INFLUENCES OF METACOGNITION-BASED TEACHING AND TEACHING VIA PROBLEM SOLVING ON STUDENTS' BELIEFS ABOUT MATHEMATICS AND MATHEMATICAL PROBLEM SOLVING

by

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ABSTRACT

The aim of the present study was to investigate the effect of metacognition-based teaching and teaching mathematics via problem solving on students' understanding of mathematics, and the ways in which the students' beliefs about themselves as doers and learners of mathematics and about mathematics and mathematical problem solving were influenced by the instruction. The 60 hours of instruction occurred in the context of a day-to-day mathematics course for undergraduate non-science students, and that gave me a chance to teach mathematics via problem solving. Metacognitive strategies that were included in the instruction contributed to the students' mathematical learning in various ways. The instruction used journal writing, small groups, and whole-class discussions as three different but interrelated strategies that focused on metacognition.

Data for the study were collected through four different sources, namely quizzes and assignments (including the final exam), interviews, the instructor's and the students' autobiographies and journals, and class observations (field notes, audio and video tapes).

Journal writing served as a communication channel between the students and the instructor, and as a result facilitated the individualization of instruction. Journal writing provided the opportunity for the students to clarify their thinking and become more reflective. Small groups proved to be an essential component of the instruction. The students learned to assess and monitor their work and to make appropriate decisions by working cooperatively and discussing the problems with each other. Whole-class discussions raised the students' awareness about their strengths and weaknesses. The discussions also helped students to a great extent become better decision makers.

Three categories of students labeled traditionalists, incrementalists, and innovators, emerged from the study. Nine students, who rejected the new approach to teaching and learning mathematics were categorized as traditionalists. The traditionalists liked to be told what to do by the teacher. However, they liked working in small groups and using manipulative materials. The twelve incrementalists were characterized as those who propose to have balanced instruction in which journal writing was a worthwhile activity, group work was a requirement, and whole-class discussions were preferred for clarifying concepts and problems more than for generating and developing new ideas. The nineteen other students were categorized as innovators, those who welcomed the new approach and utilized it and preferred it. For them, journal writing played a major role in enhancing and communicating the ideas. Working in small groups seemed inevitable, and whole-class discussions were a necessity to help them with the meaning-making processes.

The incrementalists and the innovators gradually changed their beliefs about mathematics from viewing it as objective, boring, lifeless, and unrelated to their real-lives, to seeing it as subjective, fun, meaningful, and connected to their day-to-day living. The findings of the study further indicated that most of the incrementalists and the innovators changed their views about mathematical problem solving from seeing it as the application of certain rules and formulas to viewing it as a meaning-making process of creation and construction of knowledge.

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CHAPTER 1

INTRODUCTION.

Mathematics, many researchers believe, is a natural mode of human thought. As concluded by the report of the National Research Council (NRC), the acts of pattern making by children of all ages and making guesses after they observe things are natural ways of doing mathematics (NRC, 1989). People use and need to use mathematics for different purposes. A wide range of fields of inquiry from social studies to technology and computer science are dependent on mathematics as a driving force. The NRC's report (1989), stated that "between now and the year 2000, for the first time in history, a majority of all new jobs will require post-secondary education" (p. 39) which means almost everyone, one way or another needs to learn mathematics. Therefore, problem solving as a "lifeforce of mathematics instruction" (Driscoll, 1982) should be taken seriously into account. As emphasized by Stanic and Kilpatrick (1988) "problem solving really is for everyone" (p. 20), and the core of mathematical learning is the problem solving process (e.g., Confrey, 1987; Thompson, 1985; von Glasersfeld, 1983; Nicholls, Cobb, Wood, Yackel, & Patashnick, 1988).

Overview of the study

Over the past decade, problem solving has been recognized widely as a central component in the teaching and learning of mathematics. The National Council of Supervisors of Mathematics' position paper (1977) stated, "Learning to solve problems is the principal reason for studying mathematics" (p. 20). Similarly, in the National Council of Teachers of Mathematics (NCTM) Agenda for Action in the 1980's it was emphasized that, "problem solving must be the focus of school mathematics in the 1980's" (p. 1). More recently, the *Curriculum and Evaluation Standards for School Mathematics* (1989) focused on the urgent need for mathematical literacy among students. It stated, "(1) that they [students] learn to value mathematics, (2) that they become confident in their ability to do mathematics, (3) that they become mathematical problem solvers, (4) that they learn to communicate mathematically, and (5) that they learn to reason mathematically" (p. 5). Furthermore the *Curriculum and Evaluation Standards* described mathematical literacy as "an individual's ability to explore, to conjecture, and to reason logically, as well as to use a variety of mathematical methods effectively to solve problems. By becoming literate, their mathematical power should be developed" (p. 6).

Despite all the emphasis on the importance of problem solving in teaching and learning mathematics, the nature of problem solving and its place in school mathematics require more detailed scrutiny. The NCTM's (1991) *Professional Standards for Teaching Mathematics* suggests mathematical tasks, discourse, learning environment, and analysis as the "central dimensions of teaching mathematics" (p. 9) in which the goal is to develop students' mathematical power. Teaching mathematics via problem solving helps students to gain such power (Schroeder & Lester, 1989). A number of studies have been done to identify the ways that students solve problems. They make conjectures about the possible explanations for students' weaknesses in problem solving, and they also offer suggestions to overcome some of those weaknesses.

A review of the literature suggests a number of important factors regarding teaching and learning of mathematical problem solving. These factors are categorized by Lester, Garofalo and Kroll (1989a; 1989b) as knowledge, control, affect, beliefs, and socio-cultural factors. In order to make sense of what students do when they are engaged in problem solving activities, Schoenfeld (1987a; 1987b) suggests that it would be helpful to look at the students' cognitive resources, their knowledge of problem-solving strategies, metacognitive or managerial skills, and belief systems. Traditional classrooms with teachers lecturing most of the time and students passively taking notes are not flexible enough to allow an inquiry of that kind. Therefore, creating an appropriate learning environment with a problem-solving approach seems to be essential for allowing such detailed investigation. This approach to the teaching of mathematics would help students to become engaged in a number of problem solving activities, as well as providing opportunities for teachers and investigators to look at the kind of factors that are mentioned by Lester, Garofalo, and Kroll (1989b) and Schoenfeld (1987a).

Justification of the Study

Mathematics educators have been interested in problem-solving instruction for many years. Central to any question regarding the teaching of problem solving is the primary question of understanding what people actually do when they solve problems. Within the past decade, investigation of the problem-solving process has led to a new focus in some of the research literature. As explained by Silver (1982), "any serious learner and teacher of mathematics recognizes that ability to solve problems involves much more than acquiring a collection of skills and techniques . . . . The ability to monitor progress during problem solving and awareness of one's own capabilities and limitations are at least as important" (pp. 56-57). He characterizes these abilities as metacognitive ones.

Metacognition, the knowledge and control of cognition, has gained a great deal of attention from the research community since mathematics educators have realized that non-cognitive aspects of problem solving-performance are as important as cognitive ones. Therefore, there has been a growing interest in the study of the role of metacognition in mathematical problem solving. A number of studies have been conducted to investigate the role of metacognition-based instruction to help students to become better problem solvers. Lester, Garofalo, and Kroll's (1989a) study of seventh graders' mathematical problem solving was of this nature. They concluded that "instruction is most likely to be effective when it occurs over a prolonged period of time and within the context of regular day-to-day mathematics instruction (as opposed to being a special unit added to the mathematics program)" (p. v).

Considering this suggestion, in the Summer of 1991, I conducted a study in the context of a mathematics content course for undergraduate, non-science students, using a metacognition-based approach to instruction. Although the course was a content course and not a problem-solving course *per se*, choosing appropriate problems as a means to introduce and develop the mathematics concepts was an approach that made the most sense for metacognition-based instruction to enhance students' cognitive and metacognitive abilities.

Description of the Instruction

The approach to instruction was teaching mathematics via problem solving in which the mathematical concepts were developed through discussing and solving appropriate problems. The instruction used journal writing, small groups, and whole-class discussions as three different but interrelated strategies that focused on metacognition. These strategies provided the opportunity for the students to become more reflective on their own actions in class as well as to think about their thinking and talk about their feelings. They helped the students to increase their self-awareness as well as to gain in self-confidence, self-respect, and self-regulation that ultimately resulted in better understanding of the mathematics concepts. A similar finding was observed by Lester and Kroll (1990). "Control processes and awareness of cognitive processes develop concurrently with the development of an understanding of mathematics concepts" (p.156).

Journal Writing

There was no structure for the students' writing. They were free to choose topics of their own interest. They had to write at least two or three entries per week. However, the majority of them wrote about many issues situated in the course context. Journal writing also served as a communication channel between the students and the instructor that facilitated the individualization of instruction.

In general, for many students, writing was an occasion of personalizing and internalizing the learning processes. Journal writing provided the opportunity for the students to be more reflective and gave me a chance to study possible changes in the students' beliefs. I collected the journals every Friday and returned them the following Monday giving students feedback and raising some issues with them. However, I need to say that I spent at least thirty hours each weekend to keep my promise of delivering the journals to the students.

Small Groups

The students learned to assess and monitor their work and to make appropriate decisions by working cooperatively and discussing the problems with each other in small groups. I constantly asked the *what, why,* and *how* questions that Schoenfeld asked his students (1985a, 1987a) while they were working in the groups. My role as an external monitor (Lester, Garofalo, & Kroll, 1989a; 1989b) was to coach them in a way similar to that described by Schoenfeld (1987a), which was to help them to become aware of their own resources, to appreciate them, and to use them proficiently. Small groups created the opportunity for me to point out to students a variety of problem solving strategies (heuristics).

Whole Class Discussion

I asked the students to share their thoughts about given problems with the whole class after spending between ten and twenty minutes discussing them in their groups. I did not try to lead the students to *the* solution. On the contrary, I constantly encouraged them to come up with as many different ideas about solving the problems as they could. They were expected to participate in the class discussions and be actively involved in the process of solving problems. My role at this stage was to coordinate the discussions, to help the students to utilize what they knew, and to help them become more reflective. The students' responsibility was to try to make sense of the problems, to speculate, to make conjectures, and to justify them. The class discussions helped them a great deal to become more self-regulated and better decision makers.

I gave the whole class a problem and asked each group to work on the same problem simultaneously. This strategy was very useful when bringing the whole class together while discussing a problem. As a result of this activity, the students gained more cognitive and metacognitive knowledge of problem solving and a better sense of responsibility. They eventually realized that they would learn more by analyzing and solving one problem thoroughly, than by doing many problems without really understanding them. It was pleasant and reassuring for them to see that although their approaches and solutions were different, they were acceptable and valid. Besides, they had to decide which approach made the most sense to them, since they had the opportunity to examine many possible alternatives. These events were used as a vehicle to promote self-regulation.

Research Questions

The purpose of this research study was to address four interrelated questions:

1. What are students' usual beliefs about themselves as doers of mathematics, and about mathematics and mathematical problem solving?

2. How are these beliefs prior to instruction associated with students' approaches to problem solving?

3. In what ways do students' beliefs about mathematics and about themselves as doers of mathematics change during metacognition-based instruction and teaching mathematics via problem solving?

4. In what ways do students' beliefs about mathematical problem solving and their approaches to problem solving change during metacognition-based instruction and teaching mathematics via problem solving?

Limitations of the Study

The study was a case study to investigate the influences of metacognition-based teaching and teaching mathematics via problem solving in the context of a day-to-day mathematics course. Generalizations from case studies are analytical and not numerical generalizations (Yin, 1989). Analytical generalization is based on a replication logic in which the researcher compares the theory he or she has used to frame the study with the empirical results from the study. Generalization therefore is governed by the nature of the research enterprise. This study has demonstrated the applicability of metacognition-based teaching and teaching via problem solving. Those who share similar views about teaching and learning could generalize the findings to their own practice.

The study was limited to the instruction of MATH 335. The syllabus was specifically tuned to interesting and engaging problems to make connections between mathematical ideas and develop them via those problems rather than putting much emphasis on procedures. Students who had registered for the course introduction to mathematics (MATH 335) in summer of 1991 provided data for this study. The course was required for those who wished to apply for the Elementary Teacher Education Program at the University of British Columbia (UBC) and who had not taken a mathematics course. However, it was open to all non-science students. The course material and the textbook were chosen by the Mathematics Department. The case study methodology enabled me to deal with a wide variety of evidence including students' work, interviews, and classroom observations.

The study was not intended to determine beliefs about mathematics and mathematical problem solving held by students in general. However, it was designed to characterize the beliefs held by students who participated in this study before and after the instruction. The aim was to investigate the interactions between the students and the instructor and among themselves, the kinds of approaches that the students adopted in solving problems, and the kinds of cognitive and metacognitive strategies that they used in order to solve problems. The data analysis focused on the effect of the instructional strategies that were used in this course rather than generalizing about instruction and students. Nevertheless, the study provided detailed information regarding classroom interactions, classroom culture, students' actions while solving problems, and the nature of instruction that could potentially contribute to the research and teaching in this area.

Organization of the Chapters

The study is presented in six chapters. Chapter One is an introduction giving an overview of the study. A review of related literature is presented in Chapter Two. Chapter Three gives the plan and procedures used in the study. The Fourth Chapter describes the nature of the interactions within the small groups and in the whole class. Findings of the study are discussed in Chapter Five. The final chapter presents conclusions of the study.

CHAPTER 2

REVIEW OF RELATED LITERATURE

Mathematics educators have conclusively expressed the importance of problem solving in mathematics instruction. From the *Agenda for Action* (NCTM, 1980) to the *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989), the emphasis has always been on problem solving. The latter stated that "problem solving should be the central focus of the mathematics curriculum" (p. 23). However, problem solving in mathematics still needs a greater scrutiny. Lester, Garofalo, and Kroll (1989a) have discussed two possible reasons for difficulties with problem-solving competence. The first reason is the fact that problem solving is a complex cognitive activity which is influenced by non-cognitive factors. "A second reason why so many students have trouble becoming proficient problem solvers is the fact that they are not given appropriate opportunities to do so" (p. 3).

The review of literature gives a summary of research in problem solving in general, and about metacognition in particular, to give a better understanding of the ways in which the present study has been designed and developed.

Summary of Research about Problem Solving

This section gives a summary of research regarding problem solving. The review will help to examine some of the crucial aspects and views about problem solving. First an overview of how different researchers define a "problem" will be given.

What is a Problem?

Although there is general agreement on the importance of problem solving among mathematics educators, the same agreement does not exist on the nature of a problem. For example, in Halmos' view, "the major part of every meaningful life is the solution of problems." He states that "it is the duty of all teachers, and teachers of mathematics in particular, to expose their students to problems much more than to facts" (Halmos, 1980, p. 523). On the other hand, according to von Glasersfeld (1987) problems are definable only by the subject (a student) and only arise when someone tries to overcome the barriers to achieve a goal. In this sense, most routine examples used by mathematics instructors could hardly be called "problems." Halmos' position expresses a view that the definition of a problem lies with the teacher whereas von Glasersfeld views the definition of a problem as resting with the problem solver.

Many others have included the motivation and action of the problem solver as an essential part of their definition of problems. For example, in Polya's view, to have a problem means "to search consciously for some action appropriate to attain a clearly conceived, but not immediately attainable aim. To solve a problem means to find such action" (1962, p. 117). Similarly, in Henderson and Pingry's view (1953), the existence of a question is necessary in order to have a problem, yet it is not considered as sufficient. As they say, "The additional conditions pertain to the individual who is considering the question." What may be a problem for one individual may not be a problem for another. A problem for a particular individual today may not be a problem for him tomorrow. In line with Henderson and Pingry's view of "a problem for a particular individual," Lester (1980) states that "in order for a situation to be a problem for an individual, the person must: (1) be <u>aware</u> of the situation, (2) be <u>interested</u> in resolving the situation, (3) be unable <u>to proceed directly</u> to a solution, and (4) make a <u>deliberate attempt</u> to find a solution. A <u>mathematical</u> <u>problem</u>, then, is simply a problem for which the solution involves the use of mathematical skills, concepts, or processes" (p. 30, emphasis in original). Schoenfeld (1981) points to the different individuals' perceptions of the problem as well, since he takes into account the views of mathematicians, mathematics educators, and cognitive psychologists. The view that the problem solver acts to define a problem also is shared by a number of other researchers such as Radford and Burtons (1974), Skinner (1966), and Newell and Simon (1972), all of whom are cited in the *Alberta Education monograph* (1987).

As early as 1953, Henderson and Pingry's summary of the characteristics of a problem and the role of a problem solver in defining the problem could encapsulate the current writings about the nature of a problem. They state that:

1. The individual has a clearly defined goal of which he is consciously aware and whose attainment he desires.

2. Blocking of the path toward the goal occurs, and the individual's fixed patterns of behavior or habitual responses are not sufficient for removing the block.

3. Deliberation takes place. The individual becomes aware of the problem, defines it more or less clearly, identifies various possible hypotheses (solutions), and tests these for feasibility. (p. 230)

Classification of Problems

During the process of learning mathematics, students are asked to solve a host of problem exercises. Mathematics textbooks have been and continue to be an important source of problems. If the aim of presenting problem exercises to students is to help them to learn mathematics and problem solving, repetitive mathematics problems are not sufficient. Beyond *doing* the repetitive mathematics problems, those that hold no "surprises," which Schoenfeld (1981) calls "exercises," students need to be confronted with more challenging and more interesting problems. Students often learn algorithms or procedures for solving "textbook" problems but they cannot transfer this skill to a real problem setting. Textbook problems are often good examples of "routine problems," those that could be solved by substituting given data into a familiar problem that is solved generally or by following step by step algorithms.

Students' perceptions of a problem solution are associated with their conceptions of problems. Seeing a problem solely as a textbook exercise gives legitimacy to students' assumption that there is only one solution to each problem. According to Schoenfeld (1987b), and the results of the Third National Assessment of Educational Progress

(Carpenter, Lindquist, Matthews, & Silver, 1983), the majority of junior high and secondary school students believe that mathematics is mostly memorization, that there is usually one right way to solve every mathematics problem, and that mathematics problems should be solved, if at all, in a few minutes or less.

Polya classifies mathematical problems into "problems to find" and "problems to prove" which are the extension of two distinguishable propositions of Euclid. "The aim of the first kind (the Latin name is problema) is to construct a figure; the aim of the second kind (the Latin name is theorema) is to prove a theorem" (Polya, 1962, p. 119). The principal parts of a problem to find are "unknown" and "data" that are linked together by "condition." An example of this type of problem is to find the rate of change of quantities that are related. The principal parts of a problem to prove are hypothesis and conclusion. "Prove that the bisectors of the base angles of an isosceles triangle are equal" is an example of a problem to prove.

Polya's Influence on Problem-solving Research and Instruction

The problem-solving literature has been greatly influenced by the ideas of Polya. Although there are different interpretations of how to use his ideas, Polya's influence on the last 30 years of study and research in problem solving is unquestionable. This influence is reflected in the vast number of researchers who refer to Polya as one of the sources from which their own studies were developed, such as Hoz (1979), Kulm (1979), Lester (1980), Silver (1982; 1985), Kilpatrick (1985), Noddings (1985), de Lange, (1987), and Schoenfeld (1985a).

The intensity of the praise of Polya's work expressed by leading mathematics educators reflects a reverence for the man and his work. According to Lester (1980), Polya's work is ". . . perhaps the most lucid thinking about mathematical problem solving" (p. 32). In 1945, Polya published *How to Solve It*, in which he introduced his four-stage model as a method of solving problems. Schoenfeld (1987b) stated that that book is "a tour de force, a charming exposition of the problem solving introspections of one of the century's foremost mathematicians" (p. 30, emphasis in original).

Instruction in general processes of problem-solving strategies was uncommon prior to Polya's fascinating book. But as Polya acknowledged, problem solving and the study of heuristics have had a long past, and he hoped for a long future. Polya's work, in his own view (1973), is in the spirit of Pappus', Descartes' and Leibniz' attempts to devise rules for the direction of mind, a universal method to solve problems.

In Polya's view, mathematics has two faces: (a) the systematic deductive science of Euclid, and (b) an experimental inductive science. Polya notes that both of these aspects are as old as the science of mathematics itself. However, mathematics in the process of "being invented," which means the experimental inductive science, has never been introduced to the students, the teachers, and the general public. Polya introduced problem solving as a new way to understand mathematics. He and a few other mathematicians, among them Kline, questioned the validity of the formalist view in which mathematics is characterized as a closed deductive system (look to Dieudonne's address to O.E.E.C., 1961 as typical of this approach). Lakatos (1976) also objected to the formalists' view as the only approach to mathematics and talked about informal mathematics as well. For Lakatos, "informal mathematics is a science in the sense of Popper; it grows by a process of successive criticism and refinement of theories and the advancement of new and competing theories" (cited in de Lange, 1988, p. 144). Mathematics education has been influenced by informal mathematics ever since and has become more process, rather than product, oriented in relation to problem solving and mathematical modeling.

Polya's plan was based on his view of mathematics as an inductive, experimental science. He stressed that mathematical work consists largely of observations and experiments; guessing at what might be true; feeling what ought to be true; testing hypotheses; looking for analogies; and building mental pictures. Polya described problem-solving as "a practical skill like, let's say, swimming. We acquire any practical skill by imitation and practice . . . . Trying to solve problems, you have to observe and imitate what other people do when solving problems, and, finally, you learn to do problems by doing them" (1973, pp. 4-5). Lester (1980) held the same view using baseball as an example.[1]

Heuristics

In order to help students become better problem solvers, Polya (1973) suggested a list of heuristics which helps students solve problems. Heuristics are plausible rather than strict, and as Polya said, heuristic is about the rules and methods of discovery and invention. Thus, heuristics, which are tools available to the problem solver and which exist independent of problem content, can be used efficiently as long as one distinguishes and maintains the distinction between these two kinds of reasoning, namely, heuristic reasoning and formal mathematical arguments.

In general, problem solving research evidence supports the premise that teaching students heuristics enhances problem solving performance (Suydam, 1980), but the role of heuristics in the teaching and learning about problem solving is viewed in several different ways. In particular, there are those who support the view that students' problem solving performance will be increased by teaching them <u>via</u> heuristics (Schoenfeld, 1979a; Goldberg, 1975; and Lucas, 1972, both cited in Schoenfeld, 1979b). However, some curriculum developers support the view that heuristics should be taught as individual skills or techniques which are then applied to problem solving. In this regard Stanic and Kilpatrick (1988) raise the following concerns about the nature and effect of some implemented curricula:

> There are those today who on the surface affiliate themselves with the work of Polya, but who reduce the rule-of-thumb heuristics to procedural skills, almost taking an algorithmic view of heuristic . . . A heuristic becomes a skill, a technique, even, paradoxically, an algorithm. In a sense,

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1. It is interesting to see that after 28 years, even researchers' examples are from the same sporting contexts as Polya's.

> problem solving as art gets reduced to problem solving as skill when attempts are made to implement Polya's ideas by focusing on his steps and putting them into textbooks. (p. 17)

Polya's intention in introducing heuristics was, as stated by Schoenfeld (1981), "to uncover useful strategies for inquiry into (somewhat) novel situations" (p. 43). However, Schoenfeld (1987b) argues that Polya's heuristics are descriptive rather than prescriptive, and each heuristic could break down into many other strategies useful for solving problems. In Schoenfeld's view, Polya's descriptive heuristics are more useful for those who already know how to apply them. Yet these strategies are not detailed enough to be as useful for those who are not familiar with them. "Polya's characterizations were labels under which families of related strategies were subsumed" (Schoenfeld, 1987c, p. 42). Since there are many other components such as students' motivation and classroom culture (Schoenfeld, 1988a) which affect the students' performance on problem solving, Schoenfeld (1981), argues that "a teaching theory based solely on heuristics is doomed to failure" (p. 43).

Different Approaches to Problem-solving Instruction

This section deals briefly with the different approaches to teaching problem solving. Almost all mathematicians and mathematics educators believe that ability to solve problems is an essential part of learning mathematics. However, their perceptions of problem solving influence their ways of interpreting the role of problem solving and its applicability in an actual classroom situation.

One view is to consider problem solving as a goal to be reached independent of specific problems, their mathematical content and procedures or methods used to solve them. Learning how to solve a problem, in this view, is the primary reason for studying mathematics. The entire curriculum has been influenced by this view, and it has important implications for classroom practice. This view is what Stanic and Kilpatrick recognize as "problem solving as context," with five sub-themes namely: (a) problem solving as justification; (b) problem solving as motivation; (c) problem solving as recreation; (d) problem solving as vehicle; (e) and problem solving as practice in which all of these sub-themes seek to reach the valuable ends by the means of solving problems (for more detail see Stanic & Kilpatrick, 1988). This approach resembles what Schroeder and Lester (1989) call "teaching for problem solving," in which the teacher usually introduces the basic mathematical content before applying that content in finding the solution of several problems. The assumption in this view is that students should study the mathematics content first, then solve actual problems.

A second view interprets problem solving as a skill in which the specific content of a problem, kinds of problems and methods leading to solution should be considered. As pointed out by Stanic and Kilpatrick (1988), "the problem-solving-as-skill theme has become dominant for those who see problem solving as a valuable curriculum end deserving special attention, rather than as simply a means to achieve other ends or an inevitable outcome of the study of mathematics" (p. 15). Stanic and Kilpatrick continue to argue that viewing problem solving as a skill does not provide opportunity for all students to get involved in solving non-routine problems, only capable students, since the skill of solving non-routine problems as a higher level skill is acquired only by those students who have mastered skill at solving routine problems.

This approach is similar to "teaching about problem solving" as Schroeder and Lester (1989) call it. In this situation the teacher usually solves a problem and then he or she talks about the process by identifying Polya's four stages. The key in this interpretation is talking about problem solving process and engaging students in solving problems. This view leads to adding a unit on problem solving to the school curriculum. British Columbia Curriculum Guide from K-8 (1987) is a good example for this approach.

However, a third view sees problem solving as a dynamic, ongoing process. For those who hold this view, the final product is not as important as the methods, procedures, strategies and heuristics used by students to solve problems. This view resembles what Stanic and Kilpatrick call "problem-solving as art." The work of Polya gave rise to this aspect of problem solving. Polya (1980) emphasized that problem solving is the "specific achievement of intelligence, and intelligence is the specific gift of man" (p. 1), and the primary goal of education is to develop the intelligence and make teaching and learning mathematics "insightful." Polya believes that "to know mathematics is to be able to do mathematics" (Polya, 1969, p. 574), like any other practical art that is learned through "imitation" and "practice." In Polya's view, teachers should illustrate the techniques and strategies of problem solving and discuss them with the students. This view is almost identical to what Schroeder and Lester (1989) call "teaching via problem solving," in which the mathematics content is introduced through discussions and extensions of specific problems.

As a summary, "teaching via problem solving" can provide a situation in which students *do* mathematics actively and creatively. This approach gives students the opportunity to use plausible reasoning by teaching them how. It is crucial to understand how individuals learn to solve problems to be able to understand how such learning takes place in the classroom. In the next section, I will discuss research on students' thinking processes, their beliefs about mathematics, and how well they can control their actions in solving problems. These are the three main categories on which research on metacognition has focused.

Metacognition

Central to any questions regarding the teaching of problem solving is the primary question of understanding what people actually do when they solve problems. Within the past decade, investigations of the problem solving process have led to a new focus in some of the research literature. As explained by Silver (1982), "any serious learner and teacher of mathematics recognizes that the ability to solve problems involves much more than acquiring a collection of skills and techniques . . . . The ability to monitor progress during problem solving and an awareness of one's own capabilities and limitations are at least as important" (pp. 56-57). He states that these abilities are metacognitive ones. According to Flavell (1976), "'Metacognition' refers to one's knowledge concerning one's own cognitive processes and products or anything related to them. . . . Metacognition refers, among other things, to the active monitoring and consequent regulation and orchestration of these processes . . .." (p. 232). Many researchers have reached conclusions similar to those of Flavell and Silver and, as a result, metacognition and its implications for mathematics education are now widely studied.

Schoenfeld (1987a, p. 190), indicates three related but distinct categories of intellectual behavior, on which the research on metacognition has focused:

1. Your knowledge about your own thought processes. How accurate are you in describing your own thinking?

2. Control, or self-regulation. How well do you keep track of what you're doing when (for example) you're solving problems, and how well (if at all) do you use the input from those observations to guide your problem solving actions?

3. Beliefs and intuitions. What ideas about mathematics do you bring to your work in mathematics, and how does that shape the way that you do mathematics?

Schoenfeld's three categories of metacognition are reflected in the work of Garofalo and Lester (1985). Based on Flavell and Wellman's (1977) categories, Lester and Garofalo categorized knowledge about cognition according to how it influences performance in terms of the person, task, or strategy. In the person category, metacognitive knowledge consists of "What one believes about oneself and others as cognitive beings" (p. 164). Cognitive knowledge in the task and strategy categories refers to an awareness of one's knowledge about the nature of tasks and an awareness of one's thinking strategies.

The regulation of cognition concerns knowing how and when to use the knowledge described above. These metacognitive, or managerial skills, include planning strategies, monitoring these strategies and being willing to abandon them when necessary, and evaluating the outcomes for efficiency and their degree of effectiveness (Schoenfeld, 1985a; 1985b).

It is this latter aspect of metacognition that has the greatest impact on problem solving in mathematics. There is not much purpose in knowing a variety of heuristics if one does not know when to use them. Many researchers believe that it is an ability to make these executive decisions that determines whether or not an individual will be a successful problem solver. By definition, metacognitive actions affect problem solving in a global sense and failure to adequately monitor and assess one's strategies can result in an unsuccessful attempt at reaching a reasonable conclusion. This provides one means for accounting for the behavior of the person who knows the appropriate strategies needed for solving a problem but yet is unable to solve it.

Studies of Expert and Novice Problem Solvers

The discussion about the types of problem and their structure has given rise to the extensive research proposals that aim to identify the characteristics of good and poor problem solvers. Krutetskii's (1976) research on gifted students of mathematics indicates that a distinguishing factor between good and poor problem solvers is the students' perception of the important elements of problems. Differences between good and poor problem solvers help to better understand metacognition and its implications for the teaching of problem solving. Many researchers have attempted to discover what differentiates the novice problem solver from the expert. Krutetskii (1976), based on his research of gifted students in mathematics, concluded that what distinguishes between good and poor problem solvers is their perception of the important elements of problems. Driscoll (1982) referring to Krutetskii (1976), identifies characteristics of good problem solvers including

1. Good problem solvers sort problems according to their mathematical structure and not by their content.

2. Good problem solvers distinguish between relevant and irrelevant information and do not focus on single variables within the problem.

3. Good problem solvers are more active and use more problem solving strategies and processes.

Weak problem solvers in contrast will fail to exhibit the above behaviors and will also fail to look back and assess their work. While he does not explicitly refer to metacognition, Driscoll does discuss elements of metacognitive actions. His survey provides a preliminary indication that good problem solvers may use metacognitive skills that are superior to those used by weaker students.

The difficulty in doing research in metacognition, however, is that for many people metacognition is a covert process. This situation is further complicated by the fact that most students have not been taught to use metacognitive skills such as managerial skills and, therefore, it is difficult for them to articulate processes with which they are not familiar (Lester, 1986). Schoenfeld (1980) in an attempt to record and analyze metacognitive activity during mathematical problem solving, found that expert problem solvers possess efficient managerial skills while novice problem solvers do not. The experts, according to Campione, Brown, and Connell (1988) "find learning challenging and see themselves in control of their destiny. Weaker students [novices], in contrast, acquire fewer problem-solving strategies, are less aware of the utility of those strategies, and do not use them flexibly in the service of new learning. They are not convinced that they can control their performance and tend to be relatively passive in learning situations" (p. 94). Silver (1980) cites the identical situation when physics problem solvers were observed. He offers the possible explanations that expert problem solvers may have so incorporated metacognitive behaviors into their problem routines that they are difficult to observe, or that these behaviors are only needed at certain points in the development of expertise.

Metacognition and Beliefs

One issue directly related to metacognition which has been referred to previously, but not discussed in detail, is that of "beliefs." As mentioned earlier, Schoenfeld considers one's belief system to be an integral part of one's problem-solving structure, and the works of Silver (1982) and Garofalo and Lester (1985) both include beliefs in their definitions of metacognition.

According to a number of researchers, the link between metacognition and beliefs occurs because one's world view affects the decisions that one makes. One's beliefs determine the context within which one selects from available "resources" and how one chooses to use those "resources" (Schoenfeld, 1983). An individual is unlikely to pursue certain strategies if he or she does not believe that they will be successful. Therefore, although he or she may be aware of metacognitive behaviors, a person may make what seems to be inappropriate use of them if they are inconsistent with his belief systems.

In the context of mathematical problem solving, behaviors can be influenced by a variety of beliefs. A person's view of the nature of schooling and learning in general, the nature of mathematics and the learning of mathematics, the specific task itself, and one's own mathematical abilities can all play a role in how one responds to a problem solving situation. For example, Silver (1982) suggests that "a person who believes that there is an underlying structure to mathematics and that this structure is more important than the surface details, will approach the study of mathematical material quite differently than a student who does not hold this belief" (p. 21). Other beliefs which might positively influence how one approaches mathematical problem solving are the beliefs that there is usually more than one way to solve a problem, whether two methods are used to solve a problem, they should result in the same solution, and that there exists a most concise way to present a problem and its solution.

It is perhaps equally important for educators to be aware of the mathematical beliefs that students hold which might hinder the problem solving process. A set of such beliefs was reported by Lester and Garofalo (1982). Their findings indicated that many grade three and five students believed that verbal problems are more difficult than computational ones, and that the size and quantity of numbers in the problems are important indicators of difficulty. Similarly, Driscoll (1982) reported that, in an assessment of thirteen and seventeen year old students, almost fifty percent agreed with the statement that learning mathematics is mostly memorizing. Furthermore, almost ninety percent agreed with the view that there is always a rule to follow in solving mathematics problems. Driscoll (1982) suggests that these students' conceptions of mathematics grow out of a mathematical experience which usually emphasizes "memorization, regurgitation, and the conviction that the sole purpose for doing any mathematical problem is to get the right answer" (p. 63).

Schoenfeld's (1987b) experience with college freshmen has led him to establish his own set of commonly held beliefs. They are as follows: that formal mathematics has little to do with real problem solving, that mathematics problems are always solved in ten minutes or less, and that only geniuses are capable of discovering or creating mathematics.

Like Driscoll, Schoenfeld indicates that teachers must assume responsibility for the existence of these kinds of perceptions and beliefs. As much of mathematics deals with abstract structures with which students have little real-world experience, the majority of students' mathematical world views will be based on their experiences in the mathematics classroom. Many classrooms place a strong emphasis on writing mathematical arguments in a prescribed form, which contributes only to the idea that "being mathematical" means "doing your steps." Likewise, most problem solving exercise sets are limited to those which can be adequately discussed within a single class lesson (Schoenfeld, 1985b). In summary, he states that teachers too often "focus on a narrow collection of well-defined tasks and train students to execute those tasks in a routine, if not algorithmic fashion. They test the students on tasks that are very close to the ones they have been taught . . . .

To allow them, and ourselves, to believe that they understand that mathematics is deceptive and fraudulent" (Schoenfeld, 1982b; p. 29).

Thompson's (1988) research indicates that the actions of teachers are a reflection of their own views of mathematics. Therefore, not only must teachers be aware of potential student beliefs and how they are acquired, but they must also be aware of their own. Teachers' beliefs about mathematics learning and teaching influence the nature of the mathematics environment in the classroom and the type of instruction that the student receives. As beliefs cannot be eliminated, Silver (1982) suggests some questions for possible further study in this area: What beliefs are held by mathematics teachers? What are the beliefs of "excellent" teachers? and of course, What is the influence of beliefs on the teaching of problem solving?

The issues of metacognition and belief systems and their relationship to problem solving in mathematics are currently the focus of interest for many researchers, yet the implications for the mathematics teacher are still unclear. To date, most instruction in problem solving has emphasized the development of algorithms and heuristics and has almost ignored the managerial skills involved in the process. As indicated above, teachers should increase their awareness of their own beliefs and metacognitive actions.

Schoenfeld (1985b) recommends that, at a minimum, teachers should demonstrate the decision making process when working on problems in class. Questions such as "What are the possible options here?" and "Is this strategy working out?" can serve as models to students and legitimize the process in their minds.

Both Schoenfeld (1985b; 1987b) and Driscoll (1982) also encourage teachers to help students to verbalize their problem solving experiences. Schoenfeld indicates that this can be done with the class as a whole where the teacher acts as a manager. By carefully monitoring the interactions, the teacher may be able to have the class engage in a metacognitive analysis of a proposed problem. This involves questioning all decisions, even when the development of the solution appears to be proceeding well. A postmortem on the problem should also occur where a discussion of the representation of the problem, the use of control strategies and possible other solutions to the problem take place.

Schoenfeld also advocates breaking the class into small groups to work on problems while the teacher acts as consultant. The emphasis, however, is still on the students' awareness and control of their actions. Schoenfeld (1985a) has a poster in his classroom which contains the three executive questions: (a) What (exactly) are you doing? (b) Why are you doing it? and (c) How does it help you? The purpose of the small groups is to bring out into the open what would ordinarily be covert processes and to encourage the proposal and discussion of alternate options. His suggestion for determining what the students' beliefs are is the repeated use of the question *why*. It is one of the practical suggestions offered by mathematics researchers to date and is a reasonably concise method of explaining what the teacher must do. It must be noted, as Schoenfeld himself points out, that his recommendations are based on anecdotal evidence and are not the result of extensive research.

Frameworks For Metacognition

The major contributions of the study of metacognition to research on mathematical problem solving have been in the development of frameworks within which to better understand and further that research. As previously stated, Schoenfeld (1984; 1987a) has identified three broad categories, namely: (a) "resources," or an individual's knowledge, such as facts, algorithms, and heuristics which have bearing on the problem; (b) "control," or the conscious and unconscious metacognitive acts used; and (c) "belief systems," which are the determinants of an individual's behavior. As he believes that decisions at the managerial or control level may determine the success or failure of a problem solving attempt, Schoenfeld (1983, 1985a) has focused his study of protocols on this type of behavior. By parsing protocols into chunks of consistent behavior, which he calls episodes, he has developed a list of six managerial decision-points. These are reading, analysis, exploration, planning, implementation, and verification, and most of these decision-points are self explanatory. According to Schoenfeld (1985a), "it is at the transition points between these episodes (as well as other places) where metacognitive decisions, especially managerial ones, can have powerful effects upon solution attempts" (p. 9). These six decision-points function as a framework within which the broad category of "control" could be investigated.

Similarly, Garofalo and Lester (1985) have also designed a cognitive-metacognitive framework that considered it to be "directly relevant to performance on a wide range of mathematical tasks, not only tasks classified as 'problems.' It is not a list of all possible cognitive and metacognitive behaviors that might occur; rather, it specifies key points where metacognitive decisions are likely to influence cognitive actions" (p. 171). The framework contains four categories, namely: orientation, organization, execution, and verification. This framework is based on the work of Polya, Schoenfeld and others, and it is "intended as a tool for analyzing metacognitive aspects of mathematical performance"(p. 172). The framework shows that "distinctive metacognitive behaviors are associated with each category" (p. 171). Although Polya never explicitly mentioned metacognition, it seems there is a parallel between the above work and Polya's "four stage" plan.[2]

Perkins and Simmon (1989) talked about four kinds of knowledge, namely: content, problem solving, epistemic, and inquiry which, as suggested by Donn (1990), might be used as a framework to study students' misconceptions. The framework describes heuristics and metacognitive strategies inside all four kinds of knowledge. The content knowledge consists of terminology, definitions, and relevant rules or algorithms to a content. Ability to recall and to use notations are of this kind. However, problem solving knowledge includes general problem solving and managerial strategies, and beliefs about problem solving. Furthermore, the epistemic knowledge verifies and justifies the use of a particular concept or procedure. Finally, the knowledge of specific content is

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2 The "Four Stage" plan includes (1) Understanding the Problem; (2) Devising a Plan; (3) Carrying out the Plan; and (4) Looking Back.

extended or challenged by specific and general beliefs and strategies that are called the inquiry knowledge.

The Influence of Metacognition-based Instruction on Students' Belief System

Mathematics educators have been interested in problem-solving instruction and performance for many years. In particular, Lester, who has been involved in problem-solving instruction for twenty years, has recently studied young children's metacognitive awareness as an important component of their mathematical problem solving. Metacognition, the knowledge and control of cognition, has gained a great deal of attention from the research community since mathematics educators have realized that non-cognitive aspects of problem solving performance are as important as cognitive ones. Therefore, there has been a growing interest in the study of the role of metacognition in mathematical problem solving among these educators. A number of studies have been conducted to investigate the role of metacognition-based instruction to help students to become better problem solvers. The following is the summary of some of the classroom-based research regarding metacognition in the context of day-to-day instruction.

Lester, Garofalo, and Kroll's (1989a) study of seventh graders' mathematical problem solving was of this nature. This 12 week study involved students' written work, interviews of students individually and in pairs, classroom observations, and questionnaires. The teacher played different roles at each phase of the instruction; among them was his role as monitor, facilitator, and a role model for metacognitive behavior. They found that students' beliefs are mainly shaped by classroom instruction and classroom environment. They concluded that "instruction is most likely to be effective when it occurs over a prolonged period of time and within the context of regular day-to-day mathematics instruction (as opposed to being a special unit added to the mathematics program)" (p. v).

Lester, Garofalo, and Kroll's (1989a) suggestion is in line with Schoenfeld's effort in his problem solving courses. Schoenfeld (1985a; 1987a) has done extensive work to develop instructional techniques that promote different aspects of metacognition, namely awareness of one's own thinking processes, control or self-regulation, and beliefs that students have about mathematics and mathematical problem solving. He has dedicated a fair amount of time investigating the third aspect of metacognition and exploring the origins of beliefs that students bring into the classroom.

Raymond, Santos, and Massingila, (1991) taught a course for prospective elementary teachers at Indiana University. The course was developed by a team under Lester's direction (NSF Final Report # TEI-8751.478) and it was described by Raymond, Santos, and Massingila (1991) as: "teaching mathematics via problem solving by challenging students to construct and/or reconstruct their [student teachers'] understanding of mathematics" (p. 4). Their study was influenced by Lester, Garofalo, and Kroll's (1989a) of seventh graders. Raymond, et al. investigated the ways in which students' belief systems might have been influenced by the instruction. They found that "the three basic components of T 104 [a mathematics content course] instructional approach (problem solving, cooperative learning, and written reflections) have the potential to challenge students to question their mathematical belief systems regarding what it means to know, do, learn, and teach mathematics" (p. 1). The significance of the study might be its potential impact on students' future teaching actions. In conclusion, they suggested that more research is needed to study teachers' belief systems and the extend to which they will be positively influenced by mathematical instruction.

Santos (1990) investigated the students' beliefs about mathematics, as well as the effect of metacognitive instruction on students' problem solving performance at the college level. In this study, he regularly observed the class while the instructor was teaching. The instructor became familiar with metacognition and problem solving through guided readings and discussions with Santos. He also developed many problems focusing on metacognitive knowledge. Santos used both a modification of Schoenfeld's (1985) framework and Perkins and Simmon's (1989) framework for analyzing his data and to study the students' misconceptions and difficulties.

Santos' original aim for the course was to teach mathematics via problem solving and to emphasize problem-solving strategies. He believed that this approach would give students a chance to discuss their ideas, to become more engaged in problem solving processes, and to work more cooperatively in the class. Moreover, Santos developed a number of nonroutine and interesting problems for the assignments which were accompanied by a series of metacognitive questions. Santos went through all the students' responses to the assignments and the metacognitive questions. Eventually, the students' approaches to mathematical problem solving were influenced by these activities. Those students who received the instruction showed more awareness of the importance of using different approaches to solve mathematical problems. In addition, the study showed that students who were exposed to metacognitive instruction scored higher on the final exam which was the same across the college.

In conclusion, Santos suggested that teaching mathematics via problem solving should become the approach to instruction rather than only a means to be used occasionally. He also concluded that more metacognitive strategies should be developed and be adopted throughout instruction. His recommendation is to design more challenging course materials appropriate for teaching mathematics via problem solving.

Campione, Brown, and Connell (1989) considered metacognition-based instruction as an alternative for instruction. They called it "reciprocal teaching," summarized by Lester, Garofalo and Kroll (1989) as: "(a) instruction occurs in cooperative learning groups; (b) instruction takes place in the context of learning specific content; (c) the students' attention is focused on solving a specific problem, not on monitoring, regulating, or evaluating actions per se; (d) students are not protected from error in their solution efforts; (e) the teacher is allowed to be fallible; and (f) the teacher's role as a guide and model diminishes as students become more confident and competent" (p. 36).

In addition to the studies mentioned, several others have been conducted to investigate the different aspects of metacognition, concerning specific groups of people in specific settings rather than regular day-to-day mathematics course. Notable studies include: Hart and Schultz's (1985) study of three in-service intermediate teachers using Schoenfeld's framework for analyzing students' metacognitive behavior, and Defranco's (1987) study of 16 mathematicians' problem solving and metacognitive behaviors.

Using Writing in Metacognition-based Instruction

There is a growing sentiment for journal writing among researchers as a means to facilitate learning of different subject matter. Vygotsky's (1962) view of the relationship between language and thought lend support for the use of journal writing. As pointed out by Emig (1977) ". . . writing can engage all students actively in the deliberate structuring of meaning, it allows learners to go at their own pace, and it provides unique feedback, since writers can immediately read the product of their own thinking on paper" (quoted in Borasi and Rose; 1989, p. 348). Students could either have freedom to write any thought related to their mathematics or their writing could be fully structured and guided.

The results of Crowhurst and Kooy's (1985) study show that journal writing during the reading of novel indeed encourages students' thinking. "It encourages them to reflect upon and clarify their thoughts about and their responses to the literary work. Writing both requires and facilitates the clarification of thought" (p. 263). Langer and Applebee (1987) strongly recommended the use of writing in the school curriculum.

Journal writing as a learning tool in the mathematics classroom has gained popularity among the mathematics and mathematics education research community. Borasi and Rose (1989) conducted a research study using writing in a regular college mathematics course. In conclusion, they gave a taxonomy of potential benefits of journal writing for both teacher and students and show how writing can be a means for more individualized teaching. Lester, Garofalo, and Kroll (1989a) used writing to investigate the kinds of metacognitive decisions that the students made. In their study, Raymond, Santos, and Massingila (1991) used reflective writing for the same purpose as well as to document possible changes in students' beliefs about themselves as doers of mathematics.

The number of mathematicians and mathematics educators who have extended the idea of journal writing throughout mathematics courses is growing. Burkam, Britton, Talman, Hartz, Buek, Rose, and White (all documented in Sterrett, 1990) are among them. They found that "writing to learn" has potential to contribute to mathematics instruction.

The review of related research gives every reason to believe that writing gives students the opportunity to reflect upon their feelings, knowledge, and beliefs about mathematics. It will also be beneficial for teachers to read the students' journals because reading the journals would help teachers to improve their teaching. Journal writing allows for individualized instruction, and it might help to create a friendly and supportive classroom environment.

Summary

This chapter gives a review of studies about the importance cognitive and non-cognitive factors pertaining teaching and learning of mathematical problem solving. Beginning with a summary of research about problem solving, it then gives an account of metacognition and its role in mathematical problem solving. It ends with a brief description of several classroom-based research studies using metacognitive strategies. Those studies helped to identify the types of metacognitive activities that would be beneficial for enhancing students' understanding of mathematics, and the present study was influenced by them.

CHAPTER 3

DESIGN AND PROCEDURES

The review of the related literature indicates a number of important factors pertaining to teaching and learning of mathematical problem solving. Lester, Garofalo, and Kroll (1989b) categorize these factors as knowledge, control, affects, beliefs, and socio-cultural factors. Silver (1985) expressed his concern for some of the methodological, conceptual, and reporting problems with research in this area. Among them were the lack of information on what the teachers actually did in the classroom when teaching problem solving, the amount of control over the "teacher variable," the insufficient assessment of the influence of direct instruction on students' problem-solving behaviors, and the absence of any instructional theory in guiding most of the research studies. A study of the role of metacognition in mathematical problem solving in two grade seven classes by Lester, Garofalo, and Kroll (1989a) addressed these concerns and provided some insight into the research in this area. These researchers studied the effect of problem-solving instruction based on metacognitive strategies on students' problem solving behaviors and concluded that the instruction would have been more effective if it had happened in the context of regular day-to-day mathematics instruction rather than as a unit added to the mathematics program.

Raymond, Santos, and Masingila (1991) taught a mathematics content course known as "mathematics for elementary teachers via problem solving," at Indiana University and it was required for prospective elementary teachers. They adopted a non-traditional approach to instruction and used a number of instructional techniques designed to help students become more reflective. They documented some changes in students' beliefs by investigating the students' in-class and assigned written reflections.

The purpose of the present study is to document students' beliefs about mathematics and mathematical problem solving as well as to investigate the possible changes in students' beliefs as a result of teaching through problem solving. This study is similar to the Raymond, Santos, and Masingila's (1991) study. Yet there are a number of significant differences in the variety of instructional techniques used, and the content of the course. Another aim of the present study is to address Silver's (1985) concern about the lack of information about what teachers' actually did in classrooms by documenting classroom processes through audio and video-taping classroom sessions as well as taking into account the researcher-instructor's daily journal in analyzing the data and reporting the findings.

The design of the study, the nature of the instruction, and the rationale for choosing different instruments for gathering data are discussed in the following sections.

Rationale for the Research Paradigm: A Qualitative Approach

As has been indicated by many researchers, the processes of obtaining the solution to a problem is more important than the final product. This indication gives rise to the use of qualitative approaches to research since it involves the close-up, detailed observation and documentation of the phenomenon under study (Yin, 1989). Krutetskii (1976) found that qualitative research was advantageous in studying students' individual differences during the process of solving problems. In line with Krutetskii (1976), Davis (1986) discusses the significance of qualitative data such as those obtained by task-based interviews for investigating the students' understanding of mathematical concepts. These data cannot be replaced by the results of achievement tests. "Achievement tests provide a certain kind of data, but they omit much more; they do not give a broad comprehensive picture of what is taking place" (p. 87). Qualitative methodology is an appropriate research paradigm with respect to the aims of the present study. It is crucial in studies of this nature to discuss the ways they are designed and redesigned during the natural course of the inquiry. Two pilot studies were helpful in designing the study.

Pilot Studies

Prior to the main study, two pilot studies were conducted to investigate the possibilities and the limitations of the research setting, the data collection instruments, and the interview techniques, and to revise the presentation of the course material. The summaries of the two pilot studies follow. All instances of emphasis in original materials, for example in students' responses, are indicated by underlining. Where emphasis has been added by the writer, italics are used. "I" stands for interviewer.

First Pilot Study

The first pilot study was conducted in the Fall of 1989. The pilot study had two main purposes: first to test the thinking aloud technique for the task-based interview and second to evaluate the appropriateness of the non-routine tasks chosen. Two grade eight and two grade nine students participated individually in the interview. The students were interviewed in an informal setting outside their schools. Two of them were from the same school and the other two came from two different cities. The following two tasks were chosen for the interview

Task I: What is the units digit for the following sum? $13^{841} + 17^{508} + 24^{617}$

Task II: The last digit of the product 3 x 3 is 9. The last digit of the product 3 x 3 x 3 is 7. What is the last digit of the product when thirty five 3's are multiplied?

The two cards with tasks written on them were put on the table. Students had a chance to choose either one of them to start. The reason for not writing the two problems on the same card was my hypothesis that whichever task was given first, the students would presume that there had to be a reason for it. For example, the first problem might be easier. This idea was to prevent those potential feelings.

All four interviews chose the second task first. For them, working with smaller quantities seemed to be easier. This belief is identical to what Lester and Garofalo (1982) identified in grade three and five students that problems involving larger numbers are more difficult to solve. All the interviews were audio taped and transcribed for data analysis along with the students' written work and my own notes. All four interviewees are given pseudonyms for the sake of confidentiality.

Summary of results of the first pilot study.

The aim of this section is to share a few excerpts of the students' interviews to first show the appropriateness of similar tasks for the study that focuses on students' beliefs about mathematics and mathematical problem solving, and to establish that the task-based interview was a suitable technique to be used in the main study.

Students' beliefs are shaped in part by their teachers. Teachers and textbooks are often responsible for formation of students' beliefs. Students rarely face a real "problem" in their everyday classes. As stated by Schoenfeld (1987b), "Even when students deal with 'applied' problems, the mathematics that they learn is generally clean, stripped of the complexities of the real world. Such problems are usually cleaned up in advance--simplified and presented in such a way that the techniques the students have just studied in class will provide 'solution'" (p. 37).

One of the common beliefs shared by all the interviewees came from what Schoenfeld has described. Students have had little chance to "mess up" and engage in the process of solving problems. Teachers usually prescribe certain procedures to students and often their prescriptions affect students' curiosity and their eagerness for creativity and discovery. The interviewees were embarrassed by not "remembering" the procedure to do the problem. Sara said, "You probably get someone that knows how to do it right at the beginning."

The students believed that school mathematics is mostly chasing after the "correct numerical answers." They usually see the mathematics and problem solving through their teachers' eyes. Teachers play a significant role in shaping the students' beliefs about mathematics. Mathematics could be fun or boring, depending on how it is presented to them.

I		How do you like math?

Sara		I liked it last year because I liked my teacher. . . this year math doesn't interest me. This year's teacher just goes by the book. It's really boring.

Vicky, as well, said that this year she did not like mathematics because her teacher was boring!

Vicky		He just reads us the book that says something and then tells us how to do it . . . he is boring.

I		What does he do that bores you?

Vicky		You've got to go to his class! I don't know! He just gives us the worksheets, there is nothing in it, *no excitement*, anything . . ..

All the students believed that mathematics is a set of rules and procedures that should be memorized. All of them first tried to remember how they solved the problem in class. Don even surprised me more when he said:

Don		I remember *that page* in the math book in grade 7.

I		What was in that page?

Don		There was a way to do it [task II]. My teacher showed me how to do it . I remember vaguely in my math book that they did it [task II].

In his view, problem solving could be reduced to memorization of series of rules and methods offered by teachers and textbooks, and students' main responsibility was to follow them step by step. This way of teaching would destroy students' confidence. It is hard to expect them to think about creativity and discovery when the teacher always shows them what to do and asks them to spell out the same thing again in return. Most of the students believe that "teachers know the right way of solving the problems." It is unfortunate that they think there is only *one* solution to each problem and *the* best way to do it is the teacher's way because they certainly get the correct answer that way. The students' beliefs about mathematics and problem solving could be summarized by the following:

Belief 1: Mathematics is a set of rules and procedures that should be memorized.

Belief 2: There is only one solution to each problem.

Belief 3: Mathematics is boring because it has no excitement.

Belief 4: The most important feature of a mathematical problem solving is finding the correct answer.

The task-based interviews also gave a chance to study the effect of students' beliefs about mathematics and mathematical problem solving on their approaches to problem solving. The students believed that following the teacher's way is the wisest thing to do in order to get the correct answer. They are usually told what they have to do and they have had few chances to say how they think. The transmissive mode of teaching leaves little room for students to think, to guess, to speculate, to conjecture, to reason, or to take any risks at all. These activities are not greatly valued in their classes. Students' beliefs about mathematics and mathematical problem solving directly affect their approaches to problem solving. They usually try to remember the "right" formula for a given problem. This is one of the reasons that they do not see any point in looking for various ways to solve a problem. In fact they believe that variations in methods of solving a problem indicate a weakness not a strength and there is always "the best way" for solving any problem, which is "the teacher's way." This view was expressed very clearly by Vicky when she was asked:

I If you had this problem in class, what would be your teacher's way to do it?

Vicky He would read the textbook. The book says how to do it and then we just do it.

I Does everybody have different ways of doing the problems, or are they the same?

Vicky The first one [first problem] like everybody is gonna get the different one, like second one may be more and the third one will get *better* and then like the tenth one . . . like everybody has the same thing as the teacher . . . *it depends on how the teacher* teaches.

I What do you like to do? Do you like to solve the problems in your way or do you like to do what he says?

Vicky I just do what he says.

I Why?

Vicky Because that's what he says!

I Do you have all the trust in him?

Vicky Ya, the book is right.

Ed said that it was safer for him to do what his teacher told him to do.

Ed I prefer going by teacher's method because if I go by myself and I do it my . . . well, I just might do it wrong.

The students believed that their success, which in their view was the correct answer, would be guaranteed if they followed teachers' way of solving problems. Fear of not being successful stops them from taking the risks which they cannot afford.

Ed The teacher is probably right. He knows better than me. Then if I go by my way then I might get a faster way, but I might just do it wrong.

Interestingly when an alternative way was suggested to him he responded differently.

I I was wondering whether you had a chance to do the problem in a different way with the teacher's presence . . . I mean if your teacher was supervising you in a small group, to solve the problem together and discuss it together, which way would you then prefer?

Ed I then prefer to do it my way because *I remember it longer*.

His response was amazing. He naturally liked to do it his own way. He would "remember it longer" since he had been involved in the process of doing it rather than using the teacher's finished product.

The first pilot study informed me whether teachers' approaches to teaching and learning mathematics inclined more toward negotiation than imposition (Cobb, 1988), the students would have a better chance to use their cognitive and metacognitive skills. Also their views about mathematics and mathematical problem solving would be altered as a result of the instruction.

Educational significance of the first pilot study.

It seems that if mathematics is presented to students via problem solving and if they are given a chance to think and to do, they learn more mathematics than by following teachers' footsteps. In Sara's interview, she was stuck because she could not "remember" what to do. She said, "I know I was told how to do this!" yet she forgot. When I asked her; "Can you stop thinking about what you were told? Read the problem again and see what you can do," it did not take more than a minute or so before she said, "Hold on! . . . There is a pattern!" I asked her what happened, and she said, "Well . . . figuring out . . . I mean after awhile that you multiplied a number by 3 so many times . . . I probably would not realize that there is a pattern unless I stopped and started over and *looked back* to see what I have done that makes it easier."

The pilot study suggested that students would enjoy mathematics and learn more from it if mathematics teachers valued the processes of students' problem solving and if the main goal of school mathematics were not only the finished product of students' work, a correct answer. Sara found different ways of doing the task only when she had an opportunity to try different methods. She commented, " If you say that this is the first thing that came to my head, you try it and if it gets too hard you have to go back and . . . you have to force yourself to find an easy way to do it."

It was a new thing for her to take a step back and "force" herself to find other ways of doing a problem. In fact, it would be a breakthrough for all students to gain more confidence and try to think, make decisions, and look for alternative ways when solving problems.

Second Pilot Study

The second pilot study was carried out in the Winter term of 1991. I was a teaching assistant for Mathematics 335 (Introduction to Mathematics). The purpose of the second pilot study was to assess the potential research design, to revise the students' evaluation scheme, and possibly to alter some of the course material such as worksheets. The course met for three hours of lectures (Mondays, Wednesdays, and Fridays) and two hours of tutorials (Tuesdays and Thursdays) for 13 weeks. I was in charge of the tutorial sessions, although the instructor was always present. These sessions helped me assess the effect of the instructional techniques and the instructor's role in the class and with regard to the students. Furthermore, the pilot study gave me the chance to revise my plans for student evaluation and to devise more suitable criteria for marking, since I was responsible for the marking of the students' monthly projects (three altogether) and the students' journals. The final exam was marked by the course instructor.

Summary of results of the second pilot study.

I suggested to the instructor collecting and marking the students' journals every second week. The instructor gave me the following criteria for marking the journals: (a) presentation, (b) integrity/originality, (c) completeness, (d) clarity, and (e) mathematical level. However, I realized that my perception of journal writing and its role for students' learning was different from that of the instructor. The marking scheme gave too much weight to each criterion without much flexibility. I recognized that such criteria would limit the students' creativity and structure their writing to a great extent. Also, giving interim marks for the journals based on the rigid criteria did not seem to be a useful technique if the purpose was to help students to become more reflective. The pilot also helped me design a variety of quizzes and assignments to assist the students with their learning.

I attended all the lectures and took field notes while sitting at the back of the room. Those notes included my reflections on the course material and the students' responses to the instruction in general. I regularly and constantly talked to the students and the instructor after each class. I gained insights from the students and always shared those with the instructor. Some of the course material for the main study underwent small yet significant changes as a result of those reflections and the interactions. For example, one of the first worksheets that the students got was titled "Two easy questions about angles." The title disturbed many of students. They were concerned about why they had so much difficulty and could not solve those problems if they were "easy." One student told me that she felt so stupid because the problem was supposed to be easy and still she could not do it. These things might seem to have little or no significance on the whole teaching and learning process. However, they sometimes cause great dismay and result in lower self-confidence for the students.

I noticed that during the "lecture" hours, the students were not involved in the teaching process. Only students in the first few rows raised questions every once in a while. The rest of the class were mainly taking notes and did not interact with others. However, in the tutorial sessions more students interacted with each other and with us (the instructor and I) while working in small groups. In those sessions, the problems helped to clarify some of the mathematical concepts which were taught in the lectures, but which students had difficulty understanding.

The pilot study gave me the confidence to teach mathematics *via* problem solving rather than teaching mathematics *for* problem solving. The results showed that students would be more engaged and better motivated if the mathematical concepts were presented to them through problems.

Methodological Elements of the Main Study

The results of the two pilot studies gave considerable insight into redesigning different parts of the proposed study. These studies supported a qualitative method as suitable for the purpose of the study. The following sections give a description of the methodological elements of the main study, an account of the trustworthiness, and an explanation of the process of analysis.

Subjects of the Study

Students in the course introduction to mathematics (MATH 335) at the University of British Columbia (UBC) in the summer of 1991 were the subjects of this study. Forty-three students registered for the course; however, three students withdrew. One student found a good job that she could not refuse. Before the first quiz, a second one said, "I am a perfectionist, and I know that I can not do it perfectly; therefore I want to withdraw from the course." A third one withdrew after the second week of the course because it interfered with her other courses due to the difficulty that she had with the registrar office. The remaining forty students filled out the consent forms (see Appendix B) giving permission for their work to be used as the data for the study after they read my letter describing the study and informing them that pseudonyms will be used for the sake of their confidentiality (see Appendix A). Every student except one consented to be audio- and video-taped in the classroom. None of the students except one, had ever successfully finished any university mathematics course; yet a number of them had already earned their bachelor's degrees in various fields, and the rest of them were in either the third or the fourth year of their program. Ten of the students were males, and thirty were females. The students' ages ranged from early twenties to mid-fifties. Most of the older students had enrolled in the course following a decision to change their careers and become elementary school teachers. None of the students except one, took the course as an elective; they all clearly said that they had to take the course because they needed it, not because they liked it. At least half of the students worked long hours and were extremely tired when attending the class in the evenings.

Description of MATH 335

Introduction to mathematics is a mathematics content course which is offered by the Mathematics Department and designed for students who have a university education that does not include a mathematics course. MATH 335 is a required course for students who wish to apply for the University of British Columbia elementary teacher education program and who have no other university mathematics course. However, the course is open to all students with various background and educational purposes. The goal of the course has been to give the flavor of interesting and important mathematical issues without using calculus (see Appendix D for a copy of the course outline). The course content is divided into three sections: geometry, numbers, and probability. The designer of the course has hoped to introduce the concepts in a more intuitive way to give students a chance to become more actively involved in doing mathematics. The emphasis of the course is not to "stuff" students with rules and facts but to engage them in the process of doing mathematics in a more concrete manner.

Sequence of the Instruction

The instruction took place Monday to Friday from 4:30 to 6:30 p.m. for six weeks. It started on May 6 and ended on June 14, 1991. This is in contrast to Winter term (the course is not offered in Fall term), when it is taught for 50 minutes per day, five days per week, including three hours of lecture and two hours of tutorial for 13 weeks. Teaching assistants are normally in charge of tutorials and marking, except the final exam. In the Summer, I did not separate the lecture from the tutorial. The teaching assistants and I worked with students all the time. The amount of lecturing was limited to 15 to 20 minutes each day. The rest of the time was spent on working in small groups and whole-class discussions. There were also office hours for those students who needed extra help outside of class time. However, only a few students showed up at those hours, since most of the students preferred to stay a little longer each night and ask their questions. I was happy to spare half an hour after the class time was over. The students were allowed to take a ten-minute break, but only a few of them stopped working and left the classroom.

The first two weeks of the course were devoted to geometry, the next two weeks to numbers, and the next week and a half to probability. The course was reviewed in the second half of the last week, and the final exam was given on the last day of the class.

Nature of the Metacognition-based Instruction

The metacognition-based instruction used three different but interrelated techniques that focused on metacognition. These techniques are somewhat similar to, and yet distinct from the ones that have been used in the other studies mentioned in the second chapter. These techniques were (a) journal writing, (b) small-groups problem solving, and (c) whole-class discussion. Along with these, two other less used but significant techniques were (d) teacher as role model for metacognitive behavior, and (e) using videotapes. A short description of each of these techniques and the rationale for using them are given in the following sections. The social norm of the class is also discussed in some detail.

Journal writing.

Journal writing was one of the main activities in this course. The students were asked to write at least two or three entries each week. There was no assigned structure for writing the entries. Students were free to write about anything that they wished regarding the course. At the beginning of the course, I explained journal writing as a communication channel between the students and the instructor that facilitates individualized instruction. This potential benefit of journal writing was even more important considering the diverse background of students.

I had chosen a number of students' journal entries from the previous year to read in class to give students a better sense of what I meant by journal writing regarding a mathematics course. However, I changed my mind and let students interpret it by themselves. I thought the chosen entries might become a norm for their writing. Moreover, the students asked me to give them some idea for writing in a mathematics course since they had never done writing in respect to mathematics. I therefore made some suggestions of possible writing themes such as writing about (a) your feelings toward mathematics, (b) reflection on the course and day-to-day activities in the class, (c) why you did what you did in order to solve a problem and how you did it, (d) what helped you to understand a concept or a problem or what hindered you from doing so, (e) your frustrations and difficulties regarding different issues in the course as well as your joyful and happy moments of discovery and satisfaction.

As a result of loosely structured writing assignments, I got a great variety of students' responses which I would not have gotten from them otherwise. Some of the students wrote mainly about their difficulties with problems that they solved either individually at home or cooperatively in the class, but others mostly reflected on their own feelings and beliefs about everyday activities in the class without quite putting them into the course context. However, the majority of them wrote about many issues situated in the course context. I collected the journals every Friday and returned them the following Monday. It took me at least thirty hours each weekend to thoroughly read the journals and respond to them accordingly. The evaluation of the journals is described in a separate section under the heading "Course Evaluation."

Small-group problem solving.

Journal writing was designed to give students the chance to become more reflective on their own actions in class as well as the opportunity to think about their thinking and talk about their feelings. For many students, writing provided an occasion for personalizing and internalizing the learning processes. On the other hand, it was expected that students would learn to assess and monitor their work and make appropriate decisions by discussing the problems among themselves in small groups. Therefore asking students to work in small groups seemed a reasonable way to achieve this goal and to help students to become more reflective. I will give a brief description of why I chose small groups as one of the main instructional techniques.

As I said earlier, the intention for the instruction was teaching via problem solving (I will explain it under the same heading). However, I believed that it was impossible to teach mathematics through problem solving without having students interact with each other in the class. "Bucket- filling" teaching does not leave much room for discussion and creativity. As Dewey once said, a child (and any student for that matter) is not a cup to be filled. Students are full of interesting ideas. One of the important roles of every teacher is to cultivate those ideas, to allow them to grow and develop. Literature is overflowing with research findings regarding the role of small groups in students' learning (e.g., Schoenfeld, 1989; Good, Grouws, and Mason, 1990; Maher, and Alston, 1985; Noddings, 1985; Dees, 1985). There is great variation in the ways in which different researchers and teachers have used small groups in their studies and their classrooms.

Vygotsky's (1976) ideas give a strong justification for the role of small groups in students' development. He argued that working in collaboration with others helps students to reach the "Zone of Proximal Development (ZPD)." Vygotsky (1962) explained that a child might be able to function up to a certain level, but working cooperatively with others who are more capable might help him or her to function at a higher level. The potential ability that a child (or person at any age) has and that could be cultivated by some assistance but not in isolation, is the ZPD (also see Schoenfeld 1985a, 1987a).

In MATH 335 classes, students were asked to work cooperatively when solving problems. Some of the students reflected on this issue in their journals and expressed their concerns. However, not all the concerns were of the same nature. Some of the students were afraid of everyone else in the group finding out about their "stupidity," yet the other students felt insulted by the idea. Melisa voiced that concern in her journal as, "I was extremely taken aback by the suggestion that not only was co-operative learning okay, but that it was *required* for this class." However, after about two weeks of debating and experiencing, almost everyone felt comfortable about working in small groups, and they eventually found a suitable one. The small groups were not structured by any means, and their sizes varied from three to seven or even eight members. The students liked the unstructured approach to formation of these groups. For example, Clara, in an interview at the end of the course, said, "It was nice in the class the way that groups kind of formed. Everybody had their own little group that they worked well with." Many of the students changed their groups a number of times until they found a "right" one to work with. When I asked how they chose their groups and whether it just happened or they changed their groups at all, Clara said, "I did change it [my group]. I was working with Kent first, but I found that he was [pause]. . . he knew too much for me at first, so he kind of talked beyond me, but later on I could deal with him. I couldn't have at first because . . . I had no idea what he was talking about." The interesting phenomenon is that after Clara felt confident working with him, she did so, and they worked together from that time to the end of the course.

Whole-class discussions.

The pilot study showed that discussing what went on in small groups and sharing the ideas was a necessary activity in the class. Whole-class discussions was one of the most helpful strategies to enhance the students' understanding of mathematics, and to influence their beliefs about mathematics and mathematical problem solving. However, it took more effort to establish a social norm for whole-class discussions in which every student felt comfortable to get actively involved in the meaning-making processes, and to eventually become more self-regulated and a better problem solver.

In the first three weeks of the course, I gave the students worksheets relevant to the concepts that had been or were to be discussed by means of those problems. Later I will explain the ways in which they were prepared. The worksheets presented relatively rich problems, ones which could be approached differently and which required some planning and using of different problem solving strategies. However, different groups worked at different paces, and this fact created some difficulties regarding whole class discussions. To be specific, the teaching assistants and I felt that the worksheets were becoming the focus of the class discussions. No matter how many times we told them that those problems were not the end in themselves and that the important thing was the process of solving problems not just getting the correct answer, they were still worried about not finishing those worksheets during the class time. I therefore decided to give the whole class a problem and ask every group to work on the same problem simultaneously. This technique was very useful in bringing the whole class together while discussing a problem. It also relaxed the students since they were not worried about finishing the problems as before or falling behind the others. Gradually they gained more cognitive and metacognitive knowledge of problem solving, a better sense of responsibility, greater self-confidence, and ultimately the motivation to do some of the worksheet problems at home rather than holding the class responsible to do every single problem for them. The class observation and the students' work provided evidence to support this claim. The students eventually realized that they would learn more by analyzing and solving one problem thoroughly, than by solving many problems without really understanding them.

After they all talked it over in their groups, one person from each group shared the solution or approach of his or her group to each problem orally with the class while I was writing it on the board (or they themselves wrote it on the board). Sometimes a problem was solved or approached in four or even five different ways depending on its nature. The pilot study and my previous teaching experience led me to believe that such activities would be extremely helpful in enhancing the students' metacognitive abilities. They had to make decisions after they were confronted with the variety of approaches to the same problem. By examining different possibilities, they had a chance to see how one strategy could work better than the others and *why*. Everyone was encouraged to ask the person or group that solved a problem on the board for explanations. The debating and dialogue helped the class to extend their understanding of the problem and consequently the relevant and related concepts, as well as to gain self-confidence and self-respect. It was pleasant and reassuring for them to see that although their approaches and solutions were different, they were acceptable and valid. Besides, they had to decide which approach made the most sense to them since they had the opportunity to examine many possible alternatives. This process also helped self-regulation which is one of the crucial issues regarding metacognition.

However, sometimes nothing seemed to work, and no approach yielded a solution. This could lead to the students' frustration and disappointment. In such instances, I took control of the class discussions by summing up what went on in those sessions. The intention was to help the students realize why they got nowhere, what possibly went wrong, and what could be done. This activity helped them to see both the pitfalls and the light through the awareness and the knowledge that they gained by examining the different approaches to the problem. Then they were better able to put everything in perspective and decide what to do to solve the problems. These events served as a vehicle to promote self-regulation.

Class discussions provided good opportunities to discuss the students' beliefs about themselves, mathematics and mathematical problem solving. It was interesting to see how those beliefs affected the ways in which the students approached the problems. I will discuss those in detail in the next two chapters.

Teacher as role model for metacognitive behavior.

Schoenfeld (1987a) once in a while used the "teacher as role model for metacognitive behavior" technique to model a good problem solving behavior (see also Lester, 1988). His intention was to go through the solution processes in detail as if he were confronted with the problem for the first time. He then discusses about the advantages and shortcomings of this technique. One of the benefits for students is that they see appropriate problem solving behaviors that they might adapt. The main disadvantage in his view is that "this kind of modeling approach is artificial and must be used sparingly" (p.200). Lester (1989a) taught the seventh graders as part of the "Metacognitive based instruction" project. He, too, occasionally adopted a similar technique. At the end of the project, he expressed the same concern as Schoenfeld did, noting that there was hardly any new problem at that level (seventh grade) which he had not seen before. When he pretended to act like someone who was just confronted with the problem and wanted to model good problem solving behavior, he admitted that his action was indeed artificial.

I thought about the ways in which this technique might be utilized in my course less artificially. I came to the conclusion that this strategy would be embedded in the whole class discussion to some extent. In whole class discussions I did not limit the students to the assigned problems, and when the students or I were confronted with a new problem directed to us from other students, we indeed had to demonstrate our problem solving behavior. Sometimes these problems were extensions of the given problems, and occasionally they were "what-if" questions which deserved genuine, rather than artificial problem solving processes. Since the students were expected to discuss and speculate on the varieties of possibilities regarding the problems, I eventually encountered unexpected situations that required "on-the-spot" thinking and decision making. As Schoenfeld (1987a) and Lester, Garofalo, and Kroll (1989b) mentioned, neither they nor I used this technique frequently. However, occasionally such incidents happened unplanned and proved to be beneficial.

<u>Using videotapes.</u>

Another technique that was adopted by Schoenfeld is the use of videotapes of other students working on the problems. He describes that his students, after watching the video, empathized with students in the video and said "that could be me." Schoenfeld (1987a) notes, "That's precisely the point. It's a lot easier to analyze behavior when it's someone else's, and then to see that the analysis applies to yourself" (p. 199). Such events could help students to become more aware of their own thinking processes which is one of the main aspects of metacognition. This technique was used only once in the course, and I was quite surprised by its effectiveness.

Before the course, I thought of using such a technique, but I was not sure where to get the appropriate videos suitable for the course. Fortunately something happened during the course that helped me to find one. In the second week of the course, Paul Cobb (1991) visited UBC and made two presentations. In the second one, which was mainly for teachers, he wanted to show a video clip of elementary students working on some multiplication tasks. I decided to take all the students to that presentation (with their consent). Knowing Cobb through his work (Cobb, 1983; 1988; 1989), I believed that the talk would help the students to become more reflective and would possibly stimulate the class discussion.

Amazingly, many students reflected on that video clip and identified themselves with students in the video. They talked about that situation mainly in their journals and rarely in the class discussions. This activity helped them to become more aware of metacognitive issues. The point that I am trying to make is that this technique was indeed helpful. However, finding and choosing relevant video clips is not an easy task.

Teaching Via Problem Solving

The instruction was not only based on metacognition, it also adopted the approach Schroeder and Lester (1989) call teaching via problem solving. The course coordinator and I tried hard to find interesting and engaging problems and activities that would develop mathematical ideas as students solve those problems. The problems were chosen in a manner to make connections between mathematical ideas and to lead one idea to the other. In teaching via problem solving, I gave no formula or procedures to students unless we as a whole group arrived at them.

It is impossible to discuss every problem and activity that we did in class, but in this section I would like to clarify what I mean by teaching mathematics via problem solving using examples from three major parts of the course, geometry, numbers, and probability, and then discussing one problem related to exponential growth in a greater length.

The course started with two puzzles (Appendix E). In the dodecagon puzzle, we deductively proved that the pieces of dodecagon could be rearranged to make a perfect square without losing the area. In the other example, the students saw that by rearranging the pieces into rectangle they lost or gained one unit. Some of them thought that there was either an error sight or something wrong with their cutting. They tried everything they could. A number of them noticed that if they fixed the corners to make 90 angles, there was either an overlap or a gap in the middle. Kent suggested making sure if the diagonal was indeed a straight line by finding its slope at different parts. His insight led the class to see that the figure was not a perfect rectangle. Some of the students noticed that by cutting the pieces of these squares and reassembling them, the area of so called rectangles were always one unit more or one unit less than their corresponding squares. They finally realized that there was an interesting pattern between the side lengths and the areas of the squares and the side lengths and the areas of their corresponding imperfect rectangles. Those patterns involved the Fibonacci numbers. The students were fascinated to see that many natural phenomena correspond to those numbers. These puzzles helped us to have better understanding of mathematical reasoning. They were extremely helpful in taking the class to the geometrical proof.

The manipulative materials helped us to understand the concept of area and volume. The whole class developed the ways in which to find the areas and the volumes of different polygons and polyhedrons. We eventually came up with working procedures and formulas for those areas and volumes. Dissecting a cube into three irregular pyramids showed them that why the volume of a pyramid was 1/3 of the volume of the cube, and by extension, any prism. Using Cavalieri's Principle, the same factor 1/3 relates the cone to the cylinder. Therefore, the 1/3 factor was not a mystery to them any more because they actually saw it. We finally found the volume of hemisphere together, and all I did was to record and organize their ideas. We then moved on to scaling and Cavalieri's Principle. Then finding the area of the unit circle took us to the section on numbers.

Finding π by repeated doubling of the number of sides in a regular polygon was an exciting and engaging activity. The students were happy to calculate π as 3.141592 by themselves. Many of them said π had always been a magic number for them prior to the course. Using Newton's method to find square roots of numbers was also engaging. It was presented as a way of changing a rectangle into a shape more and more like a square, by successively averaging its sides. As Peggy said, activities like that helped the students to have a greater appreciation for mathematics. Periodic and terminated decimals approached via Zeno's Paradox, and the ways in which to convert them into fractions gave students a chance to find many interesting patterns, and they eventually established formulas to facilitate their calculations. We found out where the numbers in logarithm and trigonometric tables came from using graphs. For instance, we got the fractional powers of 10 by filling the gap between $10^0 = 1$ to $10^1 = 10$ using calculator. We then took one step further and constructed the graph of $a = 10^{n/32}$. The graph helped enormously to unfold the mystery of logarithms (Appendix E contains a copy of the graph). The students got a sense of empowerment knowing how the keys on calculators operate and where those magic numbers come from.

I gave them simple problems to start a section on exponential growth and decay, including a problem involving doubling one's money everyday for a month, compound interest, growth of bacteria, and carbon dating. The students had special interest in these problems since their contexts were familiar to them. For example, there were great discussions in the class regarding interest rates, inflation, budget deficits, cancer cells, the validity of carbon dating in identifying the age of artifacts, environmental issues, and so on. Problems of these kinds were extremely helpful in developing the mathematical concepts and formulas. The following is an example of a problem regarding exponential growth. (Appendix E contains two problems regarding exponential decay.)

The cost of real estate in a certain city has gone up at a rate of 10% per year since 1975. If this trend continues, approximately what will a house be worth in 1995 if it was purchased in 1985 for $120,000?

Most of the students had little difficulty solving the problem because they were familiar with the context; they simply multiplied each year's total price by the factor of 1.1. I then suggested that they make a systematic list. Therefore, rather than multiplying the obtained price by 1.1 to get the new price, we multiplied $120,000 by 1.1, then the next year's price was increased by the same factor as ($120,000x1.1)x1.1, and so on. They eventually better understood the exponential growth, and they were able to easily arrive at the formula to facilitate the calculations when large numbers were involved. The section included more and more complicated problems. I was surprised to see that their understanding of the concepts allowed them to solve those problems without much difficulty.

In teaching mathematics via problem solving I as the teacher had a special role to play. I orchestrated the class discussions and monitored the students' work in small groups. The course material included many interesting problems and activities to develop mathematical concepts. However, I was constantly looking for interesting and suitable problems everywhere I could to accommodate the students' need. I tried to include a range of activities considering the students' special interests. Small groups were particularly appropriate to take those interests into account. For example, the relation between mathematics and painting, woodworking, baking, and games were among them. The students' journals provided me a unique opportunity to consider the students' needs concerning different mathematical issues and to stress them during the instruction. I spent an enormous amount of time regularly to take every student's concern into account. However, these efforts made the study a "qualified success story."

All these activities required many hours of hard work and continuous reflection on what went on in the class to be able to improve the situation. However, this hard work paid off enormously, and added much joy to my teaching.

Many practical problems helped us developing the concepts and formulas in probability. Without any lecturing or imposition in my part, those problems helped to establish the concepts of combinations and permutations, and led us to develop formulas to facilitate the solution of problems involving larger numbers. Many students expressed their concerns about this section of the course in the beginning. However, many of them ended up liking this last part more than other parts of the course. This section also made possible the use of a number of problem solving strategies.

Social Norm of the Class

The most suitable environment--or "classroom culture", as Davis (1989) calls it--for teaching and learning mathematics, is one which is more natural and less artificial. A number of mathematics educators who have done classroom-based research (e.g., Schoenfeld, 1985a, 1987a, 1987b; Lester, Garofalo, & Kroll, 1989a, 1989b; Raymond, Santos, & Masingila, 1991) have concluded that appropriate settings help students to become more aware of their thinking processes and to develop positive beliefs towards mathematics. For example, it is easier for students to analyze another student's behavior while solving problems, and be able to associate themselves with that when he or she is solving a problem for the whole class and when everyone else has chance to see what he or she is doing. Schoenfeld (1985a; 1987a) talks about the creation of a "microcosm" of mathematics in a way similar to the way that mathematicians do mathematics. Lester, Garofalo and Kroll (1989a) found that students' beliefs are mainly shaped by classroom instruction and the classroom environment. These findings led to the creation of a "social norm" of the class which was an appropriate environment for metacognition-based teaching and teaching via problem solving.

It was not easy for the students or me to set up such a social norm. It was difficult for the students to change their views of the teacher's role. Many of them expected me to tell them what to do all the time. In the beginning, a majority of them preferred "spoon feeding" instruction since they were used to being taught that way more often. They reflected on the teaching method in their journals and expressed their dislikes or their concerns about this issue. Comments such as "Please give us a formula to solve the problem . . . ," "tell us which is *the* way of solving the problem because multiple ways of solving it is confusing and a waste of time . . . ," "why are you so mysterious? Just tell us what to do," were not exceptional in the first two weeks. However, they diminished as the course moved forward. My goal was to help students realize that their involvement and participation was a crucial contribution to the teaching and learning process, so I reflected on their comments in reasonable detail and attempted to address those concerns both through their journals and in the class.

I was hoping that they would become independent and conscientious learners through discourse and interaction with each other and that they would reflect on what they did and why they did it. I wanted to show them that the teacher is not *the* authority in the class who is always "right." On the contrary, my aim was to confront them with a teacher as a moderator, facilitator, monitor, friend, and role model who is also *fallible*, that he or she indeed experiences the ecstasy of solving a problem after spending lots of time and effort, as well as the agony of not knowing and being frustrated and confused.

I tried hard to make sure that the students were actively involved in all the class activities. I did everything I could to convince them that their participation in the class discussions was indeed beneficial to themselves and to other students. I talked to them individually, wrote about it in their journals, and spoke to the whole class hoping that most of the students would finally come along and participate in group discussions. Like most typical classes, only a few students were really engaged in the class discussion in the beginning. Sometimes the active involvement of those students and their strength of reasoning intimidated the other students. A number of them reflected on this issue in their journals. They argued that since those students were better able to solve the problems and explain them to others, why should they "waste" the class time with their "wrong" solutions and "inadequate" explanations. My aim was to show them that there is not always one correct way of solving a problem. Besides, everyone would learn a great deal from each others' ideas and even mistakes. I talked to them about the ways in which most mathematicians did mathematics to came up with world shaking ideas. I challenged the myth that "some people have it and some people don't" concerning mathematics. The goal was to let them to believe that everyone can do mathematics to some extent and that the best way of doing it is to get involved and take responsibility for one's own learning.

My role as a facilitator and monitor was crucial, especially in small-group work. I had to make sure that every single student was engaged in mathematical activities in the groups. I believed that working in small groups would be effective, if and only if the students were all negotiating the meaning of what they were working on. The students had the responsibility to make reasonable decisions after discussing the problems in their groups. I knew that in group work, there was always a tendency for some people to passively follow the others in the group without being involved in the meaning-making process. To avoid situations like that, I reserved the right for myself to intervene and make sure that everyone knew his or her own responsibility.

As the course moved forward, more and more students participated in the class discussions and felt less intimidated by others' ability. A major breakthrough would happen when the students' fears dissolved, and they took responsibility for being part of the class discussions in a respectful and friendly manner. I did not expect all the students to come along and be part of the class discussions immediately since I believed that "old habits die hard." However, I was hoping that more students would become interested to at least try it. There were only a few individuals who never showed such an interest. "Why don't you go to the board?" I asked one of them once. He laughed and said

"Because I am a chicken!" He was a private person and although he was in a small group, he never really engaged in the group's activity. My intention was to encourage the students as much as I could, but I did not want to force them to work cooperatively in the groups. I would be glad if only a few of them changed their beliefs about the issue. However, I was certain that the class atmosphere had a major role to play in making the students feel comfortable expressing their ideas.

One day the class was discussing a problem on probability. I wrote every small-group's responses on the board, and then I asked the students to discuss them and see which ones made more sense and why. Nina argued that her response made sense to her because "It sounds mathematical!" "I don't want to create any friction," said Jack, "but I have an argument against it." This was an example of the social norm that I hoped to establish. Such a social norm would provide the students the opportunity to become metacognitively aware, to take control of their own learning, and finally, to help a great deal to achieve the goal of changing the students' beliefs toward mathematics and mathematical problem solving.

In conclusion, establishing a suitable social norm for the class was indeed the most difficult task to do regarding the instruction. It took lots of effort and mutual understanding to create such a norm. My role as a teacher was crucial in the creation of such a norm. I listened to the students carefully and appreciated the suggestions and comments that I received from them. The students' suggestions and comments helped me to become more sensitive to their needs and to make decisions accordingly. It took at least ten class sessions for everyone to realize that the desirable environment was one that was not only relaxed and friendly, but it was also engaging and thought provoking.

Course Evaluation

On the first day of the class I talked about the evaluation: 50% for final exam, that was required by the Mathematics Department at UBC; and the other 50% for assignments, quizzes, and journals. I did not have control over the percentage of the final exam.

However, I left it to the students to decide the weightings for other 50%. They agreed to have 25% for quizzes and assignments (six altogether), and 25% for the journals.

None of the students was satisfied with the final exam at 50% of their mark. I did not like it either since my intention was to provide an opportunity for the students to learn without being frightened by the final exam. I took the issue to the coordinator of the course. He and I decided to change the final exam's weighting from 50% to 40%, because the final was a two-hour exam in the Summer term as opposed to in the Winter term a three-hour exam (worth 50%). Therefore the weightings of journals and assignments (including three quizzes) changed from 25% each to 30% each.

Except for part of the second and third quizzes, which were marked by one of the teaching assistants, quizzes and assignments were marked by me. The evaluation of the quizzes and assignments was not based on the "correct answer," but on the gist of the students' reasoning. The students were asked to explain why they did what they did and how. The students were also asked to answer a series of metacognitive questions as part of their quizzes.

The journal entries were collected every week to be read and commented on. However, no interim marks were given to the journals, since I wanted the students to feel free and write the journal entries in any way that they wanted to. Otherwise, the marking scheme might dictate to them what to write rather than help them genuinely about their feelings, ideas, concerns, and criticism. Except for the second week's entries that were read and commented on by both the teaching assistants and me, I read and commented on all the students' writing carefully and gave them my feedback, trying not to impose my ideas on them.

The criteria used for marking the journals were the maintenance of at least two or three entries per week and volume of writing. By volume I mean the writing had to be substantial enough to make a point. It was not enough for the students to write about their "likes" and "dislikes" without any explanation. However, they were not denied any credit for the mechanics or viewpoints expressed in their writings.

One of the main purposes of journal writing was for the students to reflect on the existing issues regarding the course, so it was crucial for the students to write regularly every week. Marks were taken off for late submission of the journals. For example, Tony never wrote anything in his journal until the last day of the course when he handed the whole thing in all at once. "I could not do it regularly because I am working," he said. He just wrote to fulfill one of the requirements of the course. This kind of writing could not help him to become more aware of the issues and be reflective, because it was only story telling, full of fancy words without much meaning attached to them. Therefore, he got 15% out of 30% for his journal.

Towards the end of the term, I asked the students to evaluate their journals themselves and write about it in their journals. They had to justify the self-given mark with good reasons. After reading their journals, I was surprised with some of the students who gave themselves marks that were lower than what I thought they deserved, and I was disappointed with few others who gave themselves full marks (30 out of 30) and whose only justification was that "I worked hard and I deserve it" or "I really need to pass the course." Furthermore, a majority of the students had a kind of analytical approach and their self-given marks were well funded. However, experiencing this, I told them I said that I would reserve my right to adjust the marks if they were not appropriate referring to those who asked for a mark without any justification.

Summary

A metacognition-based instruction was designed to facilitate teaching and learning mathematics via problem solving. A review of related literature gave strong justification for the use of metacognitive strategies in a mathematics classroom to enhance students' mathematical understanding. Those strategies provided an opportunity for the students to become actively involved in the meaning-making process through reflection and interaction.

In addition, as has been suggested by Cobb (1988), the idea of focusing instruction on negotiation rather than imposition was fostered. A wide variety of problems and activities were prepared prior to the instruction, but as in the study by Cobb, Yackel, and Wood (1989) of second graders' mathematical problem solving, those activities needed to be developed and modified as the study progressed.

Data Collection Procedures

Data for the study were collected through four different sources, namely quizzes and assignments (including the final exam), interviews, the instructor's and the students' autobiographies and journals, and class observations (field notes, audio and video-tapes).

Task-based Interviews

Six volunteer students in MATH 335 (introduction to mathematics) participated in two task-based interviews, one in the first two weeks and the other in the last week of the instruction. These six students were chosen after the collection of students' first assignments and journals. These assignments and journals were used to judge the range of the students' mathematical abilities (low, medium, and high). The purpose of the task-based interviews prior to the instruction was to identify some of the students' initial beliefs about mathematics and mathematical problem solving. The ways in which the students solved problems were also documented. In addition, students' knowledge of heuristics and their ability to apply them in solving problems were investigated. The post-instruction interviews were aimed at investigating whether the students' views about mathematics and problem solving as well as their approaches to problem solving had changed as a result of instruction. Two other students were interviewed at the end of the instruction in addition to the six students who participated in both interviews. They were interviewed because of the substantial changes in their beliefs about themselves as doers of mathematics and about mathematics and mathematical problem solving as a result of the instruction.

Interview technique.

The thinking-aloud technique was introduced to volunteer students prior to interviews. Perkin's (1981) method for thinking aloud was fostered. The students were asked to say whatever was in their minds, whether it was a hunch, a guess, a wild idea, or an image. They were expected to speak continuously and audibly. Krutetskii (1976) warned that the thinking aloud is different from explaining the solution aloud. He also recommended that the students should know that in thinking aloud they are not explaining anything to anyone, but talking to themselves. They could do this by pretending that there was no one in the room except themselves.

Appropriate hints such as "read the problem again," "try another approach," and "think about a smaller number" were provided if the students needed them. The aim was to help the students go as far as they possibly could in revealing their understandings and beliefs regarding mathematics and mathematical problem solving. The researcher did not pre-structure the direction of inquiry. The tasks provided the opportunity for students to talk about their beliefs and different approaches that they had toward problem solving in doing mathematics, as well as to demonstrate their managerial skills. A friendly reminder such as "will you please keep talking" was helpful to encourage them to talk while working on a task

All fourteen interviews were audio-taped and eleven of them were also video-taped. The first three interviews were not video-taped because they were carried out in my office which did not have enough room to set up the camera. However, an arrangement was made to conduct the rest of the interviews in the classroom before or after the regular class hours. All the students signed a consent form for being audio-taped, and all but one did so for video-taping. The person who did not agree to be video-taped said that she was shy and might get nervous having the camera present. No other person was present while carrying out the interviews. Pseudonyms were used for the sake of the students' confidentiality.

Nature of the tasks.

The tasks were two challenging and non-routine problems, namely "the hand-shake" and "the Fibonacci problem about the rabbits" or their extended problems (the problems are presented in Appendices O and P). These tasks were chosen to provide the students with a chance to speculate, to pursue their hunches, and to conjecture. There was usually more than one reasonable approach to these tasks, and the students were expected to consider and use different strategies to do them. I thought the tasks could help me to identify the mathematical processes and operations that the students applied in order to solve them. The tasks required planning and reflection that would give me a chance to observe the students' cognitive and metacognitive behaviors in action.

In the pre-interviews, three students did the "hand-shake" problem and the other three did the "rabbits." In the post-interviews those who did "hand-shake" problem in the pre-interviews, were asked to do "rabbits" problem for the post-interviews and vice versa. The two students who only participated in the post-interviews, did both tasks in one session.

Use of Journal Writing

This study used journal writing as a means of communication between each individual and the instructor. I took this opportunity to write extensive comments (feedback) on students' journals hoping that the feedback would help students to reflect on what they did and to become more aware of their own thinking. It should be mentioned that journal writing was part of the course evaluation.

The students' journal entries provided the main part of the data for the study. They helped me to monitor changes in the students' beliefs about themselves as doers of mathematics, and about mathematics and mathematical problem solving, since the students wrote them regularly and reflectively after classes.

The data reduction regarding journal entries was an integrated part of the data analysis. Although I had read the journals regularly during the course of the instruction, I read them again and again to make a better sense of the data. I summarized the students' writing in terms of the ways in which they utilized their journals, namely; if the students only addressed cognitive issues, metacognitive issues, or both. The next step was to document the changes (if any) in the students' beliefs about themselves as doers of mathematics and about mathematics and mathematical problem solving, and the nature of those changes. Different themes emerged from the students' responses, and they are discussed under the heading "data plan" later in this chapter.

In addition to the students' journal entries, my own reflective journal provided data for the study as well. I wrote regularly during the course of the instruction. My journal was particularly useful in analysis of the classroom processes. The journal entries helped me to make more sense of the audio and video-tapes of the classroom processes.

Moreover, the students' autobiographies that they wrote at the end of the first class became an important part of the data for the study. They gave valuable information regarding the students' demographic and mathematical background. The autobiographies provided a unique opportunity for me to monitor changes in the students' beliefs about themselves as doers and learners of mathematics, and about mathematics and mathematical problem solving.

Class Observation

Two doctoral students in the Department of Mathematics and Science Education were hired by the Mathematics Department at UBC as teaching assistants for the course. They both had extensive teaching experience in Canada and abroad. Hazel (pseudonym) assisted me in the first four weeks of the course, and Roy (pseudonym) in the last four weeks of the six-week long course. They both observed the class and Hazel took field notes occasionally. In addition, Roy more frequently contributed to the class discussions while Hazel's contribution was mainly in small groups (as it happens without a specific plan). In particular, Roy gave a lecture to introduce logarithms. Many students reflected on his teaching method in their journals. Samples of these reflections will be discussed in Chapter 5.

The aim of the class observation was to document the class interactions when solving problems as a whole-group or in small groups. I also kept my own journal and wrote immediately after each class about my observations and understandings of the classroom processes. In addition, the assistants and I talked after each class to reflect on what went on in the classroom and to discuss the relevant issues regarding the instruction and the students. The discussions with the assistants and my reflections on classroom processes provided valuable data for the study. The purpose of documenting the classroom processes was to investigate the ways in which the students solved problems, the extent to which they used problem solving strategies, the amount of control that they had in choosing or abandoning the strategies, the time allocation for each problem, and the nature of their participation.

The last four weeks of the class sessions were audio-and video-taped using two portable tape-recorders and one camera which was set up in the back of the class and operated by the teaching assistants and myself. The procedure was to get the whole class in the range of the camera as much possible, and to focus it on the different groups while the group members were discussing a problem. I moved the tape-recorders from group to group as the students were working together. Towards the end of the course, I asked each group to say the name of its members aloud, to be taped while writing them on a piece of paper. This helped me to match the students' voices with their written work.

The class sessions were video-taped for two reasons. The first was a technical one to ensure the "fidelity" of the data, which according to Lincoln and Guba (1985) is "the ability of the investigator later to produce exactly the data as they become evident . . . the greatest fidelity can be obtained using audio or video recordings" (p. 240). The second reason was an educational one in the sense that Davis (1990) discussed. Videos could show to teachers and student teachers the possibilities for a wide range of classroom behaviors . They would acquire a "powerful kind of knowledge" by viewing the tapes. He went on to say that "giving this knowledge to students (or to other researchers) is a major part of the job of mathematics education" (p. 21). It is worth mentioning that all the students except one signed a consent form letting their audio and video-tapes to be used for the educational purposes (see Appendix C). The one who did not consent to be video-taped, always sat in one corner, and I made sure that she was not in the range of camera.

Quizzes, Assignments, and the Final Exam

All the students did three "take-home" assignments and took three "in-class" quizzes which were worth 30% of their final mark. The aim was not to test the students for what they did not know. On the contrary, the goal was to engage them in the process of solving a number of challenging and non-routine problems, ones that did not have immediate solutions. These non-routine problems helped me to investigate possible improvement in the students' methods and approaches while they received the metacognition-based instruction. Two of the assignments and the three quizzes each consisted of only one non-routine problem or task. The non-routine problems could offer the opportunity to identify the students' ability to transfer their content knowledge to other contexts. The quizzes and assignments along with the metacognitive questions that accompanied them, provided data regarding the students' beliefs, awareness, and managerial skills. (See Appendix F for a copy of the metacognitive questions.) The quizzes in particular, offered valuable data for the study, since the students' work was evaluated under time constraint, which is more often the case in normal classroom settings. The following is the short description of the ways in which the quizzes and assignments were selected.

The first assignment which asked for a proof of the Pythagorean Theorem (see Appendix G), was given on the first Friday of the course and was collected on the following Monday. This assignment helped me to investigate the students' conception of proof, as well as their ability to do a proof. I talked to those who did not understand the proof at all and asked them to do it again after we discussed it until they were convinced that they understood it. I changed the marks of those who did a better job in the second time to encourage them to seek understanding and to try to make sense of the mathematical concepts. However, I made it clear that the same procedure would not continue and my intention was to modify it for the other quizzes and assignments.

The first quiz, the one about "length and volume" (see Appendix H) was given on the second Friday of the class and returned on the following Monday. The first half hour of the class was planned for this quiz, but it took about one hour. The quiz consisted of one rich problem with five assumptions. The problem was taken from the resource book (Jacobs, 1982) required for the course and was one of the problems in the third "worksheet." While the students were working on this problem in their small groups, I noticed a wealth of different approaches to this problem and decided to use this problem for the quiz to collect all those interesting solutions. It also would help the students to be relaxed since they had already seen the problem. The students were allowed to discuss the problem in their small groups. However, they had to write it individually. Working together gave them confidence to overcome the fear of an exam, which most of them had commented on in their journals.

The problem for the second quiz, about "volume and proportionality" (see Appendix I), was taken from Schoenfeld (1987b), and was given to the students in the first half hour of the third Friday of the course. Like the first quiz, the students worked in their small groups and then wrote the solution individually. This quiz was also marked over the weekend and returned to the students on the following Monday.

The second assignment (see Appendix J) was given on the fourth Monday and was returned to the students on the following Friday. The task was to prove that the square root of ten was irrational using indirect proof. Some of the students who got low marks for this assignment, asked permission to do it again. I told them that I did not mind at all going through their assignments for the second time. However, I would average their new marks with the previous ones to obtain the mark for their second assignment.

The third quiz was a problem with two parts on the content that had been covered in the probability section (see Appendix K). It was given on the fifth Friday of the class. A week before the test, the students were told that they had to solve and write the last quiz individually. Many of them complained and expressed their concerns. However, they appreciated the idea after I explained that the purpose was to help them get ready for the final exam which had to be written individually.

The third assignment was a special one since the students were free to choose any problem or topic that seemed interesting to them. I suggested the following: (a) apply a mathematical concept to a real life situation, (b) formulate your own problem and try to answer it, (c) choose any problem or topic related to the course that you are particularly affected by or disappointed by and explain why, (d) evaluate the course, (e) critique the teaching method, and others. However, it was totally up to them to decide whatever they wanted. This variation of the students' responses added richness to the data. Appendix L contains a table showing students' selections for the third assignments.

The final exam was given on the last day of class, and the students had to write it individually. The exam consisted of essay questions and multi-step problems, and it provided data for the study (see Appendix M). The intent was to evaluate the students' selections of strategies, their approaches to problems, and the coherency of their written responses. These aims were achieved, for example, by selecting open-ended problems that ask for students' explanations. Appendix N contains a table showing students' selections of the final exam.

All the assignments, quizzes, and the final exam provided data for the study. They added to my insight about the students and helped me to better understand the students' problem solving behaviors under different constraints and conditions.

Data Analysis

Data for the study were provided through four different sources, namely quizzes and assignments (including the final exam), interviews, the instructor's and the students' autobiographies and journals, and class observations (field notes, audio and video-tapes). Several cognitive-metacognitive frameworks that research has shown to be appropriate for analyzing the data emerging from a study of this type, such as Garofalo and Lester's

(1985), and Schoenfeld's (1985, 1987a), were considered. However, those frameworks only served as helpful and general guidelines and were by no means exact or limiting.

In the early stages of analysis, I tried to make sense of the data that emerged from the students' written tests and assignments, their journals, the interviews, and the classroom processes separately, and then put them together to draw final conclusions. When this proved unfruitful, I took the advice of Strauss and Corbin (1990) to look at the data from a different angle. I got the idea of looking for common themes that might emerge by looking at student by student data.

I made a new file for each student consisting of the student's autobiography, the journal entries sorted by dates, the student's assignments and quizzes, the final exam, notes on classroom interactions taken from video-tapes, and the excerpts of his or her interview, if applicable.

This new scheme gave me a chance to look at each individual's changes from the beginning of the course to the end. Despite individual differences, the students appeared to be falling into three distinct categories. The first category is representative of those students who not only showed little or no interest in different ways of teaching and learning mathematics, but were also resistant to them. The second category consists of students who were inconsistent. They sometimes showed enthusiasm for new ways of looking at mathematics and mathematical problems. However, they were concerned that it might be too late for them to look at things from different perspectives. Finally, the third category represents a group of students whose beliefs about themselves as doers of mathematics, as well as their beliefs about mathematics and mathematical problem solving, became more positive during the course of instruction.

The description of each category, the range of the students they represent, the nature of changes in the students' beliefs, and the ways in which the changes happened are described in Chapter 5.

Many common themes emerged from the students' responses considering various sources of the data. The analysis focused on the following themes: (a) instruction and its components, (b) students' beliefs about mathematics and about themselves as doers and learners of mathematics, (c) students' beliefs about mathematical problem solving, (d) mathematical connections, (e) and tests or other forms of evaluation. The students reflected on different components of the metacognition-based instruction, namely: (a) journal writing, (b) small groups and whole-class discussions, (c) time, (d) the teacher's role, (e) and classroom environment.

The findings of the study are organized based on these themes. However, each theme is not necessarily discussed within each category. The analysis consists of the effects of the metacognition-based instruction and the instructional techniques as well.

Data Reduction

My approach to the data reduction was first to study each student's file separately. Every student was like a puzzle, and different sources of the data were the pieces of those puzzles. The more I studied the sources of the data, the better I was able to put the pieces together.

The journal entries were the most important set of data, since they were mainly the reflections of the students on everything else that was going on in the course. In addition, the students' autobiographies proved to be a valuable source of the data considering the purpose of the research that included the study of change in each individual from beginning of the course to the end. My first attempt after reading the journals was just to cluster them, which involved rewriting most of them and using different colored markers to write the themes that kept emerging from the students' writing. The next step was to make sense of what the students wrote and look for commonalities and differences among them.

The assignments, the quizzes, the final exam, and the metacognitive questions that were part of the last two quizzes were helpful in investigating the ways in which the students solved mathematical problems, the problem solving strategies that they used, and the effect of the exams on their self-confidence. Different solution approaches were also clustered.

The following example shows how the data that were provided by the first quiz were reduced for the purpose of analysis. The students used a variety of different approaches in finding the area of hexagon, octagon, and dodecagon. The students discussed the problem in groups and wrote it individually. I first collected all the different approaches to finding those areas, first in terms of each student, and then across the groups. I was interested to see how students inspired each other in groups and negotiated the meaning and at the same time asted as decision makers. Working in small groups in exam situations showed how well students understood the essence of group work and their responsibility and their active involvement in that regard.

The students' assignments that they did individually at home gave me a chance to look at their work in a totally different situation. The first assignment asked students to prove the Pythagorean theorem mainly using cut outs and manipulatives, and the second one involved proof by contradiction. I clustered the students' approaches to these two different kinds of proofs based on the nature of their argument, and their conceptions of mathematical proof.

The data from the third assignment, which was an open one, were organized in terms of the students' selection of problems and issues. I was interested to find out how students might possibly pose problems on their own and discuss the solution processes. I reduced the data in terms of the students' selection of problems.

Responses to each final question were collected and then reduced according to the problem solving strategies that students used, their understanding of mathematical proof, the generalizability of the proof, and so on. The managerial skills that students demonstrated in their final exam were also clustered.

Procedures for analyzing the students' interviews followed the same format as those for the video-tapes of the classroom processes. I used some parts of the interviews to clarify and enrich the findings from the other sources. For example, I transcribed the ways in which the students solved the tasks, the kind of problem solving strategies that they used, and the evidence of managerial skills. I also talked to them about their beliefs about themselves as doers of mathematics, and about mathematics and mathematical problem solving. I compared and contrasted the data from both pre- and post-interviews to help me in analyzing the possible changes in students' beliefs about mathematics and mathematical problem solving.

Chapter 4 provides a report of the data that were obtained through the audio- and video-tapes of classroom processes and my reflective journal that I wrote each night after class. I watched the video-tapes of the classroom processes several times until I felt comfortable with my understanding of the processes. The next step was to transcribe the parts that most clearly showed the nature of the whole-class discussions and captured the essence of the cooperation within small groups. I regularly contrasted the video-tapes with my reflective journal. This exercise proved to me the authenticity of these two sets of the data.

Trustworthiness

Lincoln and Guba (1985) argue that different research paradigms require different criteria for trustworthiness. They explain the basic issue in relation to trustworthiness as: "How can an inquirer persuade his or her audiences (including self) that the findings of an inquiry are worth paying attention to, worth taking account of? What arguments can be mounted, what criteria invoked, what questions asked, that would be persuasive on this issue?" (p. 290). In general, they used the term "trustworthiness" to discuss credibility, transferability, dependability, and confirmability of qualitative studies. These terms correspond roughly to internal validity, external validity, reliability and objectivity in quantitative studies.

The results of the present study are based on the students' ideas and responses and that gives "credibility" to the study. All the claims and categories in the analysis are supported by the students' input coming from various sources of data, namely: (a) interviews, (b) journals, (c) classroom interactions, and (d) assignments, quizzes, and tests. This triangulation enhances the internal consistency of the study and shows that the findings are based on the analysis of the data which were gathered through different instruments.

Before the study was even finalized, some of the results and the findings were applied to the same course or other mathematics courses through informal discussions with other mathematics instructors. This applicability is known in qualitative studies as "transferability" (Lincoln & Guba, 1985). The purpose of this chapter has been to give as complete a description of the study as possible to help other interested researchers to conduct similar studies and possibly to shed more light on the research in this area. This description would increase the transferability of the study.

Studying and scrutinizing a number of relevant studies (e.g., Schoenfeld, 1987a, 1988b; Lester, Garofalo, and Kroll, 1989a; Raymond, Santos, and Masingila, 1991) led to the selection and development of the methodology and the data collection procedures for the present study. Furthermore, the appropriateness of the techniques for the analysis was attested to by the pilot studies. The record of the data gives the opportunity to any independent observer who is familiar with the nature of the study to examine their authencity and the validity of the collection procedures. This accessibility of the data and the procedures accounts for the "dependability" of the study.

"Confirmability" refers to the agreement with the findings on the part of mathematics education researchers familiar with similar studies. Furthermore, the views of mathematics instructors about the findings are crucial, since the study was carried on in a context of mathematics instruction. I believe that I am able to give adequate grounds to refute possible disagreement about any part of the findings.

CHAPTER 4

CLASSROOM PROCESSES

In Grade One the teacher split us into three groups; the Beavers, the Robins, and the Ducks. I was a Duck and I have never forgotten that feeling. I also realize this gave me a math block or phobia. Today I can understand the theories behind physics and chemistry, yet still carry with me a fear of math. I know I will overcome this phobia of math; hopefully this class will do this. *(Lilian's autobiography)*

The main emphasis of this study was to investigate the effect of metacognition-based instruction on the students' beliefs about themselves as doers of mathematics and about mathematics and mathematical problem solving. The metacognitive techniques (or strategies) that were used to implement the instruction were described in Chapter 3. This chapter focuses mainly on the classroom processes and the students' interactions with each other and with the instructor, both in small groups and in whole-class discussions. These findings are based on the data that were provided by videotapes of classroom processes and my reflective journal that I kept every day during the course of the instruction. The chapter is divided into three sections, namely: rationale of the course, the nature of the class interactions, and a summary.

Rationale for the Course

On different occasions, the coordinator of MATH 335 and I had several conversations regarding teaching the course. We talked extensively before I started teaching in the summer of 1991. It was interesting to hear his points of view about teaching and learning mathematics in general, and about MATH 335 in particular. "What do we really want these students to accomplish" was the main theme of our discussions. I was relieved to learn that we had somewhat similar aims for MATH 335. The course was originally designed for prospective elementary teachers, but it was open to all non-science university students.

The students taking the course would soon become elementary teachers and teach youngsters. As teachers, they will be confronted with children who do not separate the subjects. They are in "school" to do things and learn things. These energetic and creative elementary students like to work with puzzles, and have fun. Mathematics is best learned when it is "fun" and appealing to students. Therefore, the course should expose the prospective teachers to various positive and enjoyable experiences regarding mathematics. Most of them were mature learners, either in their third or fourth year of an undergraduate program, or they already had obtained their baccalaureate degrees. Moreover, they have limited yet diverse experiences with mathematics. The diversity has to be acknowledged and the course designed in a manner to accommodate the students' needs.

The rationale justified the role of manipulative materials, classroom activities, small groups, and whole-class discussions as integrated parts of teaching and learning mathematics. These activities and discussions were helpful to achieve one of the goals of the instruction that was to encourage students to "do" mathematics. Hands-on activities and class discussions helped the students to encounter contradictory situations, which means enjoying the success and feeling the difficulty, and learn that they both were pleasant parts of every experience. In addition, the instruction was to bring about mathematical connections to make the mathematical teaching and learning more meaningful. Furthermore, the classroom environment allowed them to become risk-takers and see that everybody was capable of understanding some sorts of mathematics. The aim of the course was to show them that not only mathematicians "do" and "enjoy" mathematics, but also that non-mathematicians can "do" mathematics and deserve to have "fun" with it. Therefore, the course included many little fun activities and used them to explore and develop a number of subtle mathematical concepts. It gave the students a chance to see the attractive and creative side of mathematics.

Small groups and whole-class discussions helped the students to see the dialectical nature of mathematics. It helped them experience mathematics as a social activity that is learned through social interaction. The students were encouraged to create and to do mathematics by themselves without relying on outside authority, the teacher, to tell them what to do all the time, and to value the process of doing mathematics rather than being so much concerned about the finished product, the "correct answer." In conclusion, the rationale was to help the students see that they would not become musicians just by studying the theory of music, only by actually getting involved in the process of making music, and so it is with mathematics.

Classroom Interactions

Instruction was based on the students' interactions with each other and the instructor either in a whole-class setting or in the small groups. It took a long time and a lot of effort to make the students believe that it would be worth their while if they actively took part in the discussions. Some of the students were annoyed and frustrated with the noise level. Lynn and few others asked for less interruption, which meant no or less questioning, and more direct instruction and guidance on my part. I believed if they saw any benefits, they would like the idea of interacting with each other in the class. Sometimes students ended up working by themselves. In such cases, I had to ask them repeatedly to work in groups. Eventually, working in small groups and participating in whole-class discussions became a norm and the students felt comfortable participating in them. However, a few students resisted working in small groups and participating in whole-class discussions.

Bringing about some changes in terms of classroom processes was not easy and required a tremendous effort both on the students' part and mine. To avoid redundancy, I have chosen a few episodes based on video-tapes of the class and my reflective journal, to portray the interactive nature of the course. In these episodes, "I" represents me, the instructor.

Episode 1, May 8

The following happened on May 8 [third day of class]. Nina came to the board to do a problem. She solved the problem correctly. However, as soon as Jim asked her why she did what she did, she wiped it out.

I	Why did you wipe it out?
Nina	Because I was wrong.
I	Why?
Nina	I don't know.
I	You must have had some reason for that . . .
Nina	I did it because I had to do something with these numbers. Okay, I can multiply 1.2 by 50 rather than dividing 1.2 by 50.
I	How did you get that 50?
Nina	I don't know!

Nina was not the only one who would say that she did not know and could not understand anything regarding mathematics because she had "math phobia." In the beginning many students would sympathize with her and thought that she was speaking on their behalf. In fact it was more acceptable to say that "math is dreadful," and wash their hands of it, without taking any responsibility for their own learning or even trying to improve the situation. The instructor's goal was to challenge this view. I aimed to show them that mathematics was not "dreadful" as they thought it was and also to help them to gain more self-confidence in doing mathematics, to be able to defend their ideas rather than feeling vulnerable. Nina's responses to Jim's question was a manifestation of her insecurity and low self-confidence regarding mathematics which was typical of the majority in the class.

Episode 2, May 27

What Kent did on May 27, was only one example showing how the students confidently expressed their ideas in class and were able to defend them. On May 27, we were discussing the irrational numbers and how to calculate π[3] . A number of students were arguing about whether a "true circle" exists and what was the concept of limit. Kent shared his understanding with the class:

Kent If you keep slicing and slicing the sides of polygons, you get smaller and smaller slivers and you won't be able to see it after a while. I can see that you can do that forever. So as π could go forever.

The students asked for more explanations, and Kent continued:

Kent If you take this part of the number line [he pointed at the interval between 3 and 4] and blow it up by a microscope, you can go this way [within the interval] forever. 3.14 is somewhere here but it can go forever.

Melisa Is it the matter of complete and incomplete numbers? Because irrational numbers never end like π. . . .

The calculation of π gave the class a chance to have a more in depth discussion of rational versus irrational numbers and decimal expansions. Jack's question about π was a good transition to "proof by contradiction." He said he was told in high school that π was equal to 22/7, so he was confused. Jack's question was discussed in class at length. It gave us a chance to more specifically discuss the idea of indirect proof. Therefore, the question was the contradiction between π as an irrational number and as being equal to 22/7 which was a repeating decimal, a rational number. In discussions like this one, students were engaged in a meaning-making process, a number of mathematical ideas were connected, and the students' mathematical understanding was developed.

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[3] There was a banner on the wall that had π to many decimal places. When we wanted to calculate π, Nina looked at the wall and said: "Here is π. We don't need to do it any more."

In general, there was a big difference between what Kent did on May 27, and what Nina did on May 8. Nina immediately wiped out what she wrote on the board when her solution was questioned either by the students or the instructor. Nina (and many others for that matter) thought she was wrong, since she was not confident to defend what she did. As the course progressed, the students gained more confident, since discussing their ideas and defending them was the day-to-day practice in class. Class environment, or social norm played an important role in helping the students to become more independent and self-reliant, and consequently more confident. Therefore, what Kent did became a norm rather than an exception.

The development of the social norm of the classroom was an interesting phenomenon. After the first half of the course, the students found it easier to work together in the small groups and on different problems. Therefore, it was difficult to attract all the students' attention to the same problem when it was discussed in class as a whole. After the first half of the course, I gave the whole class the same problem to work on for 10 or 15 minutes. We then discussed the problem in class. In this way, I was able to pull the whole class together and more students interacted with each other. The following few episodes describe the nature of those interactions.

Episode 3, May 30

I put on the following problem on the board and asked the students to spend 15 to 20 minutes on it in small groups.

If someone offers you a job that pays one penny on the first day and doubles the money every day after that for 30 days, would you accept the offer?

After all the groups had a chance to discuss the problem among themselves, I then asked every group to put their solution attempts on the board to be discussed by the whole class. I constantly told them that the purpose was to discuss ideas and not necessarily to judge final products. The following dialogue is a snapshot of the interaction regarding the above problem.

I Now I would like each group to share their ideas with the whole class.

Clara I will just tell you what we've done in our group. But we haven't got the answer.

I That's fine. We don't want a finished product. We just want to hear from each other.

Clara People worked it out on calculator as 2x2x2x . . .. Second way that we did was . . . and then we took like for the first day it was only one penny and then the second day it doubled. It was 30 days. The second day was 2, and that was for 29 days. So we put 2^{29}, and it worked out on calculator, and both answers were the same.

(What Clara's group did was to multiply two 29 times and then add one. The other way was to use x^y function on the calculator and add one at the end).

I So did you get the sum of the money the person got at the end of the 30th day?

Clara Yes.

Melisa [Raising her hand] What we did [another group] was make a table having three columns. The first column was a penny and then second column was doubling that money and the third column was accumulated pay. What we noticed was for example, on 11th day, you earned $10.24 a day and the sum of the first 10 days which was for previous days, so . . . like the 30th day, you'll be actually earning 2^{30} but minus one cent. That would be the sum of what you've earned from the first day to the 30th day.

I Thanks . . . so what about you [pointing at another group]?

Shirley I didn't quite understand what the 10th day has anything to do with it.

Melisa	Oh, just to give an example.
Patrick	[From Melisa's group] Well on the 10th day you earn $5.12 and the accumulative pay on the 10th day was $10.24 and on the 11th day you make $10.24. You make one penny more than you've already made.
Melisa	That was accumulative pay from the 1st day to the 10th day which is equal to the pay on next day. But you actually make one more penny.
Patrick	So, on the 10th day, you make 2^9. That's how you earn $5.12. The total pay was $10.23 which was $2^{10} - 1$ . . . So, on the 16th day, I used 2^{15}. The accumulated pay was $655.36 which is 2^{16} - 1. So the formula is 2^X- 1 penny.
Jack	[from another group] We did it as: on day one you make 1 penny, on day 2 you make 2 pennies, day 3 you make 4, so we noticed the pattern on the right hand column. The 1st day is 2^0, 2nd day is 2^1, 3rd day is 2^2, . . . , so after 30 days, that's equal to so many cents . . . is equal to 2^{29}. When you punch that into calculator, that was how many cents you would earn.
Nina	[Same group as Patrick] So more like what you guys did [pointing to the Clara's group].
Patrick	That's how much money you made on the 30th day.
Jack	The problem we had and I've still seen is that the power is one less at the day, and originally we had 2^{30}. But because you start at 0, you don't start at one, that's where the difference comes in.
Nina	Could I say something? If we said 2^{29}, is that the same as if we just said that it's 30 days, so it's 1^{30}? Would that give us the same answer? It wouldn't!? Would it?!!

Lana and few others said "NO! NO!" and explained why it was not correct to do such a thing.

Nina So we have to make sure that you got the first accumulative pay before you add the other powers. So if you make 5 cents on the first day, the 2nd day you make 10 cents, it will be 10^{29}, what you would make. . . right?

The interesting point was that she wanted to extend the pattern and check to see whether it would work for different problems with the same structure. However, the class did not discuss Nina's proposed problem at length, since they first wanted to finish the problem at hand. Furthermore, we discussed what she said within her own group. Shirley then continued the discussion.

Shirley If it had been 31 days which you've gone to 2^{30}. So you just automatically lost the top one.

Lana [Same group explaining to Shirley] . . . on 1st day you get one penny, the next day you gonna get an increase, so on the 2nd day you get 2 pennies. So to get down to 29 . . . that's your pay for 29th day, but you have to add it to everything that you've accumulated . . . so that comes up to $2^{30}- 1$ That's all we did.

Patrick I don't argue with that. 2^{29} is how much you get on the 30th day.

Melisa But your question was much you get on the 30th day if you keep the job . . . well if I was making that much money, I'd better take the job.

Patrick Did I ask Zahra if you want the total amount of money that you'll be paid for a month is work?

Carmel So the answer does it mean the last day of everything or we have to add up every day on the top of that?

I It depends on how you interpret the problem. Are you looking for the last day's pay or for the accumulated pay?

Carmel That was the question. What did you want? Last payment?

I That's one way of interpreting it. You could either do that or find the sum for the salary.

Roy [TA] In the real life situation, what do we mean by salary or wage?

Jim Normally we only get paid after each day of work.

Roy So in that case you gone interpret as

Patrick So in that way it would be 2^{29}.

The discussion continued for a while. The students shared their ideas about the ways in which to carry out a plan for either interpretation of the problem. The class discussed different ideas that were expressed by small groups. I then talked a few minutes about the problem to conclude the discussion.

I So overall, is there any question about this problem?

Nina What is the correct answer?

I It depends on how you look at the problem.

Nina But there must be a right way of looking at the problem . . . I don't know. I wasn't given any answer.

Leah Why?

Nina [Laughing] I don't know! I was looking for a correct answer.

I I mean there are two answers to this depending on how you would interpret the problem.

Nina So you either think of it as . . . [interrupted by Patrick],

Patrick . . . amount of money that you get paid on the last day or all the money that [you] get for 30 days.

Nina Okay! I got it.

I Everything is relative. I mean [the] correct answer depends on how you conceive the problem.

Nina What if we get the question on the final exam? How do we really interpret it?

(Unfortunately, the fear of the exam sometimes interfered with the meaning-making process. However, this chapter is not the suitable place to discuss the purpose of "evaluation.")

This problem was an interesting activity to introduce the idea of exponential powers and logarithms. Many students expressed their concerns about the topic immediately after they received the outline of the course. Some of them talked about logarithm as being "dreadful." However, the problems of this kind provided an opportunity for the students to better understand the topic and even enjoying it without dreading it. For the students, a solution to this problem was conceivable. If nothing worked out, they could always punch 2 and hit the multiplication button 30 times on their calculators to get the answer. However, the problem stimulated many thoughts regarding exponential powers and logarithms, and the students did not feel bored.

In the second pilot study, the instructor posed the same problem. Most of the students thought that the problem was difficult and beyond their grasps, because it was related to exponential powers and logarithm. Moreover, the instructor solved the problem by himself, without any interactions with students except students' amusement at starting with only a penny and ending up with such a large number. He spend a great amount of time calculating 2^{30} by breaking it down into numbers with smaller exponents. For many students, calculation became the main focus of the problem. They therefore suggested using a calculator to save the class time, as the aim was only to calculate 2^{30}. However, if students in that class had had a chance to discuss the problem, and struggle with it, they would probably have gotten much more out of it. Nevertheless, they silently took notes to make sure that they could produce the solution (and answer for that matter) on the exam.

This problem stimulated good class discussions, as well as providing an opportunity for them to use a variety of problem solving strategies (heuristics) to solve it. For example, a group of students thought of a similar problem with smaller numbers to make a better sense of the problem, and another group made a systematic list, while third group was looking for a pattern. In the meanwhile, they had a chance to see how things were growing exponentially. Furthermore, the students learned how to use logarithm and x^y functions on their calculator rather than punching 2 and hitting the multiplication button 30 times.

Episode 4, June 4

On this day, I started the class with a problem to introduce the "fundamental counting principle" and then continued with a True or False question to talk specifically about permutations and combinations.

In how many ways can you answer a T or F exam that has three questions?

I posed the problem and asked the students to spend 10 to 15 minutes discussing it in their groups. After group-discussions, every group presented its solution attempts to the class. I wrote all those suggesting ideas while students were presenting them. Then, the class as a whole discussed each solution. The discussion helped the class to reject the solutions that did not make sense to them and keep those which did. The following excerpt is taken from the class interaction on June 4:

Sandra 2^6, because there are two ways of answering [each question].

I What do you think about it [asking the whole class]?

Nina It sounds mathematical [every one laughed].

Jim *I don't want to create friction here. I only have an argument against it.* I think that's the number of possible answers that you can have when you consider [pause] like in each case you are only allowed to have three answers out of the six possibilities. That's [referring to 2^6] suggesting that you are able to have six answers out of six possibilities. But you have to eliminate three because of the fact that you can only have three answers in such an exam. You can't have for example, . . . the questions can not be

True and False. They either have to be True or False. Does that make any sense? I don't know if that's clear.

This episode shows how we were able to establish a social norm for the class, in that every one had a voice and we all respected each other's opinion. We, as a whole, gradually developed a mutual understanding among ourselves. Jim's response to Sandra was interesting: "I don't want to create friction here. I only have an argument against it."

After a good discussion about why 2^6 could not be an answer to the problem posed, I said:

I	Now, is it O.K. to say that this one [2^6] is not correct?[interrupted by Jim]
Jim	Sandra! The way of looking at it would be to say that if you want to look at the number of possibilities, then that would be appropriate, . . . because then you would have six T, F, T, F, T, F. But when you look at the possible answers, then you have to do it with 3.
I	How different are they? You said the number of possibilities and number of answers.
Jim	What? You mean the distinction that I made?
I	Yes.
Jim	Well, the fact is that there are six; there are 6 possible responses: T,F,T,F,T,F [pause] but you can have one answer for each one. So that's . . . ya!
Nina	So 2 possibilities for each question.
Jim	Ya! 2 possibilities for each question.
Nina	And 3 questions, so there are only 8 possible . . . [interrupted by Jim]
Jim	I'm only trying to make distinction between possibilities and answers. You can only have three answers, but there are 6 possibilities.

Jim could relate to Sandra and understand why she thought of 2^6. It would be possible simply to tell her that she was wrong and that would be the end of the discussion.

However, the whole-class discussion gave Sandra a chance to see her confusion, rather than letting those ideas go underground (Cobb, 1991). Jim had no difficulty understanding the problem, instead he talked about a hypothetical situation to make Sandra realize where she went wrong. Then Carmel talked about the reality of True or False questions and they could only be answered either "True" or "False" and not both.

Kent extended the problem to an exam with 3 choices (as opposed to only True or False), and said:

Kent	In other words, [if] you can say True, False, or True/False, you could! Then it logically tells us that there are 3 possible choices and three questions, then it would be 3^3. Am I right?
Class	Ya!
Kent	Okay, then it seems we've got some principle here . . .. You could have multiple choice questions with 5 possible answers and 3 questions, then we have 5 x 5 x 5. Three questions and 5 possible choices.
Lana	Are you saying that there are 3 choices?
Kent	No! That would be 3 x 3 x 3.

Kent continued to explain a problem with 5 questions and 5 different answers very clearly. After a good discussion about this problem Kent said:

Kent	Well, I'm just trying to find the underlying structure of the logic of this and seeing if I plugged in other numbers from different questions and still it could be true.
I	So 2^6 doesn't work for the True or False exam with 3 questions?
Kent	If it were 6 questions, it would.
Patrick	And with 2 possible responses for each questions.

It was interesting to see how the class developed an understanding of the counting principle. Kent and Patrick discussed a situation in which 2^6 was an appropriate response. Furthermore, I turned to the problem of having an exam with 3 questions and 2 responses each, to address Patrick's concern about 2^3, one of the solutions to this problem. Patrick argued that in a problem involving 5 shirts and 3 pants, the number of choices that someone could make was 5x3. However, in this problem we were talking about the same principle but the solution seemed different. "What does the 2 in 2^3 stands for? What about 3?", I asked the class.

Many students got involved in the discussion to find the reason for the difference between those two responses. They knew there were 2 choices to make for each one of those 3 questions, therefore the total number of choices was 2 x 2 x 2, the same argument for the number of choices relating to 5 shirts and 3 pants was 5 x 3. However, the presentation of a solution for the first problem (that was also suggested by the students) confused some of the others including Patrick. Patrick expected to have the same presentation for the problem involving shorts and pants since they both had the same underlying structure. I told them 2^3 was only a short cut to present the number of choices, otherwise we did exactly the same thing in both problems, that was multiplying the number of choices for each question, or for pants and shirt, to find the total number of choices to be made: $2^3 = 2x2x2$ ways of answering 3 questions with 2 choices for each, and 5x3 ways of combine 5 shirts and 3 pants.

Solving problems like this might not take more than 5 minutes in a regular class. However, if we did not discuss the problem, I would not realize the difficulties that students were facing in understanding a problem that might look simple and easy. It would be easier to introduce the concept (in this case counting principle), solve a few examples showing the class how to apply the certain rules and formulas, and then ask students to solve a list of similar problems. However, in such situations Sandra's answer would simply be "wrong" and the "correct" answer would be put on the board for display. Of course, students would copy it down in their notebooks. Patrick had to make sure to write the total number of choices in an exponential form if each case had the same number of choices (similar True or False question), and to multiply them if they were different (similar to shirts and pants).

Nevertheless, the metacognition-based instruction provided an opportunity for the students to freely express their ideas and discuss them both in small groups and in class. We only solved a few problems in each class, we developed many mathematical concepts through those few problems.

Summary

The class interactions were stimulating and there were many interesting episodes to talk about. However, it is not possible to address all the problems that were discussed in the class. The purpose of giving these few snapshots was to portray the nature of interactions, and to show the nature of the class environment-- its social norm. Such an environment allowed the students to participate actively in the meaning-making process. Many mathematical concepts were developed via problem solving. In addition, a number of problem solving strategies were discussed within the contexts of interesting problems, whenever it was appropriate. For example, in a problem related to the counting principle, many students used one or more of the following problem solving strategies: Looking for a similar problem, using smaller numbers, acting out the problem, guessing and checking, making a table or systematic list, looking for patterns, and others.

In general, the students left the class feeling more confidence in their mathematical understanding. They realized that everyone was capable of doing mathematics to some extent, and that no one deserved to be a "duck." Self-confidence and self-reliance played important roles in the students' learning. The students learned to be critical regarding mathematical problems, and as Ben said at the end of the final exam: "I learned to never accept anything without asking 'why'." He continued: "That's the best thing I've learned from this course. I *learned to learn*. I got my confidence back, and that's just great!"

CHAPTER 5

FINDINGS

The findings of the study are based on 40 students who enrolled in MATH 335 (introduction to mathematics) in the Summer of 1991 at the University of British Columbia. The main purpose of the study was to monitor changes in the students' beliefs about themselves as doers of mathematics, and their beliefs about mathematics and mathematical problem solving as a result of metacognition-based instruction. The chapter is organized into six sections, namely; (a) general demographic description of the students, (b) description of three categories of students, (c) descriptive statistics concerning all the students, (d) findings within each category and comparison of the findings across categories, and (e) summary of the chapter.

Direct quotes are taken from the students' work, including their journal entries, to facilitate the interpretation of the data. The students' quotes are underlined whenever the student underlined, or added emphasis in some other way. Italics indicate that emphasis has been added by the writer.

Direct quotes from the students' journals are shown by date (e.g., Jim's May 13 journal entry is shown as "Jim, May 13" except for the last journal entries that are shown as "Jim's last journal entry." Some of the students dated their entries in terms of weeks that they were in the course. In that case, I have used the date-range of the week to avoid ambiguity. For example, the first week entries are dated May 6-10. In a few instances the dates of the entries were not clear, and in these instances I have used a question mark instead of the date.

General Demographic Description of the Students

This section presents academic status, age range, mathematical background, and work status of the students. The section ends with a summary.

Academic Status

The students who took MATH 335 were all non-science students. Of the 40 students, 26 provided information about their specific academic status. Twenty one out of 26 students had baccalaureate degrees in either Arts (B.A.), Physical Education (B.P.E.), or Education (B.Ed.). One student had already finished her first year in the magisterial program in Educational Psychology. The other five students were in their third or fourth year of an undergraduate program.

Age Range

The students' ages ranged from early 20's to early 50's. Those who were 40 and above were updating their teaching credentials. The dates of the last mathematics courses that they had taken ranged from 1966 to 1988, except for one student who took Algebra 11 at night school in a local college in 1989.

Mathematics Background

Some of the last mathematics courses these students took before this course were as follows: Grade 10 Mathematics, Algebra 11, Algebra 12, Grade 12 Mathematics, Grade 13 Mathematics, High School Mathematics (as they called it), Calculus, and Finite Mathematics. The mathematics background of some of the students is not known, data for 36 of the 40 students were available for this purpose. Three of these 36 students failed their last high school mathematics courses once, and one student failed it three times. One student took "Mathematics for Elementary Teachers," at a university in England in 1966, to fulfill the requirements for a general degree in education (B. Ed.). One student took Calculus in 1966 and needed MATH 335 to update her teaching credentials. Only one student successfully finished first year calculus at a local college. He did not need MATH

335 to enroll in the Elementary Teacher Education Program. However, he took the course to enrich his mathematical background. Two students took Finite Mathematics at the University of British Columbia and they both failed the course.

Work Status

I could not quantify the data for this section due to the nature of the information. I collected the information in a narrative manner. Personal information about the students was obtained through their journals and after-class discussions. Most of the students either had full-time jobs, or other courses to attend, or both. Some of them were single parents. Their jobs varied from working full-time at a cemetery to performing on stage every night after class. There were five performing artists and three substitute teachers in the class.

Summary

The students were distinctly different in many ways, including their academic status and mathematical background. However, they shared a common goal of wanting to work with children. Some of the students chose the teaching profession because they considered it a rewarding mission. Those students wanted to help the youngsters since the future of the nation is in their hands. However, some other students wanted to enter the Elementary Teacher Education Program because of the job market and economic situation.

Description of Categories of Students

Three categories of students emerged from the data collection and data reduction process. Each category consists of the students who shared similar beliefs about themselves as doers of mathematics, and about mathematics and mathematical problem solving. Also the members of each category had common beliefs about the ways in which the metacognition-based instruction either altered their approaches to problem solving or failed to make significant changes. Three categories of students ranged from resisting change to accepting change to embracing change. The three names, "traditionalists," "incrementalists," and "innovators" have been chosen based on the common characteristics, as labels for the categories. The following is a description of the three categories and a discussion of the idiosyncratic cases within each one of them.

First Category: Traditionalists

Nine students belonged to this category which is characterized by those who either rejected the new approach to instruction and explained why it did not work for them or those who were never influenced by it for reasons not related to the instruction. A description of the former group is given here.

Dan was a visual artist. He related all his journal entries to his art: "I also have a particular interest in the geometry section as it relates to painting." He mainly participated in the activities that he enjoyed the most. For example, he did well in geometry because he was fascinated by the subject, but he hardly participated in activities regarding rational and irrational numbers. He said, "Ordinarily I would be uncomfortable with dealing with money and numbers. . . . Numbers drive me crazy!" I could not judge the influence of the instruction on him. However, the instruction gave him a chance to utilize mathematics in the context of his art.

Joan had a full-time job: "I wasn't here at all this week as I had to work." She missed almost half of the classes due to her job situation, and she came to classes late. She left early because of another class after the mathematics course. She always complained and never tried to do anything to improve her school situation. Several times she made appointments to meet me in my office, but she failed to keep them. Joan handed in most of her assignments and journal entries late. She kept saying she just wanted to pass the course. She did not have time to "think" about and reflect on what we did in class. Her responses to the metacognitive questions, which were part of the second and third quizzes, showed her lack of interest in mathematical issues. When asked, "Would you restate the problem using your own words?" Joan wrote; "The way the problem is stated is good."

Like Joan, Tony was often absent or late, and in addition, did not have time to keep his journal entries. He produced the journal in one day to meet the requirements of the course. Therefore, I gave him only one third of the journal mark without any feedback, except one page of comments explaining the purpose of journal writing. His journal had nothing to do with "reflection," it was a story written in a rush.

Sarah was the only student who did not agree to be video taped during the class sessions. She was a private person. She wrote very little in her journal entries, and her work was not reflective.

Lynn had a full-time job and other courses that made her extremely tired and late most of the time. However, unlike Joan, she hardly missed any classes.

Nina was unique in many ways. She talked all the time and she always insisted on saying that she did not know anything and she could not understand anything related to mathematics. However, she was quite capable of doing and understanding mathematics when she changed her attitude.

Pamela looked at things in a mechanical way. Journal writing was only a requirement that she had to fulfill. In her last entry she wrote, "I think I'm all 'journalled' out." Fifteen minutes into the final exam she wanted to leave. I asked her to read the problems carefully and not to give up so soon. It worked and she managed to respond to many questions. I told her that she could keep the copy of the exam if she wanted. She refused and said, "I don't want to look at it ever again."

In general, the traditionalists were resistant to change. They preferred to have more structured instruction even though some aspects of the instruction, such as working in small groups and using manipulative materials, helped the majority of them to gain more self-confidence.

Second Category: Incrementalists

The twelve students in this category expressed their appreciation for some of the components of the instruction. However they questioned its lack of "direction" and "guidance." These students considered the instruction as a series of teaching strategies that should or should not be adapted depending on the situation. Beth's last journal entry captured the views of the people in this category: ". . . I believe this with any teaching method; it must be used in moderation."

Kayla, Lilian, and Terri expressed their "fear" of mathematics. However, Teresa had never had "any serious difficulty with the subject." Shirley "liked" it, and Jim, whose hobby was designing buildings (his post-interview), "enjoyed" mathematics although he "never got good grades in high school."

Some of the incrementalists were optimistic about doing well in the course. Beth, Leah, Carmel, Lana, Keith, Diana, and Lilian hoped to become "more confident," "understand mathematical processes and thoughts," and "gain more confidence in math computation" by taking the course.

Lilian, the only student to fail the course, tried very hard but was working long hours and had another course to attend after the course. She gained more self-confidence as she said, but needed more time and more responsibility towards her own learning.

Overall, the incrementalists were more homogenous than the students in the other two categories. In fact, there was not an idiosyncratic case in this category to be discussed separately. The students in this category asked for a more "balanced" instruction that adapted some aspects of the metacognition-based instruction and some "structured" instruction, as they called it.

Third Category: Innovators

All nineteen members of this category were definitely influenced by the instruction. With the exception of Patrick who said, "Math always came easy to me. The only problem I had in the class was with identities. I know why, however; it was because I simply did not memorize those ugly things (identities)" (Patrick's autobiography), the eighteen others expressed their concerns about mathematics at the end of the first class. Clara's response was typical: "I have to say I approach this course with my stomach in a bit of turmoil" (Clara's autobiography), but she then continued, "I hope I will do it! . . . I also hope that I will enjoy it, not just tolerate it to get a grade." The most important factor helping the innovators to benefit from the instruction was their willingness to try and overcome their fears. "I am nervous about this course," said Barbara, "but I am prepared to work hard. I even hope to overcome some of the blocks to understanding" (Barbara's autobiography).

The innovators were different in many ways. The variation among them is worth mentioning.

The youngest and the oldest students in the class were among the innovators. The interviewees were chosen on the basis of their performance on their early work in this course, and on their beliefs about mathematics. Two of the interviewees who were initially among high achievers fell into this category. Furthermore, an interviewee from the medium range ended up in this category. In addition, I interviewed two more students who showed noticeable changes from the beginning to the end of the course. Therefore, five interviewees were eventually characterized as innovators. There were two performing artists (musician and stage actor) and one painter in this category. Their approaches to problem solving were articulate and interesting. Their interest in art helped them to efficiently utilize the mathematics they learned. They wanted to learn by "doing" not just memorizing the facts and formulas.

Patrick was unique in his own way. He consistently showed positive attitudes toward mathematics. At times, Patrick was frustrated and confused, but he was never disappointed. Melisa wrote frequently in her journal about how had she wanted to construct her own meaning since she was in elementary school, but the system did not let her. "I used to ask [my grade 4 teacher] 'but why?' The teacher said, 'It doesn't matter why Melisa. Just do it as I did'" (Melisa's autobiography). Patrick and Melisa were more enthusiastic about new ideas, studying the mathematical connections, and striving for further extensions of problems and concepts. They shared some of those ideas through their written work, and their journals became a channel for individualized instruction through reflection and interaction. Their eagerness was well described by Melisa, "I became quite excited because of the whole new can of worms that opens up in my mind."

Clara's journal was superb. She solved every problem using multiple approaches. The managerial skills were well manifested in her solution processes. While she was the only student in the whole class who got 100% on her journal, assignments, and quizzes, she froze on the final exam and could not perform. She got 57% on her final exam and her overall mark was 83%.

In general, the innovators were all willing to try, and that was crucial. Sometimes they were upset and worried and sometimes they were happy and satisfied. Metacognition-based instruction provided them an opportunity to gain more self-confidence and to realize that they were capable of doing mathematics. They also changed their beliefs about problem solving since they experienced multiple approaches to solve a problem. These students adapted the metacognition-based instruction as a new teaching approach. In fact, they shifted their paradigm and saw the teaching and the learning of mathematics from a new perspective. That was the reason for referring to the members of this category as "innovators", those who were open to change and were actively involved in the meaning-making processes.

Descriptive Statistics

The present study was qualitative in nature. Its main purpose was to monitor changes in the students' beliefs about themselves as doers of mathematics and about mathematics and mathematical problem solving. Nevertheless, descriptive statistics will help the readers to have a better picture of the individuals in the class. The information from each category is presented in table form. The following three tables give a descriptions of the background of the students in the three categories. In addition, attendance patterns, final exam scores, and final course grades are shown.

Table 5.1

Descriptive Statistics Regarding the Traditionalists

Name	Age	Math	Date	Degree	In Ed	Exam	Final	Attend
Pam	20's	G12*	-	-	-	46	66	+
Nina	20's	H.S	-	B.A.	2Yr	55	75	+
Lynn	20's	-	-	-	-	34	63	L/A
Lora	20's	G12	-	-	2Yr	59	71	+
Sandra	20's	-	-	-	-	60	71	+
Dan	30's	H.S	1976	B.A.	UPD	63	67	+
Joan	30's	-	-	-	Apply	39	58	L/A
Marion	20's	G11	-	B.A.	Apply	68	75	+
Tony	20's	G12	-	B.PE.	Apply	54	52	A/L
N0 = 9			Average Exam = 53%			Average Total = 66%		

Notes.

- = No data are available

* = Failure in the course

Math = mathematics backgound

HS = High School Mathematics

Date = date of last mathematics course

In Ed = in Elementary Teacher Education program

2Yr = Already in the Elementary Teacher Education Program

UPD = Updating teaching credential

Apply = applying for the Elementary Teacher Education program

Exam = Final exam score

Final = Overall grade/course final mark

Attend = Students' attendance

L/A = Mostly late, a few absences

A/L = Mostly absences

+ = Full attendance without tardiness

Table 5.2

Descriptive Statistics Regarding the Incrementalists

Name	Age	Math	Date	Degree	In Ed	Exam	Total	Attend
Keith	20's	Calculus	-	3rdYr	Apply	66	76	+
Lana	30's	Al.11	1980	B.A.	Apply	93	89	+
Teresa	20's	Al.12	-	3rdYr	-	86	84	+
Kayla	20's	-	-	B.A.	2Yr	55	68	L/A
Leah	30's	H.S	1978	B.A.	-	61	74	+
Terri	20's	Al.12	1986	B.A.	2Yr	63	78	+
Lilian	20's	H.S	1979	-	-	22	47	A/L
Beth	20's	Al.11	1984	-	2Yr	65	75	+
Jim	30's	-	-	B.A.	-	80	85	+
Shirly	40's	G13	60's	B.?	2Yr	39	64	+
Carmen	20's	Finite*	-	-	-	69	71	+
Diana	20's	Al.12	-	B.A.	Apply	81	79	L
No = 12			Average final = 67%			Average total = 74%		

Note. Codes for this table are on the next page.

Notes for Table 5.2

- = No data are available

^ = Repeat high school mathematics at night school

* = Failure in the course

Math = mathematics backgound

Al = Algebra

HS = High School Mathematics

Date = date of last mathematics course

In Ed = in Elementary Teacher Education program

Apply = applying for the Elementary Teacher Education program

2Yr = Already in the Elementary Teacher Education Program

3rdYr = Third year undergraduate program

Exam = Final exam score

Total = Overall grade/course final mark

Attend = Students' attendance

L/A = Usually late, a few absences

A/L = More absences, Usually late

L = Late

+ = Full attendance without tardiness

Table 5.3

Descriptive Statistics Regarding the Innovators

Name	Age	Math	Date	Degree	In Ed	Exam	Total	Attend
Penny	20's	G12	1988	3rdYr	-	60	77	+
Linda	20's	G12*	-	B.A.	2Yr	44	70	+
Dave	20's	G11	1984	-	2Yr	49	63	L
Melisa	20's	G12	-	B.A.	-	80	83	+
Barbara	40's	H.S	1966	B.?.	2Yr	81	83	+
Kent	30's	G12	-	B.A.	-	94	90	+
Rita	40's	Calculus	1966	B.Sc.	UPD	70	79	+
Carmen	20's	Al.11	1981	-	-	47	64	+
Patrick	30's	G12	1979	B.A.	-	93	93	+
Rose	40's	-	1968	B.Ed.	UPD	85	85	+
Peggy	50's	College	1966	B.Ed.	UPD	69	84	+
Sophia	20's	H.S	1981	-	-	91	91	+
Bob	40's	G12	1968	B.Ed.	UPD	98	90	+
Sally	20's	H.S	1985	B.A.	-	68	78	+
Clara	40's	Al.11	70's	-	2Yr	57	83	+
Sarah	20's	Finite*	-	-	2Yr	58	78	+
Jack	20's	Al.12*	1986	4thYr	-	84	85	+
Ben	20's	-	-	-	Apply	72	87	+
Sheila	30's	G12*	-	-	Apply	78	81	+
No. = 19		Average Exam = 73%			Average = Final = 81%			

Note. Codes for this table are on the next page.

Notes for Table 5.3.

- = No data are available

^ = Repeat high school mathematics at night school

* = Failure in the course

Math = mathematics backgound

Al = Algebra

HS = High School Mathematics

Finite = Finite Mathematics at university level

Date = date of last mathematics course

In Ed = in Elementary Teacher Education program

Apply = applying for the Elementary Teacher Education program

2Yr = Already in the Elementary Teacher Education Program

UPD = Updating teaching credential

3-4Yr = Third or fourth year undergraduate program

Exam = Final exam score

Total = Overall grade/course final mark

Attend = Students' attendance

L/A = Usually late, a few absences

A/L = More absences, Usually late

L = Late

+ = Full attendance without tardy

Findings Within Each Category

A number of different themes emerged from the students' work. Those themes helped to organize the findings and give partial answers to the research questions. The excerpts from the students' written and oral responses are generally representative except for several idiosyncratic responses that are discussed separately.

The main themes that emerged from the students' responses were their reaction to the instruction and its components including (a) small-group work, (b) whole-class discussions, (c) journal writing, and role of the teacher and the classroom environment. Almost everyone talked about the role of small groups and whole-class discussions in their learning. Many students did not write about the role of journal writing per se, nevertheless, journal writing played an important role since it gave them an opportunity to discuss their point of views.

Students journal entries were categorized as dealing mainly with cognitive issues, or metacognitive issues, or both. Seven students focused primarily on cognitive aspects of the course and went through the solutions of the problems discussed in class to make sure they understood those problems. Eleven students reflected mostly on metacognitive issues such as their beliefs about themselves and about mathematics. However, twenty students kept a *balance* between cognitive and metacognitive issues in their journals. These students wrote about their awareness and beliefs, and used the problems to discuss metacognitive issues in context. This way of utilizing the journals helped them to improve their managerial skills to a great extent. Appendix R contains a sample of Clara's journal entry regarding an "exponential decay" problem. Clara wrote questions that she asked herself including "What do I know," "[This] is what we know, this is what we want to know," and so on. The way in which she solved the problem is a good example of those who used their journals to address both cognitive and metacognitive issues.

The following three tables show the number of traditionalists, incrementalists, and innovators who used their journals in cognitive, metacognitive, or both [balanced] ways.

Table 5.4

Ways of Utilizing Journals, the Traditionalists

Cognitive	Metacognitive	Balanced	Total
3	1	3	7*

Note.
*Two journals did not fit into this classification. One was written at the end of the course and it was more as a report and not as a reflection on what went on in the class, and the other journal was specifically written about mathematics and painting in the thirteen century.

Table 5.5

Ways of Utilizing Journals, the Incrementalists

Cognitive	Metacognitive	Balanced	Total
1	5	6	12

Table 5.6

Ways of Utilizing Journals, the Innovators

Cognitive	Metacognitive	Balanced	Total
3	5	11	19

The roles of manipulative materials and the teacher will not necessarily be discussed at length in all the three categories.

The other themes that became apparent and that the students kept referring to were their beliefs about mathematics and mathematical problem solving and themselves as doers and learners of mathematics, mathematical connections, and tests in a mathematics course. The following gives an overview of the different themes and sub-themes that are specifically discussed within each category.

The traditionalists (first category) discussed three themes in particular, namely (a) the instruction and its components, (b) students' beliefs about mathematics, (c) and students' beliefs about mathematical problem solving. The incrementalists and the innovators also discussed these three themes at great length.

The incrementalists (second category) reflected on some of the aspects of the instruction in particular. Those aspects of the instruction that are discussed in the second category are as follows: (a) time, (b) the teacher's role, (c) journal writing, and (d) small groups and whole class discussions. "Mathematical connections" is another theme that is discussed under a separate heading only in this category.

The innovators (third category) talked about class environment as a main component of the instruction. Furthermore, the innovators reflected on themselves as doers and learners of mathematics which is discussed under the heading with "students' beliefs about mathematics." "Tests" as a separate theme is discussed only in this category.

Since most of the responses were the result of the students' reflection on what happened in the course, I do not discuss "reflection" as a separate theme but as one that is embedded throughout the whole discussion.

Findings from the First Category, the Traditionalists

An interesting phenomenon that emerged from the traditionalists' responses was an embedded definition of "traditional teaching." The students' responses will be discussed, and a "student-based" definition of particular kinds of instruction will be provided.

The Instruction and its Components

> *I liked being told what to do instead of discovering it for myself by trial and error. Again this must be a legacy which I have inherited from my own school days.* *(Nina, May 30)*

Journal writing served many purposes including better communication between the students and the instructor. "I had written honestly and tried to express my understanding and concerns. I appreciate the opportunity to do this," wrote Lynn in her last journal entry. On a number of occasions, students asked for help with their journals regarding the difficulties they had with specific problems. However, some of them did not consider writing as a reflective process to enhance their learning. These students did not utilize their journals fully. For three of them, journal writing was just an added requirement to be fulfilled. Pamela, in her last journal entry, said "I think I am all journalled [sic] out!" By that Pamela meant she had a certain number of entries to reach the criterion (I did not ask them to write for the sake of writing). Joan's entries were not reflective. They were short and only reported, without discussion, the topics that were covered in some of the classes. In her entry of June 5 she wrote, "I wasn't here all three days (whole of probability), but I did get the *sheets* to be done." The interesting point was that she knew how much effort she made for those entries. "I really didn't have time to complete in the way I should have. Many times I had to write it 3-4 days late and thus it was not as *accurate* or up to date as it could have been. Also I didn't write very much each time." This was her last journal entry and yet her understanding of journal writing was the same as when she started the course: to do the *sheets* (hand-outs) and do them *accurately*. This shows her mechanical approach to the idea of writing as opposed to using writing as a reflection on what she did, why she did it, and how those things happened. She did not even commit herself to this mechanical approach due to her other engagements irrelevant to the course.

The traditionalists did not feel the necessity to elaborate and discuss mathematical ideas. Joan's responses to the metacognitive questions regarding the second and third quizzes were typical, short and inarticulate. For example, the third question asked: "Would you restate the problem using your own words?" She wrote, "The way the problem is stated is good," as if the question asked for a correction. She did not think about the question as a metacognitive activity to promote awareness.

The traditionalists were generally impatient with whole-class discussions, with the exception of Nina who kept talking unless other students asked her to be quiet so that other students could speak. Lynn's response on May 29, sheds more light on this claim. "Roy took control of the learning environment for over an hour. He lectured so that the *usual incessant questioning* by the students, was unnecessary and generally unacceptable." Lynn did not see the need for any kind of discussion. She even devalued the others' eagerness. "This portion of the lesson was only confusing when the class became obsessed with debating their viewpoints. I find this a waste of time and frustrating to 90% of the other class members who would like to move forward. There is a little bit of a 'star' syndrome happening" (Lynn, June 4). Lynn did not respect the others' viewpoints as genuine interactions of ideas, but rather she reduced all their effort and enthusiasm to a "star syndrome" and a sign of "obsession." Another point is that she did not have much interaction with other students to make such a generalization about 90% of the students' who "liked to move forward." (She usually left immediately after each class due to her full-time job about which she frequently wrote in her journal). Finally, I could not figure out why she wanted to "move forward" since she knew the students were responsible only for the material covered. She sometimes lost patience with the other students and became "angry" with them. "It makes me angry that people choose to make the class difficult by creating extra variables. . . . Am I unreasonable? . . . This was a very straightforward question that was needlessly dragged out," she continued. Her journal entry shows that she did not like any speculation and prediction regarding mathematical problem solving, and secondly, that she allowed herself to identify with other students' needs without interacting with them. Nevertheless, her journal entry of June 6 revealed her concern for whole-class discussions. "All the arguing that goes on often leaves me more confused about things I was very sure of in my life previous to this class!!!!" She was more confused because she deliberately did not want to take part in the debating and the discussion process. "This 'matching' question was awful. I immediately set up a matching question on my sheet. I probably would have demonstrated this to the class. However, I got so frustrated with those who do waste class time with their 'self-important' . . . explanations that I could be hypocritical!!". So for her and others like her, whole class discussions were nothing but confusing and frustrating.

Although many other students in this category were impatient with whole class discussion, they acknowledged the benefit of working in small groups. In her May 7 journal entry, Lora expressed her gratitude for working in small groups. "I liked the idea of working in groups. I find by explaining my method/solution to others, I better understand it myself. It also helps me to lessen my anxiety." Marian, wrote in her last entry, "One thing I have learned from the course is the importance of working together." She repeated the same statement in her post-interview.

Many students expressed difficulties that they had with geometry prior to the course. However, working with manipulative materials helped the majority of the students to gain a better understanding of geometry. "I enjoyed participating in today's class. I find it very helpful to use cut-outs/manipulative when doing mathematics. I find it easier to comprehend the purpose and steps of the activity," wrote Lora in her May 7 journal entry. Sandra addressed this issue after the class went through the proof of the Pythagorean theorem: "It is much easier to imagine relationships using a model (3-dimensional) rather than using only your mind to imagine a shape." In general "hands-on" activities, as the students called it, gave them a chance to see mathematics as a "fun" activity. "I feel that I actually learned something, and that math is not as hard as it seems. . . . Thanks for a great mathematical experience. It was a lot of work but a lot of fun too," wrote Lora in her last journal entry.

The Students' Beliefs about Mathematics

The traditionalists shared a common belief about mathematics. Lynn said, "Math is not interesting enough to persevere." For Nina, mathematics was a subject that she did not like much. Marian was a bit "apprehensive" about it, and Lora had "anxiety/phobia when it comes to math." All of them expressed these feelings in their autobiographies at the end of the first class. These thoughts showed the traditionalists' belief about mathematics prior to the instruction. Pamela explained how she developed such belief:

> I guess you could say I have a phobia about math. In elementary school I enjoyed it very much and did very well, but once I got to high school I seemed to struggle with it from the very beginning. Once I reached Grade 12 I hated it and I had to repeat Math 12 three times before I finally passed (just barely). I still panic whenever I am in a situation where I have to do any calculating of any sort. If there is any way to get out of a situation where I have to do math I will find it! I want to be comfortable in this area so I hope this class will help me with it.
>
> (Pamela's Autobiography)

Pamela's description of the development of her "hatred" of mathematics was amazing. It made me wonder how students' beliefs were influenced by the ways in which mathematics was presented to them. She was honestly voicing many other students' beliefs. However, she was hoping that the course would help her and others by "being told" as opposed to being actively involved in the sense-making and meaning-making process. Nina's response on the second day of the course is representative: "It seems that I need somebody to constantly explain things to me. So far I find it very hard to understand things on my own."

The Students' Beliefs about Mathematical Problem Solving

There are different definitions for "procedural learning" and "traditional teaching" in the literature. Nevertheless, the traditionalists defined these terms clearly and independently. A descriptive checklist for "traditional teaching" based on the students' responses will be given at the end of this section.

The emphasis of the metacognition-based instruction was to develop the mathematical concepts through problem solving. The trend was to start the class with a problem. Then by discussing it within the small groups and collectively in the class, the students would develop an understanding of the concepts involved in solving that problem. This teaching method required active involvement of the students. They were always encouraged to ask questions and to see the dialectical nature of mathematics. My role as a teacher was more as a monitor and a facilitator and not as the class authority. The students in this category rationally rejected this approach to teaching mathematics and asked for more "direction," "lectures," "structure," and "guidance." Marian voiced this clearly in her May 9 journal entry as well as in her post-interview: "I am having a bit of difficulty in an *unstructured* class. I suppose I have to get used to a lab-oriented class rather than a *lectured* class setting." Lora felt she was on her own "with little *guidance*. I did not even know where to begin on most questions." On the second day of the course, Lynn wrote;

> I'm afraid at this phase of my education I need fewer instructions and more direct knowledgeable guidance. It is not easy for students to be overaggressive in their beliefs and unfortunately can confuse ingenuity or innovation with correctness. *Am I obsessed with rules*? I really feel I need more concreteness! Sorry. . . . For those who are trying to hold their 'shaking' interest in math in check, side points such as discovering Fibonacci in a very specific case are distracting. (Lynn, May 7)

The important thing is that she questioned her dependency on the rules. "Am I obsessed with rules?" but she did not want to find a way out of the situation. Nina first wrote how she "liked to be told," and then talked about the roots of her dependency and reliance on rules. She held the same belief all through the course as did the other traditionalists:

> I must admit one thing. I found Roy's instruction exceptional and extremely helpful. *I liked being told what to do* instead of discovering it for myself by trial and error. Again this must be a *legacy* which I have inherited from my own school days. In high school our math instructor stood in front of the class and told us mathematical facts, much like Roy did today, and that is how we consequently learnt. I know that there is a better way to teach math by allowing students to have a period of exploration and discovery. However, there is only six weeks to teach a math class with so many topics, I think *direct instruction* is a better way to go. I may be wrong but for myself this way would work better.
>
> (Nina, May 30)

The point is that although I categorized these students as traditionalists, they rejected the approach mainly for pragmatic reasons. They were afraid of "wasting time" as Lynn said, and of not having enough time to finish the course material. They were not risk-takers, and sometimes felt that it was too late to change. "I need a lecture reviewing the steps" said Lynn, who continued, "Too much pride and stubbornness and frustration results from adults being responsible for their own learning! I really want to understand this. Please provide more guidance!". It is true that old habits die hard. She held the same belief at the end of the course, ". . . There was a method to your madness Zahra! I still wish I was taking this class as a 12 year old, I might have been a math major!!" wrote Lynn in her last journal entry.

Traditional Teaching"; the Traditionalists' Perspective

The traditionalists defined "traditional teaching" in mathematics based on their beliefs about mathematics and mathematical problem solving. They wanted me to first teach the concept, then solve a few problems on the board step by step, and then give them more exercises to practice the method of solving the other problems. "I really hoped to see more examples worked out step by step on the board. I feel I'm wasting too much time doing the 'wrong' thing" wrote Lynn on the third day of the class. She later expressed her belief more bluntly and even asked for a panacea. "I want to know the formula that will help me solve a task. I'm not saying that there is only one method of solving every type of question, but I would like to be able to confidently employ a method" (May 24).

The main characteristics of "traditional teaching" regarding mathematics, based on the Traditionalists' views, were as follows:

-Teacher to be in "control of class" and to give "guidance" and "direction" to students

-There is no need for "incessant questioning"

-Students "like to be told" by teachers what to do

-"Frustration" is a sign of weakness and forces students to give up

-Time should be used "wisely"

-Problems should be solved by teachers on the board "step by step"

-The "correct answer" is the main emphasis of problem solving

Summary

Although the traditionalists did not change their beliefs as a result of the metacognition-based instruction, they definitely were influenced by it. Even their old beliefs were challenged by it at times. Journal writing was a communication channel between most of them and myself, writing gave them a chance to become more reflective. For example, Pamela solved a problem in class and reflected on it in her journal. "Somehow that doesn't make sense. If the pyramid is only 2.5 cm tall and 5 cm width, how can it hold something that is 20.82? I am confused" (Pamela, May 22). Marian, Lora, and Lynn reflected on different issues in their journals frequently. I tried to stimulate discussion as much as I could by relating to them through their journals. The benefits of working in small groups and with manipulative materials were acknowledged by most of them.

Generally, traditionalists liked a "direct instructional approach" as it was called by the students, or some of them referred to it as "procedural knowledge." Marian wrote about this approach in her last assignment, and talked about it in her post-interview: "It is difficult to get from A to C if one does not know B. What I am saying is that teachers need to refresh students' memories more, particularly on areas they have forgotten. Sometimes going back a few steps sheds light on the present situation." In this approach, the teacher is the sole authority and has the final word on directing students step by step. This approach considers mathematics to be cut and dried, with concepts presented in isolated fashions with more emphasis on rules and procedures. In this view, mathematics has little or no connection with day-to-day life and mathematical concepts are treated as isolated facts. What Nina wrote in her May 9 journal entry illustrates this view. "I found it very difficult to work backwards because I am so conditioned to applying the rule or the theorem to the *exact* situation. . . . I always thought that when the problem was finished it was over and done with and we didn't have to depend on it any longer. . . . I always thought that when we are given a rule, unusual things could not happen." For her, rules have an absolute role which do not leave room for generalization and extension in mathematics.

Findings from the Second Category, the Incrementalists

The students in this category, the incrementalists, were influenced by the metacognition-based instruction to some degree. They liked some aspects of the instruction which worked well for them, and were opposed to the other aspects that seemed unrealistic to them. They regularly compared traditional and the metacognition-based instruction regarding teaching and learning of mathematics. The incrementalists concluded that a "balanced" instruction using these two approaches would be more practical for teaching and learning mathematics. This section illustrates some aspects of this duality.

The Instruction and its Components

> *Most of us were taught mathematics by a direct instruction approach where the emphasis was on getting the right answer. If you know the correct formula your answer would match with the answer in the back of the textbook. Math was never something that could be explored or speculated about, which produced a great deal of anxiety.*
> *(Kayla, May 8)*

Kayla beautifully portrayed her beliefs about mathematics and mathematical problem solving, and the ways in which they were influenced by the kind of instruction that she described. She and many others tried to look at the teaching and learning of mathematics from a different perspective. At times they thought that the interactive instruction that allows discussion and values innovation was the way to go. "I guess the biggest gift a teacher could give to a student is the ability to look at, think about, and solve any problem on his or her own. . . . My understanding is that the method discussed would be steering children towards thinking for themselves and working things out, and nothing could be more valuable" (Leah, May 21, reflecting on a video presentation by Paul Cobb). However, there were times when they became frustrated and disappointed. On May 23, Carmel reflected on the same presentation and wrote;

> I can see that this method of instruction is being used in our mathematics class. To be honest, at times I find it frustrating. I'm not sure what I should be working on, I also feel like I'm far behind everyone and not understanding the lessons fully. Maybe I just need to get used to it, but after 3 weeks you'd think I would already be adjusted. *I guess changing just isn't easy.* (Carmel, May 23)

It is absolutely true that "changing just isn't easy" for anyone, but it is conceivable. Besides, change was less likely to happen if people were satisfied with the current situation, or they did not question it. "Let's talk about the possible ways that would help you to catch up and not feel frustrated or left behind (as you said). Please come and see me," I wrote in her journal. We sat down and talked about these issues at length. I believe interaction and communication helped to reduce frustration, and promote awareness and understanding.

Carmel voiced many other students' opinion when she said "changing just isn't easy." Lana expressed this difficulty in her May 24 journal entry, "I have 'moments' of understanding the logic of the reasoning behind the example we went through. But then after a while it all starts sounding like some philosophical game of words and semantics. I guess I see it this way because I find it just beyond my grasp and can't appreciate the process. I find this learning method frustrating because it is so foreign to me." Nevertheless, she tried to make sense of the "learning method" and become more familiar with it. "When we do a problem in class, it makes sense, but this understanding is always 'a posteriori.' It is like sitting in front of a work bench full of tools but not knowing which ones to use for a given project. If someone said, 'this is what you want to do . . . ,' then I might have a better idea of what tool to pick up in mathematics . . .." Concepts, solution methods, and formulas were among those mathematical tools that Carmel wanted to learn more about. Familiarity with those mathematical tools was a necessity; however, Carmel was waiting to be told what to do with those tools. Furthermore, the next step would be to learn how to use those resources efficiently, that is to know when to use them, why, and for what reason, skills that are called "managerial."

The incrementalists sometimes tasted "moments of understanding", as Lana said, and at other times became tired and gave up: "I'm not sure why we weren't just told the principle in the beginning. Why are we made to guess the theory? . . .. Personally I like to be told the right thing the first time, then practice using it in equations," wrote Carmel in her June 5 journal entry. Furthermore, she explained why: "Children may like the 'guessing style,' but we're adults." Nonetheless, the "guessing style" helped Beth (May 15) to have a "concrete understanding" of the "formula or method."

> I find that in this class we are working backwards compared to the way I was taught in high school math class and this may be part of my confusion

> and misunderstanding at times. In high school we were just given a particular formula then asked to apply it in various situations. However, in this class I feel we are going through a guess and check process in order to create a formula and in the end we have a more concrete understanding of the formula or method rather than just memorizing it. (Beth, May 15)

Incrementalists were in a constant conflict between their previously held beliefs about teaching and learning mathematics, and the new approach that they were confronted with. Terri's journal entry of May 13 illustrates this conflict. "I still feel that I could benefit from more direct instruction, as I find the exercises a bit difficult to undertake based on my notes taken during class." Furthermore, she compared the class with her high school Algebra class where any talking was "no! no!" and said, "My mind got a much needed break from memorizing facts once again." Terri continued the same discussion in her May 22 journal entry: "We need to get away from assigning students page after page of boring questions to answer. . . . We need to show them math can be fun." The metacognition-based instruction created a quandary situation for the students. On one hand, traditional instruction was straightforward and easier to deal with--predictable but "boring." On the other hand, the new approach gave them "a much needed break" to move away from facts and formulas and see that "math can be fun." Furthermore, they talked about "balanced" instruction and how to incorporate certain aspects of the two approaches into instruction.

The incrementalists were influenced by the metacognition-based instruction but they were not ready to make a transition. The ways in which the incrementalists synthesized the metacognition-based instruction was to consider it as a set of teaching strategies, and not necessarily a different way of looking at teaching and learning processes. Shirley explains; "It would have been helpful if we were told to hand our homework in the beginning. I found it difficult to see concepts through the many calculations we had to do at times" (Shirley, June 14). She wanted me to first teach the concepts, then give them rules and formulas, and the students would apply those rules and formulas in the homework problems. Developing those concepts through problem solving was difficult for her and other incrementalists. The point was that their understanding of the instruction was "teaching mathematics *for* problem solving," which meant understanding the concepts, learning the rules and the formulas, and then "applying" them to a problem or a question. Kayla's comment in her June 5 journal entry shows her expectations of the instruction. "The worksheet questions are much more difficult than they need to be. If the goal is for us to apply what we know to a problem or a question, it should be made very clear."

Most of the other incrementalists were clear in expressing their views. They knew what the goal of the instruction was. However, they tried to compromise and solve the conflict by proposing a "balanced" and "modified" instruction. By that they meant to adapt some aspects of traditional instruction and some aspects of the metacognition-based instruction. "This class gets frustrating because although I can see the benefits of group discussion and letting the students share their ideas and discover math for themselves, it seems as if *there's a lack of guidance and clear instruction*. I think that there needs to be a *balance* between instruction and 'self-discovery' (for the lack of a better term) . . .." That was Teresa's solution (May 30) for a more "balanced" instruction to accommodate their needs. Beth's advice was more elaborate:

> I think the teaching method of helping the students realize their thoughts can sometimes be over used. I believe this with any method, *it must be used in moderation*. In this situation in particular, as a student, it can be frustrating to be asked 'what do you think?' in response to a question, it is like answering a question with a question. In a hypothetical situation a student may be completely confused and not have a clear idea at all. . . . I think that when a child obviously has no clear conception of something, asking them 'what do you think?' may only increase the frustration which may affect their attitude towards mathematics. I do see this method useful though, once the child/individual has developed an understanding of the basics involved in a particular math problem. My response to this opinion is taken strictly from personal experience. If I know when I am going to ask a question and the response is going to be 'what do you think?' (or a form of this question) I am very hesitant to ask and this is when I feel I need the most help. (Beth, June 6)

There are a number of interesting issues in this passage. She was in a quandary situation. Her entries clearly show the conflict. She was struggling with different beliefs and that was the main distinction between the incrementalists and the traditionalists. The traditionalists were characterized by those who rejected the new approach and did not show much enthusiasm for looking at mathematics from a different perspective. However, the incrementalists were moved by the new approach. On one hand it made them feel fulfilled and satisfied, while on the other hand they felt that it was not practical. This conflict was a sign of change that was taking place, but "changing just isn't easy." (Carmel, May 23). The instruction helped Beth to some extent but it did not cause a drastic change in her beliefs about mathematics and mathematical problem solving. The instruction (or method as she called it) was to help the students understand better. It was not something to apply after they had developed an understanding because the "method" was not an end in itself. Constructing meaning by individuals through reflection, interaction, and discourse was a great activity to acquire knowledge and to develop further understanding.

There are more examples to show the characteristics of incrementalists. For example, Keith in his May 9 journal entry wrote, "I personally like this method because you are able to discuss each others' ideas and come up with a single answer. I wish more classes were set up this way to give students the opportunity to express themselves." However, he started his June 10 journal entry writing that "According to Zahra . . . " when he wanted to discuss a statistics problem and then continued to solve that problem himself. The emphasis of the instruction was on the sense-making process, that no one should accept anything because someone else said it is correct but because it seems reasonable and makes sense. Nevertheless, by referring to me, he wanted to legitimize what he was doing.

The instruction influenced their beliefs about mathematics and mathematical problem solving and gave them a chance to learn mathematics and be more "critical" about mathematical problems. Carmel explained in her last journal entry:

> I guess after all I got a lot out of this course. More than I can recall learning in any other mathematics class. *At first I didn't like this teaching style but I now feel that my views have changed.* What I have learned in this class is solid in my mind. I can be critical and really think out the mathematics problems. . . . Although I must admit some aspects of the course slipped by. (Carmel's last journal entry)

Shirley did not say anything about change, but she got a chance to see a community of people who showed some positive views about mathematics. "It was an enormous amount of material for that length of time . . . I would not wish this on a friend although I met some wonderful people in the class and enjoyed their enthusiasm for mathematics" (Shirley, June 14). Teresa, in her last journal entry, commented on the interactive nature of the instruction, an opportunity she had not experienced in her previous mathematics classes. "It was nothing at all like what I expected. It was definitely not your traditional math course! I liked it in that the atmosphere was relaxed and unlike the usual lecture format that so many . . . university courses have, there is a lot of opportunity to interact with others in class." Moreover, Lilian, who knew she would fail the course, expressed similar feelings about it in her June 3-10 journal entry: "Believe it or not, taking math again has had a rather therapeutic effect on my life."

Time.

Some people were concerned about "time." Jim wrote "a constructive critique of mathematics 335" as his last assignment. In that critique he talked about the positive and the negative aspects of the instruction. "The teaching method of this course gave students an opportunity to get to know each other, to realize that not everybody goes about solving a problem in the same way, and to challenge one another." He then continued, "The drawback in implementing such a method is that *it is often time consuming*. In certain classes we only managed at times to cover two problems. Therefore, one has to watch one's timing. Another drawback is that in many classes we failed to actually go over many of the problems that were given to us. A solution to this would be to make certain that we go over all of the problems related to a particular topic before moving on to another topic." I responded to this concern in his journal, "It *is* time consuming, but time is not wasted. I think if you understood the two problems well, you'll be able to do the rest of them on your own. But if you didn't thoroughly understand them, it doesn't help you if even 10 of them are solved in the class, because still the 11th one gives you trouble." What Beth wrote was a good reply to Jim's concerns about "time."

> Some people may feel that much time may be spent on one problem, but they have to understand that a fully developed concept can be extended and further developed more than a *superficially learned concept.* So in reality, the same amount is learned to a deeper understanding if promoted by the method in which children discuss their problems and share their ideas.
> (Beth, May 21)

This journal entry reflected on a video clip from Paul Cobb's presentation at the University of British Columbia. Interestingly enough, the visual aspect of the presentation helped her and many others (who all commented on the presentation) to relate to the children involved in the video presentation and enable them to see the benefit of similar approaches to teaching and learning mathematics. In fact she believed that the investment in time helped them gain a better understanding of the mathematical concepts.

<u>Teacher's role.</u>

The "teacher's role" was another issue that directly related to the instruction. Some of the incrementalists expressed their gratitude for group work and discussion in class, but when it came to specifics, they preferred to be told. Teresa, in her June 4 journal entry explained. "I know that these kinds of questions [referring to the second quiz] are meant to get one thinking, solving, and explaining, but I am just not sure about <u>what it is I'm supposed to think</u> and explain!". "What do you mean by that" I wrote, and I continued, ". . . I can't tell you what to think, but I may ask you to give an explanation of what you did and, why you did it, and how you arrived at and these questions force you to think!" My purpose was to challenge her dependency on external authority to tell her what she should do and when. Holding such a view interferes with the process of becoming an independent and active learner.

Beth had a different view about the teacher's role in class. In response to a passage from the "Republic" by Plato, she wrote in her May 8 journal entry, "The dialogue was interesting, but I felt Socrates, the teacher, did too much talking. Perhaps he could have encouraged the boy to come up with the facts he had the boy acknowledge." Beth's expression showed her reflection on the teacher's role; what it is and what it should it be. She was moving away from teacher-centered instruction towards more student-centered instruction, that teacher was more a facilitator, monitor, and most importantly a friend rather than a distant authority in the class.

Lilian's reflection on her second quiz on May 27 was interesting; it showed a mutual trust between two friends: student and teacher. "The last quiz [second] I walked away with a feeling of guilt because at our table three out of the five seated knew how to solve the problem. I mainly copied for the first little bit (round in square). Soon however, the problem made sense." I could speculate that good communication between the students and myself was partly responsible for this honesty and openness. It showed my relative success in establishing a friendly relationship between the students and myself.

Journal writing.

The idea of journal writing in a mathematics class was new to many students. However, some of the incrementalists like Lana wanted to give it a chance.

> This idea of keeping a journal may be a good one. I've never thought of exploring how I feel about math in my regular diary. Guess I assume I always have to write about 'bigger' thoughts. However, exploring these smaller, 'micro' thoughts and behaviours may be a more productive and successful way to deal with the larger problem of confidence in my academic abilities . . . we shall see. (Lana, May 7)

Diana misunderstood the whole purpose of journal writing. She said some people write more extensively because they don't understand the discussed problem as opposed to those who do understand, therefore: "they do not have very much to comment about . . . It is more honest and it is a *better reflection* of how a student really feels if they simply state that they understand and don't have much to comment on" (Diana, June 6). Her description of journal writing suggests the only purpose of writing is to give either a report of what they do not understand or to ask for clarification of several minor points. I asked Diana to talk about "better reflection" and discuss it in more detail, since I wondered how a person can judge the "betterness" of a reflection. However, Diana did not respond to my feedback, in either writing or verbally, and since she usually came to class late and left early, I did not have a chance to discuss this matter with her.

With the exception of Diana, all the other incrementalists utilized journal writing more than the traditionalists. A majority of the incrementalists used journal writing as means of individual instruction. I would not have been able to help them the way I did, if those incrementalists like Keith did not address the difficulties that they had regarding specific topics. "The entire process was very confusing for me because I wasn't too *sure* as to why we started at bar graph #8 and not for example bar graph #2?" (Keith, June 10-11). He then discussed another topic in detail in the same journal entry and wrote, "Please correct me if my definition is incorrect! . . . " (Keith, June 10-11). I responded to his request by going through the problem in his journal in detail, and wrote, "This is your responsibility to make sure that your problems are solved in the class. Please don't let it go unless you are *sure*!" The journal writing helped him and many other incrementalists to reflect on what we did in class, and whether the solution processes made sense to them or not. Lilian even wrote in her June 12 journal entry that the course and journal writing had a "therapeutic" effect on her life. Terri's views about the role of journal writing was representative:

> I asked you my questions in my journal entries. In this respect journal writing has been very effective, there are several types of journals and not only is this journal a personal journal, it is also a learning journal. Journal writing has been a very effective means of learning and helping the teacher

> teaches better in this class. Therefore, I am definitely going to encourage my students to write in their journals after math classes. . . . Teachers more like to give each and every student individual attention but, with today's overcrowded classrooms this isn't possible. So if the students use their journals as a means to reflect on how they are feeling as well as to ask questions which they didn't get a chance to do during the class, the teacher can give each student individual help. This type of journal also enables the teacher to monitor the class progress rather than always testing the students through quizzes or exams. . . . At first we all moaned and groaned about having to write journal entries 2-3 times a week, but I'm glad that we had to, because I learned a lot from the experience. It was also nice to get your feedback and I'd like to thank you for your time and effort.
> (Terri, June 12)

Small groups and whole class discussions.

A number of incrementalists appreciated the role of whole class discussions more than others, but some were confused by it. "I was confused when everyone was arguing about all the different ways people can sit in chairs [referring to a problem concerning permutations that we acted out in the class] . . . I think it is a great idea to back up one's ideas when discussing the equations" wrote Shirley in her June 5 journal entry. She got confused but she appreciated the role of discussion. However, Teresa did not have the same appreciation for the whole class discussions.

> I really don't think that this group method works for everyone. Some people just work better on their own (because working in groups frustrates them or confuses them.) But it's difficult to give everyone a choice in this matter I suppose. I guess it's just that sometimes I feel as if I almost know what I'm doing but when hearing other people speak, it only serves to confuse rather than clarify. (Teresa, May 30)

In general, the incrementalists wrote briefly about the role of whole-class discussions and were less reflective in this regard.

Except Beth, Teresa, and Lilian, the incrementalists rarely commented on the specific role of small groups. For Beth, working in small groups gave her assurance. "It was interesting to see how different people attacked the problem (different thinking patterns have always interested me!). I found that when I was doing the problem I was not entirely sure that I was correct, but when I explained it to someone else, I would either notice my errors or realize that I was correct in solving the question" (Beth, May 15). However, Teresa was afraid that she had become dependent on the others. "I think that working math problems in groups, etc. has been a good idea because we can learn from each other; but one thing that it is missing is that I found I was starting to rely on someone else to show me how to do something instead of sitting down and trying to think of a way to do it myself" (May 26). Lilian raised another important issue in this regard: "I have noticed that in group situations I feel awkward and stupid if I can't contribute to the team effort or offer anything of value. When I felt this way I became quiet and sometimes, I've noticed, I try to avoid the task at hand" (May 23-30). I commented on Lilian's concern and told her that every "why" that she asked would contribute to the "meaning-making" process of the group.

All the incrementalists acknowledged the role of manipulative material in their better understanding of the mathematical concepts especially geometry. Kayla's journal entry on May 8 is representative; "I found that exploring shapes by actually cutting them out and manipulating them a very interesting activity."

The Students' Beliefs about Mathematics

The first and the third research questions concern the students' beliefs about mathematics and possible changes in their beliefs as a result of the instruction. The purpose of this section is to study the effect of the instruction on the incrementalists' beliefs and to discuss the nature of the change.

All twelve students in this category, wrote in their autobiographies at the end of the first class, about their views and their experiences with mathematics. Terri, Shirley, and Jim all said they enjoyed mathematics although they never got good marks in their mathematics courses in high school. However, Kayla "never enjoyed it", and Beth said "Math has never been one of my strengths . . . ," as did Keith. Leah did not see any point to studying mathematics at all. "I also know that 'you' don't need math in the real world, or at least the one I was in. Not with calculators and computerized cash registers," but she continued to say "I would like to understand mathematical processes and thoughts" (Leah's autobiography). Lana got good marks for her mathematics courses but: "despite these successes I am convinced I cannot understand the principles behind algebra. So far, I feel I have gotten by strict rote memorization of rules and formulas." Carmel took Finite Mathematics and failed the course; however, she said: "I hope to achieve more confidence with my mathematical ability. I'm looking forward to the next six weeks." Diana was hoping as well to overcome her "fear" of mathematics. For Lilian, the mission was more difficult since she had "math phobia," but she was "determined to overcome this phobia." Teresa never had "any serious problems in the subject and enjoyed some aspects of it but was never interested enough to continue studying it." Her belief about mathematics was firm and clear, she believed that mathematics was "more objective" than other subject matters. Therefore, it was easier to teach mathematics than, let us say, literature.

Teresa's beliefs about mathematics like many others, altered as the course developed. She wrote about the process in her May 15 journal entry.

> Before this class, I always felt that math was a purely objective subject where one could only be right or wrong. In a way, this is true. But I think the problem is that many people think math to be like this, which in turn, leads them to believe that not only must there only be one *correct answer*, but there must only be one *correct method* in which to reach that answer. We're puzzled when our first try at a problem does not work, and when we find a solution that seems too 'simplistic,' we lack confidence in our answer. But it's difficult to opt out of that frame of mind. (Teresa, May 15)

Shirley expressed her views and wrote, "I am becoming a little more intrigued than tortured . . . well I am getting more interested in math but still finding it difficult. . . . I seem to need to go over things many times. . . . I enjoy the inspirational messages" (June 10). Everything is relative in this world, and change is a matter of degree.

After the first few classes when we were all working with concrete materials, Terri wrote in her May 9 journal entry. "If you have positive experience with math through manipulative materials, then you experience the same degree of success." Kayla who "never enjoyed math" before, felt that she was making some progress in overcoming her mathematics anxiety. "I feel I am making better progress this week. I think I am probably overcoming some of the math anxiety I was experiencing" (Kayla, May 17). Kayla wrote in her last journal entry, "It is important for me to leave behind my negative feelings so that students don't pick up on my negativity." She directly addressed the role of a teacher in the students' beliefs about mathematics. "I can honestly say my attitude about math has dramatically improved since the beginning of the course!" This alone can partially answer the third research question that the students' beliefs about mathematics were influenced by the metacognition-based instruction, for as Teresa wrote in her last entry "math can be enjoyable."

<u>The Students' Beliefs about Mathematical Problem Solving</u>

> *I have to feel a certain amount of success in problem solving to want to continue.*
>
> *(Kayla, May 9)*

Some of the incrementalists showed a considerable change in their beliefs about mathematical problem solving. The aim of this section is to portray that change.

The incrementalists, like the traditionalists, started the course asking for *the* method of solving problems. "I wish we could have one *direct approach or formula* to use. That the *problem solving could be more economical and fast.* I understand we have many ways of approaching problem solving but I am concerned about not being able to <u>recognize</u> what to do when given area or height to find," Lilian wrote in her May 13-17 journal entries along with other examples[4]. Lilian's view was as extreme as Lynn's view, a traditionalist who was also looking for a panacea to solve *all* her problems. They both were so concerned about *time* that they could not relax to enjoy the activities in the class.

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[4] Lilian, the only one who failed the course, missed many classes due to her full-time job and other courses that she was taking. She took the same course in the following year (1992) and earned an "A." Lilian said she started the course the second year with different beliefs and that helped her to get through easily. She even "enjoyed" the mathematics course.

However, Teresa's belief was less extreme and more representative of the incrementalists. "All that is needed is the knowledge of basic rules . . . and an application of these rules, plus the information given about the problem pieces together to know the completion or answer to what one is trying to solve" (Teresa, May 7). The incrementalists, as Teresa described, were struggling with the idea of mathematics as being objective, or mathematics as a creative endeavor. Sometimes they felt that they could save time by seeking *the* formula or method to solve problems since they thought it was "*more economical and fast*" as Lilian described.

There was a constant battle between these two different views about mathematical problem solving among the incrementalists. They often made a comparison between traditional and the metacognition-based instruction and discussed different aspects of both approaches. They showed some interest in being taught in the latter way, but were more conservative regarding "new" ideas. Lilian and a few others were exceptions among the incrementalists who were persistent in their views about mathematical problem solving. They were afraid they would be more confused if dealing with a different approach.

> More confusion followed me when I would have to see the many ways of problem solving on the board. I would tend to find this frustrating because ultimately what I would want is <u>one direct way</u> of problem solving. However, even when we did get formulas I would have difficulty in memorizing the procedure or know when to apply what and where.
> (Lilian, June 10-14)

Lilian was in such conflict that it could be a sign of change! On one hand, "one direct way" looked an easy way out, but on the other hand, she felt the need for certain skills that were metacognitive in nature, such as the ability to use her resources efficiently.

On the contrary, for the majority of the incrementalists the traditional approach to mathematical problem solving did not make much sense. Keith, in his May 8-9 journal entries explained, "I think math is not just only applying a law with a bunch of numbers, but rather, one must understand the reasoning behind the law in order to fully answer the question properly." Furthermore, the metacognition-based instruction offered them a new perspective, but it was not enough for them to make a transition.

The incrementalists were happy to understand the reasons for what they did while solving problems. Kayla wrote in her May 20 journal entry, "I found it important to keep asking 'why do you think it is this way?' or 'what is another way to do this?' . . . I realize how important it is to understand why." The new approach had something to offer to Kayla and other incrementalists and helped them alter their beliefs about mathematical problem solving, Kayla's case was an illustrative example. On the first day of the course she said she "never enjoyed" mathematical problems. Later on, she wrote on her third quiz: "Once I got started and was able to explain what I was doing and why, I began to enjoy solving the problem." This was a big achievement for her, and an answer to the fourth research question that, indeed "the students' beliefs about mathematical problem solving were changed as a result of the metacognition-based instruction."

Students' approaches to mathematical problem solving was shaped by their beliefs about mathematics. For those who saw mathematics as "objective" as Teresa described it in the beginning of the course, mathematical problem solving was as straightforward as applying certain formulas or following a method to arrive at some answers. This approach did not leave any space for reasoning and creativity. In this view, there was a one-to-one correspondence between mathematics and numbers, while everything else seemed "unmathematical." Teresa's journal entry of May 13 was illustrative:

> When trying to find the area of the dodecagon, we figured a way of finding an answer, but we were doubtful because it seemed to be a slightly too <u>unmathematical,</u> way to solve it, however, in comparing our answers with other groups who did the problem another way, the answers were the same. I've also realized that there are many different ways of approaching math problems.

In fact their way of finding the area of the dodecagon was innovative and interesting. They constructed the dodecagon using pattern blocks (one hexagon, six squares, and six triangles,) and found the area of each one of them, and then added them up to get the area of the dodecagon. However, her group members had a hard time seeing that their solution could be mathematically sophisticated and elegant. (Appendix Q contains a copy of Teresa's work.) Shirley's last entry shed more light on this view. Shirley said that she could do many things but she was worried about "the actual math, not the ideas," by that she meant calculations, using formulas, and plugging in numbers. It is sad that "the ideas" of solving a mathematical problem in her view did not count as "actual math."

The incrementalists gradually saw the benefit of having different approaches to problem solving. They experienced the importance of problem solving processes as opposed to emphasis on the "correct answer." "I know that getting the answer correct is not the most important factor when figuring out problems, . . ." wrote Shirley in her June 5 journal entry, "the process is more important." They started to see that the multi-approach to problem solving gave them "strength" and increased their mathematical power. "I think my strengths are that I try and look at the problem from several perspectives . . . I also can see things spatially" (Kayla, May 14). In addition, she was also aware of her "weakness." "My weakness is that if I experience some difficulty I have a hard time trying to overcome this." That awareness was the reason that she was able to overcome that "weakness." The emphasis that traditional teaching and evaluation puts on the finished product, namely the "correct answer" forces students to be prudent and avoid taking risks.

<u>Mathematical Connections</u>

"Mathematical connections" was a theme that the incrementalists talked about frequently. They said that they had difficulty relating the mathematics that they had learned in their schooling, namely a set of rules and procedures, to anything outside the mathematics class. However, most of the incrementalists were excited to see the "real-life" application of mathematics that they were doing in class. In the beginning of the course, what Leah wrote in her autobiography was typical. "I also know that 'you' don't need math in the real world, or at least the one I was in, not with calculators and computerized cash registers." The instruction provided an atmosphere for the students to allow them to see the applicability of mathematics in their daily lives, and to see mathematical connections. Keith tried to link the mathematics that he learned in class to everyday life around him. In his May 13 journal entry Keith explained how he measured his bedroom's height applying a method of measuring height that only needed a sheet of paper and a pencil. "I tried the formula at home to determine the height of my bedroom and found it to be very accurate. *Thus, math isn't something you only apply in the classroom, but it's also used in your everyday life.*"

Terri was excited to see that she could use the mathematics that she learned in class to do a real problem concerning ordinary people. "It's neat little tricks like this [finding the probability of having one boy out of seven children] which keep me interested in and liking math" (Terri, June 6). Mathematical connections increased their knowledge and their appreciation of mathematics. "Today we learned about sines and cosines. It's great to finally know what those buttons on my calculator are actually for . . . " (Lana, May 27), in her last assignment Kayla explained what gave her a greater appreciation of mathematics. "Math has seemed to me unrelated to most things, when I can relate a concept or a theory to an actual person and period in history it holds greater meaning. I gain a greater appreciation and interest in solving the problem when I have something to relate it to" (Kayla's last assignment).

For some incrementalists, mathematics made sense only if it was practical and relevant to their lives. Diana wrote about the "practicality" of mathematics in response to one of the metacognitive questions that was part of the second quiz, "how did you like solving the problem?" Her response to the question was that "I enjoyed it because it was challenging for me and *practical* in that it is a problem that could be applied in everyday life to be more efficient." However, her response to the same question regarding the third quiz was that "I really didn't like it because it is confusing and mind-boggling, not only to figure out just what the question is asking but also to come up with such *large numbers*." For Kayla, Diana, and others in their camp, "large numbers" were not practical. Therefore, they disliked them and saw no point of dealing with such problems:

> I still don't fully understand the question on page 387 [Jacobs, 1982]. A major reason why I think I have difficulty is because I don't see the importance in finding out the answer. I am asking myself why do we want to know how many pages to produce a 100,000,000,000,000 . . . poems. To be honest *I don't care!* I think because the *nos. are so huge*. . . . I just don't find this problem interesting. (Kayla, June 6)

It was important to see the connection and the relevance between mathematics and real life. However, one had to be careful not to reduce the necessity of the sense-making process of looking for connection and relevance, to a strictly utilitarian view of mathematics. It would be difficult to enjoy anything if you always looked for the immediate benefit. This attitude hindered further innovation and creativity.

Teaching Mathematics *for* Problem Solving: The Incrementalists' Perspective

The incrementalists defined "teaching mathematics *for* problem solving" in their own words which were based on their beliefs about mathematics and mathematical problem solving. They wanted me to teach the concept first, solve a few problems on the board step-by-step, and then give them more exercises to practice the method of solving similar problems. However, this approach is more flexible than "traditional teaching" in that students are involved in the meaning-making process but to a limited degree. The main characteristics of the "teaching mathematics *for* problem solving" based on the incrementalists' views were as follows:

-A "balance" between traditional instruction and metacognition-based instruction

-"Discovery-learning"

-Active participation in small groups

-"Guided inquiry"

-"Time" is wisely spent

-All problems should be solved in the class

-Mathematical connection is an important part of mathematical learning

-The teacher's role is more like a mentor than the sole authority in the class

Summary

The incrementalists were in a quandary situation. The majority of them like Jim and Beth solved the conflict by proposing "balanced instruction, "meaning to adopt some aspects of traditional and some aspects of the metacognition-based instruction. Nevertheless, the other incrementalists were not able to resolve the conflict easily. The instruction did not change their world views, although it influenced their views or at least made them re-think teaching and learning mathematics.

The incrementalists utilized their journals to discuss both cognitive and metacognitive issues. They conclusively benefited from working in small groups one way or the other. However, they barely commented on the whole class discussion. They talked about mathematical connections at great length. Although most of them were not happy about writing the exams, only Keith, Lilian, and Shirley reflected on this and discussed it in their journals. The incrementalists did a great job in elaborating on the exams and answering the metacognitive questions which were part of the second and third quizzes. They were reflective in their problem solving, and their journals and exams provided an opportunity for us to communicate with each other in a reflective manner. The reflective communication helped them to become more aware and acquire better managerial skills.

Findings from the Third Category, the Innovators

I think math has been taught like 200 years ago, militant, deductive, no discourse. All we are trying to do is to find the right answer. It doesn't really matter anything else, how to get there. That's a very Victorian attitude. *(Marris's pre-interview)*

The nineteen students in this category, the innovators, were all influenced by the metacognition-based instruction to different degrees. For some of them, the instruction was a way of teaching and learning mathematics that they had always asked for but never got. For the others, they did not mind a different way of approaching mathematics that could help them understand the mathematical concepts better. Although the innovators did not reject the new approach, there was a considerable uncertainty and frustration among several of them. It took longer for some of them to see the effect of the instruction on their learning. The majority of innovators wrote about how the instruction helped them to gain more self-confidence and self-worth. They conclusively reflected on the role of the journal writing in their mathematical learning. Many of them wrote about the effect of the written and oral feedback on enhancing their understanding, self-confidence, and self-worth. The classroom environment was a major theme that emerged from this category. Unlike the traditionalists and incrementalists, the innovators frequently wrote about the effect of the classroom environment on their learning. The establishment of the social norm of the class was partly due to the constructive suggestions that the students made through their journal entries.

This section gives details concerning the different themes that emerged from the students' work in this category. A student-based definition of "teaching mathematics via problem solving" will be given later.

The Instruction and its Components

For a number of students, the instruction was the way of teaching and learning mathematics that they had longed to have. They always wanted to be involved in the meaning-making processes in their mathematics classes. The innovators' desire was to construct meaning by themselves and enrich their understanding by seeking answers to their "whys." This had not necessarily been encouraged in their past: "I used to ask [my grade 4 teacher] 'but why?' The teacher said: 'It doesn't matter why Melisa. Just do it as I did'" (Melisa's autobiography).

The instruction for most of them was "all new." Melisa wrote in her May 21 journal entry, "I realized a while ago that part of the thrust of this class isn't just us learning about math. This class is also a model of a teaching method. From my perspective this is all new" (Melisa, May 21). Nevertheless, experiencing a "new" way was what Patrick was afraid of: "The concern I have with this course stems from an experience my mother had at UBC. . . . She nearly failed . The course approached music in a way *she was not used to*. I hope I do not have similar problems" (Patrick's autobiography). However, Barbara was curious: "I find the approach so different. I am very interested whether it will help to overcome the difficulties I have in math. I am typical of the poor math student" (Barbara, May 10). Furthermore, her curiosity was gradually replaced with endorsement. "Alternative methods that I had to try to bring about my understanding were suddenly acceptable as ways of doing problems. There was no longer just <u>one</u> correct way. Talking to one's peers was also acceptable" (Barbara, May 13-17).

The innovators were able to make a comparison between the "new" instruction and what they had already experienced in their schooling. "I didn't find Roy's teaching method all that wonderful. I felt like I was back in high school. I much prefer to discover stuff for myself. It takes longer but I think I understand better in the long run" (Melisa, May 30). The innovators were therefore more influenced by the instruction than the traditionalists and the incrementalists, since they somehow got what they were looking for: "It's too sad that schools don't teach kids to think about <u>why</u> they are doing what they are doing in math . . . it makes it more accessible and understandable to more students. . . . I know when I went to school . . . <u>the</u> right answer was the only important thing . . . and I <u>never</u> did well at it" (Clara, May 22). Barbara mentioned the role of interaction, "Thus it was a delight to discover the new approach to teaching math in a concrete and more individualized way . . . not only through direct teaching but also through interaction with materials and their peers" (Barbara, ?). Clara discussed the possibility of having such instruction in the elementary schools:

> This new way encourages personal thought and I like that. I can really see the benefit of this approach to math in the classroom . . . from the very beginning. Children don't need to memorize facts and procedures . . . rather they need to understand why . . .. We are having to explain the why of things . . . The why is really important . . . this really makes it different from a high school course . . . (Clara, May 22-26)

Being encouraged to ask "why" and find answers provided an opportunity for the students to overcome their "fears" of mathematics: "It's wonderful the way the [my] fears are being put aside via being taught the 'whys' of math. . . . That's what I really like about this class, nothing is accepted because a book says that it is true" (Melisa's last journal entry).

Some of the students were concerned but hopeful that the instruction would work for them. "The first class, was different from what I expected. I assumed it was going to be the same as we took in high school. Instead it was a fact finding mission. I'm not sure I understand the significance of it. It is interesting and *different and not unpleasant,* just different. I just hope I get it all" (Sheila, May 7). Sheila's positive feelings helped her to be patient with this "different method." "I still find the method unusual and different than what I'm used to. I'm used to structure. I realize that this class makes you think on your own. Great concept! but again, I have to learn to be patient" (Sheila, May 8th). However, her patience was fragile, "I didn't understand what was expected or what I had to do. The more I got frustrated, the more I gave up. I don't like feeling this way. I'm not sure how to deal with it. Maybe I'm just not too smart when it comes to math" (Sheila, May 9). The main purpose of the instruction was to alter students' old beliefs--not to reinforce them. I wrote to her: "But your feeling has nothing to do with your smartness. You got your feelings from your experience. As you said, you are not 'used to' this way of learning." Her frustration was a legacy from many years of "bucket filling" teaching, "tell me what to do, I will do it." The question was whether the instruction could help her to "deal" with it or not: "I'm still a little concerned about what is wanted" (Sheila, May 14). She was still struggling with her previously held beliefs that were challenged by a different "method." It took a while for Sheila and some others including Peggy, to feel confident with the "new method." Peggy wrote on her May 14 journal entry, "I am finally beginning to feel a little more confident in the class . . . though there are still times when I feel I am 'sinking.'" The innovators tried hard and gained a lot. They were active in class and their contributions to the small groups and whole-class discussions were highly appreciated. The innovators wished they had learned mathematics through problem solving from the beginning: "I wish I had learned math this way the first time" (Sheila's last journal entry). Perhaps it would have helped them develop a stronger mathematical background. Peggy explained her difficulty in the absence of a strong mathematical background: "My problem, I can see, is my great lack of math background, so I have nothing to draw on. I can now work out *what I need to do,* but I don't have the math knowledge to do this, which I find frustrating" (Peggy, May 22). Peggy found an answer to what she needed and why, but she did not have enough resources to know what frustrated her. However, she was getting better at experiencing success in the course, since her awareness and her confidence were increasing. She was able to identify her emotions towards mathematics, her knowledge about it, and most importantly what she would need to deal with these issues. One of the contributions of the instruction was to help the students gain more confidence and feel more comfortable as the course progressed. Reflecting on Paul Cobb's video presentation, Peggy explained how that happened.

> The lecture [Cobb's presentation] yesterday was very interesting and I can now see why you have been "instructing" us in a similar manner. The idea that we "discover" for ourselves . . . rather than being told a way, to do it this way also involves more discussion between people in groups. Also people solve problems in different ways. I certainly feel more comfortable in this type of situation and not afraid of making mistakes but trying different ways, though at times I still get frustrated! (Peggy, May 22)

Sophia pointed out a fundamental difference between this "new method" and the "traditional math class" as she called it. "It is certainly not like a traditional math class as I remember them from high school, when *the instructor would work out long equations on the black board and all the students would follow along and copy and accept things to be true, as given.* It seems that in this class we are given a lot of time to discover things ourselves and draw conclusions. I'm finding that it's a more creative and understanding way of learning," wrote Sophia in her May 8 journal entry.

Traditionally, students would do what Sophia described: passively take notes while the instructor was "lecturing" without much interaction. Most of the questions raised were for clarification of a few minor things that students were unable to read on the board or did not hear properly. Ben was not at all sure about the way the instruction was developing away from the "old way." "I know it is good to *ask questions for clarification* but I really believe a lot of people would benefit from taking a couple of minutes to look at the example, think about it, and reflect on what was done and why" (Ben, May 13). They said that dialogue and debate were not usually a part of traditional teaching and students did not play a major role in the meaning-making process. However, as the course progressed, he saw the benefit of interactive instruction and said, "I liked Roy as a teacher, very interesting. But I felt like I was back in grade 13. The teacher doing the talking, the *students just copying the work down*" (Ben, May 29th). In such a setting, he would listen, "copying the work down", and only ask questions "for clarification," not for further development and understanding.

On June 3, Sally did what Ben experienced in his grade 13: took notes without being really involved in the class activity. However, she did not like it and reflected on that in her journal. "Today, I felt like I was writing down notes frantically without understanding what I was writing. So now, when I reread my notes it is very difficult to translate them into something comprehendable" (Sally, June 3). What Sally did was important. She tested the efficacy of what she called a "new method" against what seemed to be a trend in most of the mathematics classes. Sally realized that she could comprehend more with the "new method" and was glad to find out that the "new way of teaching was being carried out in elementary schools." By reflecting on Paul Cobb's video presentation, she explained why she thought the "new way" was more productive.

> I had no idea that this new way of teaching was being carried out in elementary schools. And, I am very glad that it is because the way I was taught math was not a success (I've always been afraid of math, and not good at it). Therefore, since a lot of people seem to feel this way, it is good that the teachers are opening up to new ways of teaching . . .. There . . . will be people who are not looking far enough into the future to see the benefits of this method. These people might get angry at the fact that the teacher does not simply tell the children to memorize. But, I feel that it is important for children to be able to *think* for themselves. I know that in my math history I could never do this, and never really had to. Therefore, I find it hard now (in this class) to *think out* how to solve a problem on my own. (Sally, May 21)

The way Sophia dealt with the instruction was interesting. She was able to clearly discriminate between two different approaches to instruction. She then internalized the instructional processes and wrote more specifically about how that worked for her. The following excerpt shows how she and others in her camp empirically concluded that the instruction would work for them. "I was very pleased by the way that this class progressed *for me* today. . . . The problems assigned to us . . . gave our whole group a good sense of accomplishment. I think I'm now getting used to the methods of learning that we'll be using. At first I was feeling like it took us an awfully long time to grasp just one idea, but as long as we're progressing at the rate expected, I guess that's fine" (Sophia, May 9).

The innovators were more critical and more observant about issues regarding the instruction. They talked about a variety of issues concerning the course and outside of it. For example, almost all of them reflected on Paul Cobb's video presentation and compared it with what was going on in the course:

> The video and discussion on math teaching methods was very interesting. I am glad I have experienced at least two weeks of this method as it is completely new to me . . . I can see there are many things being learned including social skills such as listening, sharing, communication, working in groups without intimidation, etc. The method seems to be time consuming and I feel a teacher would have to be very careful to maintain the goal of the lesson as well as dealing with unexpected points of value that come up . . . I feel it is important for children to work on their own as well . . . we need to learn independence and self reliance. This method is so much more fun than the old way . . . So briefly, I think a combination of methods may help. The final analysis is that an effective teacher has to be able to use various methods and to know her students well. (Rita, May 21)

Rita was the only innovator who talked about variation in methods. However, what she said was slightly different from what the incrementalists said. Her main point was flexibility and spontaneity in teaching. I told her that I truly believed teaching was an art and there was no "method" written in stone to be followed by all teachers, and teachers have to be flexible enough to switch from one way to another depending on situations and students' needs. However, it does not mean that teachers can switch their world views.

The majority of the incrementalists asked for more "structure" and expressed their concerns about "time." However, except Sophia and few others who sometimes thought that discussing problems took longer than they expected, Ben was the *only* innovator who was *worried* about it. "I think there should be more structure so we can go through them quickly and get on with the day's new work" (Ben, May 15). I pointed out to him that there was no reason for him to be worried about time. He knew the final exam would only include the material covered. I wrote that Ben should stop to ask himself what would be the importance of "the day's new work" if he did not develop much understanding.

In the beginning of the course, Penny said she preferred to be told by the teacher what to do. "It wasn't clear to me, what the teacher expected from us . . . I did not know what she [teacher] wanted us to do" (Penny, May 8). However, her last journal entry showed that she changed her views about the instruction as the course progressed. "The part I enjoyed the most was working with actual figures [manipulative/models] instead of the pen and paper method where the *teacher is the only one that taught.* I liked the way that we taught each other things" (Penny's last journal entry).

The students were pleased to learn how certain formulas were derived. "This course is so neat the way it has gone behind the formulas we used in high school without questioning, and now we are learning how they actually came about" (Bob, May 23 journal entry referring to calculation of π). He talked about mathematical connections and the purpose of teaching mathematical concepts to young children and adults. Rose wrote about the emphasis on applying a formula to a certain situation and finding "the correct answer" as the main activity in her previous mathematics classes: "I remember some of these [mathematical concepts] were taught when I was in high school. . . . We just applied the formula and found out the answer. We would never know or understand why the formulas are applied and worked out that way. But now we do not concentrate on merely getting the correct answers. We need to understand what these formulas mean and discover how they are set" (Rose, May 24).

"Models help me to do better. When I have got them in my hands, I can do way better," said Penny in her post-interview. The role of manipulative materials was significant in making sense of the formulas being developed and used. "I must admit that this afternoon was fun. To have the 'hands on' experience was excellent and much more meaningful than trying to understand the theory by writing down information. I hope many of the lectures will focus on the 'practical' aspect of math" (Peggy, May 7). Roslin's comment on the role of manipulative materials was representative: ". . . Another implication is that in solving this kind of complicated problem, it is vital to have the object in hand. Then you can have a clearer idea of what it looks like. It is even better to draw it or make one ourselves. Then we can feel it, reset it, or rearrange it to understand better. . . . In other words, we have to look at the object perspectively [sic] and imaginately [sic]" (Rose, May 24).

The teacher's role.

The traditionalists wanted me to direct them all the time and tell them what to do. As a matter of fact, they considered limited "lecturing" and my non-authoritarian role in the class as a weakness rather than a strength of the instruction. However, the innovators had different views about the teacher's role: "Even the instructor, from time to time, is found emerging [immersing?] herself among the groups, giving the appropriate limited guidance hints. She does the least talking and instructing. In fact, there is no lecturing" (Rose, May 10). A number of innovators elaborated on this issue. They liked to "do it" by themselves. "This was a neat lesson because the teacher could have merely told us this fact but instead she let us 'do it'" (Linda, May 7). Rose's observation about the teacher's role is representative. She wrote more specifically about traditional and metacognition-based instruction in general and the instructor's role in both settings in particular:

> It is not in the traditional and conservative way as we learned before, using and memorizing the multiplication table, or being told by the teacher what was right or wrong for the answer. . . . The teacher is not just going to sit back and relax. In fact, she has an important role to play in helping and advising the children as required. She is like a <u>guide</u> or <u>consultant</u>, going round the pairs and listening in. She observes their difficulties and mistakes, both individual and general problems. In the light of this, she will be able to shape both in the class discussion later and activities in further lessons. . . . I, personally, advocate this method and I think it is a worthwhile attempt. After all, as E. Stevick says: "Students are not a power supply, in such a way that the power used by one diminishes the voltage available to the rest." (Rose, May 24)

Journal writing.

All the students in this category considered the journal writing to be significant part of their learning. Many of them received individualized instruction through their journals. However, in the beginning of the course, there was confusion, uncertainty, and even resistance towards keeping journals in the mathematics class. "I disagreed out and out with the use of journal for this course. I felt that such a tool would be useless in our situation. What good would writing our feelings down about math, for anyone except make them feel worse?" wrote Melisa in her "critique of the course." She then explained how the uncertainty and resistance phased out and she was soon able to see the benefits of keeping a journal. "I began with reluctance, and found it difficult to do. But then I realized that I could write about the things I did not understand and in doing so I further realized that things became more clear if I did this. I utilized my journal to solve problems" (Melisa's last journal entry).

Each student benefited from journal writing in a unique way. Clara wrote about her own writing experience, "I guess that writing the journal will be helpful because in thinking about what I was going to write . . . I suddenly realized that a few things had become clear" (Clara, ?). Journal writing for Ben was a means of self-assessment. "The journal was my way of assessing how well I understood a new concept. It also provided me the opportunity to express my views on different events/occurrences in the classroom" (Ben's last journal entry). For some people, journal writing served as a review and organizer. "It [journal writing] has helped me to review what we have been doing, and sort things out in my own mind, this I have found very helpful" (Peggy, June 13).

Sandra was the only student in the entire class who asked to have more than 30% of the final mark for journal writing, but no one agreed with her. Her journal was extremely lengthy and detailed, it even had a "table of contents"! She wrote, "Personally, this journal has been a guide for me which has aided me in focusing on my strengths and building my confidence in math" (Sandra, June13). In response to my request for the self-evaluation of their journals at the end of the course she wrote, "As my journal . . . shows my progress and development and the effort I have put into it, I feel I have earned a mark of $28/30 = 93.33333...\%$! which is a rational number, sorry no time to give you a proof!" (Sandra, June 13) I promised her that "I'll be rational too!"

The innovators were the only ones who wrote extensively about my written comments in their journals and how that feedback helped them to develop a better understanding of mathematics.

> Feedback is likely to enhance students' motivation because it allows them to evaluate their progress, to understand the level of their competence, to maintain effort toward realistic goals, to correct their errors and to receive encouragement from their teachers. Sometimes it even develops a better understanding between the teacher and students because they can communicate to each other more deeply if the situation is not allowed in class due to lack of time and opportunity in the classroom. Thus it creates friendship and fosters a better relationship between them.
>
> (Rose's last journal entry)

Jack talked about an important aspect of the written communication between the students and the instructor. ". . . The feedback is important because the instructor needs to know the areas of concern of the student, and the student can be helped through written information or comments by the instructor" (Jack, June 10).

Journal writing has become a bandwagon. Many students shared a negative experience regarding journal writing in some courses. They were asked to keep their journals in many classes, but instructors did not really go through them in detail and what they usually got was a "check mark" on their journals. The fact that I responded to their journal entries pleased them enormously. Rose explained:

> Being a teacher for many years, I know it is much easier to give grades and marks than giving some written feedback on a piece of work. Besides, it is very time-consuming and painstaking. To me, the journals we write is just like the channel for communication and the feedback we receive is more than an extrinsic kind of motivation, like marks, grades, school reports, test result and teacher approval. There is understanding, assurance, comfort, and console as well. Success at it helps build up a prestige in my own eyes and in the eyes of others. (Rose's last journal entry)

I wrote to her, "I truly believe that students don't learn from their grades, rather they learn by reflecting on their doing and understanding. I hoped that the feedback would serve such a purpose, to open up a dialogue between the students and the teacher and to create a more friendly environment that learning is fun and natural rather than being boring and artificial."

For the majority of the innovators, writing was a personal endeavor that they enjoyed doing. They did not take it as "another requirement" to be done, as the traditionalists did:

> I really liked writing in my journal. It helped me sort out my own problems and thoughts, and provided a form of communication between you and me. I appreciate that you took the time to read our journals and respond to them all! You seem to care about how your students are doing in your class (whereas practically all of my previous prof's don't seem to care).
> (Sally's last journal entry)

The students' comments helped me to know the students better and become more aware of their needs. They also made constructive suggestions in terms of instructional decisions. "So it seems I would like it a bit more if you (Zahra) would sum up the ideas or problems that we've been working on at the end of the class" (Sophia, May 9). Such comments made me more conscious of the class needs in general. I carefully considered them in order to improve the learning environment. As Peggy mentioned, this was a "two-way exercise" in which both the students and the instructor were actively involved in the meaning-making process. Peggy's observation was representative:

> I had always thought that the comments you wrote on the journals were good, relevant, and showed that you obviously read them and cared. Also that you were concerned for us as people not just bodies in the class. For me the comments were encouraging and helpful. I am very grateful for the time and effort that you put into your comments. If there were little or no comments that would make me feel that you weren't interested or maybe didn't even look at them. So therefore why should I put time and effort into the project. I think that it is a two-way exercise, if you show interest then I will put the effort into doing something. (Peggy, June 11)

Small groups and whole class discussions.

"Another significance of this course is that activities dominate the entire two-hour lesson. Learning is not a passive process. If students are to learn, they must be actively engaged in doing, not just listening but comprehending" (Rose, May 10). However, a few students like Sally initially had reservations about the group work. "I am quite nervous about taking the class, especially since we have to do our work in groups so everyone can see how much I don't know" (Sally's autobiography). Soon, as the course progressed, they realized that they should not have been worried about "not knowing" since no one was a "real whiz." "I like working in groups, that's been very helpful and it makes us all feel like we're in the same boat and that none of us are real whizzes" (Sophia, May 9). Besides, they all had something to share. "Talking about it [mathematical concepts and problems] in groups helps enormously. We all have something to contribute even if it is only our ignorance" (Barbara, May 10).

The innovators actively participated in the small groups and the whole class discussions. For them, the group activities were crucial components of the instruction and not just the classroom routines. "It [the course] taught me how to work in groups, sharing ideas and suggestions. While I have done group work in many different classes, the work done here was much more intense and definitely thought provoking. Concentration and patience skills were also worked on in class," wrote Ben in his last journal entry. The group work helped the innovators to develop a deeper understanding of mathematics. "It was not hard to figure out the mechanics of it [a problem], but I didn't really know why I'd done it, but as we talked about it in class . . . it became clearer" (Clara, June 10).

The group work provided an opportunity for the students to hear different ideas and discuss every possibility. "I like the way you are going through each possibility and making it clear why one explanation works and the other doesn't" (Barbara, May 31). The discussion helped them realize their conceptual difficulties as well. "I would not have seen my mistake if there had been no class discussion (Patrick, June 12).

Peggy explained how the cooperative nature of the group work helped her group members with their quizzes. "It was good to work as a group because we generated the idea together on how to solve it then worked on our own to find the answers . . ." (Peggy, May 28). Sophia wrote how a diversity of ideas enriched their discussions and increased their understanding. "Our group is working well because it seems that different people are good at different things and we've all been able to help each other out" (Sophia, May 23).

The small groups had the potential to make some of the students dependent on the others. However, the best way to overcome that possible difficulty was to resolve this within the groups. The innovators gradually became more aware of their responsibilities and took the matter in their hands. Rita was an enthusiastic and active participant in all the class discussions. Her group worked together very well. She and her group members reflected in the second quiz on what that the students did in their groups and addressed this issue. Rita's reflection was coherent and typical:

> We received the quiz question and very shortly I suggested that we find the difference in area left unused . . . The others [in group] agreed that this seemed logical so we proceeded to do that. I didn't even know I was thinking about it until I woke up in the night as if something had bit me and the thought in my mind was oh no! we didn't go far enough . . . so we should have gone one step farther to show the ratio of square/circle to circle/square. . . . I felt really stupid and embarrassed that although I had a good suggestion, I was too careless to carry it far enough. But then I also felt a bit defensive in that one of the others could have cued in too. Group work can be stressful if you mislead someone. (Rita, May 27)

Rita raised a number of interesting issues in her journal entry. The main point was the way in which she reflected on what she did in class. Reflection on what went on in the class gave her a chance to have more insights about the problem and be able to see where she went wrong and why. Another point was the role of the small groups. She felt "stupid and embarrassed" because she did not solve the problem properly. However, every member had the right and duty to argue with others in the group if something did not make sense. The group work and the regular reflections helped them to realize that group work meant sharing ideas and not following a leader. As Rita wrote in her May 21 journal entry, "We need to learn independence and self-reliance."

Whole-class discussions helped the innovators to become better decision makers since they were constantly confronted with different ways of doing things. They had to decide which ways would make more sense to them and why. "On Tuesday we started learning about factorials and the class went crazy! It was all rather incredible to see a roomful of people who were perceiving a situation in two quite separate ways, and to further see how difficult it was for each side to explain its view to the other" (Melisa, June 5). The discussions helped the innovators to see that there was not only one "correct way" of doing a variety of problems but several ways. The discussions gave them a chance to examine a diversity of approaches to same problems. They saw a host of reliable and "correct" ways of solving mathematical problems that were all different but "correct." Barbara's response to my comments in her May 13-17 journal entries was interesting: "Perhaps I wasn't stupid after all, that problems I have experienced in trying to learn math may have been as a result of thinking that there was only <u>one</u> correct way to solve a problem, and that everyone (or most people) somehow had access to that method while I didn't."

The group work, as Barbara mentioned, helped the students to alter their views about themselves as doers of mathematics and to overcome their fear of mathematics. In addition, Linda re-emphasized Barbara's views when she said, "I was dreading this class . . . but it turned out to be quite fun. It wasn't as difficult as I thought it would be, but I think the sole reason for this is the class from the beginning developed a cooperative nature. By this I mean everyone felt insecure with math but we all shared our knowledge together to come up with the answers" (Linda, June 13).

Melisa's "in-depth evaluation of the course" summarized the different feelings expressed by different students in this category about the role of group work in the students' learning. Therefore, I would like to end this section with an excerpt from her work:

> I was extremely taken aback by the suggestion that not only was co-operative learning ok, but that it was <u>required</u> for this class. I felt insulted, because it's seemed to me that co-operative learning was for people who did not understand, or who were <u>not</u> clever. The first week I stewed when

> someone asked me to explain something until when I realized that I didn't think any less of them, and that they understood my explanation. I turned immediately and asked someone to explain something I already understood. This was illuminating because I got to understand the problem in a whole new manner, which enriched my understanding. I would advocate that co-operative learning was essential to a student's success in this class. You learn so much faster this way, and I think you learn more. It is quite exciting intellectually to bounce far fetched ideas off each other only to find out that your idea isn't so far out and that other people have similar ideas. . . . Talking out our ideas facilitates our understanding.
>
> (Melisa's last assignment)

Class environment.

One of the main characteristics of this category was the students' views about the instruction. The innovators did not consider it as a set of teaching strategies to be adopted separately depending on the circumstances. They thought about it as a new way of approaching the teaching and learning of mathematics. The innovators realized that along with the instructor, they played an important role in the meaning-making process. They also were aware of the importance of the classroom environment in metacognition-based instruction. The "relaxed" atmosphere let them feel comfortable. ". . . A very 'friendly and warm' class; one in which I felt very comfortable and at ease. I was afraid of being 'the only one who didn't understand' but this wasn't the case in this situation" (Peggy, May 7). The "friendly" class gave them a chance to become more confident: "The classroom atmosphere was very relaxed; you are eager to help us understand, and it was fun to learn in groups. Also, I really like the way you don't pressure us in any way (i.e. you don't ask people to come up to do their answers on the board, instead the people volunteer) . . .. This relaxed atmosphere allows me to admit to not understanding something (without feeling like an idiot) so that something can be done so that I do understand" (Sally, June 13). This was an interesting expression by Sally since she was "nervous" about working in groups in the beginning of the course. This was certainly a change in her beliefs about herself, and she attributed the change to the relaxed atmosphere of the classroom.

A number of innovators, among them Clara, apologized to me for the lack of order in the class: "However, I think in a sense, with these hands-on experiments and opportunities to discuss problems, we [the students] sometimes forget our manners as we become excited . . . So I would like to apologize for our apparent rudeness . . .. It really is excitement in a sense" (Clara, June 1). It was very considerate of her to feel like that, and I told her in any interactive situation this was inevitable. "I believe that this is a non-traditional way of teaching, therefore it requires a non-traditional class environment. Besides, it is not "rudeness" and you shouldn't apologize for that! I take it more as "excitement" as you said. I think if we accept to do group work and whole class discussions, we shall gradually establish the *social norm* which is suitable for this kind of learning environment . . ." (my comment on her June 1 journal entry). The reality was that all of us together helped to establish such a "social norm."

<u>The Students' Beliefs About Mathematics and About Themselves as Doers of Mathematics</u>

> *I suppose one of my biggest gains from this course has been an increased appreciation for what mathematics is all about.(Bob's last journal entry)*

All the innovators had experienced some difficulty concerning mathematics with the exception of Patrick to whom "math always came easy." The innovators' beliefs towards mathematics were intertwined with their beliefs about themselves as doers of mathematics. Some of them did not do well since they believed they were not capable. Barbara's story was representative. "I am a mature student. . . . I have always thought I was deficient in my thinking about math . . . my thinking processes are good in other areas . . . my academic experience with math is abnormally low" (Barbara's autobiography). "I think that I thought of my brain as split into thinking about words and thinking about numbers and that somehow the latter part was damaged" (Barbara, May 6-10). Her effort in doing well in mathematics failed as she developed such a belief about herself, "After taking math for a few years I stopped believing that I could do it and stopped trying" (Barbara, May 28). Later in the course she explained why: "I was afraid that I couldn't solve any of the problems. . . . Eventually I approached all problems with the secret belief that I couldn't do them, that I had to fail because I was *stupid* in math" (Barbara's last journal entry).

Furthermore, the instruction helped Barbara to change her beliefs about herself as a doer and learner of mathematics.

> I think I have always equated slowness in assimulating ideas with lesser intelligence. I think this belief is generally held in this culture as well. . . . But if we all end up in the same place but at varying times, if our learning styles are met, then ultimately what is the difference. . . . I have since discovered that I am capable even if I am slower than most.
> (Barbara's last journal entry)

The major gain was that she no longer considered her performance (marks) as the sole indicator of her capability regarding mathematics:

> I would like to say that regardless of how well I do in this course, I think that I have learned a lot, the most important thing being that *I no longer believe that I am stupid or deficient in mathematics understanding*. I honestly never thought I could understand as much as I have . . . even though I originally hated the thought of taking this course, I have found it to be of great value. (Barbara's last journal entry)

A number of innovators did "poorly" in mathematics because they did not like it. "Math has never been my favorite subject, in fact I've always done poorly especially in high school. I am a classic example of the girl who, once she becomes a teenager, has great difficulty with math. I failed Grade 11 and Grade 12 Math. Therefore I took it in summer school" (Linda's autobiography). She did not explain why she considered herself a "classic example," but the point was that she had difficulty with mathematics and it was not her "favorite subject." Later on Linda explained that mathematics was difficult for her because she could not relate to it. "I think math can be fun (in the sense of challenging) *if* and only *if* I know what I am doing. Math is a difficult subject for me because sometimes I cannot see things visually, henceforth I get frustrated and give up." She then gave an example to clarify her point: "When I first looked at the figure . . . I was intimidated. But once I looked at the figure I realized that I could use my previous knowledge of the Pythagorean theorem to solve the question" (Linda, May 14). Being involved in the meaning-making process helped her to change her beliefs about mathematics. "This sure was a switch for me explaining math instead of sitting there getting confused and frustrated. I liked it, and I hope most of the lessons will work like that. *Then maybe I will not dread math and begin to like it*" (Linda, May 9). As the course progressed she wrote, "I enjoyed working on this [problem related to volumes]" (Linda, May 16); and in her May 23 journal entry, "I'm not totally in the dark, and I am becoming more confident with math." Her reflective writing showed that she gained more self-confidence when she was able to "do" mathematics, and that played an important role in altering her beliefs towards mathematics.

"I swore that I would never look at math/algebra again but I guess I was wrong," wrote Penny in her autobiography. Some of the innovators like Penny were afraid of mathematics and did not want to face it again because their experiences with mathematics were generally unpleasant. Clara and Penny described how the lack of success in mathematics left them fearful. "I guess I have math phobia. I am worried about taking this course and wondering if I will be able to understand it. My memory of math is that it was the first and only subject I 'failed,'" wrote Clara in her May 8 journal entry. It was difficult to start a course with these bitter memories. However, more mathematical understanding helped her to leave those memories behind. "It is with great fear and dread that I began it [the course], and while math is still not easy for me, a lot more is understandable that I ever thought!!" (Clara's last journal entry). More importantly, she changed her belief about herself as a doer and learner of mathematics. Clara realized that good marks were not the indicators of her mathematical ability. "Well, no matter what happens in this course, one thing that has been good about it is that it has made me look at why I thought I couldn't do math. I have discovered, in fact, that *I can do a lot more mathematically than I ever thought possible*" (Clara's last journal entry).

Dave realized that his involvement in mathematical activities would lead him to a better understanding of mathematics. "Coming into this class, I had real concerns about my ability to do math. . . . I still feel very 'shaky' about it. It was a relief to hear that this course is a 'hands on' approach which allows me to see and to do math, not just listen" (Dave's autobiography). Previously mathematics had been presented to many students as a purely objective and rule-oriented subject. Rose was surprised to see that "mathematics is not dull and boring as my first impression of it" (Rose, May 9). Dave was glad to hear that he could "see" and "do" mathematics as opposed to memorizing the facts and formulas.

> Coming to this class on May 6, I was very worried that this would be another math failure for me. It was a pleasant surprise to have many of these fears alleviated. All through elementary school I was terrified of math and would literally sweat when the teacher said to open our mathematics books. On May 7 we began to 'play' with a couple of puzzles and do a kind of math I have not really experienced before. It was actually enjoyable and calming to cut and tape in math class! . . . We spend most of the time doing the math. . . . This may be an incorrect thought but I get the feeling that we are free to experiment and come up with different answers. If they are wrong (the answers) then there seems to be a lot of time given for each of us to discover the answer. I like that! (Dave, May 7)

Doing mathematics as Dave described, was enjoyable for innovators, yet many of them had not done it for years. Clara wrote, "Which part of the brain is involved in mathematical thinking? I can feel the wheels creaking into action . . . long unused" (Clara, May 9). However, they did better when they were actively involved in the meaning-making processes and were able to "accomplish" something. "I was not frustrated because I felt I actually accomplished something (even though I was only able to complete one question)" (Ben, May 16). Furthermore the sense of "accomplishment" gave them more self-satisfaction and they gained more self-confidence. ". . . [I was] feeling quite pleased with myself and obviously having a great sense of achievement and satisfaction" (Peggy, May 16). As Bob wrote in his journal, "This course is so much like running long distance, it's incredible;" but he also said, "No gain without pain!!" (Bob, May 30).

Jack felt "uneasy" in the beginning. "During the first class of Math 335 I felt uneasy and somewhat *nervous*. . . . I feel my mathematical background is weak" (Jack's autobiography). Furthermore, the class activities helped Jack to become "easy" with mathematics and even "enjoy" it. "I am enjoying the work much more now because I am able to understand why specific questions give certain results" (Jack, May 23).

In general, the innovators held beliefs about mathematics similar to the other students in the class. Their beliefs were shaped by the ways in which mathematics had been presented to them--abstract with no practical aspect. Besides, some of them believed that there were people who could learn mathematics, there were others who could not, and that was the end of the story. "The instructor did well at making us feel that . . . *anybody* can learn math. It is not something you are born with as the *previously accepted notion that settled into my brain from high school.* I actually felt a little excited about math for the very first time" (Jack, May 7). However, some of the innovators were optimistic. As Ben expressed in his autobiography, "I am hoping that I will become more comfortable with the whole math process." The others were willing to try their best to do well in the course. "I passed the course (Algebra 11) with a great deal of stress and nightmares. However, that was ten years ago, and I hope that I have reached a higher level of maturity which will enable me to apply myself fully" (Carmen's autobiography). A number of them were even more persuasive. "This is so confusing, but I'm not about to give up yet. I've got this far so I'm sure if I persevere, I will make it through to a better understanding" (Ben, June 5). Sandra's "determination" was exceptional. "I am a visual learner. I know that writing about my difficulties and successes with math will be beneficial. I need to learn strategies and how to approach math. I am willing to learn as much as I can . . . I am *determined* to apply what I can to learn more about math" (Sandra, May 7).

The students' work showed that their beliefs about mathematics were shaped by the ways mathematics was presented to them. When Bob wrote about the influence of the teacher and instruction in his success or failure in mathematics was typical:

> In my background, math has gotten me very *depressed* at times as well as very *uplifted* . . . A combination of poor attitude (adolescent) and *questionable teacher*, resulted in my failure of the course [Grade 11 Math] and the need to repeat it in the following year. . . . I was quite *scared* to take Grade 12 Math to complete my university entrance. Well I had such a good teacher for math in Grade 12 that I ended up getting an honors mark at the end. After feeling quite *discouraged* about [mathematics], I ended up feeling very good and *positive* towards math. (Bob's autobiography)

Later in the course, Bob raised an important question about the nature of mathematics. "We are now into π and how it was *formed, developed, discovered;* what's the right word?" (Bob, May 23). I wanted him to respond to his question. "What do *you* think?. . . ," I asked. These questions were crucial, since different approaches to teaching and learning of mathematics were influenced by people's beliefs about the nature of mathematics. The students were actively involved in the formation, development and discovering of mathematical ideas.

The metacognition-based instruction provided an opportunity for the social and individual construction of mathematical knowledge. The students learned mathematical concepts by "doing" not "memorizing." The majority of innovators were apprehensive about geometry and probability in particular. "Geometry presents the greatest fear. I have always just memorized the necessary information and then copied it out without ever developing a true understanding" (Ben's autobiography). Furthermore, being able to "do" things in class and using manipulative materials helped the students to see the "fun" side of mathematics. "I was very apprehensive about taking this course, as I haven't done secondary math for over 20 years. I must admit this afternoon was *fun*" (Peggy, May 7). It also helped them to gain a better understanding of geometry and probability.

". . . Anyways, probability was quite a dark cloud over my head a week ago and now it's gray with a few sunny periods" (Bob, June 11, referring to whole class discussion and the demonstration regarding probability problems). Sophia said, "I really enjoyed the lesson on Pascal's triangle. It's really fascinating to discover the relationship between the numbers and how the numbers can be used to help solving problems" (Sophia, June 6). It was a pleasure to see the "fear" of mathematics gradually replaced by fascination as the course progressed.

Kent and Patrick, musician and stage player respectively, reflected particularly on their mathematical background. These two artists both expressed their appreciation for geometry. Kent wrote: "Geometry . . . I enjoyed very much and algebra . . . I did not enjoy. Perhaps the visual aspect of geometry and its seemingly more tangible characteristics was more appealing. Algebra seemed more abstract and 'on paper' and also a less creative endeavor" (Kent's autobiography). Patrick also found that mathematics always came easy to him. The similarity of their mathematical interests and their reasoning was interesting. Kent and Patrick were well aware of their likes and dislikes regarding mathematics. They were eager to learn and gain more mathematical power. "I think that my mathematical tower is increasing more in width than it is in height. It was very narrow before I started this class," wrote Patrick in his May 24 journal entry.

Melisa reflected on her beliefs towards mathematics prior to and after the course. "This was without a doubt, one of the more difficult courses in my university life. This was primarily because the class was math (and I had a hang-up about the subject) and that it was being taught very differently from any class I've been taught before. *I came to the class hating math, dreading math, and now I rather like it*" (Melisa, June 10).

Melisa's enthusiasm was phenomenal. At her work, she utilized the mathematics she was learning in class and she was excited. She wrote, "I found myself today at work, in the midst of doing a display, trying to figure out how much paper I needed to do a display, and at what angle to cut it. Well!! Pythagoras came right in and I knew exactly what/where to cut!" (Melisa, May 6-10). She then continued, "It isn't awful or scary, I like it! I am having some difficulty with working and doing this class, but I find myself thinking about what we are discussing in the class during work." Melisa regularly used her journal to extend the problems discussed in the class.

Patrick and Melisa were similar in certain ways. They were eager to extend the problems and concepts that were discussed in class. I was happy to help them develop new ideas through their journals. "I had a fantastic moment of realization over the weekend," wrote Patrick in his June 10 journal entry when he found out why in binomial situations the Mean and the Standard Deviations were $n/2$ and $\sqrt{n}/2$ respectively. I wrote to him, "Isn't that great to discover things like this? I really admire your perseverance, enthusiasm, and endeavor in learning mathematics." Patrick was also enthusiastic after he developed the formula for the area of an n-sided polygon by himself. He did an excellent job of extending what he knew and came up with a subtle equation: "Before this class began I did not know what a mathematician meant by *'this is a beautiful equation!'* I now know what is meant by such a statement. It has to do with the area of polygons and circles $A_n = (\sin \Phi_{360/n})\, 1/2n$" (Patrick's last assignment). Melisa told me that she was so happy to discuss the mathematical ideas in a family gathering that was dominated by many "male scientists!" (Melisa's post-interview).

One of the characteristics of the innovators category was that its members were risk-takers who welcomed challenges. They were persuasive and eager. As Ben said they were not "about to give up." Patrick's example was not unique: "Boy was I ever confused today--the matching test question. It's a scary feeling but I'm glad it happened. I now know the feeling of total confusion and I think that will help me with my teaching. I now appreciate what others go through. Peggy, Melisa, Ben, Rita, Barbara, Kent, and Clara all reported that they woke up in the middle of the night experiencing similar incidents.

The course offered the students a new perspective. The innovators' beliefs about mathematics were influenced by the metacognition-based instruction. "This has been the most unique course that I have taken. *I have never thought of math as being creative endeavor*, and yet I am convinced that what we did for last six weeks was definitely creative" (Carmen's last journal entry). "I suppose one of my biggest gains from this course has been an increased *appreciation* for what math is all about and how I need to bring this enthusiasm into the classroom," wrote Bob in his last journal entry. It was rewarding for Bob to extend his scope and to increase his "appreciation" for mathematics. Bob explained how class discussions and interactions helped him to "see the light." ". . . I've been amazed when I think back and just the way the communication took place was quite something. Math seems like that. You're in the dark not knowing what to do and then one word is said and the light comes on and it's like the darkness has completely disappeared" (Bob, May 13).

Carmen summarized her beliefs about mathematics prior to and after the metacognition-based instruction. Her summary was fairly representative of innovators' views of mathematics and themselves as learners of mathematics. I will end the section with it.

> What I am saying is that math should be meaningful. *It is meaningful, and it should be taught in a meaningful way.* Up until this course, *math had been a terrifying ordeal of numbers* . . . I couldn't translate words into numbers. That is what math is all about. It is the ability to see the universe speak the language of math. If you want to learn the secret of it, you must first learn to speak the language. This math class has served as a trip to another country. At first everything is foreign, and you ask yourself why am I here? Why could I have not gone to some place where they speak English and isn't so foreign, and feels more comfortable. But once you have stayed in the country for a while, the language becomes a little easier to understand. You begin to understand a bit of the language and become one step closer to understanding the whole culture of the country. That's what this course has been. It has been a chance to experience another foreign part of this world that makes up our journey. It has been nice traveling with you Zahra. Thank you; you have served as an English to

math dictionary. Without you I would have been really lost in space.
(Carmen's last journal entry)

The Students' Beliefs About Mathematical Problem Solving

The difference between a true understanding and knowledge of procedure is the difference between following directions on a map without any idea of where you are going or what the map really represents, and understanding what the map is and why the symbols are used in a particular way and also having at least some idea of where you are supposed to end up . . .. I think you can apply this analogy to mathematics problems. *(Barbara, May 13-17)*

Generally, the innovators *came* with the same belief about mathematical problem solving as the traditionalists and the incrementalists, that usually there is only "one correct way" to solve problems. Moreover, they believed that the main goal of problem solving was the "correct answer" and that it could be obtained by applying certain rules and procedures and plugging in formulas. This belief was shaped by their experiences regarding teaching and learning mathematics: "When I did math . . . ten years ago, we were given formulas and told to apply them in a certain way to arrive at some abstract conclusion" (Carmen's last journal entry). However, the innovators were distinctly different from the traditionalists. They were willing to try another approach to mathematical problem solving since they felt the "traditional approach" was not adequate.

However willing they were to try a different approach, it was not easy for some of them to leave behind what they grew up with. After all, it was easier for them to solve problems using certain formulas. "I am good at plugging in formulas and solving the problem, but when it comes to written problem solving questions, I have difficulty" (Penny, June 3).

The static view of mathematical problem solving and the dependency on formulas made Linda wonder whether there was *a* "solution" to solve *all* the problems. "Well, today was interesting. I was honestly hoping we would not have to construct geometrical shapes from other shapes but unfortunately I was wrong. I am absolutely hopeless . . . because I just cannot decipher them. Is it possible for the brain not to see where the *solution* lies?" (Linda, May 7). I was curious to know why she was thinking that way. "What is a solution?" I wrote to her and continued, "Why do you think that the solution lies in the brain? Don't you think that you are solving problems by going through processes rather than just calling them from your brain or your memory?"

The students' beliefs about themselves as doers of mathematics and toward mathematics and mathematical problem solving were all tied together. Change in one's beliefs about mathematics would result in a change in the beliefs about himself or herself as a doer of mathematics and towards mathematical problem solving. For example, they would not feel "stupid" anymore if they were not able to comprehend "one" way of solving a problem because they were aware that there are other possibilities. "Perhaps I wasn't stupid after all, that problems I have experienced in trying to learn math may have been as a result of thinking that there was only <u>one</u> correct way to solve a problem, and that everyone (or most people) somehow had access to that method while I didn't" (Barbara's response to my comments on her journal entries for May 13-17).

Most of the innovators liked the practical approach to mathematical problem solving. They became interested in doing things. "I feel better about the math course today. . . . I could actually <u>do</u> some questions. Of course one of the big discoveries is that there are different ways to prove things . . . I like the <u>hands-on</u> approach!" (Clara, May 9). The innovators' involvement in class activities facilitated their understanding of the processes of doing problems rather than memorizing facts and formulas. "I feel that without understanding the ideas and concepts of math, the calculations won't make sense. There is no point in memorizing a formula or concept if it is not initially understood" (Sandra, June 11). Barbara wrote about how it would feel applying formulas without really understanding the processes. "If I were doing a problem by rote with a formula, I would do the first few steps on a journey to a destination. But I would always reach a point where I couldn't take the next step because the way was dark. I just couldn't make a connection between the previous steps and the next one" (Barbara, May 21-24).

The innovators, unlike the traditionalists, became eager to find out "why" they did what they did. It was no longer sufficient for them to learn "how" just by following what the instructor told them to do. Barbara reflected on her previous experience with mathematical problem solving. "I understood how to do things but not why I did them. And I often forgot how to solve problems or mixed various methods up, because I'd forget the procedures and formulas or get them mixed up" (Barbara's responses to my comments on her May 28 journal entry). Therefore, they benefited more, as Peggy explained, when they were actively involved in the sense-making process. "I think this is much more beneficial to students than 'learn by being told,' as you remember much more when you actually do it yourself" (Peggy, May 7). Learning about different approaches to solve problems helped them to gain more confidence. Sandra explained, "Since I feel more confident and sure of myself when I approach a math question, I don't get fazed as easily as when I started the course. As I don't get as easily frustrated as compared to at the beginning of the course, I find it interesting to learn how other people approach figuring out the areas" (Sandra, May 17). Their confidence helped them to become better problem solvers: "This is the first math course in which I have been starting to feel confident. I can do mathematical problems that would have been inconceivable a few years ago" (Dave, May 13).

Most of the innovators were amazed to see different approaches to solving the problems. As Jack said, "It is amazing to me that there are so many different ways of solving problems" (Jack, May 23). Yet, Carmen observed, "Everyone tackled the problems in different ways" (Carmen's last journal entry). Learning about different ways to solve mathematical problems broadened their perspectives. "I know now that there are many ways to get the same answer and it is not just a matter of using a formula. One has to know <u>why</u> they are doing something and not just plug in formula after formula" (Linda, June 13). The more they actively participated in the meaning-making processes and experienced various approaches to solving problems, the less they became "stuck." As Barbara explained, "For me, I don't seem to get 'stuck' in the way that I did earlier in the course. . . . I think that is because I . . . read the problem carefully, write all the important information down, relate the question to what we have been learning in class, try to simplify the problem" (Barbara, June 6). The innovators acquired managerial skills and became better decision-makers when they were confronted with the many approaches to solving problems, and they gradually became better problem solvers. Peggy's work was typical. "It took quite a while to work out what the problem was, what I already knew, and how I could solve it!" (Peggy, May 14).

The innovators enjoyed learning mathematics through games and puzzles: "It's been really great to learn by doing puzzles. . . . *Is this really math??*" (Sophia, May 16). Mathematics through games and puzzles helped them to get away from a view considering mathematics to be everything but *fun*. The games and puzzles also helped them to understand the mathematical problems better. "Calculating puzzles is a lot more fun than the traditional pen and paper method. Cutting and pasting the different shapes challenged my intellect to a certain degree, and at the same time made me learn more about geometry. By being more practical allowed for more learning" (Penny, May 8). Rose reflected on the creative aspects of mathematics. "As soon as I entered into the classroom, I found it was like a handicraft lesson. . . . After all, I have found the lesson is full of fun and creative and it can be introduced through games, puzzles or paradoxes. But most important of all, they are not used only for entertainment but to draw the learners to the acquisition of the fundamental ideas of math" (Rose, May 9).

As the course progressed, the innovators became more aware of their own thinking and learning styles. Their managerial skills improved tremendously, and their beliefs about mathematics and mathematical problem solving changed considerably. The students' work showed that the metacognition-based instruction helped them to gain metacognitive knowledge and become better decision-makers. "My problem is that I have trouble deciding which way to attack the problem, which method to use, but after working through some of the questions on the sheet you gave me . . . it is beginning to sort itself out in my mind" (Peggy, June 6)

A number of innovators solved their final exam problems by reflecting on what they did, why they did it, and how. This was a big achievement considering their disappointment at having to write a "stressful" final exam. Kent's and Jack's examples were illustrative. There was one problem (B5) on the final exam that was new. That meant that the class had not gone through a similar problem. Kent chose this problem. The ways in which he solved it were interesting. He wrote questions that he asked himself, "Will this help me? Can I use this?" He tried to reason it out and then decide. (Appendix S contains a copy of his work.)

Jack (referring to the same problem) first got the shaded area using the Pythagorean theorem involving long calculations three times. He then crossed all three out and gave an elegant solution and wrote, "Mark this." Jack was very pleased about it and after the final exam he told me, "This is good for your study," and laughed. These two solutions showed his awareness and his managerial skills to help him come up with a new and better solution.

The instruction provided the students with an opportunity to learn different problem solving strategies and to use a variety of approaches to solve problems. "The question that you gave us to work on about the book of poems was very interesting and it seemed very difficult, until I made it into an easier question by saying that if each page had three lines and there were five pages in the book, each line would have 5 combinations, 5x5x5" (Peggy, June 7). She then solved the problem beautifully. Sophia wrote about the strategies that she used to solve some of the problems. "The most helpful way of solving problems for me is to determine whether the problem is similar to one that we've done in class or to reduce the numbers that are easier to work with or to visualize" (Sophia, June 6). The interesting point is that she realized what was helpful *for her*. She did not say she had to do it because it worked for others, but because it worked for her.

In *teaching mathematics via problem solving* the mathematical concepts were developed through the mathematical problems. Finding volumes of different solids, using the concept of scaling, and developing concepts in probability are only a few examples. Teaching via problem solving also created an opportunity for the students and me to use a number of problem solving strategies and use them efficiently. Innovators acknowledged and used those strategies more than other students. The ways Peggy and Patrick solved the problem of "one hundred thousand billion poems" are two examples of how the innovators used different problem solving strategies including, (1) using smaller numbers, (2) breaking problems into different parts, and (3) using a systematic list. In teaching mathematics via problem solving, they learned those strategies in a natural way, rather than out of context in a hypothetical situation which would have been more artificial and less appealing.

Experiencing a variety of approaches to solve problems increased their confidence regarding problem solving. "[When] I get confused by irrelevant information, . . . I think I should just start [solving the problem], and if I don't figure it out, try another approach" (Sandra, May 10). This experience increased the innovators' ability to explain things to each other more productively and to feel challenged rather than hopeless or stuck. "If one explanation doesn't work, I'm challenged to think a new way to explain the problem" (Patrick, May 24). He was getting away from the traditional teaching that repeated the same old thing, in the same old way, until it sank in.

The students were constantly encouraged to ask themselves what they were doing, why they were doing it, and how it could help them. This helped them to become reflective at all times and give reasons for what they did. It also increased their sense of responsibility towards their own learning. Melisa's reflection on her second quiz was a good example. "Actually it was really interesting, and in retrospect, even though I got the right answer, I think I won't do well because I didn't raise my concerns about the question, and we were encouraged to do this" (Melisa, May 27) In her journal, she went through the quiz again and responded to the metacognitive questions that were part of the quiz.

Most of the innovators expressed their dislike for exams of any sort. However, they were reflective even under exam pressure. Peggy wrote about her feelings towards final exams and how she tended to panic in such situations. However, she found a way to help herself, and she wrote, ". . . I must remember, what is the problem! What do I already know? How can I solve it? and not go and do things that are not required in the question and make it more difficult for myself" (Peggy, June 7).

Penny's example was interesting. She felt a responsibility for explaining everything that she did, even the formulas that she used to solve the problems on her final exam. She used the formula for finding the volume of the cone (A1 on the final exam) and she wrote, "The reason for this is that in class we observed Zahra put water into a cone and we could see that the water going into the cone was exactly 1/3 of the cylinder." She constantly said that she liked to see things to be able to make sense of them. It was rewarding that on the final exam she could relate to what she "saw" in class. Besides, she felt that she could not even use a formula without explaining "why."

The innovators practiced the same activity even when they were taking notes. "I found that one thing I have to do with my notes is to write out the details of <u>what</u> I am doing and <u>why</u>" (Peggy, May 17).

Melisa's reflection on her beliefs about mathematical problem solving after the instruction summarizes what the innovators have said in this regard. "This class has been like an explosion of ideas about what I know and feel, not just about math but about all things. When we break down the barriers of how we know, and acknowledge that there are other ways of knowing, we learn quicker, and I think, adapt more easily to new learning strategies" (Melisa, May 26).

Tests

In the end all the enjoyment of discovery is buried because I'm so worried about whether I'll do the [final] exam okay. *(Clara, June 7)*

The innovators raised a number of issues concerning "tests" in general. Among them were: the purpose of evaluation, fear of exams, and the students' perception of exams. "I have never been able to overcome in a test situation, and it is one reason why I think I did so badly in math in elementary school. So much of the education was based on test scores that whatever real understanding I might have had was not seen" (Barbara, May 28). They complained that the nature of metacognition-based instruction and the exam did not match. "Throughout this course we have been encouraged to talk over everything, work together, don't worry about the right answer, understand the process, etc. Then we are given a final exam, and in two hours without the benefit of our notes or interacting with others, we are expected to pass the course" (Clara, June 6).

The students knew they could pass the course if they did a reasonable job during the course, and that the final exam would not be any different from what they had done in the class. However, the exam situation was stressful. The difficulty was that a few of them felt desperate that they could not "remember" on the exam. "I have to try and remember for the exam. I get very nervous and anxious" (Peggy, May 30). Clara's case was extreme and she was ready to leave the final exam after 15 minutes. I encouraged her to stay and do as much as she could. She was the most articulate and hard working student in the class. Her contribution to the teaching and learning processes was enormous. Clara knew she could not fail since she had earned full marks for her journal, assignments, and quizzes before the final, that was 60% of her total mark. Nevertheless, she was so overwhelmed on the final exam that she could hardly utilize what she knew.

However, Peggy and many other innovators tried to put their fears toward exams behind them. "I must get myself back into a more positive frame of mind. At the moment I am feeling really overwhelmed by this test paper. I can't sleep at night worrying over the

. . . final exam. I know that this is not good and will not help me, but at the moment this is how I feel" (Peggy, June 4). Melisa had better success in this regard. "The final is still the big scary test that it's always been! And I get queasy every time I think of it. But, you know, I really feel that I've done the best I could in this course. . . . I'm not sure why, but I'm not worried about the exam any more" (Melisa's last journal entry).

Many innovators considered exams to be something different and necessarily difficult--an insurmountable task. Sandra explained, "The [first] math quiz today was very fair. I was expecting a question I had never seen before" (Sandra, May 17). Kent reflected on the same quiz and wrote, "My math quiz went extremely well. In fact, I do not recall any other time after a test feeling so good" (Kent, May 17).

The majority of the innovators also believed that exams focused mainly on finished products, which meant getting "correct answers," more than on problem solving processes. Some of them were annoyed to see the process problems on the exam. "Today we had a quiz in math. I found the quiz to be quite confusing because one could perceive the question in many different ways. I think that the questions should have been clearer. I did not know if Zahra wanted me to do calculations in figuring out the answer or just writing statements to describe what I thought was the right answer" (Penny, June 8). The students were used to having everything "set" and "fixed" in mathematics, so when the question was flexible, they got confused. They were surprised to learn that their solution processes were worth more than the product, the "correct answer." "I'm glad we get credit for our ideas of how and why we calculate and solve the questions" (Sandra, June 8).

Carmen's honest reflection on her third quiz questioned the purpose and the validity of evaluation in general. "I had frightened myself so much that the negative reinforcement manifested itself in fear. . . . *I managed to do well on the quiz* but I'm not 100% convinced that I understood what I was doing? Is that possible?" (Carmen, June 10).

There are a number of interesting issues in this short passage that require discussion. The first issue is the old and ever existing question of evaluation: why do we evaluate students, what do we evaluate them for, and how? (A partial answer to this question would requires another doctoral dissertation!). The second is the fact that some people are good exam takers, "I managed to do well," as Carmen said, and the marks they get do not necessarily reflect their conceptual understanding of mathematics. Last but not least is Carmen's honesty. I was impressed by the way she talked about her mark. I could conclude, based on her writing, that the "relaxed environment" and a friendly relationship between the students and the instructor allowed her to express herself openly.

Teaching and Learning Mathematics via Problem Solving: The Innovators' Perspective

A working definition for "teaching and learning mathematics via problem solving" emerged from the innovators' reflection on the instruction. The innovators welcomed many aspects of the instruction. They were tired of experiencing their failure regarding mathematics. The innovators were not satisfied with the ways in which mathematics had been presented to them. The course offered the innovators a new perspective that allowed them to gradually see the more innovative and practical side of mathematics. They finished the course believing that mathematics could be a shared knowledge and that it has mostly been developed out of practical needs. The innovators were happy to have an opportunity to develop their own ways of solving problems, and to see the importance of processes in problem solving rather than mere emphasis on finished products, "correct answers." They realized the necessity of having dialogue and discourse in meaning-making processes regarding mathematics through group work and whole-class discussions. A better communication with the instructor was another vital aspect of mathematical learning, and journal writing made that possible. They felt comfortable developing the mathematical concepts through problem solving. The main characteristics of the "teaching mathematics via problem solving" based on the innovators' views were as follows:

-A "new approach" to teaching and learning mathematics

-Dialogue and discourse are integrated components of the instruction

-Students are involved in the "meaning-making" process

-Whole-class discussions are not "structured" but "monitored" by teachers

-Mathematical problems can be approached in various ways

-Problems can be solved in different ways

-Problem-solving processes are more important than finished products

-Mathematics would be more "fun" and "meaningful" if it is presented through games and puzzles, and be connected to day-to-day life

-The teacher is a facilitator, monitor, and friend and not a sole authority in class

-The teacher is fallible

Summary

The innovators' beliefs about mathematics and mathematical problem solving prior to the instruction were not very positive. Many of them considered themselves "deficient" in mathematics and thought they were just not capable of doing mathematics. Their beliefs were influenced by the ways in which mathematics had been *presented to them*, without connections to their day-to-day life.

However, the innovators expressed their dissatisfaction of the ways in which mathematics had been taught in the past. They were willing to try a "new approach" that might help them to become more comfortable regarding mathematics. The innovators considered the metacognition-based instruction a whole new perspective to teaching and learning mathematics. They reflected on different components of the instruction extensively and consciously. The innovators' reflection on the instruction showed that their beliefs about themselves as doers of mathematics, and about mathematics and mathematical problem solving were changed to a great extent as a result of a metacognition-based instruction.

Summary of the Findings

The traditionalists rejected the "new approach" to teaching and learning mathematics. They either did not care for a new experience and felt more comfortable with the "traditional" approach, or they said they were too old to try a different instructional approach. However, their rejection was not the sign of an appreciation for the "traditional" instruction. Rather, they were used to an old habit that would die hard. The traditionalists were not risk-takers, they preferred to stick to the tradition as it was more predictable and less innovative.

The incrementalists were in a conflicting situation. The majority of them solved the conflict by proposing "balanced instruction," the adaptation of some aspects of "traditional" and some of the "metacognition-based" instruction. The instruction did not change their world views drastically. However, their beliefs about mathematics and mathematical problem solving were influenced.

The innovators' beliefs about themselves as doers of mathematics, and towards mathematics and mathematical problem solving were changed as a result of the metacognition-based instruction. It was a great achievement for them to become comfortable in a mathematics class and gain enough confidence to discuss the mathematical ideas and make appropriate decisions. However, the process of change was not easy. In many instances, there was a great deal of frustration and confusion.

Overall, although the students enjoyed working in small groups, whole-class discussions did not appeal to all of them. The traditionalists did not see the necessity for class discussions. However, those discussions helped the incrementalists to develop their understanding of mathematics, while the innovators considered discussions as an integrated part of the teaching and learning process.

The incrementalists and the innovators utilized their journals to facilitate their learning, but the traditionalists' view of journal writing was more mechanical. They merely considered the journal writing another requirement to be met.

Each student, without exception, expressed his or her appreciation for using manipulative materials to enhance mathematical learning one way or another.

Postscript

The students fell into three distinct categories based on the data that were gathered through different sources. Out of 40 students, 9 of them belonged to the first category; the traditionalists, 12 of them fell into the second category; the incrementalists, and the other 19 students; the innovators, were in the third category. The following event at the end of the course confirmed the selection of categories to a great extent.

The 36 students responded anonymously to a questionnaire about evaluation of course and instructor on June 13, 1991, one day before the final exam. Surprisingly, the number of students falling into the same three categories based on the nature of their responses matched closely with the number of students that belonged to those categories that emerged from the analysis of the data. In fact, 19 students' responses had the same characteristics as the innovators, 11 as the incrementalists, and 6 as the traditionalists. This last source of data was valuable to attest to the validity of the categories. The following anonymous excerpts crystallize the range of the students' responses.

A majority of the students talked about the positive aspects of the course, and how it gave them a chance to "try to do math without being pressured. The course allowed for many possible <u>ways</u> to get answers."

However, there were two sharply distinct opinions on the ways in which the course/instruction affected different groups of students. For the majority of the students, it was a "good course to cure math phobia,"[because] . . . it made [them] face [their] math anxiety and to some extent overcome it. It made [them] see that [they were] able to solve mathematical problems." The instruction gave the students an opportunity to "see" and to "do" mathematics rather than memorizing facts and formulas, using them for a period of time, and then forgetting them: "It is a visual course. You can *see* how things are applied and *work out* as opposed to just *memorizing formulas*." They expressed their gratitude for having a chance to discover things by themselves. "I really appreciated being able to discover things in class as opposed to being told what something was [for example] . . . π and deriving π." Many of the students wrote that the instruction (the course) offered them a new perspective regarding teaching and learning mathematics. However, the course and the instruction was so foreign for a number of them, (the traditionalists), to the degree that it was ". . . frustrating and an insult to [their] intelligence."

CHAPTER 6

CONCLUSIONS AND DISCUSSIONS

Mathematics educators have been interested in problem-solving instruction for many years. Central to any question regarding the teaching of problem solving is the primary question of understanding what people actually do when they solve problems. Within the past decade, investigations of problem solving processes have led to a new focus in some of the research literature. Metacognition, the knowledge and control of cognition, has gained a great deal of attention from the research community since mathematics educators have realized that non-cognitive aspects of problem-solving performance are as important as cognitive ones. Therefore, there has been a growing interest in the study of the role of metacognition in mathematical problem solving.

This chapter gives a conclusion of the study, including answers to the research questions. Its educational significance and implications for future practice and research are also discussed.

Conclusions

The aim of the present study was to investigate the effect of metacognition-based instruction and teaching via problem solving on students' understanding of mathematics, and the ways in which the students' beliefs about themselves as doers and learners of mathematics and about mathematics and mathematical problem solving were influenced by the instruction. Metacognitive strategies that were included in the instruction contributed to the students' mathematical learning in various ways. The instruction used journal writing, small groups, and whole-class discussions as three different but interrelated strategies that focused on metacognition.

Lester, Garofalo, and Kroll (1989a), expressed their concerns about the lack of data regarding students' backgrounds, and about the credibility of students' self-reports, in their study. In the present study, an effort was made to overcome these concerns. The students' autobiographies provided much valuable data regarding the students' backgrounds and the knowledge and beliefs that they brought to the class. In addition, this data source was extremely helpful in analyzing the possible changes in the students' beliefs about themselves as doers of mathematics, and about mathematics and mathematical problem solving. Furthermore, the students' journal entries provided the rich and authentic data used in analysis. Moreover, the second and third quizzes were accompanied by a set of metacognitive questions to partially address the difficulty regarding students' self-reports mentioned by Lester, Garofalo, and Kroll (1989a).

Journal writing served as a communication channel between the students and the instructor, and as a result facilitated the individualization of instruction. In general, for many students, writing was an occasion of personalizing and internalizing the learning processes. Many of them appreciated the opportunity to write about related cognitive and metacognitive issues as the course progressed. Journal writing provided the opportunity for the students to clarify their thinking and become more reflective. Journals were also an excellent data source for studying possible changes in the students' beliefs about themselves and about mathematics and mathematical problem solving.

Small groups proved to be an essential component of the instruction. The students learned to assess and monitor their work and to make appropriate decisions by working cooperatively and discussing the problems with each other. In addition, small-groups created an opportunity for me to offer the students a variety of problem solving strategies (heuristics) and to help them to become aware of their own resources, to appreciate them, and to use them proficiently. Most importantly, as one of the students wrote, "working in groups really helped to take the frustration out of the course because it is math."

Whole-class discussions raised the students' awareness about their strengths and weaknesses. The discussions also helped students to a great extent to become better decision makers. These discussions provided an opportunity for the students to become more reflective of their own actions in class as well as to think about their thinking and talk about their feelings. Whole-class discussions helped the students to increase their self-awareness as well as to gain more self-confidence and self-respect, and that ultimately resulted in a better understanding of mathematical concepts and increased their mathematical power.

In general, the metacognition-based instruction gave me a chance to teach mathematics *via* problem solving. Starting the class with a problem that all groups could work on simultaneously was a worthwhile activity for developing mathematical concepts. This strategy was useful for bringing the whole class together while discussing a problem. As a result of this activity, the students gained more cognitive and metacognitive knowledge of problem solving, and a better sense of responsibility for their own learning. The students eventually realized that they would learn more by analyzing and solving one problem thoroughly, than by solving many problems using only a set of rules and procedures, and by plugging in formulas, without really developing a deep understanding of problems and concepts. It was pleasant and reassuring for the students to see that although their approaches and solutions were different, they were acceptable and valid. Besides, they had to decide which approach made the most sense to them since they had the opportunity to examine many possible alternatives. These activities were used as a vehicle to promote self-regulation.

Three categories of students labeled *traditionalists*, *incrementalists*, and *innovators*, emerged from the study. Nine students, who rejected the new approach to teaching and learning mathematics were categorized as traditionalists.. The traditionalists liked *to be told* what to do by the teacher. They did not see any point in spending time and energy developing mathematical ideas. Instead, the traditionalists preferred learning certain rules and procedures and becoming more fluent in applying formulas in specific situations. However, they liked working in small groups. The twelve incrementalists were characterized as those who propose to have *balanced* instruction in which journal writing was a worthwhile activity, group work was a requirement, and whole-class discussions were preferred for clarifying concepts and problems more than for generating and developing new ideas. The nineteen other students were categorized as innovators, those who welcomed the new approach and utilized it and preferred it. For them, journal writing played a major role in enhancing and communicating the ideas. Working in small groups seemed inevitable, and whole-class discussions were a *necessity* to help them with the meaning-making processes. The students in each category shared common beliefs about mathematics and mathematical problem solving.

The incrementalists and the innovators gradually changed their beliefs about *mathematics* from viewing it as objective, boring, lifeless, and unrelated to their real-lives, to seeing it as subjective, fun, meaningful, and connected to their day-to-day living. Since they were actively involved in the meaning-making processes, the innovators saw mathematics as shared knowledge that could be constructed by themselves to a great extent. They realized that mathematics was not as straightforward as they thought, that they could guess, speculate, and make conjectures regarding mathematics. As the course progressed, the incrementalists and the innovators did not feel hopeless dealing with mathematical issues. On the contrary, most of them believed that everyone could understand and do mathematics to some extent.

The findings of the study further indicated that most of the incrementalists and the innovators changed their views about mathematical *problem solving* from seeing it as the application of certain rules and formulas to viewing it as a meaning-making process of creation and construction of knowledge. They learned that there were many different approaches to most of the mathematical problems and that they could all be sound and promising. The students developed certain managerial skills to decide which approach made the most sense to them and why. However, the traditionalists did not like the ways in which the course progressed. They rejected the teaching approaches throughout the course except for working in small-groups and using manipulative materials, that all students liked to do.

Answering the Research Questions

The findings provided partial answers to the four interrelated research questions. The following summarizes the answers to these questions.

1. What are students' usual beliefs about themselves as doers of mathematics, and about mathematics and mathematical problem solving?

Generally speaking, prior to the instruction, the students believed that mathematics was objective, difficult, often unreachable and incomprehensible, a set of isolated facts and formulas that had no connections with real-life situations. That some people were good at it and some others were deficient when it came to mathematics. Many students even thought that some people could do mathematics and others could not. Except for a few individuals, mathematics had always been scary and boring.

2. How are these beliefs prior to instruction associated with students' approaches to problem solving?

With the exception of a few, the students considered mathematical problem solving as difficult and dry. They mostly believed there was only one correct solution and one correct answer for every problem. Some of the students even wished to find *a formula* to solve all the problems. They thought of problem solving as nothing but knowledge of procedures. They did not see much flexibility in solving mathematical problems. They considered problem solving to be imposed on students rather than being negotiated. Many of the students did not attempt to solve any mathematical problem on their own, since they believed that they were deficient in mathematics and solving mathematical problems was an unfeasible task.

3. In what ways do students' beliefs about mathematics and about themselves as doers of mathematics change during metacognition-based instruction and teaching mathematics via problem solving?

The beliefs of the incrementalists and the innovators about mathematics were changed to a great extent, as a result of metacognition-based instruction that aimed to teach mathematics via problem solving. Teaching mathematics *via* problem solving provided an opportunity for them to see mathematics as shared knowledge and a "human endeavor." They eventually believed that mathematics could be fun, practical, engaging, and an interesting activity that everyone can do and understand to some extent. Moreover, the innovators' appreciation for mathematics increased as they got more involved in the meaning-making processes with mathematical ideas. However, the traditionalists did not like to get involved in many activities towards construction of mathematical knowledge. They continued preferring to be told how to use the formulas and plug in the numbers to get the correct answers.

4. In what ways do students' beliefs about mathematical problem solving and their approaches to problem solving change during metacognition-based instruction and teaching mathematics via problem solving?

Teaching mathematics *via* problem solving provided an opportunity for the incrementalists and the innovators to "see" and to "do" mathematics instead of only learning certain procedures and formulas. They became more critical and more sensitive to mathematical problem-solving. Many of them, especially the innovators, tried to justify their reasoning by answering what they did, why they did it, and how they did it while solving mathematical problems. Teaching mathematics via problem solving provided the opportunity to present mathematical concepts in the context of engaging problems. Furthermore, the solution processes helped them to develop those concepts in a meaningful way. In addition, their knowledge of mathematics was extended and their appreciation for mathematics was increased after they had a chance to see a host of different approaches to solving the same problems. Most of the incrementalists and the innovators eventually learned that those approaches could all be equally valid and correct as long as it made sense to them and they could justify those approaches with solid reasoning.

Educational Significance

The present study benefited from the recommendations of Lester, Garofalo, and Kroll's (1989) research on the role of metacognition in mathematical problem solving of seventh graders. They wrote about two fundamental premises of their study that, "metacognitive processes develop concurrently with the development of an understanding of mathematical concepts (assumption 1) and that metacognition instruction is more likely to be effective if it takes place in the context of learning mathematics (assumption 3)" (p. 118). Taking these premises into account, they reflected on their study and addressed the following concern for further research in this area of research. "More particularly, the instruction was largely isolated from the regular mathematics curriculum . . . For the most part, the problem-solving sessions had little or no direct relation to regular mathematics instruction and many students did not view them as a being central part of their mathematics class" (p. 118).

I sincerely thought about these two assumptions and realized that it was indeed my intention and my goal to carry out research in "regular mathematics instruction." However, I realized that it would be more work and I would probably face more difficulty and less success, considering all the constraints that exist in a regular day-to-day mathematics class. Nevertheless, I was interested to try and see whether it was at all possible to do such a study under those circumstances.

The present study was about teaching an undergraduate mathematics course. The study was unique in that it had many constraints that classroom teachers face in their regular day-to-day teaching. I chose neither the course syllabus nor the textbook. The class size of 40 students was beyond my control. The course had a mandatory final exam that was worth 40% of the students' final marks. However, the study showed that metacognition-based instruction was applicable considering all those constraints. There were other constraints or limitations that some of students were concerned about. For example, no one was happy to come to class for two hours every weekday in hot summer evenings, for six weeks. Many students expressed their dismay about the intensity of the course, especially since most of them had full-time jobs, and some of them had other courses to finish in that summer. The study was carried out while facing all those constraints. In fact, I welcomed those constraints hoping that the findings of the study might contribute to teaching practice more broadly. Classroom teachers also do not have much control over the conditions similar to those I had in respect to teaching this mathematics course.

I have had many conversations with different teachers about bridging a gap between research and practice. On a number of occasions, they have complained that the findings of various studies have not reflected the reality of their classrooms. These teachers were argued that the day-to-day teaching has limited their flexibility in terms of trying different approaches to instruction and doing more qualitative evaluation of students' work. Their point was that the research settings were usually well chosen and researchers were not accountable for students' achievement on exams. These teachers were also doubtful about the feasibility of group discussions with too many students in their classes. This study was conducted in a regular mathematics classroom in which the teacher/researcher was limited by the same constraints as these teachers were. Therefore, the findings of the study might give practicing teachers some idea about the ways in which metacognitive strategies could be adopted in their own classrooms.

Implications for Future Practice

The present study employed several metacognitive strategies to teach mathematics *via* problem solving. Those strategies proved to be useful in enhancing students' understanding of mathematical concepts and promoting their self-regulation and self-confidence.

The study showed that journal writing was an effective means to enhance students' awareness. It also helps teachers to become more aware of their students' beliefs about themselves, mathematics, and mathematical problem solving. Students' writing provides a unique opportunity for teachers to acknowledge individual differences among students, and to help them in more effective ways. Journal writing could be utilized in various ways. For example, students can correspond with each other and reflect on each others' writing. It could be on-the-spot reflection on certain issues, whether it is structured, unstructured, or semi-structured. However, the main point is that teachers should read students' journal entries regularly and respond to them accordingly. The most discouraging action of a teacher is to ask students to write without giving students feedback. Students in the study appreciated the fact that their writing was valued, and their writing was as important to the teacher as it was to them. It is obvious that going through students' writing is extremely difficult and requires lots of time and effort. However, as was expressed by Rishel (1990), "although the workload may increase by a linear factor with this kind of teaching, the rewards are sure to go up also--by an exponential factor" (p. 33). Therefore, on the basis of the present study, I strongly recommend the use of journal writing in mathematics classes. I also advocate working in small groups and having whole-class discussions for facilitating students' mathematical learning, by providing detailed information about the role of these two activities in the study. Furthermore, the classroom culture (social norm) which teachers establish plays an important role in facilitating and monitoring studetns' learning and has an extremely important effect on students' learning.

The following gives an example of how I have implemented the findings of the study in my own teaching practice: I had a chance to teach MATH 335 again in the summer of 1992. This teaching assignment provided an opportunity for me to reflect on my study and to use metacognitive strategies in a more systematic way.

I started the course by giving the whole class the same problem to work on, and I consistently did the same thing throughout the course. The study showed that, despite the richness of the selected problems, worksheets had the potential to become the focus of the course instead of being a means to enhance students' mathematical learning. To overcome the difficulty in the second summer, I gave the students one problem at a time, rather than giving them worksheets with many problems. In this way, I was able to pull the whole class together to work on the same problem. I also was more experienced in applying certain metacognitive strategies namely, journal writing, small-group work, and whole-class discussions.

However, the most important contribution of the present study to my teaching in the following year was a greater insight that I got about students' beliefs about themselves as doers of mathematics, and about mathematics and mathematical problem solving. I also learned how those beliefs would influence students' mathematical understanding. Although the second class was different from the first one in terms of students' characteristics and personalities, one thing seemed to be unchanged: there was the same range of beliefs as I observed in the study. The study taught me to be more appreciative of students' difficulties and to be more sensitive to the variety of conceptions (some would say misconceptions) that students had about some of the mathematical concepts.

The study showed that students were frustrated when they did not see much connection between mathematics and their real world. This finding encouraged me to collect more mathematical problems dealing with day-to-day life to connect the students' mathematical world to their daily experiences.

To conclude this section, I would like to share my observation regarding the teacher's role in conducting classroom-based research. Teachers should have a chance to see more emphasis on metacognition in their training program. Metacognitive activities are teachable and learnable. Teachers and researchers should work together to develop cognitive-metacognitive frameworks considering social and cultural differences of their work places. Crosswhite (1987) suggested that researchers can provide practicing teachers with a "ready-to-wear" framework for promoting the growth of metacognition among students. However, there is no panacea to give to teachers, and any research without active involvement of teachers is doomed to failure in the long run. The complex area of thinking and reflecting cannot be reduced to a set of algorithms. There have been a number of promising research studies that focus on real classroom settings and have offered different practical models. These models can be adopted and used depending on different socio-cultural background of students and the knowledge that they bring into the classrooms.

Implications for Future Research

The three metacognitive strategies and many other strategies could be adopted, developed, and extended to enhance students' cognitive and metacognitive skills. The following includes a number of suggestions for further classroom-based studies in this research area considering students' socio-cultural backgrounds.

A number of studies have been conducted to investigate the use of journal writing in matheamtics courses. *The ways in which journal writing are utilized might vary considering students' age and socio-cultural backgrounds*. Therefore, more research is required to study the possibilities and limitations of journal writing across age and cultural groups. In addition, further research is needed for finding suitable criteria for evaluating journals since *journal evaluation is a difficult task and research in this area is in its infancy*.

As was suggested by Lester, Garofalo, and Kroll (1989a), the best setting in which to conduct such research is the context of a regular day-to-day mathematics course. Small groups and whole-class discussions proved to be essential components of the instruction. One area that requires great attention is the establishment of a suitable social norm for the class in which students have the chance to work cooperatively in small groups and to discuss the ideas with the whole class. The establishment of such a social norm, or classroom culture, was among the hardest tasks to achieve in the study. However, its establishment paved the way for adopting a number of metacognitive strategies fruitfully and productively. Further research is needed to refine the processes of conducting class discussions and establishing classroom culture (or social norm) in which students develop better managerial skills and to gain more self-confidence. In addition, *the teacher's/researcher's role in conducting whole-class discussions and establishing classroom culture is crucial and requires more detailed scrutiny in regular classroom settings.*

Moreover, further research is needed to investigate *the ways in which students' beliefs about themselves as doers of mathematics, and about mathematics and mathematical problem solving might be influenced by video presentations using a variety of approaches to teaching and learning of mathematics.* The study showed that video presentations provided a unique opportunity for students to analyze other students' behaviors in the presentations while they are involved in the problem-solving processes, and relate to them. These video presentations give them a chance to become more reflective on their own problem solving behaviors.

Final Note

The study of covert behavior of human beings is difficult and complex, but it is feasible and exciting. Researchers need to focus on the problem of bridging between theoretical and practical aspects of the role of metacognition. If they work more closely together, researchers and practicing teachers might unveil these covert behaviors to a great extent. Students' beliefs are mainly shaped by instruction and the classroom environment. Teachers' beliefs about mathematics and mathematical problem solving have a major effect on students' metacognitive awareness. I therefore, wish to see more research being done in this research to shed more light onto this complexity of human learning.

Teaching mathematics through problem solving and using metacognitive strategies is time consuming, but it is enjoyable and rewarding. The teacher's role in this approach as facilitator, monitor, coach, and role model is significant. In fact, teachers are key elements in such an approach. It is crucial for us as teachers, to genuinely listen to our students and appreciate the difficulties they have regarding mathematical understanding. Review of the literature shows that different means may be used to help students to become more metacognitively aware. However, means and strategies that other researchers have described can only serve as useful but general guidance, and they are by no means certain and definite.

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APPENDICES

APPENDIX A

Researcher's Letter to Students

Dear student;

With the permission of the Mathematics Department, I am doing a study on "The influence of selected instructional strategies on students' understanding of mathematics and problem solving ability." The study is being conducted as part of my dissertation for the doctoral degree in mathematics education.

The study includes two parts. One consists of asking volunteers to participate in individual task-based interviews before and after the course of the instruction. Those of you who agree to participate will be asked, during the interview, to discuss with me how you solve mathematical problems. Each interview will not take more than one hour and will be audio and video taped. The second part of the study consists of documentation of instruction and class interactions by taking field notes and video-taping the class sessions. Also your written work and exams will contribute to the data to be analyzed for the study.

Participation or non-participation in the study is voluntary, and will not affect your marks for the course. You can withdraw your data from the study at any time without any consequence to your class standing. The data obtained in this study will be kept strictly confidential. Pseudonyms will be used in any report of the study. The data will be destroyed after the culmination of the study.

In working with students last year in this course, they reported that this experience was most beneficial, to their understanding of mathematics and their problem solving skills.

Please indicate your consent or refusal, to participate in the study in the attached form. It would be helpful if you return the form as soon as possible. If you have any questions, you may contact my research supervisor Dr. Owens at 822-5318 or myself (Zahra Gooya) at 822- 2351. I appreciate your consideration of this request and I thank you in advance for your input into the study.

Sincerely,

Zahra Gooya
Graduate student
Department of Mathematics and Science Education
University of British Columbia

APPENDIX B

Student Consent Form

I received a copy of Ms. Gooya's letter describing the project.

I understand that my agreement or rejection will not in any way affect my status in classroom interaction or in academic assessment.

I do agree to participate in the study of the influence of instructional methods on students' understanding of mathematics and problem solving by offering myself as informant in the task-based interviews.

Signature--

I will give permission to Zahra Gooya to use my written work as part of the data for her study.

Signature--

I agree to be in the range of the video camera while the class sessions are being video-taped.

Signature--

Date--

APPENDIX C

Student Consent Form

(Regarding Video tapes)

I received a copy of Ms. Gooya's letter describing the project.

I understand that my agreement or rejection will not in any way affect my status in classroom interaction or in academic assessment.

I will give permission to Zahra Gooya to use the video tapes of class interactions with me in it for professional development, but not commercial purposes.

Signature--

Date--

APPENDIX D

Course Outline-Summer 1991

MATH 335 (Introduction to Mathematics)

1. *From Meno to Pythagoras*. The first math lesson on record: Socrates shows how to double a square. Gradually generalizing the method, we finally arrive at the theorem of Pythagoras. No numbers yet: Lengths and areas need not always be quantified.

2. *Diagonals and Heights*. If you can find square roots, Pythagoras facilitates indirect measurement. Concentrating on lengths related to regular polygons, but also trying some three dimensional objects. Making use of scales and proportion.

3. *Area and volumes*. Stacking strips or slabs, you can use heights to determine areas and volumes. Prime example: triangles and pyramids. Thence (regular) polygons and polyhedra.

4. *Cavalieri's Principle*. Reinterpreting slices of pyramids, we find the area under a parabola. Comparing a hemisphere (slice-by-slice) with a cratered cylinder, we find the volume of a sphere.

----

5. *Rational vs. Irrational Numbers*. We compute square roots by repeatedly averaging the sides of rectangles (Newton), and approximate π by repeatedly doubling the number of vertices of a polygon (Archimedes). We show that periodic decimals represent rational numbers (cf. Achilles and tortoise), and prove that many square roots (e.g. $\sqrt{2}$) are irrational.

6. *The Lore of Large Numbers*. We find spatio-temporal representations for several very large quantities, like the national debt. Scientific notation and orders of magnitude. The technique of "counting factors" gives easy estimates for products and quotients.

7. *Fractional Powers: Logarithms*. Using square roots, we make a table of 10^r, where $r = k/32$ with $0<k<32$. The resulting graph gives a way of writing any positive number as 10^a, with obvious benefits for multiplicative calculation.

8. *Growth and Decay*. Many quantities evolve according to a scheme of the form $Q(t) = Q_0 a^t$ at, where $a>1$ (growth) or $a<1$ (decay). We look at a typical crop of such problems.

9. *Multiplicative Counting*. Multiplying the number of choices available at each stage of a composite task. Counting with or without replacement. Permutations. Probabilities as relative counts.

10. *Binomial Coefficients*. If k elements become interchangeable, the total count is divided by k!. Reshuffling "words" with repeated letters. Binary words. More probabilities.

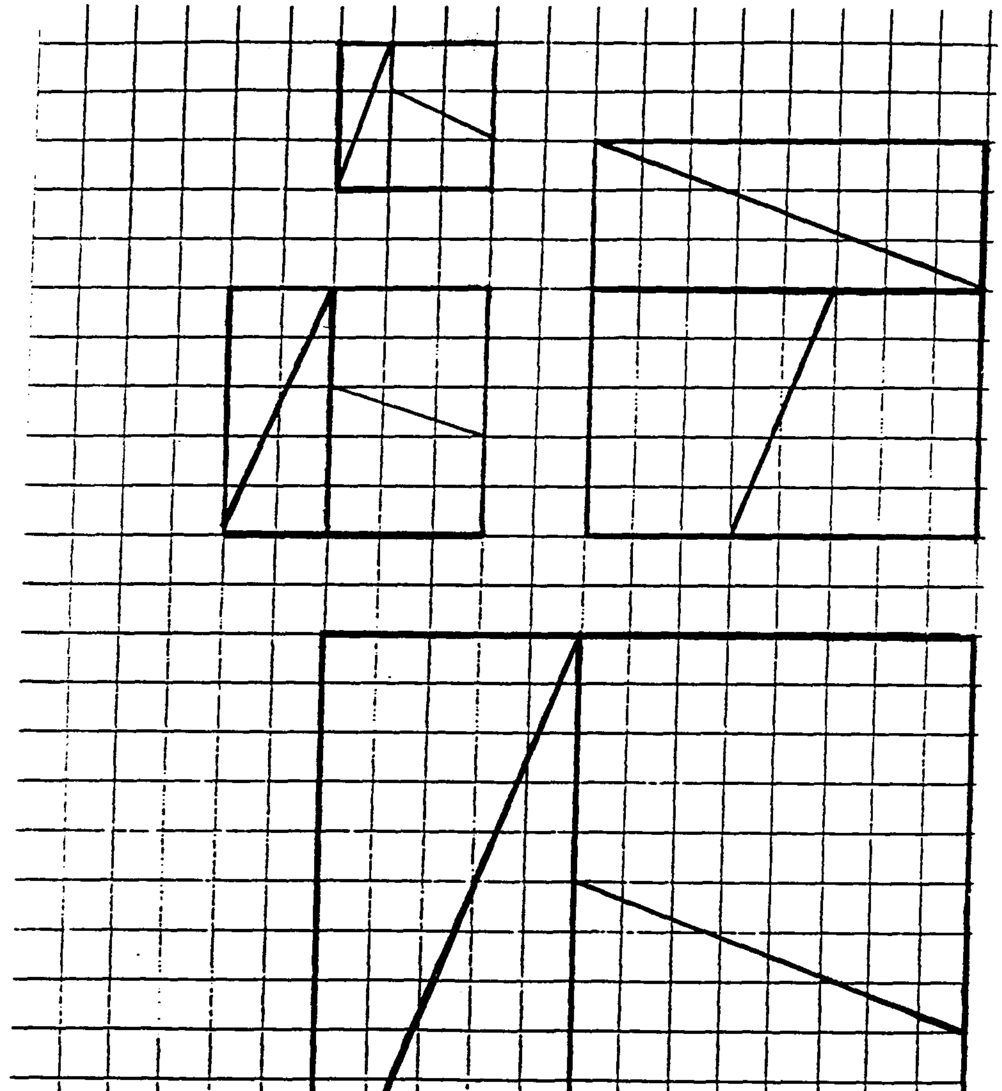
11. *From Pascal to Gauss*. Histograms derived from different lines of Pascal's triangle have a strong family resemblance. Extrapolating the limiting case. Probabilities interpretation.

12. *Testing and Sampling* . The normal curve shows how the values of many (but certainly not all) random variables tend to cluster. Of course, it is also useful in computing binomial probabilities, especially when the number of "trials" is large. Standard Deviation versus sample size.

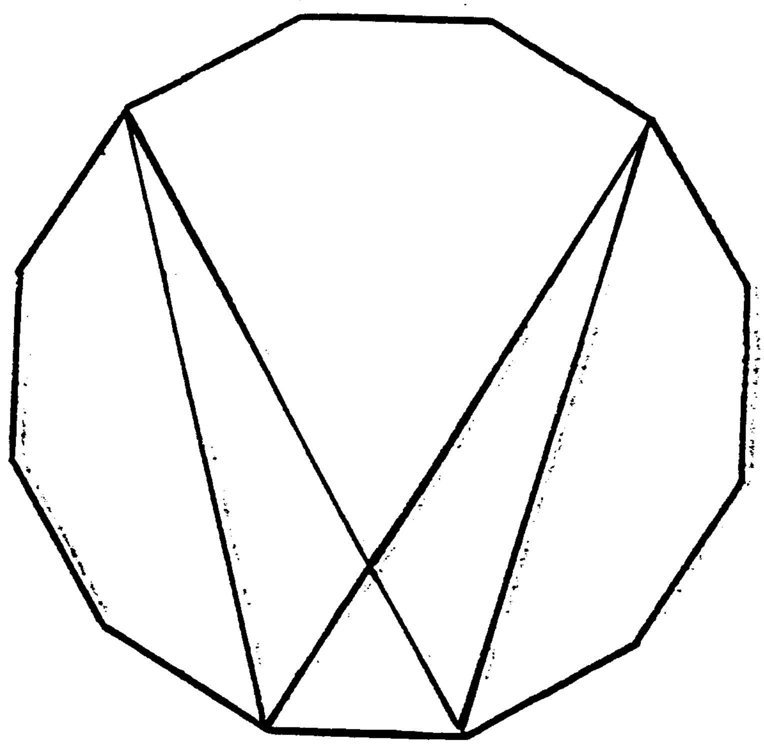
APPENDIX E

Course Materials

First Dissection Puzzle

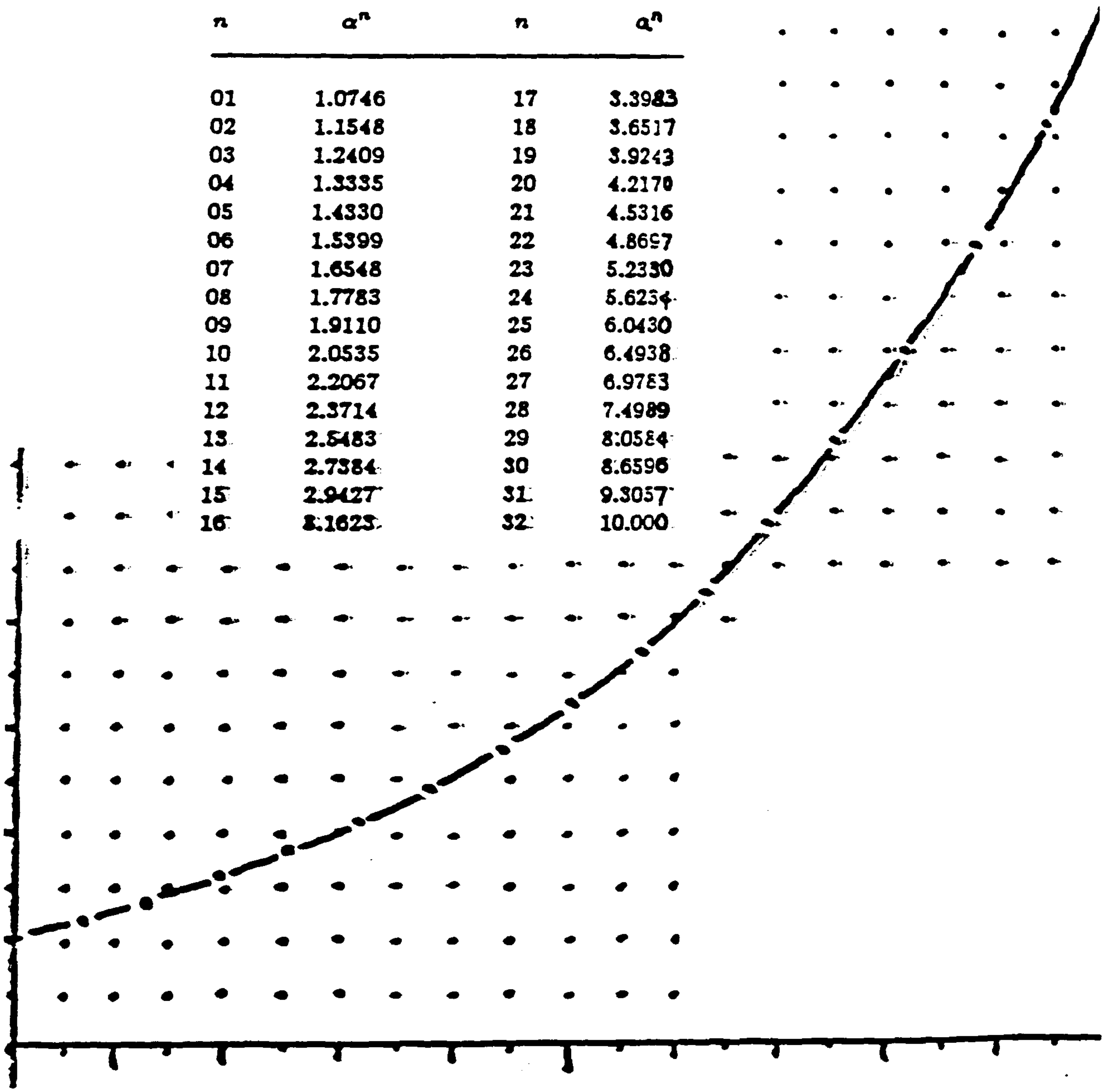


Second Dissection Puzzle



Graph of $a = 10^{n/32}$

n	a^n	n	a^n
01	1.0746	17	3.3983
02	1.1548	18	3.6517
03	1.2409	19	3.9243
04	1.3335	20	4.2170
05	1.4330	21	4.5316
06	1.5399	22	4.8697
07	1.6548	23	5.2330
08	1.7783	24	5.6234
09	1.9110	25	6.0430
10	2.0535	26	6.4938
11	2.2067	27	6.9783
12	2.3714	28	7.4989
13	2.5483	29	8.0584
14	2.7384	30	8.6596
15	2.9427	31	9.3057
16	3.1623	32	10.000



APPENDIX F

Sample of Metacognitive Questions

Name--
Date--

ANSWER THE FOLLOWING QUESTIONS REGARDING THE SECOND QUIZ[5] (it takes about ten minutes):

1. What information helped you to solve the problem?

2. What difficulties did you have in solving the problem?

3. Would you restate the problem using your own words?

4. What do you think you needed to know in order to solve the problem?

5. Would you state a problem which is related but more general than this problem?

6. How did you like solving the problem?

[5] These questions were included in both second and third quiz.

APPENDIX G

Math 335, First Assignment

(May 10, 1991)

Proof of the Pythagorean Theorem

1) State the Pythagorean Theorem;

2) prove the Pythagorean Theorem (As you know, there is not only *one* way or *the* way prove this theorem. It is up to you to prove it in a way that makes the most sense to you.)

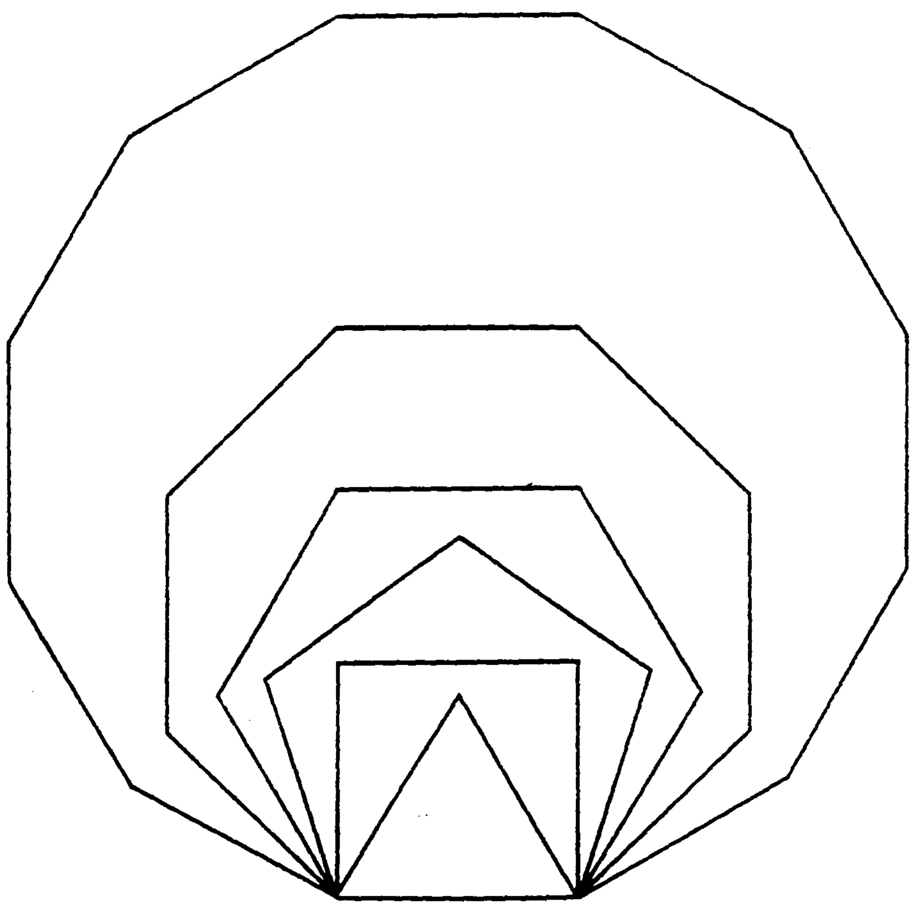
3) What do you think a mathematical proof is? What do you think about the generalizability of a proof?

APPENDIX H

Math 335, First Quiz

(May 17, 1991)

Find the area of each of the following polygons. Assume that all sides have unit length and work out the answers in the corresponding square units. You have to show all your work and explain that WHY you did WHAT you did and HOW would it help you. You can work in groups and use your notes. However, you have to write the solution up individually.



APPENDIX I

Math 335, Second Quiz

(May 27, 1991)

Which fits better, a square peg in a round hole or a round peg in a square hole?

You should explain that WHAT are you doing? WHY are you doing it? And HOW is it helping you?

APPENDIX J

Math 335, Second Assignment

(May 31, 1991)

(a) Show that the equation $x^2 = 10y^2$ can not be satisfied by any two whole number x and y.

(b) What does this say about $\sqrt{10}$?

APPENDIX K

Math 335, Third Quiz

(June 7, 1991)

How many different license plates are possible if each contains three letters followed by three digits? How many of these license plates contain no repeated letters and no repeated digits? You have to show all your work and explain that *why* you did *what* you did and *how* would it help you.

Appendix L

Students' Selections for the Open Assignment as Percents

	Geometry	Numbers	Counting	Critique	Pose
*Traditionalists (9)	22	45	11	11	11
*Incrementalists (12)	18	9	64	9	8
*Innovators (19)	16	32	42	11	5
Class (40)	18	28	42	10	7

Note.

Pose = Students' self-posed problems

*These categories of students are defined in Chapter 5.

APPENDIX M

Final Exam--Spring 1991

Mathematics 335

Instructor: Zahra Gooya **Time: 2 hours**

Part A

Select TWO of the following four topics. Each counts for 20% of your mark.

A1. Explain Cavalieri's Principle. How could it be used to find volume of a hemisphere with a circular base of radius R.

A2. State the Pythagorean theorem. Show why it is true and how it is useful by the means of an example.

A3. How can Pascal's triangle be used to determine binomial probability? What is the relation between binomial probability and the bell curve?

A4. Choose a topic in the course which struck you as particularly interesting or useful (it must be a topic that you have not or will not be using in any other parts of this examination.)
(a) Set a question or a problem on this topic.
(b) Solve the question or the problem, giving clear explanations and showing all the steps used to arrive at the answer.

Part B

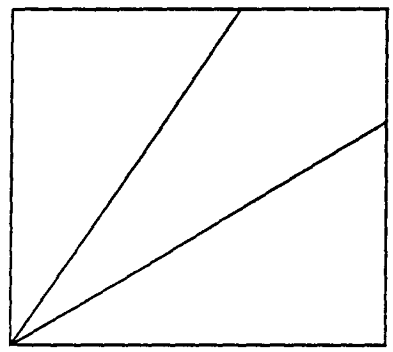
B1. A toy in the shape of a hemisphere on top of a cone. The cone has sloping edge of 13 cm. Its base matches the top of the hemisphere, which has radius 10 cm.
(a) Make a sketch of the toy.
(b) Find the height and the volume of the toy.

B2. The number 1/7 and 0.3333... are both rational numbers. Is the number 0.012345679012345679... a rational number? If you agree that it is a rational number, write it as a fraction of two integers. If you disagree that it is a rational number, prove that it is an irrational number.

B3. A committee of 5 is to be chosen from a group of 7 females and 6 males. In how many ways can the committee be chosen such that:
(a) Any 5 are chosen, irrespective of sex.
(b) All 5 are females.
(c) All 5 are males.
(d) 3 are females and 2 are males.

(e) There is at least one female in the committee.

B4. A couple just had a new child. How much should they invest now at 12%
(a) Compounded *monthly* in order to have \$60,000 for the child's education 16 years from now?
(b) Compounded *annually* in order to have \$60,000 for the child's education 16 years from now?
(c) Which one is a wiser investment?
B5. Find the area of the shaded region if the vertex angle of a square with sides measuring 4 units is trisected.



Appendix N

Students' Selections of Problems on the Final Exam as Percents

	A1	A2	A3	A4	B1	B2	B3	B4	B5
*Traditionalists (9)	22	33	44	33	67	44	100	54	33
*Incrementalists (12)	50	83	42	17	75	25	92	33	75
*Innovators (19)	42	100	32	26	74	26	84	47	68
Class (40)	40	80	38	25	73	30	90	44	62

Note.

*These categories of students are defined in Chapter 5.

APPENDIX O

Interview Question A

Ten delegates from ten countries meet at a conference on world peace. If each delegate shakes hands with the delegate from each of the other countries, how many handshakes would take place?

APPENDIX P

Interview Question B

A pair of rabbits one month old are too young to produce more rabbits, but suppose that in their second month and every month thereafter they produce a new pair. If each new pair of rabbits does the same, and none of the rabbits die, how many pairs of rabbits will there be at the beginning of each month? What happens in the first six months?

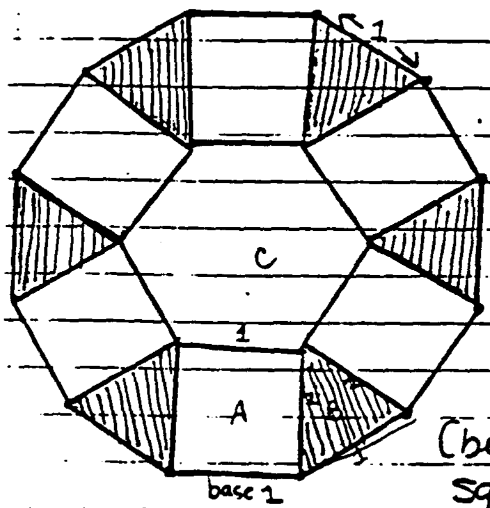
APPENDIX Q

Sample of Teresa's Work

(Finding Area of the Dodecagon)

AREA of DODECAGON

To find the area of the dodecagon, we can either scale it (with the dodecagon of worksheet #1) OR, as I will do, carve it up into different shapes:



if we start with a square with the dimensions 1x1x1x1 (according to the base - we end up with six such squares with six equilateral triangles in between with the measurements 1x1x1 (because they share 2 sides w/ the square - and the base/or side of dodecagon is known as 1)

Then, resulting in the centre is a hexagon, with the dimensions of 1 on each side (we know this because its sides are shared by the squares' sides).

We only need to add up these areas.

- AREAS A (Squares) = Area of Square as in #2
  6 X 1 sq unit = 6 sq. units.
- AREAS B (Triangles) = Area of Triangle as in #1
  6 x .433 sq units = 2.598 sq. units.
- AREA C (hexagon) = Area of hexagon in #3
  = 2.598 sq. units.

TOTAL Area of Dodecagon =

6 + 2.598 + 2.598 = $\boxed{11.20 \text{ sq units}}$

(height is 1 + 1 + 1.732 = 3.732)

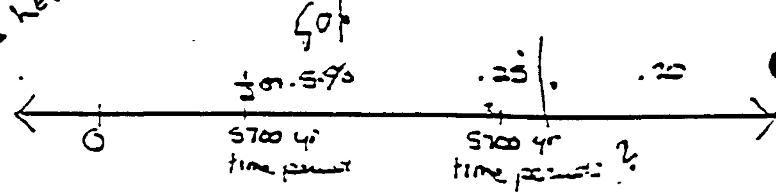
APPENDIX R

Sample of Clara's Work

Clara solved the following problem on her June 5 journal entry.

5. The carbon of a fossil bone contains only 20% of the normal amount of C_{14}. Approximate the age of the bone.

How these time periods would help you?



Only 20% left of what there originally was

rate $(\frac{1}{2})^t = .2$ why = Rate = $\frac{1}{2}$ in 5700 years

Here t is the number of years because we don't know how many years at that rate it will take to decay 20%

$(.5)^t = .2$

$\frac{1}{2}^t = .2$

$(.5\times10^{-1})^t = (2\times10^{-1})$ no. of time periods

$(10^{.699}\times10^{-1})^t = 10^{.301}\times10^{-1}$ Why you multiply, you add up the exponents. Isn't it?

$(1\times.699)^t = 1\times.301$

$.699t = .301$ $10^{.699-1}$ $10^{.301-1}$

$t = \frac{.301}{.699}$ $-.301t = -.699$

$\boxed{t = \frac{.699}{.301} = 2.32}$

I'm getting totally confused. I only know what I'm doing and I keep losing my chain of thought!!

Why? Any specific reason.

This cannot be correct the large number should be on the top I think   Try again!!

5. Question 5 again and again

Okay C^{14} — half life of 5700 years --
After 5700 years 50% of the original
left, after another 5700 years 50% of
the 50% will be left - after another
5700 - 50% of the previous 50% will
be left -- that's why 50% gone in
5700 — totally gone in 11,400 doesn't
$\frac{5700}{2}$ 4500
work because half life always means
fifty % of the remaining amount --
so the thing (whatever is decaying
exponentially rather than in a linear
way (i.e. gone in 11.400 yrs for C^{14}
so 50% of 50% or $\frac{1}{2} \overset{of}{\times} \frac{1}{2} = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8} \times \frac{1}{2}$
$\frac{1}{16} \times \frac{1}{2} = \frac{1}{32} \times \frac{1}{2} = \frac{1}{64}$ and on and on . . .
Now that I've got that straight let's
tackle no. 5 . . .

1 Time Period
5700 years

50% gone $\frac{1}{2} \times \frac{1}{2}$

2 time periods
11,400

$\frac{1}{4}$ 25% gone

3 time periods

.8781 = 600 mg dosage

17,100

$\frac{1}{8}$

12.5%

So the answer lies somewhere between 17,100 yrs and 11,400 years
(12.5%) (25%)

So $\left(\frac{1}{2}\right)^{t} = 20\%$
(half life)

$.301\,t = 1.301$?

$t = \frac{1.301}{.301}$

So $\frac{1}{2}^{t}$ unknown $= 20\%$
(half life)

~~.301 t = 1.301 4.3 x 5700~~

Now can I use the formula

$\frac{1}{2}^{t} = 2.00$?

$-.301(T) = .301$ $t = 1$?? $\times 5700 = 5700$

Question 5 - logs

~~50~~ t is $\frac{1}{2}t = 20\%$

No, this is not working what is wrong??

~~$10^{.699}$ .5 x10 = 10.3621~~ how did you get this?

$\frac{1}{2}t = 20\%$

$.5t =$

$.301t = .698$

$\frac{.301}{.301}t = \frac{.698}{.301} = 2.32$ times units of 5700 years

$= 5700 \times 2.32 = 13,225$ years

Well that seems reasonable You bet!

Clara solved the following problem on her June 7 journal entry.

8. To anesthetize a dog, 30 mg of sodium pentobarbital is required for each kilogram of body weight. If the drug is eliminated exponentially from the blood stream, with a half-life of 4 hours. Approximate the dose which will keep a 20 kg Alsatian anesthetized for 45 minutes.

Now I finally understand this question

We want the dog to have at least 600 mg of S.P. in his system at 45 minutes -- so we must give him more: We need to find out - how much we need to give him - given the rate of decay so he'll have 600 mg at 45 minutes

half life 4 hours -

In four hours 50% is gone

In Eight hours ~~50%~~ $\frac{1}{2}$ of 50% is gone = 25%

In twelve hours $\frac{1}{2}$ of 25% 12.5%

None of the above is very helpful to me so

$\frac{1}{2}^{(t)} = 600 \text{ mg}$ What do I need to multiply by rate to get 600 mg???

$\frac{1}{2}^{\left(\frac{45}{240}\right)} = 600 \text{ mg}$

4 hours = 60 × 4 = 240 - In 45 minutes

$\frac{45}{240}$ of the half life will have been used

$\frac{1}{2}^{301/1.65} = 600 \text{ mg}$ - Scrap that and try again. (over)

2.38

$\left(\frac{1}{2}\right)^{\frac{45}{240}} t = 600\ mg$

$.5^{\frac{1.65}{2.37}}$

Right way

$.3466\ t = 600\ mg$

$\frac{1}{2}^{.1875} = 6.187811$

$t =$ this is not getting me anywhere

$\frac{37}{60} = \frac{600}{x}$

45 minutes = 3/4 of and hour

4 hours = 4 hours

$\frac{3/4}{4} \times 4 = .75$ at 87 x-dose

$\frac{37}{60} = \frac{600}{x}$

why t again.

$37x = 60000$

$x = 68.9\ mg$

$.5 \quad \frac{1}{2}^{.75} t = 600$ four t is 3/16 ?

$.5^{.1249}\ t = 600$

$.301^{.1249}$

$.301 \times .1249 = .0375\ t = \frac{600}{.0375} = 1.0025$

t

700.25 mg

Having said I understand it -- I still did it wrong $\frac{3/4}{4}$ is not .75 $\frac{3}{4} = .75$ of 1

but $\frac{3/4}{4} \times \frac{1}{4} = 3/16$ So now I'll try again

I was lying in bed, after I finished this when I suddenly realized $\frac{3}{4}$ is .75 of 1 not 4

~~It must seem funny to you that anyone would make such a simple mistake because really it's so obvious !!~~

$\frac{1}{2}^{\frac{3}{16}(t)} = (600)$

$\frac{1}{2}^{\frac{6}{32}(t)} =$

$t(.301)^{\times\frac{3}{16}} = .778$

$t\ .056 = .778$
$\overline{.056}$

$t = \frac{.778}{.056}$

$t = 13.89$

What???

What went wrong?

Do you know? Isn't it again

What is your t? having

$3/16$? Then why already

when you substitute for that?

So is the total dosage 613.89?? That sounds wrong -- I'm really not clear about what I'm doing wrong. Zahra said these questions could all be done using the equation $\frac{1}{2}t = p$ No! I said using $a^{t} = p$
with a be either growth or
decay factor ($a>1$ growth factor
and $a<1$ decay factor)

But this equation is different because we have an extra fraction in the left side of the equation

So the equation $\left(\frac{1}{2}\right)^{\frac{3}{16}} x = 100\ mg$?? = Why 100 mg?

or $.301 \times \frac{3}{16} x$

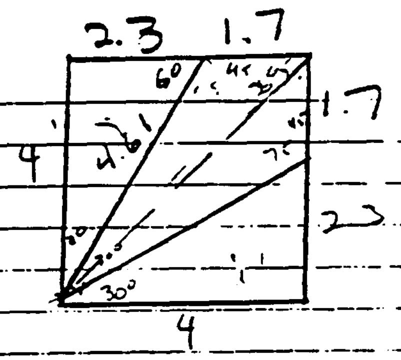
$.0564 x = 600$

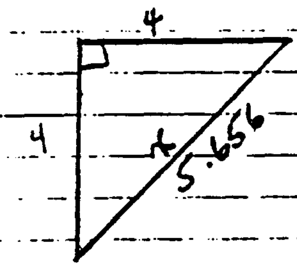
$\frac{.0564}{.0564} x = \frac{600}{.0564} = x = \frac{.778}{.0564}$

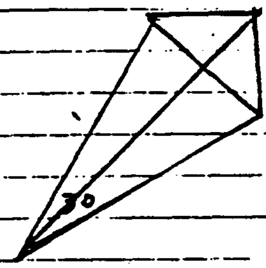
APPENDIX S

Sample of Kent's work

(Regarding Problem B5 on the Final Exam)







$4^2 + 4^2 = N$

$16 + 16 = N$

$32 = N$

$\sqrt{32} = N$

$5.656 = N$

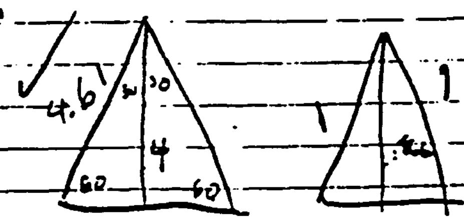
What do you need this for?

FROM THE DODECAGON PROBLEM WITH A SIDE LENGTH FOR THE SLICE THE SHORT SIDE OF THE SLICE = .517 IT HAS AN ANGLE OF 30°! WILL THIS HELP ME... CAN ~~YOU~~ I USE THIS AND FACTOR?

Very interesting!

NO....

THE 2 Δ'S MAKE AN EQUIL. Δ

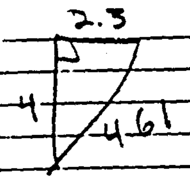


$a^2 + \frac{1}{2} a^2 = 4^2$

$\frac{.866}{1} = \frac{4}{?}$

$? = 4.67$

GOT IT



$21.33 = 16 + b^2$

$5.33 = b^2$

$\sqrt{5.33} = b$

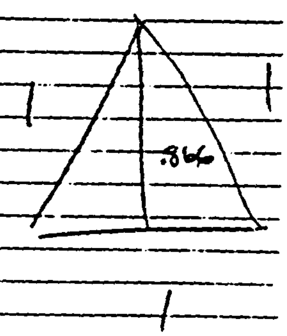
2.30

NOT ENOUGH TIME !!

(you already did it!)

BUT I COULD DO IT FACTOR USING THE △ FROM THE DODECAGON

PROBLEM



APPENDIX T

Thank You Letter to the Students at the End of the Course

Dear

I wanted to take this opportunity to thank you for participating in my study. The study could not become a reality without your involvement. I know that there were moments of frustration and dissatisfaction while we, as a whole class were trying to negotiate the meanings for better understanding. Yet there were moments of enjoyment and discoveries that could hardly happen if I simply imposed my ideas on you. I greatly appreciate your patience while working with me for six hard working weeks. I can only talk about myself to say that I truly enjoyed these exhausting weeks! I will dedicate my study to you as a potential teacher and I am sure that you will make a wonderful one. I hope that your future students will have many opportunities to express themselves and will not accept anything until they find answers to their "why's"!

I will miss you!
Zahra Gooya