Constructing shapes & building images:

The spatial understandings of kindergarten-aged children

by

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Abstract

This study focuses on the spatial understandings of kindergarten-aged children. Its goal is to discover how students of this age demonstrate and verbalize their understanding of the physical attributes of 2- and 3-dimensional shapes through comparison and construction. Three children were chosen from the original group of six students and their videotaped interviews are described and assessed in detail. In order to design appropriate tasks and identify significant features of the children's dialogue and activities, three theories are emphasized. The use of Jean Piaget's topological primacy thesis shows that children of this age are able to distinguish shapes according to all three levels of description identified by him in his early work. An examination of data in accordance with the Pirie-Kieren theory for growth of understanding reveals that with varying degrees of prompting, kindergarten-aged children use unique images to note and compare specific properties among different shapes. Thirdly, Stuart Reifel's developmental progression for construction is utilized to show that within a small group of children, a range of complexity in building structures can be identified. Suggestions are given for the application and extension of this study's interview tasks and subsequent analysis for further research and use in the classroom.

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Chapter I

Introduction

Five years ago, an interest in Howard Gardner's theory of multiple intelligences (Gardner, 1983) led me to join a group of kindergarten teachers initiating a creative building program focusing on young children's visual/spatial abilities. The children look forward to their weekly opportunities to freely construct structures with collections of objects such as blocks of various size, shape, colour and texture (referred to by mathematics teachers as "manipulatives"), lego, two- and three-dimensional shapes, and attachable building sets. Over five years of offering this activity, I have noticed that as early as the first week of kindergarten these children's individual and group constructions consistently show geometric aspects such as pattern (for example: the use of alternating colours or shapes), symmetry, shape, and tessellation (a design in which all of the objects fit together with no space in between). I became interested in finding out more about the knowledge and experiences behind the children's tendencies to display these features in their designs.

Statement of the Problem

The National Council of Teachers of Mathematics defines spatial sense as "an intuitive feel for one's surroundings and the objects in them" (NCTM, 1989: p.49). The NCTM's recommendation that students be given opportunities to develop spatial sense illustrates "the increasing attention of mathematics educators to spatial imagery and spatial thinking" (Yackel and Wheatley, 1990: p. 52). In 1983, Gardner foresaw the development of this mathematical concept as a research focus when he identified that:

> Though the centrality of spatial intelligence has long been recognized by researchers who work with adult subjects, relatively little has been definitively established about the development of this set of capacities in children (p. 178).

In the time period following Gardner's book, "Frames of Mind", many studies have been conducted in an attempt to further understand young children's spatial abilities. A brief survey of such studies revealed several studies of children aged seven years and older as well as studies centered on computer-generated spatial tasks (for example: Clements et al, 1997, Vasu and Tyler, 1997 respectively). An additional study included four and six year old children, but required its participants to choose a single correct answer for each spatial task (Rosser, 1994). Other studies in spatial sense relied on quantitative data such as that gathered by intelligence tests (Brown and Yakimowski, 1987), centered on the relationship between artistic ability and visual perception (Cox and Ralph, 1996), or served to inform the teaching of mapping skills (Liben and Downs, 1993). What these studies do show is that researchers believe there is a link between spatial ability and curricular goals. What they do not encompass, however, is a particular approach to assessing spatial skills that is appropriate for five year olds. When educators apply traditional assessment methods such as standardized tests or single-answer performance measures, they risk underestimating children's abilities and knowledge. Clements and Battista (1992: p. 457) further support the notion that an opportunity for subsequent study exists: "Research is needed to identify the specific cognitive constructions that children make at all age levels, especially in the context of supportive environments (for example, those including manipulatives, computer tools, and engaging tasks.)" There are three elements characterizing the emerging research problem that interest me: age level, manipulatives and engaging tasks. The purpose of my study is to determine what understandings about space and shape can be revealed when kindergarten-aged children are challenged with open-ended (more than one correct answer) manipulative tasks requiring spatial thinking.

Research Questions

Broad Question: What can be learned about the spatial understandings of kindergarten-aged children through specific, open-ended learning tasks with 2- and 3-dimensional shapes?

Specific Questions:

1) How do kindergarten students verbalize the physical attributes of 2-dimensional shapes?
2) How do kindergarten students show their understanding of size through construction with 2-dimensional shapes?
3) How do kindergarten students symbolically represent objects from their world using 2- and 3-dimensional shapes?

Educational Significance of the Study

Children arrive in kindergarten with a unique collection of experiences, interests, and skills they have developed while operating in a variety of contexts. A developmentally appropriate program for five year old children is one in which "each child is viewed as a unique person with an individual pattern and timing of growth" (Bredekamp, 1987: p.67). A curricular program consistent with such a philosophy would be an active pursuit in which "teachers guide children's involvement in projects and enrich the learning experiences by extending children's ideas, responding to their questions, engaging them in conversation, and challenging their thinking" (Bredekamp, 1987: p.67). In previous studies, the spatial abilities of kindergarten-aged students were assessed via close-ended, paper and pencil, and intelligence-based tasks and tests. This study seeks to extend such task-centred work toward a more child-centred evaluation. In order to appropriately assess the abilities of participants who are five years old, I examine and question children about shapes and structures focusing on what they are able to do in a familiar low stress environment. One example of such an age-appropriate open-ended activity is asking a child to build a model of his or her house using an assortment of two- and three-dimensional shapes. While children often display aspects of symmetry and pattern in their buildings, it is necessary to discuss their structure with them in order to find out whether or not they recognize those properties and can explain them verbally. A Soviet study conducted by Anne Leushina supports the notion of listening to children speak about objects in their world to determine their level of knowledge about geometry: "The children's knowledge of the various dimensions should be properly reflected in their speech...The children should also apply their knowledge in various activities" (1991: p. 298). Leushina further justifies the importance of exposing children to explorative activities with shapes: "Thus, familiarity with geometric solids strengthens the children's cognitive ability, enriching their notions about life around them, and is also reflected in their productive activity (drawing, modeling with clay, building, and their accounts of what they observe)" (p. 305).

By engaging children in regular building activities with and without specific criteria and taking a few moments with each child to ask open-ended questions about their structures, teachers can learn a great deal about a child's understanding of shape and space. Ultimately, this study seeks to provide insight as to the nature of spatial knowledge children of this age may have so as to inform classroom decisions about the type of instruction needed to reveal, support and further that understanding.

Chapter II

Review of Related Literature

The National Council of Teachers of Mathematics identifies geometry and spatial sense as a curriculum standard and gives several reasons for its importance:

> Spatial understandings are necessary for interpreting, understanding, and appreciating our inherently geometric world....Children who develop a strong sense of spatial relationships and who master the concepts and language of geometry are better prepared to learn number and measurement ideas as well as other advanced mathematical topics (1989, p.48).

Bruni and Seidenstein (1990: p.203) outline more specifically why the above NCTM standard is relevant to early childhood education,

> "Children's first experiences in trying to understand the world around them are spatial and geometric as they distinguish one object from another and determine how close or far away an object is. As they learn to move from one place to another, they use geometric and spatial ideas to solve problems and make decisions in their everyday lives."

Jean Piaget conducted several studies of the development of children's spatial understandings (Gardner, 1983: p. 178). In his opinion, spatial intelligence is "part and parcel of the general portrait of logical growth" (Gardner, 1983: p. 178). Wheatley (1990: p.10-11) supports the relationship Piaget makes between spatial ability and logic, explaining that children with high spatial ability are more successful with problem solving, analytic reasoning, and integral calculus. Many other researchers have also established a link between spatial ability and mathematics achievement: "Positive correlations have been found between spatial ability and mathematics achievement at all grade levels" (Fennema and Sherman, 1977; Fennema and Sherman, 1978; Guay and McDaniel, 1977 cited in Grouws, 1992: p. 443).

The Nature of Spatial Sense

In order to develop spatial sense, the NCTM claims that "children must have many experiences that focus on geometric relationships; the direction, orientation, and perspectives of objects in space... (1989: p. 49). The notion of the direction an object takes in space is consistent with the first half of Piaget's conception of mental imagery as the ability to, "appreciate the paths of objects as they are moved through space" (Gardner, 1983: p. 178). The idea that spatial sense is also defined by the orientation of an object in space is expressed in the second half of Piaget's mental imagery definition: the ability to "find one's way between various locales" (Gardner, p. 178). In his research, Piaget discovered that children aged four to six years old are able to perform reversible mental operations such as imagining events without actually looking at them, picturing how an object would look to someone else seated in a different place, and rotating objects in their minds (Gardner, 1983: p.179). Piaget further explains that while school age children know their way around an area or neighbourhood, they still have difficulty representing this knowledge verbally or through drawing: "Representing their piecemeal knowledge in another format or symbol system proves an elusive part of spatial intelligence" (Gardner: p. 180). It would therefore be worthwhile to investigate alternate methods for children to visually represent familiar areas such as their houses, schools, and neighbourhoods. Gardner points out the opportunity for further investigation presented by Piaget's early work:

> Piaget provided the first general picture of spatial development and many of his observations have stood the test of time. For the most part, however, he restricted himself to paper and pencil or to desk-top measure of spatial ability, and so largely

ignored the child's understanding of the broader spatial environment. (p. 180)

In his investigations of spatial perception with two to seven year olds, Piaget (1956: p. 4) defined three developmental levels of imagination or "representational thought". The first level, called "topological", refers to how features in an environment are related and covers such concepts as open/closed, near/far, and inside/outside. The second level, called "projective", relates to properties of objects such as curved or straight. The third level, called "Euclidean" describes the length, area, and angle size of objects. Piaget found that children aged four to six years old distinguish shapes through active exploration according to topological and projective aspects. Children of this age are not, according to Piaget, capable of recognizing Euclidean properties such as the difference between a triangle and a diamond (Fuys and Liebov, 1992: p.199). Fuys and Liebov support Piaget's findings and explain their concern for current instructional methods: "This progression of spatial thinking from topological to projective and then to the Euclidean, known as the "topological primacy thesis", is the opposite of the one used in most primary school programs, which typically begin with measurement aspects of geometry" (Fuys and Liebov: p. 199). A study of 600 first graders cited by Fuys and Liebov appears to contradict Piaget's thesis. The study found that seventy percent of the participants correctly answered topological questions without specific instruction in such concepts. The study also found that students were able to grasp Euclidean concepts without instruction in topology (p. 199). Considering that first grade children are six or seven years old, it is possible that the majority of them already possess the topological and projective knowledge necessary to perform Euclidean tasks through their experiences. While Fuys and Liebov use these study results to make recommendations for grade one geometry instruction, they do not discuss its implications for kindergarten-aged children. Effective instruction in geometry for five year olds would be dependent on knowing at what stage of development students were according to Piaget's "topological primacy thesis" in order to ensure that students weren't expected to understand Euclidean concepts before they had experience with topological and then projective tasks. An examination of the British Columbia kindergarten/grade one mathematics curriculum (B.C. Ministry of Education, 1995: p. 22) reveals Euclidean learning outcomes such as "select an appropriate non-standard unit to estimate, measure, record, compare, and order objects and containers" and "classify, describe, and arrange objects using comparative language to compare length, size, area, weight, and volume." While it is expected that kindergarten students be given much opportunity to explore manipulative objects such as shapes in their play, the curriculum guide for mathematics gives no specific instructions as to the order that concepts should be explored or ways that teachers can measure their students' interest in learning mathematical concepts.

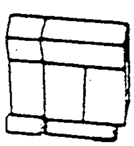
Many researchers recognize Piaget's contribution to our understandings of the development of spatial intelligence and have provided a variety of contexts in which to continue his work. In a comparative study of four and seven year olds, Reifel (1984) explored how constructing with blocks reveals children's thinking around spatial concepts. Reifel points out that children think as they play, create patterns and models without adult interaction, represent objects from their past experience, and share aspects of their designs with one another. These factors are consistent with the National Association of the Education of Young Children's recommendations that effective instruction be active, social as well as individual, and offer a variety of activities (Fuys and Liebov, 1992: p. 215). Reifel further points out that as children build, they rely on their topological spatial knowledge to place objects beside, under, on top of, around, and in other objects (Reifel, 1984: p. 62). In comparing the topological understandings of four (pre-kindergarten) and seven year olds, Reifel's findings were consistent with Piaget's in that both researchers concluded that children's abilities to symbolically represent their world increase with age. Further, by allowing his participants to represent familiar objects such as the layout of a house with blocks of various shapes rather than just verbally or by drawing, Reifel was able to gain an even more detailed picture of the children's mental images than that described by Piaget. Specifically, Reifel observed that the constructions of the two age groups differed in complexity, scale (ie: relative size), and separation of exterior versus interior space (p. 66). By engaging his participants in open-ended tasks such as using blocks to show a story they had heard, Reifel was able to identify eight specific developmental stages of spatial representation that children feature in their constructions (shown in figure 1). Reifel's research illustrates that much insight can be gained by engaging young children in specific open-ended tasks with concrete materials. For this reason, I chose to apply Reifel's model to kindergarten students involved in a regular building program by having them represent a familiar environments (their own bedroom) rather than more abstract objects such as those they recall from a story in order to characterize the development of these children's spatial understandings.

In their study of learning difficulties in geometry, Dina and Pierre van Hiele created a five level model to illustrate how children pass through various levels of understanding. Children in kindergarten are said to be at level 0, meaning that they judge shapes by their appearance as a whole (Fuys and Liebov, 1992: p.205). A study in the constancy of shape by Vurpillot (Del Grande, 1990: p.16) confirms that five year olds are operating at level 0, a stage in which children conceptualize the overall appearance but not the definitive properties of shapes. Vurpillot found that while four and five year olds could identify a square as such when shown one right side up, these children claimed that that same shape was no longer a square when the researcher rotated it to stand on one corner. At level 1, children are able to analyze and describe the properties of shapes such as recognizing that a square has four sides. However, they do not see that particular properties are necessary for defining a shape by a certain name. For example,

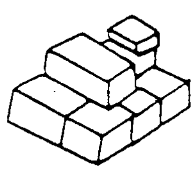
Figure 1: Reifel's developmental progression

a. stack, for *on* (vertical)

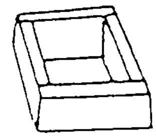
b. row, for *by* (horizontal)



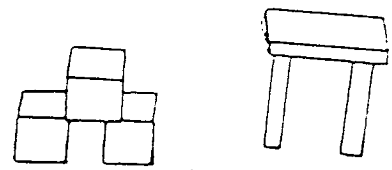
c. stack and row combination



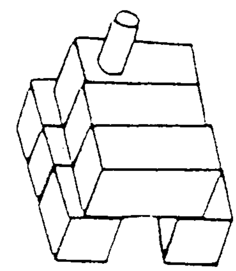
d. pile, three dimensions with no interior space



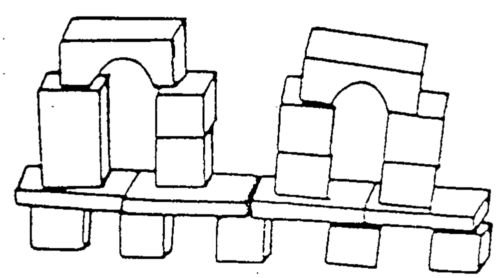
e. enclosure (flat)



f. enclosure (arches)



g. enclosure (combination)



h. combinations of many forms

(Reifel, 1984: p.63)

level 1 children would still identify a rectangle as a square. At the higher levels, Euclidean concepts such as angle size and abstract properties are understood. The van Hieles' philosophy for applying their model in the instruction of primary-aged children is summarized by Fuys and Liebov (p. 205):

> Geometry in grades K-4 involves thinking mainly at levels 0 and 1… Young children need experiences that develop their global understanding of geometric objects such as constructing and drawing shapes, fitting 2- or 3-D shapes together and looking for shapes in their home and school environment.

Recommendations for instruction in geometry made by the National Council for the Teaching of Mathematics (NCTM) are consistent with the van Hieles' advice. The Council advocates drawing shapes and constructing models of shapes from various materials such as straws. There is one significant difference between the theories of the van Hieles and Piaget. While Piaget and Reifel both concluded that progress in children's understanding of geometric concepts was developmental (increased with age), the van Hieles advocate that it is instruction that most strongly influences mastery. The van Hiele method proposes that the key to learning in geometry is talking to the student to determine what level he or she is at and engaging that student in the active exploration of topics appropriate to that level (Fuys and Liebov, 1992: p. 205). The first level in Piaget's topological primacy thesis which covers such contrasting relationships as open/closed, resembles the van Hieles' level 0 understandings of the overall appearance of a shape. Piaget's second level, projective or descriptive properties, corresponds with level 1 of the van Hiele model. The importance the van Hieles place on discussion with students and active learning is supported by Piaget's constructivist view that "children develop spatial concepts by acting on objects and reflecting on their actions, not simply by looking at the objects" (Fuys and Liebov: p. 199).

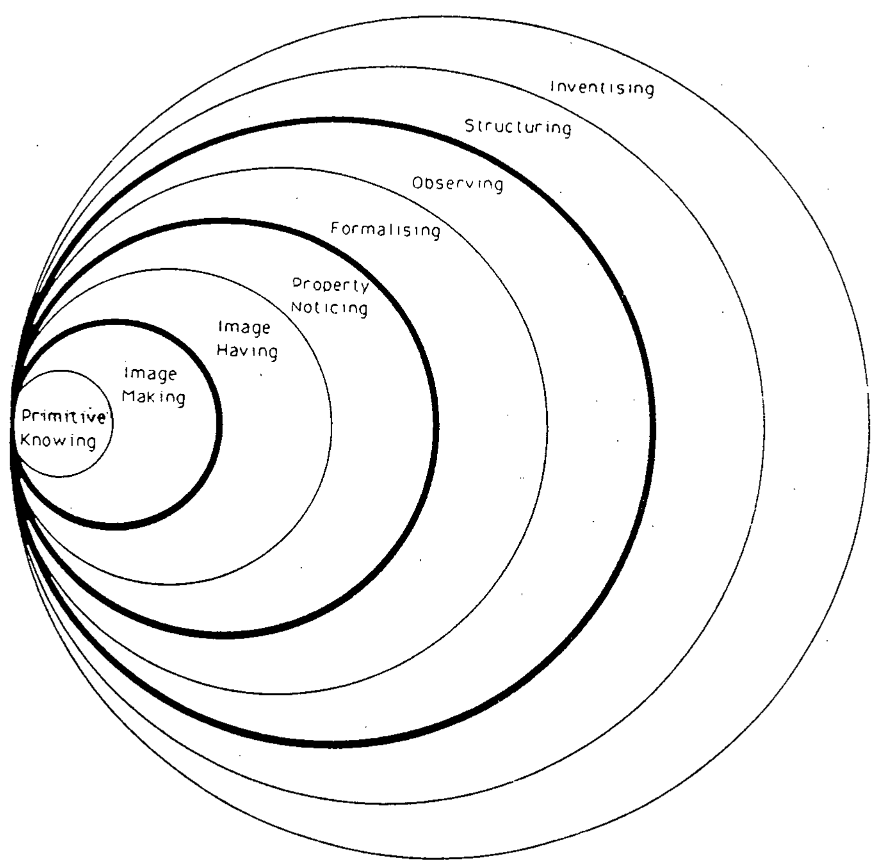
Recommended Tasks

Researchers in early childhood education have produced many documents with suggested activities for fostering growth in understanding and expressing spatial concepts. The February 1990 issue of *Arithmetic Teacher* is dedicated to spatial sense and contains many activities appropriate for kindergarten-aged students. In this issue, Yackel and Wheatley recommend a series of exercises requiring primary students to look at one or a few shapes placed in a design for three seconds, discuss how they saw the shape(s) and then make a drawing to show the image they have in their heads (p. 53). Since drawing mental images with specific properties (eg: a triangle with three sides) would be difficult for kindergarten-aged children due to their still developing fine motor coordination, this activity could be adapted by having the participants re-create what they saw by choosing from a collection of assorted shapes. This exercise would help the researcher to determine if the children were able to visualize and maintain a specific shape in their memory. A similar article, by Werner Liedtke, begins with topological tasks that ask primary children to focus on the location of objects in space such as imagining what a block would look like from a helicopter's point of view (1995: p. 13). Next, Liedtke focuses on helping students understand projective qualities of objects by suggesting that children hold a 3-dimensional shape behind their backs and indicate which shape in a set in front of them is similar to the one behind their back. Finally, Liedtke's activities move children toward Euclidean thinking by having them focus on finding objects that are similar and different in size. The process Liedtke describes respects Piaget's primacy topological thesis. Since the purpose of this particular article was to put theory into practice so as to inform instruction in the primary grades, there exists an opportunity to gather data as to how kindergarten-aged children in particular respond to these tasks. An earlier article by Liedtke (1975) recommends engaging kindergarten students in specific teacher-directed tasks with blocks in order to determine what knowledge of spatial relationships they have already constructed. The advantages to Liedtke's method are that children are actively involved in open-ended tasks and interviewing is used to gain greater insight than observations alone. Liedtke's articles also provide an opportunity for educators to gather their own findings using assessment tools recommended by researchers in the field.

About Understanding

Since I am addressing the notion of children's *understanding* it is relevant to review current theories of understanding to discover their usefulness in gaining information about young children's spatial sense. At present there is much interest in the composition and acquisition of mathematical understanding. Richard Skemp is credited with initiating interest in this area via his 1976 article published in *Mathematics Teaching* in which he distinguishes between relational and instrumental understanding. According to Skemp, relational understanding is "knowing what to do and why" while instrumental understanding is "rules without reasons" (p.29). Skemp emphasizes that these two types of understanding are very different and teachers needs to be aware that a gap may exist between the type of understanding they are intending for students to gain and the actual understanding that students display. Anna Sierpinska's research with secondary mathematics students led her to theorize that understanding in mathematics is often obstructed by misconceptions students have for good reasons which she calls "epistemological obstacles" (1987: p.371). Sierpinska distinguishes between a lack of understanding and misunderstanding by describing how students can be misled when they apply a known strategy in the wrong place such as at a higher concept level. Alan Schoenfeld's theory of understanding contrasts the actual subject matter or standard knowledge with what the educator's understanding of the student's knowledge at various levels of specificity (1989: pp. 108-110). To apply this theory, the educator describes the aspects of a task or problem she would expect a student to comprehend in order of difficulty and then contrasts this scale with what the student is actually able to do. In his "reflective abstraction" model, Dubinsky (1991) explains understanding as the possession of dynamic non-linear schema containing actions, processes, objects and structures. Each mathematical concept can be broken down into these elements in order for the teacher to make decisions about instruction. The above theories refer heavily to a standard body of knowledge, as presented by the teacher, and how students negotiate that curriculum. A study of young children's spatial understandings via open-ended geometric tasks, would describe what a child already knows and how he or she is able to express that knowledge—a child-centred rather than subject- or teacher-centred endeavour. Also, within many of the articles describing these and other theories of understanding, the researchers cite examples from secondary level mathematics. While it is possible to apply their ideas at a lower level of mathematics, this action does not provide as rich a description as the model for growth of understanding presented by Susan Pirie and Thomas Kieren (1992). Intended as a theory *for* rather than *of* understanding, the Pirie-Kieren model focuses intensively on the images the student is possibly holding as he or she negotiates problems and concepts. The Pirie-Kieren model sees understanding as a layered rather than leveled dynamical process in which students move in multiple directions between various modes of understanding (see figure 1). The Pirie-Kieren model is appropriate for a study of kindergarten-aged children because it allows for specific rich description even with a seemingly simplistic task. The layers of this model that are relevant for mathematical tasks with young children include primitive knowing (background knowledge); image making (doing an activity to get an idea); image having (using a mental construct about a topic without having to continue image making); property noticing (noticing connections between images); and formalizing (drawing a common quality from the observed properties) (1992: pp.245-247). Another important feature of this model are the "don't need boundaries" represented by the solid rings on the diagram in figure 2 which convey the notion that once the outer ring is reached, the specific understanding of the inner ring is no longer needed. In addition, Pirie and Kieren put forward the idea that students who lack understanding when faced with a problem at a given level will "fold back" to an inner level in order to work through that task.

Figure 2 (Pirie & Kieren, 1992: p.247)



Summary

There is much research to justify the importance of spatial skills in negotiating problems in mathematics, science, and technology. Many researchers have built on the foundation laid by Jean Piaget to operationalize the concept of "spatial sense" and recommend ways of fostering students skills in this area. There exists some current studies of kindergarten-aged children's spatial perception when given close-ended paper and pencil tasks to work through as well as many studies of older children's spatial abilities. Since the NCTM's 1989 curriculum standards placed new emphasis on the importance of spatial sense, much work has been done to provide research-based instructional activities educators can use to improve young children's spatial abilities. In particular, many tasks appropriate or adaptable for kindergarten-aged children have emerged. These tasks are characterized as active, open-ended, specific, and involving the use of engaging concrete materials. What I chose to pursue, then, is an assessment of kindergarten children's topological, projective, and Euclidean understandings, through optimal tasks inspired by those described in the section, Recommended Tasks" above. As indicated earlier then, my broad research question is: What can be learned about the spatial understandings of kindergarten-aged children through specific, open-ended learning tasks with 2- dimensional shapes? Specifically, I will employ Piaget's topological primacy thesis to examine how kindergarten students verbalize the physical attributes of 2-dimensional shapes. Secondly, I will analyze how kindergarten students show their understanding of size through construction with 2-dimensional shapes with the help of the Pirie-Kieren model for growth of understanding. Finally, I will pursue a third investigation as to how kindergarten students symbolically represent objects from their world using 2- and 3-dimensional objects with the aid of Reifel's eight level developmental scheme.

Chapter III

Methods and Procedures

The Participants

The sample population for my study consists of children enrolled in kindergarten. Their ages range from 5 .0 to 6.5 years. The data was gathered between mid January and early March, 2000. I chose six children, three girls and three boys from my kindergarten classes. Two children are enrolled in English kindergarten and the other four are in French Immersion. These particular children were asked to participate because they met the following criteria: English is their first language; they are talkative and outgoing; they have a positive attitude toward school and building activities in particular; they are able to come after school (ie: an adult was able to drop them off and return thirty minutes later. It was important to me that my participants were able to verbalize their ideas as much as possible in their first language without becoming frustrated or uncomfortable. I chose children from my own kindergarten class because my experience with kindergarten-aged children is that they are much more at ease and talkative with their own teacher in a familiar setting. Also, I selected a balance of girls and boys since I wish to focus on characteristics of children working at particular tasks at a certain grade level rather than gender differences.

The Researcher

I am a classroom teacher in a lower mainland school district in my seventh year of teaching. This is my fifth year working with kindergarten children. I teach both English and French immersion programs. This study is part of my work toward a Master of Arts degree in Curriculum and Instruction.

The Interview Protocol

In order to explore each of the three aspects of my broad research question I have chosen some specific open-ended tasks modified from the various journal articles referred to earlier as well as tasks I have developed from my own teaching experience. To investigate the first question, "How do kindergarten children verbalize the physical attributes of 2- dimensional shapes?" I asked the children to compare attribute block shapes (same, different). In order to probe further, I had the children build a model of a square and then a rectangle and show me how their constructed shapes are similar to the attribute shapes. To investigate the second specific question, "How do kindergarten students show their understanding of size through construction with 2-dimensional shapes?", I invited my participants to build structures that are longer, shorter, bigger and smaller than a sample structure and ask them to explain how they know that their construction satisfies my request. Also, using their bigger and smaller structures, I asked the children if their shapes are taller, shorter, wider and narrower than the sample structure and seek explanations for their descriptions. For the final question, "How do kindergarten children symbolically represent objects from their world using 2- and 3-dimensional objects I invited the children to build a model of their bedroom using a variety of 2- and 3-dimensional shapes. After the students declared that they are finished building, I asked them to tell me about their structure. Throughout these tasks, I asked the children if there is anything else they wish to tell me about their constructions.

The Interview Procedure

The three specific research questions above were divided into three sessions each consisting of two tasks. Each session took place after school in my own classroom for twenty to thirty minutes. I conducted session one with all of my participants individually before moving on to session two, and session two with all participants individually before moving on to session three. Since the tasks are not progressive and I did not teach a formal unit on geometry before or during this time period, I do not feel that the two month time span has affected the children's "performance" in any way. All sessions were videotaped and later transcribed as needed in data analysis. Also, I photographed the final product of each participant in the third session so that I could refer to the children's products when describing their actions and commentary. A detailed description of each task and its purpose can be found in Appendix A.

Noteworthy Features of the Study

For each task I chose, a few corresponding questions were also pre-determined for the purpose of exploring each child's thinking on specific criteria. However, most of the conversation between the child and me are led by the child's responses. If I was unclear about the child's meaning, I would ask further questions and introduce new conditions into the task so as to obtain a deeper understanding. As a result of this student-centred approach, each conversation and related series of actions with the six participants are unique. Although common elements among interview participants can be identified, given such a small sample size I was more interested in what could be learned about each individual child through the chosen tasks and analytic theories.

It is also important to note that I am the current teacher of the interview participants. Considering the young age of the participants (kindergarten) it is vital that they feel comfortable and familiar with their surroundings in order for them to focus on the tasks and explain their thinking clearly. The familiarity between the interviewer and participant created a relaxed atmosphere which aided greatly in communicating to the child that the I was truly interested in what he or she has to say. The students were naturally curious about why they were asked to participate in this study. I explained to each child that I am a student myself and I would like them to teach me about their thinking. My students took this role seriously and were highly motivated to share their ideas about the tasks with me.

The Classroom Program

Beginning in September and continuing throughout the school year, the children have been involved in weekly free building and exploration of the materials used in the study. At the time of my data collection I had not provided formal instruction in the names of geometrical shapes and solids nor the characteristics which distinguish one from another. My students have the materials as an option during free choice times along with non-mathematical centres such as house and theatre. Once each week, I partner the students up and ask them to build anything they wish using a choice of stacking materials and geometric shapes and solids. This weekly building session is a time of unstructured exploration and was in no way intended as an instructional time to prepare the children for my study tasks.

Sample Analysis

In order to provide a deep analysis of what each of the three tasks reveals about my participants' spatial understandings, I chose to concentrate on three participants. By narrowing my focus to three students, I was able to apply the theories of Piaget and Pirie and Kieren at a detailed level. The goal of my analysis was to utilize these theories to explain how children show their understandings of spatial concepts when they are engaged in specific open-ended tasks. The criteria I used for choosing the three children for in-depth data analysis were three-fold: The three children needed to: (1) show a high level of interest and confidence as they responded to my instructions and questions; (2) represent a diversity of strategies or styles in their constructions and explanations; (3) articulate their ideas and proofs clearly. The three children I chose were assigned the pseudonyms Anna, Brian, and Chris. All three children are in my French immersion kindergarten class. I will describe each child briefly.

Anna was five years and one month old at the time of our first session. She is a highly articulate and independent child who often invoked humour into our conversations. Anna was highly curious about my project and asked many questions. She seemed genuinely interested in teaching me about her thinking. Brian was five years and ten months old at the time of our first session. He is also a very independent child who enjoys building and drawing elaborate designs. Brian's quick responses to my questions and his requests for harder tasks indicated to me that he was a confident thinker who was comfortable interacting with me during our sessions. Chris was five years and eight months old at the time of our first session. He is an energetic and highly kinesthetic learner who was calm and attentive throughout all three sessions. Chris was able to describe his thinking and prove his claims to a high degree of detail. His explanations provide information about what he knows and where he learned it. While all of the participants were given identical tasks and consistent instructions, the data analysis chapters of this study (4, 5, & 6) recite the actions and dialogue of three unique individuals.

Data Analysis

The use of a video camera as a data recording tool carries many advantages and challenges. While the camera is able to capture all of the dialogue and physical activity between researcher and participant, the action often moves very quickly making it difficult for the researcher to note all salient details when reviewing the video data later for analysis. For this reason, I watched the video tape and transcribed the dialogue word for word recording our actions in brackets for all sessions. In order to be accurate, I watched each session many times until I felt that I had an exact record of conversation and an accurate description of our corresponding actions. I followed this process for all six of my original participants. After examining the video and transcripted notes, I narrowed my focus to the three participants described earlier and began my analysis.

In analyzing the data from the first session, my purpose was to describe each child's understanding of geometric concepts in terms of Piaget's topological primacy thesis. While I anticipated that my participants would not necessarily fit perfectly into one of the three levels described by Piaget, I was interested in finding out what open-ended tasks revealed in reference to these levels about their conception of shape and space. I analyzed the data by making notes on the transcription using the descriptors from Piaget's primacy thesis to highlight specific comments or actions. A sample transcript with notes can be found in Appendix B. As I wrote chapter four which describes the first session tasks, I frequently reviewed the video data to ensure that I was accurately depicting the child's words and actions.

For the second set of tasks, I again worked very closely with the video data to obtain exact quotes while making notes as to actions and comments that possibly related to the Pirie-Kieren theory. While it was helpful to produce a transcript, when it came to describing my participants' understanding using the language of the Pirie-Kieren theory, I found that I needed to work directly with the video in order to capture even the subtlest movements and reactions of my participants to the tasks that I set before them. As I wrote a description of each child's comments and actions, I paused frequently to review the video tape and integrate the Pirie-Kieren theory into the data. The video data is therefore an essential companion to the text analyzing the second session tasks found in chapter five.

After transcribing the data from our third session, I reviewed my participants' comments and circled key spatial vocabulary such as "inside", "beside" or "on" and "by". Secondly, I used the still photographs I had taken of their constructions and noted examples of pieces of their constructions that reflected the levels of Reifel's developmental scheme.

Chapter IV

Results for Session 1: Comparing Similar Shapes

In this chapter, each child's responses to the three tasks conducted during the first session are described and analyzed according to Piaget's topological primacy thesis.

Anna

A: Same and Different

We began our first session by discussing the triangle and circle attribute blocks. I ask Anna how the two shapes were the same and she immediately engaged the shapes in a moving image: "You put this one rolling down here it looks like a snowball falling down a mountain." When I ask if they are alike she responds, "No. This one looks like a ball (circle) and this one (triangle) looks like a block." In discussing how they are different, Anna places the circle first to the left of the triangle, then above, to the right and finally below stating "they're not the same" at each of the four positions. Anna is displaying that she recognizes the shapes do not resemble one another no matter where one is placed in reference to the other. I probe for further detail by asking what it is about the two shapes that is not the same. Receiving no response, I say, "What do they look like and how is that different?" Anna then places the circle on the top point of the triangle and says, "Well if you put them like this, it looks like a party hat, toot!" It seems as though the phrase *look like* has stimulated Anna to describe other objects in the world that feature circles and triangles. Next, I separate the two shapes and ask Anna to tell me how one was different from the other. She traces her finger around the sides of the triangle and says, "This one is a triangle" and then traces the circle stating, "This one is a circle. So they're not alike."

Anna's thinking is clearly topological. One shape is a triangle, the other a circle, so they cannot be alike. I continue by asking Anna to tell me about the triangle. She stays close to her definitions, "This triangle is shaped different. This one is shaped like a circle and this one (triangle) is shaped sort of like a octagon but it's not." I query, "What is an octagon?" Anna then raises her finger and traces a six-sided shape in the air (she draws six lines changing directions slightly five times in a circular direction). Anna then goes on to explain a computer game called, "Candyland" in which the goal is to place some mixed shapes in the right place.

Next I give Anna a square and a triangle and go through the same series of questions. After quickly telling me that both objects are blue, Anna places the triangle on top of the square and says simply, "House." When asked if there was anything about these two objects that is the same, Anna says, "No. This one is shaped like a square and this one is shaped like a triangle so they're not the same." Once again, Anna's conclusions are topological in nature. When asked about how the shapes are different, Anna says, "Because this is shaped different from this." As I probe further Anna begins to play with the shapes placing the triangle upside down, placing the square on one corner on top of the triangle, making the shapes dance and so on.

Moving on to the last set of shapes, a square and a rectangle, Anna immediately responds that the two shapes are not the same because they are "shaped different". Pointing to the rectangle, she says, "This one is shaped like a ...... What do you call it again?" I prompt, "Do you remember?", pause for about five seconds and then suggest, "A rectangle?" Anna responds very excitedly, "Yeah!" and then continues her train of thought, "This one is a rectangle and this one is a square because they're different names."

I review, "Why are they different?" Anna replies, "Because they have different names and different colours." Again, Anna's reasons for same and different lie in the topology of the shapes. Anna then begins to play with the shapes by placing the rectangle to the left of the square (see figure 3, position A), "This looks like a sideward T," above the square (figure 3, position B), "Put them like this it doesn't match," and finally to the right of the square (figure 3, position C), "Even if you put this one like this it doesn't match." Anna

Figure 3:

A.

B.

C.

maintains her notion that no matter how you place the shapes, you cannot change the fact that they are different from one another. When asked why they don't match, Anna said, "Because they're shaped different; I told you that already." At this point I wanted to prove conclusively that these differences didn't go beyond a topological sense of these shapes. Anna's use of the phrases, "shaped differently" made me wonder if she was able to describe the shapes projectively. I review her words, "You said they were shaped different, how are they shaped different?" Anna replies, "They're shaped different because this one's a rectangle and this one's a square so if you put it on this side (places the square above the rectangle and points to the top side of the rectangle as shown in figure 4) it's longer (stretches out the word longer) and these sides (traces her finger around the

Figure 4:

square) are shorter." Then pointing to a short side of the rectangle, Anna says, "This side (A) is shorter than this one (B) and this side (C) is longer than this one (B)."

This final portion of our discussion led me to conclude that Anna does have a projective understanding of the difference between a square and a rectangle. The fact that it took a fair bit of prompting for her to use some projective descriptions suggests that for Anna, shapes are things that she sees in her world as individual images with clear labels or as differently shaped objects that can be combined to compose pictures.

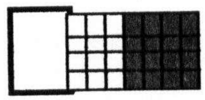
B: Build a Square

Giving Anna the square attribute block once more, I ask her to build that shape using one inch colour tiles. I do not tell her the dimensions of the tiles. Anna builds a four by four square, right next to the attribute block. The attribute square is very slightly (about one eighth of an inch) shorter than four tiles. While Anna does not mention this difference here, it does stimulate some discussion later. I then ask her to tell me how the shape she built is like the shape I gave her. Anna responds without hesitation, "This is a square and this is a square so they're alike. They don't have the same colour but they have the same shape." I probe further, "Show me where they're alike." Anna pauses and then says, "They're not alike because they're different colours." Anna is able to consider different attributes of the shapes independently. In dealing with shape, the two objects are alike because they are both squares. In dealing with colour however, the objects are not alike.

C: Build a Rectangle

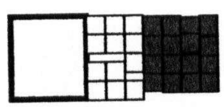
I next ask Anna to build a rectangle by showing her a rectangle attribute block and asking her to make that shape. Anna builds a rectangle 3 tiles wide by 4 tiles long right next to the attribute block and she has lined up her tiles with the top side of the attribute block. She builds the rectangle right next to her constructed square (figure 5):

Figure 5:



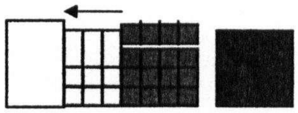
Anna's constructed rectangle is about one quarter of an inch (one quarter of a colour tile) shorter than the attribute block rectangle. When I ask Anna if her shape is the same as the one I gave her, it is evident that she can see the height difference, "It's a little bit smaller but I think I can change it." Anna then proceeds to move her shape down beginning with the first row which she slides even with the bottom of the attribute block leaving some space in the centre of her constructed rectangle such that the top of her rectangle still lined up with the top of the attribute block. She moves only five tiles so that her construction now looks like this (figure 6):

Figure 6:



She explains as she works, "So if I put one down here (points to the bottom right corner of the attribute block where it touches the left corner of her constructed rectangle) instead of up here (points to the top right corner of the attribute block where it touches the top left corner of her tiled rectangle) it would make it more bigger and it would go up to here (points to a spot about an inch below the bottom of her constructed rectangle)." Needing to confirm that she was talking about adding another tile to her construction to make it taller, I ask, "If you added another tile?" Anna confirms, "Yeah."

Our next task was to compare her constructed square with her constructed rectangle to see if she was able to identify some specific size differences such as the amount of tiles per side that she used in each construction. I begin by confirming that she believes her constructed rectangle is indeed a rectangle, "Is it the same shape?" Anna nods, "Uh hm." I probe further, "What is it called again?" Anna pauses for about ten seconds and I hint, "Rrrrrrr." Another pause and then, "A riangle?" I correct, "Rectangle. Is this (pointing to her constructed rectangle) a rectangle too?" Anna responds, "No, because if I put one up here (points to the top of her rectangle) it would make it more bigger. I can't do it very really." I confirm, "So, this shape (constructed rectangle) is not a rectangle?" Anna replies, "Its sort of like one." We begin a comparative discussion, "Is this shape (constructed rectangle) more like this shape (constructed square) or more like this shape (attribute rectangle)?" Anna then moves the bottom left tile of her constructed square down so that it is level with her constructed rectangle at the bottom and comments, "If I change it to here." I interrupt, "But the way it is now." Anna stops moving tiles and replies, "This one (points to constructed square) is like this one (points to attribute rectangle)." While I am asking her about her constructed rectangle, Anna responds by saying that her constructed square was like the attribute rectangle. Anna continues moving the tiles of her constructed square until the bottom three rows are level with the other two shapes leaving the top row even with the top of the attribute rectangle (see figure 7). Even when I place the attribute square next to her constructed square and

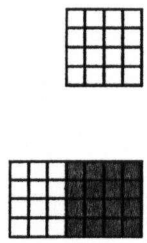
Figure 7: 

remind her that this shape is what I asked her to build she still claims that her constructed shape is more similar to the attribute rectangle: "This is the shape you were building. So is this shape (constructed square) more like this shape (attribute square) or more like this one (attribute rectangle)?" Anna replies, "More like this (points to attribute rectangle) because it's longer and it doesn't go like to here (draws an imaginary line from the top left corner of her constructed square to the top right corner of the attribute rectangle as shown by the arrow). What is unclear to me at this point is that although she labels her constructed square a square and her constructed rectangle a rectangle, she maintains that her constructed square is more like the attribute rectangle than the attribute square it was modeled after. Anna's statement that her constructed square is "longer" seems to indicate that she has noticed that her constructed square is slightly taller than the attribute square and the same height as her constructed rectangle. I continue to seek clarification, "Is this shape (constructed square) a square or a rectangle (I point to the attribute rectangle)?" Anna replies, "A square." "How do you know it's a square?" I ask. Anna says, "Because it's a different colour." I press, "So, squares are blue and rectangles are yellow?" Anna slaps her hand against her head and says, "I forgot the blue one!" At this moment, Anna seems to think that she should have made her constructed square the same colour as the attribute square I gave her. I reassure her that the one she made is fine and continue questioning her. The reason for her thinking in this way becomes clear as our conversation continues. I review once again, "Is this (pointing to constructed square) a square or a rectangle?" Anna responds, "A square." I ask, "How come?" Her reply is confusing, "Because it doesn't look like this one (points to constructed square) but it looks like this one (points to constructed rectangle)." I ask her to name all four shapes one last time. She labels all four correctly. Then, I ask her why the two squares are squares and the two rectangles rectangles. Anna's response shows clearly that she was simply following my directions and that perhaps the names she knows are assigned to each shape do not mean that they are different: "Because you asked me to build this kind (points to attribute square) so I builded it and I did it in this sort of shape (points to constructed square) and you asked me to build this one (points to constructed rectangle) next and then it looked like this shape (points to attribute rectangle) but it's not." This last phrase "but it's not" seems to indicate that she now feels her constructed rectangle doesn't really resemble the attribute rectangle. Anna's words seem to contradict earlier statements in which she appeared to believe that two shapes with different names are not the same. When Anna's constructed rectangle and constructed square are placed side by side she is able to see that they are the same height and this observation seems to be important enough to her that it overrides the different names of the shapes and renders them the same. Throughout this section of the discussion, as I make inquiries as to how the shapes are the same and whether they are still squares or rectangles, Anna is busy moving tiles around to make her constructed shapes level at the top or bottom with the attribute shapes. She appears to pay quite a lot of attention to the height of the shapes and is clearly aware that the attribute rectangle is slightly taller than her two constructed shapes. Her attention to the relative sizes of these shapes as well as the discovery that when placed side by side, her constructed square is the same height as her constructed rectangle show that Anna is beginning to be aware of the Euclidean properties of shapes. Anna's observation that her constructed square is more like her constructed rectangle than the attribute square suggests that Anna places importance on the fact that her two constructed shapes are the same height. Despite this realization, Anna maintains that her constructed square and constructed rectangle are named as such by recalling that I asked her to build those two shapes while pointing out the attribute square and rectangle. In comparing her constructed square and rectangle, Anna does not reconcile the paradox of the sameness of their height (Euclidean focus) with their different topologies. In other words, it is not clear how Anna is managing the realization that two different types of shapes can be the same height. Anna does not yet seem to be aware that it is the relative sizes of the dimensions of these shapes (a Euclidean concept) that serve as criteria for their labels. Specifically, Anna does not express that all four sides of her square are the same size, and that while her rectangle is as tall as her square it is not as wide.

Our next few exchanges further illustrate Anna's developing understanding of squares and rectangles. Taking away the attribute shapes, I ask, "How are these two shapes different?" Anna replies, "They're not different. They're the same." I confirm, "Are they exactly the same?" Anna explains, "Yeah, because they have the same shape (points to both shapes). I mean they have the same size (Anna is lining up the top rows of the tiles to make them perfectly even)." I ask, "Is this shape (constructed rectangle) exactly like this one (constructed square) except the colour?" Anna confirms, "Um, except the colour but it's exactly like this shape." While's Anna's criteria for sameness requires that she deduce the Euclidean characteristic of relative height between objects, she does not seem to notice the difference in the width of the two shapes. Her focus on their same height is so strong that it overrides the different labels she firmly assigns to the shapes and renders them "exactly the same."

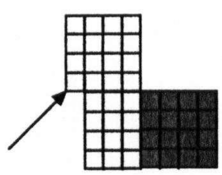
I continue to explore this paradox by constructing another four tiles high by four tiles wide square and placing it above her constructed square (see figure 8). I ask Anna if

Figure 8:



my constructed square is the same as her constructed rectangle. She immediately says, "It's the same colour but not the same shape." When I ask her why, she indicates that she would like to move my shape and I allow her to do so. She moves it on top of her constructed rectangle as shown in figure 9. I ask, "Is this shape (indicating my yellow

Figure 9:

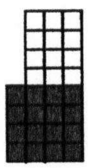


constructed square) the same as this shape (pointing to her constructed rectangle)?" Anna replies, "No because this one is bigger (pointing to the left column of my constructed square)." I then ask, "Is this shape (my yellow square) the same as this shape

(Anna's red square)?" Anna responds, "Yes, but except the colour." I then move my constructed square back to its original position (see figure 8) and ask, "Is this red shape (her square) the same as this yellow shape (my yellow square) except the colour?" Anna pauses here for about ten seconds before responding affirmatively and then provides an unclear comparison when I ask why the two shapes are the same, "Because it looks like me because look it isn't crooked it is straight."

In our final few exchanges, I move my yellow constructed square above her constructed rectangle once more (see figure 9) and Anna maintains that my square is "longer than this one (her rectangle)." She also maintains that her constructed rectangle and constructed square are the same. When I move her constructed rectangle above her constructed square (see figure 10) and ask if the two shapes are the same, Anna says, "On the top it's not; on the side it is." When I ask, "Why not?" Anna points to the bottom of

Figure 10:



her rectangle and says, "Because this side is shorter because it looks like a…what was it called? (reaches for the attribute rectangle) This shape?" I hint, "A rec.." and Anna guesses, "Rectagon?" I correct, "Rectangle" and ask, "Is this (pointing to her rectangle) a rectangle when it's over here (pointing beside her square)?" Anna says, "Yes," and then explains by revealing her Euclidean reasoning, "When this (her rectangle) is over here (points to the top of her square) it's not the same because this short side of the rectangle it's short side and this is a long side and this (indicates side of square) is long as it." I query, "This is long as what?" and Anna uses her finger to show me that the left vertical side of her square is "long as" the right (vertical) side of her rectangle. While it is clear from this exchange that Anna is aware of the fact that rectangles have short and long sides, she appears to focus on length and width as two separate criteria for sameness rather than generalizing that the two shapes are different because of the relationship of length and width featured in each shape. I conclude our discussion by pretending to move her rectangle beside her square and asking. "But when I move this over here are they the same?" Anna confirms, "Yes." Although the labels *square* and *rectangle* do not change in different positions as evidenced by my questioning earlier, the two shapes are considered to be the same if their position reveals that one of their dimensions is equal in size.

Brian

A: Same and Different

Brian communicated to me early on in this first session that he wanted to be challenged with "something hard". He pursues each task in a confident manner and often uses the phrase, "That's easy" before revealing his observations about the shapes. For the first exercise, Brian relates the triangle to the circle by showing me that they can be placed together to make a face with a hat. When I ask more specifically if there is anything about what the shapes look like that is the same, Brian demonstrates projective reasoning, "Well they have sides. This is a side (pointing to one side of the triangle) and all of this is a side (tracing the edge of the circle with his finger). When asked what is different about the shapes, Brian uses similar reasoning but provides more detail, "That's easy. This one (points to the triangle) has corners. This one (circle) only has none. This one's (triangle) got three straight lines. This one (circle) has 1 straight line around it. It is interesting to note that Brian is the only student out of the six participants who uses the term "sides" and describes the nature of the sides of these shapes.

When comparing the square with the triangle, Brian expresses the following projective properties: "This one's (square) got corners and so does this one (triangle). This one's (square) got sides and so does this one (triangle)." Brian also refers to the sides and corners of the square and triangle to answer the question of how they are different: "Ooh! 'Cause you can see there's three of these (points to each side of the triangle) and there's four of them (slides his finger along each side of the square) and there's four corners (points to the corners of the square) and only three corners (triangle).

While reviewing this task on videotape, it is clear that Brian does not indicate that each shape has different topological labels. In fact, he never uses the terms "triangle", "circle", or "rectangle". The term square is not used until he is comparing the square with the rectangle. Even then, Brian only uses this word once. In the final comparison, Brian's use of Euclidean reasoning further shows that he is able to perceive the salient attributes that distinguish these basic shapes. The absence of topological terminology in Brian's discourse should not necessarily be interpreted as a lack of knowledge of their geometric names. Brian's focus on projective aspects of the shapes is indicative that he perceives topological differences as well. If he did not believe that the shapes look the same or different, then he would not be able to give specific examples of how they are the same or different. Given his extensive vocabulary, experience in pre-school and passion for building with 2- and 3-dimensional shapes, I am quite certain that Brian knows the terms "circle", "square", and "triangle" in reference to those shapes. I am uncertain as to why Brian does not use these terms. However, his focus on corners and sides is consistent throughout our discussion suggesting that he considers these aspects of the shapes to be significant.

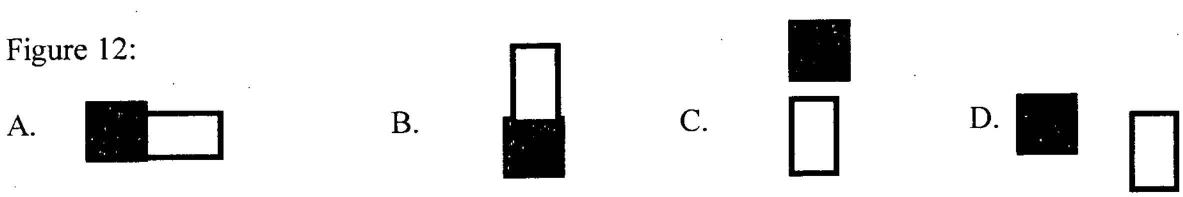
As soon as I give Brian the attribute rectangle and square, he places them together in two positions as seen in figure 11. With the shapes in position B, Brian responds to my

Figure 11:

question about how the square and rectangle are the same using the following Euclidean terms to compare them: "This one's (rectangle) just shaped like it except only a tiny bit bigger." When I again ask Brian to tell me how they look the same, he moves the shapes back to position A and describes how he would alter the two shapes to make them the same: "If it (rectangle) was this tall (points to the top of the square) it could be a shape and if you cut this off (slides his finger along the dotted line on the rectangle pretending to cut the shape) and it was about that high (touches the top of the square) it could be the same square, except only yellow."

Brian is able to compare the length and width dimensions of the two shapes without having to move them around such that their relative length and width are easier to perceive. It is becoming clear that Brian is quite perceptive of the subtle differences of square and rectangles. When asked how the two shapes are different, Brian briefly reviews his prior projective arguments. "That's kind of easy. It's going to be the same corner thing or it's going to be the same side thing." When prompted for more information about the "corner and side thing", Brian places the shapes in four positions in the order shown in figure 12 and provides Euclidean comments at three of the positions:

Figure 12:



"These corners (traces the length of the rectangle vertically from top to bottom corners with a finger on each side with shapes in position C) are wider and these ones (traces the vertical sides of a square with a finger on each side) are closer together and because about these (places shapes back in position A and traces the length of one side of the rectangle) well they're bigger and sometimes these ones (points to one length of the rectangle) are bigger (moves to position D) than these ones (points to one side of the square) and these ones (points to the width of the rectangle) are smaller than these ones (points to one side of the square)." Brian's comments and his use of the term "sometimes" seem to indicate that he realizes that four-sided, four-cornered shapes occur in different sizes. Also, while Brian observes that the sides of the rectangle are longer than the sides of the square, this does not necessarily mean that he is aware that these two shapes have different names. It is possible that he sees them as two differently-sized squares. The label "rectangle" is one that I observe children of this age seldom use without guided investigation of the difference between squares and rectangles. At this point in our classroom math program, we had not examined squares and rectangles for the specific purpose of comparison. Brian has again made no reference to the names of the shapes but rather focuses his comparison on projective and Euclidean aspects. Since Brian does describe these specific differences, it is likely that he perceives more general topological differences as well.

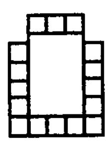
B: Build a Square

When given the attribute square and asked to build that shape with colour tiles, Brian builds the same four by four tile square constructed by Anna. In order to prove that his is the same as the attribute shape, he places the attribute shape on top of his tiled square and says, "Because if you put it like this like you can't see except it might be a little crooked, if you put it like this you won't see anything." Brian is the only student to take this approach. When prompted for further information about how the two shapes are the same, Brian does not offer any other evidence.

C: Build a Rectangle

In building a shape similar to the attribute rectangle, Brian begins by building a column of tiles against the right side of the attribute rectangle. As he is adding a fifth tile to this column, Brian discovers that the fifth tile will make his shape taller than the attribute rectangle: "I'm going to need a little thing like about that big (uses his thumb and forefinger to show me the space that is left between the fourth tile and the top of the attribute rectangle) because if I do it this way (adds the fifth tile) it won't match." I encourage Brian to build it as close as he can. He continues to build around the attribute rectangle and states that he has finished when his construction resembles figure 13.

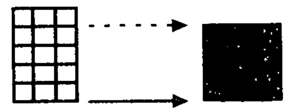
Figure 13:



I probe, "How is the shape you built like the shape I gave you?" As Brian begins to respond, he takes the rectangle out of the middle, moves the two columns closer together and adds a column in the middle changing his shape to a three by five tiled rectangle. He then covers his tiled rectangle with the attribute rectangle and says, "Because if you put it on you won't see a bit." When I push for more information, he repeats this phrase.

In the final segment for this first session, I ask Brian to compare his constructed rectangle with his constructed square. I begin by asking him to tell me how the two shapes he constructed are different. His two constructed shapes are side by side separated by about five inches of space as displayed in figure 14. Brian's explanation demonstrates

Figure 14:



his awareness that the two shapes have different heights and widths: "Because it goes to here (traces the solid red arrowed line at the bottom of the two shapes as in figure 14) and it only goes up to here (traces the dotted arrowed line between the two shapes) and if you want this one (points to constructed rectangle) you're going to have to take away these (points to the last column of tiles on the right side of his square) and make it a bit taller (points to the top of his square)." In response to the question, "Is there any way that they are the same?", Brian again describes how they could be transformed into the same shape: "Well kind of. If you take these things away (points to the top row of the rectangle) and you put four on this (points to the inside column of the rectangle) you'll make a square." Brian's description shows his awareness that the two shapes differ in height and width by one row and one column of tiles respectively. Brian does not move the shapes at all as he makes these comments showing that he can perceive relative width differences without needing to line the shapes up vertically.

Chris

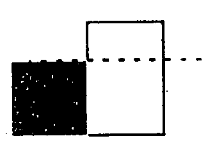
A: Same and Different

When we sat down together at our first session Chris seemed slightly nervous and restless. This condition faded as he became more engaged in the tasks. Throughout our discussion, Chris takes his time before responding and he sometimes appears to be unsure. Once he begins his explanations, his thoughts are well-supported and he is able to demonstrate his thinking with the materials. When I ask Chris the very first question about how the triangle attribute block is the same as the circle he responds, "There's no way they're the same because this one (picks up the circle) is just like round and plain and this one (picks up the triangle) is a different colour and a different shape so it can't be like this one." Chris has given the two topological attributes of colour and shape to justify his claim that the two shapes are different. When I press for more information by asking how they are different, Chris states that there are lots of ways in which they are different. He repeats the colour difference (the circle is yellow and the triangle is blue) and the fact that they are both a different shape. When I ask him to describe how each of them is shaped, Chris continues to use topological language by responding, "like a circle" and "like a triangle". Seeking projective justification, I ask Chris to tell me about circles: "What are they like?" In a similar way to Anna's response to my use of the term "like", Chris identifies objects from the world that are like circles and triangles, "Circles, they're like wheels. Triangles, they're like mountains."

Next I give Chris the square and triangle attribute blocks and question him about how they are the same. Chris states that he doesn't think they are the same "because they are different shapes." When asked how they are different, Chris points out projective features of the two shapes, "There's only one way they're different. This one (points to the square) has more corners than this one (triangle)." When I ask him how many corners the square has, he quickly states four without counting them in any obvious way. When I ask him to show the corners to me he points to only one corner of the square which may indicate that he is aware that one corner looks just the same as the others. When I ask him how many corners there are on the triangle, he says, "Trois" (three). His demonstration of those corners is again limited to just one which he points to with his finger.

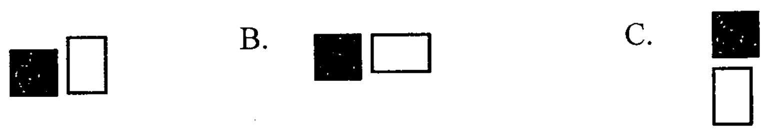
Finally, I ask Chris about the square and the rectangle. When asked how the two shapes are the same, Chris immediately recognizes topological aspects of colour and shape: "But they're really different because this one's a different shape and a different colour." Before I respond, Chris adds a projective level to his explanation, "But they both have the same amount of corners." As I press Chris to explain how they are shaped differently, he seems to be deep in thought as he moves the shapes around on the table in different configurations. After a long pause (about 30 seconds), Chris says the he doesn't know. I try to be specific, "What would this shape (pointing to the square) have to be like to be the same as this shape (pointing to the rectangle)?" Chris has lined the shapes up side by side and his precise Euclidean response surprises me, "You would have to cut it in half like this (uses one finger to "cut" the rectangle as shown in figure 15)." I then ask

Figure 15



Chris how the two shapes are not the same and he seems to have regained his confidence: "This one (rectangle) is bigger than this one (square). Anyway you put it it's bigger." As he is making this statement, Chris begins to move the shapes into different positions repeating his last phrase, "Anyway you put it it's bigger" each time he moves the shapes into a configuration. Figure 16 shows the various positions Chris uses to demonstrate that the rectangle is bigger. He cycles through the positions several times.

Figure 16: A. B. C.



It is important to note here that in positions B and C the greater width of the square can be perceived. Chris does not take notice of this subtle difference (or he chooses not to comment on it) and he continually returns to position a after demonstrating position B or C. The pattern of configurations he shows is ABABAC. During this exercise, it appears that Chris is focusing solely on the differences in height (or length) of the two shapes which he expresses by using the term "bigger" when the two shapes are in position A. It is also possible that Chris is able to see this length difference in all of these positions. Having begun his explanation of how the two shapes are different by referring to the rectangle's superior size, Chris seems to stay with this observation as he manipulates the shapes to prove his claim. Since Chris seems to be considering only one dimension of the two shapes (length) in each of the above orientations his claim that yellow shape is "bigger" seems to be more generic and is thus perhaps more projective than Euclidean. By contrast, when Brian compares these two shapes, he notes their height and width differences, and demonstrates how the corners are wider apart on one shape (the rectangle) than the other (the square).

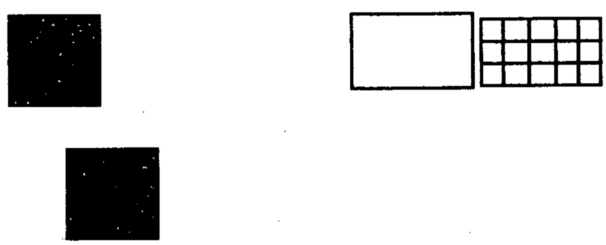
B: Build a Square

In the second task of the first session Chris demonstrates that he is able to support his projective explanations with specific numeric proof. Given the task of building a model of the attribute square with one inch colour tiles, Chris begins by saying, "I have to copy it to build it," and he proceeds to line the tiles up with the bottom of the attribute square. His process is very interesting. For the first row of his square, he places four colour tiles under the attribute square and then moves them down about an inch as if he has measured the attribute square and is satisfied that he is on the right track with his own square. Next, he adds another four-tiled row on top of his original row, carefully straightens the tiles and lines these two rows up under the attribute square as if to check that they match the model. He then moves these two rows down about an inch and adds a third four-tiled row again on top of the other two rows chatting all the while with me about the castle he hopes to build for me later. At this point, it is evident that Chris is aware that the attribute square is equivalent to four colour tiles in width. As soon as he has completed his third row, he moves the attribute square beside his tiled square, adds two tiles to the top (fourth row) and states, "I just need two more." When he has finished, I ask him how his shape is the same as the attribute shape. While I suspect that his building procedure indicates that he perceives a similarity in size, his explanation reveals topological and projective reasoning: "It's like it because it has the same shape and the same colour and it has the same amount of corners."

C: Build a Rectangle

For the task of building a rectangle with attribute blocks, Chris follows a very similar procedure as he used to construct the square. He begins by placing the attribute rectangle on the table horizontally and then lines up five colour tiles underneath. When he has finished the first row, Chris slides it down about an inch and adds the first tile of a second row on top. As he begins the second row, he moves the attribute rectangle beside his construction perhaps to check how close his constructed rectangle is to matching the width (vertical dimension) of the rectangle. The final stage of construction is captured in figure 17. It is interesting to note that when he was constructing the square, Chris did

Figure 17:



not check the vertical dimension until he had completed three rows. Thus Chris' verification of his rectangle at two rows is possibly evidence that he perceives a width difference. After his second row is completed, Chris immediately adds a third row to the top and declares that he has finished. When I ask him how the shape he has made is the same as the shape I gave him he responds with three criteria: "It has the same amount of corners, it has the same shape, and it's always the same colour."

In an effort to discover if Chris will use quantitative means to compare the two constructed shapes, I ask him how his square is the same as his rectangle. I point out the shapes rather than using their labels because I want him to look beyond topological names. Chris replies simply, "Well they have the same amount of corners." When I ask him how the two shapes are different he provides three topological criteria concerning their differences as well as an additional projective attribute for sameness: "It's just because they're different shapes, different colours, different looks like. And they don't have three corners." Chris' use of the term "three corners" suggests that he is referring to the triangle we discussed earlier in this session. Since neither the square nor the rectangle resemble a triangle, this is something that they have in common. His response to my next question, "Is this shape (pointing to constructed rectangle) the same as this shape (pointing to constructed square)?" is an emphatic, "No way." I ask, "Why not?" and Chris seems unsure, "Because I don't know how...because." I review: "You said they're different shapes. What exactly is different about their shapes? If they both have four corners how can they be different shapes?" At this point Chris slides the attribute square on top of the attribute rectangle which is in a horizontal position and explains, "This thing (touches the attribute rectangle) is longer any way you put the square. Anyway you put the square this thing is longer." Chris again shows me position A in figure 16 as he is speaking. Seeking more specific proof, I ask Chris if his constructed rectangle is longer than his constructed square. Chris replies affirmatively and then provides Euclidean justification. The following conversation ensues:

Researcher: Is this (constructed rectangle) longer than this (constructed square)?

Chris: Yes.

R: How do you know?

C: About an inch longer.

R: How do you know?

C: Because I've seen the shapes before and I've measured them in my head.

R: Which shapes? These shapes (pointing to the attribute shapes)?

C: Yeah. I measured them in my head.

R: Is this (constructed rectangle) longer than this (constructed square)?

C: Yes.

R: How do you know? You just built it?

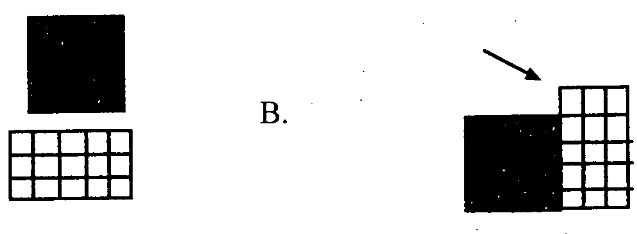
C: I knew it two years ago. I knew it when I was three. I just started knowing it when I was three.

R: Can you show me how this (constructed rectangle) is longer?

C: (Moves constructed square above constructed rectangle until they are positioned at A in figure 18).

Figure 18:

A. B.



C: Can you look at this? See? This thing (constructed rectangle), it's longer. And if I put it this way (moves rectangle into position B in figure 18) it will be still longer. See it's an inch longer.

R: Where's the inch?

C: Right here (points to the top left tile of his constructed rectangle).

Chris then begins to slide his square up as he points out that the inch could be at the bottom if the two shapes were lined up at the top. As he is moving his tiles back down, he appears to count one row of his square, pointing to each tile and quietly whispering. It becomes clear that Chris has a very strong grasp of the quantitative dimensions of the two shapes as evidenced by this final section of our discussion:

C: See how a square is four inches?

R: A square is four inches? Oh.

C: 'Cause one of these is an inch, right?

R: How do you know about inches? (We are indeed using one inch square tiles but I wish to find out how Chris knows this.)

C: My dad showed me an inch once.

R: Did he show you with these tiles?

C: No. He showed me how long they were like this long (shows me an approximate inch with the thumb and index finger of one hand on the table and then on a tile). He told me I think last Saturday I mean Sunday when I was at my grandma and grandpa's cabin skiing.

R: So this square is four inches?

C: Yeah.

R: What about the rectangle?

C: It's about five inches.

R: Where is that? Where's the five inches. Can you show me why you thought five inches?

C: I just looked at it and I thought it was an inch longer see?

While Chris does not count the tiles for me to prove his measurement figures, his calculation that five is one inch longer than four clearly shows that he is aware of their relative size difference. He is able to verbalize this Euclidean observation and show that his understanding holds true no matter how the two shapes are arranged. Anna's conclusion was similar. In addition, both Anna and Chris seem to focus on one dimension, either length or width, at a time rather than making generalizations about the relationship between the two dimensions featured in each shape. Finally, although Anna and Chris both display Euclidean understandings in showing that the two shapes are different in size, Anna's explanations do not provide the quantitative detail expressed by Chris.

Chapter V

Results for Session 2: Comparing Constructions

As described in chapter 3, I have chosen to analyze the children's activities and descriptions for the second set of tasks using the Pirie-Kieren model for growth of understanding. In using this model, my primary intention is to access its language as a way of talking about my participants' understanding. I am interested in *how* not *what* they understand. My analysis will focus on the image-making and image-having layers of the model by describing the images or mental representations I think each child is working with as he or she negotiates the questions and tasks I pose. Specifically, I will seek to describe what these images possibly indicate about each child's understanding of the concepts inherent in the tasks. I will also note any connections between images or property noticing the children might be engaging in. While as Pirie and Kieren point out, one can never be certain about the nature of a child's background knowledge, or "primitive knowing" (1992: p. 245), I will look for evidence that my participants have possibly "folded back" (Pirie & Kieren, 1992: p. 248) to previously-formed images such as those discussed in our first task to help them make sense of the new experiences in task 2.

To illustrate how the Pirie-Kieren model can be applied to children's workings and explanations in order to describe the images they are accessing, it is important to realize that specific concepts can carry several different images. For example, the concept of subtraction carries two distinct notions: *take away* and *difference between.* The problem, "What is 6 take away 3?" requires different understanding than "How many more cats does Joe have than Ruth?" Taking something away is not the same image as comparing two amounts. While a student could answer both questions correctly by subtracting the smaller number from the larger, this would not necessarily mean that he or she has understood the conceptual difference between the two problems. In a similar light, the concepts involved in the second session of my research carry a number of different potential images a child may use to make sense of the tasks.

This second set of tasks involves the construction of four products that can be compared in some way to the researcher's constructions. I am interested in the child's understanding of comparative size concepts such as longer, shorter, bigger, taller, wider, and narrower as displayed in his or her constructions and explanations.

The following description of Anna's constructing and related comments is quite dense and non- linear. The ideas behind Anna's images seem complex and she does not always appear to be confident in her responses. In an effort to understand how Anna understands the concepts involved, we often revisit previous thoughts she has voiced when new statements seem contradictory. Through this circular dialogue, this section focuses on evoking the images Anna possesses as a means to understanding in an effort to determine what properties if any she is able to notice about the shapes and patterns involved.

Possible Images

With respect to the notions *longer, shorter, bigger, smaller, taller, wider and narrower* that are dealt with in this chapter, there are three main images a person might possibly call upon when comparing the size of two 2-dimensional objects:

(1) Visual-the general appearance of the shapes, such as which "looks" wider.

(2) Quantitative-counting the number of tiles across a dimension of the shapes.

(3) Measurement-gauging the differences in size using a standard unit (for example a child might say that the shape is about 10 centimetres wide) or a non-standard unit such as the number of thumbs long a shape is.

A: Longer and Shorter

The Researcher's Chain

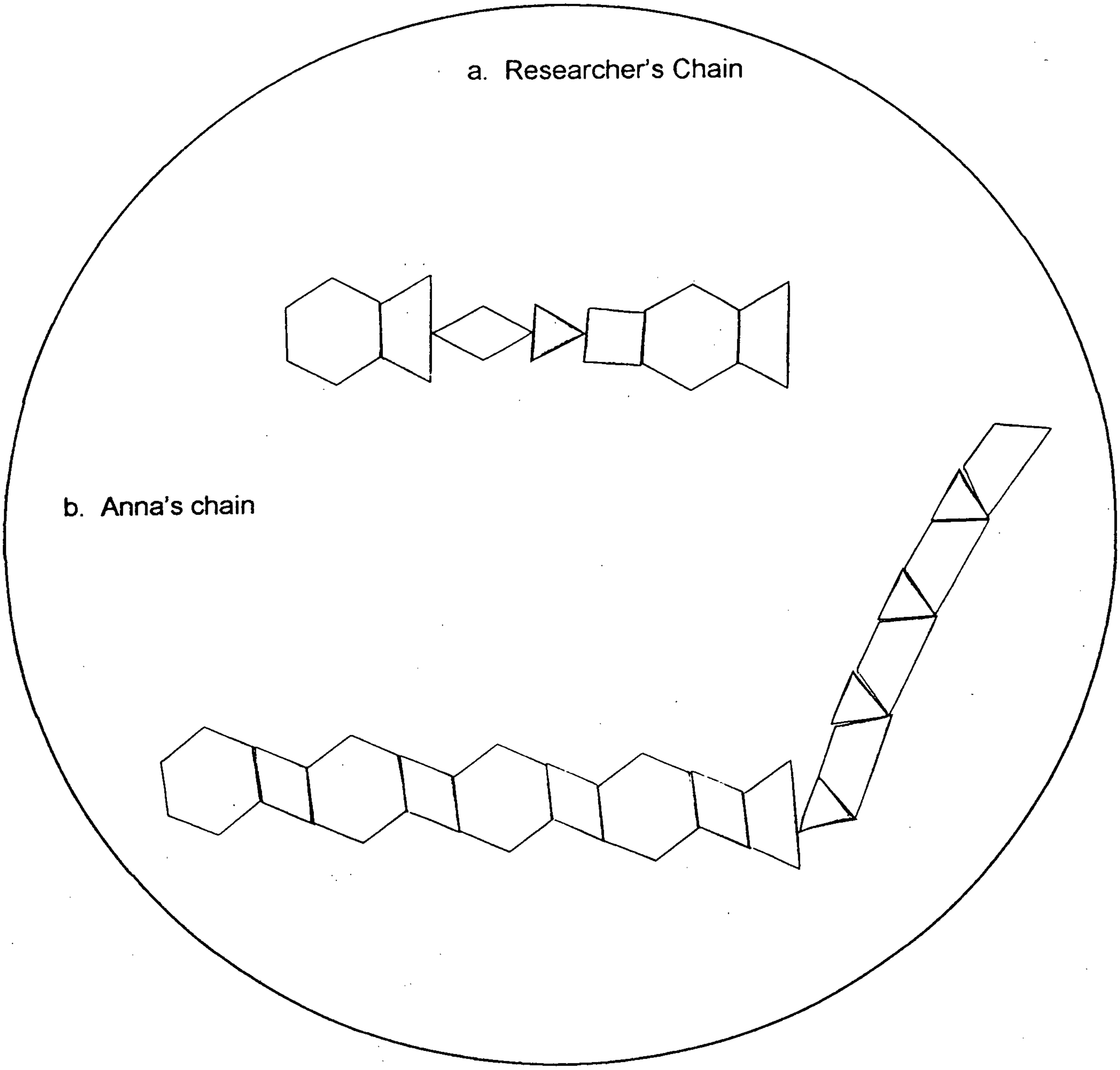
The researcher's chain is composed of seven pattern blocks lined up in a straight line. The shapes were placed in the following order: hexagon, trapezoid, rhombus, triangle, square, hexagon, trapezoid. The chain is placed on a table as displayed in figure 19.

Anna

Longer chain

When asked to make a chain that is longer than mine, Anna lines up pattern blocks from one end of the table to the other. Her chain consists of two different patterns each sixteen links long. Both patterns are AB patterns, one of hexagons and rhombi and the other of trapezoids and triangles. Each shape in the patterns is touching and the two patterns are connected and placed beside my chain as shown in figure 19. When invited to tell me about her chain, Anna says, "Because you made some little of these and you didn't

Figure 19:



even make it to the bottom of the table and I made it to the top and the bottom but you didn't." I reply, "So what does that mean?" and Anna says, "That means that mine is taller." Anna's explanation reveals her possible reliance on a visual image that longer objects start and end beyond shorter objects. Her use of the entire table strengthens the visual proof that she provides. Anna's use of the term "taller" perhaps indicates that in this context, "taller" and "longer" have the same meaning for Anna. When I ask her if her chain is longer than mine she replies without hesitation, "Yes." In an effort to discover if Anna will make use of a quantitative image to prove that her chain is longer by counting the number of shapes each of us has used, I ask Anna, "Is there any other way that you can prove to me, tell me, that your chain is longer than mine." Anna says confidently, "Yes, I can prove it. 'Cause look see it (pointing to my chain) goes all the way down to here (traces its length) and mine goes all the way up to here (traces its length)."" I try one more time, "Is there any *other* way you can tell me that it's longer?" Anna ends our discussion, "No." Once again, Anna relies on her visual image of longer as *beginning and ending beyond* to satisfy the problem that is posed to her. At this point, I wondered if perhaps Anna didn't count the shapes because either her chain was so obviously longer or there were so many shapes that she didn't want to count them all.

Shorter Chain

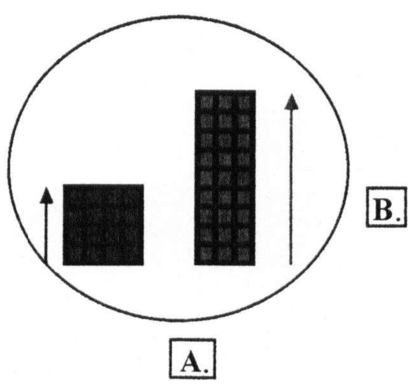
When asked to make a chain that is shorter than mine, Anna lines up four shapes (hexagon, square, hexagon, square) in a row. She again uses a visual comparison to prove that her chain is shorter, "Because your chain goes all the way up to here (traces its length) and mine goes all the way up to here (traces its length)." I again prompt, "Is there any other way you know it's shorter?" Anna replies, "No." I try a new term, "Is it bigger?" Anna says, "Smaller." Even when comparing two chains only seven and four links in length, Anna still does not utilize quantitative reasoning to demonstrate their different lengths. Therefore, I would conclude that Anna likely has a visual image for comparing lengths but not a quantitative image.

B: Bigger and Smaller, Taller and Wider

Build Bigger

When asked to make a shape that is *bigger* than my blue four by four tiled square, Anna makes a red rectangle that is quite tall (3x9) as displayed in figure 20. When asked if her shape is bigger than mine, Anna says, "Yes" and when asked how she knows, she

Figure 20:



traces the height of both shapes as illustrated by the arrows and explains, "Because your shape goes all the way up to here and my shape goes all the way up to here." Perhaps, *bigger* means *taller* for Anna. When asked if her shape is taller, Anna again says, "Yes" and points to the top of her shape when asked to show where it is taller. Anna responds to the words "bigger" and "taller" in the same way. It is possible that Anna has the same image for "bigger" as she has for "taller" because objects that are superior in height are so often referred to as "bigger" such as a child versus an adult, an elephant versus a giraffe, a house versus an apartment building and so on. While the width of these taller objects may also be greater, their relative height seems to be the more obvious dimension. Anna's experience with "bigger" (which she may be accessing from her primitive knowing) is possibly characterized by these types of comparisons.

<u>Wider</u>

In the next section of video, Anna and I are engaged in a discussion about whose shape is wider and why. When asked if her shape is wider than mine, Anna responds, "No, I didn't make it wider." Looking for more information about Anna's understanding of the term "wider" I ask, "What would it look like if it was wider?" Anna says, "A square." I continue, "Where is wider on the shape? Where is the wider part?" Tracing the length of her shape, Anna says, "These parts, they're wider see?" I repeat, "Is your shape wider than mine?" Anna affirms, "Yes." At this point, it appears that Anna is giving conflicting answers. However, on closer examination, it becomes clear that Anna has shifted position around the table from point A to point B as shown in figure 20. Thus, it is more obvious from her new vantage point, that her rectangle is wider than the square. Since I did not realize that Anna was looking at the shapes from a different perspective during the session, her answer puzzled me and I decided to find out more about her comment that if her shape was wider it would look like a square. I seek clarification, "You said at first that it would be wider if it was a square." Anna corrects, "I didn't say that. I said, that if it would be wider it would look like your square. It would look like a big, big square." The emphasis Anna places on "big, big square" suggests that she recognizes that in order to make her nine tile high rectangle into a square, it would be quite a large square compared to my four by four shape. I then ask Anna to make her shape wider. She begins to count out more tiles from the bin and I check her understanding of the task, "You're making yours wider?" Anna responds, "Yeah, very wide." Her use of the term "very" confirms her goal of making a "big, big square."

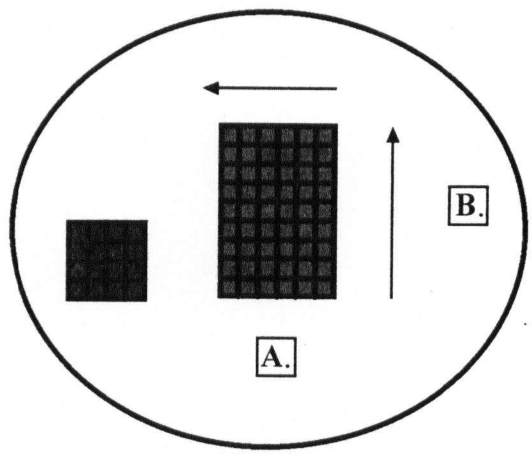
Once Anna adds on a fourth column to her rectangle, rendering it the same width as my square, I ask her if her rectangle is wider than my square and she says, "No. I haven't even got all the squares on." She does not point out the fact that the two shapes are the same width either because she is not aware of it or she does not consider it to be an important marker in her quest for a "big, big square" that is wider than mine. As she completes a fifth column, I again ask, "Is yours wider now?" Anna says, "Not yet." I push for details, "Show me where it's not wider." Anna takes her finger and traces a vertical line about an inch to the right of her shape and says, "It has to be at least there." I continue to question her to see if she recognizes that her shape is now wider than mine: "Is mine still wider right now?" Anna pauses and looks puzzled. I try not to pressure her, "I'm just asking. I don't know." Anna says, "I don't know either".

It is clear that Anna has only one image for comparing the two constructed shapes - a visual one. Her visual image of wider is consistent with the "real-world" definition of wide as *having great extent between sides*. Mathematically speaking, the width of a four-sided object is the measurement of its shorter sides. In this context, Anna is able to correctly make her shape wider because the base of her rectangle is horizontal. Therefore, extending her rectangle horizontally does increase its mathematical width since the base is shorter than the height. Her sole reliance on comparing the two shapes visually does not help Anna to successfully solve the problem because she does not have any other method of determining if she has reached the goal of making her shape wider. At no time throughout this clip does she either count the tiles or use some kind of standard or non-standard unit to estimate the relative widths.

In an effort to invite Anna to use another image to compare the width of the two shapes, my next question is, "Look at yours and look at mine -is there any way to *figure out* whose is wider?" Anna responds emphatically, "Mine will be if I finish this line!" At this moment, Anna has completed the fifth column and is proceeding to add on a sixth. I press, "Right now mine's wider than yours?" Anna sings, "Yeah. But it's not going to be!" Anna continues to build her shape so that she can distinguish them visually. From her perspective at point A as in figure 20, the widths of the two shapes cannot be easily compared visually. Anna's observations during our previous session suggest that if the shapes were aligned vertically at this point with her five tile wide rectangle above or below my four tile wide square she *would* recognize that her shape is wider. At the end of the first session as described in chapter 4, Anna was asked to compare a 4x4 tiled- square with a 4x3 tiled-rectangle. When placed side by side, Anna described the shapes as being the "same size" despite the fact that their widths were different. However, when placed one on top of the other, Anna was able to clearly see that there widths were different. Because Anna only has a visual image, the positioning of objects plays a crucial role in her comparison of dimensions. Also, it is possible that the vast tallness of her shape makes it more difficult for her to see the subtle difference in width.

When Anna finishes the sixth column of her rectangle, she is satisfied that her shape is wider than mine. Her explanation is puzzling, "Because this part (traces the length of her shape (nine tiles) along the right vertical side) is longer than this part (traces the six tile width of her shape along the top as shown in figure 21)." She then describes the attributes of a rectangle, perhaps illustrating that she is aware that her shape is still not a square. Using her finger to draw a rectangle in the air, Anna questions, "What do you

Figure 21:



call those things like this?" I ask, "Like what you made?" and Anna continues, "Yeah. The sides are short and the bottom is big and the top is big and the sides are short." I suggest, "A rectangle?" Anna responds excitedly, "Yeah!" I paraphrase, "So you're saying that your shape is wider than mine because this part (tracing the length her rectangle) is longer than this part (tracing the width of her rectangle)?" Anna quietly says, "Yeah." I confirm, "Is that what you said? Is that right?" Again, Anna says, "Yeah."

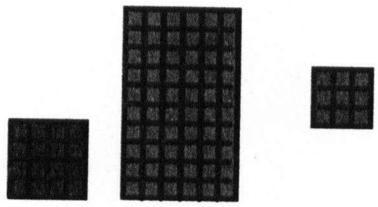
During this final exchange, Anna has again shifted around the table to point B as in figure 21. Thus it appears as though she is comparing the length of the rectangle (mathematical definition, the longer side) with the width of the square from that perspective. From point B, the nine-tile length of the rectangle is quite evidently longer than the four-tile width of the square. It is also possible, that Anna is focusing only on the rectangle and affirming that the rectangle is "wider" on its nine-tile side than on its six-tile side. Whichever conclusion is correct, it appears that Anna has not reconciled her image of wider as a "big, big square" with her observation that her shape has the attributes of a rectangle.

Interestingly, Anna never refers back to her original notion that in order for her shape to be wider than mine, it must be a square. The fact that she adds tiles on to the width dimension of her rectangle in order to make it wider seems to indicate that she has a horizontal dimension image for wider. Anna' s prediction that doing so would create a "big, big square" establishes a relationship between the horizontal dimension image and the big square image. Anna's efforts to build a big square by increasing the width of her rectangle also suggest that she has some generalized notion that squares are as wide as they are tall (property noticing). Despite these images, it appears as though in order for Anna to conclude that one shape is wider than another, the two shapes need to be lined up one on top of the other as in perspective B, figure 21 and also as discussed near the end of our first session (figure 10). Since Anna does not count the tiles to determine if one shape is taller or wider than the other, she must rely on the visual images that she either creates spontaneously or draws from previously-formed notions in her primitive knowing.

Build Smaller

In response to the challenge of building a shape that is smaller than my four by four tiled square, Anna builds her own three by three tiled square. She places her new shape next to her nine by six rectangle as seen in figure 22 and answers my inquiry about how she knows that her shape is smaller than mine

Figure 22:



with another visual illustration: "Because your shape is up to here (traces the height of my square) and my shape is up to here (traces the height of her square) see?" Anna's notion of *smaller* seems to involve the height of the shape just as *bigger* and *taller* were paired in the previous task. Once again I press her to move beyond her visual image, "I think mine looks about the same as yours. How can you prove to me it's not the same?" Anna surprises me with a quantitative image, "Because yours is bigger than mine and I only have (tracing each column of her square from top to bottom) three, three, three and you have four, four, four, four." I decide to check her image of the horizontal dimension, "Is your shape narrower than mine?" Anna suggests a more familiar term, "Smaller than yours." I try a different word, "Is it skinnier?" Anna laughs, "Skinnier? What does skinny mean?" I contrast, "Is it fatter?" Again, Anna is amused, "What do you mean fatter?" I clarify, "Does it look fatter?" Anna does not respond.

Switching to a new line of questioning I ask, "Is your shape *taller* than mine?" Anna responds affirmatively and shows me how her three by three square is "…up to here" while my four by four square is "way up to here so mine is shorter than yours." She then elaborates by exposing her quantitative image: "I only used three, three and three." I ask her to show me where she sees the three, three, and three and Anna counts the three tiles in each of the three columns starting from the bottom. When I ask if my shape is *wider* than hers, Anna says, "Yes." When she is asked to point out the wider part, she points to each side of my square stating, "This part" four times. I had expected Anna to recall the horizontal image for *wider* that she displayed in the previous task and apply the quantitative reasoning she used in counting the height of our two squares to this dimension. I then ask her to point right on the shape to show me the wide part and she puts her finger on one tile of each side stating, "Here and here and here and here." Perhaps it is because the shapes are as wide as they are long that Anna is not able to isolate the image of *wide*. When comparing a nine tile "wide" rectangle with a four tile wide square from point B in figure 21, the concept of wide is much clearer than in the case of two squares whose width differs by only one unit. This portion of our discussion thus further illustrates that Anna's image for *wider* is dependent on visual perception and is still not as stable as her image for *taller*.

This final activity of our second session served to illustrate that when given shapes with fewer tiles, Anna is able to use a second, quantitative image to compare the size of two shapes. Her image of small is clearly tied to her image of *tall* while her image of wide seems to be multi-directional. It is also clear that Anna does not have an image to relate ideas such as *narrow, skinny* or *fat* to the context of shapes. At this point, I cannot be certain that Anna's understanding of the concept of wide is any more secure than the image-making stage. Her understanding seems tentative in that she still appears to be working at getting ideas. In particular, when I give her specific words such as *narrow, skinny,* or *fat*, she does not find them useful in this context indicating that she has not yet developed images for these words that can be applied to describe shapes.

Brian

A: Longer and Shorter

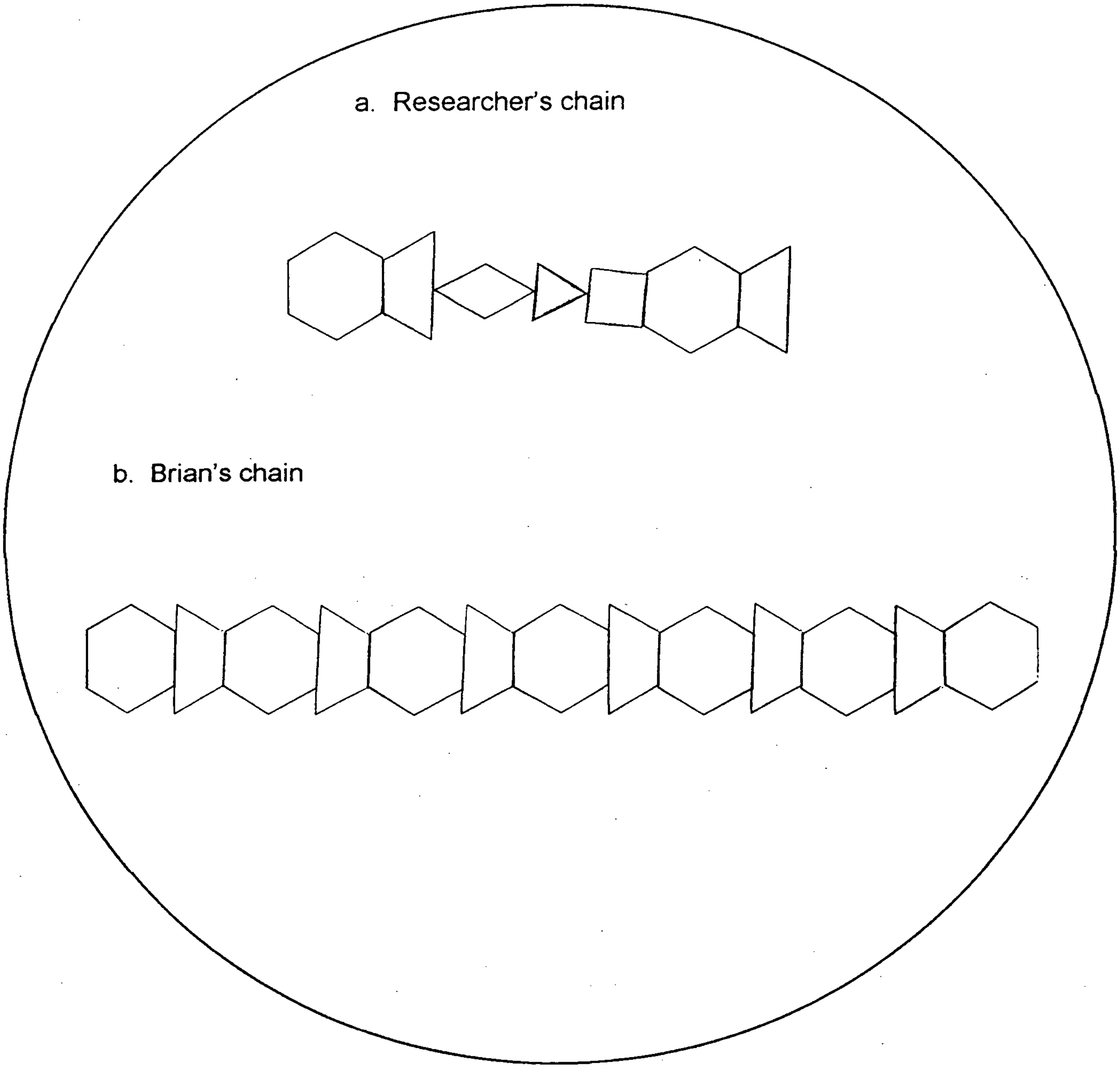
Longer Chain

As Brian was building a chain longer than mine, he counted the seven pattern blocks in my chain. When he was finished building his chain, he counted thirteen shapes (a hexagon and trapezoid AB pattern, see figure 23) and I asked him why he was counting our chains. He replied, "So I can make sure it's longer; so I can keep track of them." When I asked if his chain was longer his tone of voice seemed to imply that the answer was obvious: "Well yeah, I've got thirteen and you've only got seven." Brian's image for longer appears to be *more units in a row*. As in the previous task, Brian does not point out the topological features of the shapes he is working with but rather relies on a more detailed Euclidean image.

Shorter Chain

When asked to build a chain shorter than mine, Brian removes nine shapes from his chain. I ask, "What amount did you take away?" and Brian replies, "I took away thirteen and now I only got 1, 2, 3, 4." Brian appears to be engaged in image-making for subtraction. In order to represent less shapes, he conceptually removes the entire set of thirteen shapes and replaces it with four shapes. In actuality, he has removed nine shapes but he either does not have or does not use the image for quantifying the amount that he took away from thirteen to make four. Another possibility is that he does not wish to focus on the amount he took away but is more interested in the amount that is left as the

Figure 23:



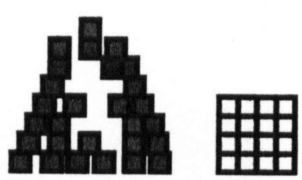
goal of the problem was to make a shorter chain. When I ask Brian if he knows how many he took away he says, "No." and I encourage him to guess. Brian hesitates only a few seconds before responding, "Ten." The close approximation of Brian's guess in conjunction with the fact that Brian is able to create equations for numbers to 30 using a number line as well as through mental computation suggests that Brian's image-making for addition and subtraction is progressing toward a mental construct. When I ask Brian if his chain is shorter than mine, he again uses a quantitative image: "It's shorter 'cause yours has seven and mine only has four."

B: Bigger and Smaller, Taller and Wider

Build Bigger

When I introduce my four by four tiled square by saying, "I have made a shape." Brian interrupts, "And it's called a square." When I ask him to make any shape that is bigger than mine, he proceeds to make a triangular shape with assorted colours. Brian's shape is depicted in figure 24. I ask Brian if his shape is bigger than mine and he responds

Figure 24:

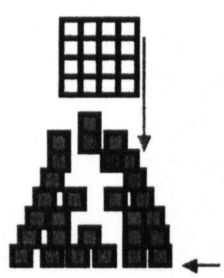


affirmatively. When I ask him to show me where it's bigger, Brian replies, "Well, I put these blocks up—the amount." I seek clarification, "The amount?" and Brian confirms, "Yeah, their amount." I press, "How do you know yours is bigger?" and Brian touches the top of each of our shapes saying, "Because this one (his triangle) is up to here and this one (my square) is only down to here." I press for other images, "Is there any other way to tell me yours is bigger?" Brian calls upon some images of real-life objects to describe how his structure is bigger than mine: "Yes because if I put them in a row like it is now (slides a finger along the bottoms of our two shapes to show me that they are lined up) this one (points to his triangle) is up to the top of here (touches the peak of his triangle) like a building and this one (touches my square) is like a house." Brian's image for *bigger* appears to be *taller*. Included in Brian's images for the concept of "building" seems to be the notion of a structure that is taller than a house.

<u>Wider</u>

Next I ask Brian if his structure is wider than mine. He says, "Yes," and when I ask him to show me where his shape is wider he slides my square above his triangle so that it is centred and comments, "Because if I put it way up here you can see...(straightens my tiles) this one only goes down to here (draws a straight line with his finger from the right side of my square down to where it touches his triangle as shown by the red line in figure 25) but this one goes to about down to here (points to the bottom right tile of his triangle

Figure 25:



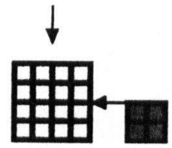
as indicated by the blue arrow in figure 25)." It appears that Brian's image for *wide* is consistent with the real-life definition of horizontal extension. This supposition is confirmed by Brian's actions with respect to my next question, "OK, so show me where wide is." Brian places one hand at each bottom corner of his triangle (the widest points) and says, "About right here." He then places his hands on either side of my square and says, "This one isn't wide."

At this point I am satisfied that Brian has a strong visual image for "wider" and I press for other images, "Is there any other way you can prove to me that yours is wider than mine besides showing me with your hands?" Brian then places one hand on either side of his triangle and slides them upwards explaining, "If my hands go like that it just misses." I ask, "So that means…" and Brian finishes, "So that means that it's wider." In response to my last invitation for more information, Brian creates his own criteria: "About how it's different. See this one (points to my square) has four sides and this one (indicates his triangle) only has three. And this one (my square) has four corners (holds his open hand over my square but doesn't touch it) and this one's only got three corners." In this exchange Brian recalls our earlier conversation about how two shapes are the same and different and assumes that this is the kind of information I am looking for when I ask, "Is there anything else to tell me?"

<u>Build Smaller</u>

For this task, Brian builds what he calls a "baby square" that is two tiles by two tiles in area. When I ask if his square is smaller than mine, Brian replies, "Yup." When I ask where his shape is smaller Brian once again uses his hands to show me how my shape is "…about like up to here" while his shape is "…only down to here." To strengthen his

Figure 26:



position, Brian draws a line with his finger from the top of his shape (figure 26) to its height on my square and says, "Like if I put my hand like that it won't match. It's like it has to be here (points to the top of my shape) to be the same size." It appears that *smaller*, like *bigger*, conjures up an image of height for Brian.

Narrower

Our conversation then turns to the subject of whose shape is wider and why. When I ask Brian if his shape is narrower than mine, he appears puzzled, "Narrower, what does that mean?" He answers my next question immediately and with confidence: "Is mine wider than yours?" Brian responds, "Well yeah," and moves his square below mine as in figure 27. I ask, "Can you show me where it's wider?" Brian begins by saying, "On the sides," and then he takes two fingers, traces them on *my* shape along the red lines in

Figure 27:

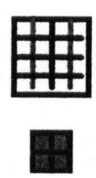


figure 27 and says, "Because that's how wide I am and that's how wide you are (places his hands on either side of my square)." Brian's words and actions demonstrate his awareness that his width is a proportion of my width.

Hoping to confirm that Brian has two different images for *wide* and *tall,* I ask Brian to show me the tall part. He places one hand above the top horizontal row of my shape and says, "This is the tall part." I review, "And where is the wide part?" Brian places one hand on either vertical side of my shape and I am satisfied that he has two different images.

In our final conversation for this task, I decide to take a different approach in order to stimulate Brian to access a quantitative image when comparing the two shapes: "Can you tell me how much taller mine is than yours?" Brian moves his shape back beside mine so that they are level at the bottom and touching on the inside and surprises me with a non-standard measurement image: "A baby toe or one big toe (running his finger along the side of my square where it is taller than his)." When I ask Brian how much wider my shape is than his, he moves his square back underneath and in the centre of my square and again measures in baby body parts: "About one baby's…if a baby was just born, that's how big it's baby finger." Brian's choice of such a unique non-standard unit with which to quantify the difference in height and width of our two shapes is interesting. Perhaps Brian chooses this measurement image because he has a little brother at home and he has recently noticed how much bigger his own hands and feet are from his brother's. Alternatively, perhaps Brian wishes to make the problem more challenging for himself as indicated by his confident tone of voice and use of the phrase, "That's easy." Of further interest is Brian's inclination to move his constructions into various positions relative to mine suggesting that as with Anna, orientation plays an important role in comparing the widths of two shapes.

It is interesting to note that Brian's conception of *tall* is the vertical dimension beginning from a level base as indicated by the fact that he lines up the bases of our two shapes to compare their height. However, when Brian is asked if his shape is as wide as mine, he centres the shapes suggesting that his image of *wider* is extension in both right and left directions.

Brian appears to have a fairly strong image for distinguishing between the concepts of tall and wide when working with both three- and four-sided shapes. He is also able to apply a method of measurement to support his visual comparisons. Brian's use of non-standard units places his understanding at the property noticing level because he has made a connection between the length of objects from various contexts. The use of standardized measurement images would place Brian's understanding at the formalizing level.

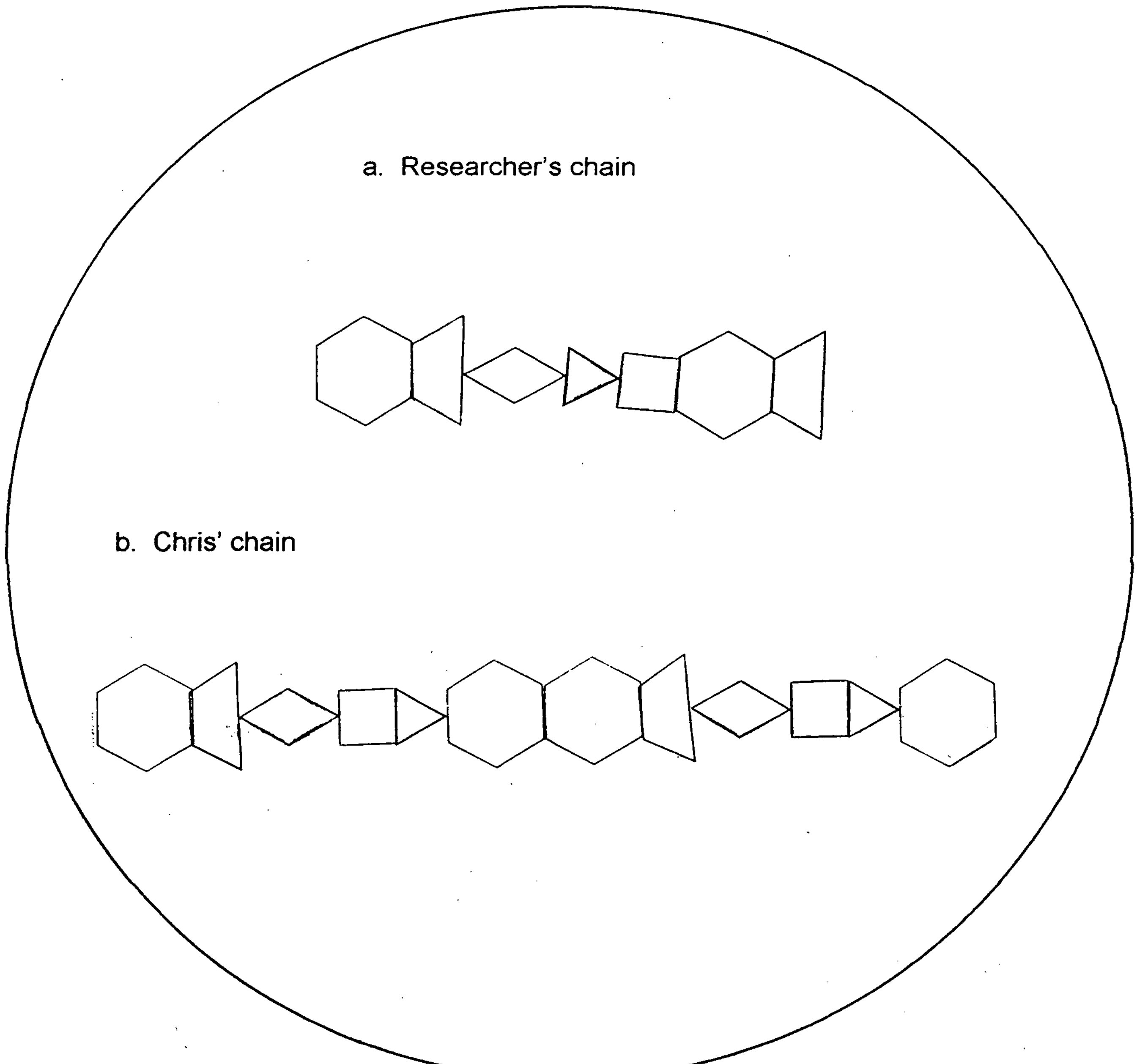
Chris

A: Longer and Shorter

Longer Chain

When challenged to build a chain that is longer than mine, Chris takes notice of my pattern and builds a similar chain but with eleven elements (see figure 28). He explains how he knows that his is longer: "'Cause I did two of each ones and you didn't and I did two of the patterns, two each of one I did two patterns of two and each ones of these." Since my pattern involves five different shapes and my chain is only seven shapes long, only the first two elements are repeated. In Chris' pattern, he repeats all five elements and also adds an extra hexagon in the middle. Chris' image for *longer* appears to involve quantification as a multiplicative image . He does not express his chain as two whole patterns but instead seems to conceive of his chain as having "two of each one"—two of each shape for every one shape that I have. Chris is the only child of the six participants to use a multiplicative image to create a longer chain. When I probe for other images by asking Chris if there is any other way he can show me that his chain is longer than mine, Chris responds, "Probably not." I decide to be more explicit: "How much longer is yours than mine?" Chris straightens his chain and adds a hexagon to one end so that it is now symmetrical. His pattern is as follows: Hexagon, trapezoid, rhombus, square, triangle, hexagon, hexagon, trapezoid, rhombus, square, triangle, hexagon. After a fairly long pause during which Chris runs his finger up the side of his chain and appears to be counting although he is not pointing to each individual shape, he declares, "About one foot and five inches." The phrase "how much longer" triggers Chris to express "how

Figure 28:



long" his chain is in a way that makes me curious about his experience with standard measurement:

Researcher:	How do you know about feet and inches?
Chris:	Because I know what shapes they are and everything.
R:	Can you show me how long a foot is?
C:	About this long (holds his hands over the first section of his chain).
R:	How many inches are in a foot?
C:	I might have forgotten.
R:	Is five inches bigger than a foot or smaller than a foot?
C:	Way smaller.

Evidently Chris has a strong image for the length of a foot and also understands that inches are a proportion of a foot that can be used for objects that aren't quite as long as a whole foot. Chris' understanding of measuring objects that are longer than one foot is still developing as evidenced by the fact he sees the first repetition of his pattern as one foot long but the second as only five inches long despite the fact that he has used the same number of elements in each. He doesn't appear to make a connection between his idea that five inches is "way smaller" than a foot and the fact that he has consciously doubled what he estimates to be a one foot pattern to make his longer chain. Perhaps Chris sees my seven pattern chain as one foot, the equivalent of seven inches. Working with this image, Chris' conclusion that his twelve element chain is one foot and five inches would suggest that he is equating each element to one inch. Thus the first seven block pattern is one foot and the second pattern is five inches. This theory is tested when Chris makes a second shorter chain. At this point, it appears that Chris is able to negotiate the concept of "make longer" with a multiplicative image and "how much longer" with a standard measurement construct. Both images appear to be under development and as our discussion continues, I soon learn more about Chris' interest in feet and inches.

Shorter Chain

For his shorter chain, Chris builds the following five block chain: hexagon, trapezoid, rhombus, hexagon, trapezoid. When I ask him how he knows that it is shorter, he holds his thumb and forefinger about an inch apart and slowly slides them up his chain. After about fifteen seconds, Chris says, "Six and a half inches?" While I had expected Chris to express his chain as five inches since it contains five elements, the fact that he slides his fingers up the chain in a fairly fluent manner rather than inches them up incrementally, indicates that he is not equating each element with one inch. Since Chris has told me the length of his chain rather than explained how much longer his is than mine, I refer him to my chain for comparison, "Yours is six and a half inches? What about mine?" Chris again moves his thumb and forefinger along the chain moving from the bottom to the top and then says, "Nine inches and a half." I remind him, "So whose is longer?" and he immediately replies, "Yours." Surprised that he still has not simply counted the blocks in each of our chains I try one last time to draw him in that direction by asking him if there is any other way that he can show me that his chain is longer than mine. He responds, "I don't think so. Probably not."

Chris' use of the phrase "and a half" reveals that his image of inches as a unit of measurement includes the notion that just as feet are comprised of inches inches can also be further divided into smaller units. It also appears as though Chris understands that "half" is the term used to mean not quite a whole. While it is not clear that Chris has the mathematical image of one half as *one out of 2 equal pieces*, he does appear to have the everyday image of a half as *a part or piece* of something.

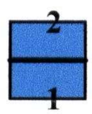
The fact that Chris does not focus on the shapes as units either by counting them or even using the term "shapes" but rather chooses to use the standard measurement units of *feet* and *inches* is evidence that he is working at the formalizing level of the Pirie-Kieren model. At the formalizing stage, the student is able to use a common method to abstract properties from a given problem or task (Pirie & Kieren, 1992: p. 247). Chris is able to use his concept of feet and inches to measure chains of various lengths as well as the square and rectangle he constructed in the first task. He estimates a twelve shape chain to be "one foot and five inches", a seven shape chain to be "nine inches and a half" and a five shape chain to be "six and a half inches". His predictions are logical in that longer chains are estimated to be more inches. While he states that he is unsure how many inches there are in a foot, he does seem to have a visual image of the length of a foot since he estimates my seven shape chain to be nine and a half inches and his twelve shape chain to be one foot five inches. Perhaps he conceives of a foot as a length somewhere between a seven and twelve pattern block chain.

B: Bigger and Smaller, Taller and Wider

Build Bigger

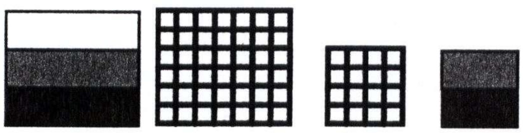
Chris watches attentively as I build my four by four square and when I have completed two rows of four tiles, he comments, "That's a rectangle right?" And then, as he sees that I am not finished, he adds, "It could turn out to be a square." After I add the third row, I ask, "What is it now?" Chris predicts, "It's gonna be a square." Once I have finished my four by four square, I ask, "How do you know it's a square?" Chris reminds me of the projective attributes he expressed in the first session, "Because. Look at it. It has four corners and it's not this short (uses two fingers to make a line dividing the top two rows from the bottom two rows) so it can't be a rectangle and you need two rectangles that are short to make it." Chris' visual image of my square is that of two short rectangles as displayed in figure 29. When I ask him if he can see two rectangles on my

Figure 29:



shape he says, "Not right now (places one hand on each side of my square) because they're all together," indicating perhaps that he considers my shape to have been fully transformed into a square. When I ask him to show me the rectangles, he places his hands over the top two rows and then the bottom two rows explaining, "This is one of them. And this is the other one." I then give Chris his instructions for making a shape bigger than mine and perhaps picking up on my phrase, "any shape you want" he immediately comments, "You can't make a triangle out of these." I query, "Why not?" Chris places a row of four tiles on an angle and ponders, "Because it would look sort of like this. You could sort of make one." He then appears to accept this challenge, "OK, I'll make a triangle bigger." He then proceeds to build a large square at a forty-five degree angle from my square. As he is almost finished he comments that he can make a triangle and I seek clarification, "You can make a triangle? Why did you say that?" Chris explains, "Because look I made like half a triangle…but it's gonna be a square when I'm done. See?" When he stops building I ask, "What did you make?" Chris has made a rectangle seven tiles long and six tiles wide which he perceives as, "A square." When I ask if his shape is *bigger* than mine he says emphatically, "Way bigger." In order to answer my next question about how he knows that his shape is bigger, Chris moves my shape next to his lining the two shapes up at the bottom as shown in figure 30 and then comments, "Two inches bigger. See you made two rectangles on this one and I made three". When I

Figure 30:



ask him to show me the rectangles, he points out two horizontal rows of his rectangle at a time working from bottom to top and commentating, "Here is one, here is a other one, here is a third one." Chris' bigger shape is both longer and wider than mine perhaps indicating that as he was building he was thinking about "bigger" as an *area* image, extending in both horizontal and vertical directions. However, his response to my request that he show me how he knows that his shape is bigger involves a visual image for the height dimension only. When I inquire whose shape is *taller* Chris again refers to the height of his shape by showing me how the third rectangle (comprised of the top two rows) is the tall part, "Here where the third rectangle is."

Wider

When our conversation turns to the relative widths of our two shapes, we are working with the real-life definition of wider as a horizontal dimension despite the fact that the width of Chris' shape is actually the vertical side as it is positioned in front of Chris on the table. When asked if his shape is wider than mine, Chris answers positively and when I ask him how he knows he reveals an image that can be described as both quantitative (the number of tiles) and measured (length in inches): "Because see how much this is. It's four inches (points to my square but doesn't count the tiles out loud) and this is (counts the tiles horizontally across his rectangle) this is seven inches." When I ask Chris how tall my shape is he looks at it briefly and says simply, "Four inches." In our first session, Chris also uses the term "inches" to compare the size of two tiled shapes he has built. When I ask how he knows about inches he recalls a recent conversation he had with his father. Chris' recollection suggests that he is folding back to a related experience that resides in his primitive knowing.

The fact that Chris uses the word "inches" rather than "tiles" to express the unit of the numbers he quotes suggests that he has possibly combined the notion of counting tiles with measuring in inches. At the end of our first session in which we are discussing Chris' four by four square made of colour tiles, he comments, "'Cause one of these is a inch right?" Once again Chris' use of the standard unit of inches in various contexts suggests that he is formalizing because he seems to have an understanding of measurement as a tool that can be applied in many situations. Further evidence that Chris has crossed a don't need boundary from property noticing into formalizing is provided by the fact that Chris does not count individual tiles aloud to make comparisons. He is able to abstract a universally applicable method of measurement using one inch tiles (without ever being told that they are one inch square) as well as non-standard shapes such as pattern blocks.

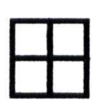
<u>Build Smaller</u>

Chris satisfies my request to build any shape smaller than my four by four square by constructing a square that is a proportion of mine. Pointing to the four tiles at the centre of my square as illustrated in figure 31 , Chris comments, "OK. I'll make a square

Figure 31:

that's only this small." The first image of "smaller" that Chris reveals is that smaller objects can be contained within larger ones. Chris quickly builds a two by two tiled square and as if anticipating my questions comments, "All I need is four pieces. It's only two inches wide and two inches tall." When I ask Chris to show me where the tall part of his shape is, he points out what appears to me to be the bottom row of his square. Hoping to clarify my understanding of his image for "tall" in this context, I ask him to show me the wide part. He traces the top row of his square and seems unsure, "Here, I guess." At this point I realize that because his square is so small, Chris is able to look at it from two angles at once. When he looks at his square from point A in figure 32 he sees *tall* and

Figure 32:



B.

A.

wide as the reverse of the image he has from his perspective at point B. When I ask him to count on his shape to show me that it is two inches wide as he stated before, he appears to be working from point A and he counts the tiles vertically from top to bottom, "One, two." When I ask him to show me how he counted the wideness, Chris counts horizontally across the top row of his square from left to right confirming that he does indeed have an accurate image of the real-life concepts of "tall" and "wide" which he can quantify in standard units (inches).

These final pieces of dialogue from our second session illustrate that Chris is engaging in property noticing with respect to his images for *tall* and *wide*. In dealing with the concepts of tall and wide, Chris is able to notice these two properties in shapes of different dimensions. When faced with a very small square of equal height and width, Chris is still able to recognize the vertical dimension of *tall* and the horizontal dimension of *wide*. In the context of a rectangle, in order for Chris to be working at the formalizing level for images of height and width, he would need to display understanding of the mathematical sense of width as the measurement of the shorter sides. For the purposes of this task however, Chris was not asked about his understanding of the mathematical term "width" but rather the concept of "wider" in its real-life sense of horizontal extension.

Chapter VI

Results for Session 3: Building From Reality

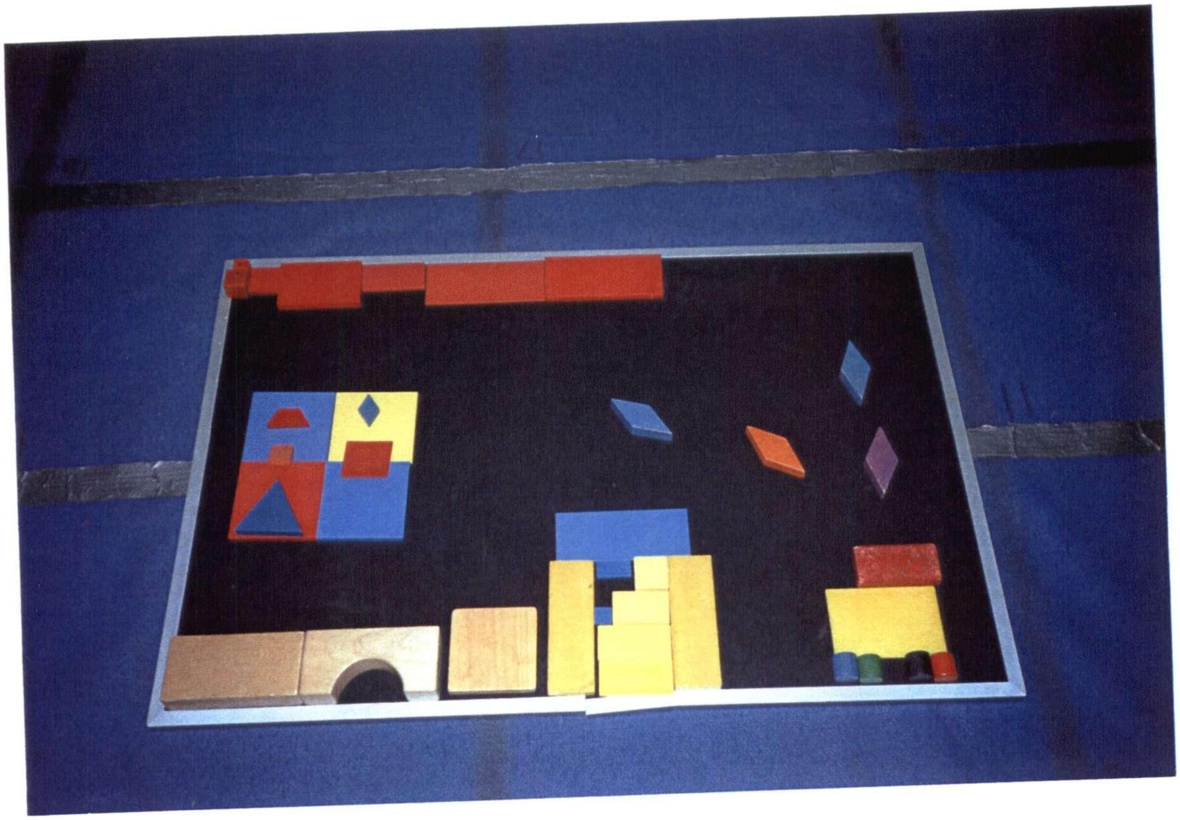
Build Your Room

In order to construct a model of their rooms, the children were provided with six tubs of building materials: large wooden solid blocks, pattern blocks, small wooden solids, small coloured wooden cubes, 2-dimensional coloured wooden shapes, and plastic attribute shapes. Each child was asked to build his or her room on a black rectangular felt board one metre long by 75 centimetres wide for the purpose of focusing their thinking on a particular physical space. Without such a guideline, I was concerned that the children would have difficulty visualizing and re-creating the rectangular or square space their rooms consist of. The children were given as much time as they needed to complete this task and were given one instruction: "Build your room using as many of the materials as you like." In this chapter I will describe the children's buildings by referring to Reifel's eight-level developmental progression as described more thoroughly in chapter 2.

Anna

As seen in the photograph (figure 33), Anna's room takes up much of the perimeter space of the felt board. She took approximately 30 minutes to complete her room. Anna picked out each piece carefully, searching through the tubs of materials until she found the shape she had in mind. When I asked Anna to describe her room for me, she began by telling me about her bed which is located in the bottom right corner of the

Figure 33:



photograph and consists of a yellow attribute rectangle, four round blocks, and a red rectangular prism. Anna explains that the yellow rectangle is her bed, the round blocks the legs, and the red rectangular prism her pillow. It is interesting to note how Anna has laid out the components of her bed in a 2-dimensional fashion. The legs of her bed are not under the block representing the mattress holding it up but rather are beside it, lined up in pairs with some space between as if to signify that there are two legs at the top and two legs at the bottom of her bed. The pillow is also separated from the mattress lying beside it toward what may be the top of her bed rather than right on top of it. The construction of Anna's bed is consistent with level b in Reifel's progression in which objects are lined up horizontally "by" one another. Anna's tendency toward placing objects beside one another in 2-dimensional space is consistent throughout the model of her bedroom.

The next object Anna describes is her chest of drawers in which she keeps her clothing. Moving one object to the left of her bed, Anna points out what she keeps in each drawer. Beginning with the smaller shapes below the large blue rectangle in the photograph, she explains the contents of each drawer moving toward the bottom of the photograph: "This is my sock drawer, and this is my panty drawer, and this is my shirts and pants." Having arrived at the last rectangle at the edge of the felt board (bottom of the photograph) she stops and asks, "What else do you wear?" I suggest, "Pajamas?" and she confirms, "Yeah. Pajamas. My pajamas are always at the bottom." Anna's reference to the "bottom" drawer clarifies that she is looking at her dresser from the angle shown in the photograph and describing the drawers from top to bottom. Anna also explains that the larger yellow rectangles are "the sides" and the larger blue rectangle "the top". Anna does not use any term such as "dresser" to describe the piece of furniture that she has built. Once again, Anna's construction seems consistent with the *row* level (b) in Reifel's schema. Rather than stacking the shapes vertically as her drawers are configured in real life, Anna represents her dresser in 2-dimensional space.

Moving again to the left in the photograph, Anna describes her train set which features, "the umbrella so people don't get wet." The wooden arch block together with the rectangular prism beside it provide a wall beside which the trains ride. Anna slides her hands along the flat edges of the two shapes on the inside where they touch the felt floor to show me that, "the train can only ride this way and that way…that much only." Anna describes the corners of the blocks as "dead ends", the limit of the trains' range of movement. Anna's train set is a brief example of the *row* level described by Reifel.

Above the train set, we find four attribute squares placed in square formation with smaller shapes on top. Anna labels this arrangement her, "painting board" and describes it in the following way, "my board where all my paintings go when I paint them. These (points to the smaller shapes on top) are my paintings and these squares are my board." Anna's painting board features a combination of horizontal and vertical configurations though at a very basic level. It is not clear if her painting board is flat on the floor as shown in her construction, against a wall, or hanging from an easel of some sort.

Next Anna explains that the series of red rectangles lined up in a row at the top of the felt board are her red curtains: "These are my curtains 'cause they're very red." Curious about her use of the two small cubes on the left end of her curtains, I ask Anna what they are and she simply says that they are part of her curtains. Anna's use of *row* as a spatial representation is perhaps most clear in the instance of her curtains.

Finally, I question Anna about the parallelogram shapes that seems strewn in the middle of her room. She animatedly describes their significance: "Those are the diamonds of my night lights so I won't get scared because I'm scared of the dark." I ask Anna where the night lights are and she explains that they are on the wall. When I ask her where the walls are she states, "The wall is all the black spots." This last statement confirms my prediction that Anna's vision of her room has been somehow transformed into two dimensions. If her walls are the "black space" then her furniture is sitting upon them. Also, there is no reference made to the constraints of the space of her room such as the floor and ceiling.

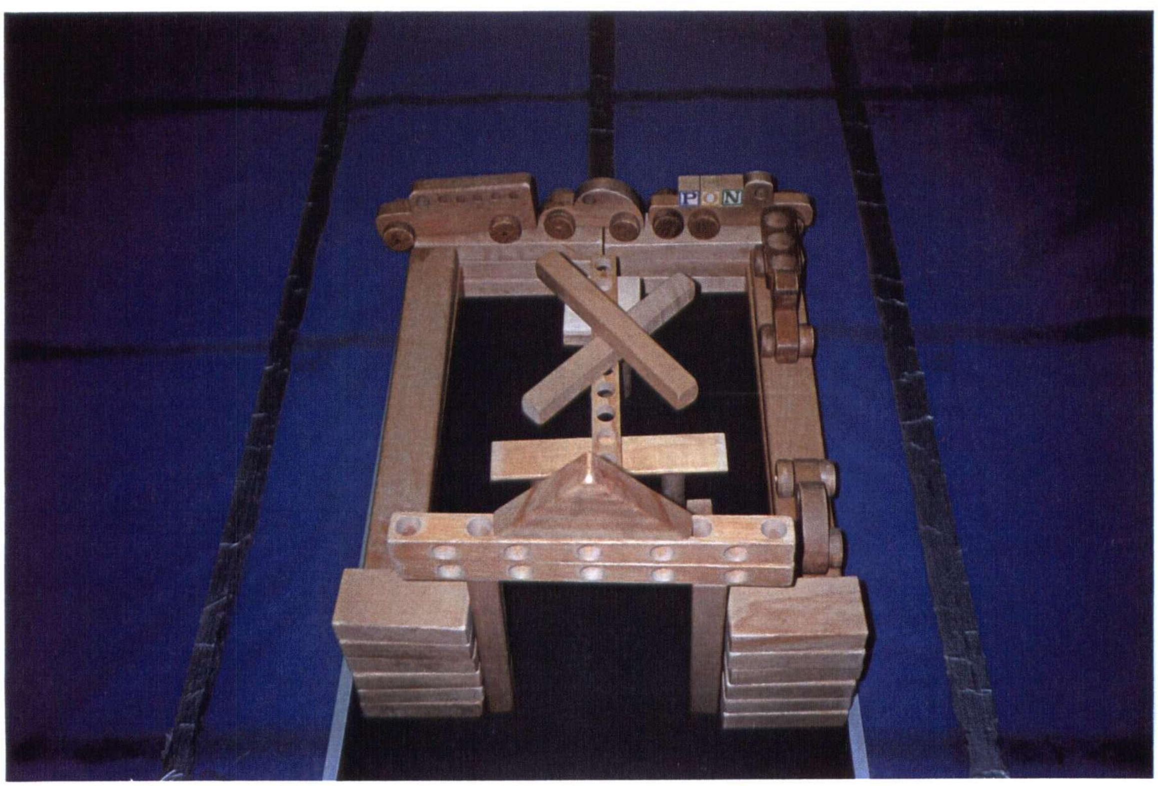
As Anna and I were discussing her room, I noticed that she did not use many prepositions such as *under, in, beside, on top of* to describe the components of her room. Rather than describing one shape in relation to another shape, Anna talked about each shape separately. For example, rather than saying that the red block is her pillow which is *on* her bed and the four round blocks are the legs which are *under* the bed, Anna simply said, "This *is* my bed," or "Those *are* the sides."

Although Anna's constructions are only categorized at the second of the eight-level developmental progression described by Reifel, it is important to notice the high degree of detail featured in her model. Anna has taken great care to ensure that every detail of her room is recorded, from the four legs of her bed to the diamond shapes of her night lights. While Anna's constructions may not be complex, she seems remarkably able to visualize the details of her room and represent them clearly in space using geometric shapes. As her experience with shapes grows, her constructions will likely feature "a wider range of spatial forms to represent [her] impressions of spatial configurations" (Reifel, 1984: p.62).

Brian

Brian's room takes significantly less time to construct than Anna's. He seems quite certain about how to go about representing such an area with blocks. According to Brian's mother, he spends a great deal of time in his room building with a variety of objects including blocks and lego alone and with friends. Brian constructs his room with four walls as seen in the photograph (figure 34). The entrance to his room is clearly marked by the doorway which is framed by two columns supporting a block overhead with a triangular prism centred on the very top. The walls on either side of the door are each six blocks high rendering this front wall symmetrical. In fact, looking at Brian's room from the doorway, it is clear that all of the construction is symmetrical. Each of the

Figure 34:



two side walls consist of three stacked long wooden blocks. The back wall is made of two stacked piles three blocks high placed in a horizontal row. At the centre of Brian's room is a structure that he describes as a reconstruction of a project that he and a friend were building in his room just before he came to meet with me for this session (Brian is in the morning class and our session was at 3:10 pm). Brian explains the significance of the "X" on the top: "We had to smash it so that's why I put an X on it." Brian also shares that the structure represented by these blocks was originally constructed out of lego.

Brian also provides some detail to explain the significance of the pieces he chose to represent the doorway into his room. The triangle on the very top of the door is described in the following way: "It's a house thing. It's on the roof and it keeps lightening from bouncing on our heads. It hits and then it bounces off; it's like in the middle of the house." When Brian locates this triangle as the "middle of the house" I wonder how he has arrived at the decision to place the triangle on top of his door and so I ask him where his room is. Brian seems uncertain, "...mine's on the end, the side of the house, the back of the house." It appears as though Brian simply wants to show the lightening rod in his structure because it interests him. He does not appear to be concerned with the accurate location of this object in relation to his room. By placing the lightening rod at the highest point of his room, however, he is perhaps acknowledging that it is on top of the roof of his room. Brian also explains the significance of the holes in the blocks on top of his door: "You see the holes in my room, the holes on my top where the door is...? You know why there's holes because that's the way it was 'cause we had little stairs up here so we could see who's coming up the stairs. We got little window things you just push the thing and then you look out. As his mother explained later, Brian's door is in the process of being repaired and there is some space between the door frame and the wall. When Brian is playing in his room, he is able to look out through these gaps at the stairway even when his door is closed. Again, Brian brings additional details to his structure that help to establish the location of his room in relation to other features of his house.

On the back wall of Brian's room, he has placed three wooden vehicles with some alphabet blocks on the truck at the very right. Brian tells me that these objects represent the car alphabet wallpaper he used to have. He points out that although he does not have enough letter blocks to show me in his structure, the cars in the wallpaper were "carrying the whole alphabet". Consistent with the symmetry of his room, even the vehicles placed on top of this back wall (a bus and truck surrounding a smaller car) suggest an attempt at balance.

The model of Brian's room features a variety of spatial representations. The walls themselves are stacked (Reifel level a) and together they form a flat enclosure (e). The front wall features an arch (f) surrounded by two stacks (a) on either side. The structure at the centre of Brian's room is also composed of an arch and a stack. Considering the whole structure together, I would conclude that Brian is able to use a range of forms (h) to represent the spatial configurations of this familiar setting. While Brian uses a greater variety of spatial relationships to build his room model, he does not choose to build the furniture and other objects in his room to show how they are arranged in relation to one another. It is not surprising that, given the unique personalities and interests of these students, Anna and Brian focus their constructing on different aspects of their rooms. While Anna is concerned with the details of her furniture and belongings, Brian has a more global view of his room as a space contained within four walls situated inside a larger building that is his house.

Chris

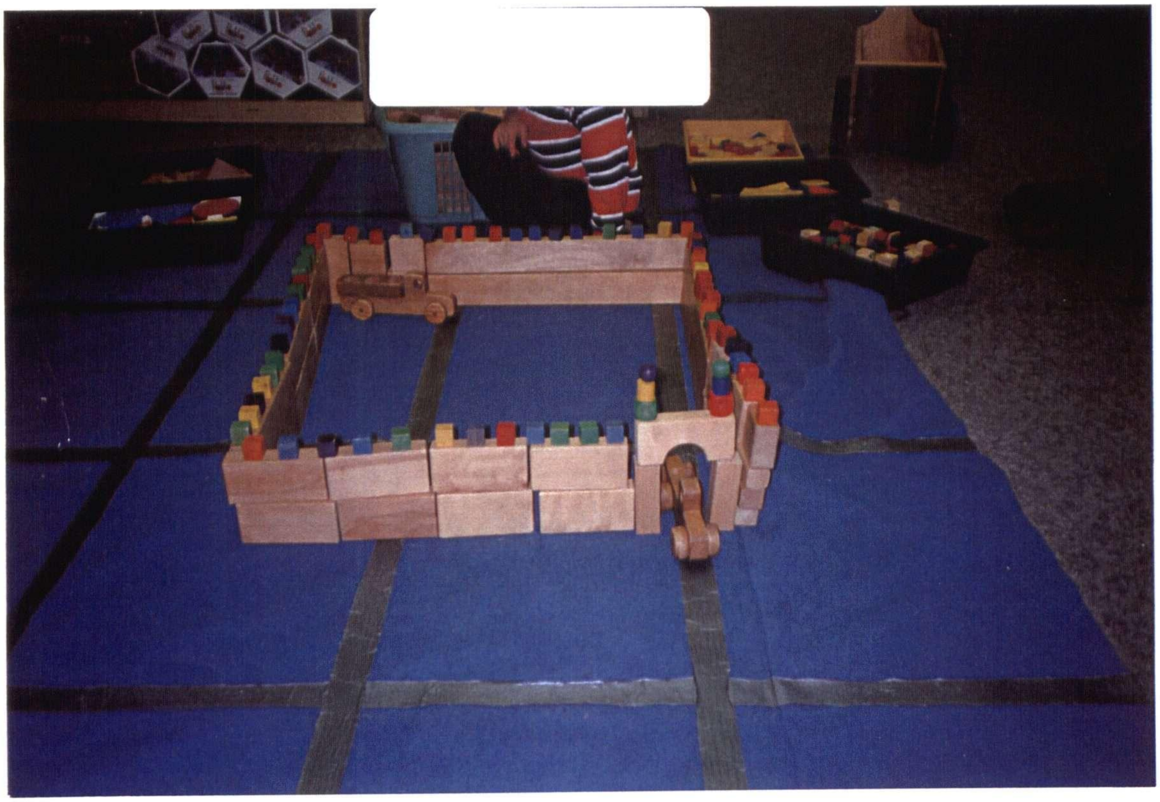
Chris began constructing his room by creating a model of his bed using attribute blocks. When he was finished, he decided that given the large size of his bed, he wouldn't be able to fit the rest of his room on the felt board. When I ask him to show me how much room he would need to build his entire bedroom, Chris walks around the felt board tracing a perimeter about four inches wider than the board with his foot. Despite my encouragement to do his best to build what he could of the rest of his room, he is satisfied with his bed as the sole representative structure of his room. Chris' bed is shown in the photograph in figure 35. He took great care to ensure that the shapes that form the mattress were supported by the legs underneath and he persevered until the structure

Figure 35:

could be held up on its own. When I ask him to tell me about his bed, he begins with the shapes at the centre: "To finish I put these things on so I wouldn't fall down the middle when I slept." When I ask him what is holding up his bed, Chris says, "Those circle things". While he does not use the terms *legs, mattress, or headboard*, his structure clearly features those elements and he places them in relation to one another exactly as they are in real life, creating a miniature model of a bed. Chris' distinction between the head and foot of his bed is expressed in the following way: "These parts (points at the red and blue rectangles standing on their sides as the top of his bed) are to make sure that when I lean up further I just bonk a pillow." Reviewing Reifel's developmental progression, Chris' model shows a combined enclosure (level g) as it features horizontal rows (the mattress), stacking (the legs), and some enclosed space (underneath the bed). While Chris does not choose to show any other objects and thus I am not able to learn about his conception of the layout of his room, his bed model is a clear indication that he is able to take his visual image of an object and use shapes to create a very accurate 3-dimensional representation.

Following this activity, I asked Chris to build anything he wanted for the purpose of seeing what other spatial representations he might display in his building. A photograph of Chris' construction is shown in figure 36. He describes his structure as "a wall around a parking lot," and when I ask him where this wall might be he seems unsure, "In China? I can't remember the city." His explanation of his model is quite descriptive: "I have built a castle wall around a parking lot with a little store in the middle and there's a little park there (points to an empty spot on the carpet in the middle of the enclosure) and there's

Figure 36:



washrooms and there's a boy one here (points on the carpet just inside the doorway) and there's stairs in the store; you can go underground to a subway train that goes underground all around the area." Chris' explanation reveals more features and ideas than he has actually represented with the blocks. He then stands up and walks around the felt board showing me where the train goes when it leaves the parking lot area. Chris' structure is composed of stack and row combinations (level c) to form the walls as well as an enclosure (f) to mark the doorway into the parking lot. Taken as a whole, Chris' parking lot is a combination of many spatial representations (h).

When I ask Chris about the coloured shapes that sit on top of his parking lot walls, he refers to a television show in which he saw a similar wall surrounding a parking lot and then provides some mathematical description: "You know how walls all the time they have these little things that stick up about a couple of feet apart? If it was the Great Wall of China I'd have to put them this far apart (holds his hands about one foot apart)." I confirm, "Farther apart?" and Chris replies, "Yeah like a foot." When I ask him how far apart his coloured blocks are he says, "One inch. Real ones are usually 50 feet apart." Chris' earlier point that he didn't have enough space to build the rest of his room combined with this last statement contrasting his model to the vast size of a real wall seem to indicate that Chris is aware of the scale considerations that come into play when one is building a small model to represent a real life object.

Chapter VII

Conclusions & Implications

Session One

In the first session, "Same and Different" described in chapter 4, my purpose was to listen to how my participants verbalize the physical attributes of 2-dimensional shapes in order to place each child's descriptions on Piaget's 3-level primacy thesis by having them compare attribute shapes and construct a square and a rectangle with colour tiles . In his research, Piaget claimed that children between the ages of four and six years old are not able to note Euclidean properties (Fuys and Liebov, 1992: p. 199). Although I don't wish to generalize the results of my study of three kindergarten students, my analysis does reveal that while all three children focused on different aspects of the materials which they described in unique ways, elements of topological, projective, and Euclidean concepts can be identified within each child's comments. The data for session one shows that each child revealed topological, projective and Euclidean-type thinking in his or her own way. For example, Brian's comments can be characterized as projective and Euclidean with only brief topological references (p. 41). While Brian does not use the common names of the shapes, his projective comparisons are taken as evidence that he perceives topological differences. When a child points out specific projective differences in two shapes, it seems reasonable to assume that he or she is aware that the general appearance of the two shapes is not the same. Both Anna and Chris begin with topological features and eventually describe Euclidean aspects but at varying levels of complexity. In fact, all three children share their Euclidean understanding in various ways. For example, when Anna's four tile-wide square is placed above her three tile-wide rectangle, she points out that her square is bigger than her rectangle (p. 32). She doesn't appear to notice this difference unless the shapes are placed in vertical alignment. In addition to moving the constructed shapes around, it was also necessary to question Anna at length about her ideas and remind her of previous statements she had made before she began to use Euclidean terminology to describe her observations. For Brian, he is able to see differences in the height and width of his constructed shapes while they remain in one position (side by side) and also describe how a certain number of the tiles could be moved around to transform his square into a rectangle (p. 44). At yet another level of Euclidean-type description, Chris uses standard measurement terminology to predict that the colour tiles are one inch tiles (they are), describe his square as "four inches" (p. 54), and compare his constructed rectangle with his constructed square, "it's an inch longer."

Piaget's claim that children progress from topological to projective and then Euclidean understanding sequentially is not negated by my findings. These kindergarten-aged children appear to be very different from the time period in which Piaget's studies were conducted, almost fifty years ago. Today, many children attend pre-school programs in which spatial concepts are explored through games, puzzles, direct teaching of the names of shapes, arts and crafts and creative play using building materials. All six children in my study either attended pre-school or were home-schooled with a pre-school curriculum. Also, many children in kindergarten play on computers at home and school that feature games involving matching and other spatial tasks. It is therefore not surprising that these children readily display topological and projective knowledge as they work through the tasks. During this first session, for example, Anna mentioned the word "octagon" and when I asked her where she learned that word she described a computer game she often plays involving different shapes. Prior to conducting my data collection, I asked the participants' parents to fill out a questionnaire describing their child's favourite activities at home and during leisure time. While Anna's interests tend toward artistic activities such as drawing and painting, Chris and Brian enjoy building with blocks and lego as well as playing on the family computer. Both Chris and Brian's awareness of projective and Euclidean properties is therefore not surprising given their interest and experience in construction. This background information serves to contextualize the spatial knowledge these sessions revealed in each of the children. While a single assessment method is not sufficient to make judgements about the children's spatial intelligence, their "performance" on all of the tasks in combination with information from their parents serves to provide clues about how their experiences and interests strengthen their understanding of spatial concepts. Anna's tendency toward more topological comments is perhaps due to her appreciation for aesthetic properties while Chris' Euclidean references to inches clearly results from memorable experiences and conversations he has had with his father. Back in 1956 when Piaget's topological primacy thesis was published, it is likely that the children in his study may have had very different experiences prior to entering school. Today, children between four and six years old are more likely to be exposed to topological and projective concepts prior to kindergarten. Therefore, their pre-school experience combined with their play interests equip them at varying degrees of readiness to observe and comprehend Euclidean notions.

Implications

In order to further investigate the implications of Piaget's theory in the context of the 21st century, the spatial understandings of children younger than five years old needs to be examined. A worthwhile question to pursue would be: "How do pre-school children, aged three and four years old, show their understanding of the physical attributes of 2-dimensional shapes?" The children's comments and actions could be examined in the context of the pre-school curriculum and their play interests at home to determine their exposure to topological, projective, and Euclidean notions.

Session Two

In session two, entitled, "Longer, Bigger, Taller, Wider" and described in chapter 5, we pursued the question of how kindergarten students show their understanding of size through open-ended construction with 2-dimensional shapes using the Pirie-Kieren theory for growth of understanding as a descriptive language. Concerning the concepts *longer*, *shorter*, *taller*, *wider*, and *narrower*, I identified three possible types of images a child might hold or construct. Anna's reliance on a single (visual) image placed her at the image-making level of understanding and much discussion was needed in order for the researcher to gain a clearer understanding of Anna's concept of *wider* . Brian's use of visual and non-standard measurement images allows him to describe the size property of several shapes (property noticing). Chris' use of a standard measurement image (inches) places him at formalizing because he is able to abstract a measurement unit and apply it to evaluate and describe objects in all three sessions. The Pirie-Kieren theory has thus allowed me to identify the images the students appear to be using and draw conclusions as to how those images help them to make useful generalizations and understand the broader concepts contained within the problems that were posed to them.

It is not always necessary for students to describe all three images in order to conclude that they are able to notice properties (relative length, width, height) of the constructions with which they are working. For example, when Brian claims that his chain of pattern block shapes is longer than mine he never refers to a visual image; he counts the elements and states that thirteen is more than seven. He seems to have a general understanding that more objects equals a longer chain (property noticing). Chris also never refers to a visual image to compare our chains, squares or rectangles. He uses his standard measurement image in all situations. Chris' reliance on Euclidean concepts to describe his image should not be interpreted to mean that he cannot see the topological or projective differences in the shapes. For Anna as well, we cannot assume that she does not know that she could count the blocks or tiles to find out which object is longer, taller, or wider. However, her frequent reliance on visual comparison despite the significant time spent prompting her to use another image, suggests that her use of Euclidean descriptors for the images she holds is still under construction.

Implications

An important aspect of my study that these second session results bring forward is the varying amounts of scaffolding that the children needed in order to share a certain level of description. While some children seemed to have already developed tools for analyzing shapes and comparing their size, others needed to be led there through extended discussion and manipulation of the materials. Teachers need to take note of the amount and type of intervention their students need when working with problems so as to better understand what pieces of information may be required to make the images a child is working with useful in helping him or her to understand concepts in mathematics. Scaffolding has many forms. In this study, I reviewed my participants' previous statements and asked confirming questions to bring contradictory statements to light. The wording of questions is another critical factor in evoking particular responses. It is important to be aware of how the teacher's questioning may possibly limit or enhance the child's understanding of the concepts under investigation.

Session Three

In our final session, entitled, "Build Your Room" and described in chapter 6, I was interested in finding out how kindergarten-aged students symbolically represent objects from their world using 2- and 3-dimensional shapes. Reifel's eight-level developmental schema was a useful tool to analyze the structural complexity of each child's construction. Focusing on Reifel's model, would lead me to conclude that Anna's construction is the least complex since she never builds beyond the "b" or row level. Also significant according to Reifel, is Anna's lack of spatial terminology. Chris and Brian both achieve an "h" level (range of forms) rating for their complex constructions.

Implications

While Reifel's model provides a language and a scale for evaluation, I would hesitate to rely on it as a solitary tool for assessing the development of students' spatial sense in the classroom. Such an analysis fails to capture some very important aspects of the children's constructed models. There are three other important aspects of my participants' structures that distinguish them from one another and fully capture their unique experiences and perspectives:

(1) subject matter: what they choose to represent with the materials

(2) symbolization: how they represent aspects of their room.

(3) perspective: how the student views his or her room

While Anna's room is not as vertically developed as Brian or Chris', she chooses to represent many details and her construction and accompanying commentary provide a very detailed picture of her bedroom. For example, while she doesn't use enclosures in creating her bureau of drawers, she does use larger shapes to represent the top and sides of her dresser and smaller rectangles for each drawer whose contents she also describes. While Anna's model is two-dimensional, further examination of the way in which her furniture is placed and constructed shows her awareness of the vertical and horizontal dimensions of her room. For example, she places her dresser, bed, and curtains right against the edges of the felt board because these objects are against the walls in her room. The features of her bed are lined up vertically in order with the legs at the bottom, mattress in the middle, and pillow on top. The construction of her dresser likewise shows her awareness of top, bottom, and sides. Each child's construction should be examined as a representation of his or her unique perspective. Reifel's model therefore has limited use and a broader analysis of the tasks serves to confirm that interest and experience are likely significant contributors to complexity in construction. For example, Chris and Brian's complex constructions are reflective of their interest in building towers and models of buildings which they often pursue during creative play times at home and school. In order to assess the activity, "Build Your Room" a descriptive anecdotal recording of the child's commentary along with a detailed drawing or colour photograph would provide information about each child's unique perspective, conception of 2- and 3-dimensional space, and experience and interest in building with various materials.

General Implications

A valuable aspect of this project was the quality of information that resulted from a one-to-one interview in a familiar and comfortable setting. The opportunity to sit with a young child who is engaged in a task that interests them and ask them open-ended questions about their thinking is a rare occurrence for most primary teachers. In order for such an endeavour to be successful, several conditions might prove useful to replicate. First, the tasks were engaging, interesting and appropriate for a child of this age. Second, the problems were given in small increments, one challenge at a time so that the child is clear about what is being asked. Third, the tasks were open-ended so that the child's thinking is not limited by a single answer or possibility. Finally, the tasks were extendible so that possibilities for future research could be identified. For instance, in order to extend the tasks in my study, I would be interested in finding out how children would react to the challenge, "Build a shape that is wider than my shape," rather than asking them to build a bigger shape and then asking if that shape is wider. I also wonder if the question, "How tall/How wide is your shape?" would stimulate children to count the objects more than the less specific question I asked, "How do you know that your shape is taller?" Thirdly, I wonder how children of this age would approach a request to build a shape that is in between two other shapes. Given a small constructed shape and a large constructed shape, would kindergarten-aged students be able to build a "medium" shape and how would they explain how they know that it is in between?

Usefulness as an Assessment Tool

In order to gather information about the development of students' spatial sense, I recommend the provision of a wide variety of building materials, regular building times throughout the week, and the preparation of a list of conceptual questions in which the children's responses can be noted anecdotally. For example, when a child is building a tower with blocks, the question, "How tall do you think your tower is?" would yield a variety of possible responses. A child might place his or her hand on top of the structure and say, "About this tall"; he or she might estimate that the tower is almost as tall as the table; or, the child might use a measurement image such as "two feet tall". I believe that this approach to finding out about children's conceptions of spatial ideas is an effective and valid assessment technique because it occurs as the children are engaged in the learning situation. Also, students can be given specific tasks such as those featured in this study in small groups or individually and while the rest of the class is engaged in independent activities, the teacher can note the children's actions and explanations.

The Role of Theory in Assessment

Piaget's topological primacy thesis and the Pirie-Kieren theory for growth of understanding are both useful tools for educators in that they provide a language and a rating scale for assessing the development of student's spatial knowledge. Using these theories, educators can make decisions about the types of experiences students may need in order to help them acquire the Euclidean learning outcomes present in curriculum documents and necessary for understanding more difficult concepts in geometry. It is also important for teachers to be aware of students who show well-developed images for concepts so that they can be challenged with appropriate activities.

In reporting students' progress in mathematics to parents, teachers can describe the images each student seems to be working with and explain what further images that child needs to understand or acquire in order for him or her to fully grasp the concepts explored that school term. The Pirie-Kieren theory is useful for identifying levels of understanding within a mathematical notion. In this study for example, on the topic of size notions such as tall and wide, Chris' formalized understanding would be considered quite advanced for students of his age range. Teachers using this model as an assessment tool would need to translate some of the theoretical terminology into language that would have meaning for parents.

Describing Images

Finally, the most interesting aspect of this study for me is the critical link between image and explanation. For every task that I offered my participants, each child had at least one image to help him or her make sense of the concepts involved. Some of these images were clearly illustrated by the children's workings and commentaries while others became obscured by the transition from mental picture to verbal description. In order to identify the images a child has constructed to make sense of particular concepts, the child is asked to translate his or her internal understandings into some form of external communication. In observing the children as they work through the tasks, it is important to realize that many of the children's thought and ideas will not be expressed verbally. In reviewing the video tapes, I had the advantage of noting the children's gestures, facial expressions, and manipulation of the materials as they explained and built. When the children were asked about their constructing, they seemed to make choices about which aspects on which to comment. Or, perhaps they comment only on those aspects that they are able to explain. For example, when Chris is asked to build a shape bigger than my four by four tile square (session two), he builds a square that is bigger both horizontally and vertically (six tiles high by seven tiles wide) indicating that perhaps he has an area image for *bigger*. When he is asked to show me how his shape is bigger, he focuses only on the horizontal dimension. Anna's workings also seem to reflect more understanding than she is able to express in words. When discussing whether her bigger constructed shape is also *wider* than mine (session 2), I have difficulty following her reasoning. However, when Anna proceeds to make her shape wider than mine, her understanding of *wider* is clearly displayed as she proceeds to add on columns of tiles in a horizontal direction to make her shape wider. Throughout the second session, I have the sense that there was much more to Anna's thinking than I was able to elicit from her verbally. At one point, Anna drew a rectangle in the air in front of me as she described how some sides are short while the others are "big". In order for educators to gain as full a picture as possible about a child's understanding of a problem at which he or she is working, we need to use a variety of tools and be attentive to different kinds of clues. Photographs, drawings (by the observer and the child), gestures, and hand motions in reference to the materials, on the table, and in mid-air all provide pieces of information about the images that contribute to the child's understanding. Further research is necessary to investigate the role that language development plays in communicating mathematical knowledge and to identify non-verbal methods that young children use in order to express their understanding. While recording these bits of evidence to the degree of detail necessary to fully describe the children's understandings takes a great deal of practice, such a process is key to validating the rich knowledge that kindergarten children possess in the context of spatial activities.

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APPENDIX A

Description of Tasks

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Description of Tasks

SESSION 1: **COMPARING SIMILAR SHAPES** (Attribute Blocks, Pattern Blocks)

Purpose:
-to determine if the child is able to see differences in two similar objects when his or her focus is limited to just those two objects

Data Analysis:
At what Piagetian level of description (topological, projective, or Euclidean) is the child able to verbalize the differences he or she sees in the objects?

- **TASK 1A: SAME AND DIFFERENT**

  Place a triangle and circle on the table.

  Question 1: How are these two shapes the same? Are they the same in any other ways? How?

  Question 2: How are these two shapes different? Are they different in any other ways? How?

*Repeat with: (2) triangle and square (3) square and rectangle

- **TASK 1B: BUILD A SQUARE**

  Place container of colour tiles on the table. Show the square attribute block.

  Question 1: Can you make this shape with these tiles?

  Question 2: Tell me how the shape you made is the same as the shape I gave you.

- **TASK 1C: BUILD A RECTANGLE**

  *Repeat Task B with rectangular attribute block.

<u>SESSION 2</u>: COMPARING CONSTRUCTIONS (attribute blocks, colour tiles)

<u>Purpose:</u> (1) to determine the child's conception of Euclidean concepts such as *longer, shorter, taller, wider,* and *narrower* through the language of the Pirie Kieren model for growth of understanding.

- **2A: LONGER AND SHORTER**

The researcher makes a chain of shapes.

Question 1: Make a chain with anything you wish. Make your chain longer than mine.

Question 2: How do you know that yours is longer?

Question 3: Now make a chain that is shorter than mine.

Question 4: How do you know that yours is shorter?

- **2B: BIGGER AND SMALLER, TALLER AND WIDER**

Question 1: Researcher builds a 4X4 square using colour tiles. Build a shape with these tiles that is bigger than mine. Show me how it is bigger:

Question 2: Is your shape taller than mine? Show me how you know.

Question 3: Is your shape wider than mine? Show me how you know.

Question 4: Build a shape with these tiles that is smaller than mine. Show me how it is smaller.

Question 5: Is your shape narrower than mine? (skinnier?)

SESSION 3: BUILDING A FAMILIAR SETTING

Purpose: to examine how the child is able to represent objects in his or her experience with familiar building materials such as blocks and shapes of various sizes according to Reifel's developmental schema for construction with blocks.

- **TASK 3: BUILD YOUR ROOM**

  The child is invited to build a model of his or her bedroom and then to describe its features to the researcher.

*Researcher: Listens attentively and prompts, "What else can you tell me?" Points to various features and asks what they are once the child says there is nothing else to tell.

*N.B.: Concerning tasks 2A and 2B, these building tasks are open-ended in that the child can build any structure he or she desires. It is in the child's explanation of his or her structure and how it meets my single criteria that I will determine the child's understanding of relative size.

Chronology of Tasks:

Session 1: Tasks 1A, 1B & 1C   Session 2: Tasks 2A & 2B   Session 3: Task 3

APPENDIX B

Sample Coded Transcripts

T = topological
P = projective
E = Euclidean

Sample Transcripts

Brian, Session 1A: Same and Different

Researcher:	How are these two shapes the same? (triangle and circle)
Brian:	They were in the same box. You can build something with them. Like a face with a hat. That's all. T
R:	Is there anything about the actual shapes…about what they look like?
B:	Well they have sides. This is a side (points to one side of the triangle) and this is a side (another side of the triangle) and this is a side (traces his finger around the circle). P
R:	Anything else?
B:	No.
R:	How are these shapes different?
B:	That's easy. This one (triangle) has corners. This one (circle) only has none. and because this one's (triangle) got three straight lines. This one (circle) has one straight line around it. P
R:	How are these two shapes the same? (square and triangle).
B:	That's easy. They're both blue. T
R:	Is there any other way that they're the same?
B:	This one's (square) got corners and so does this one (triangle). This one's (square) got sides and so does this one (triangle). P
R:	Anything else?
B:	You can build a house with it (places triangle on top of the square standing them up). T
R:	Anything else?
B:	You can pile them up (places one on top of the other flat on the table). T
R:	Is there any way that those two shapes are different?

B: Ooh! Ah! 'Cause you can see there's three of these (points to each side of the triangle) and there's four of them (points to each side of the square). And there's four corners (square) and only three corners (triangle). P

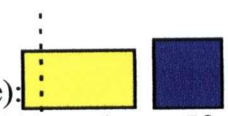
R: Anything else?

B: No.

R: How are these two shapes (square and rectangle) the same?

B: This one's (rectangle) shaped just like it except only a bit a tiny bit bigger. E And you can build it like it's a lollipop T (places square on top of rectangle).

R: How do they look the same?

B: (Places rectangle beside square):  If it (rectangle) was this tall (points to top of square) it could be a shape. If you cut this off (dotted line) E and it was about this high (top of square) it could be the same square except only yellow.

R: How are these two shapes different?

B: The same corner thing, or it's going to be the same side thing. These corners (rectangle) are wider. And these ones (square) are closer together. (Takes two hands and slides them along either side of rectangle) This is from E here to here 'cept and this one (slides hands along side of square) here to here. And because about these (long side of rectangle) well they're bigger and sometimes these ones (long side of rectangle) are bigger than these ones (side of square) and these ones (short side of rectangle) are smaller than these ones (side of square).

Chris, Session 1B & 1C: Build A Square/Build a Rectangle

Brian has been asked to build the attribute rectangle using colour tiles. Once he declares that he has finished, our dialogue is as follows:

Researcher: How is the shape you built like the shape I gave you?

Chris: It's just like it because it has the same T shape and the same colour T and it has the same amount P of corners.

R: Can you show me the corners?

C: See? 1, 2, 3, 4 (points on his constructed square) and 1, P 2, 3, 4 (attribute square).

Chris then builds the attribute rectangle with the tiles.

R: How is the shape you built like the shape I gave you?

C: It has the same amount of corners P, it has the same T shape and it's always the same colour. T

R: How is this shape (his constructed square) the same as this shape (his constructed rectangle)?

C: Well they have the same amount of corners. P

R: Is there any way that they are different?

C: Plenty. It's just because they're different shapes, different colours, different T looks like. And they don't have three corners. P

R: Is this shape (attribute square) the same as this shape (constructed rectangle)?

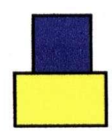
C: No way.

R: Why not?

C: Because…I don't know how…because …

R: You said they're different shapes. What exactly is different about their shapes? If they both have four corners how can they be different shapes?

C: (Slides the attribute square above the attribute rectangle): This thing (rectangle) is longer any way you put the square.



E

R: Is this (constructed rectangle) longer than this (constructed square)?

C: Yes.

R: How do you know?

C: About an inch longer. E

R: How do you know?

C: Because I've seen the shapes before and I've measured them E in my head.

R: Which shapes? These shapes (point to attribute shapes)?

C: Yeah. I measured them in my head.

R: Is this (constructed rectangle) longer than this (constructed square)?

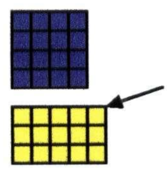
C: Yes.

R: How do you know? You just built it?

C: I knew it two years ago. I knew it when I was three. I just started knowing it when I was three.

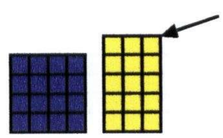
R: Can you show me how this (constructed rectangle) is longer?

C: (Moves constructed square above constructed rectangle)



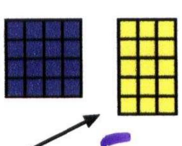
R: What are you trying to do?

C: I'm trying to get the corners together. Can you look at this ? See this thing (points to the fifth column of the rectangle) it's longer and if I put it this way (moves the rectangle beside the square) it's still longer. See, it's an inch longer. E



R: Where's the inch?

C: Here (points to the top row of the rectangle). Or if you put it like this (moves his square so that the top is level with the top of the rectangle) it could be right here (points to the bottom row of the rectangle).



E

R: What?

C: The inch. See how a square is four inches? 'Cause one of these is an inch right?...

R: How do you know about inches?

C: My dad showed me an inch once.

R: Did he show you with these tiles?

C: No. He showed me how long they were like this long (shows me with his thumb and forefinger on the table) …He told me I think last Saturday I mean Sunday when I was at my grandma and grandpa's cabin skiing.

R: So this (constructed square) is four inches? What about this (constructed rectangle)?

C: It's about five inches. E

R: Where is that? Can you show me why you thought it was five inches?

C: I just looked at it and I thought it was an inch longer. See (points to the top row of his rectangle)? E

R: An inch longer than four is five?

C: Yeah.