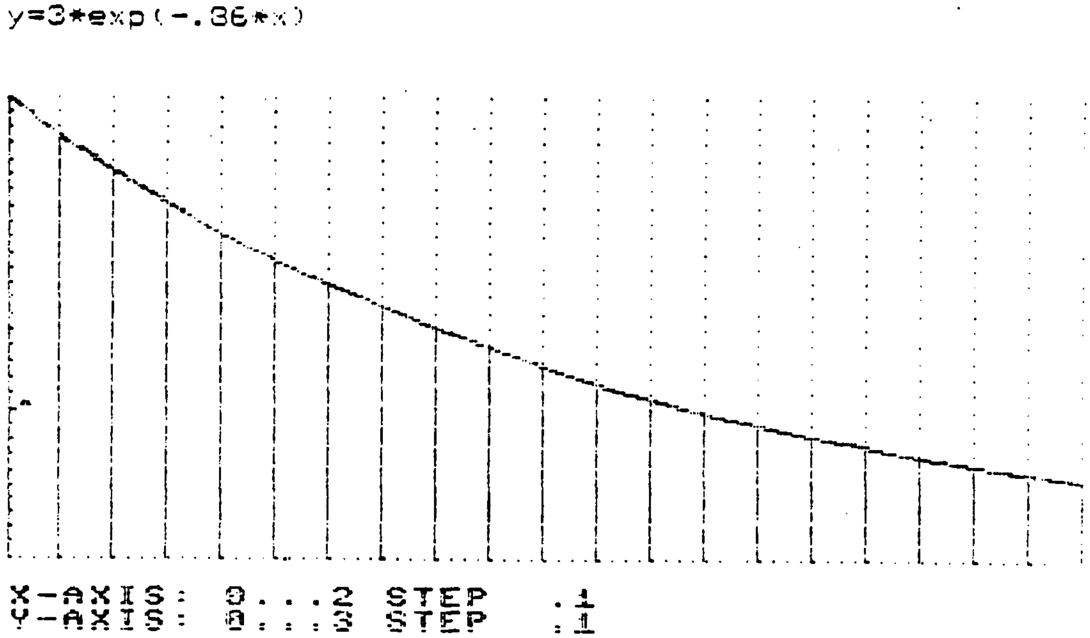


Figure 50. Average value of a function

$$\int_{a}^{b} \pi * [f(x)]^2 \, dx,$$

where a, b are the bounds of the plane region rotated. For example if the curve y = exp(x) between x = 1 and x = 2 is rotated about the x axis the shape generated has volume, given by

$$\int_{-1}^{2} \pi * [\exp(x)]^2 \, dx = 86.1.$$ See Figure 51.

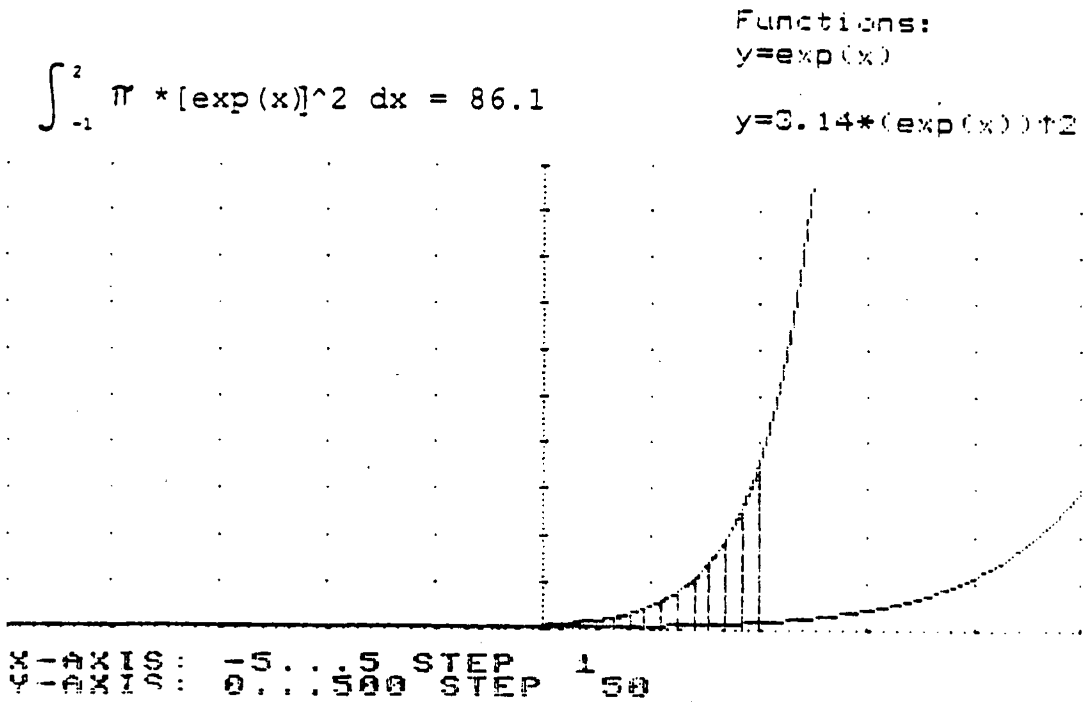


Figure 51. Volume of revolution

Summary

The examples in this chapter have shown that the use of the computer represents the possibility of

1. turning the mathematics curriculum on its head.
2. providing powerful techniques for the use of all students (not just the 10% who meaningfully complete Algebra 12 now).
3. introducing meaningful applications of mathematics immediately.

The computer aided, numerical approach to calculus described in this chapter is aimed at the following:

1. Enabling students to see the world through mathematical eyes and to perceive mathematics as useful intellectual training.
2. Engaging mathematical ideas with ideas from other disciplines and with ideas in the student's active cognitive repertoire to combat the tendencies of fragmentation, isolation and alienation of knowledge from the working beliefs of the learner in daily life.
3. Actively involving the learner in the construction of knowledge by offering meaningful examples, reducing the emphasis on brute calculation and increasing student independence and experimentation.
4. Networking new knowledge to past experience through applications with a view to integrating, rather than further fragmenting knowledge.
5. Allowing the construction and acceptance of new knowledge to take place through numerical, concrete experience and experimentation. New knowledge gradually becomes believable, rather than just memorizable, as it is used, tested and elaborated upon by the learner. Finally, rather than initially, the knowledge can be formalized.

The perceptivist-constructivist theory of computer integrated mathematics provides the insight needed to start working systematically toward the realization of the kind of calculus program suggested by the NCTM standards document, which in part suggests:

> Computing technology makes the fundamental concepts and applications of calculus accessible to all students. The area under a finite portion of a curve, for example, can be approximated geometrically by partitioning ...
>
> All students could use a graphing utility to investigate and solve optimization problems, including the maximum-minimum problems traditionally associated with the first college-level course in calculus, without computing a derivative. ...
>
> Using interactive graphing utilities, college-intending students could examine other characteristics of the graphs of functions, including continuity, asymptotes, end behavior (i.e. behavior as x -> ∞) and concavity ...
>
> Computing technology also permits the foreshadowing of analytic ideas for college intending students. From a computer-graphics perspective, for example, a differentiable function can be viewed as a function having the property that a small portion of its graph, when highly magnified, approximates a line segment....
>
> Instead of devoting large blocks of time to developing a mastery of paper-and-pencil manipulative skills, more time and effort should be spent on developing a conceptual understanding of key ideas and their applications. All students should have the benefit of a computer enhanced

> introduction to some of the types of problems for which the calculus was developed.
> (NCTM, 1989, pp.182-183)

The realization of this program is facilitated to the extent that perceptivism can show what to do, constructivism can show how to do it and the computer can show how to make the computational complexities manageable. The student becomes actively involved, through real life applications, in the process of seeing the world in terms of the calculus. This represents more than schooling--it represents education.

CHAPTER VII

REPORT ON A FIELD TEST OF THE CALCULUS UNIT

The aim of this chapter is to report on field tests of the Perceptivist-constructivist calculus unit outlined in Chapter VI. The unit was given to 41 grade twelve students at Templeton Secondary School in Vancouver, B.C., from September 1988 to April 1989. This chapter will first discuss the purpose of the field tests. The instructional sequence given to the students will then be outlined. Following this, teacher notes and student questionnaire results will be summarized and discussed in terms of the extent to which they suggest Perceptivist-constructivist principles were realized in the classroom.

Purpose of the field test

The major goal of this paper has been to show how Perceptivist-constructivist principles can inform the effective utilization of the computer to improve the meaningfulness of mathematics education in secondary schools. Perceptivism suggests that curriculum and instruction must aim beyond mere factual recall and concept mastery to actual perception and use. The philosophy also suggests that students must become proficient in the discriminate selection and skilled use of the different modes of perception. Constructivism suggests that students must become actively involved in the construction of meaning, that they must integrate past and present experience in an ongoing, meaningful way and that students must challenge new concepts to survive competition for belief and use with older, established concepts. The purpose of the field test was to determine the extent to which the teacher and students could see the theoretical principles being realized in the computer mathematics classroom.

The author designed an instructional unit in calculus, based in part on applied calculus exercises adapted from Bittinger and Morel (1982). Notes were kept by the author and a student questionnaire was given to determine if the teacher and students could see a valid experimental Perceptivist-constructivist treatment taking shape in the classroom. This, in turn, could set the stage for later experimental work in which the performance of a computer assisted experimental group could be compared with that of a noncomputer control group. This study stops short of this comparison. M. Kathleen Heid (1988) has in fact made a comparison type study of performance using the computer as a tool to teach applied calculus. Although no mention of Perceptivist-constructivist principles is made, her program meets most of the criteria set out in the five principles. Heid summarizes her study of the effect of computer motivated resequencing of skill and concept development on student performance in the following manner:

> During the first 12 weeks of an applied calculus course, two classes of college students (n=39) studied calculus concepts using graphical and symbol-manipulation computer programs to perform routine manipulations. Only the last 3 weeks were spent on skill development. Class transcripts, student interviews, field notes, and test results were analyzed for patterns of understanding. Students showed better understanding of course concepts and performed almost as well on a final exam of routine skills as a class of 100 students who had practiced the skills for the entire 15 weeks.
>
> (Heid, 1988, p.3)

Instructional sequence

The usual topics in applied differential and integral calculus were presented to the students. They worked on slopes, rates of change, minimum-maximum, area under a curve, area between two curves, average value of a function, volume of revolution and antiderivative applications. A major difference from traditional, noncomputer courses was the "concepts first" approach to curriculum and instruction. The teaching of concepts, and the perceptual possibilities they open up, preceded the teaching of routine calculation skills in sequence as well as priority. Problems from a wide range of disciplines were used to develop student understanding of the important concepts. Graphing and calculations were done by the computer. Set up, reporting and analysis were done by the student. Students were later shown how to handle only polynomial functions by hand. Hand-in assignments and exams were used for evaluation.

Instruction could more immediately involve the student in problem solving and seeing the ideas of calculus in action. This was because:

1. Computers facilitated computation, especially graphing, differentiation and integration.
2. Computers easily provided data for discussion so that the development of ideas didn't need to take place in the abstract.
3. Computers made realistic problems manageable through information processing such as parameter changing, generating tables of values, equation solving and curve fitting.

Formal, abstract ideas like limits were delayed until some intuitive understanding is present. Students were actively involved in exploration from the start because they did not need to simply rely on the teacher to transmit knowledge. New concepts were developed, merged and assimilated with old as the computer manages the technical calculations.

It might be argued that giving students functions on which to operate and then having the computer do the calculations is likely to leave the students with a sense of magic and mystery as bad as that generated by the blind manipulation of formulae. Indeed it might be preferable, in the best of all possible worlds, to derive all functions from first principles, or at least using regression. But this approach is counterproductive with beginners due to its complexity. The goal here was to proceed as directly as possible, past factual recall and concept mastery, to actual perception. The temptation to proceed using a premature formalism and insincere promise or relevance later has demonstrated its shortcomings. The balance must be shifted back toward relevance, meaning and perception first with formalism, for those who need it, to be delivered later.

Field notes

The author regularly collected field notes while walking around the classroom observing and helping students. The results of these notes are summarized here. The results should be interpreted as the substance of an hypothesis generating study. What is offered is a preliminary set of observations, which may be confirmed or not by future, neutral observers.

Because the computer did the calculations, students got right to solving problems. The teacher spent little time instructing formally. Most of the teacher's time was spent observing and helping individual students with problems. Formal lessons were usually less than ten minutes in length. Freed from the duty of lesson presentation the teacher can become more involved in the learning process with the individual, offering specific help at required times. Observation suggests that the teacher will spend the majority of class time engaged in this activity. Rather than "something-for-everybody" lessons, teaching was directed to the overcoming of particular obstacles to learning. Because the teacher was actively involved in the actual learning process as it occurred, formative evaluation was an ongoing process.

Perceptivist principles were observable as translating into practise in the classroom. Students were observed discussing temperatures, populations, profits, etc., as these ideas related to the relevant mathematics. One student described the course as "easy to relate to because of the use of actual things, for example, divorces, memorization, dosage of medicine." Another student explained, "It was much easier to catch onto the concept initially when we could visualize the problem. The applications to other areas like economics, business, population, especially helped in the understanding process." On one occasion, two students were observed talking about the difference between the population of a city and its rate of change. The nature of the problems that could be set demanded a perception of what was going on before the relevant mathematics was brought to bear. The problems also demanded students engage different ideas to solve a problem, ideas from ordinary language, science, mathematics, economics, etc. The trajectory, for example, sparked numerous discussions about footballs, streams of water, etc. and the ideas needed to explain their behavior such as velocity, path, distance, arc, etc. The learning situation then became cooperatively experimental as students tried out their ideas to see if the computer would verify their hunches. Some students felt guilty. One a student remarked, "I found it too easy to rely on the computers for many questions when I could easily have done the calculations in my head."

Constructivist principles could also be observed emerging in practise. Independent learning through active construction was promoted in that little time was spent giving information. Because problem solving was student directed, premature formalism was not a concern. A student remarked, "It was beneficial because it encouraged students to think for themselves." For example, with the exercise concerning the fundamental theorem of algebra, the problem was simply set with the only hint being--use the rational roots theorem to work backward from the roots to the equation. The calculating power of the computer then empowered student initiated investigation. One student explained, "The computer was a great teacher aid. By playing with the computer, I did not know that I was actually learning calculus." (The computer made learning of calculus easier.) The same situation applied with graphing curves to make (see Figure 36) faces. Ideas can be tried out; the computer shows whether they work. Past experiences were drawn upon and concept assimilation was promoted through dialogue. A cooperative construction of meaning was observable as students reflected on computer calculations. One student commented, "The students can draw the graphs on the screen easily. Then we can spend more time to think about what the information that the graphs is (sic), instead of waste time to draw the graph." In function modelling exercises, for example, past experience could be drawn on and conflicting views simulated on the computer. Students were frequently observed discussing details of problem situations.

Questionnaire results

The questionnaire (Table 5) was given to the students at the end of the unit. The questions aim at determining the degree to which students are aware of the five basic principles of the Perceptivist-constructivist program being realized in practise. (see pp.105-107) The students had no explicit knowledge of the principles themselves, however. Question 1, 2, 3 and 17 concerned actual perception, principle 1. Questions 4, 5, and 18 concerned synthesis of knowledge, principle 2. Question 6, 8, 9 and 7 concerned active construction of knowledge, principle 3. Questions 11, 12, 10 and 19 concerned relation to past experiences, principle 4, and questions 13, 14, 15, 16 and 20 concept assimilation and accommodation, principle 5. Thirty student replies were received.

A) Analysis of mean response to each question

The mean response to each question was calculated together with the standard deviation and t-score (critical value 2.16). The questions are given in Table 5. Results are summarized in Table 6. H_0 was that student response was a neutral 3. There seems to be some evidence to suggest that students see principles one, two and five as being partially realized in practise. The null hypothesis seems acceptable however with regard to principles 2 and 3.

Consider the items for principle one, actual perception (see Table 6). H_0 was rejected for each of items 1, 2, 3 and 17. The alternative is accepted for each item and consequently for the group. Thus it is accepted that the students realized the principle of actual perception.

For principle two, synthesis of knowledge, H_0 was rejected for each of the items 4, 5 and 18. Thus, it would appear that this principle was realized by the students. This provides evidence, together with the results concerning principle one that perceptivist principles were realized in the classroom. In particular, it provides some evidence that the computer facilitated the setting and solving of meaningful calculus problems which promoted actual perception and encouraged interdisciplinary thinking.

For principle three, active construction of knowledge, H_0 was accepted for items 6, 8 and 9 while the null hypothesis was rejected for item 7. Students did not realize this principle. This was disappointing in the face of the goal of this unit. Apparently, the "black box" aspect of the computer was bothersome to some students who felt guilty or short changed by not actually doing the calculation for themselves (see field note p.138, for example). This suggests further work might need to be done in reaching a compromise between computer calculation and manual calculation.

For principle four, relation to past experience, H_o was rejected for items 10 and 19 but accepted for items 11 and 12. The evidence here is mixed. The positive results on items 10 and 19 do suggest students were able to relate to the work the negative results on items 11 and 12 may suggest they don't realize why.

For principle five, concept assimilation and accommodation, H_o was accepted for item 13 but rejected for items 14, 15, 16 and 20. The preponderance of evidence indicates the principles were realized. Of special importance is the evidence that students seemed to feel that the computer allowed them to become involved with ideas without relying so much on the authority of the teacher. Evidence for principle five, together with evidence for principles three and four to a lesser extent, suggest constructivist principles were partially realized in the computer classroom.

Table 5
Student Questionnaire - Computer Calculus Unit

Directions

1) Please circle the number of the response which most closely matches your agreement or disagreement with the statements as you understand them.
2) There is definitely no right or wrong, favourable or unfavourable response - just your honest opinion counts.
3) Please think about and review to yourself what happened in the computer classroom. Base your responses on what you recall actually happened in the computer classroom.

Questions

1) There were many real world examples used to show the uses of calculus.
2) The questions used made me think about how situations would actually look if I were to experience them myself.
3) The computer took care of most of the calculations for me.
4) The computer questions made me use knowledge from different sources e.g. physics, business, common sense.
5) When I could get problems that required knowledge from different sources it helped me understand the mathematics.
6) It was easier for me to get involved in learning calculus with the help of the computer.
7) I found the calculus unit boring. I couldn't relate to it.
8) When the computer did the calculations I could concentrate on what the results meant.
9) Not knowing the details of how the computer made its calculations left me with an incomplete understanding.
10) I didn't enjoy the calculus unit because it was too abstract and incomprehensible.
11) I had to draw on knowledge I already had to answer many of the calculus problems.
12) Mathematics is no more nor less comprehensible with the computer than without it.
13) The teacher spent a lot less time, compared to a regular classroom, making class presentations, lectures etc.
14) I worked more on my own or in student groups in the computer classroom instead of relying on the teacher for instruction.
15) The computer encouraged me to discuss things more with other students to try to figure out what was going on.
16) The use of the computer in solving different problems helped me understand and believe in calculus rather than accept it on the teacher's authority.
17) Calculus does not have any uses.
18) Calculus is not related to any subject areas outside of mathematics.
19) Calculus is not useful in relating things in my life past, present or future.
20) By introducing different subject areas into calculus problems I got confused. It didn't help me understand.

Responses

	Strong Agree	Agree	Neutral	Disagree	Strong Disagree
1)	5	4	3	2	1
.					
20)	5	4	3	2	1

Table 6
Analysis of mean response to each question
(critical t=2.16)

Question Score	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Mean	4.0	3.6	4.4	4.0	3.5	3.3	2.4	3.2	2.9	2.4	3.4	3.1	3.1	3.7	3.6	3.5	2.0	1.7	2.3	2.4
S.D.	.7	.9	.6	.7	.8	1.3	1	.9	1.4	1.1	1.1	1	.8	.8	.8	1	.9	1	9	1
t6.49	3	10.76	3.26	2.77	1.03	-2.72	1	-.32	-2.4	1.6	.45	.55	3.88	3.33	2.77	-6.25	-7.22	-4.38	-3.33	
H_o/H_1	H_1	H_1	H_1	H_1	H_1	H_o	H_1	H_o	H_o	H_1	H_o	H_o	H_o	H_1	H_1	H_1	H_1	H_1	H_1	H_1

B) Analysis of responses by individual students

Positive responses (4-5) on questions 1, 2, 3, 4, 5, 6, 8, 9, 11, 12, 13, 14, 15, 16 (group 1), positively stated questions, indicate the realization of principles in practise. Negative responses to 17, 18, 7, 10, 19, 20 (group 2), negatively stated questions, also offer evidence for the realization of principles. Students scores were analyzed on this basis, the results are summarized in Table 7. H_o, the null hypothesis, is that a student's response is a neutral 3. H_1 is that student response is not neutral. Ten student's scores were H_1 on both positive and negative questions. Six students had H_1 on positive but H_o on negative. Eight students had H_o on positive and H_1 on negative. This latter group of fourteen students gave some indication of seeing Perceptivist-constructivist principles being realized in practise. Six students scored H_o on both group of questions. As tenuous as the results are, there is some evidence to suggest that some students see Perceptivist-constructivist theoretical principles being realized in the computer classroom.

There is, in summary, tentative evidence for the realization of some perceptivist-constructivist goals in the computer-mathematics classroom. Field notes suggest that a teacher should be able to observe the realization of some of the five principles in the classroom. The questionnaire results indicate that students in total see some principles realized and that a majority of individual students see perceptivist-constructivist principles partially realized in the computer-mathematics classroom.

Table 7
Analysis of responses by individual students (critical t=2.57)

Score / Student	Group 1 Mean	S.D.	t	H_o/H_1	Group 2 Mean	S.D.	t	H_o/H_1
1	3.8	.7	4.28	H_1	2	.6	-4.0	H_1
2	4.0	.8	4.76	H_1	3	.8	0	H_o
3	4.0	1	3.7	H_1	1.6	.7	-4.83	H_1
4	3.2	1.2	.63	H_o	2.6	.5	-2	H_o
5	3.3	.9	1.25	H_o	1.5	.8	-1.52	H_o
6	4.0	.9	4.16	H_1	1.5	.5	-7.5	H_1
7	4.0	.9	4.16	H_1	1.5	.5	-7.5	H_1
8	4.0	1	3.7	H_1	1.1	.4	-11.88	H_1
9	3.3	.9	1.25	H_o	1.8	.7	-4.13	H_1
10	3.7	.8	3.33	H_1	3.0	.6	0	H_o
11	3.2	1.1	.69	H_o	1.6	.7	-4.8	H_1
12	3.4	.7	2.1	H_o	3.8	.4	5.0	H_o
(contradict)								
13	3.5	.8	2.38	H_1	2.1	.7	-3.1	H_1
14	3.5	.8	2.4	H_1	3.0	.6	0	H_o
15	4.2	.6	7.5	H_1	2.1	.9	-2.43	H_1
16	4.2	.6	7.5	H_1	2.5	1.1	-1.1	H_o
17	3.2	1.1	.69	H_o	1.5	.5	-2.5	H_1
18	3.5	.7	2.63	H_1	2.1	.4	-5.62	H_1
19	3.3	.7	1.57	H_o	2.0	0	-2.50	H_1
20	3.6	.6	3.75	H_1	2.6	.7	-1.37	H_o
21	3.5	.7	2.63	H_1	2.5	.8	-1.52	H_o
22	3.5	.5	3.84	H_1	2.1	.4	-5.62	H_1
23	2.7	1.1	-1.03	H_o (Contradict)	1.6	.7	-4.82	H_1
24	3.2	1	.74	H_o	2.3	.7	-2.41	H_o
25	3.9	1.1	3.10	H_1	1	0	-5.0	H_1
26	3.7	1.3	2.00	H_o	1.5	.8	-1.5	H_o
27	3.4	.9	1.67	H_o	2.5	.8	-1.5	H_o
28	3.2	1	.74	H_o	3.6	.5	+3.00	H_o
(contradict)								
29	3.1	.6	.62	H_o	3.5	.4	3.12	H_o
(contradict)								
30	3.4	.9	1.66	H_o	2.3	.9	-1.89	H_o

CHAPTER VIII

CONCLUSION AND RECOMMENDATIONS

The main thesis of this paper has been that a major way to improve secondary mathematics education is to translate Perceptivist-constructivist principles into practise by emphasizing curriculum that actively involves the learner as constructor of knowledge and that leads the learner to actual perception. The traditional problems with this type of pragmatic, utilitarian agenda have been that a large amount of information transmission, from teacher to student, was assumed necessary and that the computational difficulties inherent in perceptions of reality were overwhelming. The information age technological innovation that mitigates the effects of both of these problems is the personal computer with appropriate software. Theoretical arguments, practical examples and a report on a field test have been provided in support of these claims.

In this final chapter conclusions will be stated and discussed. The conclusions focus on a) the five basic principles of a Perceptivist-constructivist computer integrated mathematics program and b) the extent to which the principles were observed as translating into practise by a teacher and students in a grade 12, computer calculus field test. Suggestions for further research will follow. Once organized, informed computer usage is practical in secondary mathematics, many questions suggest themselves. A selection of ten such questions, will be discussed. Finally, some recommendations will be offered. These recommendations are in three areas of concern. (a) Professional development must equip teachers to deal with new classroom complexities. The competent professional must be able to deal with philosophy, learning theory, mathematics and computer technology. (b) Organized programs should be developed to systematically exploit the computer in secondary mathematics. Random or "friday fun" usage wastes a valuable resource and is unlikely to do much good. (c) Finally, boards and ministries of education need to rethink curricula in terms of active construction, constructivism, and actual perception, perceptivism. In the long term, the entire curriculum needs to be redone along Perceptivist-constructivist lines.

Conclusions

The writer of this paper has shown how a philosophy of education, perceptivism, a theory of knowledge acquisition, constructivism and their joint application to the problem of computer usage in secondary mathematics, can suggest ways to improve secondary mathematics education. Detailed examples of how the theory might actually translate into practice have been given. Five basic principles are developed in the paper.

1. Aim at actual perception of real world situations. In so doing the learner is encouraged to get beyond factual recall and concept mastery to perception with, and use of, concepts. This process goes beyond mere academic schooling to utilitarian education in that knowledge becomes applied. The computer facilitates this completion of the educational process in mathematics by managing the computational complexities. Refer to Figure 1 in Chapter I, for example.
2. Promote synthesis and interaction of knowledge domains. By forcing different modes of perception, and the languages that symbolize them to interact, a fluid dynamic conceptual repertoire is encouraged. Rather than having a specialized, fragmented academic approach to knowledge the learner works toward an eclectic, integrated, utilitarian one. The learner manages information by choosing from and applying the best of a collection of well tested conceptual systems, as

the situation dictates. The computer facilitates this process by boosting information processing capability. Refer to Table 1 and Figure 3 in Chapter II for an example.

3. Encourage the active construction of knowledge by the student rather than its passive reception. Active reconstruction, re-invention or rediscovery is not optional for understanding, it is necessary. With the computer, relevant practical applications of mathematics are manageable at once so that the student can relate to the concepts more immediately and begin reconstruction, re-invention and/or rediscovery. Computer handling of algorithms can make students less dependent on the teacher and more dependent on their own resources. Refer, for example, to Figure 36 graphing a face in Chapter V,

4. Facilitate the incorporation of past knowledge and experience into the learning situation. By so doing the teacher encourages the student to draw on knowledge from ordinary language or other domains. This process exploits existing competence, promotes the integration of experience and so empowers students to use all their

abilities. Refer, for example, to the trajectory project in Chapter IV.

5. Allow for conceptual conflict and accommodation to help clarify ideas. By plunging into applications right away, the computer enables students to explore different ideas as their understanding is taking shape. Competing ideas can vie for belief. The computer is exploited here to carry out this doing-is-believing exercise. The computer empowers students to do things with concepts and so to believe in, retain and reuse them. As an example, see the fundamental theorem of algebra exercise in Chapter V.

Evidence has been supplied to indicate that the theory can actually be translated into practise. Details of a computer calculus field test has been given and discussed. The evidence here is tentative and with limitations but does give some preliminary support to the hypothesis that the theory can actually be translated into practise. The program's lessons and instructional techniques need to be developed further. What the writer has promoted is the idea that theory can help practise and that there is reason to believe that theoretically motivated suggestions work in the classroom.

Suggestions for further research

Many questions suggest themselves as the possibility of extensive computer usage in the mathematics classroom becomes a real possibility. Certain theoretically motivated claims have been made that would need empirical verification. For example, more field work is needed to provide evidence relevant to answering the following questions.

1. Can the computer shift mathematics away from computation toward application? Can this be done without lowering performance levels? Heid (1988), for example, has shown how first year computer calculus students do almost as well as traditionally instructed students on basic skills and better on conceptual understanding. Further research needs to be done to determine the benefits of computer use and the trade-offs computer usage may necessitate.
2. By introducing interfering, competing ideas, does use of applications clarify or confuse? Do students benefit from conceptual conflict and competition involved in applied computer mathematics or do they simply suffer information overload? It may well be necessary to differentiate curriculum and instruction.

Different ability levels may tolerate different levels of conceptual conflict.

3. Can teachers develop and teach an applied mathematics curriculum? Under what conditions can teachers be prepared to handle the philosophy, instructional theory, computer proficiency and understanding of applications needed to teach Perceptivist-constructivist mathematics? The adequacy of present teacher, present teacher training and in-service programs needs to be analyzed.
4. Whenever results favourable to Perceptivist-constructivist computer mathematics are obtained, do they represent genuine substantive improvement or just Hawthorne effect? Evidence is required over the long term as to whether or not the computer is just a technological fad that produces results because of its novelty.
5. Is there more student time-on-task with the computer exploration approach than with the traditional teacher lecture approach? The expectation might be that the computer can help to engage the student as an active constructor of knowledge and, as a result, yield more time-on-task. But perhaps simple transmission is superior

because of direct teacher control of the learning environment. Student investigation may be chaotic and off task.

6. Do students draw on and integrate past experience while using Perceptivist-constructivist based computer mathematics activities more than with traditional lecture presentations?
7. Are alternate conceptions more easily resolved with computer help than with lecture presentations? Is the freedom from the burden of computation exploited by the student to investigate different approaches?
8. Is there more cooperative learning with computer mathematics than with traditional methods? Does the public nature of the computer encourage student cooperation?
9. Is interdisciplinary computer learning retained longer than learning with traditional methods? Does interdisciplinary computer learning, by increasing realism and relevance, help the student to relate to the mathematics, incorporate it into his active cognitive repertoire and so retain it longer. Or is the "realism and relevance" perceived as introducing more complexities, making an already difficult subject impossible?

10. Is interdisciplinary computer learning accepted and used more than learning from traditional teaching? Ford (1979) wrote about the prevalence of students "going through the motions" but believing little. The way the computer empowers the student to see the world in a new way would hopefully motivate the student to feel a sense of ownership. Whether this is so needs empirical investigation.

Recommendations

On the theoretical level, mathematics teachers who wish to use the computer need a background in Perceptivism and Constructivism, or some other educational theories, to provide a basis for understanding what they will be trying to do with the computer. Professional in-service and teacher training must provide opportunities in this area.

Action plans need to be developed to mobilize mathematics teachers into using the computer. Hardware, software, programs of computer integration and sequences of lessons should be developed and made available. Pilot programs need to be developed so that comparisons between traditional and computer integrated, Perceptivist-constructivist mathematics classes can be made.

Only when the computer is systematically and knowingly exploited, using sound theoretical principles, can computer mathematics legitimately be empirically compared to traditional. Programs such as outlined in Table 8 are needed as are classroom tested computer lesson packages. The programs are needed to systematically exploit the various computer capabilities in an attempt to achieve Perceptivist-constructivist goals. Table 8 suggests one configuration of lesson sequences that might be offered in grades 8-12. This would constitute, in outline form, an initial attempt to, as Brauner (1987) suggested, restructure the curriculum in terms of the perceptions students are required to attain. The computer would be not just an add-on to such a curriculum but an integral part of it. Refer to Figure 3.

Table 8 would hopefully be useful in the development of a coordinated plan for the improvement of mathematics education involving philosophy, mathematics and the computer. The basic computer capabilities, listed horizontally, across the table, would be systematically exploited, throughout grades 8-12, to deliver curriculum. The curriculum would aim at achieving actual perception. Curriculum topics, and the actual perceptions they aim to produce, would gradually fill the table as educational consensus is reached concerning the practical implications of the Perceptivist-constructivist principles. The completed table would be a design specification on a Perceptivist-constructivist, computer integrated mathematics program. In grade 8, number might be a central concept and various computer capabilities could be exploited to enable actual perceptions involving number (e.g. mean, decimal repeater, bar graph). In grade 12, function might be a central concept and various computer capabilities such as graphing and symbolic manipulation as outlined in Chapter IV, could be used to boost student information processing quickly to the level at which perception is possible. The importance of the recommendation here lies not in the specific details in the table but in the call for consensus and cooperation to reconcile the demands of philosophy, mathematics and the computer to produce a contemporary, information age mathematics program.

Boards and ministries of education need to realize that the potential of the computer is not confined to mathematics. The insights provided by Perceptivism-constructivism indicate that the information age promise of the computer involves the growing interdependence of different knowledge domains and the information management function of technology in this interdependence. Meaningful mathematics requires meaningful education as a whole. A curriculum based on the modes of perception and aimed at students achieving actual perceptions should be the long term goal of computer assisted, information age, educational reform.

Table 8
Computer Math: Possible program of Activities.

Grade	PROGRAM ASPECT: Programming Basic	Programming Logo	Applications software	Graphing	Symbolic manipulation	Prob. and statistics	Simulation & problem solving	Enrichment
8	Evaluate expressions Mean Fraction—decimal Table values Geometry quiz	Angles Lines Triangles	Trajectory angle Decimal repeater Math 8 disk Dynalist	Bar graph				Houghton Mifflin (Games) Software (Math)
9	Square root Pythagoras Equation solving Operations on integers		Saga $y=ax+b$ Math 9 disk Balance Algebra Attack Algebra Tutor	Saga			Single machines	Graph to solve any equation Interest
10 A	Formula evaluation	Angles, lines, triangles Polygons Circles Co-ord. geo	Measurement program Proofchecker Logo Brochmann Logo	Saga linear functions			Heredity Dog	
B	Quad. equation Herons Sim. equation solver		10B disk Alg. practice Class spreadsheet	Saga as equation solver			Ohm's Law	Spreadsheet Linear regression
11	Quad. equation		Equation booklet	Mathgrapher		P & S package	World population	Problem-solving booklet
12	Triangles	Functions booklet	Proofchecker (Govt. exam)	Mathgrapher				

BIBLIOGRAPHY

Albers, D. J., Anderson, R. D. & Loftsgaarden P. (1987). Undergraduate programs in the mathematical and computer sciences the 1985-1986 survey. Washington, D.C.: Mathematical Association of America.

Ayers, T., Davis, G., Dubinsky, E., & Lewin, P. (1988). Computer experiences in learning composition of functions. Journal for Research in Mathematics Education, 19(3), 246-259.

Bittinger, W. and Morel, A. (1982). Applied Calculus. New York: Addison Wesley Co.

Blais, D. M. (1988). Constructivism--a theoretical revolution for algebra. Mathematics Teacher, 81(8), 642-631.

Brauner, C. J. (1988). Perceptivism: A new philosophy of education. A brief to the royal commission on education. Manuscript. University of British Columbia, Faculty of Education, Vancouver.

Brauner, C. J. (1987). Perceptivism: Seven modes of perception. Manuscript. University of British Columbia, Faculty of Education, Vancouver.

Brauner, C. J. (1986). Perceptivism: A narrative philosophy in progress. Manuscript. University of British Columbia, Faculty of Education, Vancouver.

Brauner, C. J. (1964). American Educational Theory. Englewood Cliff, N.J.: Prentice-Hall.

Brochmann, Harold. (1988). Secondary school calculus. Vector, 30(1), 45-53.

Bruner, J. (1986). Actual minds, possible worlds. Cambridge, MA: Harvard University Press.

Burns, H. and Brauner, C. J. (1962). Philosophy of Education. New York: Ronald Press Co.

Burns, H. and Brauner, C. J. (1965). Problems in Education and Philosophy. Englewood Cliff, N.J.: Prentice-Hall.

Carpenter, T.P., Brown, C., Kouba, V., Lindquist, M. M., Silver, E. A. & Swafford, J. O. (1987). Results from the fourth mathematics assessment of the natural assessment of educational progress. Reston, VA.: National Council of Teachers of Mathematics.

Carson, G. (1913). Essays on mathematics education. Massachusetts: Ginn Co.

Cassirer, E. (1923). Einstein's Theory of Relativity. New York: Dover Publication Inc.

Cassirer, E. (1944). An Essay on Man.. New Haven: Yale University Press.

Cassirer, E. (1923). The Philosophy of Symbolic Forms. New Haven: Yale University Press.

Considine, D. (1976). Van Nostrand's scientific encyclopedia. New York: Van Nostrand Reinhold Co.

Cooke, D.W. (1987). A computer assisted instruction unit on exponential and logarithmic functions. Dissertation Abstract International, 47A, 2933. (University Microfilms No.8626265).

Cooney, T. J. (1988). The issue of reform: what have we learned from yesteryear. Mathematics Teacher, 81(5), 352-363.

Copeland, B. D. (1988). Introduction to perceptivism: A philosophy of education. Unpublished

Copeland, B. (1985). A Proposal Concerning the Application of the Principle of Complementarity in Philosophy. M.A. thesis, University of British Columbia. Unpublished.

Copeland, B. D. (1986). How might constructivism and Hewson's theory of conceptual change affect the science classroom? Unpublished.

Copeland, B. (1988). Functions as models of reality. Vector. 30(1) 34-46.

Copeland, B. (1989). Computer-assisted exploration of the exponential function. Vector, 30(2), 44-47.

Copeland, B. (1988). Computer Integrated Mathematics. Vector, 29(4), 34-42.

D'Ambrosio, U. (1986). From Reality to Action: Reflections on Education and Mathematics. Sao Paolo: Summas Editorial.

de Cecco, J. P. (1968). The Philosophy of Learning and Instruction. Englewood Cliff, New Jersey: Prentice-Hall Inc.

Dewey, J. (1974). The child and the curriculum. In John Dewey on Education. edited by R.D. Archambault. Chicago: University of Chicago Press.

Dolcetti, I.C., Emmer, M., Falcone, M., & Vita, S.F. (1988). The impact of new technologies on teaching calculus: A report of an experiment. International Journal of Mathematical Education in Science and Technology, 19(5), 637-657.

Elder, H. (1979). Epochs of culture. Unpublished.

Erickson, G. (April, 1987). Constructivist epistemology and the professional development of teachers. Paper presented to the American Educational Research Association.

Feigl, H. (1967). The Mental and the Physical. Minneapolis: Minnesota Paperbacks.

Fey, J. T. and R. A. Good. (1985) Rethinking the sequence and priorities of high school mathematics curricula. The secondary school mathematics curriculum, pp.42-52, Virginia, NCTM.

Ford, N. (1979). Study strategies, orientations and "personal meaningfulness" in higher education. British Journal of Educational Technology, 24(1), 143-155.

Goldenberg, E. P. (1988). Mathematics, metaphors, and human factors: Mathematical, technical, and pedagogical challenges in the educational use of graphical representations of functions. Journal of Mathematical Behaviour, 7, 135-173.

Heid, M. K. (Nov., 1983). Calculus with MuMath: Implications for Curriculum Reform. The Computer Teacher, 18(1), 46-49.

Heid, M. K. (1988). Resequencing skills and concepts in applied calculus using the computer as a tool. Journal for Research in Mathematics Education, 19(1), 3-25.

Hewson, M. G. et al. (1986). Teaching and Learning in Medicine. Unpublished paper.

Hewson, P. W. (1985). Diagnosis and remediation of an alternate conception of velocity using a microcomputer program. American Journal of Physics, 53(7), 684-690.

Hewson, P. W. Epistemological commitments in the learning of science: Examples from dynamics. European Journal of Science Education, 7(2), 163-172.

Hirsch, C. R. and M. J. Zweng. (1985). The secondary school mathematics curriculum - 1985 yearbook, Reston, Virginia: NCTM

Hofstadler, D. (1979). Gödel, Escher, Bach: an eternal golden braid. New York: Basic Books.

Holton, G. (1973). Thematic Origins of Scientific Thought-Kepler to Einstein. Cambridge, Mass.: Harvard University Press.

Hook, S. (1956). Note on a Philosophy of Education. Harvard Educational Review, XXVI (Spring 1956), 145-148.

Howson, G. (1986). School mathematics in the 1990's, Cambridge: Cambridge University Press.

Hynd, C. and Alvermann, D. C. (1986). The role of refutation text in overcoming difficulties with science concepts. Journal of Reading, 29(1), 440-446.

Jagi, G. (1986). The uses of calculators and computers in mathematics classes in twenty countries: Summary report. Second International Mathematics Study. Urbana, IL: University of Illinois. (ERIC Document Reproduction Service No. ED 291589).

Jauch, J. M. (1973). Are Quanta Real? Bloomington, Indiana: Indiana University Press.

Johnson, G. (1986). School mathematics in the 1990's. Cambridge: Cambridge University Press.

Johnson, J. L. (1987). Microcomputers and secondary school mathematics: A new potential. Focus on Learning Problems in Mathematics, 9(2), 5-17.

Kerr, D. (1976). Educational Policy; Analysis, Structure and Justification. New York: David McKay Co. Inc.

Keuper, Sandra. (June 1985). An annotated bibliography of the effectiveness of the computer used as a tool to learn mathematics in the secondary schools. (ERIC Abstract ED 257 679).

Kilpatrick, J. (1987). What constructivism might be in mathematics education. In J. C. Bergeron, N. Herscovics & C. Kieran (Eds.), Proceedings of the eleventh international conference, psychology of mathematics education, Vol.I, pp.3-27. Montreal: University of Montreal Press.

Kine, M. (1953). Mathematics in western culture. Oxford. Oxford University Press.

Klassen, John. (1988). The Balmoral School Project. North Vancouver, B.C. Unpublished manuscript.

Kraushaar, I. (1982). Introduction to mechanics, matter and waves. New York: Addison Wesley.

Kuhn, T. (1970). The structure of scientific revolutions. Chicago: The University of Chicago Press.

Lapointe, A., Mead N., Phillips G. (1989). A World of Differences - an International Assessment of Mathematics and Science. Princeton, N.J.: Educational Testing Service.

Leitzel, J. and Osborne, A. (1985). Alternatives for college preparatory students. In C. Hirsch & M. Zweng (Eds.), The Secondary school mathematics curriculum, pp.150-165. Reston, Virginia: NCTM.

Lochhead, J. (1985). New horizons in educational development. In Review of Research in Education. Washington, D.C.: American Educational Research Association.

MacPherson, E. (1985). The themes of geometry: Design of the nonformal geometry curriculum. In C. Hirsch & M. Zweng (Eds.), The secondary school mathematics curriculum, pp.65-80. Reston, Virginia: NCTM.

Mathematical Association of America and the National Council of Teachers of Mathematics (1980). A Sourcebook of Application of School Mathematics. Reston, Va.: NCTM

McDonald, J. L. (1988). Integrating spreadsheets into the mathematics classroom. Mathematics Teacher, 81(8), 615-622.

McKnight, C., Crosswhite, F., Dossey, J., Kifer, E., Swafford, J., Travers, K. & Cooney, T. (1987). The Underachieving Curriculum. Champaign: Stipes Publishing Co.

Minstrell, J. (1982). Explaining the "at rest" condition of an object. Physics Teacher, 20 10-14.

Morante, E. Phillips, T. and Lorenz, J. (1984). The New Jersey basic skills assessment program. Journal of Development and Remedial Education 7, 2-32.

National Commission on Excellence in Education (1983). A nation at risk. Washington, D.C.: Government Printing Office.

National Assessment of Educational Progress (1979). The second national mathematics assessment. Denver: Education commission of the states.

National Council of Teachers of Mathematics (1989). Curriculum and evaluation standards for school mathematics. Reston, Virginia: The Council.

Piaget, J. (1973). To Understand is to invent. New York: Grossman Publishers.

Pollak, H. (1987). Notes from a talk at the Mathematical Sciences Education Board. Frameworks Conference, May 1987, at Minneapolis.

Posner, G. J., Strike, K., Hewson, P., Gertzog, W. (1982). Accommodation of a scientific conception: towards a theory of conceptual change. Scientific Education, 66(2), 211-227.

Project to Increase Mastery of Mathematics and Science (1988). A Review of Selected Microcomputer Software Packages with Lessons for Teaching Mathematics Grades 8-12. Connecticut: Connecticut Dept. of Education.

Ralston, A. The really new college mathematics and its impact on the high school. In C. Hirsch & M. Zweng (Eds.) The Secondary School Mathematics Curriculum, pp.29-42. Reston, Viriginia: NCTM.

Santos, M. (1988). Reflections on mathematics learning." Vector 29 (3), 40-44.

Schofield, J.W. & Verban, D. (1988). Computer usage in the teaching of mathematics: Issues that need answers. In D. A. Grouws & T. J. Coonney (Eds.), Effective mathematics teaching (pp.169-193). Reston, VA: The National Council of Teachers of Mathematics.

Second International Mathematics Study (SIMS) (1986). Detailed report of the United States. Champaign: Stipes Publishing Co.

Steen, L. (1987). Power to the People. Focus: Newsletter of the Mathematical Association of America, 7(1), 5-6.

Steffe, Leslie (1988). Mathematics curriculum design - a constructivist perspective. Paper delivered at ICME, Budapest Hungary.

Strike, K.A. (1982). Educational Policy and the Just Society. Urbana: University of Illinois Press.

Swetz, Frank. (1987). Mathematical Modelling in the School Curriculum. Harrisburg: Pennsylvania State University. National Science Foundation Grant No. TEI - 8550425.

U.S. Dept. of Education, Office of Technology Assessment (1989). Technology and the American Transaction. Washington: U.S. Government Printing Office.

Usiskin, Z. (1985). We need another revolution in secondary school mathematics. In C. Hirsch & M. Zweng (Eds.), The secondary school mathematics curriculum, pp.1-21. Reston, Virginia: NCTM.

Voelker, A. (1982). The development of an attentive public for science: implications for science teaching." What research says to the science teacher. Vol.4. Washington, D.C.: National Science Teachers Association.

von Glasersfeld, E. (1986). Steps in the construction of "others" and "reality": a study in self regulation. In R. Trapp, Power, automony, utopia, pp.107-116. New York: Plenum Publishing Co.

von Foerster, H. (1984). On constructing a reality. In Watzlawick, P. (Ed.), The invented reality: How do we believe we know? London: W.W. Norton and Co.

Waits, B. and Demand F. (1989). Computers and the rational-root theorem--another view. Mathematics Teacher, 82(5), 124-12.

Waits, B. and Demand F. (1989). Using Computer Graphing to Enhance the Teaching and Learning of Calculus and Precalculus Mathematics. Dept. of Mathematics: The Ohio State University.

Walker, D. (1984). Promise, potential and pragmatism: Computers in high school. Institute for Research in Educational Finance and Governance Policy Notes, 5(3), 3-4.

APPENDIX

The role of mathematical realism in perceptivism

In order to facilitate explanation, Figure 1 on page 19 simplifies a theoretical point of considerable importance. A version of mathematical realism is implicit in the diagram to the extent that it appears to be possible to perceive mathematical aspects of reality. The computerized mathematical applications supplied by the writer are meant to demonstrate that the meaning of mathematics is revealed to the student through its use in the perception and description of aspects of reality.

In discussions with the writer, Brauner (personal communication, September 15, 1989) has pointed out that Perceptivism is not the handmaiden of mathematical realism as the writer might have implied. Brauner insists that Perceptivism says nothing about reality. This writer finds Brauner's Perceptivism compatible with mathematical realism whereas Brauner himself does not. Brauner seems to be content with a conventionalist position that sees mathematics as a self consistent logical system. Whether that system accounts for things in the world of experience is viewed as having nothing to do with mathematics as such.

In Table 1 on page 25, along side the heading "examples", Brauner provides examples of perceptions in the different modes. He is careful to distinguish between seeing in mythic, thesistic, theoretic, standard, primal, primary and operational perception and identifying or recognizing with regard to thematic and relational perception respectively. In the latter two cases Brauner wants to hold that "actual perception" is not possible but at best only the identification or recognition of some conventional pattern in pre-existing perceptions is possible.

What this writer finds objectionable in Brauner's characterization of the two "exceptional cases" is the implicit view that "direct, uninferred" perception is possible in the other cases, but not in relational or thematic perception. But perception, seeing, is never direct. At best only an inferred reality is observable. To the extent that mathematical concepts and theories are used and confirmed in the public cognitive forum, mathematical realism would hold that they give at least probable knowledge about aspects of reality, just as concepts and theories from other domains of knowledge. A free choice is available, from among the various accepted conceptual and theoretical, perception-enabling, world builders but they are all on a par with respect to epistemological immediacy except on some dubious or arbitrary grounds. None can be assumed to give a more "direct" perception of reality than any other. For this reason the writer holds that it is reasonable to claim that mathematics can give at least probable knowledge of aspects of reality. A more detailed discussion of this issue of realism is available in Copeland (1985).

The extension the writer gives to basic Perceptivism, in the direction of mathematical realism, is based on both philosophical and pedagogic considerations. The pedagogic considerations center on the simplicity and directness of realism as a way to motivate and explain the uses of mathematics to students. Philosophically, the view stands up nicely to the competition. Even in standard perception concepts of ordinary language such as "table" or "horse" mediate perception. In mathematics the same thing happens. What mathematical realism interprets is the view that the relations perceived are actual relationships in the world. Quine, as for example, argues that

> From among the various conceptual schemes best suited to these various pursuits, one--the phenomenalistic--claims epistemological priority. Viewed from within the phenomenalistic conceptual scheme, the ontologies of physical objects and mathematical objects are myths. The quality of myth, however, is relative; relative, in

> this case, to the epistemological point of view. This point of view is one among various, corresponding to one among our various interests and purposes.
> (Quine, 1953, p.19)

There is no fact, Brauner agrees, apart from some theoretical point of view. Mathematical knowledge can have as much or as little immediacy to reality as any other form of knowledge. Beth, in arguing for a new version of mathematical realism, compares mathematical perception, which Brauner does not accept, with scientific perception in physics, which he does.

> Deductive theories cannot, in general, provide an adequate description of mathematical structures; therefore, it seems likely that our knowledge of such structures has, at least partly, an intuitive, an immediate character ... It does not follow of course that all mathematics consists of such knowledge, or that the deductive theories of modern mathematics should be judged exclusively by their conformity to our mathematical intuition. It seems wiser to suppose, with Bernays, that mathematics results from our rationally remodelling those fundamental insights which originate from our mathematical intuition. In the respect mathematics might be reasonably compared to physics.
> (Beth, 1959, p.643)

What this writer is claiming is that a realistic interpretation of Perceptivism is a philosophical and pedagogic possibility that provides a useful way of thinking about mathematics, mathematics education and the mathematical uses of the computer. Other forms of Perceptivism may also be defensible and useful. This would be neither surprising nor upsetting since the debate between realism, idealism, intuitionism and formalism is still an open question.

Appendix References

Beth, E. W. (1959). The foundations of mathematics. New York: Harper and Row.

Brauner, C. T. 1989. Personal communication, September 15, 1989. (M.A. Thesis defense).

Copeland, B. (1985). A proposal concerning the application of the principle of complementarity in philosophy. M.A. thesis, University of British Columbia. Unpublished.

Quine, W. V. (1953). From a logical point of view. Cambridge: Harvard University Press.