BETWEEN THE STEPS ON "THE MIND'S STAIRCASE": INDIVIDUAL PATHWAYS TO THE DEVELOPMENT OF YOUNG CHILDREN'S MATHEMATICAL UNDERSTANDING

By

SUSAN NEDDA LOEWEN

B.A. The University of British Columbia, 1974

Professional Certificate, Ministry of Education, Science and Technology, 1980

M. A. The University of British Columbia, 1996

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

in

THE FACULTY OF GRADUATE STUDIES

(Human Learning, Development and Instruction)

THE UNIVERSITY OF BRITISH COLUMBIA

June 2007

Abstract

This study explored how individual, 4- to 5-year-old children, who displayed average to above-average mathematical ability for their age responded to an instructional program designed to facilitate construction of the mental counting line. Case's (1996a) neo-Piagetian theory, Case's (1996b) model of the process of structural change, and Porath's (1991b) model of the intellectual development of academically advanced children provided the theoretical framework for the study. The microgenetic approach advocated by Catán (1986a) was used to explore change and variability in the developmental pathways of each of the children in the study.

Three girls (aged 4.0, 4.1, 4.8 years) and 1 boy (aged 4.11 years), who had not yet constructed the mental counting line, participated in the study. An instructional program (7 weeks long) was used to stimulate the development of the mental counting line. The Quantitative Reasoning subtests of the Stanford-Binet Intelligence Scales, Fifth Edition (Roid, 2003a) and four measures of conceptual understanding (Number Knowledge test, Balance Beam, Money Knowledge and Birthday Party tasks) were administered prior to instruction. The measures of conceptual understanding were readministered following instruction. A qualitative analysis of the children's pretest and posttest scores, descriptive microgenetic quantitative and qualitative analyses of the children's responses to the instructional program, and a trend analysis of the children's performance were conducted.

The results indicated the children progressed from the pretest to the posttest. Intra-individual and inter-individual differences in the rate and the pattern of construction of the mental counting line were apparent. The results provide evidence for individual pathways to development as children negotiate the critical transition between the predimensional and dimensional stages of Case's (1996a) theory. The results support Case's (1996a) neo-Piagetian theory, Case's (1996b) model of the process of structural change, and Porath's (1991b) model of the intellectual development of academically advanced children. The results are consistent with the results of studies from Case's (1996a) neo-Piagetian theoretical perspective, the results of studies from other neo-Piagetian theoretical perspectives (Case, 1996c; Siegler, 1996b), and the results of studies on the development of children's mathematical understanding (Abbott, Berninger, & Busse, 1996; Robinson, Abbott, Berninger, Busse, & Mukhopadhyay, 1997) .

Table of Contents

Chapter 6: Discussion 225

List of Tables

List of Figures

Acknowledgements

I thank my supervisor, Marion Porath and the other members of my committee, Janet Jamieson, and Ann Anderson for their guidance and support. I also thank other members of the department who took an interest in and assisted me with my project.

I thank my husband Ronald for his encouragement, help, and support, without which this project would not have been possible. Finally, I thank my children, Claire, John, Katherine, and David for their patience and understanding.

Chapter 1: Introduction

Individual Pathways to Development

Research has shown that children of the same age attain similar levels of cognitive development through a variety of different developmental pathways (Case & Edelstein, 1993; Larivée, Normandeau, & Parent, 2000; Siegler, 1996a). Different development pathways have been detected both within and across major stages of development (Case, 1996b; de Ribaupierre & Rieben, 1995; Hoppe-Graff, 1993; Inhelder, Sinclair, & Bovet, 1974; Knight & Fischer, 1992; McKeough & Sanderson, 1996; Okamoto, Case, Bleiker, & Henderson, 1996). For example, in a series of studies investigating the cognitive processes responsible for producing changes in children's cognitive structures, Inhelder et al. found that individual children displayed differences in both the rate and pattern of their development. In a study investigating the development of specific reading skills, Knight and Fischer found that individual children progressed along one of three different developmental pathways. In a longitudinal study investigating children's responses to a variety of Piagetian tasks, de Ribaupierre and Rieben found that individual children showed a preference for one of two different pathways or modes of cognitive processing.

Cognitive development is a constructive process influenced by a variety of factors internal and external to each individual child (Bidell & Fischer, 1992; de Ribaupierre & Rieben, 1995). The complex interaction of different biological predispositions, physical environments, social experiences and constructive processes produce individual developmental trajectories for each individual child (Bidell & Fischer, 1992; Case, 1996b; de Ribaupierre & Rieben, 1995).

Central Problem in the Study of Individual Pathways to Development

The importance of addressing the problem of individual differences within the framework of developmental theories has been recognized (Case, 1998a; Case & Edelstein, 1993; Larivée et al., 2000; Siegler, 1996a). The importance of describing the processes responsible for producing changes in children's thinking has also been recognized (Case, 1998a; Case & Edelstein, 1993; Larivée et al., 2000; Siegler, 1996a). However, the problem of how to describe the specific processes that individual children use to bring about changes in their thinking has not been adequately addressed (Case, 1996b, 1996d, 1998a; Case & Edelstein, 1993; Fischer, Knight, &

Van Parys, 1993; Larivée et al., 2000; Siegler, 1996a). Traditional approaches to development focused primarily on describing and confirming the existence of one universal pathway or set of processes that leads to development. The research methods that were used were designed to assess single pathways to development (Case & Edelstein, 1993; Fischer et al., 1993; Siegler, 1996a). Few detailed models describing the processes that individual children use to bring about changes in their thinking have been developed (Siegler, 1996a). There is also controversy over the kinds of processes that may be responsible for producing changes in children's thinking. Researchers working within the information processing perspective suggested that changes in children's thinking occur in a specific, continuous fashion in response to factors in the contexts in which children's learning actually occurs. Researchers working within the Piagetian theoretical perspective suggested that changes in children's thinking occur in a general, discontinuous fashion in response to age-related biological changes in the human brain (Case, 1996b; Inhelder et al., 1974; Siegler, 1996a).

Why the Process of Change in Individual Pathways to Development Should be Addressed

An understanding of the process of change in individual pathways to development is important for theoretical and practical reasons. There is a lack of understanding of how general and specific processes interact to produce changes in children's thinking (Case, 1996b, 1998a; Efklides, Demetriou, & Gustafsson, 1992; Larivée et al., 2000). There is the lack of an integrated description (in terms of both structure and process) of the development of children's thinking (Case & Edelstein, 1993). There is a lack of understanding of how the thinking of individual children develops and changes (Larivée et al., 2000). There is a lack of understanding of the abilities children use when performing training and transfer tasks and how these abilities interact with training procedures (Efklides et al., 1992). There is a need to develop instructional programs and methods of assessment that are consistent with the way individual children's thinking actually develops (Biggs, 1992; Efklides et al., 1992; Ginsburg, Klein, & Starkey, 1998).

Young Children's Mathematical Understanding

By 4 to 5 years of age most children have acquired a number of informal mathematical understandings (Case & Mueller, 2001; Griffin, 2005). They can subitize and count small sets of objects (Baroody, 1987; Gelman & Gallistel, 1978; Ginsburg et al., 1998; Griffin, 2005; Mix, Huttenlocher, & Levine, 2002). They can touch each object once and only once when counting and can repeat the last number word said when asked how many objects are in a set (Baroody, 1987; Gelman & Gallistel, 1978; Griffin, 2005). They know that adding objects to sets increases the size of the sets and that taking objects away from sets decreases the size of the sets (Baroody, 1987; Baroody & Wilkins, 1999; Gelman & Gallistel, 1978; Ginsburg et al., 1998; Griffin, 2005; Mix et al., 2002). They can detect large differences between sets and can describe these differences in general terms (more than, less than, bigger than, smaller than) (Griffin, 2005).

By 6 years of age most children have acquired a set of more sophisticated mathematical understandings (Case & Mueller, 2001; Griffin, 2005). They can subitize and count larger sets of objects and can recite larger portions of the number word sequence (Baroody, 1987; Case & Mueller, 2001). They can count backwards as well as forwards (Baroody, 1987; Case & Mueller, 2001). They understand that numbers have magnitude and that each successive number in the number sequence has a greater relative magnitude (is one more) than the number that precedes it (Case & Mueller, 2001; Griffin, 2005). They can solve addition and subtraction problems by mentally counting up and down the number sequence and can make precise numerical judgements between sets in a variety of quantitative dimensions (weight, height, length, and musical tonality) (Case & Mueller, 2001; Griffin, 2005).

Why the Process of Change in Individual Pathways to the Development of Young Children's Mathematical Understanding Should Be Addressed

More research is needed to understand how early mathematical understandings develop and change (Ginsburg et al., 1998; Mix et al., 2002; Okamoto & Case, 1996). The relationship between conceptual and procedural understanding is not well understood (Okamoto & Case, 1996; Rittle-Johnson, Siegler, & Alibali, 2001). The relationship between children's understanding of number and "verbal tags" (Okamoto & Case, p. 56) that describe aspects of conceptual structures such as the mental counting line (Case, 1996a, 1998a; Resnick, 1983) has not been investigated (Okamoto & Case). More research is needed to understand how children make the critical transition from the informal mathematical understandings they acquire during the preschool years to the formal mathematical understandings they acquire after they enter school (Mix et al., 2002).

Additional research is needed to understand the full range of individual differences in the development of early mathematical skills and understandings. Research has focused primarily on the development of the mathematical thinking of average ability children (Ginsburg et al., 1998; Robinson, Abbott, Berninger, & Busse, 1996) and children with specific learning disabilities

(Robinson et al., 1996). The development of the mathematical thinking of mathematically precocious young children has not been investigated (Robinson et al., 1996).

Research on the development of young children's mathematical understandings will provide teachers with a better understanding of how children's mathematical thinking changes and develops (Ginsburg et al., 1998; Griffin & Case, 1997). It may also lead to the development of instructional programs and assessment instruments that are consistent with the way children's mathematical thinking changes and develops (Ginsburg et al.).

Studying the developmental pathways of academically advanced children will provide insight into the development of expertise (Horowitz & O'Brien, 1986; Masten & Coatsworth, 1998). It will also lead to the development of a mathematical curriculum that is developmentally appropriate and is matched to particular talents and needs (Waxman, Robinson, & Mukhopadhyay, 1996).

Organization of the Chapter

In the remainder of this chapter, conceptualizations of development and change within the Piagetian theoretical perspective, information processing perspective, and neo-Piagetian theoretical perspectives are summarized. Strengths and weaknesses of the Piagetian theoretical perspective and information processing perspective and strengths of the neo-Piagetian theoretical perspective are highlighted. Case's (1991c; 1996a; 1998a) neo-Piagetian theory of cognitive development, Case's (1996a; 1996b; 1998a) model of the process of structural change, Porath's (1988; 1991b) model of the intellectual development of academically advanced children, and the microgenetic approach are described. Strengths of Case's (1991c; 1996a; 1998a) neo-Piagetian theory of cognitive development, Case's (1996a; 1996b; 1998a) model of the process of structural change, Porath's (1988; 1991b) model of the intellectual development of academically advanced children, and the microgenetic approach are highlighted. The purpose of the study is stated and a rationale for using the microgenetic approach to explore individual pathways to the development of young children's mathematical understanding is provided.

Piagetian Theoretical Perspective

The Piagetian theoretical perspective represents one traditional approach to the study of cognitive development (Case, 1998a). According to Piaget, infants are born with simple perceptual and motor capabilities (Piaget, 1952, 1983). As infants and children observe and manipulate the objects in their environments, schemas or mental representations are formed (Piaget, 1952, 1983). As children acquire more experience, their schemas or mental representations are coordinated to form sets of operational structures that act across all domains (Piaget, 1952, 1964, 1983).

These sets of operational structures are formed by a combination of biological, experiential and auto-regulative factors (maturation, physical experience, social experience, and the process of equilibration) (Piaget, 1964, 1983). They develop through a universal and invariant sequence of four major stages (sensorimotor, pre-operational, concrete operational and formal operational) in all children, across all domains, at approximately the same age (Piaget, 1964, 1983). Each stage represents a more complex or sophisticated form of thinking than the stage that comes before it (Piaget, 1964, 1983).

Although children are seen as actively constructing their own understandings, the operational structures on which these understandings depend develop in a manner that is relatively free of the contexts in which learning actually occurs (Piaget, 1964). These structures are conceptualized as "a coherent set of logical operations that can be applied to any domain of human activity and to which any cognitive task in the domain must ultimately be assimilated" (Case, 1996a, p. 1). Consequently, children's understanding within domains is a result of the experiences they have within domains and the properties of the operational structures they apply to the content of the domains (Case, 1991a).

Strengths of the Piagetian Theoretical Perspective

The primary contribution of the Piagetian theoretical perspective was its focus on children's active construction of knowledge (Case, 1991a, 1998a). Children were seen as active participants in the development of their own cognitive structures rather than as passive recipients of information from the environment. Physical maturation and experience were seen as essential components of development. Development was conceptualized as an internal biological process that was influenced by environmental factors, but was not solely dependent upon environmental factors for stimulation (Case, 1991a, 1998a).

Piaget's theoretical perspective provided a comprehensive description of the organization and development of children's conceptual understanding (Case, 1991a). Differences in the level of children's conceptual understanding were investigated across a wide range of tasks and subject domains and explanations were provided for a variety of empirical results (the finding that children's conceptual understanding developed in a similar fashion within and across different cultural groups; the finding that young children were unable to solve many of the logical problems posed by Piaget; the finding that children were able to solve many of the logical problems posed by Piaget, without training, during middle childhood) (Case, 1991a).

Limitations of the Piagetian Theoretical Perspective

Although Piaget (1952; 1964; 1983) acknowledged that a variety of factors influenced development, these factors were not specifically addressed and were not accounted for or explained within the framework of his theoretical perspective (Bidell & Fischer, 1992; Case & Edelstein, 1993; de Ribaupierre, 1993; Larivée et al., 2000). Piaget was primarily interested in describing the general form children's thinking took at different stages in their development rather than investigating individual differences in development (de Ribaupierre, 1993). Individual differences were conceptualized simply as variations in rate of development along a single developmental pathway (de Ribaupierre, 1993; Fischer et al., 1993; Larivée et al., 2000).

Piaget described children's thinking in terms of relatively context-free, logico-mathematical operational structures that acted across all domains (Piaget, 1952, 1964, 1983). However, research has shown that children's cognitive structures develop within the context of their everyday experiences and are influenced by the contexts within which they develop (Bidell & Fischer, 1992). The abstract, logico-mathematical structures Piaget described were based on the principles of symbolic logic rather than on the contexts within which children's thinking actually develops (Case, 1991c). As a result, these structures did not accurately reflect the way children solved problems and articulated their thinking to others. For this reason, they provided no information about how children's thinking was organized or about how specific aspects of children's thought might be accessed for instructional purposes (Bidell & Fischer, 1992).

Piaget's theoretical perspective provided a better account of the nature of children's thinking at different stages of development than of the processes that were responsible for producing changes in children's thought (Siegler, 1996a). Although Piaget conceptualized children as actively constructing their own cognitive structures and addressed the issue of cognitive change, the processes he assumed were responsible for producing changes in children's thought were described in a very general way (in terms of assimilation, accommodation and reflexive abstraction). In the absence of detailed descriptions of the processes responsible for producing changes in children's thought, it is difficult to understand how general and specific processes interact and lead to the construction of different pathways to development (Siegler, 1996a).

Studies conducted from the Piagetian theoretical perspective were designed to assess a single developmental pathway (Fischer et al., 1993; Siegler, 1996a). These studies included controls that reduced the influence of individual and environmental factors that contributed to the development of individual differences (de Ribaupierre, 1993). When individual differences were investigated, they were limited to variables that influenced the rate of children's development along a single developmental pathway (Case & Edelstein, 1993; de Ribaupierre, 1993; Fischer et al., 1993). The assessment instruments that were used were designed to assess a single, linear sequence of behaviours (Fischer et al., 1993). The tasks included in these assessment instruments were based on this single, linear sequence of behaviours. Individual differences in performance were interpreted as data that disconfirmed the existence of this sequence of behaviours (Fischer et al., 1993).

Group-based, cross-sectional, or longitudinal designs were frequently used (Siegler, 1996a). Groups rather than individual children were the focus of the designs. Differences in children's thinking were averaged across age groups. The processes that were thought to be responsible for producing changes in children's thinking were inferred by comparing the average performance of the children in each age group. Although the longitudinal designs that were used assessed the children's performance at different points in time, the length of time that elapsed between these assessments was too great to observe the processes responsible for producing changes in the children's thinking (Siegler, 1996a).

Information Processing Perspective

The information processing perspective represents another traditional approach to the study of cognitive development (Case, 1998a). Researchers working within this perspective view the mind as an information processing system (Kail & Bisanz, 1992; Klahr, 1989). Cognition is described in terms of mental representations (internal codes) and information processing mechanisms (processes) (Kail & Bisanz, 1992). Incoming information (stimuli) is encoded symbolically and is transformed, stored, and retrieved by information processing mechanisms (Kail & Bisanz, 1992; Klahr, 1989). Learning is conceptualized as change within the information processing system (change that is brought about by the information processing mechanisms). Learning is thought to lead to development (Klahr, 1989). Research is focused on "the processes that manipulate symbols and symbol structures" (Klahr, 1989, p. 135). Development is conceptualized as the way in which children encode, transform, store, and retrieve information within this system at different points in time (Klahr, 1989).

Changes in children's thinking are attributed to a continuous rather than a discontinuous, age-related process (Kail & Bisanz, 1992; Klahr, 1989). Over time, the interaction of simpler cognitive processes leads to the emergence of more sophisticated forms of thought (Kail & Bisanz, 1992; Klahr, 1989). External influences such as instruction, modelling, and specific environmental factors are seen as playing a strong role in bringing about changes in children's thought (Kail & Bisanz, 1992). Restrictions on children's working memory capacity are recognized. However, restrictions on children's working memory capacity are thought to be overcome through learning (Klahr, 1989).

Strengths of the Information Processing Perspective

The primary strength of the information processing perspective is its focus on detailed and comprehensive descriptions of the processes, symbols, and structures that children actually use when solving problems (Case, 1998a; Klahr, 1989). These processes, symbols, and structures are organized into coherent systems and are explicitly represented in the form of scripts, schemas, flow charts, and diagrams. These forms of representation make it possible to determine what the parameters of these systems are and make it possible to see how the components of these systems are related. The detailed task analysis that is a prominent feature of this perspective makes it possible to test hypotheses about the components of these systems and answer questions concerning the mechanisms of learning and cognitive development (Case, 1998a; Klahr, 1989).

Another strength of the information processing perspective is its focus on the environments in which children's problem-solving actually occurs (Case, 1998a; Klahr, 1989). Researchers working within this theoretical perspective recognize that children acquire knowledge within the context of specific problem-solving situations. Task-specific factors and task-related experience are seen to be important determinants of performance. Problem-solving situations are described in greater detail than in the Piagetian theoretical perspective. Detailed descriptions of this kind make it possible to answer questions about the kinds of task-specific conditions that lead to variations in children's performance. Individual differences in development are acknowledged and are explained within the framework of this perspective (Case, 1998a; Klahr, 1989).

Limitations of the Information Processing Perspective

Studies conducted within the information processing perspective share some of the same methodological problems as studies conducted within the Piagetian theoretical perspective (Siegler, 1996a). Many studies conducted within the information processing perspective use group-based, cross-sectional designs. Groups rather than individual children are the focus of these designs. As with studies conducted within the Piagetian theoretical perspective, the problem-solving strategies children use at different points in their development are described and assessed in terms of a single developmental pathway (Siegler, 1996a).

A primary criticism of the information processing perspective is that it presents a picture of children's cognitive development that is overly task- and context-specific (Case, 1991a). Researchers working within this perspective focus on children's performance on specific tasks in narrowly defined domains. They do not always consider how tasks in these domains are related to one another or to the cognitive system as a whole. A related criticism of the information processing perspective is that it places an overly strong emphasis on learning and factors external to the child rather than on development and factors that originate within the child (Case, 1991a).

Neo-Piagetian Theoretical Perspective

The neo-Piagetian theoretical perspective developed in response to the limitations of the Piagetian theoretical perspective and the information processing perspective (Case, 1991c, 1998a). This theoretical perspective integrates the core assumptions of the Piagetian theoretical perspective and the information processing perspective in order to preserve the strengths and eliminate the weaknesses of the Piagetian theoretical perspective (Case, 1991c, 1998a). Researchers working within this theoretical perspective retained Piaget's notion of developmental stages and introduced a stronger focus on domain- and task-specific factors that influence the development of children's thinking (Case, 1991c, 1998a; Larivée et al., 2000).

Although neo-Piagetian theorists differ in how they define and organize cognitive structures and how they conceptualize processes of cognitive change, they all agree in the following ways: (a) children's conceptual understanding is based on the development of domain-specific cognitive structures (schemes or systems of schemes that represent symbolic or conceptual aspects of thought) (Case, 1996a, 1996d, 1998a), (b) children's cognitive structures develop in response to the experiences children have within particular domains (Case, 1996a, 1996d, 1998a), (c) increases in the complexity of children's cognitive structures produce qualitative changes in children's conceptual thought (Case, 1996a, 1996d, 1998a), and (d) the complexity of children's cognitive structures is held in check at certain ages by system-wide limitations in children's information-processing capacity (Case, 1991c, 1996a, 1998a). The importance of both domain-general and domain-specific processes is recognized in the development of children's cognitive structures (Case, 1991c, 1998a). Greater emphasis is placed on understanding the processes of change (Case, 1991c, 1998a). Task requirements are specified in greater detail than in the Piagetian theoretical perspective (Case, 1998a). More attention is paid to the description and explanation of individual pathways to development (Case, 1991c, 1998a).

Strengths of the Neo-Piagetian Theoretical Perspective

The neo-Piagetian theoretical perspective combines features of both the Piagetian theoretical perspective and the information processing perspective (Case, 1991c, 1998a). Children's cognitive structures are seen as developing within specific contexts and are described in a fashion that is in keeping with the way children actually solve problems and articulate their thinking to others (Bidell & Fischer, 1992; Case, 1991c, 1998a). Changes in children's cognitive structures are attributed to processes of both development and learning (Bidell & Fischer, 1992; Case, 1991c, 1998a).

Individual differences in children's cognitive development are addressed and explained and issues linking cognitive development and educational interventions are actively investigated (Bidell & Fischer, 1992; Case, 1991c, 1998a). Children's cognitive structures are described in greater detail than in the Piagetian theoretical perspective (Case, 1998a). The process of structural change is investigated in a more substantive way and a number of transition models have been proposed (Case, 1998a).

Tasks reflect the complexity of children's thought at each major stage of development and the problem situations children engage in at school and in other aspects of their lives (Bidell & Fischer, 1992; Case, 1991c). Children's representations and problem-solving strategies are made explicit and are related to the information processing capacity that is required (Case, 1998a). This makes it possible to describe individual pathways to development and analyze and predict where individual children are along a particular developmental pathway (Case, 1991c, 1998a; Fischer et al., 1993).

Case's Neo-Piagetian Theory of Cognitive Development

Case's (1991c; 1996a; 1998a) neo-Piagetian theory of cognitive development is a multi-level theory of cognitive development that integrates both the Piagetian theoretical perspective and the information processing perspective. In this theory, Case (1996a; 1998a) attributes the development of children's understanding in different domains to the development of domain-specific central conceptual structures. Central conceptual structures are mid-level (modular or domain-specific) representational structures (schemas) (Case, 1991b) that combine both domain-general (biological, age-related) and domain-specific (task specific: cultural, instructional, contextual, motivational, experiential, talent-related) aspects of thought (Case, 1991b; Case & Sandieson, 1991).

Central conceptual structures are composed of a network of conceptual understandings and the relations that exist between them (Case, 1991b; Case & Sandieson, 1991). They develop in particular domains in response to the experiences children have within these domains and are constrained at particular points in their development by system-wide, age-related changes in children's working memory capacity. Central conceptual structures are "the product of children's central processing: although the content they serve to organize is modular, the structures themselves reflect a set of principles and constraints that are system-wide in their nature" (Case, 1996a, p. 5).[1] They form the conceptual core from which a variety of specific problems are understood at each developmental level and are the conceptual basis from which further understandings within a domain develop (Case, 1991b, 1996a; Case & Sandieson, 1991).

There are four major stages in the development of central conceptual structures: (a) sensorimotor (0-1½ years), (b) relational (1½-5 years), (c) dimensional (5-11 years), and (d) vectorial 11-19 years) (Case, 1991c, 1996a, 1998a). Each stage is characterized by "its own distinctive cognitive operation and structure" (Case, 1991c, p. 18). There are three substages within each major stage: (a) substage one - unifocal coordination, (b) substage two - bifocal coordination, and (c) substage three - elaborated coordination (Case, 1991c, 1996a, 1998a).

As children move from one major stage to the next (from substage three of one major stage to substage one of the next major stage), they experience a qualitative change in their conceptual understanding. As children move from one substage to the next of each major stage (from substage one to substage three of each major stage (Case, 1991c, 1996a, 1998a) the conceptual understanding that is characteristic of each major stage becomes more integrated and complex (Case, 1991c, 1996a, 1998a).

Transitions from one major stage to the next occur as a result of the hierarchical integration of two previously separate cognitive structures (Case, 1991c, 1996a, 1998a; Griffin, 2004b). Two lower order cognitive structures (schemas) are "activated by the system at the same time or in immediate sequence" (Case, 1991c, p. 29). The functional relationship between the two structures is perceived. The content of the two structures is reorganized and a new qualitatively different, higher order cognitive structure is "practiced, until it is consolidated as a new unit in its own right" (Case, 1985, p. 278).

---

[1] Central processing in involved in general learning.

Transitions from one substage to the next occur as a result of the elaboration and coordination of a new higher order cognitive structure (Case, 1991c, 1996d). One cognitive structure is coordinated in substage one (unifocal coordination). A second cognitive structure of the same type is added and tentatively coordinated in substage two (bifocal coordination). The two cognitive structures that were tentatively coordinated in substage two are more fully coordinated in substage three (elaborated coordination) (Case, 1991c, 1996d; Griffin, 2004b).

Transitions from one major stage to the next and from one substage to the next are related to system-wide, age-related increases in children's working memory capacity (Case, 1991c, 1996a, 1996d). Working memory capacity is the number of elements children can pay attention to or hold in their working memory while processing information (Case, 1991c). Maturation of the neurological system and increases in the speed of information processing (due to practice) lead to increases in working memory capacity (Case, 1985, 1991c, 1996d).

Increases in working memory capacity enable children to construct increasingly complex cognitive structures (Case, 1985, 1991c, 1996d). Research on children's working memory capacity for numbers (Griffin, 1994, as cited in Case, 1996d) supports the notion that increases in working memory capacity facilitate the construction of central conceptual structures (Case, 1996d). In a series of instructional studies, Griffin (1994, as cited in Case, 1996d) found that children who had a working memory capacity of less than two units were less able to benefit from instruction than children who had a working memory capacity of two units or more (Case, 1996d).[2]

Empirical support has been found for the domain-specific development of central conceptual structures in three domains of mathematical understanding: the domain of whole numbers, the domain of rational numbers, and the domain of functions (Griffin, Case, & Sandieson, 1991; Kalchman & Case, 1998; Moss & Case, 1999; Okamoto & Case, 1996). Positive correlations have been found between measures of working memory capacity and measures of conceptual complexity in studies on the development of narrative structures and spatial representations from Case's (1991c) theoretical perspective (Dennis, 1991; McKeough, 1991). Positive correlations have been found between children's performance on measures of numerical working memory capacity and their responsiveness to instruction in training studies

[2] Two units of working memory capacity are required for hierarchical integration to occur. Two units of working memory capacity allow children to focus on two lower order cognitive structures at the same time (Case & Mueller, 2001).

from Case's (1996a) theoretical perspective (Griffin, 1994 as cited in Griffin & Case, 1996). A diagram of the stages, substages, and corresponding working memory capacity described in Case's (1991c; 1996a; 1998a) neo-Piagetian theory of cognitive development is shown in Appendix A.

Case's (1991c; 1996a; 1998a) neo-Piagetian theory of cognitive development also integrates research on cognitive development and research on learning and instruction (Case, 1998a; Case, Griffin, & Kelly, 2001; Griffin, 2004b; Griffin & Case, 1997; McKeough, 1991). Central conceptual structures have important implications for learning and instruction (Griffin, 2004b). They specify conceptual understandings that underlie successful performance within domains, describe how these understandings are constructed and represented at different ages and suggest activities that move children to higher levels of conceptual understanding (Griffin, 2004b, 2005). This makes it possible to assess where children are developmentally, provide developmentally appropriate activities, and plan instruction that follows the natural developmental sequence (Griffin, 2004b, 2005).

Strengths of Case's Neo-Piagetian Theory of Cognitive Development

Case's (1991c; 1996a; 1998a) neo-Piagetian theory of cognitive development provides a description of cognitive development that integrates both domain-general (system-wide) and domain-specific aspects of children's thought. Although central conceptual structures develop within specific domains and form the conceptual core of children's thought within these domains, they "reflect a set of principles and constraints that are system-wide in their nature and that change with age in a predictable fashion" (Case, 1996a, p. 5).

Case's (1991c; 1996a; 1998a) neo-Piagetian theory of cognitive development describes both structural and behavioural aspects of children's conceptual thought, provides an explanation for transitions in the development of children's conceptual thought, and outlines social and environmental factors that contribute to the development of children's conceptual thought (Griffin, 2004b). Central conceptual structures represent the skills, concepts, and conceptual relations that underlie children's conceptual thought at particular points in their development (Case, 1996a). They form the conceptual core of children's understanding within particular domains and are the foundation from which more sophisticated levels of understanding are constructed (Case, 1996a).

Case's (1991c; 1996a; 1998a) neo-Piagetian theory of cognitive development provides a solution to the problem of how to design effective instructional programs. Instructional programs that support children through the critical transitions that occur in central conceptual structures in the mathematical (Griffin & Case, 1996, 1997; Griffin, Case, & Carpenter, 1992; Kalchman, Moss, & Case, 2001) and social/narrative (McKeough, 1991; McKeough & Sanderson, 1996) domains indicate that the knowledge that is represented in these central conceptual structures can be successfully taught (Griffin & Case, 1997; Kalchman & Case, 1998; McKeough, 1991; McKeough & Sanderson, 1996; Moss & Case, 1999).

Case's (1991c; 1996a; 1998a) neo-Piagetian theory of cognitive development may also provide one solution to the problem of how to account for individual pathways in development within the context of a structuralist theory of cognitive development (Case, 1996d). The central conceptual structures that develop in the mathematical, social/narrative, and spatial domains and the steps involved in the construction of these central conceptual structures represent "a 'preferred developmental pathway' for a large class of individuals" (Case, 1996d, p. 211). Because these central conceptual structures represent "a 'preferred developmental pathway' for a large class of individuals" (Case, 1996d, p. 211), it may be possible to track individual variation in the development of these central conceptual structures within the context of Case's theory (Case, 1996d). Case (1996d) noted that between 5 and 6 years of age, each of the central conceptual structures in the mathematical, social/narrative, and spatial domains is composed of approximately 50 separate elements and that each of these separate elements may be acquired independently (Case, 1996d). Although the independent acquisition of each of these separate elements may not lead to the development of a large number of different developmental pathways (Case, 1996d), the independent acquisition of each of these separate elements may lead to variations in the rate and the pattern of construction of the preferred developmental pathways (central conceptual structures) that develop in the mathematical, social/narrative, and spatial domains.

Case's Model of the Process of Structural Change

Transitions from one major stage of development to the next occur as a result of the hierarchical integration of two previously separate cognitive structures and lead to the formation of new higher order central conceptual structures (Case, 1996a). Transitions between major stages of development are important for both theoretical and practical reasons. Transitions between major stages of development move children to new, higher levels of conceptual thought. The manner in which children negotiate these transitions determines the course of development of children's conceptual thought (Case, 1996a).

Case's (1996a; 1996b; 1998a) model of the process of structural change describes how new central conceptual structures are formed and how central conceptual understandings are transferred to specific tasks and problem situations. Case's (1996a; 1996b; 1998a) model is based on the notion of the hierarchical learning loop, a dynamic, feedback mechanism that connects learning that occurs in specific situations and learning that leads to central conceptual understanding. This model describes the hierarchical learning loop and explains how the operation of the hierarchical learning loop facilitates the construction of new central conceptual structures and the transfer of central conceptual understandings to specific tasks and problems situations (Case, 1996a, 1998a).

Two pairs of processes are involved in the operation of the hierarchical learning loop. These pairs of processes are: (a) associative learning (C-learning) and conceptual learning (M-learning)[3] and (b) specific learning (learning that occurs in specific situations) and general learning (learning that leads to central conceptual understanding). Each pair of processes is connected by a feedback loop (Case, 1996a).

The feedback loop that connects associative learning and conceptual learning in specific learning situations facilitates each type of learning in an iterative fashion.[4] This feedback loop is embedded within a second, broader (hierarchical) feedback loop that facilitates specific learning and general learning in an iterative fashion (Case, 1998a).

The hierarchical learning loop is "a two-level structure in which the emerging top level reads patterns that are present within or across lower levels, and then feeds back the results to the lower levels themselves, thus facilitating the ongoing dynamic that is present at that level between associative and attentional learning" (Case, 1998a, p. 792).[5] The action of the

---

[3] C-learning is a slow, conditioned type of learning that occurs when children are exposed to activities such as modelling or direct instruction (Case, 1991d, 1996a). M-learning is a fast, flexible, attentionally mediated type of learning that occurs during active problem-solving (Case, 1996a). In Case's (1996a, 1996b, 1998a) model of the process of structural change C-learning and M-learning work together in an iterative fashion (C-learning strengthens connections and makes attention available for M-learning; when attention expands, more opportunities open up for further C-learning and so on).

[4] Iteration "is a process that takes its output as its new input, produces new output, which it takes as input, and so on" (van Geert, 1994, p. 14). The iterative feedback loop that connects associative learning (C-learning) and conceptual learning (M-learning) is present in both specific learning and general learning (Case, 1996a). The iterative feedback loop that connects specific learning and general learning (hierarchical learning loop) is involved in central conceptual learning (Case, 1996a).

[5] The "emerging top level" that "reads patterns that are present with or across lower levels" (Case, 1998a, p. 792) is the mechanism that is involved in the integration (mapping) of two separate schemas (Case, 1998a). "Depending on the external conditions (facilitory versus inhibitory) and the internal conditions (multiplicative versus additive interaction among levels)," the operation of the hierarchical learning loop can lead to schematic

hierarchical learning loop insures that "children's developing general structures exert a positive influence on their specific learning" and "specific learning also exerts a positive influence on children's level of general understanding" (Case, 1996b, p. 159). A diagram of the hierarchical learning loop (Case, 1996a) is shown in Appendix B.

As the action of the hierarchical learning loop continues, "a process of intellectual 'snowballing' or 'bootstrapping'" occurs (Case, 1996b, pp. 159-160). This produces a uniform rate of growth across the specific situations that are influenced by particular central conceptual structures (Case, 1996b). It also accelerates the overall rate of growth within developmental levels up to the maximum allowed by children's working memory capacity (Case, 1996b).

The effects produced by the bootstrapping process have important implications for the development of individual pathways to development (Case, 1996b, 1996d).[6] These effects imply that children who have different types of specific experiences (children from different cultural groups) will attain similar levels of conceptual understanding. The understandings children acquire in specific situations where they have a lot of practice or experience will be transferred to other specific situations where they have little or no practice or experience through the influence of their general conceptual understanding (Case, 1996b, 1996d). These effects also imply that children who have little or no practice or experience across specific situations (children from disadvantaged backgrounds) will attain an overall lower level of conceptual understanding. The lesser amounts of practice or experience these children receive will be magnified and averaged across all specific situations by the influence of their general conceptual understanding (Case, 1996b, 1996d). In addition, these effects imply that children who have a great deal of practice or experience across all specific situations (children from advantaged backgrounds) will attain an overall higher, though still age-appropriate, level of conceptual development. The greater amounts of practice or experience these children receive will be amplified and averaged across all specific situations by the influence of their general conceptual understanding, but constrained by age-related, biological limitations in their working memory capacity (Case, 1996b, 1996d).

---

integration that is gradual and follows a wave-like pattern (children will move through an intermediate phase between lack of understanding and understanding) or schematic integration that is abrupt and follows a step-like pattern (children will move rapidly from lack of understanding to understanding without moving through an intermediate phase between lack of understanding and understanding) (Case, 1998a).

[6] The bootstrapping process was modelled using a set of mathematical equations (Case, 1996b). The results of the modelling were tested against existing data from Case's theoretical perspective.

Strengths of Case's Model of the Process of Structural Change

Case's (Case, 1996a, 1996b, 1998a) model of the process of structural change integrates elements of the information processing perspective, the neo-Piagetian theoretical perspective and the sociohistoric theoretical perspective (Case, 1996d, 1998a; Keating, 1991) and adds a dynamic aspect to Case's (Case, 1991c, 1996a, 1998a) theoretical perspective. This model articulates the process of structural change (Case, 1998a) and describes one way in which general and specific processes might interact to produce more sophisticated levels of conceptual understanding (Case, 1998a). This model provides a coherent and testable explanation for the differences and similarities that have been found in the developmental levels of children from different social classes and cultural groups (Case, 1996b, 1996d). This model may also be used to investigate other individual differences, such as differences in academic advancement or differences in developmental delay (Case, 1996b).

A mathematical version of this model has been developed (Case, 1996a, 1996b). The interaction between understanding in specific contexts or situations (physical causality, telling time, money and transactions with money, social causality and school math) and more general, conceptual understanding (understanding of whole number) (Case, 1996a, 1996b) has been modelled and assumptions regarding the effects of the bootstrapping process have been explored (Case, 1996a, 1996b).[7] When the mean scores of subjects from different social classes and cultural groups (on three separate test batteries: numerical, narrative, and spatial) were plotted on the same graph as growth curves generated by the mathematical model, it was found that at each age level the specific growth curves generated by the mathematical model were similar in shape and spread to the specific growth curves obtained from the empirical data (Case, 1996b). Although these findings do not provide definitive support for this model, they indicate that there is a good fit between the theoretical model and existing empirical data from Case's (Case, 1991c, 1996a, 1998a) theoretical perspective.

Porath's Model of the Intellectual Development of Academically Advanced Children

Porath's (1988; 1991b) model of the intellectual development of academically advanced children is derived from Case's (Case, 1991c) neo-Piagetian theory of intellectual development. This model describes general conceptual understandings and domain-specific skills associated

[7] This is a preliminary model. Data were not collected to test the model (Case, 1996c).

with outstanding intellectual performance within particular domains. This model suggests academically advanced children are similar to children of more average ability in the development of conceptual understandings, but different from children of more average in the acquisition of domain-specific skills. In general, academically advanced children move through the developmental stages outlined in Case's (1991c; 1996a; 1998a) neo-Piagetian theory of cognitive development at approximately the same rate as children of more average ability (Porath, 1988, 1991b). Advanced levels of performance on measures of conceptual understanding rarely exceed the children's chronological age by more than two years or one substage in Case's (1991c; 1996a; 1998a) theory. However, unlike average ability children, academically advanced children learn domain-specific skills at a more rapid rate and apply the thinking that is characteristic of each developmental stage in a more elaborate and flexible way (Porath, 1988, 1991b).

Strengths of Porath's Model of the Intellectual Development of Academically Advanced Children

Porath's (1988; 1991b) model of the intellectual development of academically advanced children provides an age-appropriate description of academically advanced children's thought, a framework from which to investigate the relationship between domain-general and domain-specific aspects of academically advanced children's thought, and a base from which to plan instruction (Porath, 1991a). This model also provides support for the notion of developmental differences within the context of Case's (1996a) theory of cognitive development and a framework from which to investigate one type of individual developmental pathway (Case, Okamoto, Henderson, & McKeough, 1993).

Microgenetic Approach

The microgenetic approach has been used to study variability and change in conceptual understanding as individuals negotiate critical transitions in the development of their conceptual thought (Catán, 1986a; Lee & Karmiloff-Smith, 2002; Siegler & Crowley, 1991). The microgenetic approach focuses on changes in individuals' verbal and nonverbal behaviour as these changes actually occur (Catán, 1986a; Parziale, 2002; Siegler, 1997). A critical feature of the microgenetic approach is the continuous observation of an individual's behaviour (moment-to-moment, trial-to-trial, or session-to-session) throughout an entire period of developmental change (Lee & Karmiloff-Smith, 2002; Parziale, 2002; Siegler, 1997). The continuous observation of an individual's behaviour throughout a period of developmental change provides a detailed description of an entire sequence of developmental change (Lee & Karmiloff-Smith, 2002; Siegler & Crowley, 1991; Siegler & Jenkins, 1989). A detailed description of an entire sequence of developmental change enables researchers to discover how and why changes in an individual's thinking may have occurred (Lee & Karmiloff-Smith, 2002; Parziale, 2002; Siegler, 1997).

The microgenetic approach advocated by Catán (1986b) was used in this study. This microgenetic approach focuses on miniaturizing and modelling "large-scale developmental processes" over short periods of time (Catán, 1986b, p. 49). An instructional procedure is used to stimulate, accelerate and externalize the processes of change. Trial-to-trial and session-to-session analyses of the participant's responses to the instructional procedure are conducted (Catán, 1986b). The developmental process is observed from its point of origin, through to the complete development of a cognitive structure. Frequent observations are employed so that the dynamics of the developmental process can be observed and described (Catán, 1986b). Mechanisms of change are directly inferred from the participant's responses to the instructional procedure (Werner & Kaplan, 1957, as cited in Catán, 1986b).

Strengths of the Microgenetic Approach

The microgenetic approach advocated by Catán (1986b) provides a detailed description of an entire sequence of developmental change (Lee & Karmiloff-Smith, 2002; Siegler & Crowley, 1991; Siegler & Jenkins, 1989). The continuous trial-to-trial and session-to-session analyses of the participant's responses to the instructional procedure make it possible to discover how and why changes in the participant's thinking may have occurred (Lee & Karmiloff-Smith, 2002; Parziale, 2002; Siegler, 1997). The description of the contexts within which the changes occurred make it possible to speculate about which environmental factors may have contributed to the changes that occurred (Catán, 1986b; Siegler & Crowley, 1991).

The microgenetic approach is a suitable method for studying variability in development (Chen & Siegler, 2000). The large amount of quantitative and qualitative data produced make it possible for researchers to "identify differences in the types of strategies that children use initially, in the benefits they derive from various types of experiences, and in the path of change that their thinking follows" (Chen & Siegler, 2000, p. 13).

Werner's argument for the "unity of cognitive and behavioural processes" (Catán, 1986b, p. 44) makes the microgenetic approach a particularly suitable method for investigating the processes of change in very young children (Chen & Siegler, 2000). Werner argued that observable behaviour (verbal and nonverbal behaviour) was directly linked to internal thought processes (Catán, 1986b). Although young children often have difficulty expressing themselves verbally (Chen & Siegler, 2000), they display a considerable amount of nonverbal behaviour (Catán, 1986b). Werner's argument for the "unity of cognitive and behavioural processes" (Catán, 1986b, p. 44) allows researchers to make inferences about young children's strategy use (Chen & Siegler, 2000) or underlying thought processes (Catán, 1986b) directly from their nonverbal behaviour.

The dense data collection and trial-to-trial analysis of the microgenetic approach make it a suitable method for studying the thinking of young children (Chen & Siegler, 2000). Young children have short attention spans, making it difficult to hold their attention for extended periods of time. The dense data collection and trial-to-trial analyses of the microgenetic approach make it possible for researchers to obtain a great deal of information about young children's thinking in short periods of time (during a short session or a series of short sessions) (Chen & Siegler, 2000).

Purpose of the Study

The purpose of this study was to explore how individual, 4- to 5-year-old children, who displayed average to above-average mathematical ability for their age, responded to an instructional program that was designed to facilitate the construction of the mental counting line. The mental counting line is a central conceptual structure that develops in the domain of whole number understanding as children negotiate the critical transition between the relational and dimensional stages of Case's (1991c; 1996a; 1998a) neo-Piagetian theory of cognitive development. The microgenetic approach advocated by Catán (1986b) was used to observe and to describe intra-individual and inter-individual variability in the rate and the pattern of construction of the mental counting line and the transfer of the understanding represented in the mental counting to new tasks and problem situations.

Case's (1991c; 1996a; 1998a) neo-Piagetian theory of cognitive development, Case's (1996a; 1996b; 1998a) model of the process of structural change, and Porath's (1988; 1991b) model of the intellectual development of academically advanced children provided the theoretical framework for this study. The microgenetic approach advocated by Catán (1986b) was used to explore change and variability in the developmental pathways of each of the children in the study. Changes in the children's thinking as they constructed the mental counting line and the learning processes hypothesized to cause these changes to occur were inferred from the children's responses to the instructional program (Catán, 1986b; Lee & Karmiloff-Smith, 2002) (Siegler & Crowley, 1991).

Neo-Piagetian studies investigating the development of the mental counting line have suggested this central conceptual structure is of central importance in the teaching and learning of elementary and middle school mathematics (Griffin & Case, 1996, 1997; Griffin, Case, & Capodilupo, 1995; Griffin et al., 1991; Griffin, Case, & Siegler, 1994; Kalchman et al., 2001). When the components of this structure are taught, children show improved levels of performance on tasks that require an understanding of number and numerical relationships and are more able to take advantage of the mathematical instruction offered in the early years of elementary school (Case & Griffin, 1990; Case & Sandieson, 1991; Griffin & Case, 1996, 1997).

Rationale for Using the Microgenetic Approach

The microgenetic approach advocated by Catán (1986b) was used for the following reasons. There is a need to conduct detailed analyses of the process of change in central conceptual structures at the individual level (Case, 1996c; Jackson, 1993; Siegler, 1996b). There is a need to link Case's (1991c; 1996a; 1998a) neo-Piagetian theoretical perspective, Case's (1996a; 1996b; 1998a) model of the process of structural change, and Porath's (1988; 1991b) model of the intellectual development of academically advanced children to behavioural data at the individual level (Case, 1996c; Siegler, 1996b). There is a need to integrate the results of studies conducted at the individual level (microgenetic studies) with the results of existing macrodevelopmental studies (Case, 1996c; Siegler, 1996b).

A microgenetic analysis of the process of change during the transition from the relational to the dimensional stages of the mental story line, the central conceptual structure that develops in the domain of social/narrative understanding, has been conducted (McKeough & Sanderson, 1996). A preliminary mid-level analysis of the process of change during the transition from the relational to the dimensional stages of the mental counting line, the central conceptual structure that develops in the domain of whole number understanding, has also been conducted (Case, 1996a, 1996b). However, a microgenetic analysis of the process of change during this critical transition has not yet been done.

A microgenetic analysis of the process of change during the critical transition from the relational to the dimensional stages of the mental counting line will provide greater insight into the circumstances and mechanisms of change (Siegler, 1996a), will provide a deeper understanding of individual pathways to development (Case & Edelstein, 1993; Fischer et al., 1993; McKeough & Sanderson, 1996), and will lead to improvements in the instructional program that was designed to facilitate the construction of the mental counting line (McKeough & Sanderson, 1996). Linking Case's (1991c; 1996a; 1998a) neo-Piagetian theory of cognitive development, Case's (1996a; 1996b; 1998a) model of the process of structural change, and Porath's (1988; 1991b) model of the intellectual development of academically advanced children to behavioural data at the individual level (Case, 1996c; Siegler, 1996b) to behavioural data at the individual level will provide evidence of the validity of Case's (1996a; 1996b; 1998a) theory of cognitive development, Case's (1996a; 1996b; 1998a) model of the process of structural change (Siegler, 1996b), and Porath's (1988; 1991b) model of the intellectual development of academically advanced children (Siegler, 1996b). Integrating the results of a microgenetic study of the construction of the mental counting line with the results of existing macrodevelopmental studies from Case's (1996a; 1996b; 1998a) neo-Piagetian theory of cognitive development will lead to a more comprehensive description of the process of change (Lee & Karmiloff-Smith, 2002) and will enhance the credibility of the results of existing macrodevelopmental studies from this theoretical perspective (Case, 1996c; Jackson, 1993).

Definition of Terms

Rate of construction: refers to how quickly the children moved through the sequence of steps involved in the construction of the mental counting line. Rate of construction was rapid when: (a) there was an abrupt change (a change of more than one level) in the children's level of response; and (b) there was a higher frequency of independent correct responses subsequent to the first independent correct response; or (c) there was a consistent pattern of independent correct responses subsequent to the first independent correct response. Rate of construction was slow when: (a) there was no abrupt change (a change of one level) in the children's level of response; or (b) there was a lower frequency of independent correct responses subsequent to the first independent correct response.

Pattern of construction: refers to the level, trend, and variability or stability in the children's responses and the frequency of different types of responses generated prior to and subsequent to the first independent correct response.

Transfer: refers to the generalization of the understanding represented in the mental counting line to new tasks and problem situations.

Intra-individual variability: refers to differences within the children in the rate and the pattern of construction of the mental counting line and the transfer of the understanding represented in the mental counting line to new tasks and problem situations.

Inter-individual variability: refers to differences between the children in the rate and the pattern of construction of the mental counting line and the transfer of the understanding represented in the mental counting line to new tasks and problem situations.

Variability in the children's responses: refers to differences in the level of conceptual understanding from one response to the next. Level of conceptual understanding is represented by the different types of responses the children provided for the tasks presented during each instructional session.

Individual pathways: refers to the individual developmental pathways of each of the children in the study. The mental counting line and the sequence of steps involved in the construction of the mental counting line represent "a 'preferred developmental pathway' for a large class of individuals" (Case, 1996d, p. 211). Variability in the rate and the pattern of construction of the mental counting line and in the generalization of the understanding represented in the mental counting line to new tasks and problem situations results in the construction of individual developmental pathways.

Outline of the Study

Research on the process of conceptual change and individual pathways to cognitive development from the Piagetian theoretical perspective, information processing perspective, and neo-Piagetian theoretical perspective and studies on mathematically precocious young children are reviewed in Chapter 2. The central conceptual structure that develops in the domain of whole number understanding between 5 and 6 years of age (the mental counting line), the instructional program that was designed to facilitate the construction of this central conceptual structure, and the microgenetic approach are described in Chapter 2. A rationale for combining Case's neo-Piagetian theoretical perspective and the microgenetic approach is also presented in Chapter 2. The study design, study participants, pretest and posttest measures, instructional program, and method of data analysis are described in Chapter 3. The coding scheme is presented in Chapter 4. Qualitative and descriptive microgenetic quantitative and descriptive microgenetic qualitative analyses of the study's results are presented in Chapter 5. A discussion and interpretation of the study's results from the perspective of Case's (1991c; 1996a; 1998a) neo-Piagetian theory of cognitive development, Case's (1996a; 1996b; 1998a) model of the process of structural change, and Porath's (1988; 1991b) model of the intellectual development of academically advanced children are presented in Chapter 6. The educational implications of the study's results and suggestions for future research are also presented in Chapter 6.

Chapter 2: Review of the Literature

Neo-Piagetian theory resulted from the integration of the Piagetian theoretical perspective and the information processing perspective. The Piagetian theoretical perspective and the information processing perspective each present a different view of the process of change and variability in cognitive development. Case's (1991c; 1996a; 1998a) theory of cognitive development and Case's (1996a; 1996b; 1998a) model of the process of structural change reflect the theoretical integration of these different theoretical perspectives (Case, 1998a). In order to understand why Case's theoretical perspective offers a promising approach to the study of the process of change and variability in cognitive development, it is important to examine the contribution of each of these theoretical perspectives.

There are two parts to this literature review. Research on the process of conceptual change and individual pathways to cognitive development from the Piagetian theoretical perspective, the information processing perspective, and the neo-Piagetian theoretical perspective and studies on mathematically precocious young children are reviewed in the first part of this review. The central conceptual structure that develops in the domain of whole number understanding between 5 and 6 years of age (the mental counting line), the instructional program that was designed to facilitate the construction of this central conceptual structure, and the microgenetic approach are described in the second part of this review. A rationale for combining Case's neo-Piagetian theoretical perspective and the microgenetic approach is also presented.

Part 1

Piagetian Theoretical Perspective

Piaget viewed the child as actively constructing his or her own cognitive structures. Through the adjustment and readjustment of the processes of assimilation and accommodation, simpler, lower order cognitive structures were differentiated and coordinated to form new, qualitatively different, higher order cognitive structures at specific points in children's development (Piaget, 1964). The construction of new higher order cognitive structures was conceptualized as a "spontaneous" internal process that was part of the natural biological process of growth and development (Piaget, 1964, p. 8). Learning was conceptualized as a separate, task-specific process brought about by circumstances in the individual's environment (circumstances external to the individual). Learning was not seen as affecting or explaining development in any way. However, development was seen as necessary for certain types of learning to occur (Piaget, 1964).

Although Piaget's model of the process of conceptual change was accepted, it was felt his model provided too global a description of the process of conceptual change. Piaget focused only on changes that occurred in children's conceptual thinking. Specific processes responsible for producing these changes were not described (Case, 1998a; Fischer et al., 1993; Hoppe-Graff, 1993; Larivée et al., 2000). Changes that occurred in children's thinking as a result of learning in specific contexts were not explored or described in detail (Case, 1998a; Fischer et al., 1993; Hoppe-Graff, 1993; Larivée et al., 2000) and the issue of how learning in specific contexts might influence cognitive development was not addressed (Case, 1996a). There was also a need to show how general developmental and task- and domain-specific processes interacted to produce changes in children's thinking (Case, 1998a).

Piaget's model described a universal set of processes that were not connected in specific ways to the contexts in which children's thinking actually developed. All children were assumed to use the same set of processes and construct the same logical mathematical structure in the same way. Individual differences in development were not accounted for or explained (Case, 1998a; Fischer et al., 1993; Hoppe-Graff, 1993; Larivée et al., 2000).

Research on Individual Pathways to Development

Despite Piaget's focus on a universal pathway to development and universal processes of change, the issue of individual differences in development has been investigated within the Piagetian theoretical perspective (Case & Edelstein, 1993). In a series of studies, Inhelder (1943) explored the level of conceptual understanding and cognitive processes of 10- to 24-year-old developmentally delayed children and adults on a series of conservation of substance tasks. She found developmentally delayed individuals reasoned at the level of normal ability 5- to 7-year-old children on these tasks. She concluded developmentally delayed children move through the same sequence of stages as normal ability children. However, they move through these stages at a slower rate and remain at a given stage for longer periods of time than normal ability children.

Other researchers (Droz, 1970; Larivée, 1980, 1988; Tissot, 1979, as cited in Larivée et al., 2000) also explored the developmental level and cognitive processes of children who displayed a variety of developmental delays and psychiatric disorders in order to determine the level of conceptual understanding and cognitive processes used by these children. Although the work of these researchers did not lead to the study of individual differences within the Piagetian theoretical perspective, their work focused researchers' attention on environmental and psychological factors thought to produce differences in the rate of children's cognitive development in a number of different domains (Case & Edelstein, 1993; Larivée et al., 2000).

A few researchers explored the impact of environmental factors on children's developmental level. For example, Hollos and Cowan (1973) investigated the effect the amount of verbal interaction had on children's level of performance on two types of Piagetian measures (six measures of classification and conservation of objects and three measures of the ability to adopt another's perspective or point of view). Children in three rural Norwegian communities (farms, a village and a town) that differed in the amount of verbal interaction the children were exposed to were assessed. Forty-eight 7-, 8- and 9-year-old children were assessed in each community. Factor scores on the Piagetian measures were calculated for each child. A two-way analysis of variance was used to assess the effects of age and social environment for each factor. The results showed the amount of verbal interaction the children were exposed to had a significant effect on their ability to adopt another's perspective or point of view. Children from the town and village performed at a higher level than the children from the farms on this measure. However, the children from the farms performed at a higher level on the measures that assessed the ability to classify and conserve physical objects. The authors concluded the two factors were differentially affected by the amount of verbal interaction to which the children were exposed.

In a more recent longitudinal study, Edelstein, Keller, and Schroder (1990) explored the inter- and intra-individual variability of Icelandic children from 9 to 15 years of age who differed in terms of cognitive ability, gender, and social class on two Piagetian measures, a measure of cognitive development (syllogistic reasoning) and a measure of social cognitive development (friendship reasoning). They found the level of syllogistic reasoning was not influenced by gender. However, level of syllogistic reasoning was significantly influenced by cognitive ability and social class, with cognitive ability presenting a more significant influence than social class. They found level of friendship reasoning was influenced by the context in which the development occurred. The children displayed more advanced reasoning on some aspects of friendship than on others. They also found level of friendship reasoning was influenced by social class and cognitive ability. Children from higher social classes and children with higher levels of cognitive ability displayed higher levels of friendship reasoning. They concluded that although similarities existed in children's patterns of development in the cognitive and social cognitive domains both context and individual differences contribute significantly to individual variability over time.

Although the aforementioned studies led to a focus on the psychological rather than the epistemological nature of the development of children's conceptual understanding, they did not resolve the theoretical problem of how to investigate individual differences within a general theory of cognitive development (Case & Edelstein, 1993; Larivée et al., 2000). Researchers continued to focus on one universal developmental pathway. Individual differences were investigated only in terms of differences in rate of development along this single developmental pathway in response to internal, biological or external, environmental factors that were unique to each child (de Ribaupierre, 1993). Development along different developmental pathways was often interpreted as not being in accord with Piaget's theory (Larivée et al., 2000).

Research on the Processes of Change

In response to the problem of explaining findings not in accord with Piaget's theory, Inhelder and her colleagues, within the Genevan School, shifted from a focus on the study of mental structures children displayed at different points in their development to the study of processes children used to solve particular classes of problems. The primary goals of the research were to investigate why children passed Piagetian measures that assessed the same underlying construct (Piagetian measures that assessed conservation of weight and conservation of number) at different ages and to better understand the processes that were involved (Case & Edelstein, 1993; Inhelder et al., 1974; Larivée et al., 2000).

Inhelder et al. (1974) conducted a series of microgenetic teaching studies to understand the processes involved in solving a variety of conservation problems (numerical, physical and spatial) and to explore the relationship that existed between development and learning. Four- to 9-year-old children were included in the study. All of the children were of average to above-average intelligence. However, variations in socio-economic status and IQ were not taken into account. The goal of this study was to understand the problem-solving processes the children used rather than to investigate individual differences in the way the children responded to the instructional procedure. A variety of pretests and posttests were administered (all Piagetian measures of conservation) and a qualitative analysis of the children's responses to the instructional procedure and the pre- and posttests was conducted. Inhelder et al. found the children progressed through the stages hypothesized by Piaget and through some intermediate stages as well. They also found the instructional procedure generally accelerated the progress the children made in attaining the various concepts of conservation. However, the amount of progress the children made depended upon their initial level of development. This finding was interpreted as confirming Piaget's contention that a general mental structure must be in place before certain types of learning could occur. Although the environmental contribution was stressed and variations in the children's processes were acknowledged, this study and other studies conducted by Inhelder and her colleagues maintained the earlier focus of investigating the development of children's conceptual understanding along one universal developmental pathway (Case & Edelstein, 1993; Larivée et al., 2000).

Although Inhelder and her colleagues' process-oriented work did not resolve the theoretical problem of how to integrate and explain the existence of individual developmental pathways within Piaget's theory of cognitive development, their work provided additional evidence to support the growing awareness that this issue had to be addressed (Larivée et al., 2000). This led other researchers working within the context of Piaget's theory and the information processing perspective to conduct detailed studies of the processes involved in solving particular tasks and attempt some form of theoretical integration (Case & Edelstein, 1993; Larivée et al., 2000).

Information Processing Perspective

Several researchers working within the information processing perspective proposed models that described and explained individual pathways to development and the process of conceptual change. Unlike the Piagetian theoretical perspective, which conceptualized changes in children's conceptual thought as a series of discontinuous, qualitative shifts in children's thinking, the information processing perspective conceptualized changes in children's conceptual thought as occurring incrementally, in a continuous fashion in response to experiences children have within their particular environments (Case, 1998a).

Research on Individual Pathways to Development and the Processes of Change

McClelland (1995) proposed a connectionist or parallel-distributed processing (PDP) model to explain individual differences and the processes of change within the information processing perspective. This model was based on the architecture of the human brain. This model consisted of input units that were activated directly by environmental experience, output units that produced responses to this experience and intermediate or hidden units that connected these input and output units. Information was represented in the brain in the form of "graded connection strengths" between stimuli processed in the input units and responses generated by the output units (McClelland, 1995, p. 171). Learning occurred as a result of a simple associative process (the back-propagation rule) that matched predictions made by the model with representations of the way things actually occurred in reality. Differences between what was predicted by the model and what actually occurred were used to adjust the weights between these connections. Changes in children's thinking resulted from gradual adjustments between these connections (McClelland, 1995).

Computer simulation studies showed that when this system was presented with information, forced to make choices or decisions, and provided with feedback on the sufficiency of these choices or decisions, it solved problems the same way children solved problems (Case, 1998a). In one study, the balance beam task was simulated using this model (McClelland, 1995). The model was presented with 625 problems representing various combinations of weight and distance on both sides of the balance beam. When the model was tested using 24 problems developed by Siegler (1981, as cited in McClelland, 1995), it was found the "model's performance corresponded to one of Siegler's rules about 85% of the time" (McClelland, 1995, p. 176). McClelland (1995) concluded that, at least, in a general fashion, the model was able to replicate the sequence of development that had been found in children between 4 and 10 years of age.

McClelland's (1995) connectionist model demonstrated in a mechanistic fashion how a series of small continuous changes, based on the building up of associative links between discrete units of information, could lead to qualitative shifts in children's conceptual understanding. However, one criticism of this model was that the type of learning it portrayed took too long. Many trials were required to produce the type of learning this model portrayed. The associative learning of real children occurred at much faster rates (Case, 1998a). Another criticism of this model was that it was limited in what it could accomplish. Shultz and Schmidt (1991) noted that in the simulation of the balance beam task there was "a strong bias in the training patterns favoring equal distance problems, a local binary representation of weight and distance information and a forced segregation of weight vs. distance information in connections to the hidden units" (p. 635). There was also vacillation between levels at higher levels of understanding (rules 3 and 4) and the highest level of understanding was never achieved (Shultz & Schmidt, 1991).

To address the limitations of connectionist models such as McClelland's, Shultz and Schmidt (1991) proposed a connectionist model that incorporated a modification of the learning process specified by McClelland (1995). This learning process, referred to as the Cascade-

Correlation (Case, 1998a; McClelland, 1995), built "its own network topology by recruiting new hidden units as it learns to solve problems" (Shultz & Schmidt, 1991, p. 635). During the input phase of the learning process new hidden units were recruited. The new hidden units received input from both the input units and existing hidden units. The hidden unit whose pattern best fit the output errors was then selected. The authors suggested the modifications they incorporated into their model provided an explanation for both the qualitative and quantitative changes that occurred in children's thinking. Qualitative or discontinuous changes may result from the formation of new hidden units and quantitative or continuous changes may result from the adjustment of the connection weights (Shultz & Schmidt, 1991). Simulations using this model showed this model learned the rules involved in solving increasingly abstract balance beam problems more effectively and at a much faster rate than the model proposed by McClelland (1995). This model also displayed "spurts" in learning similar to the qualitative shifts described by Piaget (Case, 1998a, p. 776).

The connectionist models described above provided an explanation for one way in which changes in children's thinking occurred as a result of experience. However, these models had limitations. They replicated the functioning of the human brain in a general way and provided only a coarse-grained analysis of the processes children used to solve problems (McClelland, 1995). With the exception of the Cascade-Correlation model proposed by Shultz and Schmidt (1991), metacognitive knowledge was not modelled. These models relied primarily on associative procedures that stemmed from experience and from learning based on trial and error. Novelty and generalization were rarely achieved in these models (Crowley, Shrager, & Siegler, 1997). The input representations were highly structured and feedback from human operators was necessary to build connections between input and output units of these models (McClelland, 1995). However, despite these limitations, connectionist models provided one promising way of modelling human learning (McClelland, 1995).

Siegler's (1996a) research focused on change and variability in young children's cognitive development. His primary research interest was the strategies 4- to 6-year-old children discovered and used as they learned to solve a variety of simple addition and subtraction problems. He proposed an overlapping waves model, based on associative learning and rule-based learning (Case, 1998a), to describe and explain the changes that occurred in the development of children's thought (Siegler, 1996a). Siegler's (1996a) theory was based on observations of how children discovered and used different addition and subtraction strategies (the min strategy, a strategy that entails counting up from the larger addend to obtain answers to addition problems). Using a microgenetic approach, he discovered children used a variety of different strategies as they learned to solve problems and children retained and used a repertoire of more and less sophisticated strategies as their learning progressed. He also found the frequency of children's use of any one strategy changed with age and experience. Children discovered new strategies and ceased to use existing strategies as their learning progressed. Siegler (1996a) conceptualized development as "a gradual ebbing and flowing of the frequencies of alternative ways of thinking, with new approaches being added and old ones being eliminated as well" (p. 86). Changes in conceptual thought were viewed as "continuously changing frequencies of alternative ways of thinking, rather than as a substitution of one way of thinking for another (Siegler, 1996a, p. 87).

In a computer simulation of a model of strategy choice (Strategy Choice and Discovery Simulation [SCADS]), Shrager and Siegler (1998) showed how associative and conceptual or metacognitive processes interacted to produce changes in children's strategy use. Both associative and metacognitive learning mechanisms were included in the model used in Shrager and Siegler's (1998) study. The addition of the metacognitive mechanism made it possible to monitor and evaluate the strategies that were used. In this study Shrager and Siegler found the SCADS model used the same strategies as 4- to 5-year-old children, acquired the strategies in the same order, and discovered the strategies without resorting to trial and error or experiencing prior failure. The model also used a variety of different strategies, chose effective strategies, and used these strategies to successfully solve similar types of problems (Shrager & Siegler, 1998). Like young children, the model discovered the sum strategy and used this strategy to discover the min strategy. The authors concluded both associative and metacognitive processes were necessary to discover new strategies and make effective choices between existing strategies.

To complement Shrager and Siegler's (1998) computer simulation, Rittle-Johnson, Siegler and Alibali (2001) presented an iterative model to describe and explain how conceptual understanding and procedural understanding might interact to produce changes in children's thinking. In this model, they hypothesized increases in conceptual understanding would lead to more accurate problem representations and more accurate problem representations would lead to increases in children's understanding of procedures. They also hypothesized the resulting increases in children's understanding of procedures would lead to further increases in conceptual understanding.

Two microgenetic studies were conducted to evaluate this iterative model (Rittle-Johnson et al., 2001). Both studies focused on the development of the conceptual and procedural understandings necessary to solve decimal fraction problems. The first study investigated the iterative link between conceptual understanding, problem representation, and understanding of procedures. The second study generated causal evidence to support this hypothesized relationship. Seventy-four students (33 girls, 41 boys, mean age 11.8 years) participated in the first study. Fifty-nine fifth grade students (33 girls, 26 boys, mean age 10.6 years) and 58 sixth grade students (28 girls, 30 boys, mean age 11.6) participated in the second study. The children received either conceptual instruction or procedural instruction, combined conceptual and procedural instruction, or no instruction in the first study. The children also received individual instruction that enabled them to construct more accurate representations of a series of number line problems in the second study. The children's performance was evaluated prior to and following instruction in both studies.

The model's predictions were confirmed in both studies. The posttest administered in the first study indicated the majority of the children demonstrated improved procedural and conceptual understanding. The posttest administered in the second study indicated the instructional support that was provided led to increases in the children's procedural understanding. The authors concluded conceptual and procedural understanding developed in an iterative fashion and the problem representations children constructed were an important component of this process. They also concluded individual differences in conceptual knowledge at the beginning of the study were related to "improvements in problem representation" and "amount of improvement in problem representation predicted individual differences in acquiring procedural knowledge" (Rittle-Johnson et al., 2001, p. 360).

However, the Rittle-Johnson et al. (2001) study had one major limitation. Although the regression analysis conducted in the first study showed conceptual understanding predicted the development of procedural understanding, this analysis also showed that when both the conceptual and procedural knowledge pretests were included in the analysis, the pretest that assessed procedural understanding did not predict increased success on either the conceptual or the procedural posttests. The pretest designed to assess procedural understanding may have assessed conceptual understanding rather than procedural understanding. Scores on the pretests and posttests designed to assess procedural understanding both appeared to be related to initial conceptual understanding. More work is needed to investigate this relationship further.

Studies from the information processing perspective showed associative learning processes could lead to the development of sophisticated levels of conceptual understanding on specific tasks, especially when they were combined with some sort of higher-order mechanism that allowed for the evaluation of the products of the associative learning process (Case, 1998a). However, the type of learning modelled in this perspective was local and task-specific. The changes observed in children's thinking were changes that resulted primarily from children's responses to specific environmental stimuli rather than from the action of more general conceptual cognitive processes within the child (Case, 1996b).

Neo-Piagetian Theoretical Perspective

Starting in the 1970s, researchers working within the neo-Piagetian theoretical perspective began to revise aspects of Piaget's theory in an attempt to account for the variability observed in the development of children's thinking and provide a more detailed description of the processes that led to changes in children's thinking (Case, 1985, 1998a; Case & Edelstein, 1993). One of the first researchers to attempt the neo-Piagetian revision of Piaget's theory was Pascual-Leone (1989). Pascual-Leone (1989) analyzed different classes of Piagetian tasks in terms of their functional structure (in terms of the information processing schemes or units of information that must be activated to produce a correct response - a semantic-pragmatic analysis). He explained cognitive growth in terms of M-capacity (the number of units of information children could hold in their working memory while solving problems), and suggested Piaget's stages (qualitative differences in children's conceptual thought) could be interpreted in terms of the number of units children could hold in their working memory at different ages (M-capacity increases with age).

Research on Individual Pathways to Development

Pascual-Leone (1989) suggested changes in children's thinking occurred as a result of two types of learning processes - slow, conditioned learning processes (associative learning processes) and more rapid, attentionally-mediated, logical learning processes (learning processes that abstracted schemes generated by the associative learning processes). Conditioned learning processes resulted from the automatization of associative connections. These learning processes were not related to or constrained by age-related growth of M-capacity and were closely tied to stimulation children received from their learning environments. Logical learning (L-learning) processes resulted from the application of children's attentional resources and conscious thinking processes to solve particular problems. These learning processes were related to and were constrained by age-related growth of M-capacity. M-capacity was necessary for these processes to develop (Pascual-Leone, 1989). Pascual-Leone also suggested an increase in M-capacity opened the way for "a new wave of L-learning" that led to system-wide changes in children's conceptual thought (Case, 1998a, p. 780). Case {, 1985 #344} described a series of studies that tested Pascual-Leone's hypotheses of the relationships between M-capacity and L-learning. The results confirmed Pascual-Leone's hypotheses. The results showed that the M-capacity available to children was related to the complexity of the control structures children were able to construct.

Pascual-Leone (1970; 1989) explained variability in children's performance on Piagetian tasks in terms of the number of units of information children were able to mobilize or attend to in order to solve the problems the tasks presented. He also made a distinction between children's maximal M-capacity and children's functional M-capacity. For example, older children were able to mobilize more units of information than younger children. Children who used a particular set of processes to solve certain types of problems were able to mobilize more units of information than children who used a different set of processes to solve these problems even though their total M-capacity was the same. From this analysis he concluded different children (field-dependent and field-independent children) respond differently to certain classes of Piagetian tasks (conservation tasks) because they learned to respond to the demands presented by the tasks (facilitating or misleading cues) with different sets of processes that required the mobilization of different amounts of M-capacity (maximal as opposed to functional M-capacity) (Pascual-Leone, 1989). In this way Pascual-Leone was able to make a distinction between cognitive style and cognitive development and suggest a way in which individual differences could be predicted on a variety of Piagetian tasks (Pascual-Leone, 1969, 1974, as cited in Case, 1998a).

In a more recent longitudinal study, de Ribaupierre and Rieben (1995) confirmed Pascual-Leone's contention that it was possible to articulate and describe different developmental pathways within the context of Piaget's theory and provided a detailed description of two new developmental pathways through which children's conceptual thought might progress. Individual differences in children's responses to a variety of Piagetian tasks and the effect these differences had on their later school careers were investigated in three phases, over a period of 11 years. A battery of eight Piagetian tasks was administered to 154 6- to-12-year-old children in the first phase of the study. The tasks were readministered to all but the oldest age group (12-year-olds) three years later, in the second phase of the study. The school records of 105 of the children who participated in the second phase of the study were reviewed eight years later, in the third phase of the study. Factor analyses were performed on the data. A general factor and two group factors emerged. The general factor corresponded to developmental level (the complexity of the children's cognitive structures). The two group factors corresponded to two different modes of processing (developmental pathways) proposed by the authors, an analogical mode of processing and a propositional mode of processing. Although the study showed individual differences in developmental pathways exist, it did not clearly show how these differences were related to children's later school careers. Comparisons between the children's processing modes and the schools they attended after completion of secondary school showed the children who later attended university ranked higher on the general factor and tended to be more propositional in their mode of processing. However, these comparisons also showed children who later attended art and technical schools also ranked higher on the general factor and tended to be more propositional in their modes of processing. The authors concluded a variety of factors other than performance on Piagetian tasks probably accounted for choice and success of later school careers.

Pascual-Leone (1970; 1989) introduced methodological innovations that made it possible to investigate individual pathways to development within the Piagetian theoretical perspective (Case & Edelstein, 1993). He identified several alternate pathways to development and proposed a model of change that accommodated associative and attentionally mediated learning (Case & Edelstein, 1993). However, he did not suggest a new way of characterizing children's thought in each of the major stages described by Piaget. He also did not differentiate between processes that explained how children's thinking changed as they moved from one major stage of development to the next and processes that explained how children's thinking changed as they progressed within each major stage of development (Case & Edelstein, 1993). De Ribaupierre and Rieben (1995) strengthened Pascual-Leone's notion of individual pathways within the Piagetian theoretical perspective by confirming the group factors previously been identified by Pascual-Leone (Larivée et al., 2000). They also identified two additional pathways through which children's conceptual thought might develop (Larivée et al., 2000). However, they also did not explain how changes in children's thinking came about as they moved from one stage of development to the next (Larivée et al., 2000).

Research on Individual Pathways to Development and the Processes of Change

Other researchers working within the neo-Piagetian theoretical perspective also introduced methodological innovations that made it possible to incorporate a focus on individual pathways to development within the Piagetian theoretical framework and provided a more detailed and realistic characterization of children's thought within each major stage of development. However, unlike Pascual-Leone and de Ribaupierre and Rieben, these researchers provided explanations for how children's thinking changed as children moved from one major stage of development to the next.

Fischer (1980) combined a detailed description of the skills children used to solve problems in specific contexts with a general description of how children's thinking changed from one major developmental stage to the next. He conceptualized development as the construction of increasingly complex skills and emphasized characteristics of both the child and the context in the construction of these skills (Fischer, Hand, Watson, Van Parys, & Tucker, 1984). He suggested Guttman-type scaling methods (methods that used ordered sets of tasks) (Fischer et al., 1993) should be used to track and describe individual differences in the developmental pathways of different groups of children as they acquired intellectual (Knight & Fischer, 1992), social (Fischer et al., 1984) and psychological (Fischer et al., 1997) competencies.

In a study on learning to read words, Knight and Fischer (1993) described how POSI (Partially Ordered Scaling of Items) analysis (a statistical technique based on Guttman scaling) and pattern analysis were used to detect individual differences (differences in the order of skill acquisition) in the developmental pathways of primary school children as they learned to read specific words (integrated the visual and phonetic skills involved in reading words). When the children's performance profiles were compared, three different developmental pathways emerged (the pathways of good readers, bad readers, and a mixed group of good and bad readers) (Knight & Fischer, 1992).

In an earlier series of studies, Fischer et al. (1984) described how skill theory (1980) was used to detect individual differences (differences in developmental level or developmental range) in the developmental pathways of preschool children as they constructed an understanding of social roles. When the children's levels of performance were compared across different situations and contexts (situations and contexts with high and low support), individual differences in developmental range were consistently found (Fischer et al., 1984).

Fischer et al., (1993) also suggested process methods should be used to track and describe individual differences in developmental pathways at the molecular level. In a study on building bridges, Parziale (1997; 2002) described how skill theory (Fischer, 1980) and the microdevelopmental approach were used to detect individual differences in the developmental pathways of ten pairs of fifth grade and seventh grade children as they constructed bridges (using marshmallows and toothpicks) during a single forty-five minute session. He found the children used three different mechanisms (shift of focus, bridging and distributed cognition) throughout the session to generate ideas about how to construct their bridges and proposed a bi-directional model to explain how these mechanisms facilitated the development of the children's ideas (Parziale, 1997, 2002).

Van Geert (1994), in collaboration with Fischer, proposed a dynamic growth model (combined skill theory with mathematical growth cures) to explain how skills acquired in specific situations might contribute to the development of children's conceptual thought. In a preliminary investigation of this model (using group data), they showed points in a developmental transition where different specific variables (modelled graphically as s-shaped curves) contributed to changes in the level of children's conceptual thought (van Geert, 1994). This model has since been used to explore how high-support and low-support social contexts contribute to variability in the development of the self-descriptions of Korean adolescents (Fischer & Kennedy, 1997).

Case (1996a; 1996b; 1998a) also proposed a dynamic growth model (combined the notion of central conceptual structures with mathematical growth curves) to explain how general and specific aspects of thought might contribute to the development of children's conceptual thought. Case's (1996a; 1996b; 1998a) model of the process of structural change described a hierarchical learning loop facilitated both "the original assembly and consolidation" of new higher order central conceptual structures and " the gradual expansion of tasks to which a new structure is relevant" (Case, 1998a, p. 791) within the working memory capacity children had available to them at particular stages in their development. The hierarchical learning loops described by Case (1996a; 1996b) provided a more detailed description of how the process of structural change and the transfer of conceptual understandings occurred.

The bootstrapping process that was set up by the reciprocal relationship between specific learning (learning that occurred in specific situations) and general learning (learning that led to general conceptual understanding) made it possible to make predictions about cultural and social class differences in the development of children's central conceptual structures. A prediction that could be made about social class differences was that the overall rate of children's development in particular domains would be either accelerated (children from advantaged backgrounds) or decelerated (children from disadvantaged backgrounds) as a result of the standardizing influence of central conceptual structures in these domains on different specific tasks in these domains (tasks on which children had different amounts of practice or experience) (Case, 1996a, 1996b). A prediction that could be made about cultural differences was that children from different cultural backgrounds (children from Japan and America) would show similar rates of development in particular domains (quantitative domain) as a result of the standardizing influence of central conceptual structures in these domains on different specific tasks in these domains (tasks that received different degrees of emphasis in different cultures) (Case, 1996a, 1996b).

Although Case's model of the process of structural change (1991c; 1996a; 1998a) is still in the early stages of its development, a good fit has been found between data on the development of children's conceptual understanding in different cultures (Case, Okamoto, Henderson, McKeough, & Bleiker, 1996; Case, Stephenson, Bleiker, & Okamoto, 1996) and different social classes (Case, Okamoto et al., 1996), and theoretical growth curves derived from the mathematized version of the model. This indicated the model may be used to explain the reciprocal relationship hypothesized to exist between understandings that develop in specific contexts or situations (physical causality, telling time, money and transactions with money, social causality and school math) and more general, central conceptual understanding (understanding of number) (Case, 1996b, 1998a). For example, adjusting the model to reflect the development of children's central conceptual understanding in different cultures (increasing growth of different specific tasks by 5%) resulted in a negligible change in the growth of the central conceptual structure (Case, 1996c). This was consistent with the finding that although Japanese children received greater amounts of practice in certain specific mathematical understandings, their overall level of conceptual development in the numerical domain was similar to that of American children who received comparable amounts of practice on other specific understandings (Case, Stephenson et al., 1996). Adjusting the model to reflect the development of children's central conceptual understanding in different social classes (reducing growth of all specific tasks by 1%) resulted in a noticeable change in the growth of the central conceptual structure (reduced growth) (Case, 1996c). This was consistent with the finding that the lower levels of practice children from the lower-class backgrounds received on all specific tasks resulted in a lower level of development of their conceptual understanding as compared to children from middle-class backgrounds (Case, Okamoto et al., 1996). The preliminary work suggested Case's (1996a; 1996b; 1998a) model of the process of structural change could be used to explore the effects of developmental variables that underlie giftedness, since giftedness is known to produce variations in the development of children's central conceptual structures.

Porath (1991b) proposed a model to explain the intellectual development of academically advanced children. She suggested academically advanced children were similar to children of more average ability in terms of the development of their conceptual understandings. However, they were different from children of more average ability in terms of the rate at which they acquired specific understandings within particular domains. In a series of studies, Porath (1991b; 1993; 1996a; 1996b) consistently found the general conceptual understandings of academically advanced children did not exceed the general conceptual understandings of more average-ability children by more than 2 years. However, more specific understandings were acquired at a much more rapid rate and the thinking characteristic of each developmental stage was applied in a more elaborate and flexible way. For example, in a study that compared the narrative development of verbally precocious and average ability 6-year-old children, Porath (1991b; 1996b) found the verbally precocious children's central conceptual understanding in this domain was only moderately advanced as compared to the central conceptual understandings of the average-ability children. However, she found significant elaboration of their central conceptual understanding and significant advancement in the development of specific verbal skills (a large vocabulary, the use of mature words, the use of a variety of different words, and the use of more complex grammatical structures); skills not influenced by the development of central conceptual understanding (Porath, 1996b).

A microgenetic study was also conducted from Case's (1991c; 1996a; 1998a) neo-Piagetian theoretical perspective. In this study McKeough and Sanderson (1996) used the microgenetic approach advocated by Catán (1986a) to explore individual differences in the developmental pathways of five, 4-year-old children as they responded to an instructional program that was designed to facilitate the construction of the mental story line, a central conceptual structure that develops in the domain of social/narrative understanding. They found all of the children in the study integrated and consolidated the mental story line. However, individual differences were apparent in the children's use of stereotypic plot sequences, stereotypic elaborative sequences, and original event sequences (McKeough & Sanderson, 1996). They also found the children's learning progressed in variable rather than in a linear fashion. When support was provided the children's learning increased. When the demands of the tasks exceeded the children's information processing capacity, the children's learning decreased (McKeough & Sanderson, 1996). The finding that the children's learning progressed a variable fashion was supported by a time series analysis of seven (from a total of 22 children in the experimental group) Grade 1 children's story telling performance in a study comparing the effectiveness of two instructional programs (a developmental approach and a process oriented approach to teaching narrative) (McKeough, Davis, Forgeron, Marini, & Fung, 2005).

The work of Fischer, Parziale, Case, Porath, and McKeough adds to and extends the work of Pascual-Leone and de Ribaupierre by investigating a wider variety of individual pathways to development and proposing mechanisms to explain how these individual pathways are constructed. The work of Fischer and his colleagues (Fischer et al., 1997; Fischer & Canfield, 1986; Fischer et al., 1993) contributes further by highlighting the importance of both individual and contextual factors in development. The work of Case and his colleagues (Case, 1996a, 1998a; Case, Griffin, McKeough, & Okamoto, 1991; Case, Stephenson et al., 1996; Porath, 1991b) contributes further by confirming the existence of central conceptual structures, mid-level conceptual structures that represent preferred developmental pathways within particular domains.

Although the work of Pascual-Leone and de Ribaupierre is limited in some ways, the changes these researchers made to Piaget's theory and the methods they used to investigate individual differences (detailed task analyses, a focus on individuals rather than groups, and the use of longitudinal designs) also contribute significantly to the study of individual pathways to development (Larivée et al., 2000).

Research on Mathematically Precocious Young Children

Few studies have investigated the mathematical capabilities of young (preschool and kindergarten) mathematically precocious children. Most of the work in this area has focused on the mathematical capabilities of older school-aged children (Robinson et al., 1996). However, two studies (both studies were included in a larger longitudinal study) investigated differences in the mathematical capabilities of young mathematically precocious children.

Robinson, Abbott, Berninger, and Busse (1996) investigated a number of questions related to individual differences in the mathematical capabilities of mathematically advanced kindergarten and preschool aged children. Three hundred and ten children (61 girls and 78 boys in kindergarten; 77 girls and 94 boys in Grade 1) participated in the study. The children were selected on the basis of a score of 98% or higher on the Arithmetic subtests of the Wechsler Preschool and Primary Scale of Intelligence-Revised (WPPSI-R) and the Kaufman Assessment Battery for Children (K-ABC). A battery of quantitative, verbal, and spatial measures was administered. Two quantitative measures (the Number Knowledge Test and the Word Problems subtest) and two working memory measures (the Counting Span Test and the Visual-Spatial Span Test) from Case's (1991c; 1996a; 1998a) neo-Piagetian theory of cognitive development were included in the battery of measures. The results showed all of the children performed at advanced levels on all of the measures that were administered (mathematical and non-mathematical). The results also showed that relationships between cognitive factors were similar for both the boys and the girls. However, the boys performed at higher levels than the girls on the quantitative measures and the visual-spatial working memory measure. The boys also showed a higher correlation between the verbal and spatial factors ($r = .36$ as compared to $r = .07$). The authors concluded it was possible to identify advanced mathematical capabilities in very young children and educational accommodations must be made as soon as the children entered school. They also suggested more work needed to be done to investigate gender differences found in the study.

Robinson, Abbott, Berninger, Busse, and Mukhopadhyay (1997) investigated the "stability and modifiability" (p. 146) of the development of the mathematical capabilities of two groups of mathematically advanced young children (kindergarten to Grade 2). Two hundred seventy-six children participated in the study. The children were selected on the basis of the criteria used in the Robinson et al. (1996) study. Half of the children received Saturday enrichment classes in addition to regular classroom mathematics instruction and half of the children received regular classroom mathematics instruction. The measures used by Robinson et al. (1996) were used in this study. Pre- and posttests were administered. The entire battery of measures was administered in the pretest and again in the posttest. The results showed children in both the treatment and the control groups either maintained their previous level of performance or showed improved levels of performance on all of the measures. However, the boys made greater gains than the girls on the quantitative and visual-spatial measures. The correlation between the verbal and visual-spatial factors was also higher for the boys. The children in the treatment group benefited from the enrichment classes. Their level of performance on the quantitative measures increased. The correlation between the verbal and the quantitative factors was also higher for the children in the treatment group at the end of the study. The authors concluded mathematically precocious young children maintain high levels of mathematical performance after they enter school and benefit from enrichment in this domain. They also concluded more work needs to be done to determine the causes of the gender differences also found in this study.

These studies indicated that for some children a great deal of learning occurred in the mathematical domain before they started school. These studies also indicated higher levels of performance in this domain were maintained as the children continued in school.

Part 2

Central Conceptual Structure in the Domain of Whole Number Understanding

Two separate cognitive structures underlie 4- to 5-year-old children's mathematical understandings: a counting schema that is verbal, digital, and sequential and a global quantity schema that is spatial, analogical, and non-sequential (Case, 1996a, 1998a, 1998b; Griffin, 2005; Kalchman et al., 2001). Children use the counting schema to subitize and count small sets of objects, touch each object once when counting, and say how many objects are in a set. They use the global quantity schema to transform small sets of objects (add objects or take objects away) and determine which of two sets is bigger or smaller (Case, 1996a, 1998a; Case & Mueller, 2001; Griffin, 2005). Diagrams of the counting schema and global quantity schema (Case, 1996a) are shown in Appendix C.

Between 5 and 6 years of age, as children make the transition to a more sophisticated level of mathematical understanding, the content of their schemas for counting and estimating quantities becomes more elaborated and differentiated (additional units of information are added to each schema; individual units of information are viewed as separate entities) (Case & Mueller, 2001; Griffin, 2005). As the content of children's schemas for counting and estimating quantities becomes more elaborated and differentiated, the content of these schemas is gradually linked together and mapped to form the mental counting line: a higher order cognitive structure that integrates the verbal, digital and sequential components of the counting schema and the spatial analogical, non-sequential components of the global quantity schema (Case, 1996a, 1998a, 1998b; Case & Mueller, 2001; Griffin, 2005).

The development of the mental counting line transforms children's schemas for counting and estimating quantities into a single cognitive structure. This cognitive structure represents the core elements of 6-year-old children's mathematical understanding in the domain of whole number understanding (Case, 1996a, 1998a; Griffin & Case, 1997). The development of the mental counting line permits children to understand quantity in numerical terms (as a continuous series of points expressed as numbers) (Griffin et al., 1994) and use counting and the number sequence to solve a variety of mathematical problems (solve addition and subtraction problems and make precise numerical judgements between sets in a variety of quantitative dimensions) (Case, 1996a; Case & Mueller, 2001; Griffin, 2005). An investigation of the correlational patterns of questions related to counting and numerical magnitude has provided empirical evidence for the existence of the mental counting line. This investigation found questions related to counting and numerical magnitude loaded on two separate factors prior to the construction of the mental counting line and loaded on a singe factor following construction of the mental counting line (Y. Okamoto, personal communication, September, 1998 as cited in Case & Mueller, 2001). A diagram of the mental counting line (Case, 1996a) is shown in Appendix D.

Instructional Approach Designed to Facilitate the Construction of the Central Conceptual Structure

Instructional Approach

The instructional approach used in this study was designed to facilitate the construction of the mental counting line (Griffin & Case, 1997; Griffin et al., 1994). This instructional approach involves: (a) knowing where children are at in terms of their current mathematical understandings; (b) providing activities that incorporate and build on children's current mathematical understandings; (c) providing activities that follow the natural developmental sequence of the mathematical understandings represented in the mental counting line; (d) providing activities that encourage children to move back and forth between different representations of the mental counting line; and (e) using physical props (board games) that integrate the digital, verbal, and sequential and spatial, analogical, and non-sequential components of the mental counting line (Griffin & Case, 1997; Griffin et al., 1994).

A test (Number Knowledge test) is used along with instruction to determine children's current level of mathematical understanding and plan instruction that will move children to the next level of mathematical understanding (Griffin & Case, 1997). Teachers guide children's learning by selecting developmentally appropriate activities and by asking questions that encourage children to move to the next level of mathematical understanding (Griffin & Case, 1997). Children learn by participating in group activities, by manipulating instructional materials, and by sharing their developing mathematical understandings with teachers and peers (Griffin & Case, 1997).

The Rightstart and Number Worlds mathematical programs are based on this instructional approach (Griffin, 2004b). The Rightstart mathematical program was initially developed to teach the mathematical understandings represented in the mental counting line to low-income children who had not yet acquired these mathematical understandings (the Rightstart mathematical program was later revised and expanded and the name of the program was changed to Number Worlds) (Griffin, 2004b; Griffin & Case, 1997; Griffin et al., 1992). The instructional units in the Rightstart and Number Worlds mathematical programs follow the natural developmental sequence of the mathematical understandings represented in the mental counting line (Griffin, 2004b; Griffin & Case, 1997; Griffin et al., 1992). The games and activities included in the instructional units are designed to help children construct and integrate each of the mathematical understandings represented in the mental counting line (Griffin, 2004b; Griffin & Case, 1997; Griffin et al., 1992).

Studies using the Rightstart and Number Worlds mathematical programs have shown the understanding represented in the mental counting line can be successfully taught. Low-income kindergarten children who participated in these mathematical programs in large urban centres in Ontario, Massachusetts, and California outperformed children who did not participate in these programs on all measures that required some form of quantitative understanding (the children performed at the 6-year-old level on tasks designed to assess the mathematical understanding represented in the mental counting line) (Griffin & Case, 1996; Griffin et al., 1995; Griffin et al., 1994). A follow-up of this study indicated the children who participated in the Rightstart and Number Worlds mathematical programs were better able to profit from the traditional mathematics instruction offered at the end of kindergarten and in Grade 1 (Griffin & Case, 1996; Griffin et al., 1995; Griffin et al., 1994). A second, longitudinal study indicated the children who participated in the Number Worlds program from kindergarten to the end of Grade 2 consistently made gains on the Number Knowledge test (a test used throughout the study to assess the understandings represented in the mental counting line) and eventually outperformed children who did not participate in the Number Worlds program (Griffin & Case, 1996).

Advantages of the Instructional Approach

The instructional approach used in this study is based on research on the development of the mental counting line and research on learning and instruction (Case & Griffin, 1990; Griffin, 2004b; Griffin & Case, 1997; Griffin et al., 1994). Research on the development of the mental counting line provides a detailed description of the mathematical understandings children acquire as they negotiate the critical transition from the relational to the dimensional stages of Case's (1991c; 1996a; 1998a) neo-Piagetian theory of cognitive development (Case, 1998a; Case & Griffin, 1990; Griffin & Case, 1997; Griffin et al., 1994). Research on learning and instruction provides principles and techniques for the design of effective instruction (Case & Bereiter, 1984).

The instructional approach used in this study combines well-articulated knowledge objectives (knowledge objectives based on the facts, skills, concepts, and procedures represented in the mental counting line) with a hands-on, child-centred approach to teaching the mathematical understandings represented in the mental counting line (Griffin, 2004b; Griffin & Case, 1997; Griffin et al., 1994). Developmentally appropriate materials and activities are used (Griffin, 2004b; Griffin & Case, 1997; Griffin et al., 1994). The sequencing of these materials and activities follows the natural developmental sequence of the mathematical understandings represented in the mental counting line (Griffin, 2004b; Griffin & Case, 1997). Learning is conceptualized as a process of helping children build connections between mathematical understandings that were previously unrelated (understandings of quantity and counting represented in the counting schema and global quantity schema) (Griffin, 2004b). The materials and activities (board games; number lines) are designed to help children make these connections (make connections between quantities and number symbols) (Griffin, 2004b; Griffin & Case, 1997; Griffin et al., 1994). There is a focus on dialogue, play, problem-solving, and reflection (Griffin & Case, 1997; Griffin et al., 1994). Teachers are encouraged to listen to children and to build on children's current mathematical understandings (Griffin, 2004b; Griffin & Case, 1997; Griffin et al., 1994). Children are allowed to learn at their own pace and are encouraged to participate in a variety of independent and group activities (Griffin & Case, 1997; Griffin et al., 1994).

Microgenetic Approach

The microgenetic approach "is a promising tool" (Lee & Karmiloff-Smith, 2002, p. 249) for investigating the process of change (Lee & Karmiloff-Smith, 2002; Siegler & Crowley, 1991) and individual pathways in development (Fischer et al., 1993; Siegler, 1996b). The microgenetic approach focuses on individual children, critical transitions in development, and intra-individual and inter-individual variability in development (Lee & Karmiloff-Smith, 2002; Siegler, 1996a). A focus on individual children and critical transitions in development enables researchers to obtain detailed accounts of the process of change (Lee & Karmiloff-Smith, 2002; Siegler, 1996a). A focus on intra-individual and inter-individual variability in development provides researchers with "meaningful information about developmental change" (Lee & Karmiloff-Smith, 2002, p. 249). Increases in intra-individual variability suggest children may be moving through developmental transitions (Lee & Karmiloff-Smith, 2002; Siegler, 1996a). Increases in inter-individual variability suggest children may be following different developmental pathways as they move through developmental transitions (Lee & Karmiloff-Smith, 2002).

The microgenetic approach was first used by Sander (a member of Felix Kreuger's German Gestalt School of psychology) in the latter part of the 19th century. Sander believed perceptual events " are realized gradually, over time, through the operation of lawfully organized mental activity" (Catán, 1986a, p. 253). He developed microgenetic techniques that artificially created perceptual events in laboratory settings (in single experiments) so the development of these perceptual events could be observed and described (Catán, 1986a). The microgenetic approach was later adopted by Werner in the early part of the 20th century. Werner believed psychological phenomena that developed over longer periods of time could be miniaturized and modelled over shorter periods of time (Catán, 1986a). He developed microgenetic techniques that artificially miniaturized and accelerated the development of psychological phenomena that typically developed over longer periods of time so the origin and development of these psychological phenomena could be observed and described (Catán, 1986a). Although Sander and Werner used the microgenetic approach in different ways, both Sander and Werner viewed the microgenetic approach as a legitimate way "to visibly actualize or externalize the development of internal representations and the mechanisms whereby they were constructed" (Catán, 1986a, p. 256).

Current researchers have used the microgenetic approach in two different ways (Catán, 1986a, 1986b). Different assumptions and purposes underlie each way of using the microgenetic approach. The first way of using the microgenetic approach is similar to the way Werner used the microgenetic approach (Catán, 1986a). Major developmental changes, such as those that occur during developmental transitions, are the focus of the research. An instructional procedure is used to miniaturize and accelerate developmental phenomena that would naturally develop over a longer period of time (Catán, 1986a). Inhelder, Sinclair, and Bovet's (1974) study investigating the schemes children used during equilibration is an example of the first way of using the microgenetic approach. The second way of using the microgenetic approach is similar to the way Sander used the microgenetic approach (Catán, 1986a). The use of a concept, skill, or strategy within one or more observational sessions is the focus of the research. Steps in the use of the concept, skill, or strategy are observed as each step naturally occurs (Catán, 1986a). Siegler and Jenkin's (1989) study investigating how young children acquired addition strategies is an example of the second way of using the microgenetic approach (Catán, 1986b).

Catán (1986a) suggested the second way of using the microgenetic approach focused more on learning than on development and was, therefore, a misinterpretation of the microgenetic approach. However, this suggestion may be somewhat extreme. As Siegler and

Crowley (1991) pointed out, it is often difficult to distinguish "neatly between phenomena that reflect development and phenomena that reflect learning" (Siegler & Crowley, 1991, p. 607). Also, both ways of using the microgenetic approach provide detailed accounts of change, and good quality information about change is better than no information at all (Siegler & Crowley, 1991). Long-term change and short-term change may also share common features (Werner, 1948 as cited in Siegler & Crowley, 1991). Understanding what these common features are and how they contribute to both types of change depends on detailed accounts from both sources of information about change (Siegler & Crowley, 1991).

Both ways of using the microgenetic approach share the following characteristics (1997). These characteristics are (a) observing participants' behaviour from the beginning to the end of a period of conceptual change, (b) observing participants' behaviour at frequent intervals during a period of conceptual change (dense sampling), and (c) basing inferences about changes in participants' conceptual understanding on trial-to-trial analyses of the participants' response (1997). Dense sampling on consecutive occasions throughout a period of conceptual change is an important aspect of the microgenetic approach (1997). Dense sampling allows researchers to (a) observe how particular conceptual changes occurred, (b) determine which contextual factors may have contributed to the changes that occurred, and (c) make inferences about mechanisms that may be responsible for the changes that occurred (1997; Siegler & Crowley, 1991).

Rationale for Combining Case's Neo-Piagetian Theoretical Perspective and the Microgenetic Approach

Case's (1991c; 1996a; 1998a) neo-Piagetian theory of cognitive development, Case's (1996a; 1996b; 1998a) model of the process of structural change, the instructional approach designed to facilitate the construction of the mental counting line (Griffin & Case, 1997; Griffin et al., 1994), and Porath's (1991b) model of the intellectual development of academically advanced children provide a context within which Catán's (1986b) version of the microgenetic approach can be applied. Case's (1991c; 1996a; 1998a) neo-Piagetian theory of cognitive development describes "large-scale developmental processes" (Catán, 1986a, p. 46) from their point of origin to the complete development of a cognitive structure. The sequence of changes that results in the construction of the mental counting line is one of these developmental processes (Case, 1996a, 1998a). Case's (1996a; 1996b; 1998a) model of the process of structural change explains how central conceptual structures such as the mental counting line are formed. Porath's (1988; 1991b) model of the intellectual development of academically advanced children describes one alternate pathway in the construction of the mental counting line. The instructional approach designed to facilitate the construction of the mental counting line (Griffin & Case, 1997; Griffin et al., 1994) miniaturizes and accelerates the changes involved in the construction of the mental counting line and provides consecutive occasions on which these changes can be observed.

Case's neo-Piagetian theoretical perspective and Catán's (1986b) version of the microgenetic approach were combined for the following reasons. There is a need to conduct a detailed analysis of the process of change at the individual level as children construct the mental counting line (Jackson, 1993; Siegler, 1996b). A microgenetic analysis of the process of change during the critical transition from the relational to the dimensional stages of this central conceptual structure has not yet been done. There is a need to link Case's (1991c; 1996a; 1998a) neo-Piagetian theory of cognitive development, Case's (1996a; 1996b; 1998a) model of the process of structural change, and Porath's (1988; 1991b) model of the intellectual development of academically advanced children to behavioural data at the individual level (Case, 1996c; Siegler, 1996b). There is a need to integrate the results of studies conducted at the individual level (microgenetic studies) with the results of existing macrodevelopmental studies (Case, 1996c; Siegler, 1996b).

A microgenetic analysis of the process of change during the critical transition from the relational to the dimensional stages of the mental counting line will provide greater insight into the circumstances and mechanisms of change (Siegler, 1996a), will provide a deeper understanding of individual pathways to development (Case & Edelstein, 1993; Fischer et al., 1993; McKeough & Sanderson, 1996), and will lead to improvements in the instructional program that was designed to facilitate the construction of the mental counting line (McKeough & Sanderson, 1996). Linking Case's (1991c; 1996a; 1998a) neo-Piagetian theory of cognitive development, Case's (1996a; 1996b; 1998a) model of the process of structural change, and Porath's (1988; 1991b) model of the intellectual development of academically advanced children to behavioural data at the individual level will provide evidence of the validity of Case's (1996a; 1996b; 1998a) theory of cognitive development, Case's (1996a; 1996b; 1998a) model of the process of structural change, and Porath's (1988; 1991b) model of the intellectual development of academically advanced children (Siegler, 1996b). Integrating the results of a microgenetic study of the construction of the mental counting line with the results of existing macrodevelopmental studies from Case's (1996a; 1996b; 1998a) neo-Piagetian theory of cognitive development will lead to a more comprehensive description of the process of change

(Lee & Karmiloff-Smith, 2002) and will enhance the credibility of the results of existing macrodevelopmental studies from this theoretical perspective (Case, 1996c; Jackson, 1993).

Chapter 3: Method

Research Questions

The research questions were based on Case's (1996a; 1996b; 1998a) model of the process of structural change. More specifically, the research questions were based on the "requirements that would have to be met" (Case, 1998a, p. 778) to construct a new higher order central conceptual structure. Case's (1996a; 1996b; 1998a) model of the process of structural change describes the changes involved in the construction of new higher order central conceptual structures (differentiation, linking across, mapping, and consolidation), the learning processes hypothesized to cause these changes to occur (C-learning or associative learning and M-learning or attentionally mediated learning), and the dynamic relationship that exists between "children's learning in specific situations" and "their more general structural understanding" (Case, 1996a, p. 21). The "requirements that would have to be met" (Case, 1998a, p. 778) to construct a new higher order central conceptual structure specify in detail the changes that occur in children's cognitive structures as children construct a new higher order central conceptual structure, such as the mental counting line, from two existing lower order conceptual structures (the counting schema and global quantity schema).

The requirements are:

1. The content of children's schemas must be elaborated and differentiated (linkages within children's existing schemas must be created).
2. The content of children's schemas must be linked together (linkages must be created across children's existing schemas).
3. The content of these schemas must then be mapped (the content of children's existing schemas must be reorganized).
4. Children must acquire a second-order symbol system (written numerals) to represent the elements of the new structure that has been created (the elements of the new structure must be represented as objects).
5. The new structure that has been created must be used to solve new tasks and problem situations (the new structure that has been created must be applied to specific problem situations).

The research questions are:

1. When and how will the children elaborate and differentiate the content of their schemas? (When and how will the children come to know that adding one unit to the number 4 gives you the number 5 in a counting sequence?)
2. When and how will the children make linkages between the content of their schemas? (When and how will the children come to know that the number 5 involves a greater quantity than the number 4?)
3. When and how will the children map together the content of their schemas in order to abstract a new higher-order numerical principle? (When and how will the children come to know that adding one unit to a set of objects will result in a number that is one step further along in a string of numerals and that taking away one unit from a set of objects will result in a number that is one step behind in a string of numerals?)
4. When and how will the children apply the mental counting line to new tasks and problem situations?
5. When and how will the children acquire the written symbols used to represent the elements of the mental counting line?

A further question that is related to the working memory capacity that is necessary for the structural changes that have been described to occur is:

6. Will the children in the study show an age-related correspondence between their performance on the relational and dimensional measures of conceptual understanding in the domain of whole number understanding and the relational and dimensional measures of working memory capacity?

The research questions were based on the "requirements that would have to be met" (Case, 1998a, p. 778) to construct a new higher order central conceptual structure for the following reasons: (a) the changes specified by these requirements describe specific steps in the construction of the mental counting line (describe a specific developmental sequence), (b) the changes specified by these requirements have been translated into behaviours that have been operationally defined (Griffin & Case, 1997), and (c) the changes specified by these requirements and the learning processes hypothesized to underlie these changes can be inferred from the behaviours that have been operationally defined (Catán, 1986b).

Study Design

An embedded (more than one unit of analysis), multiple-case design (Yin, 1994) was used for this study.[8] The microgenetic approach advocated by Catán (1986a; 1986b) was used to collect and analyze the data. The researcher was the "primary instrument for gathering and analyzing the data" (Merriam, 1998, p. 20). Intra-individual and inter-individual variability in the rate and the pattern of construction of the mental counting line and the transfer of the understanding represented in the mental counting to new tasks and problem situations were observed and described (Lee & Karmiloff-Smith, 2002; Yin, 1994).

An instructional program was used to "implement the assumptions and principles of the microgenetic approach" (Lee & Karmiloff-Smith, 2002, p. 253). The instructional program stimulated and accelerated the construction of the mental counting line (Catán, 1986b; Inhelder et al., 1974; Lee & Karmiloff-Smith, 2002; McKeough & Sanderson, 1996). Artificially stimulating and accelerating the construction of the mental counting line made it possible to observe the changes that occurred in the children's thinking over a much shorter period of time (Catán, 1986b; Inhelder et al., 1974; Lee & Karmiloff-Smith, 2002; McKeough & Sanderson, 1996).

The instructional program spanned the entire period from the beginning to the end of the construction process (Catán, 1986b; Inhelder et al., 1974; Lee & Karmiloff-Smith, 2002; McKeough & Sanderson, 1996) The activities included in the instructional program were designed to bring about the changes that occurred in the children's thinking in as natural a way as possible (Catán, 1986b; Inhelder et al., 1974; Lee & Karmiloff-Smith, 2002; McKeough & Sanderson, 1996). The children's pre-instructional levels of whole number understanding were taken into account and the skills and conceptual understandings that make up the mental counting line were introduced in the order in which they are naturally acquired (Griffin & Case, 1997; Griffin et al., 1995; Griffin et al., 1994). The instructional sessions were based on the kinds of activities thought to promote the construction of the mental counting line outside the context of an instructional setting (Case, 1998a; Griffin, 2004a; Griffin et al., 1995; Griffin et al., 1994).

Dense sampling was used to obtain a detailed account of the construction process (Catán, 1986b; Lee & Karmiloff-Smith, 2002; McKeough & Sanderson, 1996). Data were collected each

[8] This was a case study. Microgenetic studies examine the performance of individual children (Lee & Karmiloff-Smith, 2002).

time the children were given an opportunity to respond (Lee & Karmiloff-Smith, 2002). Changes in the children's cognitive structures as they constructed the mental counting line and the learning processes hypothesized to cause these changes to occur were directly inferred from the children's responses to the instruction that was provided during each instructional session (Catán, 1986b; Lee & Karmiloff-Smith, 2002; McKeough & Sanderson, 1996).

Pretest and posttest measures were administered. The pretest measures included a screening measure, two measures of domain-specific mathematical skills, four measures of conceptual understanding, and two measures of working memory capacity. With the exception of the screening measure, the posttest measures were the same as the pretest measures.

The screening measure was administered to determine whether the children displayed average to above-average mathematical ability for their age and to assess individual differences in the children's mathematical ability. Since the goal of the study was to explore how individual children responded to the instructional program, individual differences in the children's mathematical ability were assessed.

The measures of domain-specific mathematical skills were administered to assess the children's level of domain-specific mathematical skills at the beginning and the end of the instructional program (Kaufman & Kaufman, 1983; Wechsler, 1989). One assumption of the microgenetic approach is that "a child's developmental history may affect subsequent development" (Lee & Karmiloff-Smith, 2002, p. 249). For this reason it was important to obtain a precise measure of the children's domain-specific mathematical skills before the children participated in the instructional program and after the children completed the instructional program.

The measures of working memory capacity were administered to determine whether there was a relationship between the children's working memory capacity and their level of conceptual understanding at the beginning and the end of the instructional program. Studies from Case's (1985; 1991c; 1996a) neo-Piagetian theory of cognitive development have shown a correlation between children's working memory capacity and their level of conceptual understanding (Case, 1985; Griffin & Case, 1996; McKeough, 1991; Porath, 1991b).

The measures of conceptual understanding were administered to ensure that the children had not yet constructed the mental counting line at the beginning of the instructional program and to determine the extent to which the children had constructed the mental counting line at the end of the instructional program (Case & Sandieson, 1991; Inhelder et al., 1974; McKeough & Sanderson, 1996). The Number Knowledge test directly assessed the children's progress in the construction of the mental counting line (items on this measure reflected the content of the instructional program) (Griffin & Case, 1996, 1997). The remaining three conceptual measures assessed the children's ability to apply (transfer) the conceptual understanding represented in the mental counting line to novel tasks and problem situations (items on the remaining three conceptual measures did not reflect the content of the instructional program) (Griffin & Case, 1996).

The screening measure and the measures of conceptual understanding were used to select the children for the study (McKeough & Sanderson, 1996). Only children who had average to above-average mathematical ability for their age and who had not yet constructed the mental counting line were selected for the study.

The researcher assumed primary responsibility for collecting, analyzing, and interpreting the data (Lee & Karmiloff-Smith, 2002; Merriam, 1998). When the researcher assumes primary responsibility for collecting, analyzing and interpreting the data "all observations and analyses are filtered through that human being's worldview, values, and perspective" (Merriam, 1998, p. 22). For this reason, the researcher must be sensitive to the biases or preconceptions that he or she brings to the study (Merriam, 1998).

My biases or preconceptions are both personal and professional. They are reflected in my roles as a mother, teacher and researcher. As a mother and a teacher, I have had experience caring for and educating young children. As a researcher, I have become aware of the contributions cognitive psychology has made to education and child development. As a result of my position, I have become sensitive to both individual differences in the development of young children and the need for developmentally appropriate instruction. I feel my position has contributed important knowledge and experience that assisted me in designing and carrying out this study.

Pilot Study

A pilot study was conducted to determine how the children would respond to the pretest and posttest measures and to the tasks presented during each instructional session. The pretest measures included: (a) a screening measure, the Basic School Skills subtest of the Early Screening Profiles (Harrison, 1990); (b) two measures of domain-specific mathematical skills, the Arithmetic subtest of the Kaufman Assessment Battery for Children (Kaufman & Kaufman, 1983) and the Arithmetic subtest of the Wechsler Preschool and Primary Scale of Intelligence-Revised (Wechsler, 1989); (c) four measures of conceptual understanding, the Number

Knowledge test (Griffin & Case, 1997), the Balance Beam task (Case, Okamoto et al., 1996; Marini, 1991), the Money Knowledge task (Case, Okamoto et al., 1996) and the Birthday Party task (Case, Okamoto et al., 1996; Marini, 1984); and (d) two measures of working memory capacity, the Counting Span test (Case, 1985) and the Visual Spatial Span test (Crammond, 1991). With the exception of the screening measure, the posttest measures were the same as the pretest measures.

Seven children, 6 girls and 1 boy, who had not yet constructed the mental counting line participated in the study. The children ranged in age from 3 years 10 months to 4 years 6 months. A non-random selection procedure was used. The children were selected on the basis of the following criteria: (a) a score in the average or above-average range on the Basic School Skills subtest of the Early Screening Profiles; (b) a score below 1.5 on the Number Knowledge test; and (c) a score below 1.5 on at least two of the three remaining conceptual measures, the Balance Beam task, Money Knowledge task and Birthday Party task. Scores below 1.5 on the Number Knowledge test indicated that children had not yet constructed the mental counting line or were just beginning to construct the mental counting line (Case, Okamoto et al., 1996; Case & Sandieson, 1991). Scores below 1.5 on two of the three remaining conceptual measures indicated that children had not yet consolidated the mental counting line and were not yet able to use the mental counting line to solve a range of specific problems that depended upon the understanding represented in the mental counting line (Case, Okamoto et al., 1996; Case & Sandieson, 1991).

The selection/pretest measures were administered in the 2- to 3-week period prior to instruction. The measures were administered on separate days, 1 to 2 days apart. One measure was given per day. The Basic School Skills subtest of the Early Screening Profiles was administered first, followed by the Number Knowledge test, and the Balance Beam, Money Knowledge, and Birthday Party tasks. The Counting Span test (Case, 1985) and the Visual Spatial Span test (Crammond, 1991) were administered next, followed by the Arithmetic subtest of the Kaufman Assessment Battery for Children and the Arithmetic subtest of the Wechsler Preschool and Primary Scale of Intelligence-Revised. The instruction began 2 days after administration of the last selection/pretest measure.

The instructional program consisted of five instructional units. Each instructional unit consisted of four 10- to 15-minute instructional sessions. The instructional units were taught over a 5-week period. One instructional unit was taught each week. The children participated in one instructional session a day, 4 days a week. The posttest measures were administered 2 to 3 days after completion of the last instructional session. The posttest measures were administered on separate days, 1 to 2 days apart. One measure was given per day. The Number Knowledge test was administered first, followed by the Balance Beam, Money Knowledge and Birthday Party tasks, the Visual Spatial Span test (the Counting Span test was dropped), the Arithmetic subtest of the Kaufman Assessment Battery for Children, and the Arithmetic subtest of the Wechsler Preschool and Primary Scale of Intelligence-Revised. All of the measures were administered individually, by the researcher. All of the measures were presented verbally. Verbal and nonverbal responses were required.

The administration of the selection/pretest and posttest measures indicated that some of the measures would have to be dropped from the study. The administration of a screening measure, along with two measures of domain-specific skills, four measures of conceptual understanding and two measures of working memory capacity was too demanding for children between 4 and 5 years of age. As a result, the Basic School Skills subtest of the Early Screening Profiles (Harrison, 1990) (the screening measure), the Arithmetic subtest of the Kaufman Assessment Battery for Children (Kaufman & Kaufman, 1983) and the Arithmetic subtest of the Wechsler Preschool and Primary Scale of Intelligence-Revised (Wechsler, 1989) (the measures of domain-specific skills) were dropped from the study and were replaced with the quantitative component (Verbal and Nonverbal Quantitative Reasoning subtests) of the Stanford-Binet Intelligence Scales, Fifth Edition (SB5) (Roid, 2003a).

The lower levels of the quantitative component (Verbal and Nonverbal Quantitative Reasoning subtests) of the SB5 (Roid, 2003a) assess many of the same mathematical facts, skills, and conceptual understandings (facts, skills, and conceptual understandings related to whole number understanding) as the Basic School Skills subtest of the Early Screening Profiles (Harrison, 1990), the Arithmetic subtest of the Kaufman Assessment Battery for Children (Kaufman & Kaufman, 1983), and the Arithmetic subtest of the Wechsler Preschool and Primary Scale of Intelligence-Revised (Wechsler, 1989) (the ability to identify and name numbers, count, say how many objects are in a set, say which object is bigger, say which set has more and solve simple addition and subtraction problems using objects, pictures or numbers).[9] However, the Basic School Skills subtest of the Early Screening Profiles (Harrison, 1990), the Arithmetic

[9] The Number Knowledge test (Griffin & Case, 1997) provides a fine-grained assessment of the facts, skills, and conceptual understandings related to whole number understanding (facts, skills, and conceptual understandings that make up the mental counting line) (Griffin, 2005). Many of the items on the Number knowledge test also assess competencies indicative of number sense (computational fluency, understanding of language associated with quantity, and ability to reflect on and explain mathematical reasoning underlying problem solutions) (Griffin, 2005).

subtest of the Kaufman Assessment Battery for Children (Kaufman & Kaufman, 1983), and the Arithmetic subtest of the Wechsler Preschool and Primary Scale of Intelligence-Revised (Wechsler, 1989) place greater emphasis on verbal quantitative capabilities (Roid, 2003a). The quantitative component of the SB5 (Roid, 2003a) places equal emphasis on verbal and nonverbal quantitative capabilities (verbal and nonverbal capabilities are assessed on separate subtests that yield separate scores) (Roid, 2003a). This makes it possible to evaluate verbal and nonverbal aspects of the children's performance separately (Roid, 2003a).

The more balanced emphasis on both verbal and nonverbal mathematical capabilities fits well with Case's (1991c; 1996a; 1998a) theoretical perspective and the goals of the study. The mental counting line is made up of verbal, digital and sequential components and spatial, analogical and non-sequential components (Case, 1998a). The instructional program that was designed to facilitate the construction of the mental counting line focuses on the development of both the verbal and nonverbal components of the mental counting line (Griffin et al., 1992). The items on the lower levels of the Verbal and Nonverbal Quantitative Reasoning subtests of the SB5 (Roid, 2003a) also more closely reflect the skills and conceptual understandings that make up the mental counting line (ordering and comparing numbers and quantities from 1 to 9).

The Counting Span test (Case, 1985) and the Visual Spatial Span test (Crammond, 1991) (the measures of working memory capacity) were also dropped from the study, along with the research question that corresponded to these measures. The Counting Span test (Case, 1985) and the Visual Spatial Span test (Crammond, 1991) were difficult to administer to 4- to 5- year- old children. The children also had difficulty responding to these measures. As a result, it was not possible to obtain a reliable assessment of the children's working memory capacity.

The conceptual measures were retained because the focus of the study was the development of the children's conceptual understanding. These measures were administered during the pretest and posttest. The Verbal and Nonverbal Quantitative Reasoning subtests of the SB5 (Roid, 2003a) were administered during the pretest, but not during the posttest. The Number Knowledge test (a measure of conceptual understanding administered during the posttest) assessed many of the same mathematical capabilities as the Verbal and Nonverbal Quantitative Reasoning subtests of the SB5 (Roid, 2003a). The mathematical skills and conceptual understandings the children acquired as they progressed through the instructional program were assessed during the microgenetic analysis of the data.

The children's responses to the tasks presented during each instructional session indicated some components of the instructional program needed to be redesigned and additional components needed to be added. The instructional program was expanded from five to nine instructional units, the period of instruction was extended from 5 to 7 weeks, and the duration of the instructional sessions was changed from 10- to 15-minutes to 5- to 10-minutes.

During the pilot study it was noticed the children found it difficult to count backwards and touch an object once each time a number word was said. The children also had difficulty distinguishing between addition and subtraction. When asked to subtract, the children would add. As a result, more complex activities were broken down into simpler activities that focused on separate elements of the more complex activities. For example, a counting task that involved reciting the number words forward from 1 to 10, reciting the number words backward from 10 to 1, and touching a block each time a number word is said was broken down into three separate tasks - a task that involved reciting the number words forward from 1 to 10 (Counting from 1 to 10 task), a task that involved reciting the number words from 10 to 1 (Blast-Off task), and a task that involved touching an object once each time a number word is said (Build the Tower task). Individual understandings were also taught in separate units. For example, incrementing sets and decrementing sets were taught in separate units rather than in one unit. Incrementing sets was taught first, followed by decrementing sets.

Breaking more complex activities down into simpler activities and teaching individual understandings in separate units ensured that (a) each new skill or conceptual understanding was introduced in the order in which it was naturally acquired, (b) only one new skill or conceptual understanding was introduced at each higher level, (c) each new skill or conceptual understanding was directly related to the skill or conceptual understanding that preceded it, and (d) each new skill or conceptual understanding was more complex than the one that preceded it (Case, 1985). Introducing new skills and conceptual understandings in this way (a) made it easier for the children to acquire and automatize each new skill or conceptual understanding (Case, 1985), (b) allowed for a better fit between the research questions and the units in the instructional program, and (c) made it easier to analyze the data.

An additional component was also added to the instructional program. This new component involved mapping quantities onto the number line and basing comparisons between these quantities on the numbers that represented each quantity. Several of the children in the pilot study had difficulty counting two quantities and then comparing the resulting numbers. These children did not seem to understand that that numbers could be used to compare quantities. Previous research indicated that this understanding was difficult for many children to acquire

(Griffin & Case, 1996). This understanding was also one of several understandings considered "most crucial for success on unidimensional tasks" (Griffin & Case, 1996, p. 102).[10]

More practice (warm-up activities) was incorporated throughout the instructional program to help the children consolidate and automatize lower level skills and conceptual understandings (Case, 1985). Props (a vertical number line and a horizontal number line) were introduced to help the children acquire more difficult skills, maintain the children's interest and keep the children focused on the task. The duration of each instructional session was shortened to maintain the children's motivation.

Present Study

Participants

The participants were selected from two preschools in a large urban centre. One preschool was situated on the campus of a major university (Preschool #1). The other preschool was situated nearby, in an adjoining neighbourhood (Preschool #2). The preschool teachers were approached 6 weeks to a month before the data were collected. The purpose and nature of the study were discussed and the selection procedure was explained.

Consent forms were distributed to the parents of all of the 4- to 5-year-old children in the two preschools. This age range was chosen because previous research from Case's (1991c; 1996a; 1998a) theoretical perspective has shown that children of this age have not yet constructed the mental counting line (Case & Sandieson, 1991; Griffin & Case, 1996; Griffin et al., 1995; Griffin et al., 1991; Okamoto & Case, 1996). The consent form is shown in Appendix E.

The parents of 15 children gave consent for their children to participate in the study. However, only eleven of the children completed the pretesting. Three of the children refused to be tested. One child was absent and did not complete the testing. Of the eleven children who completed the pretesting, six children, four girls and two boys, who had not yet constructed the mental counting line were selected for the study.[11] The children ranged in age from 4 years 0 months to 5 years 1 month. Two of the children were later withdrawn from the study due to lack

---

[10] The modifications that were made to the instructional program increased the number of units in the instructional program from five to nine. The nine units were taught over a seven-week period.

[11] The children who were selected for the study met the selection criteria. The children who were not selected for the study did not meet the selection criteria. This is a case study (a multiple-case design). The children who were selected for the study are not representative of a larger group of children.

of cooperation during the instructional sessions. Four children remained in the study.[12] The children's age, gender, preschool attended and status in the study are shown in Table 1.

Table 1. Age, Gender, Preschool Attended and Status in Study of Children Selected for the Study

Participant	Age in years and months	Gender	Preschool attended	Status
Kevin	4-11	M	Preschool #2	Included
Sarah	4-0	F	Preschool #2	Included
Anne	4-8	F	Preschool #2	Included
Wendy	4-1	F	Preschool #2	Included
Lisa	4-8	F	Preschool #1	Withdrawn
Jason	5-1	M	Preschool #1	withdrawn

The children who remained in the study were from middle- to upper-middle class homes. All of the children were fluent in English. The children's parents were cooperative and enthusiastic about the study. All of the parents were interested in their children's intellectual development and were keen to have their children participate in the study. All of the children were cared for at home. Two of the children had full-time nannies. Two of the children had mothers who stayed at home. The children's homes were child-centred. All of the children were provided with a range of educational experiences, games and toys. However, only one of the children received direct, school-based instruction in mathematical skills (from her mother).

A non-random selection procedure was used. The children were selected on the basis of the following criteria: (a) a score in the average or above-average range on the Quantitative Reasoning factor index of the SB5; (b) a score below 1.5 on the Number Knowledge test; and (c) a score below 1.5 on at least two of the three remaining conceptual measures, the Balance Beam task, Money Knowledge task and Birthday Party task.

Scores in the average or above-average range on the Quantitative Reasoning factor index of the SB5 indicated that the children displayed average or above-average mathematical ability for their age (Roid, 2003a). Factor index scores are standard scores with a mean of 100 and a

[12] The children who remained in the study attended one preschool and the children who were withdrawn from the study attended the other preschool. Selection of subjects across more than one setting was attempted, but was not possible.

standard deviation of 15 (Roid, 2003a). Average scores are within one standard deviation of the mean (Roid, 2003a). Low scores are one to two standard deviations below the mean (a score of 85 is at the 16th percentile or one standard deviation below the mean; a score of 70 is at the 2nd percentile or two standard deviations below the mean) (Roid, 2003a). High scores are one to two standard deviations above the mean (a score of 115 is at the 84th percentile or one standard deviation above the mean; a score of 130 is at the 98th percentile or two standard deviations above the mean) (Roid, 2003a).

Scores below 1.5 on the Number Knowledge test indicated that children had not yet constructed the mental counting line or were just beginning to construct the mental counting line (Case, Okamoto et al., 1996; Case & Sandieson, 1991). Scores above 1.5 on the Number Knowledge test indicated that children were close to completing construction of the mental counting line (Case, Okamoto et al., 1996; Case & Sandieson, 1991). Children who were close to completing construction of the mental counting line were not included in the study (Inhelder et al., 1974).

Scores below 1.5 on two of the three remaining conceptual measures indicated that children had not yet consolidated the mental counting line and were not yet able to use the mental counting line to solve a range of specific problems that depended upon the understanding represented in the mental counting line (Case, Okamoto et al., 1996; Case & Sandieson, 1991). Scores above 1.5 on at least two of the three remaining conceptual measures indicated that the children were in the process of consolidating the mental counting line and were able to use the mental counting line to solve specific problems that depended upon the understanding represented in the mental counting line (Case, Okamoto et al., 1996; Case & Sandieson, 1991).

Of the children who remained in the study, 2 children scored within the average range (obtained average scores) and 2 children scored above the average range (obtained high scores) on the Quantitative Reasoning factor index of the SB5. All of the children scored below 1.5 on the Number Knowledge test and on at least two of the three remaining conceptual measures. The children's pretest scores on the Quantitative Reasoning factor index of the SB5, the Number Knowledge test, the Balance Beam, Money Knowledge and Birthday Party tasks are shown in Table 2.

Table 2. Pretest Scores on the Quantitative Reasoning Factor Index of the SB5, the Number Knowledge Test, the Balance Beam, Money Knowledge and Birthday Party Tasks[13]

Participant	SB5 Quantitative Reasoning Factor Index		Number Knowledge Test	Balance Beam Task	Money Knowledge Task	Birthday Party Task
	standard score	percentile rank	developmental level score			
Kevin*	114	82	1.1	1.0	1.2	1.5
Sarah*	116	86	1.0	1.0	1.0	1.0
Anne*	108	70	1.1	1.0	1.0	1.0
Wendy*	125	95	1.3	2.0	0.8	1.0
Lisa	108	70	1.2	1.5	1.0	1.0
Jason	103	58	1.3	1.0	1.2	1.0

* Indicates children who remained in the study

The children's performance on the Quantitative Reasoning subtests of the SB5, the Number Knowledge test and the Balance Beam, Money Knowledge and Birthday Party tasks indicated that they were able to: (a) say the number words from one to ten, (b) count small sets of objects, (c) say how many objects were in small sets, (d) say which of two sets was bigger or had more when the difference between the sets was large, (e) recognize the numerals from one to ten, and (f) solve simple addition and subtraction problems when they had objects to count. The children's performance on these measures also indicated that they were not able to: (a) say which of two sets was bigger or had more when the difference between the sets was small, (b) order numbers from 1 to 10, (c) solve simple addition and subtraction problems when they did not have objects to count, and (d) say which of two numbers was bigger or smaller when the difference between the numbers was small.

Preschools

The preschools the participants attended were play-based and child-centred. The curriculum of both preschools was based on a constructivist approach to instruction. Children were viewed as active learners who constructed their own knowledge and understandings. The

[13] The scores on the Number Knowledge test and the Balance Beam, Money Knowledge, and Birthday Party tasks (measures of conceptual understanding) ranged from 1.0 (Level 1 or predimensional level) to 2.0 (Level 2 or dimensional level).

learning environments of both preschools were organized around a variety of learning centres that focused on large-group, small-group, and individual activities (art centre, sand table, big blocks, dress-up centre, house centre, and story centre). The instructional programs of both preschools focused on learning through concrete experiences and active interaction with teachers and peers rather than on acquiring specific facts and skills. Stories were read and discussed at group time (some stories focused on number concepts). Materials were provided for creative, self-directed art activities and for art activities that were more structured. Games, puzzles, and art materials were set out on tables. Children were encouraged to follow their own interests and choose activities that interested them. Numbers and letters of the alphabet were displayed on the walls and were included in some of the games and puzzles (alphabet floor puzzle, number puzzle, number lotto game). Opportunities to learn number concepts were provided through play (matching patterns, counting objects, sorting objects, and measuring amounts). Spontaneous, informal math instruction was provided at group time and centre time if the children were interested and the opportunity presented itself (How many can play here?; Take one away. How many?). The development of positive social relationships was actively encouraged.

Procedure

The selection/pretest measures were administered in the 2-week period prior to instruction. The selection/pretest measures were administered on separate days, 1 to 2 days apart. One measure was given per day. The Nonverbal Quantitative Reasoning subtest of the SB5 was administered first, followed by the Verbal Quantitative Reasoning subtest of the SB5. The Number Knowledge test was administered first, followed by the Balance Beam, Money Knowledge and Birthday Party tasks. The instruction began 2 days after administration of the last selection/pretest measure. The instructional sessions were conducted 5 days a week over a 7-week period. The children participated in one instructional session each day. The posttest measures were administered 2 to 3 days after completion of the last instructional session. The posttest measures were administered on separate days, 1 to 2 days apart. One measure was given per day. The Number Knowledge test was administered first, followed by the Balance Beam, Money Knowledge and Birthday Party tasks.

The pretest/selection measures were administered individually, by the researcher in a separate room of one preschool and in a separate room or quiet corner of the other preschool. The testing sessions were scheduled at the convenience of the preschool teachers. The posttest measures were administered in a quiet room in the children's own home. The testing sessions were scheduled at the convenience of the children's families. All of the measures were presented verbally. Verbal and nonverbal responses were required. The Verbal and Nonverbal Quantitative Reasoning subtests of the SB5 were administered according to the standard procedure outlined in the Examiner's Manual, Item Books and Record Form of the SB5 (Roid, 2003a). The Number Knowledge test and the Balance Beam, Money Knowledge and Birthday Party tasks were administered according to the procedures outlined in Case, Okamoto et al. (1996). Each measure took approximately 10 to 15 minutes to administer. Verbal assent was obtained from each of the children before the testing began. The script for the children's verbal assent is shown in Appendix F.

The instructional sessions were conducted in the same locations as the testing, in both preschools for the first 6 weeks of the instructional program. The instructional sessions were scheduled in the morning, at the convenience of the preschool teachers. For the final week of the instructional program, the instructional sessions were conducted in the children's own home (the preschool program had ended for the summer). The instructional sessions were scheduled in the morning or afternoon, at the convenience of the children's parents. The children were instructed individually in each instructional session. The instruction was provided by the researcher (Lee & Karmiloff-Smith, 2002).The instructional sessions lasted approximately 5 to 10 minutes. Each instructional session was audiotaped and videotaped. The videotapes were later transcribed by the researcher (Siegler & Stern, 1998). The transcriptions of the videotapes were coded by the researcher and an independent rater who was familiar with Case's (1991c; 1996a; 1998a) neo-Piagetian theory of cognitive development.

Before each testing and instructional session the children were asked if they would like to play a number game. Every effort was made to ensure that participation in the testing and instructional sessions was a pleasurable experience.

Measures

Screening Measure

<u>Verbal and Nonverbal Quantitative Reasoning Subtests of the SB5</u>

The Verbal and Nonverbal Quantitative Reasoning subtests of the SB5 were designed to assess the verbal and nonverbal mathematical abilities of children as young as 2 years of age (Roid, 2003a). Both subtests are composed of testlets. The testlets are ordered according to age level (Roid, 2003a). Testlets at lower age levels of the Nonverbal Quantitative Reasoning subtest assess children's ability to make global judgements of quantity (say which object is bigger or which set has more), construct small sets with objects, add or subtract with objects, recognize numerals and determine the relative magnitude of objects or numerals (Roid, 2003a). Testlets at lower age levels of the Verbal Quantitative Reasoning subtest assess children's ability to recite the number words, count and say how many objects are in a set, name numerals, add or subtract with pictured objects and solve simple number word problems (Roid, 2003a). Both subtests focus on the ability to reason mathematically. However, the Verbal Quantitative Reasoning subtest focuses more on the kinds of mathematical skills acquired in an instructional setting. The Verbal and Nonverbal Quantitative Reasoning subtests are combined to form the Quantitative Reasoning factor index. The Quantitative Reasoning factor index score has a mean of 100 and a standard deviation of 15. The SB5 is reliable and valid and has been adequately standardized in the United States in terms of sex, age, race, ethnicity, socioeconomic level and geographical location (Bain & Allin, 2005; D'Amato & Johnson, 2004). The internal consistency reliability of the SB5 factor index scores is high. When reliability coefficients for the factor index scores were computed the values ranged from .91 to .98 (Bain & Allin, 2005). The value for the Quantitative Reasoning factor index was .92 (D'Amato & Johnson, 2004). The concurrent validity of the SB5 has been demonstrated by comparisons with the Wechsler Intelligence Scales, the Woodcock-Johnson Tests of Cognitive Abilities and an earlier version of the Stanford-Binet (Bain & Allin, 2005; D'Amato & Johnson, 2004). A correlation of .90 was found between the full scale IQ of the SB5 and the composite score of the Stanford-Binet Intelligence Scale: Fourth Edition. A correlation of .78 was found between the full scale IQ of the SB5 and the Woodcock-Johnson III Cognitive General Intellectual Ability (Bain & Allin, 2005) and a correlation of .80 was found between the full scale IQ of the SB5 and the Wechsler Preschool and Primary Scale of Intelligence-Revised (Roid, 2003b). The predictive validity of the SB5 has been demonstrated by comparing the factor index and IQ scores of the SB5 with the achievement subtest scores of the Woodcock-Johnson III Tests of Achievement (WJ III ACH) and the achievement subtest scores of the Wechsler Individual Achievement Test-Second Edition (WIAT-III) (Roid, 2003b). The correlation between the Quantitative Reasoning factor index of the SB5 and the Math Reasoning subtest of the WJ III ACH was .65 (Roid, 2003b). The correlation between the Quantitative Reasoning factor index and the Math subtest of the WIAT-III was .69 (Roid, 2003b). For both measures, the correlations were higher than "the expected average correlation of .60 between IQ and achievement" (Sattler, 1988 as cited in Roid, 2003b).

The Verbal and Nonverbal Quantitative Reasoning subtests were scored by the researcher. The subtests were scored according to the hand-scoring procedure outlined in the Examiner's Manual (Roid, 2003a).

Measures of Conceptual Understanding

Number Knowledge Test

The Number Knowledge test was designed to assess children's understanding of whole number relationships (children's progress in construction of the mental counting line) (Griffin & Case, 1996, 1997) The questions included in the Number Knowledge test are based on the content of the instructional program (Griffin & Case, 1996, 1997). The Number Knowledge test assesses children's knowledge of domain-specific mathematical skills (Griffin & Case, 1996, 1997).

There are four levels to this test. The four levels of the test correspond to the four substages described for the dimensional stage of Case's (1991c; 1996a; 1998a) neo-Piagetian theory of cognitive development. The understanding that is described at each level of the test is acquired during a two-year period. This two year period is represented by the age range that is assigned to each level (Griffin, 2005). The questions presented at each level of the test are based on the problem-solving strategies described at each substage of the dimensional stage (Case, Okamoto et al., 1996; Griffin & Case, 1997).

Questions presented at Level 1 (predimensional level) focus on the counting schema and global quantity schema that are constructed between 3.5 and 5 years of age (Case, Okamoto et al., 1996; Griffin & Case, 1997). Children at this level are able to use their counting schema to count small sets of objects and use their global quantity schema to say which set is bigger or has more when there is a large difference in the size of each set (make global distinctions between quantities) (Griffin, 2005). However, they are not yet able to associate numbers with quantities (Case, Okamoto et al., 1996; Griffin, 2005).

Questions presented at Level 2 (unidimensional level) focus on the mental counting line that is constructed between 5 and 7 years of age (Case, Okamoto et al., 1996; Griffin & Case, 1997). Children at this level are able to use the mental counting line to compare one-digit numbers and say which number is bigger or smaller, solve simple addition and subtraction problems using one-digit numbers, say which number comes one or two numbers after a specified one-digit number and order one-digit numbers according to size (Case, Okamoto et al., 1996; Griffin, 2005).

Questions presented at Level 3 (bidimensional level) focus on the mental counting line that was constructed between 5 and 7 years of age and a second mental counting line that is constructed between 7 and 9 years of age (Case, Okamoto et al., 1996; Griffin & Case, 1997).

Children at this level are able to use both mental counting lines to compare two two-digit numbers and say which number is bigger or smaller, say which number comes four or five numbers after a specified two-digit number, say how many numbers come between two one-digit numbers and solve two-digit addition and subtraction problems that do not involve regrouping (Case, Okamoto et al., 1996; Griffin, 2005).

Questions presented at Level 4 (integrated bidimensional level) focus on the coordination of the two mental counting lines that occurs between 9 and 11 years of age (Case, Okamoto et al., 1996; Griffin & Case, 1997). Children at this level are able to use the two mental counting lines in a coordinated fashion to compare the differences between one- or two-digit number pairs and say which difference is bigger or smaller, say which number comes 9 or 10 numbers after a specified two- or three-digit number and solve two-digit addition and subtraction problems that involve regrouping (Case, 1998a; Case, Okamoto et al., 1996). A diagram of the complexity of children's mathematical understanding at each level of the test (Case, 1996d) is shown in Appendix G.

Four to nine questions are presented at each level. Testing began at the Preliminary Level and continued until the children failed over half of the questions at a level (Case, Okamoto et al., 1996). The children's responses were recorded manually by the researcher. The Number Knowledge test and test materials are shown in Appendix H.

Balance Beam Task

The Balance Beam task was designed to assess children's understanding of how weight and distance affect the operation of a balance beam (Case, Okamoto et al., 1996; Marini, 1984). This task was also used to assess whether the children in the study had consolidated the mental counting line (Griffin & Case, 1996). Because numbers are associated with the variables of weight and distance, an understanding of whole number relationships is required for this task (Case & Griffin, 1990; Case & Sandieson, 1991).

The balance beam consists of a wooden beam with nine pegs on each side, a central support, two wooden support blocks (one under each arm of the beam) and ten metal washers (Marini, 1984). A stack of washers is placed on each side of the beam. The weight of each stack and the distance each stack is from the fulcrum determine which side of the beam will go down. Children are presented with a series of problems. The weight and position of the stacks are changed for each problem. Each time the children are presented with a problem, they are asked to predict which side of the beam will go down and provide a justification for their prediction (Marini, 1991).

There are four levels to this task. The four levels of the task correspond to the four substages described for the dimensional stage of Case's (1991c; 1996a; 1998a) neo-Piagetian theory of cognitive development. The understanding that is described at each level of the task is acquired during a two-year period. This two-year period is represented by the age range that is assigned to each level (Griffin, 2005). The problems presented at each level of the task are based on the problem-solving strategies described at each substage of the dimensional stage (Case, Okamoto et al., 1996; Marini, 1984).

Problems presented at Level 1 (predimensional level) focus on the counting schema and the global quantity schema that are constructed between 3.5 and 5 years of age (Case, Okamoto et al., 1996; Marini, 1984). Children at this level are able to use their global quantity schema to estimate the size of the stacks on each side of the beam when there is a large difference in the size of each stack (make global distinctions between the quantities) (Case, Okamoto et al., 1996; Marini, 1984). However, they are not yet able to associate numbers with quantities (Griffin, 2005; Marini, 1984).

Problems presented at Level 2 (unidimensional level) focus on the mental counting line that is constructed between 5 and 7 years of age (Case, Okamoto et al., 1996; Marini, 1984). Children at this level are able to use the mental counting line to determine the exact weight of the stacks on each side of the beam (count the number of washers in each stack) when there is a small difference in the size of each stack and the distance from the fulcrum is held constant (Case, Okamoto et al., 1996; Marini, 1984).

Problems presented at Level 3 (bidimensional level) focus on the mental counting line that is constructed between 5 and 7 years of age and a second mental counting line that is constructed between 7 and 9 years of age (Case, Okamoto et al., 1996; Marini, 1984). Children at this level are able to use the second mental counting line to determine the exact distance each stack is from the fulcrum (count the number of pegs between each stack and the fulcrum) when the weight of each stack is held constant (Case, Okamoto et al., 1996; Marini, 1984).

Problems presented at Level 4 (integrated bidimensional level) focus on the coordination of the two mental counting lines that occurs between 9 and 11 years of age (Case, Okamoto et al., 1996; Marini, 1984). Children at this level are able to use the two mental counting lines in a coordinated fashion to effect a compensation between the weight of the stacks and the distance the stacks are from the fulcrum (compute the difference in the weight on each side of the beam and the difference in the distance from the fulcrum on each side of the beam and compare the two differences or compute the sum of the weight and distance on each side of the beam and compare the two sums) when the weight and distance on each side of the beam varied and there was a small difference between the weight and distance on each side of the beam (Case, Okamoto et al., 1996; Marini, 1984). A diagram of the complexity of children's mathematical understanding at each level of the task (Case, 1996d) is shown in Appendix G.

The balance beam was placed on the table in front of the children. The blocks under the arms of the beam were removed so that the children could see how the weight and position of the washers affected the movement the beam. The children were told that the washers were all the same weight and that the pegs were all the same distance apart. They were also told that this was a fair task, that there were no tricks to the task. The children were allowed a short period of practice and were then given the following instructions: "I am going to put some washers on these pegs. When the washers are on the pegs, I want you to tell me which side of the beam will go down when the blocks are taken away. I also want you to tell me why that side of the beam will go down." (Porath, 1988). The children's answers and justifications were recorded manually by the researcher. The blocks under the arms of the beam were removed after the children's answers and justifications were recorded so that the children could see which side of the beam would go down (Marini, 1991).

A basal-ceiling method of presentation was used. Two problems are presented at each level. Testing began at Level 1 and continued until the children failed both problems at a level (Marini, 1991). The Balance Beam task apparatus and problems are shown in Appendix I.

Money Knowledge Task

The Money Knowledge task was designed to assess children's understanding of money and the handling of money (Case, Okamoto et al., 1996; Griffin et al., 1991). This task was also used to assess whether the children in the study had consolidated the mental counting line (Griffin & Case, 1996). Because the value of money is expressed in terms of numbers, an understanding of whole number relationships is required for this task (Case & Griffin, 1990; Case & Sandieson, 1991).

There are four levels to this task. The four levels of the task correspond to the four substages described for the dimensional stage of Case's (1991c; 1996a; 1998a) neo-Piagetian theory of cognitive development. The understanding that is described at each level of the task is acquired during a two-year period. This two-year period is represented by the age range that is assigned to each level (Griffin, 2005). The questions presented at each level of the task are based on the problem-solving strategies described at each substage of the dimensional stage (Case, Okamoto et al., 1996; Griffin et al., 1991).

Questions presented at Level 1 (predimensional level) focus on the counting schema and global quantity schema that are constructed between 3.5 and 5 years of age (Case, Okamoto et al., 1996; Griffin et al., 1991). Children at this level are able to use their counting schema to count small sets of pennies or loonies and use their global quantity schema to say that a car is worth a lot because it is big, a candy is worth a little because it is small, a nickel is worth more than a dime because it is bigger and a five dollar bill is worth more than a loonie because it is paper (make global distinctions between representations of quantities) (Griffin et al., 1991). However, they are not yet able to associate numbers to representations of quantities (Case, Okamoto et al., 1996; Griffin et al., 1991).

Questions presented at Level 2 (unidimensional level) focus on the mental counting line that is constructed between 5 and 7 years of age (Case, Okamoto et al., 1996; Griffin et al., 1991). Children at this level are able to use the mental counting line to compare coins and bills of different denominations, add dollars or cents to determine the total amount and subtract dollars or cents to determine the amount they will receive in change (Case, Okamoto et al., 1996; Griffin et al., 1991).

Questions presented at Level 3 (bidimensional level) focus on the mental counting line that was constructed between 5 and 7 years of age and a second mental counting line that is constructed between 7 and 9 years of age (Case, Okamoto et al., 1996; Griffin et al., 1991). Children at this level are able to use both mental counting lines to compare combinations of dollars and cents and add and subtract combinations of dollars and cents that do not involve grouping (Case, Okamoto et al., 1996; Griffin et al., 1991).

Questions presented at Level 4 (integrated bidimensional level) focus on the coordination of the two mental counting lines that occurs between 9 and 11 years of age (Case, Okamoto et al., 1996; Griffin et al., 1991). Children at this level are able to use the two mental counting lines in a coordinated fashion to add and subtract combinations of dollars and cents that involve regrouping and express combinations of dollars and cents using standard decimal notation (Case, Okamoto et al., 1996; Griffin et al., 1991). A diagram of the complexity of children's mathematical understanding at each level of the task (Case, 1996d) is shown in Appendix G.

Four to five questions are presented at each level. Testing began at Level 1 and continued until the children failed over half of the questions at a level (Case, Okamoto et al., 1996). The children's responses were recorded manually by the researcher. The Money Knowledge task questions are shown in Appendix J.

Birthday Party Task

The Birthday Party task was designed to assess children's understanding of how an individual's expectations affect his or her emotional response (Case, Okamoto et al., 1996; Marini, 1984). This task was also used to assess whether the children in the study had consolidated the mental counting line (Griffin & Case, 1996). Because numbers are associated with the quantity of presents each child wished for and received, an understanding of whole number relationships is required for this task (Case & Sandieson, 1991; Marini, 1984).

The birthday party task consists of a set of pictures of a boy and a girl named David and Cathy, respectively, and a collection of different coloured marbles. Each card shows the number of presents David and Cathy wished to receive. The presents David and Cathy wished to receive are shown in thought bubbles above their heads. The different coloured marbles are used to indicate the number of presents David and Cathy actually received. The number of presents David and Cathy wished for and received determine how happy they will be. The number of presents wished for and received are changed for each problem. Each time the children are presented with a problem they are asked to predict whether David or Cathy will be happier and provide a justification for their prediction (Marini, 1984).

There are four levels to this task. The four levels of the task correspond to the four substages described for the dimensional stage of Case's (1991c; 1996a; 1998a) neo-Piagetian theory of cognitive development. The understanding that is described at each level of the task is acquired during a two-year period. This two-year period is represented by the age range that is assigned to each level (Griffin, 2005). The problems presented at each level of the task are based on the problem-solving strategies described at each substage of the dimensional stage (Case, Okamoto et al., 1996; Marini, 1984).

Problems presented at Level 1 (predimensional level) focus on the counting schema and the global quantity schema that are constructed between 3.5 and 5 years of age (Case, Okamoto et al., 1996; Marini, 1984). Children at this level are able to use their global quantity schema to estimate the number of presents each child received when the number of presents wished for is held constant and there is a large difference in the number of presents each child received (make global distinctions between quantities) (Case, Okamoto et al., 1996; Marini, 1984). However, the children are not yet able to associate numbers with quantities (Griffin, 2005; Marini, 1984).

Problems presented at Level 2 (unidimensional level) focus on the mental counting line that is constructed between 5 and 7 years of age (Case, Okamoto et al., 1996; Marini, 1984). Children at this level are able to use the mental counting line to determine the exact number of presents each child received (count the number of presents each child received) when the number of presents wished for is held constant and there is a small difference in the number of presents each child received (Case, Okamoto et al., 1996; Marini, 1984).

Problems presented at Level 3 (bidimensional level) focus on the mental counting line that is constructed between 5 and 7 years of age and a second mental counting line that is constructed between 7 and 9 years of age (Case, Okamoto et al., 1996; Marini, 1984). Children at this level are able to use the second mental counting line to determine the exact number of presents each child wanted (count the number of presents each child wanted) when the number of presents each child received is held constant (Case, Okamoto et al., 1996; Marini, 1984).

Problems presented at Level 4 (integrated bidimensional level) focus on the coordination of the two mental counting lines that occurs between 9 and 11 years of age (Case, Okamoto et al., 1996; Marini, 1984). Children at this level are able to use the two mental counting lines in a coordinated fashion to effect a compensation between what each child wanted and received when the amount each child wanted and received varied and there was a small difference between what each child wanted and received (compute the difference between what each child wanted and received and compare the two differences) (Case, Okamoto et al., 1996; Marini, 1984). A diagram of the complexity of children's mathematical understanding at each level of the task (Case, 1996d) is shown in Appendix G.

Before the testing began the children were shown a picture of David and Cathy. They were told that David and Cathy were preparing for their birthday parties and that before their parties they each wished to receive a certain number of presents (researcher points to the thought bubbles above David and Cathy's heads). During their parties they opened their presents and this is what they received (researcher places different coloured marbles on the picture below David and Cathy) (Marini, 1984). The children were then asked: "Who do you think is happier? Is Cathy happier (researcher points to Cathy), is David happier (researcher points to David) or are both David and Cathy happy?". The children were also asked why they thought David was happier, Cathy was happier or both David and Cathy were happy (Marini, 1984). Following the pretest, the pictures of David and Cathy were presented. The children's answers and justifications were recorded manually by the researcher (Marini, 1984).

A basal-ceiling method of presentation was used. Two problems are presented at each level. Testing started at Level 1 and continued until the children failed both problems at a level (Case, Okamoto et al., 1996). The Birthday Party task materials and problems are shown in Appendix K.

Scoring of the Measures of Conceptual Understanding

Scoring for the measures of conceptual level was based on the problem-solving strategies described at each substage of the dimensional stage of Case's (1991c; 1996a; 1998a) theoretical perspective. For each measure a score of 1.0 indicated that the children's responses reflected the strategies described for Level 1(predimensional level), a score of 2.0 indicated that the children's responses reflected the problem-solving strategies described for Level 2 (unidimensional level), a score of 3.0 indicated that the children's responses reflected the problem-solving strategies described for Level 3 (bidimensional level) and a score of 4.0 indicated that the children's responses reflected the problem-solving strategies described for Level 4 (integrated bidimensional) (Case, Okamoto et al., 1996).

For the Number Knowledge test and the Money Knowledge task one point was given for each question that was answered correctly (Case, Okamoto et al., 1996; Griffin & Case, 1997). If a question consisted of two parts, both parts had to be answered correctly in order to obtain a point (Griffin & Case, 1997). Developmental level scores were obtained by computing the mean score for each level and summing the mean scores across the levels (Case, Okamoto et al., 1996). The mean score for Level 1 of the Number Knowledge test included the question presented at the Preliminary Level of this test. Intermediate scores such as 1.2, 1.5 or 1.7 indicated that the children correctly answered some of the questions above a level but not all of the questions representative of the next level.

For the Balance Beam task and the Birthday Party task a score of .5 was given for each problem that was answered correctly (Case, Okamoto et al., 1996). Balance beam task problems were answered correctly if a correct prediction was given and the justification indicated that the problem-solving strategy that was described for that level was used to solve the problem (Marini, 1984). Birthday Party task problems were answered correctly if a reasonable prediction was given and the justification indicated that the problem-solving strategy that was described for that level was used to arrive at the prediction (Marini, 1984). Developmental level scores were obtained by summing the scores for each level and summing the scores across the levels. Intermediate scores such as 1.5 indicated that the children answered both questions at a level correctly and one of the two questions at the next level correctly. The response protocols of all of the measures were scored by the researcher. The response protocols were scored according to the guidelines outlined in Case, Okamoto, Henderson, McKeough and Bleiker (1996). The scoring criteria for the Balance Beam task are shown in Appendix I. The scoring criteria for the Birthday Party task are shown in Appendix K.

Validity and Reliability of the Measures of Conceptual Understanding

The construct validity of the measures of conceptual understanding has been demonstrated by studies that have been done from Case's (1991c; 1996a; 1998a) neo-Piagetian theory of cognitive development. Researchers using a variety of conceptual measures from the mathematical domain consistently found that 4-, 6- 8- and 10-year-old children displayed similar age-related patterns of performance on these measures even though the content and procedural demands varied from one measure to the next (Case & Griffin, 1990; Case, Okamoto et al., 1996; Case & Sandieson, 1991; Griffin & Case, 1997; Griffin et al., 1991; Marini, 1991). A study conducted by Okamoto and Case (1996) showed that the questions included at Level 2 (unidimensional level), Level 3 (bidimensional level), and Level 4 (integrated bidimensional level) of the Number Knowledge test were passed at the ages described in the empirical studies. A latent structural analysis and scalogram analysis of children's responses to the questions included in the Number Knowledge test and a computer simulation of children's responses to the questions included in the Number Knowledge test confirmed that the conceptual understandings described for Level 2 (unidimensional level), Level 3 (bidimensional level), and Level 4 (integrated bidimensional level) of the test were acquired at the ages and in the order described in the empirical studies (Okamoto & Case, 1996). A factor analysis of 6-year-old children's performance on a variety of conceptual measures from the mathematical domain (Number Knowledge test, Balance Beam task, Distributive Juice task, Birthday Party task, Money Knowledge task and Time Telling task) found that these measures all loaded on the quantitative factor. Pearson product-moment correlations among these measures were statistically significant (Case, Okamoto et al., 1996).

The reliability of the measures of conceptual understanding has been demonstrated by studies that have been done from Case's (1991c; 1996a; 1998a) neo-Piagetian theory of cognitive development. Researchers using the Number Knowledge test, Balance Beam, Distributive Juice, Birthday Party, Money Knowledge and Time Telling tasks to assess children's level of conceptual understanding, in different studies, with different groups of 4-, 6- 8- and 10-year -old children consistently found that 4-year-old children performed at Level 1 (predimensional level), 6-year-old children performed at Level 2 (unidimensional level), 8-year-old children performed at Level 3 (bidimensional level), and 10-year-old children performed at Level 4 (integrated bidimensional level) on these measures (Case & Griffin, 1990; Case, Okamoto et al., 1996; Griffin et al., 1991; Marini, 1991; Okamoto & Case, 1996; Porath, 1991b). Researchers using the Number Knowledge test, Balance Beam, Birthday Party, Money Knowledge and Time Telling tasks to assess children's post-instructional level of conceptual understanding, in instructional studies designed to help 5-year-old children acquire, integrate and consolidate the conceptual understandings that make up the mental counting line consistently found that the majority of the children included in the treatment group performed at Level 2 (unidimensional level) on these measures (Case & Sandieson, 1991; Griffin & Case, 1996; Griffin et al., 1995; Griffin et al., 1994). Researchers using the Number Knowledge test to assess children's post-instructional level of conceptual understanding, in a longitudinal study designed to follow the progress of three groups of children (normative, treatment and control group) found that the children included in the treatment group and the normative group performed at Level 2 (unidimensional level) of this test at 6 years of age and at Level 3 (bidimensional level) of this test at 8 years of age (Griffin & Case, 1997).

Instructional Program

The instructional program consisted of nine instructional units. The instructional units were taught over a 7 week period. The content of the instructional units was based on the content of the Rightstart and Number Worlds mathematical programs. Each instructional unit focused on the development of a separate mathematical understanding (Griffin & Case, 1997). Each subsequent instructional unit described a more complex level of mathematical understanding (Case, 1998a).

Each instructional unit included two to four instructional sessions. The instructional sessions were conducted consecutively from Monday to Friday of each week. One instructional session was conducted per day. For example, Session 1 was conducted on Monday of Week 1, Session 2 on Tuesday of Week 1, Session 3 on Wednesday of Week 1 and so on. The instructional units, instructional sessions, content of the instructional units, and the week each instructional unit was taught are shown in Table 3.

Table 3. Instructional Unit, Instructional Session, Content of Instructional Unit and Week Taught

Unit	Session	Content of Unit	Week Taught
1	1 - 2	Reciting Number Words Forward from 1 to 10	1
2	3 - 5	Reciting Number Words Backward from 10 to 1	1
3	6 - 8	Counting Objects	2
4	9 - 11	Counting to Determine Quantity	2 - 3
5	12 - 15	Incrementing Sets	3 - 4
6	16 - 19	Decrementing Sets	4 - 5
7	20 - 23	Using Numbers to Compare Quantities	5 - 6
8	24 - 26	Comparing Quantities in Two Different Quantitative Dimensions	6
9	27 - 29	Acquiring Knowledge of the Written Numerals from 1 to 10	7

Description of the Instructional Units

The units included in the instructional program were designed to help the children integrate and consolidate the mental counting line (Griffin et al., 1992). The sessions included in each instructional unit were designed to help the children acquire the conceptual understanding that was the focus of that instructional unit (Griffin et al., 1992). A single task was presented during each instructional session. Each task operationalized the conceptual understanding that was the focus of the instructional unit (Griffin et al., 1992). Each task accommodated individual differences in rate of acquisition of the conceptual understanding (Griffin et al., 1992). The explicit objectives for each task were related to the level of understanding that was the focus of the task. The implicit objectives for each task were related to the higher level of understanding that was also incorporated into the task (Griffin et al., 1995). Variations were included for some of the tasks to maintain the children's interest and "increase the level of difficulty" of the task (Griffin et al., 1992, p. 7).

Unit 1 (Sessions 1 - 2) Reciting the Number Words Forward From 1 to 10

This unit focused on knowledge of the number words from 1 to 10 and knowledge of the position of each number word in the number word sequence (Griffin & Case, 1996; Griffin et al., 1992). This unit relates to row b in the diagram of the mental counting line (Case, 1998a) (Appendix D).

The Counting From 1 to 10 task (Griffin et al., 1992, p. 28) and Pointing and Winking task (Griffin et al., 1992, p. 31) were included in this unit. These tasks were developed for the Rightstart mathematics program (Griffin et al., 1992). The Counting From 1 to 10 task was designed to teach the number words from 1 to 10 (Griffin et al., 1992). The Pointing and Winking task was designed to teach the number word that comes immediately after each number word in the number word sequence from 1 to 10 (Griffin et al., 1992). The Counting From 1 to 10 task was played in Session 1. The Pointing and Winking task was played in Session 2.

Counting From 1 to 10 Task

Explicit Objectives

- Recite the number words from 1 to 10.
- Recite the number words up from 1 and stop at a given number word (1 to 5, 1 to 6, 1 to 7, and so on to 1 to 10).

Implicit Objective

- Introduce the concept of one-to-one correspondence (between a number word and an action).

Materials

- None.

Procedure

Task

- Sit on the floor facing the child.
- Tell the child, "Today we are going to count from 1 to 5 and from 1 to 10. I want you to count from 1 to 5. I will count and then you can count." (Researcher models the activity and then the child takes a turn).
- Repeat the task with the numbers 1 to 6, 1 to 7, 1 to 8, 1 to 9 and 1 to 10.
- Prompt the child if he or she does not know a number word.
- Introduce Variation #1 and Variation #2 in Session 1.

Variation #1

- Ask the child to count from 1 to 5 and clap each time he or she says a number word. (Researcher models the activity and then the child takes a turn).

Variation #2

- Ask the child to count from 1 to 5 and clap each time he or she says a number word, but this time the child does not say the number words out loud (Researcher models the activity and then the child takes a turn).

Pointing and Winking Task

Explicit Objectives

- Give the next number word up in the number word sequence, count on from that number word, and stop at a designated number word.

Implicit Objectives

- Introduce the understanding that numbers get bigger as you count up.
- Introduce the understanding that counting up is associated with an increase in quantity.
- Introduce the numerals from 1 to 10.

Materials

- Coloured sticky arrow.
- A 7" x 30" vertical number line showing the numerals from 1 to 10.

Procedure[14]

Task

- Sit on the floor facing the child.
- Place the number line in the vertical position. Make sure the child can see the numerals on the number line.
- Practice counting from 1 to 5, 1 to 6, 1 to 7, 1 to 8, 1 to 9 and 1 to 10.
- Point to each numeral on the number line as the child recites the number words.
- Prompt the child if the child does not remember a number word.
- Tell the child, "Today, I am going to start counting 1, 2, 3. I will stop counting and point to you. You will continue counting, 4, 5. Then I will wink at you. You will stop counting and I will start counting again 6, 7, 8."
- Count until the child makes a mistake or counts to 10.

[14] The children found it hard to stop when winked at. After the task was presented a few times the children were pointed to when it was their turn to count. The children then stopped counting on their own and pointed to the researcher to indicate that it was the researcher's turn to count. The children were prompted or corrected and allowed to continue to count to 10 each time. The sticky arrow was used when the children got to 10.

- Put the coloured sticky arrow on the number line to indicate where the child counted to each time
- Prompt the child if he or she does not remember a number word the first time the task is presented.
- If the child makes a mistake repeat the task from 1.

The materials used in the Pointing and Winking task are shown in Appendix L.

Unit 2 (Sessions 3 - 5) Reciting the Number Words Backward From 10 to 1.

This unit focused on knowledge of the number words from 10 to 1 and knowledge of the position of each number word in the number word sequence (Griffin & Case, 1996; Griffin et al., 1992). This unit relates to row b in the diagram of the mental counting line (Case, 1998a) (Appendix D).

Variations of the Blast-Off task and the Pointing and Winking task were included in this unit (Griffin et al., 1992). These tasks were developed for the Rightstart mathematics program (Griffin et al., 1992). The Blast-Off task was designed to teach the children to recite the number words backward from 10 to 1 (Griffin et al., 1992). The Pointing and Winking task was designed to teach the number word that comes immediately after each number word in the number sequence from 1 to 10 (Griffin et al., 1992). The Blast-Off task was played in Sessions 3 and 4. The Pointing and Winking task was played in Session 5.

<u>Blast-Off Task</u>

<u>Explicit Objectives</u>

- Recite the number words backward from 10 to 1.
- Recite the number words down from a given number word (5 to1, 6 to 1, 7 to 1 and so on to 10 to 1).

<u>Implicit Objective</u>

- Introduce the understanding that numbers get smaller as you count down.
- Introduce the concept of one-to-one correspondence (between a number word and an action).

<u>Materials</u>

- None.

Procedure[15]

Task

- Stand in front of the child.
- Tell the child, "We are going to pretend that we are on a rocket ship. Our rocket ship is going to blast off, but first we have to count down to blast off. I want you to count down from 5 to 1. As you say each number you will crouch down a little bit. When you get down to 1 you will touch the floor, say "Blast-Off" and jump up." (Researcher models the activity and then the child takes a turn.).
- Repeat the task if the child makes a mistake.
- Repeat the task with the numbers 5 to 1 in Session 3 (if the child is having difficulty repeat the task with the numbers 4 to 1 and 3 to 1).
- Repeat the task with the numbers 6 to 1, 7 to 1, 8 to 1, 9 to 1 and 10 to 1 in Session 4.

Pointing and Winking Task

Explicit Objectives

- Give the next number word down in the number word sequence, count down from that number word, and stop at a designated number word.

Implicit Objective

- Introduce the understanding that numbers get smaller as you count down.
- Introduce the understanding that counting down is associated with a decrease in quantity.
- Introduce the numerals from 1 to 10.

Materials

- Coloured sticky arrow.
- Vertical number line showing the numerals from 1 to 10.

[15] The children found it hard to start the task again each time they made a mistake. After the task was repeated a few times the children were prompted or corrected and allowed to continue to count down to 1 each time.

Procedure[16]

Warm-Up

- Sit on the floor facing the child.
- Place the number line in the vertical position. Make sure the child can see the numerals on the number line.
- Practice counting from 5 to 1, 6 to 1, 7 to 1, 8 to 1, 9 to 1 and 10 to 1.
- Point to each numeral on the number line as the child recites the number words.
- Prompt the child if the child does not know a number word.

Task

- Tell the child, "Today, I am going to start counting 10, 9, 8. I will stop counting and point to you. You will continue counting, 7, 6. Then I will wink at you. You will stop counting and I will start counting again 6, 4, 3."
- Count until the child makes a mistake or counts to 1.
- Put the coloured sticky arrow on the number line to indicate where the child counted to each time
- Prompt the child if he or she does not remember a number word the first time the task is presented.
- If the child makes a mistake repeat the task from 10.
- If the child is having difficulty, repeat the task with smaller numbers (5 to 1, 6 to 1, 7 to 1, 8 to 1).

The materials used in the Pointing and Winking task are shown in Appendix L.

Unit 3 (Sessions 6 - 8) Counting Objects

This unit focused on knowledge of the one-to-one correspondence between number words and objects when counting (Griffin & Case, 1996; Griffin et al., 1992). This unit relates to rows b and c in the diagram of the mental counting line (Case, 1998a) (Appendix D).

The Build the Tower task was included in this unit. This task was based on the Snapping and Clapping Game. The Snapping and Clapping Game was developed for the Rightstart mathematics program (Griffin et al., 1992, p. 34) and was designed to teach the concept of one-

[16] The children found it hard to stop when winked at. After the task was presented a few times the children were pointed to when it was their turn to count. The children then stopped counting on their own and pointed to the researcher to indicate that it was the researcher's turn to count. The children were prompted or corrected and allowed to continue to count down to 1 each time. The sticky arrow was used when the children got to 1.

to-one correspondence and introduce the concept of cardinality. The Build the Tower task was played in Sessions 6, 7 and 8.

Build the Tower Task

Explicit Objectives

- Touch each object once each time a number word is said when counting forward and backward.
- Touch each dot once each time a number word is said when counting dots on a dot-set card.

Implicit Objectives

- Introduce the concept of cardinality.
- Reinforce the understanding that numbers get bigger as you count up and smaller as you count down.
- Reinforce the understanding that counting up is associated with an increase in quantity and that counting down is associated with a decrease in quantity.

Materials

- Set of 10 3.5" square, different coloured wooden blocks.

Procedure

Warm-Up

- Sit on the floor facing the child.
- Practice counting from 1 to 10 and 10 to 1.
- Move hand up each time the child says a number word when counting forward and down each time the child says a number word when counting backward.
- Prompt the child if he or she does not remember a number word.

Task

- Place the blocks on the floor in front of the child.
- Tell the child, "We are going to build a tower with the blocks. I want you to count each block as you build the tower up. When the tower is finished I want you to take the blocks off the tower and count backward as you take the blocks off the tower." (Researcher models the activity and then the child takes a turn).
- Prompt the child if he or she does not remember a number word.
- Repeat the task with 5, 6 and 7 blocks in Session 6.
- Repeat the task with 5, 6, 7, 8, 9 and 10 blocks in Session 7.

- Introduce Variation #1 in Session 7 and Variation #2 in Session 8.

Variation #1

- Ask the child to place the blocks in a row on the floor as he or she counts forward and take the blocks away as he or she counts backward (Researcher indicates the direction of the row, but does not model the activity for the child).

Variation #2

- A set of 5 x 7 inch dot-set cards showing dots from 1 to 10 (the dots are arranged randomly on each card).
- Place the dot-set cards face down on the floor in random order.
- Tell the child, "I am going to show you a card with some dots on it. I want you to count the dots on the card. Touch each dot as you count and tell me how many dots are on the card."
- Ask the child to recount the dots on the card again if he or she makes a mistake.
- Ask the child to build the tower up the same number of blocks as there are dots on the card.
- Ask the child to count backward as he or she takes the blocks off the tower.
- Prompt the child if he or she does not remember a number word.
- Select another card from the pile and repeat the task.

The materials used in the Build the Tower task are shown in Appendix M.

Unit 4 (Sessions 9 - 11) Counting to Determine Quantities

This unit focused on knowledge of the cardinal meaning of the number words and knowledge of the cardinal values of sets (Baroody, 1989; Griffin & Case, 1996; Griffin et al., 1992). This unit relates to rows b and d in the diagram of the mental counting line (Case, 1998a) (Appendix D).

The Help the Farmer task was included in this unit. The Help the Farmer task was based on the Farm Animal Game. This game was developed by Montague-Smith (1997) to teach the concept of cardinality. The Help the Farmer task was played in Sessions 9, 10 and 11.

Help the Farmer Task

Explicit Objectives

- Count the objects in a set and say that the last object counted is the number of objects in the set.

- Count out sets of a specified size.

Implicit Objectives

- Introduce the understanding of a one-to-one correspondence between the objects in two sets of objects.
- Introduce the understanding that numbers can be used to compare quantities

Materials

- Three different kinds of plastic farm animals (cows, pigs and horses); 10 of each kind of animal.
- A 9" x 6"x 8.5" barn.
- Three 6" x 9" foam mats to represent fields.

Procedure[17]

Warm-Up

- Sit on the floor facing the child.
- Practice counting from 1 to 10 and 10 to 1.
- Move hand up each time the child says a number word when counting forward and down each time the child says a number word when counting backward.
- Prompt the child if he or she does not remember a number word.

Task

- Place the barn and foam mats on the floor in front of the child (the animals are inside the barn).
- Ask the child to take the animals out of the barn.
- Tell the child, "We are going to help the farmer sort out his animals. Then we are going to count the animals as they go into their fields. We are going to count them again as they go back into the barn."
- Repeat the task with a total of 5 of each kind of animal in Session 9.
- Ask the child to sort the animals into three piles (cows in one pile, horses in one pile and pigs in another pile).
- Ask the child to count how many animals are in each pile.
- Tell the child that each kind of animal has its own field.

[17] Sorting the pigs, cows and horses into separate piles took a long time. In subsequent sessions the researcher sorted the animals.

- Begin by asking the child to put 1 horse in the horses' field, 2 cows in the cows' field and 3 pigs in the pigs' field.
- Ask the child to count out loud as he or she puts the animals into their fields.
- For each kind of animal ask the child, "How many animals are in the field?" and "How do you know there are that many animals in the field?"
- Ask the child to count each set of animals as he or she puts them back into the barn.
- After the child has put all of the animals into the barn ask the child to take the animals out of the barn.
- Ask the child, "Do you have more pigs or more cows?", "How do you know?", "Do you have more horses or pigs?", “How do you know?" and so on.
- Also ask the child, "Which is the bigger number? and Which is the smaller number?".
- Begin again increasing the number of animals the child puts into the fields until 5 of one kind of animal is put into a field (2 cows, 3 pigs, 4 horses and then 3 pigs, 4 cows and 5 horses).
- Include a total of 9 of each kind of animal in Session 10. Begin by asking the child to put 6 horses in the horses' field, 7 cows in the cows' field and 8 pigs in the pigs' field. Increase the number of animals to 7 horses, 8 cows and 9 pigs.
- Include a total of 10 of each kind of animal in Session 11. Begin by asking the child to put 8 horses in the horses' field, 9 cows in the cows' field and 10 pigs in the pigs' field.[18]
- Introduce Variation #1 in Session 9 and Variations #2 and #3 in Session 10 and 11.

<u>Variation #1</u>

- Ask the child how many animals are left outside a field after he or she has put a specified number of animals in a field.

[18] The children began to grow tired of the game. No comparisons were made between the groups of animals at the end of Session 11.

Variation #2

- Ask the child to count out a smaller set of animals from the larger set of animals in a field when putting the animals back into the barn (2 animals from a set of 5 animals or 6 animals from a set of 10 animals).
- Ask the child how many animals are left in the field.

Variation #3

- When the child is asked if he or she has more cows or more pigs, have the child to arrange the animals in rows with a one-to-one correspondence between the animals in each row.

The materials used in the Animal Farm task are shown in Appendix N.

Unit 5 (Sessions 12 - 15) Incrementing Sets

This unit focused on knowledge of the increment rule: knowledge that when a set is increased by 1, the new set that is created is represented by the next number up in the number sequence (Griffin & Case, 1996; Griffin et al., 1992). This unit relates to row e in the diagram of the mental counting line (Case, 1998a) (Appendix D).

The Good Fairy task was included in this unit. The Good Fairy task was developed for the Rightstart mathematics program (Griffin et al., 1992). This task was designed to teach the increment rule (Griffin et al., 1992). The Good Fairy task was played in Sessions 12, 13, 14 and 15.

Good Fairy Task

Explicit Objectives

- Understand that when one object is added to a set the new set that is created is represented by the next number up in the number sequence.

Implicit Objectives

- Reinforce the understanding that numbers can be used to compare quantities.
- Introduce the understanding that when two objects are added to a set the new set that is created is represented by the number that is two numbers up in the number sequence.
- Introduce the understanding that when three objects are added to a set the new set that is created is represented by the number that is three numbers up in the number sequence.

Materials

- A small, 6.5" x 5.5" x 3", brightly coloured paper bag.
- A set of 10 counters.

Procedure

Warm-Up

- Sit on the floor facing the child.
- Practice counting from 1 to 10 and 10 to 1.
- Move hand up each time the child says a number word when counting forward and down each time the child says a number word when counting backward.
- Prompt the child if he or she does not remember a number word.

Task

- Place the paper bag on the floor in front of the child.
- Drop __ cookies into the bag (count each cookie as it is dropped into the bag).
- Tell the child, "I have x cookies in my bag" (Griffin et al., 1992, p. 97).
- Empty the bag onto the floor and count the cookies (to confirm that there are __ cookies in the bag).
- Put the cookies back into the bag.
- Tell the child, "Now the Good Fairy comes along ... and gives me one more" (Griffin et al., 1992, p. 97).
- Put one more cookie into the bag (make sure the child sees the cookie go into the bag).
- Ask the child, "I wonder how many cookies I have now?" (Griffin et al., 1992, p. 97).
- The child can predict how many cookies are in the bag, but only if he or she wants to.
- Empty the bag onto the floor and count the cookies to see how many there are now (to confirm that there are __ cookies in the bag now).
- Ask the child, "How many cookies were in the bag before?" "How many cookies are in the bag now?" "What did the Good Fairy do?" "How many cookies did the Good Fairy put in the bag?" "How do you know there are __ cookies in the bag now?".
- Repeat the task putting different numbers of cookies in the bag each time.

- Put 1 to 4 cookies in the bag and add one cookie to the bag each time in Session 12.
- Put 5 to 9 cookies in the bag and add one cookie to the bag each time in Session 13.
- Put 5 to 9 cookies in the bag and add one cookie each time and put 2 to 5 cookies in the bag and add 2 (Variation #4) or 3 (Variation #5) cookies each time in Session 14.
- Put 5 to 9 cookies in the bag and add one cookie each time and put 2 to 5 cookies in the bag and add 2 (Variation #4) cookies each time in Session 15.[19]
- Introduce Variation #1 in Session 12, Variation #2 in Session 13, Variation #3 in Session 14 and Variation #6 in Session 15.

<u>Variation #1</u>

- Drop the cookies into the bag (do not count the cookies into the bag).
- "Simply tell" the child, "I have x cookies in my bag" (Griffin et al., 1992, p. 97).
- Ask the child to repeat the number of cookies in the bag.
- Empty the bag onto the floor and count the cookies to see if the child answered correctly.

<u>Variation #2</u>

- Ask the child to predict how many cookies are in the bag after 1 cookie has been added (before the cookies are dumped out to check).

<u>Variation #3</u>

- Drop the cookies into the bag (do not count the cookies into the bag).
- "Simply tell" the child, "I have x cookies in my bag" (Griffin et al., 1992, p. 97).

<u>Variation #4</u>

- Add two cookies to the bag before asking the child how many cookies are in the bag.

<u>Variation #5</u>

- Add three cookies to the bag before asking the child how many cookies are in the bag.

---

[19] Adding 3 cookies was too difficult for the children and was dropped after Session 14.

Variation #6

- Ask the child which is the bigger or smaller of two numbers (the number of cookies in the bag before 1 or more cookies were added to the bag and the number of cookies in the bag after 1 or more cookies were added to the bag).

The materials used in the Good Fairy task are shown in Appendix O

Unit 6 (Sessions 16 - 19) Decrementing Sets

This unit focused on knowledge of the decrement rule: knowledge that when a set is decreased by 1, the new set that is created is represented by the next number down in the number sequence (Griffin & Case, 1996; Griffin et al., 1992). This unit relates to row e in the diagram of the mental counting line (Case, 1998a) (Appendix D).

The Cookie Monster task was included in this unit. The Cookie Monster task was developed for the Rightstart mathematics program (Griffin et al., 1992). This task was designed to teach the decrement rule (Griffin et al., 1992). The Cookie Monster task was played in Sessions 16, 17, 18 and 19.

Cookie Monster Task

Explicit Objectives

- Understand that when one object is taken away from a set the new set that is created is represented by the next number down in the number sequence.

Implicit Objectives

- Reinforce the numerals from 1 to 10.
- Introduce the words "before" and "after" (the number that comes "before" a specified number; the number that comes "after" a specified number).
- Introduce the understanding that when two objects are taken away from a set the new set that is created is represented by the number that is two numbers down in the number sequence.
- Introduce the understanding that when three objects are taken away from a set the new set that is created is represented by the number that is three numbers down in the number sequence.

Materials

- A small, 6.5" x 5.5" x 3", brightly coloured paper bag.
- A set of 10 counters.
- An 8" x 30" vertical number line showing the numerals from 1 to 10.

Procedure

Warm-Up

- Sit on the floor facing the child.
- Practice counting from 1 to 10 and 10 to 1.
- Place the number line face up on the floor in front of the child.
- Ask the child to place a counter on each numeral as he or she counts from 1 to 10 and remove the counter from each numeral as he or she counts from 10 to 1 in Session 16 and 17.
- Place the number line in the vertical position. Make sure the child can see the numerals on the number line.
- Prompt the child if he or she does not remember a number word.
- Present the Good Fairy task once or twice. Put 1 to 8 cookies in the bag and add one or 2 cookies to the bag in Sessions 17, 18 and 19.

Task

- Drop __ cookies into the bag (count each cookie as it is dropped into the bag).
- Tell the child, "I have x cookies in my bag" (Griffin et al., 1992, p. 97).
- Empty the bag onto the floor and count the cookies (to confirm that there are __ cookies in the bag).
- Put the cookies back into the bag.
- Ask the child to close his or her eyes.
- Tell the child, "Now the Cookie Monster comes and takes one cookie out of the bag."
- Take one cookie out of the bag.
- Ask the child, "I wonder how many cookies are left?" (Griffin et al., 1992, p. 98).
- The child can predict how many cookies are left, but only if the child wants to.
- Empty the bag onto the floor and count the cookies to see how cookies are left (to confirm that __ cookies are left).
- Ask the child, "How many cookies were in the bag before?" "How many cookies are in the bag now?" "What did the Cookie Monster do?" "How many cookies did the Cookie Monster take out of the bag?" "How do you know there are __ cookies in the bag now?".

- Repeat the task putting different numbers of cookies in the bag each time.
- Put 2 to 10 (2 to 5 cookies for the less capable child and 6 to 10 cookies for the more capable child) cookies in the bag in Sessions 16, 17, 18 and 19.
- Introduce Variations #1, #2, #5 and #6 in Session 16[20] and Variations #3 and #4 in Session 18.

<u>Variation #1</u>

- Drop the cookies into the bag (do not count the cookies into the bag).
- "Simply tell" the child, "I have x cookies in my bag" (Griffin et al., 1992, p. 97).
- Ask the child to repeat the number of cookies in the bag.
- Empty the bag onto the floor and count the cookies to see if the child answered correctly.

<u>Variation #2</u>

- Ask the child to predict how many cookies are in the bag after 1 cookie has been removed (before the cookies are dumped out to check).

<u>Variation #3</u>

- Ask the child to point to or touch each number as he or she counts from 1 to 10 and from 10 to 1 in Session 18 and 19.

<u>Variation #4</u>

- Turn the number line around and ask the child to count from 10 to 1 without looking at the number line in Sessions 18 and 19.

<u>Variation #5</u>

- Drop the cookies into the bag (do not count the cookies into the bag).
- "Simply tell" the child, "I have x cookies in my bag" (Griffin et al., 1992, p. 97).

<u>Variation #6</u>

- Remove two cookies from the bag before asking the child how many cookies are in the bag.

The materials used in the Cookie Monster task are shown in Appendix P.

[20] Instruction was individualized. Variations were introduced to meet the needs of more capable children. Variations were tailored to the requirements of each child.

Unit 7 (Sessions 20 - 23) Using Numbers to Compare Quantities

This unit focused on knowledge of relative magnitude: the knowledge that a number's magnitude is relative to the magnitude of every other number in the number sequence (Griffin & Case, 1996; Griffin et al., 1992). This unit relates to the outside brackets and the rows (b and d) contained within the outside brackets in the diagram of the mental counting line (Case, 1998a) (Appendix D).

The Animals on the Number Line task was included in this unit. The Animals on the Number Line task was based on two tasks (Lesson One: Paper Number Line Activity and Lesson Two: Paper Number Line Activity 2) developed for the Rightstart mathematics program (Griffin et al., 1992). The tasks were designed to help children map quantities from 1 to 10 onto the number line and use the numbers on the number line to order and compare the quantities (Griffin et al., 1992). The Animals on the Number Line task was played in Sessions 20, 21, 22 and 23.

Animals on the Number Line Task

Explicit Objectives

- Count the objects in a pictorial display and say how many objects are in the pictorial display.
- Map the quantities from 1 to 10 to corresponding numbers on a vertical number line.
- Use the numbers on the number line to order and compare the quantities from 1 to 10.

Implicit Objectives

- Reinforce the numerals from 1 to 10.

Materials

- A 8.5" x 11" picture with a set of 4 cats in the top row, 5 dogs in the next row down, 2 rabbits in the next row down, 3 hippos in the next row down and 1 owl in the bottom row (the sets are not in consecutive order).
- An 8" x 30" vertical number line with numbers on the right and a blank space to the left of each number.
- A contact sheet with a small, 2.5" square picture of each kind of animal.
- A sticky backing on each small animal picture.

Procedure

Warm-Up

- Sit on the floor facing the child.
- Place the number line in the vertical position.
- Practice counting from 1 to 10 and 10 to 1 (without the number line for the more capable child, with the number line for the less capable child).
- Prompt the child if he or she does not remember a number word.
- Present the Good Fairy task once. Put 1 to 8 cookies in the bag and add one or 2 cookies to the bag (allow the child to look at the vertical number line is he or she is unsure of the answer).
- Present the Cookie Monster task once. Put 2 to 10 cookies in the bag and remove one cookie from the bag (allow the child to look at the vertical number line if he or she is unsure of the answer).

Task

- Place the picture with a set of 4 cats in the top row, 5 dogs in the next row down, 2 rabbits in the next row down, 3 hippos in the next row down and 1 owl in the bottom row on the floor in front of the child.
- Ask the child to count the cats and say how many cats there are, count the dogs and say how many dogs there are, count the rabbits and say how many rabbits there are and so on, until all the animals have been counted
- Ask the child to find the animal that he or she has "only 1 of. ... 3 of; 5 of; 2 of; and 4 of (Griffin et al., 1992, p. 89).
- Ask the child, "Which animal do we have the littlest amount of?" "Which animal do we have the most of?" "How do you know?" (Griffin et al., 1992, p. 89).
- Map each set of animals onto the number line.
- Tell the child, "We want to remember how many animals we have in this picture so we're going to use this number line." (Griffin et al., 1992, p. 89). Point to the picture with the rows of animals and then point to the number line.
- Ask the child, "What's the first number on this line?... Are there any animals in this picture that we only have one of? (Griffin et al., 1992, p. 89). Point to the picture with the rows of animals.

- Tell the child, "Now, find the small picture of the animal we only have one of and put it next to the number 1 on the number line." Place the contact sheet with the small picture of each kind of animal on the floor in front of the child.
- Tell the child, "Putting the picture of the owl next to the number 1 on the number line will tell us that we have only 1 of this kind of animal."
- Ask the child "What's the next number on the number line? ... Are there any animals in the picture that we have only 2 of?" (Griffin et al., 1992, p. 89). Point to the picture with the rows of animals.
- Tell the child, "Find the small picture of the animal we only have 2 of and put it next to the number 2 on the number line." Point to the contact sheet with the small picture each kind of animal.
- Follow this procedure until each set of animals has been mapped onto the number line.
- When all of the small animal pictures have been put beside the numbers on the number line tell the child, "Now we don't have to look at our pictures to know how many different animals we have. We can look at this number line and it will tell us. We know we have 1 ..., 2 ..., etc. (as indicated on the number line)." (Griffin et al., 1992, p. 89).
- Point to the number line and ask the child, "Which animal do we have the littlest amount of? How can we tell? Which animal do we have the next biggest amount of? How can we tell?" (Griffin et al., 1992, p. 89).
- Continue to ask the child questions that relate to the numbers and the relative quantity of each kind of animal. For example, "The animals we have the most of is the dog. The next biggest amount is the cat. "How do you know?", "How does the number of dogs compare to the number of rabbits? How do you know we have more dogs than rabbits?".
- Repeat the task with sets of 1 to 5 in Session 20, sets of 6 to 10 in Session 21 and sets of 1 to 10 in Sessions 22 and 23.
- Introduce Variation #1 in Session 21, Variation #2 in Session 22 and Variation #3 in Session 23.

Variation #1

- A 8.5" x 11" picture with a set of 9 ladybugs in the top row, 10 frogs in the next row down, 7 seals in the next row down, 8 chickens in the next row down and 6 turtles in the bottom row (sets are not in consecutive order).
- An 8" x 30" vertical number line with numbers on the right and a blank space to the left of each number.
- A contact sheet with a small, 2.5" square picture of each kind of animal.
- Follow the same procedure as before for the warm-up and the task.

Variation #2

- A 8.5" x 11" picture with sets of 1 cat, 2 dogs, 3 rabbits, and 4 hippos in the top row, sets of 5 owls and 6 ladybugs in the next row down, a set of 7 frogs in the next row down, a set of 8 seals in the next row down and a set pf 9chickens in the next row down and a set of 10 turtles in the bottom row (sets are in consecutive order).
- An 8" x 30" vertical number line with numbers on the right and a blank space to the left of each number.
- A contact sheet with a small, 2.5" square picture of each kind of animal.
- Follow the same procedure as before for the warm-up and the task.

Variation #3

- A 8.5" x 11" picture with sets of 1 cat and 10 turtles in the top row, sets of 5 owls and 6 ladybugs in the next row down, a set of 7 frogs in the next row down, a set of 9 chickens in the next row down, a set of 8 seals in the next row down and sets of 3 rabbits, 2 dogs and 4 hippos in the bottom row (sets are not in consecutive order).
- An 8" x 30" horizontal number line with numbers along the bottom and a blank space above each number.
- A contact sheet with a small, 2.5" square picture of each kind of animal.
- Follow the same procedure as before for the warm-up and the task.

The materials used in the Animals on the Number Line task are shown in Appendix Q.

Unit 8 (Sessions 24 - 26) Comparing Quantities in Two Different Quantitative Dimensions

This unit focused on knowledge that the mental counting line can be used to make comparisons between sets in the dimension of number and the dimension of length (Griffin et al., 1992). This unit relates to the outside brackets and the rows (b, c, d and e) contained within the outside brackets in the diagram of the mental counting line (Case, 1998a) (Appendix D).

The Which Has More task and Let's Compare task (Griffin et al., 1992, p. 118) were included in this unit. The Which Has More task and Let's Compare task were developed for the Rightstart mathematics program (Griffin et al., 1992) to teach children to count and use numbers to compare quantities in the dimension of number and the dimensions of length (Griffin et al., 1992). The Which Has More task was played in Sessions 24 and 25 and the Let's Compare task was played in Session 26.

Which Has More Task

Explicit Objective

- Count and use numbers and knowledge of relative quantity to determine which of two containers has more or less in the dimension of number.

Materials

- A set of 20 small, 1" square different coloured blocks.
- Two 6" x 6.5" clear plastic jars.

Procedure

Warm-Up

- Sit on the floor facing the child.
- Practice counting from 1 to 10 and 10 to 1.
- Move hand up each time the child says a number word when counting forward and down each time the child says a number word when counting backward.
- Prompt the child if he or she does not remember a number word.
- Present the Good Fairy task once using fingers to indicate quantity. Hold up __ fingers to indicate the first amount and 1 more finger to indicate the amount that is added to the first amount.
- Present the Cookie Monster task once using fingers to indicate quantity. Hold up __ fingers to indicate the first amount and fold down 1 finger to indicate the amount that is taken away from the first amount.

Task

- Place the plastic jars on the floor in front of the child.
- Ask the child to close his or her eyes.
- Put __ blocks in the first jar (10 or less).

- Put __ blocks in the second jar (10 or less).
- Ask the child to open his or her eyes.
- Ask the child, "Which jar has more?", "How do you know that jar has more?".
- If the child says "I can see that jar has more.", ask the child, "How can you be more certain that jar has more?", "How can you find out?".
- Dump the blocks out of each jar. Count the blocks from each jar along with the child. Make sure the child knows there are __ number of blocks in one jar and __ number of blocks in the other jar.
- Focus on the number of blocks in each jar.
- Repeat the task with different quantities of blocks in each jar. (Researcher does not model the activity before the child takes a turn).
- Present sets with a large numerical difference between the sets (8, 2; 9, 4; 5, 10) in Session 24.
- Introduce Variation #1 in Session 25.

Variation #1

- Present sets with a small numerical difference between the sets (7, 9; 5, 6; 7, 8).
- Follow the same procedure as before for the warm-up and the task.

The materials used in the Which Has More task are shown in Appendix R.

Let's Compare Task

Explicit Objective

- Count and use numbers and knowledge of relative quantity to determine which of two chains is longer or shorter in the dimension of length.

Materials

- Two plastic chains with different numbers of 1.75" links on each chain.

Procedure

Warm-Up

- Sit on the floor facing the child.
- Practice counting from 1 to 10 and 10 to 1.
- Move hand up each time the child says a number word when counting forward and down each time the child says a number word when counting backward.
- Prompt the child if he or she does not remember a number word.

- Present the Good Fairy task once using fingers to indicate quantity. Hold up __ fingers to indicate the first amount and 1 more finger to indicate the amount that is added to the first amount.
- Present the Cookie Monster task once using fingers to indicate quantity. Hold up __ fingers to indicate the first amount and fold down 1 finger to indicate the amount that is taken away from the first amount.

<u>Task</u>

- Place two chains of unequal length on the floor. Present the chains folded into piles so that the chains appear to be the same size.
- Ask the child, "Which is the longest chain? Which is the shortest? How do you know? How can you find out without stretching out the chains?" (Griffin et al., 1992, p. 118).
- If the child is having difficulty explain to the child that he or she can count the number of links on each chain and compare the numbers to find out which chain of longest or shortest.
- Repeat the task with different numbers of links on each chain (4, 5; 4, 9; 2, 7; 8, 9, 5, 6).

The materials used in the Let's Compare task are shown in Appendix S.

Unit 9 (Sessions 27 - 29) Acquiring Knowledge of the Written Numerals From 1 to 10

This unit focused on knowledge of written numerals (Griffin & Case, 1996; Griffin et al., 1992). This unit relates to row a, rows a and b and rows a, d and the wide brackets in the diagram of the mental counting line (Case, 1998a) (Appendix D).

The Name That Numeral task, Match task and Numerals on the Number Line task were included in this unit (Griffin et al., 1992). These tasks were developed for the Rightstart mathematics program (Griffin et al., 1992). The Name that Numeral task was designed to teach the number names associated with the numerals from 1 to 10 (Griffin et al., 1992). The Match task was designed to teach the quantities associated with the numerals from 1 to 10 (Griffin et al., 1992). The Numerals on the Number Line task was designed to teach the position of each numeral in the number sequence from 1 to 10 (Griffin et al., 1992). The Name That Numeral task was presented in Sessions 27, the Match task was presented in Session 28 and the Numerals on the Number Line task was presented in Session 29.

Name That Numeral Task

Explicit Objective

- Read the numerals from 1 to 10.

Materials

- A set of 5 x 7 inch cards showing the numerals from 1 to 10.

Procedure

Warm-Up

- Sit on the floor facing the child.
- Practice counting from 1 to 10 and 10 to 1.
- Move hand up each time the child says a number word when counting forward and down each time the child says a number word when counting backward.
- Prompt the child if he or she does not remember a number word.
- Present the Good Fairy task once using fingers to indicate quantity. Hold up __ fingers to indicate the first amount and 1 more finger to indicate the amount that is added to the first amount.
- Present the Cookie Monster task once using fingers to indicate quantity. Hold up __ fingers to indicate the first amount and fold down 1 finger to indicate the amount that is taken away from the first amount.

Task

- Place the cards with the numerals from 1 to 10 face up and in random order on the floor.
- Tell the child, "I want you to pick up each card, tell me which number is on the card and hand the card to me.".
- Tell the child the numeral name if the child does not know the numeral name or gives an incorrect numeral name.

The materials used in the Name That Numeral task are shown in Appendix T.

Match Task

Explicit Objectives

- Read the numerals from 1 to 10.
- Match the numerals from 1 to 10 to the appropriate set size.

Materials

- A set of 5 x 7 inch dot-set cards showing dots from 1 to 10.

- A set of 5 x 7 inch cards showing the numerals from 1 to 10.

Procedure

Warm-Up

- Sit on the floor facing the child.
- Practice counting from 1 to 10 and 10 to 1.
- Move hand up each time the child says a number word when counting forward and down each time the child says a number word when counting backward.
- Prompt the child if he or she does not remember a number word.
- Present the Good Fairy task once using fingers to indicate quantity. Hold up __ fingers to indicate the first amount and 1 more finger to indicate the amount that is added to the first amount.
- Present the Cookie Monster task once using fingers to indicate quantity. Hold up __ fingers to indicate the first amount and fold down 1 finger to indicate the amount that is taken away from the first amount.

Task

- Place the dot-set cards face up and in random order on the floor.
- Shuffle the numeral cards and place the stack face down on the floor.
- Pick up the top card and show the child the numeral on the card.
- Tell the child, "I want you to tell me which number is on the card and find the dot card that has the matching number of dots and then give the cards to me."
- Tell the child the numeral name if the child does not know the numeral name or gives an incorrect numeral name.
- Ask the child to recount the dot-set cards if the match is incorrect.

The materials used in the Match task are shown in Appendix U.

Numerals on the Number Line Task

Explicit Objectives

- Recognize the numerals from 1 to 10.
- Place the numerals in the correct order on the number line.

Materials

- Different coloured felt numerals from 1 to 10.
- A 7.5" x 36" horizontal number line, covered in felt and marked off with 10 spaces.

Procedure

Warm-Up

- Sit on the floor facing the child.
- Practice counting from 1 to 10 and 10 to 1.
- Move hand up each time the child says a number word when counting forward and down each time the child says a number word when counting backward.
- Prompt the child if he or she does not remember a number word.
- Present the Good Fairy task once using fingers to indicate quantity. Hold up __ fingers to indicate the first amount and 1 more finger to indicate the amount that is added to the first amount.
- Present the Cookie Monster task once using fingers to indicate quantity. Hold up __ fingers to indicate the first amount and fold down 1 finger to indicate the amount that is taken away from the first amount.

Task

- Place the felt numerals randomly on the floor.
- Place the line in a horizontal position with the blank spaces facing the child.
- Tell the child, "Each time I call out a number I want you to find the number on the floor and then put the number where you think it should go on the line. If you are not sure you can move it later." (the numerals are called out randomly).
- Continue until all of the numerals have been picked up and placed on the line.
- Discuss the placement of the numerals on the line. If the numerals are ordered incorrectly ask the child, "Where do you think the numerals should go?", "What could you do to find out?".

The materials used in the Numerals on the Number Line task are shown in Appendix V.

Data Analysis

Unit of Analysis

Each child (case) was the main unit of analysis (Yin, 1994). The children's scores on the pretest and posttest measures and the children's verbal (words, phrases and sentences) and nonverbal (gestures and actions) responses to the tasks presented during each instructional session were the embedded units of analysis (Yin, 1994). Because the questions asked during each instructional session provided a context for the children's responses, both the researcher's question and the child's response were included in this unit of analysis (Granott, 1998).

Plan of Analysis

There were three parts to the analysis. The first part involved a qualitative analysis of the children's scores on the pretest and posttest measures (Inhelder et al., 1974). The second part involved descriptive microgenetic quantitative and qualitative analyses of the children's responses to the tasks presented during each instructional session (McKeough & Sanderson, 1996). The third part involved a trend analysis of the children's performance across the instructional units (Kennedy, 2005; McKeough & Sanderson, 1996).

Part 1: Qualitative Analysis of the Children's Pretest and Posttest Scores

The qualitative analysis of the children's scores on the pretest and posttest measures was conducted (a) to obtain a general description of the children's initial level of mathematical understanding, (b) to obtain a general description of the children's progress in the construction of the mental counting line, and (c) to explore the relationship between the children's initial level of mathematical understanding and the children's performance on the measures of conceptual understanding (Inhelder et al., 1974). This analysis focused on the children's pretest scores on the SB5 (Quantitative Reasoning factor index, Verbal Quantitative Reasoning subtest, and Nonverbal Quantitative Reasoning subtest) and the children's pretest and posttest scores on the measures of conceptual understanding (Number Knowledge test, Balance Beam task, Money Knowledge task, and Birthday Party task). There were three steps to this analysis: (a) the children's pretest scores on the SB5 were examined; (b) the children's pretest and posttest scores on the measures of conceptual understanding were compared; and (c) the children's pretest scores on the SB5 and the children's pretest and posttest score on the measures of conceptual understanding were compared.

Part 2: Descriptive Microgenetic Quantitative and Qualitative Analyses of the Children's Responses

The descriptive microgenetic quantitative and qualitative analyses of the children's responses to the tasks presented during each instructional session were conducted to answer the research questions (McKeough & Sanderson, 1996). The descriptive microgenetic quantitative analysis focused on the types of responses the children generated to the questions asked during each instructional session (Kennedy, 2005).[21] This analysis indicated when the children first

[21] The descriptive microgenetic quantitative analysis involved trial-to-trial and session-to-session analyses of the children's verbal and nonverbal responses within each instructional unit. Changes in the patterns of the children's responses are evidence of change over time in the structure of the mental counting line.

generated an independent differentiated, mapped, or consolidated response (an independent correct response), how the children arrived at that response and whether or not the children maintained that response.[22] This analysis involved: (a) coding the children's responses and entering the coded responses into a Microsoft Access database file (the fields in the Microsoft Access database file corresponded to the categories on the coding sheet); (b) formulating questions related to the research questions and the children's responses; (c) generating database queries to answer the questions[23] (the database queries filtered, categorized, and sequenced the coded responses and provided frequency counts of the coded responses); (d) constructing tables from the results of the database queries; and (e) generating graphs from the tables (Microsoft Excel was used to generate the graphs) (questions related to the research questions and the children's responses are shown in Appendix W). There were four steps to this analysis: (a) the graphs of the children's responses were visually inspected within each instructional session to determine when the first independent differentiated, mapped, or consolidated response occurred (Smith, Best, Stubbs, Archibald, & Roberson-Nay, 2002);[24] (b) the level,[25] trend,[26] and variability or stability of the children's responses were determined (where the data permitted) to

[22] The descriptive microgenetic quantitative analysis provided descriptions of individual pathways for each separate mathematical understanding represented in the mental counting line. The first independent differentiated or mapped response indicated when the children first acquired (or integrated) a separate mathematical understanding represented in the mental counting line. The first independent consolidated response indicated when the children first transferred the understanding that is represented in the mental counting line to a new task or problem situation. Responses that occurred prior to the first independent differentiated, mapped, or consolidated response provided evidence of progression or lack of progression to a more advanced level of conceptual understanding. Responses that occurred subsequent to the first independent differentiated, mapped, or consolidated response provided evidence of stability or regression to a less advanced level of conceptual understanding.

[23] The questions focused on the first independent differentiated, mapped, or consolidated response and the patterns of responses prior to and subsequent to the first independent, mapped, or consolidated response.

[24] When the first independent differentiated, mapped, or consolidated response occurred was determined by counting the number of responses before the first independent differentiated, mapped, or consolidated response.

[25] Level is the position (numerical value) on the vertical axis where the children's responses converge (Cooper, Heron, & Heward, 2007). The level of the children's responses was determined by calculating the mean of the children's responses prior to and subsequent to the first independent differentiated, mapped, or consolidated response and drawing mean level lines through the children's responses prior to and subsequent to the first independent differentiated, mapped, or consolidated response. The mean level lines were drawn from the point on the vertical axis that represented the average value of the children's responses (Gilbert, Williams, & McLaughlin, 1996 as cited in Cooper, Heron, & Heward, 2007). To calculate the mean level of the children's responses, numerical values were assigned to the levels of understanding shown on the vertical axes of the graphs (unrelated or no response was assigned a value of 0, supported incorrect response a value of 1, independent incorrect response a value of 2, supported elaborated response a value of 3 and so on).

[26] Trend is the direction and magnitude of the children's responses (Cooper et al., 2007). The trend in the children's responses was determined by constructing ordinary least-squares linear regression lines (McCain & McCleary, 1979; Parsonson & Baer, 1978 as cited in Cooper et al., 2007).

describe the pattern of the children's responses; (c) the most frequent types of responses the children generated prior to and subsequent to the first independent differentiated, mapped, or consolidated response were also determined to provide further information about the pattern of the children's responses; and (d) the children's performance was summarized for each instructional unit.[27] For selected units this analysis was extended across the instructional sequence.

The descriptive microgenetic qualitative analysis focused on the strategies the children used and the justifications the children provided when solving the problems presented during each instructional session.[28] This analysis provided additional evidence to support intra-individual and inter-individual variability in the rate and the pattern of construction of the mathematical understandings represented in the mental counting line. There were three steps to this analysis: (a) the strategies the children used and the justifications the children provided were examined in each instructional session, (b) the children's performance was summarized for each instructional unit, and (c) the children's performance was illustrated with examples of the children's responses (strategies the children used or justifications the children provided). For selected units this analysis was extended across the instructional sequence.

Part 3: Trend Analysis of the Children's Performance

The trend analysis of the children's performance across the instructional sequence was conducted to obtain a more comprehensive description of the construction process (Kennedy, 2005; McKeough & Sanderson, 1996). This analysis focused on the children's patterns of responses to the tasks presented across the instructional sequence (Kennedy, 2005).[29] There were two steps to this analysis: (a) the children's patterns of responses to the tasks were inspected and (b) trends in the data were identified and described.

Coding Scheme

A coding scheme was developed to analyze the children's responses to the tasks presented during each instructional session. The coding scheme was designed to detect variability in the

---

[27] The children's verbal and nonverbal responses were blocked by instructional session. The instructional sessions were blocked by instructional unit.

[28] The children's strategies provided illustrations of supported and independent elaborated, linked across, and nonconsolidated responses and supported mapped and consolidated responses. The children's justifications provided confirmation of independent differentiated, mapped, or consolidated responses.

[29] The trend analysis provided descriptions of individual pathways across the instructional sequence.

pattern and the rate of construction of the mental counting line as the children progressed through the instructional program and the transfer of the understanding represented in the mental counting to new tasks and problem situations. The "requirements that would have to be met" (Case, 1998a, p. 778) to construct a new higher order central conceptual structure provided the structure for the coding scheme. The "requirements that would have to be met" (Case, 1998a, p. 778) to construct a new higher order central conceptual structure describe the hierarchical sequence of developmental steps involved in the construction of a new higher order central conceptual structure, such as the mental counting line, from two existing lower order conceptual structures (the counting schema and global quantity schema). Each step in the sequence specifies changes that occur in children's cognitive structures and describes a more complex level of mathematical understanding (Case, 1998a). The content of the coding scheme was derived from Case's model of the mental counting line, the instructional program that was used to facilitate the construction of the mental counting line (Case & Sandieson, 1991; Griffin & Case, 1997; Griffin et al., 1995; Griffin et al., 1992; Griffin et al., 1994) and research on the development of young children's mathematical understanding (Baroody, 1987, 1989; Case, 1985, 1998b; Fuson, 1988; Gelman & Gallistel, 1978; Griffin, 2005; Resnick, Wang, & Kaplan, 1973; Schaeffer, Eggleston, & Scott, 1974). A detailed description of the coding scheme is presented in Chapter 4.

Chapter 4:
Coding Scheme

Purpose of the Coding Scheme

The purpose of the coding scheme was to detect intra-individual and inter-individual variability in the rate and the pattern of construction of the mental counting line as the children progressed through the instructional program and the transfer of the understanding represented in the mental counting to new tasks and problem situations. The code definitions were defined in terms of Case's theoretical perspective (Case, 1991c, 1996a, 1996b, 1998a; Griffin et al., 1994) in order to obtain valid and reliable data on how individual children construct the mental counting line (Miles & Huberman, 1994).

Basis for the Coding Scheme

The coding scheme is based on Case's (1998a) model of the mental counting line, Case's (1996a; 1996b; 1998a) model of the process of structural change, the "requirements that would have to be met" (Case, 1998a, p. 778) to construct the mental counting line, the instructional program that was used to facilitate the construction of the mental counting line (Case & Sandieson, 1991; Griffin & Case, 1997; Griffin et al., 1995; Griffin et al., 1992; Griffin et al., 1994) and research on the development of young children's mathematical understanding (Baroody, 1987, 1989; Case, 1985, 1998b; Fuson, 1988; Gelman & Gallistel, 1978; Griffin, 2005; Resnick, 1983, 1989; Resnick et al., 1973; Schaeffer et al., 1974).

Case's (1998a) model of the mental counting line specifies the mathematical understandings that make up the mental counting line and indicates the relationships that exist between them (Case, 1998a; Griffin et al., 1992). The mathematical understandings that make up the mental counting line arise from the elaboration, differentiation, linking across, and mapping of the nodes and relations depicted in the diagram of the mental counting line (Appendix D). These mathematical understandings develop in a hierarchical fashion. Mathematical understandings acquired early in the developmental sequence are integrated with those acquired later in the developmental sequence to form increasingly complex mathematical understandings. The successive elaboration, differentiation, linking across, and mapping each of these increasingly complex mathematical understandings result in the higher level understanding represented in the fully consolidated mental counting line (Griffin et al., 1995; Griffin et al., 1994).

Case's (1996a; 1996b; 1998a) model of the process of structural change specifies the changes that must occur in children's cognitive structures (elaboration, differentiation, linking across, mapping, and consolidation) if a new higher order central conceptual structure is to be constructed from two lower order conceptual structures and describes the underlying learning processes (C-learning or associative learning and M-learning or attentionally mediated learning) hypothesized to cause these changes to occur. This model also describes how the operation of the hierarchical learning loop facilitates the integration of new higher order central conceptual structures and the transfer of the knowledge that is represented in these central conceptual understandings to specific tasks and problem situations (Case, 1996a).

The "requirements that would have to be met" (Case, 1998a, p. 778) to construct the mental counting line add to Case's (1996a; 1996b; 1998a) model of the process of structural change. These requirements (Case, 1998a) specify in greater detail the changes that must occur (elaboration, differentiation, linking across, mapping, acquiring written numerals, and consolidation) if a new higher order conceptual structure (the mental counting line) is to be constructed from two lower order conceptual structures (the global quantity schema and the counting schema). The first three requirements (elaboration and differentiation, linking across, and mapping) describe the changes that must occur if the hierarchical integration of two previously separate conceptual structures is to occur. The fourth requirement describes the second-order symbol system (written numerals) that must be acquired in order to represent the elements of the new conceptual structure. The fifth requirement (consolidation) describes the changes that occur as children use the new higher order central conceptual structure to solve specific problems that depend upon the understanding represented in this central conceptual structure (Case, 1998a).

The instructional program translates the "requirements that would have to be met" (Case, 1998a, p. 778) to construct the mental counting line into an explicit instructional sequence. The instructional program also provides operational definitions for the mathematical understandings implied in the mental counting line (Griffin & Case, 1997; Griffin et al., 1995; Griffin et al., 1992; Griffin et al., 1994). Each unit in the instructional program focuses on the development of a separate mathematical understanding. Each of these mathematical understandings are defined in terms of specific observable behaviours (Griffin & Case, 1997). The instructional units are introduced in the order in which the mathematical understandings are naturally acquired. Mathematical understandings acquired early in the developmental sequence are introduced first.

Mathematical understandings acquired late in the developmental sequence are introduced last (Griffin & Case, 1997; Griffin et al., 1995; Griffin et al., 1992; Griffin et al., 1994).

Research on the development of young children's mathematical understanding (Baroody, 1987, 1989; Case, 1985, 1998b; Fuson, 1988; Gelman & Gallistel, 1978; Griffin, 2005; Resnick et al., 1973; Schaeffer et al., 1974) adds detail to the code definitions that make up the coding scheme and allows for a finer-grained analysis of the children's responses to the instructional program. The sublevels within each code definition describe the complete or partial mastery of the mathematical understandings defined by the code definitions. Research on the development of young children's mathematical understanding provides operational definitions for the sublevels described for each code definition.

Focus of the Coding Scheme

The coding scheme focuses on the mathematical understandings defined in the instructional units. These mathematical understandings represent the different levels of complexity apparent in the children's mathematical understanding as they construct the mental counting line. The code definitions are based on these mathematical understandings.

The code definitions focus on the children's verbal (words, phrases and sentences) and nonverbal (gestures and actions) responses to the researcher's questions during each instructional session. Both verbal and nonverbal behaviours were included because these behaviours provide two sources of evidence that can be used to substantiate the children's understanding. Also, children may express different levels of understanding in their verbal and nonverbal behaviours (a less sophisticated level of understanding in their verbal behaviour and a more sophisticated level of understanding in their nonverbal behaviour). Mismatches between a child’s verbal and nonverbal behaviour are important because they indicate variability in a child's response. Variability is an important feature of developmental change (Goldin-Meadow & Alibali, 2002). A mismatch between a child’s verbal and nonverbal behaviour may indicate that a child is "ready" to move to a new, higher level of understanding (Church & Goldin-Meadow, 1986; Perry, Church & Goldin-Meadow, 1988 as cited in Goldin-Meadow & Alibali, 2002, p. 81).

Verbal and nonverbal responses not specifically related to the researcher's questions during each instructional session were not coded (Wendy: "We, we used Nanna's dinosaur cookies." Researcher: "Oh, you did to play this game." Wendy: "Yeh." Researcher: "Oh, that's really neat."). Since understandings are linked to particular tasks or activities, the coding of understandings was restricted to the tasks and activities on which these understandings were based (Parziale, 2002).

How the Coding Scheme Was Constructed

The coding scheme was constructed deductively (Miles & Huberman, 1994). The model of the mental counting line (Case, 1998a) provided the content on which the code definitions were based. The instructional units included in the instructional program provided the behavioural categories included in the code definitions (Griffin et al., 1995; Griffin et al., 1992; Griffin et al., 1994). The "requirements that would have to be met" (Case, 1998a, p. 778) to construct the mental counting line provided the sublevels for the behavioural categories included in the code definitions. The behaviours described for each instructional unit (Griffin et al., 1992) and literature on the development of children's mathematical understanding (Baroody, 1987; Fuson, 1988; Gelman & Gallistel, 1978) provided the operational definitions for the sublevels included within the behavioural categories.

A hierarchical structure was adopted (Miles & Huberman, 1994). A hierarchical structure is consistent with the conceptual theory on which the coding scheme was based (Case, 1991c, 1996a, 1998a). Evidence to support a hierarchical structure is shown in studies from Case's theoretical perspective (Case & Sandieson, 1991; Griffin & Case, 1996, 1997; Griffin et al., 1995; Griffin et al., 1994) and literature on the development of children's mathematical understanding (Baroody, 1987; Fuson, 1988; Gelman & Gallistel, 1978).

The code definitions describe a single developmental pathway. This developmental pathway represents "a 'preferred developmental pathway' for a large class of individuals" (Case, 1996d, p. 211). However, individual children may show variations in the rate and the pattern of construction of this developmental pathway (within and across the code definitions).

The hierarchical structure of the coding scheme makes it possible to detect variations in this developmental pathway. Each code definition describes a specific point in this developmental pathway (indicates the extent to which individual children have integrated the mathematical understandings that make up the mental counting line). The sublevels within each code definition describe the complete or partial mastery of the mathematical understandings defined by the code definitions (indicate where individual children are at in terms of the mathematical understandings defined by the code definitions).

The code definitions focus on the behaviours observed during the instructional sessions (Hawkins, 1982; Rusch, Rose, & Greenwood, 1988). The behaviours were defined as clearly and completely as possible. The behaviours were defined in this way in order to reduce the level of inference and improve the level of agreement between two independent observers (Hawkins, 1982; Rusch et al., 1988).

The code definitions were refined and revised during examination of the pilot data. The code definitions were further refined and revised during the coding and analysis of the study data (Miles & Huberman, 1994).

Structure of the Coding Scheme

The coding scheme is composed of nine code definitions. The first, second, third, fifth, sixth, seventh and eighth code definitions each describe one behavioural category. The fourth code definition describes two behavioural categories and the ninth code definition describes four behavioural categories.

There are two sublevels within each of the behavioural categories of the first eight code definitions. There are two sublevels within each of the first three behavioural categories of the ninth code definition and three sublevels within the fourth behavioural category of the ninth code definition. Within each behavioural category, the sublevels are ordered from a less complex to a more complex level of understanding.

The code definitions are arranged hierarchically in the order in which the mathematical understandings are naturally acquired (Griffin et al., 1995; Griffin et al., 1994). Two exceptions are "Reciting the Number Sequence Backward from 10 to 1" and "Acquiring Knowledge of the Written Numerals From 1 to 10". Knowledge of the number sequence from 10 to 1 is typically acquired later in the developmental sequence (Baroody, 1987; Fuson, 1988). However, this mathematical understanding was introduced early in the developmental-instructional sequence because this mathematical understanding is difficult to acquire (Griffin & Case, 1996). Early introduction provided the children with more practice. Knowledge of written numerals from 1 to 10 can be acquired at any point in the developmental-instructional sequence (Baroody, 1987; Griffin, 2005). However, this mathematical understanding was introduced at the end of the developmental sequence because this mathematical understanding is not usually acquired until children start school (Griffin, 2005).

Description of the Coding Scheme

Table 4 shows how the code definitions were structured and provides a brief description of the components of each code definition.

Table 4. Structure of the Code Definitions

Components	Description
Title	Mathematical understanding (related to unit in instructional program)
I. ( ) Main Heading	Operational definition of mathematical understanding
A. ( ) Subheading	Change in cognitive structure
General Definition	Description of mathematical understanding, change in cognitive structure and critical components of mathematical understanding
Instances of the Behaviour	Operational definitions of subheading
Example	Child's response

The titles of the code definitions refer to the instructional units in the instructional program (Reciting The Number Sequence from 1 to 10). The descriptions under the titles refer to the mathematical understandings defined in the code definitions. The descriptions focus on the knowledge that underlies the mathematical understandings and list the mathematical understandings that were integrated to form these mathematical understandings. The titles are arranged hierarchically (from units introduced early to units introduced late in the instructional program).

The main headings in the code definitions (indicated by Roman numerals) refer to the behavioural categories that operationally define the mathematical understandings specified in the mental counting line (<u>Saying</u> The Number Words Forward from 1 to 10). The behaviour within each behavioural category is defined in terms of the specific verbal (words, phrases and sentences) and nonverbal behaviours (gestures and actions) associated with that behaviour (Rusch et al., 1988). The main headings are arranged hierarchically (from behaviours acquired early to behaviours acquired late) when more than one main heading is included in a code definition.

The subheadings in the code definitions (indicated by capital letters) refer to the changes that occur in children's cognitive structures (elaboration, differentiation, linking across, mapping and consolidation) as they construct the mental counting line (Case, 1998a; Griffin, 2005). The first four changes (elaboration, differentiation, linking across and mapping) represent the complete or partial mastery of each of the mathematical understandings that make up the mental counting line (Case, 1998a). The fifth change (consolidation) represents the child's ability to use the knowledge that is represented in the mental counting line to make "quantitative assessment(s)" (Griffin et al., 1994, p. 37) in a variety of quantitative dimensions (Case, 1998a). The definition under each subheading describes the specific change that occurs in the children's cognitive structures and operationally defines the level of understanding that is indicated by that subheading. The examples under each subheading provide behavioural instances of the level of understanding that is indicated by that subheading.

The subheadings are arranged hierarchically. Subheadings that refer to understandings not yet integrated are listed first. Subheadings that refer to integrated understandings are listed last.

How the Data Were Coded

The data were coded manually from the transcripts of the videotapes. The units of analysis (the questions the children were asked and the children's responses to the questions) were highlighted in the text (underlined) before the data were coded. Each unit of analysis was coded in turn. The results were recorded on a coding sheet (Appendix X) and then entered into a Microsoft Access database file.

The coding sheet consisted of categories (arranged across the top of the page) that described the components of the unit of analysis (the researcher's question and the child's response). Appropriate cells on the coding sheet were checked off as the units of analysis were coded. Qualitative observations were also recorded for each unit of analysis (Miles & Huberman, 1994). The fields in the electronic database file corresponded to the categories on the coding sheet. The categories that describe the components of the unit of analysis are defined in the General Definitions.

General Definitions

The following definitions describe the terms used in the code definitions and the categories on the coding sheet.

Terms Used in the Code Definitions

(EL) Elaborated Response[30]

The child adds new units of knowledge to schemas within his or her existing schematic repertoire (When saying the number words the child fills in missing number words in the number word sequence from 1 to 10.) (Case, 1985, 1998a).

(DI) Differentiated Response[31]

The child separates the units of knowledge within existing schemas into distinct entities (When counting, the child says each number word as a separate word.) (Case, 1996a, 1998a).

(LA) Linked Across Response

The child connects the separate units of knowledge within two existing schemas in a one-to-one fashion (The number word "five" and the pattern formed by 5 dots become connected.) (Case, 1996a, 1998a). The child may understand some aspects of a new principle, rule or conceptual understanding and may have assembled some of the components of the problem-solving strategy that is based on the new principle, rule or conceptual understanding.

(MA) Mapped Response

The child integrates the units of knowledge within two linked schemas in a hierarchical fashion (the child's schema for the number word sequence [just after relations/just before relations] has become "hierarchically subordinate to, and integrated with" his or her schema for the visual patterns [objects or fingers] associated with sets [just after relations/just before relations]) (Case, 1985, p. 264). The child has abstracted a new principle, rule or conceptual understanding ("the idea that the addition or subtraction of a unit to any canonical set always yields a number that is adjacent in the counting string") (Case, 1998a, p. 778) and has

---

[30] Elaboration is an ongoing process that occurs in all components of the children's counting and global quantity schemas (knowledge of number words, knowledge of objects, and knowledge of visual number patterns etc.). Elaboration occurs before differentiation (Case, 1996a, 1998a).

[31] Differentiation is an ongoing process that occurs in all components of the children's counting and global quantity schemas (knowledge of number words, knowledge of objects, and knowledge of visual number patterns etc.). Differentiation occurs before linking across (Case, 1996a, 1998a).

assembled a problem-solving strategy that is based on the new principle, rule or conceptual understanding (the child's counting schema is used as a means to solve simple addition and subtraction problems) (Case, 1985).

(CO) Consolidated Response[32]

The mental counting line is "consolidated in a fashion that transcends any specific context" (Case, 1998a, p. 791) and the knowledge that is represented in the mental counting line is transferred to specific problems that depend upon the understanding represented in the mental counting line. The child recognizes that the problem-solving strategy he or she has assembled (counting each set and comparing the resulting numbers) can be used to make a "quantitative assessment" (determine relative amount) in one quantitative dimension (amount) and make similar "quantitative assessment(s)" (Griffin et al., 1994, p. 37) in other quantitative dimensions (weight, distance, volume) (Case, 1998a).

(NC) Non-Consolidated Response[33]

The mental counting line has not yet been "consolidated in a fashion that transcends any specific context" (Case, 1998a, p. 791) and the knowledge that is represented in the mental counting line has not yet been transferred to specific problems that depend upon the understanding represented in the mental counting line. The child does not recognize that the new problem-solving strategy he or she has assembled (counting each set and comparing the resulting numbers) can be used to make a "quantitative assessment" (determine relative amount) in one quantitative dimension (amount) and make similar "quantitative assessment(s)" (Griffin et al., 1994, p. 37) in other quantitative dimensions (weight, distance, volume) (Case, 1998a).

[32] The problem-solving strategy has already been assembled in a specific problem-solving situation (the child's counting schema has become "hierarchically subordinate to, and integrated with, his weight-determination structure" in a specific problem-solving situation) (Case, 1985, p. 264).

[33] The problem-solving strategy has been assembled in a specific problem-solving situation (the child's counting schema has become "hierarchically subordinate to, and integrated with, his weight-determination structure" in a specific problem-solving situation) (Case, 1985, p. 264).

Categories Used on Coding Sheet

Categories that Relate to the Researcher's Question

Unit

Unit refers to the instructional units included in the instructional program. The units were numbered from 1 to 9.

Session

Session refers to the instructional sessions included in each instructional unit (two sessions were included in Unit 1, three in Units 2, 3, 8 and 9 and four in Units 5, 6 and 7). The sessions were numbered from 1 to 29.

Question Content

Question content refers to the behavioural category, indicated by the Roman numeral(s), in each code definition. There were thirteen behavioural categories: one behavioural category in each of the first three code definitions, two behavioural categories in the fourth code definition, one behavioural category in each of the fifth, sixth, seventh and eighth code definitions and four behavioural categories in the ninth code definition.

(SF) Saying the Number Words Forward From 1 To 10

(SB) Saying the Number Words Backward From 10 to 1

(TO) Touching an Object Once Each Time a Number Word Is Said

(CS) Counting Objects in a Set and Saying That the Last Number Said is the Number of Objects in the Set

(CO) Counting Out Sets of A Specified Size

(AD) Adding 1, 2 Or 3 Objects to a Set and Saying That the Last Counted Number Is the Answer

(SU) Subtracting 1, 2 Or 3 Objects From a Set and Saying That the Last Counted Number Is the Answer

(SM and SL) Saying That One Set Has More Or Less Than Another Set Because The Number Associated With The First Set Is Bigger Or Smaller Than The Number Associated With The Second Set

(UN) Using Numbers to Say Which of Two Sets Has More Or Less in Two Different Quantitative Dimensions

(IW) Identifying the Written Numerals From 1 To 10

(RW) Reading the Written Numerals From 1 To 10

(MW) Matching the Written Numerals From 1 To 10 to Their Corresponding Pictorial Displays

(PW) Putting the Written Numerals From 1 To 10 in the Correct Order on the Number Line

Opportunities to Respond

Opportunities to respond refers to the occasions, in each instructional session, when the child was asked a question that required a response (Rusch et al., 1988). Opportunities to respond were numbered consecutively for each behavioural category.

Question Level of Complexity

Question level of complexity refers to the level of difficulty of the questions asked. There were two levels of question complexity: less complex questions and more complex questions.

(LC) Less complex questions

Less complex questions focus on facts ("How many dots are on the card?"; "What did the 'Cookie Fairy' do?"; "How many cookies did the cookie fairy put in the bag?"; "How many cookies are in the bag now?"; "What's the first number on the number line?"; "What's the next number on the number line?"; Which one is the bigger number?"; "Which animal do we have the littlest amount of?").

(MC) More complex questions

More complex questions require the child to explain or justify how they arrived at the correct answer to a particular question ("How do you know?"; "How do you know there are 6 cookies in the bag now?"; "How can you tell?"; "How do we know we have more dogs than rabbits?"; "How do you know which jar has more?"; "How can you find out?"; "And how, how can you be absolutely certain that, that one has more?"; "What could you do to find out?").

Researcher support

Researcher support refers to phrases, gestures or materials used by the researcher to encourage the child to respond in an appropriate fashion (O'Donnell, Reeve, & Smith, 1997). There were four categories of researcher support: modelling, verbal prompt, gestural prompt and physical prompt. A fifth category, indicating that the child required no support, was also included.

(M) Modelling

The researcher verbally or non-verbally demonstrates to the child how to perform an activity or respond to a question before the child performs the activity or responds to the question ("We're going to build a tower and we're going to count as we go. So, I'll do it first and I'll show you what we're going to do. Watch me."; "Will you count to 10 with me?").

(V) Verbal Prompt

The researcher provides specific statements that encourage the child to respond in an appropriate fashion ("Start with 1."; "Count out loud. Start with 1."; "Okay, what comes after 9?"; "You start. What comes after 4?".). The prompt gives the child specific information about how to respond.

(G) Gestural Prompt

The researcher performs specific actions that encourage the child to respond in an appropriate fashion (The researcher moved her hand up each time the child said a number word when counting from 1 up to 10 and down each time the child said a number word when counting from 10 down to 1. When the child was asked how many is 4 plus 1, the researcher held up 4 fingers and then 1 more. The researcher pointed to the first block on the floor). The prompt gives the child specific information about how to respond.

(P) Physical Prompt

The researcher provides the child with a vertical number line showing the numerals from 1 to 10 to assist the child when counting forward, counting backward, adding, subtracting and comparing numbers. Actions and gestures made by the researcher when using the number line are part of the physical prompt.

(I) Independent

The child responds to the researcher's question in an appropriate fashion without the assistance of the researcher.

Categories that Relate to the Child's Response

Nature of Response

Nature of response refers to the manner in which the child performs a task or responds to a question. There were three categories of nature of response: verbal, nonverbal and verbal and nonverbal. A fourth category indicating that the child made no response was also included.

(NR) No Response

The child fails to perform the task or respond to the question or the response is unintelligible.

(NV) Nonverbal

The child performs the task with gestures or actions or responds to the question with gestures or actions only.

(V) Verbal

The child responds to the question with words, phrases or sentences only.

(VN) Verbal and Nonverbal

The child performs the task with a combination of words, phrases or sentences and gestures or actions or responds to the question with a combination of words, phrases or sentences and gestures or actions.

Correctness of the Response

Correctness of response refers to how the child responds when asked to perform a specific task (touch each object when saying the number words) or answer a question that requires a "quantitative assessment" (Griffin et al., 1994, p. 37) ("What number comes next?"; "How many cows are in the field?"; "How many cookies did the 'Cookie Fairy' put in the bag?"; "Which number is bigger, 2 or 6?"). There were four categories of correctness of response: correct, partially correct, incorrect and unrelated. A fifth category, indicating that the child made no response was also included.

(NR) No Response

The child fails to perform the task, respond to the question or the response is unintelligible.

(UN) Unrelated

The child performs a task that is different from the task he or she was asked to perform or responds to the question with an answer that is not closely connected to the question that was asked.

(IN) Incorrect

The child performs the task incorrectly or responds to the question with an incorrect answer and indicates, by the way in which he or she performs the task or responds to the question, that he or she has no awareness of the problem-solving strategy that is based on a new principle, rule or conceptual understanding and no awareness of what the correct answer to the question might be (Griffin, 2005). An incorrect response is a response that does not fit into the partially correct category.

(PC) Partially Correct

The child performs the task incorrectly or responds to the question with an incorrect answer, but indicates, by the way in which he or she performs the task or responds to the question, that he or she has some awareness of the problem-solving strategy that is based on a new principle, rule or conceptual understanding and some awareness of what the correct answer to the question might be (Griffin, 2005).

(C) Correct

The child performs the task correctly or responds to the question with a correct answer and uses the problem-solving strategy that is based on a new principle, rule or conceptual understanding (Case, 1998a; Griffin, 2005).

Justification

Justification refers to the explanation the child was required to give for a particular "quantitative assessment" (Griffin et al., 1994, p. 37). More complex questions such as "How do you know?", "How can you tell?" or "How can you find out?" prompted the child to reflect on the strategy he or she used to solve a particular problem or the rule, principle or conceptual understanding on which a solution to a particular problem was based.

(NJ) No Justification

The child fails to provide a justification

(J) Justification

The child provides a justification

Correctness of the Justification

There were four categories of correctness of justification: correct, partially correct, incorrect and unrelated. A fifth category indicating that the child made no response was also included.

(NR) No Response

The child fails to provide an explanation.

(UN) Unrelated

The child provides an explanation that is not connected in any way to a new rule, principle or conceptual understanding or to the problem-solving strategy that is based on the new rule, principle or conceptual understanding.

(IN) Incorrect

The child provides an explanation that is incorrect and indicates by his or her response that he or she had no knowledge of a new rule, principle or conceptual understanding and no awareness of the problem-solving strategy that is based on the new rule, principle or conceptual understanding (Griffin, 2005).

(PC) Partially Correct

The child provides an explanation that is partially correct and indicates by his or her response, that he or she has some knowledge of a new rule, principle or conceptual understanding or some awareness of the problem-solving strategy that is based on the new rule, principle or conceptual understanding (Griffin, 2005).

(C) Correct

The child provides an explanation that is based on a new rule, principle or conceptual understanding or on the problem-solving strategy that is based on the new rule, principle or conceptual understanding (Case, 1998a; Griffin, 2005).

Demonstrated Levels of Conceptual Understanding

Demonstrated levels of conceptual understanding are related to the changes that occur in the children's cognitive structures as they construct the mental counting line (elaboration, differentiation, linking across, mapping and consolidation). Elaboration, differentiation, linking across, and mapping are related to the mastery (differentiation, mapping) or partial mastery (elaboration, linking across) of the mathematical understandings that make up the mental counting line. The responses related to elaboration, differentiation, linking across, and mapping are the (EL) Elaborated Response, (DI) Differentiated Response, (LA) Linked Across Response and (MA) Mapped Response. Consolidation is related to the complete mastery of the mathematical understanding represented in the mental counting line and the transfer of this mathematical understanding to new tasks and problem situations. The response that is related to consolidation is the (CO) Consolidated Response. Non-consolidation is related to the incomplete mastery of the mathematical understanding represented in the mental counting line and the failure to transfer of this mathematical understanding to new tasks and problem situations. The response that is related to non-consolidation is the (NC) Non-Consolidated Response. A demonstrated level of conceptual understanding was assigned to each question the child responded to or each task the child performed or both.

Qualitative Description

Verbal and nonverbal behaviour (the researcher's and the child's), that contributed to the development of the mental counting line, but was not coded using the categories on the coding sheet, was included in the qualitative description.

Code Definitions

Reciting the Number Words Forward From 1 to 10

This code definition focuses on knowledge of the number words from 1 to 10 and the position of each number word in the number word sequence (Griffin et al., 1992). This code definition relates to row b in the diagram of the mental counting line adapted from Griffin and Case (1996) (Appendix D).

I. (SF) Saying the Number Words Forward From 1 to 10

A. (EL) Elaborated Response

General Definition

The child can say the number words up from 1. However, the child may not know all of the number words from 1 up to 10, the correct position of each number word in the number word sequence or the meaning of the term "after". Individual elements of the child's schema for the number words from 1 to 10 are in the process of being acquired and have not yet become differentiated (the "number just after" relations for the number words have not yet become established) (Baroody, 1987, 1989; Case, 1998a; Fuson, 1988; Griffin et al., 1992)..

Instances of the Behaviour

When asked to recite the number words from 1 up to 10, the child can say some of the number words from 1 up to 10, but cannot say all of the number words from 1 up to 10.

***Example**: There are no examples of this type of response in the study.*[34]

[34]Throughout the study, instances of behaviour were retained even though there were no examples of these behaviours in the study. These behaviours tended to indicate the lower boundaries of the codes (in operational

When asked to recite the number words from 1 up to 10, the child <u>can say</u> some of the number words from 1 up to 10 in the correct order, but <u>cannot say</u> all of the number words from 1 up to 10 in the correct order.

Example: *There are no examples of this type of response in the study.*

When asked to recite the number words up from 1 and stop at a specified number word, the child <u>can say</u> the number words up from 1, but <u>cannot stop</u> at the specified number word.

Example: *Researcher: "Okay. Now. Let's count from 1 to 7. Can you do that? You count from 1 to 7 out loud." (The researcher points to Sarah.) Sarah: "1, 2, 3, 4, 5, 6, 7, 8." Researcher: "Okay. That was to number 8, wasn't it?".*

When given two or three number words in the number word sequence, asked to recite the next number words up and then stop at a designated number word, the child <u>cannot say</u> the next number words up in the number word sequence and <u>stop</u> at the designated number word.

Example: *Researcher: "Okay. So you have to watch me. Listen, listen and watch. 1. Take this off. 1, 2." (The researcher removes the sticker from the number 10 on the vertical number line. Sarah turns and looks at the number line.) Sarah: "1." Researcher: "What comes after 2?" Sarah: "3." Researcher: "And then what comes next?" Sarah: "4." Researcher "Good. Now I'm winking at you so you have to stop. Then I say 5, 6 and you say ... What comes after 6?" Sarah: "Uhh, 8." Researcher: "No. What comes after 6? 7. Now go 8."*

When asked to give the number word that comes immediately after a specified number word, the child <u>cannot give</u> the number word that comes immediately after the specified number word.

Example: *Researcher: "What's come, comes after 7?" (The researcher points to Kevin.) Kevin: "9."*

---

terms). They often described behaviours that the children had already acquired. Including these behaviours also makes it possible to use the code in future studies.

B. (DI) Differentiated Response

General Definition

The child can say the number words from 1 up to 10. The child knows all of the number words from 1 up to 10, the correct position of each number word in the number word sequence and the meaning of the term "after". Individual elements of the child's schema for the number words have become differentiated (the "number just after" relations for the number words have become established) (Baroody, 1987, 1989; Case, 1998a; Fuson, 1988; Griffin et al., 1992).

Instances of the Behaviour

When asked to recite the number words from 1 up to 10, the child can say the number words from 1 up to 10 in the correct order.

***Example**: Researcher: "Okay. Let's count to 10. Count from 1 to 10 for me, Anne, out loud." (The researcher points to Anne.) Anne: "1, 2, 3, 4, 5, 6, 7, 8, 9, 10."*

When asked to recite the number words up from 1 and stop at a specified number word, the child can say the number words up from 1 and stop at the specified number word.

***Example**: Researcher: "Now. Let's count from 1 to 8 out loud. You do it 1 to 8, out loud." (The researcher points to Wendy.) Wendy: "1, 2, 3, 4, 5, 6, 7, 8."*

When given two or three number words from the number word sequence, asked to recite the next number words up and then stop at a designated number word, the child can say the next number words up in the number word sequence and stop at the designated number word.

***Example**: Researcher: "Okay. Let's do up to 8. 1, 2, 3." (The researcher points to Wendy.) Wendy: "4." Researcher: "Keep counting." Wendy: "5, 6." Researcher: "Okay, now I'm winking at you so you stop. And then I go 7, 8.".*

When asked to give the number word that comes immediately after a specified number word, the child can give the number word that comes immediately after the specified number word.

***Example**: Researcher: "What comes after 4?" (The researcher points at Kevin.) Kevin: "Uh. 5."*

Reciting the Number Words Backward From 10 to 1

This code definition focuses on knowledge of the number words from 10 to 1 and the position of each number word in the number word sequence (Griffin et al., 1992). This code definition relates to row b in the diagram of the mental counting line adapted from Griffin and Case (1996) (Appendix D).

I. (SB) Saying the Number Words Backward From 10 to 1

A. (EL) Elaborated Response

General Definition

The child cannot spontaneously give the number word that comes just before each number word from 10 down to 2. The child may not know the meaning of the term "before" and may not know that the number words can be recited backwards as well as forwards. The child is in the process of acquiring the "number just before" relations for the number words from 10 to 2 (Baroody, 1987, 1989; Case, 1998a; Fuson, 1988).

Instances of the Behaviour

When asked to recite the number words from 10 down to 1, the child cannot say the number words from 10 down to 1 in the correct order.

Example: ***Researcher: "Good. Now let's go back down.". Kevin: "10, 9, 8, 5." (The researcher moves her hand down.) Researcher: "7." (The researcher moves her hand down.) Kevin: "7, 5.". Researcher: "6.". Kevin: "6, 5, 4, 3, 2, 1." (The researcher moves her hand down.) Researcher: "Good for you.".***

When asked to recite the number words down to 1 from a specified number word, the child cannot say the number words down to 1 from the specified number word.

Example: ***Researcher: "Okay. Let's start with 4 this time. Let's start with 4. You. Okay. Start with 4. 4." (The researcher crouches.) Kevin: "4". (Kevin crouches.) Researcher: "What comes next? (The researcher crouches.) Kevin: "5". (Kevin stays in the same position.) Researcher: "We're going backwards, remember? 4". (The researcher stays in the same position.) Kevin: "4". (Kevin stays in the same position.) Researcher: "3." (The researcher crouches.) Kevin: "3, 2." (Kevin crouches.) Researcher: "1. Blast-off." (The researcher crouches and puts her hands on the floor as she says "1." The researcher jumps up as she says "Blast-off.") Kevin: "1. Blast-off." (Kevin puts his***

hands on the floor as he says "1.". Kevin jumps up as he says "Blast-off".)
Researcher: "Good for you.".

When given two or three number words from the number word sequence, asked to recite the next number words down and then stop at a specified number word, the child cannot say the next number words down in the number word sequence and stop at the specified number word.

Example: ***Researcher: "Okay. Let's do the counting and winking game again. I'll start. 10." (The researcher turns the number line around so that Anne can see the numbers, sweeps her hand down the number line and points to 10) Anne: "10." (The researcher points to 9.) Researcher: "My turn first. 10, 9, 8 and then ...". (The researcher points to 10, 9 and 8 and then points to Anne as she says "and then ...".) Anne: "7.". (The researcher points to 7.) Researcher: "What's that one? 6.". (The researcher points to 6.) Anne: "Oh my God. I don't remember.". Researcher: "6." (The researcher keeps her finger on 6.) Anne: "6, 5, 4, 3, 2, 1.". (The researcher points to the numbers as Anne counts from 6 down to 1.) Researcher: "Good girl."***

When asked to give the number word that comes immediately before a specified number word, the child cannot give the number word that comes immediately before the specified number word

Example: ***Researcher "What number comes before 7?"(Sarah looks away.) Sarah: "8."***

B. (DI) Differentiated Response

General Definition

The child can say the number words from 10 down to 1 in the correct order. The child can spontaneously give the number word that comes just before each number word from 10 down to 2. The child knows that the number words can be recited backwards as well as forwards and knows the meaning of the term "before". The "number just before relations" for the number words from 10 to 2 have become established (Baroody, 1987; Case, 1998a; Fuson, 1988; Griffin et al., 1992).

Instances of the Behaviour

When asked to recite the number words from 10 down to 1, the child can say the number words from 10 down to 1 in the correct order.

Example: ***Researcher: "Now go back down from 10 down to 1. "10." (The researcher puts her hand up in the horizontal position.) Wendy: "10, 9, 8, 7, 6, 5, 4, 3, 2, 1." (The researcher moves her hand down. Wendy looks at the researcher's hand as she counts down to 1.)***

When asked to recite the number words down to 1 from a specified number word, the child can say the number words down to 1 from a specified number word.

Example: ***Researcher: "Okay. Do you want to start with 3? Do you think you can do it from 3? Okay. 3." (Kevin nods his head.) Kevin: "3." (Kevin crouches.) Researcher: "What comes next?" (The researcher crouches.) Kevin: "2." (Kevin crouches slightly.) Researcher: "Good." (The researcher crouches and touches the floor with her hands.) Kevin: "1." (Kevin crouches and touches the floor with his hands.) Researcher: "Good for you.".***

When given two or three number words from the number word sequence, asked to recite the next number words down and then stop at a designated number word, the child can say the next number words down in the number word sequence and stop at the designated number word.

Example: ***Researcher: "Okay. 10, 9.". (The researcher points to 10 and 9 on the number line and then points to Wendy.) Wendy: "8, 7, 6." (The researcher points to 8, 7 and 6 on the number line and then winks at Wendy.) Researcher" "Winked at you, 5." (The researcher points to Wendy.) Wendy: "4, 3, 2." (The researcher points to 4, 3 and 2 on the number line.) Researcher: "Winked at you, 1. Okay.***

Counting Objects

This code definition focuses on knowledge of the one-to-one correspondence between number words and objects when counting (Griffin et al., 1992). Knowledge of the number words from 1 to 10 and the motor routines for counting objects are integrated to form this mathematical understanding (Griffin & Case, 1996). This code definition relates to rows b and c in the diagram of the mental counting line adapted from Griffin and Case (1996) (Appendix D).

I. (TO) Touching an Object Once Each Time a Number Word is Said

A. (LA) Linked Across Response

General Definition

The child demonstrates some awareness that when counting, an object is touched once and only once each time a number word is said.[35] Individual elements of the child's schema for the number word sequence and the child's schema for the motor routines for counting objects and actions are in the process of becoming linked (Case, 1998a). The one-to-one principle (the rule that an object is touched or an action is performed once and only once each time a number word is said) has not yet been abstracted (Case). Although the child recognizes that an object must be touched each time a number word is said, the child is unable to maintain the one-to-one relationship between the objects in a set and the number words in the number word sequence (Baroody, 1987, 1989; Fuson, 1988; Gelman & Gallistel, 1978; Griffin, 2005).

Instances of the Behaviour

When asked to build a tower up from 1 to a specified number word, the child may say the number words in the correct order when counting up from 1 to the specified number word, but may not touch a block or put a block in the correct position each time he or she says a number word.

*Example: **Researcher: "Shall we do it again? Build it up again for me and then take it down. Which is 1?" (The researcher picks the first block up and places it in front of Kevin. Kevin touches the first block and puts a second block on the first block.) Kevin: "1". (Kevin picks up a third block.) Researcher: "No. Which one is 1?" (The researcher takes the second block off the first block.) Kevin: "1". (The researcher says "1" along with Kevin. Kevin and the researcher put their fingers on the first block.) Kevin: "2, 3." (Kevin puts the second block on the first block and the third block on the second block) Researcher: "Uhuhum." (The researcher steadies the blocks with her hand.) Kevin: "4, 5, 6." (Kevin puts the fourth block on the third block, the fifth block on the fourth block and the sixth block on the fifth block)***

[35] Object refers to a concrete object or an object in a pictorial display.

When asked to take the tower down, the child may touch a block or take a block from the correct position each time he or she says a number word, but may not say the number words in the correct order when counting down to 1 from the specified number word

***Example**: Researcher: "Now take them down. Take them down." (The researcher points to the seventh block.) Kevin: "1 mo ...". (Kevin takes the seventh block down.) Researcher: "What do you start with when you go down?" Kevin: "7". (Kevin puts the seventh block back on the sixth block and then takes the seventh block down.) Researcher: "Good. Okay." Kevin: "5". (Kevin puts both hands on the sixth block) Researcher: "No. What comes ...". Kevin" "4". (Kevin keeps both hands on the sixth block.) Researcher: "6". Kevin: "6, 5". (Kevin takes the sixth and fifth blocks down.) Researcher: "Good." Kevin: "4". (Kevin takes the fourth block down.) Researcher: "Good". Kevin: "3, 2, 1". (Kevin takes the third and second blocks down and puts both hands on the first block.)*

When asked to put the blocks in a row along the floor from 1 to a specified number word, the child may say the number words in the correct order when counting up from 1 to the specified number word, but may not touch a block or put a block in the correct position each time he or she says a number word.

***Example**: Researcher: "Good for you. Okay. Now, Kevin just take uhmmm ... let me see ... that many blocks in a straight line for me." (The researcher clears the blocks away from the centre of the floor and runs her hand along the floor in front of Kevin from the left to the right.) Kevin: "Okay.". Researcher: "Count as you go. Count them. What's this one?" (Kevin pushes the first block into position on the floor. The researcher picks the block up and puts it down again.) Kevin: "1." (Kevin chooses the second block.) Researcher: "Good.". Kevin: "2, 3, 4, 5". (Kevin pushes the second block against the first block, the third block against the second block, the fourth block against the third block and the fifth block against the fourth block.)*

When asked to take the blocks away from the row, the child may touch a block or take a block from the correct position each time he or she says a number word, but may not say the number words in the correct order when counting down to 1 from the specified number word.

Example: *Researcher: "Now take them away. Count back and take them away. What's that one?" (The researcher runs her hand along the line of blocks.) Sarah: "1". (Sarah touches the fifth block.) Researcher: "5". Sarah: "5, 4". (Sarah takes the fourth block away.) Researcher: "Good". Sarah: "3". (Sarah takes the third block away.) Researcher: "Good, 2". Sarah" "2, 1". (Sarah takes the second and first blocks away.)*

B. (MA) Mapped Response

General Definition

The child understands that when counting, an object is touched once and only once each time a number word is said. The content of the linked schemas has been mapped and the one-to-one principle has been abstracted (Case, 1998a; Griffin et al., 1992). The child can say the number words in the correct order and can maintain the one-to-one relationship between the objects in a set and the number words in the number word sequence. The child uses specific strategies (pointing to or touching an object or moving an object to one side as it is counted) to ensure that an object is touched once and only once each time a number word is said (Baroody, 1987, 1989; Fuson, 1988; Gelman & Gallistel, 1978; Griffin, 2005; Resnick, 1989).

Instances of the Behaviour

When asked to build a tower up from 1 to a specified number word, the child can say the number words in the correct order when counting up from 1 to the specified number word and can touch or put a block in the correct position each time he or she says a number word.

Example: *Researcher: "Okay. Let's do it one more time." (The researcher sweeps her hand up.). "Start ...Build it up from 1. Which block is number 1?". Kevin: "1." (Kevin puts his finger on the first block.) Researcher: "Good.". Kevin: "2." (Kevin puts the second block on the first block.) Researcher: "Uhuhum.". Kevin: "3, 4, 5, 6, 7." (Kevin puts the third, fourth, fifth, sixth and seventh blocks one on top of the other.)*

When asked to take the tower down, the child can say the number words in the correct order when counting down to 1 from the specified number word and can touch or take a block from the correct position each time he or she says a number word.

Example: ***Researcher: "Good. And when you start to take it down you start with ..." (The researcher puts her finger on the seventh block.) Kevin: "7." (Kevin takes the***

seventh block down and places it on the floor.) Researcher: "7. Now what ... Good." (The researcher puts her finger on the sixth block.) Kevin: "6." (Kevin takes the sixth block down and places it on the floor.) Researcher: "Good for you.". Kevin: "5." (Kevin takes the fifth block down and places it on the floor.) Researcher: "Good.". Kevin: "4, 3, 2, 1." (Kevin takes the fourth, third and second blocks down and places them on the floor. Kevin pushes the first block aside.) Researcher: "Very good, Kevin. You're doing really well.".

When asked to put blocks in a row along the floor from 1 to a specified number word, the child <u>can say</u> the number words in the correct order when counting up from 1 to the specified number word and <u>can touch</u> or <u>put</u> a block in the correct position each time he or she <u>says</u> a number word.

Example: ***Researcher: "Good. Okay. Now. Wendy, just one more thing ... I want you to put the blocks in a row. Let's do this many. You count them as you put them in a row." (The researcher moves her hand along the floor in a straight line and pushes 5 blocks toward Wendy.) Wendy: "1, 2, 3, 4, 5." (Wendy pulls the first block toward her left, puts the second block to the right of the first block, the third block to the right of the second block, the fourth block to the right of the third block and the fifth block to the right of the fourth block.)***

When asked to take the blocks away from the row, the child <u>can say</u> the number words in the correct order when counting down to 1 from the specified number word and <u>can touch</u> or <u>take</u> a block from the correct position each time he or she <u>says</u> a number word.

Example: ***Researcher: "Good girl. Now. Count backwards as you take them away." (The researcher sweeps her hand back along the length of the blocks.) Wendy: "5.". (Wendy pushes the fifth block away.) Researcher: "Good.". Wendy: "4, 3, 2, 1." (Wendy pushes the fourth, third, second and first blocks away.) Researcher: "Good girl, Wendy.".***

Counting to Determine Quantity

This code definition focuses on knowledge of the cardinal meaning of the number words and the cardinal values of sets (Griffin et al., 1992). Knowledge of the number words from 1 to 10 and the visual patterns (objects) associated with sets from 1 to 10 are integrated to form this mathematical understanding (Griffin & Case, 1996). This code definition relates to rows b and d in the diagram of the mental counting line adapted from Griffin and Case (1996) (Appendix D).

I. (CS) Counting Objects in a Set and Saying That the Last Number Said is the Number of Objects in the Set

A. (LA) Linked Across Response

General Definition

The child demonstrates some awareness that counting can be used to determine how many objects are in a set.[36] Individual elements of the child's schema for the number words from 1 to 10 and the child's schema for the visual patterns (objects) associated with sets from 1 to 10 are in the process of becoming linked (Case, 1998a). The cardinality principle (the rule that the last number word said represents the number of objects in a set) has not yet been abstracted (Case, 1998a). The child recognizes that counting can be used to determine how many objects are in a set. However, the child does not understand that, when counting, the last number word said indicates how many objects are in the set (Baroody, 1987, 1989).

Instances of the Behaviour

When asked how many objects are in a set, the child may subitize sets of 1, 2 or 3 objects and immediately say that is the number of objects in the sets.[37]

Example: ***Researcher: "Now. How many, how many pigs are left outside the field?" (The researcher points to the pigs outside the field.) Wendy: "3". (Wendy looks at the pigs as she says 3.)***

When asked how many objects are in a set, the child may count the objects in the set and fail to repeat the last number word said.

Example: ***Researcher: "Okay. You give me this one. Okay. How many ... How many pigs do you have?". (The researcher takes a cow from Kevin and circles the pigs in the***

---

[36] Objects refers to concrete objects or objects in a pictorial display.

[37] The child may be able to subitize sets of 1, 2 or 3 objects. Subitizing is the ability to recognize small number patterns (Baroody, 1987; Fuson, 1988). Understanding of cardinality and the ability to subitize small sets of objects are separate capabilities (Fuson, 1988; Gelman & Gallistel, 1978). Because the cardinality principle has not yet been abstracted, the child's response may be related to the recognition of small number patterns rather than an understanding of cardinality (Gelman & Gallistel, 1978).

field with her finger.) Kevin: "1, 2, 3, 4.". (Kevin points to the first, second, third and fourth pigs as he counts from 1 to 4.)

When asked again, how many objects are in a set, the child may recount the objects in the set.

Example: ***There are no examples of this type of response in the study.***

When asked how many objects are in a set, the child may say a number word, but not the number word that specifies the number of objects in the set.

Example: ***There are no examples of this type of response in the study.***

When asked how many objects are in a set, the child may count the objects in the set and say a number word, but not the number word that specifies the number of objects in the set.

Example: ***There are no examples of this type of response in the study.***

When asked how do you know there are x objects in a set, the child may say …

Example: ***"because I know", "cause I can see it" or "cause I can just tell" (when the child subitizes sets of 1, 2 or 3 objects).***

B. (MA) Mapped Response

General Definition

The child understands that counting can be used to determine how many objects are in a set and that the last number word said represents the number of objects in the set. The content of the linked schemas has been mapped and the cardinality principle has been abstracted (Case, 1998a; Gelman & Gallistel, 1978; Griffin et al., 1992; Resnick, 1983; Schaeffer et al., 1974). The child recognizes that counting can be used to determine how many objects are in a set and that when counting, the last number word said indicates how many objects are in the set (Baroody, 1987, 1989).

Instances of the Behaviour

When asked how many objects are in a set, the child can subitize sets of 4 or 5 objects and immediately say that is the number of objects in the set.[38]

***Example**: Researcher: "Okay. How many horses?" (The researcher puts her hand over the horses outside the field.) Sarah: "5." (Sarah looks at the horses as she says 5.)*

When asked how many objects are in a set, the child can count the objects in the set and repeat the last number word said.

***Example**: Researcher: "How many pigs have you got left in your field?" Anne: "1, 2, 3, 4, 4." (Anne touches the first, second, third and fourth pigs as she counts from 1 to 4 and looks at the researcher as she says 4 the second time.)*

When asked how many objects are in a set, the child can count the objects in the set and emphasize the last number word said.

***Example**: There are no examples of this type of response in the study.*

When asked how do you know there are x objects in a set, the child may say ...

***Example**: "I see it" (when the child subitizes sets of 4 or 5 objects), "because I counted", "counted them" or "1, 2, 3, 4" (when the child recounts the set).*

II. (CO) Counting Out Sets of a Specified Size

A. (LA) Linked Across Response

General Definition

The child demonstrates some awareness that numbers represent quantities. Individual elements of the child's schema for the number words from 1 to 10 and the child's schema for the visual patterns (objects) associated with sets from 1 to 10 are in the process of becoming linked (Case, 1998a). The cardinal-count principle (the rule that "specifies that a cardinal term, say, 'five', would be the count tag assigned to the last item when enumerating a set of five objects (Fuson & Hall, 1983 as cited in Baroody, 1987, p. 87) has not yet been abstracted. The child recognizes that counting can be used to determine how many objects are in a set. However, the child does not recognize that the number

[38] The cardinality principle has been abstracted. The child's response may now be related to his or her understanding of cardinality as well as the ability to recognize small number patterns.

word "that designated the set would be the same as the outcome of counting the set" (Baroody, 1987, p. 87).

Instances of the Behaviour

When asked to count out a set of a specified size, the child may subitize sets of 1, 2, or 3 objects rather than count out sets of 1, 2 or 3 objects.

Example: *Researcher: "And I want you to put 3 cows here, or 3 pigs here." (The researcher touches the middle field. Wendy picks up 3 pigs and puts them in the field.)*

When asked to count out a set of a specified size, the child may count out the set, but fails to stop at the number that specifies the size of the set.

Example: *Researcher: "I'd like you to put 5 pigs in this field. Would you count them as you put them in the field?" (The researcher puts her finger on the field to Sarah's right.) Sarah: "1." (Sarah puts the first pig in the field.) Researcher: "1." Sarah: "2." (Sarah puts the second pig in the field.) Researcher: "Good. Nice and loud." Sarah: "1, 2." (Sarah puts the third pig in the field.) Researcher: "No. Start again, uhm, Sarah." Sarah: "1, 2, 3. (The researcher points to the first, second and third pigs in the field as Sarah counts from 1 to 3.) Researcher: "Good." Sarah: "4." (Sarah puts the fourth pig in the field.) Researcher: "Good." Sarah: "5, 6." (Sarah puts the fifth and sixth pigs in the field.)*

B. (MA) Mapped Response

General Definition

The child understands that each number in the number sequence represents the number of objects in a set. The content of the linked schemas has been mapped and the cardinal-count principle has been abstracted. The child recognizes that the number word "that designated the set would be the same as the outcome of counting the set" (Baroody, 1987, p. 87).

Instances of the Behaviour

When asked to count out a set of a specified size, the child may subitize sets of 4 or 5 objects rather than count out sets of 4 or 5 objects.

Example: *Researcher: "Can you put 4 cows in the field." (The researcher puts her finger on the middle field. Anne picks up 4 cows in one hand and puts them in the field.)*

When asked to count out a set of a specified size, the child can count out a set of the specified size.

Example: *Researcher: "And can you put 7 cows in that field for me please? Count out loud as you go." (The researcher takes the pigs outside the field and points to the middle field.) Wendy: "1, 2, 3, 4, 5, 6, 7." (Wendy puts the first, second, third, fourth, fifth, sixth and seventh cows in the field as she counts from 1 to 7.)*

Incrementing Sets

This code definition focuses on knowledge of the increment rule: knowledge that when a set is increased by 1, the new set that is created is represented by the next number up in the number sequence (Griffin et al., 1992). Knowledge of the number word sequence (just after relations), the visual patterns associated with sets (objects or fingers), and the manner in which they are connected (adding one gives the next pattern up in the sequence) are integrated to form this mathematical understanding (Case, 1998a). This code definition relates to rows b and e in the diagram of the mental counting line adapted from Griffin and Case (1996) (Appendix D).

I. (AD) Adding 1, 2, or 3 Objects to a Set of Objects and Saying That the Last Number Counted is the Answer

A. (LA) Linked Across Response

General Definition

The child demonstrates some awareness that counting and the number sequence can be used to solve addition problems such as N+1, N+2 and N+3. Individual elements of the child's schema for the number words (just after relations) and the child's schema for the visual patterns associated with sets (objects or fingers) and the manner in which they are connected (adding one gives the next pattern up in the sequence) are in the process of becoming linked (Case, 1998a). The increment rule (the rule that when a set is increased by 1, the new set that is created is represented by the next number up in the number sequence) has not yet been abstracted (Case, 1998a). The child recognizes that addition is an incrementing process (when something is added to a set, the set gets bigger and the set is represented by a larger number), that larger numbers are further up in the number sequence and that counting is associated with the incrementing process. However, the child does not recognize that when a set is increased by 1, the new set that is created is represented by the next number up in the number sequence (Baroody, 1987; Fuson, 1988; Griffin, 2005).

Instances of the Behaviour

When asked how many N+1, N+2 or N+3 are, the child may say that any number larger than the largest addend is the answer.

Example: *Researcher: " Now. Let's try this one. Okay. I put 3 cookies in. How many cookies did I put in?" (The researcher collects the cookies, puts 3 cookies in the bag and puts the bag in front of Wendy.) Wendy: "3." (Wendy looks in the bag.) Researcher: "And cookie fairy comes along and puts in 2. How many cookies are in the bag now?" (The researcher puts 2 more cookies in the bag.) Wendy: "I know. 4." (Wendy looks in the bag.)*

When asked how many N+1, N+2 or N+3 are, the child may count up from 1 to the first addend and say that the first addend is the answer.

Example: *Researcher: "How many fingers am I showing?" (The researcher holds up 8 fingers as she asks "How many fingers am I showing?".) Kevin: "1, 2, 3, 4, 5, 6, 7, 8." (Kevin points to each finger as he counts from 1 to 8.) Researcher: "Good. Okay. Now if I add 1 more finger. How many fingers do I have then?" (The researcher continues to hold up 8 fingers.) Kevin: "1, 2. 1, 2, 3, 4, 5, 6, 7, 8." (Kevin counts silently on his fingers.)*

When asked how many N+1, N+2 or N+3 are, the child may say that the number specified by the first addend is the answer.

Example: *Researcher: "How many did you have before? Anne: "1, 2, 3, 4, 5, 6, 7, 8. Researcher: "Good. And cookie fairy came and put 1 more in and you have how many now?" (Anne drops the cookie in her hand.) Anne: "8".*

When asked how many N+1, N+2 or N+3 are, the child may count up from 1 to the first addend and say that any number larger than the first addend is the answer.

Example: *There are no examples of this type of response in the study.*

When asked how he or she knows there are N+1, N+2 or N+3 now, the child may say …

Example: *"because", "because it's such a large number", "cause I knew that" or "because I looked at it".*

B. (MA) Mapped Response

General Definition

The child understands that counting and the number sequence (just after relations) can be used to solve addition problems such as N+1, N+2 and N+3. The content of the linked schemas has been mapped and the increment rule has been abstracted (Case, 1998a; Griffin et al., 1992). The child recognizes that addition problems such N+1 can be solved by giving the next number up in the number sequence. The child recognizes that addition problems such as N+2 and N+3 can be solved by counting up from 1 and giving the last number counted as the answer[39] or counting up 2 or 3 numbers from the larger addend and giving the last number counted as the answer[40] (Baroody, 1987, 1989; Fuson, 1988; Griffin, 2005).

Instances of the Behaviour

When asked how many N+1 is, the child can say that the next number up in the number sequence is the answer.

***Example**: Researcher: "5. Okay. Cookie fairy comes along and puts 1 more cookie in. How many cookies are in the bag now?". (The researcher puts 1 more cookie in the bag. Kevin puts his hand in the bag and picks up a handful of cookies.) Kevin: "6.". (Kevin puts the cookies back in the bag and looks at the researcher.)*

When asked how many N+2 or N+3 are, the child can count up 2 or 3 numbers from the number specified by the larger addend and say the last number counted is the answer.

***Example**: Researcher: "I'm going to put 6 cookies in the bag. Okay. Now. Cookie fairy comes along and puts 2 more cookies in the bag. How many cookies do I have now?" (The researcher puts 6 cookies in the bag. The researcher then holds up 2 cookies, puts the cookies in the bag and holds the top of the bag closed.) Wendy: "So we started from 5." (Wendy looks at the number line and puts her finger on the number 5.) Researcher: "We started with 6." Wendy: "6." (Wendy puts her finger on the number 6.) Researcher:*

[39] Counting up from 1 is a less sophisticated strategy than counting on from the larger addend. However, this strategy is also based on knowledge of the increment rule. Each count up yields a new set that is increased by 1.

[40] The child may also solve N+2 and N+3 problems by immediately giving the correct answer. This type of response is acceptable if it is accompanied by a justification indicating that the child counted up from the larger addend to get the answer.

"And cookie fairy put 2 more in." (Wendy moves her finger up to number 7 and then to number 8.) Wendy: "8." (Wendy taps the number 8 with her finger.)

When asked how many N+1, N+2 or N+3 are, the child may count up from 1 (count the first addend, then count the second addend) and say that the last number counted is the answer.

Example: *Researcher: "I put 2 cookies in the bag. Okay. Here comes cookie fairy. She put 2 more cookies in the bag. How many cookies are in the bag now?" (The researcher puts 2 cookies in the bag. Kevin looks in the bag. The researcher then drops 2 more cookies in the bag. Kevin picks up the bag, puts his hand in the bag and looks in the bag.) Kevin: "1, 2, 3, 4." (Kevin touches each cookie as he counts from 1 to 4.)*

When asked how he or she knows there are N+1 cookies in the bag now, the child may say because ...

Example: *"the cookie fairy put one more in", "added one more" or "bring one more".*

When asked how he or she knows there are N+2 or N+3 cookies in the bag now, the child may say because ...

Example: *"the cookie fairy put two or three more in", "added two or three more" or "because I counted".*

Decrementing Sets

This code definition focuses on knowledge of the decrement rule: knowledge that when a set is decreased by 1, the new set that is created is represented by the next number down in the number sequence (Griffin et al., 1992). Knowledge of the number word sequence (just before relations), the visual patterns associated with sets (objects or fingers), and the manner in which they are connected (subtracting one gives the next pattern down in the sequence) are integrated to form this mathematical understanding (Case, 1998a). This code definition relates to rows b and e in the diagram of the mental counting line adapted from Griffin and Case (1996) (Appendix D).

I. (SU) <u>Subtracting</u> 1 or 2 Objects From a Set of Objects and <u>Saying</u> That the Last Number Counted is the Answer

<u>A. (LA) Linked Across Response</u>

<u>General Definition</u>

The child demonstrates some awareness that counting and the number sequence can be used to solve subtraction problems such as N-1 and N-2. Individual elements of the child's schema for the number words (just before relations) and the child's schema for the visual patterns associated with sets (objects or fingers) and the manner in which they are connected (subtracting one gives the next pattern down in the sequence) are in the process of becoming linked (Case, 1998a). The decrement rule (the rule that when a set is decreased by 1, the new set that is created is represented by the next number down in the number sequence) has not yet been abstracted (Case, 1998a). The child recognizes that subtraction is a decrementing process, that smaller numbers are further down in the number sequence and that counting is associated with the decrementing process. However, the child does not recognize that when a set is decreased by 1, the new set that is created is represented by the next number down in the number sequence (Baroody, 1987, 1989; Fuson, 1988; Griffin, 2005).

<u>Instances of the Behaviour</u>

When asked how many N-1 or N-2 are, the child <u>may say</u> any number smaller than the larger number is the answer.

Example*: Researcher: "So we've got 4 cookies in the bag." (The researcher puts the cookies back in the bag.) "Cookie monster comes along. Close your eyes. And takes 1 cookie out of the bag." (The researcher takes 1 cookie out of the bag.)"How many do we have now?" Sarah: "2." (Sarah looks in the bag.)*

When asked how many N-1 or N-2 are, the child <u>may count out</u> the number specified by the larger number and <u>say</u> the larger number is the answer.

Example*: There are no examples of this type of response in the study.*

When asked how many N-1 or N-2 are, the child <u>may say</u> the larger number is the answer.

Example*: Researcher: "How many did we, how many did we start with? 7". (The researcher puts her finger on the number 7.) "And cookie monster took 1 away. How* *many do we have now? Look at the line. How many do we have now?" (The researcher keeps her finger on the number 7.) Anne: "7." (Anne looks at the number line.) Researcher: "Take away 1 is ..." Anne: "7".*

When asked how many N-1 or N-2 are, the child may count out the number specified by the larger number and say any number smaller than the larger number is the answer.

***Example**: There are no examples of this type of response in the study.*

When asked how many N-1 or N-3 are, the child may count out the number of objects specified by the larger number, take away the number of objects specified by the smaller number, count the objects that are left say the last counted number is the answer.[41]

***Example**: There are no examples of this type of response in the study*

When asked how many N-1 or N-2 are, the child may take away the number of objects specified by the smaller number, count or subitize (for sets of up to 4 or 5 objects) the objects that are left and say the last counted number as the answer.

***Example**: Researcher: "Good girl you've got 5 cookies. Okay. Let's put them back in the bag." (The researcher puts the cookies back in the bag.) "Close your eyes. Here comes a cookie monster. Cookie monster's going to take a cookie, 1 cookie away." (The researcher takes 1 cookie out of the bag.) "How many cookies are in the bag now?" Sarah: "1, 2, 3, 4." (Sarah looks in the bag and touches the cookies as she counts from 1 to 4).*

When asked how he or she knows there are N-1, N-2 or N-3 cookies in the bag now, the child ...

***Example**: may not respond or may say because "I know", "I guessed", "because I looked at it", "because I know", "took" or "because I saw".*

[41] Counting out the number of objects specified by the larger number and taking away the number of objects specified by the smaller number is similar to counting down. Both strategies involve taking something away (counting down 1 is taking away 1) (Baroody, 1989). However, the former strategy does not depend upon the decrement rule.

B. (MA) Mapped Response

General Definition

The child understands that counting and the number sequence (just before relations) can be used to solve subtraction problems such as N-1 and N-2. The content of the linked schemas has been mapped and the decrement rule has been abstracted (Case, 1998a; Griffin et al., 1992). The child recognizes that subtraction problems such N-1 can be solved by giving the next number down in the number sequence. The child recognizes that subtraction problems such as N-2 can be solved by counting down from the larger number and giving the last number counted as the answer.[42] (Baroody, 1987, 1989; Fuson, 1988; Griffin, 2005).

Instances of the Behaviour

When asked how many N-1 is, the child can say that the next number down in the number sequence is the answer.

***Example**: Researcher: "I'm going to put 9 cookies in the bag." (The researcher puts 9 cookies in the bag".) Researcher: "How many cookies are in the bag?" (The researcher holds the top of the bag closed.) Wendy: "9". Researcher: "Good. Here comes cookie monster. Cookie monster takes 1 cookie out" (The researcher takes 1 cookie out of the bag and holds it up for Wendy to see.) Researcher:" How many do we have now?" (The researcher holds the top of the bag closed.) Wendy: "8." (Wendy looks at the researcher.)*

When asked how many N-2 is, the child can count down 2 numbers from the larger number and say that the last number counted is the answer.

***Example**: There are no examples of this type of response in the study.*

When asked how he or she knows there are N-1 cookies in the bag now, the child may say because *"there's one gone away", "tooked away", "took one out", "the cookie monster took one", "cause there's one gone away"* or *"the cookie monster ate it".*

When asked how he or she knows there are N-2 cookies in the bag now, the child may say ...

[42] The child may also solve N-2 and N-3 problems by immediately giving the correct answer. This type of response is acceptable if it is accompanied by a justification indicating that the child counted down from the larger number to get the answer.

***Example**: because "two were taken away", because "I counted" or "the cookie monster ate two".*

Using Numbers to Compare Quantities

This code definition focuses on knowledge of the fine-comparison rule: the rule that a "number that comes after another number in the number sequence is [one] more than its predecessor" (Schaeffer, Eggleston, & Scott, as cited in Baroody, 1989, p. 102). Knowledge of the number word sequence (just after relations/ just before relations), the visual patterns associated with sets (objects or fingers), the manner in which they are connected and what this means in terms of relative quantity (the next pattern up is "one more" and the next pattern down is "one less" for every number in the number sequence) are integrated to form this mathematical understanding (Griffin & Case, 1996). This code definition relates rows b and e[43] in the diagram of the mental counting line adapted from Griffin and Case (1996) (Appendix D).

I. (SM and SL) Saying That One Set Has More or Less Than Another Set Because the Number Associated With the First Set is Bigger or Smaller Than the Number Associated With the Second Set

A. (LA) Linked Across Response

General Definition

The child understands that numbers and the number sequence can be used to make "gross comparisons" between sets (comparisons between sets "that differ by many") (Baroody, 1989, p. 99).[44] Individual elements of the child's schema for the number words (just after relations/just before relations) and the child's schema for the visual patterns associated with sets (objects or fingers), the manner in which they are connected and what this means in terms of relative quantity (the next pattern up is "one more" and the next pattern down is "one less" for every number in the number sequence) are in the process of becoming linked (Case, 1998a). The fine-comparison rule (the rule that a "number that comes after another number in the number sequence is [one] more than its predecessor" (Schaeffer, Eggleston, & Scott, as cited in Baroody, 1989, p. 102) has not yet been abstracted (Case, 1998a). The child

---

[43] This code definition represents further elaboration of the integration of rows b and e.

[44] The child can make comparisons between sets represented by numbers such as 7 or 10, 2 or 9, 4 or 6 (Baroody, 1989).

recognizes that numbers have magnitude, that numbers can be compared and that numbers that come later in the number sequence have greater magnitude than numbers that come earlier in the number sequence. The child understands the meaning of the terms bigger than/littler than and more than/less than (Baroody, 1989; Griffin, 2005). However, the child is not yet able to use the number sequence to make fine numerical comparisons between sets (Baroody, 1989).[45]

Instances of the Behaviour

When asked which animal he or she has the littlest amount of, the child can focus on the numbers that represent each set of animals, pick the set that has the smallest number and say that is the animal he or she has the littlest amount of.

***Example**: Researcher: "And which animal do we have the littlest amount of?" (Anne taps her finger on the number 1.)*

When asked which animal he or she has the biggest amount of, the child can focus on the numbers that represent each set of animals, pick the set that has the largest number and say that is the animal he or she has the biggest amount of.

***Example**: Researcher: "Which animal do we have the biggest amount of? (Wendy puts her finger on the picture of the frog.) "Good."*[46]

When asked which of two sets "that differ by many" (Baroody, 1989, p. 99) has more, the child can focus on the numbers that represent each set and say that the set that is represented by the larger number has more.

***Example**: Researcher: "Do we have more turtles or more frogs (6 turtles vs. 10 frogs)?" (Anne moves her finger up to the number 10 and taps the number 10 with her finger.) Anne: "More frogs." (Anne looks at the number line.)*

When asked which of two sets "that differ by many" (Baroody, 1989, p. 99) has less, the child can focus on the numbers that represent each set and say that the set that is represented by the smaller number has less.

[45] The just after/just before relations in the number sequence are not yet established.

[46] Wendy put her finger on a picture of a single frog, next to the number 10.

Example: *Researcher: "Which number is smaller, 5 or 10? Which number is smaller, 5 or 10?" Kevin: "5."*

When asked how he or she knows which is the biggest amount, the child may say ...

Example: *"cause it's higher"", "because there's lots" or "I can see".*

When asked how he or she knows which of two sets that differ by more than one unit has more, the child may say ...

Example: *"because it's bigger", "because they're more bigger", "cause I see more dogs", "because the dogs are more bigger and the owls are more smaller", "because 4 is bigger than 2" or "cause I know".*

B. (MA) Mapped Response

General Definition

The child understands that numbers and the number sequence can be used to make "fine comparisons" between sets (comparisons between sets "that differ by one") (Baroody, 1989, p. 99).[47] The content of the linked schemas has been mapped and the fine-comparison rule has been abstracted (Case, 1998a; Griffin et al., 1992). The child recognizes that numbers have magnitude, that numbers can be compared and that the number that comes just after each number in the number sequence has a greater relative magnitude (is bigger or more) than the number that precedes it (Baroody, 1989; Griffin et al., 1992).

Instances of the Behaviour

When asked which animal he or she has the next biggest amount of, the child can focus on the numbers that represent each set of animals, pick the set that has the next biggest number and say that is the animal he or she has the next biggest amount of.

Example: *Researcher: "And the next is the ... After 1 is the ... rabbit." (The researcher moves her finger down to the owl and then back up to the rabbit.) Sarah scratches the picture of the rabbit with her finger.) Sarah: "Rabbit." (Sarah puts her finger on the rabbit.)*

---

[47] The child can make comparisons between sets represented by numbers such as 4 or 5, 8 or 9, or 2 or 3 (Baroody, 1989).

When asked which of two sets "that differ by one" (Baroody, 1989, p. 99) has more, the child can focus on the numbers that represent each set and say that the set that is represented by the larger number has more.

***Example**: Researcher: "Do we have more chickens or do we have more seals (8 chickens vs. 7 seals)?" Wendy: "More chickens." (Wendy looks at the number line and then puts the paper hat on her head.)*

When asked which of two sets "that differ by one" (Baroody, 1989, p. 99) has less, the child can focus on the numbers that represent each set and say that the set that is represented by the set that is represented by the smaller number has less.

***Example**: Researcher: "And which is the smaller number 6 or 7?" (The researcher puts her hand on the floor beside the row of pigs.) Sarah: "6." (Sarah looks at the cows and pigs.)*

When asked how he or she knows which is the littlest, biggest or next biggest amount, the child may say ...

***Example**: "cause there's one more in all of them" or "I counted".*

When asked how he or she knows which of two sets that differ by one unit has more, the child may say ...

***Example**: "cause 4" (the bigger number), "5" (the bigger number), "we have 5" (the bigger number), 'I looked at the number", "because 3 is a bigger number", "because they've got 8", "because there are 3 and these guys are 1, 2" or "I looked at the number".*

Comparing Quantities in Two Different Quantitative Dimensions

This code definition focuses on knowledge of relative quantity in more than one quantitative dimension (dimensions of number, length, weight, etc.) (Griffin et al., 1992). At this point in the developmental sequence the mental counting line is "consolidated in a fashion that transcends any specific context" (Case, 1998a, p. 791). The understanding represented in the mental counting line is then used to make sense of new tasks and problems situations. This code definition relates to the outside brackets in the diagram of the mental counting line (the outside brackets indicate that the integrated structure represented by rows b, c, d, and e is used to determine whether there is more or less of a particular quantity in a variety of quantitative dimensions) adapted from Griffin and Case (1996) (Appendix D).

I. (UN) Using Numbers to Say Which of Two Sets Has More or Less in Two Different Quantitative Dimensions

A. (NC) Non-Consolidated Response

General Definition

Although the child previously used a counting and comparing strategy (or subitizing and comparing strategy) to determine which of two sets has more or less (assembled the counting and comparing strategy in Unit 7), the child does not recognize that he or she can use this strategy to determine which of two sets has more or less in other contexts or problem situations. The mental counting line has not yet been consolidated (Case, 1985, 1996a, 1998a). The child continues to use his or her non-quantitative strategy (global, visual assessment strategy) to determine which of two jars has more or less in the dimension of number or which of two chains is longer or shorter in the dimension of length (Baroody, 1987; Case, 1985; Fuson, 1988).

Instances of the Behaviour

When asked which of two jars has more blocks, the child will look at the blocks in the first jar, look at the blocks in the second jar, compare the amounts visually and say that the jar with the larger amount has more.

Example: *Researcher: "Which jar has more (3 vs. 7)?" (Wendy looks at the jar.) Wendy: "That one." (Wendy points to the jar to her left.)*

When asked which of two jars has fewer blocks, the child will look at the blocks in the first jar, look at the blocks in the second jar, compare the amounts visually and say that the jar with the smaller amount has fewer.

Example: *There are no examples of this type of response in the study.*

When asked which of two jars has more blocks, the child will put the blocks from the first jar in a row on the floor, put the blocks from the second jar in a row on the floor, compare the two rows visually and say that the longer row has more.

Example: *There are no examples of this type of response in the study.*

When asked which of two jars has fewer blocks, the child will put the blocks from the first jar in a row on the floor, put the blocks from the second jar in a row on the floor, compare the two rows visually and say that the shorter row has fewer.

***Example**: There are no examples of this type of response in the study.*

When asked which of two chains is longer, the child will look at the first chain, look at the second chain, compare the chains visually and say that the chain that covers the larger area is longer (the chains are shown curled up).

***Example**: Researcher: "I'm going to show you the chains and you're going to tell me which pile has more." (The researcher puts 2 chains [coiled up] with different numbers of links in front of Wendy.) Researcher: "Which one do you think has more (3 vs. 5)?" Wendy: "That one". (Wendy points to the chain to her right (the chain with 5 links.)*

When asked which of two chains is shorter, the child will look at the first chain, look at the second chain, compare the chains visually and say that the chain that covers the smaller area is shorter (the chains are shown curled up).

***Example**: There are no examples of this type of response in the study.*

When asked which of two chains is longer, the child will stretch the chains out on the floor or hold the chains up from one end, compare them visually and say that the chain with the greater length is longer.

***Example**: Researcher: "Now, how do you know that's the longest?" Kevin: "Because it's the longest." (Kevin picks up both chains, holds them full length and compares them visually.)*

When asked which of two chains is shorter, the child will stretch the chains out on the floor or hold the chains up from one end, compare them visually and say that the chain with the lesser length is shorter.

***Example**: There are no examples of this type of response in the study.*

When asked how he or she knows which of two jars has more blocks, the child may say …

***Example**: "cause that has more", "it has a lot", "because I see more blocks in it", "cause that has more", "it has a lot" or "because I see".*

When asked how he or she knows which of two chains is longer, the child may say …

Example: *"because this is so big", "cause", "cause there's more", "because it's the longest", "cause there's more", "cause it's wide and that one's not wider", "cause it's longer" or "because it's wide and that one's not wide".*

B. (CO) Consolidated Response

General Definition

The child recognizes that he or she can use a counting and comparing strategy (or subitizing and comparing strategy) to determine which of two sets has more or less in more than one context or problem situation (especially when the difference between the two amounts is not visually apparent). The mental counting line has been consolidated[48] (Case, 1985, 1996a, 1998a, 1998b). The child uses his or her quantitative strategy (count and compare strategy) to determine which of two jars has more or less in the dimension of number or which of two chains is longer or shorter in the dimension of length (Case, 1985).

Instances of the Behaviour

When asked which of two jars has more blocks, the child can subitize the amount in one jar (subitize up to 4 or 5 blocks), count or subitize the amount in the other jar (subitize up to 4 or 5 blocks), compare the resulting numbers and say that the jar with the larger number has more.

Example: *Researcher: " Okay. Close your eyes again. Which one has more now (8 vs. 2)?" (The researcher changes the quantities in the 2 jars. Kevin closes his eyes. The researcher moves the jar to Kevin's right to Kevin's left and the jar to Kevin's left to Kevin's right. Kevin looks at the jars.) Kevin: " 1, 2, 3, 4, 5, 6, 7, 8." (Kevin points to each block as he counts from 1 to 8) Researcher: "Good, That one has 8." (The researcher puts her hand over the jar to Kevin's left.) Researcher: "How many does this one have?" (The researcher puts her finger on the jar to Kevin's right.) Kevin: "2". (Kevin looks at the jar.) Researcher: "Which is the larger number?" Kevin: "8" (Kevin drops a block into the jar to his left.)*

[48] The child's differentiated and elaborated counting schema ("structure for determining relative number by counting") (Case, 1985, p. 263) has become "hierarchically subordinate to, and integrated with" (Case, 1985, p. 264) his or her differentiated and elaborated quantity schema ("structure for determining relative amount by visual inspection") (Case, 1985, p. 263).

When asked which of two jars has fewer blocks, the child can subitize the amount in one jar (subitize up to 4 or 5 blocks), count or subitize the amount in the other jar (subitize up to 4 or 5 blocks), compare the resulting numbers and say that the jar with the smaller number has fewer.

Example: *There are no examples of this type response in the study.*

When asked which of two jars has more blocks, the child can count the blocks in the first jar, count the blocks in the second jar, compare the resulting numbers and say that the jar with the larger number has more.

Example: *Researcher: "Okay. Which one has more (7 vs. 8)?" (The researcher puts the jar on Wendy's right to Wendy's left and the jar on Wendy's left to Wendy's right.) Wendy: " Mmm. 1, 2. 1, 2, 3, 4, 5, 6, 7. 1, 2, 3, 4, 5, 6, 7, 8." (Wendy looks in the jar to her right, puts her hand in the jar and touches each block as she says 1, 2. Wendy touches each block in the jar to her right as she counts from 1 to 7. Wendy looks in the jar to her left and touches each block as she counts from 1 to 8.) Researcher: "Good. Okay. Which one has more?" Wendy: "That one." (Wendy puts her hand over the jar to her left.)*

When asked which of two jars has fewer blocks, the child can count the blocks in the first jar, count the blocks in the second jar, compare the resulting numbers and say that the jar with the smaller number has fewer.

Example: *There are no examples of this type of response in the study.*

When asked which of two chains is longer, the child can subitize the number of links on one chain (subitize up to 4 or 5 links), count or subitize the number of links on the other chain (subitize up to 4 or 5 links), compare the resulting numbers and say that the chain with the larger number is longer.

Example: *Researcher: " How do you know it's that one (3 vs. 5)? What makes you think it's that one?" (The researcher points to the chain to Wendy's right.) Wendy: "There's 3 on this one and not 3 on this one." (Wendy points to the chain to her left as she says "There's 3 on this one.". Wendy points to the chain to her right as she says "not 3 on this one".) Researcher: "Okay." Wendy: "2. Okay. That one and that one are making it 6." (Wendy touches first one and then another link on the chain to her right.) Researcher:*

"Okay. Let's try. Let's count them." (The researcher picks up the chain to Wendy's right and stretches it out on the floor.) Wendy: "5. 5." (Wendy looks at the chain.)

When asked which of two chains is shorter, the child <u>can subitize</u> the number of links on the first chain (subitize up to 4 or 5 links), <u>count or subitize</u> the number of links on the other chain (subitize up to 4 or 5 links), <u>compare</u> the resulting numbers and <u>say</u> that the chain with the smaller number is shorter.

***Example**: There are no examples of this type of response in the study.*

When asked which of two chains is longer, the child <u>can count</u> the links on the first chain, <u>count</u> the links on the second chain, <u>compare</u> the resulting numbers and <u>say</u> that the chain with the larger number is longer.

***Example**: Researcher: " Which one is bigger, Kevin (8 vs. 9)?" Kevin: "Oh. This. 1, 2, 3, 4, 5, 6, 7, 8. 1, 2, 3, 4, 5, 6, 7, 8, 9." (Kevin picks up the chain to his left, holds it up full length and looks at each link as he counts from 1 to 8. Kevin put the chain back down on the floor. Kevin picks up the chain to his right, holds it up full length and looks at each link as he counts from 1 to 9.) Researcher: "So which one is bigger?" (Kevin holds the chain with 9 links in his right hand and the chain with 8 links in his left hand and looks at the researcher.) Kevin: "9." (Kevin flings the chain in his right hand in the air.)*

When asked which of two chains is shorter, the child <u>can count</u> the links on the first chain, <u>count</u> the links on the second chain, <u>compare</u> the resulting numbers and <u>say</u> that the chain with the smaller number is shorter.

***Example**: There are no examples of this type of response in the study.*

When asked how he or she knows which of two jars has more blocks, the child <u>may say</u> . . .

***Example**: "cause there's 4 and I don't know ... I know", "There's 4 (the bigger number) in there", "count them", "because I counted", "because it has 8 and that one has 7" or "I'll count".*

When asked how he or she knows which of two chains is longer the child <u>may say</u> . . .

***Example**: "there's 3 on this one and not 3 on this one", "cause there's not 1 there", "because this one has 4 chains and this one has 3 chains", "I counted them", "cause 1, 2, 3 and more" or "because I counted".*

Acquiring Knowledge of the Written Numerals From 1 to 10

This code definition focuses on knowledge of written numerals (Griffin et al., 1992).[49] This mathematical understanding is formed when the written numerals are linked to separate components of the mental counting line (Case, 1998a; Griffin, 2005). This code definition relates to row a, rows a and b, and rows a and e in the diagram of the mental counting line adapted from Griffin and Case (1996) (Appendix D).

I. (IW) Identifying the Written Numerals From 1 to 10

A. (EL) Elaborated Response

General Definition

When the names of the written numerals from 1 to 10 are supplied by someone else, the child may identify some, but not all, of the written numerals or may confuse the written numerals that share defining characteristics (Baroody, 1989). Defining characteristics are "the component parts and how the parts fit together to form a whole" (Gibson & Levin, as cited in Baroody, 1989, p. 161). The child's knowledge of the written numerals from 1 to 10 has not yet become differentiated or linked to his or her knowledge of the number words from 1 to 10 (Case, 1998a).

Instances of the Behaviour

When the names of the written numerals from 1 to 10 are called out in random order, the child fail to pick up the corresponding written numerals.

***Example**: There are no examples of this type in the study.*

When the names of the numerals from 1 to 10 are called out in random order, the child may confuse the written numerals that share defining characteristics (2 and 5, 6 and 9, 8 and 9) and fail to pick up the corresponding written numerals.

***Example**: Researcher: "And find the 8. Where's the 8?" (Anne looks at the floor and puts her hand on the number 5. Anne then picks up the number 9 and shows it to the researcher.)*

---

[49] Knowledge of written numerals (row a) is not an essential part of the mental counting line. This understanding is acquired later and is linked to the separate mathematical understandings that make up the mental counting line: knowledge of the number words and the number word sequence (row b) and knowledge of cardinality (rows b and d) and knowledge of relative magnitude (rows b and d [just after/just before relations] and wide brackets) (Case, 1998; Griffin, 2005).

B. (DI) Differentiated Response

General Definition

When the names of the written numerals from 1 to 10 are supplied by someone else, the child can identify all of the written numerals from 1 to 10 (Baroody, 1989). Individual elements of the child's schema for the written numerals from 1 to 10 have become differentiated (Case, 1998a). However, the child's knowledge of the written numerals from 1 to 10 has not yet become linked to his or her knowledge of the number words from 1 to 10 (Case, 1998a).

Instances of the Behaviour

When the names of the written numerals from 1 to 10 are called out in random order, the child can pick up the corresponding written numerals.

***Example**: Researcher: "Can you find the number 5?" (Wendy picks up the number 5.)*

II. (RW) Reading the Written Numerals From 1 to 10

A. (EL) Elaborated Response[50]

General Definition

When shown the written numerals from 1 to 10, the child may be able to give the names of some, but not all, of the written numerals or may confuse the written numerals that share defining characteristics (Baroody, 1989). Although the child's knowledge of the written numerals from 1 to 10 has become linked to his or her knowledge of the number words from 1 to 10 (Griffin, 2005; Resnick et al., 1973), individual elements of the child's schema for the written numerals from 1 to 10 have not yet become fully differentiated (Case, 1998a).

Instances of the Behaviour

When shown the written numerals from 1 to 10, the child fails to give the names of the corresponding written numerals.

***Example**: There are no examples of this type of response in the study.*

[50] This response is labelled an elaborated response because the child is not yet able to identify all of the written numerals from 1 to 10, even though the child has already linked the number words from 1 to 10 to the written numerals from 1 to 10.

When shown the written numerals from 1 to 10, the child may confuse the written numerals that share defining characteristics (2 and 5, 6 and 9, 8 and 9) and fail to give correct names for these written numerals.

***Example**: Researcher: "What's this one? Look at this one." (The researcher shows Anne the card with the number 9.) Anne: "6."*

B. (LA) Linked Across Response

General Definition

When shown the written numerals from 1 to 10, the child can give the names of all of the written numerals from 1 to 10 (Baroody, 1987, 1989). The child's knowledge of the written numerals from 1 to 10 has become linked to his or her knowledge of the number words from 1 to 10 (Griffin, 2005; Resnick et al., 1973).

Instances of the Behaviour

When shown the written numerals from 1 to 10 and asked to give the names that correspond to the written numerals, the child can give the names that correspond to all of the written numerals from 1 to 10.

***Example**: Researcher: "What's this one?". (The researcher holds up the card showing the number 10.) Kevin: "10." (Kevin looks at the card.)*

III. (MW) Matching the Written Numerals From 1 to 10 to Their Corresponding Pictorial Displays

A. (EL) Elaborated Response

The child may be able to give the names of some, but not all, of the written numerals or may confuse the written numerals that share defining characteristics (Baroody, 1989). Although the child's knowledge of the written numerals from 1 to 10 has become linked to his or her knowledge of cardinality (Griffin, 2005; Resnick et al., 1973), individual elements of the child's schema for the written numerals from 1 to 10 have not yet become fully differentiated (Case, 1998a).

When asked to match the written numerals from 1 to 10 to their pictorial displays, the child may count the objects in the pictorial displays or subitize (pictorial displays of up to 3 objects), fail to read the corresponding written numerals, and fail to match the written numerals to their corresponding pictorial displays.

***Example**: There are no examples of this type of response in the study.*

B. (LA) Linked Across Response[51]

General Definition

The child demonstrates some awareness that the written numerals from 1 to 10 represent sets of a specified size (Fuson, 1988). The child recognizes that counting can be used to determine how many objects are in a set. However, the child does not understand that the last number word said indicates how many objects are in the set (Baroody, 1987). The child's knowledge of the written numerals has become linked to his or her incomplete understanding of cardinality (Griffin, 2005).

Instances of the Behaviour

When asked to match the written numerals from 1 to 10 to their pictorial displays, the child may read the written numerals, count the objects in the pictorial displays and fail to match the written numerals to their corresponding pictorial displays.

***Example**: There are no examples of this type of response in the study.*

When asked to match the written numerals from 1 to 10 to their pictorial displays, the child may read the written numerals, count the objects in the pictorial displays, say a number word, but not the number word that specifies the number of objects in the pictorial displays and fail to match the written numerals to their corresponding pictorial displays.

***Example**: There are no examples of this type of response in the study.*

C. (MA) Mapped Response[52]

General Definition

The child understands that the written numerals from 1 to 10 represent sets of a specified size (Fuson, 1988; Resnick et al., 1973). The child recognizes that counting can be used to determine how many objects are in a set and that the last number word said indicates how many objects are in the set (Baroody, 1987). The child's knowledge of the written

[51] This response is labelled a linked across response because the child uses his or her incomplete understanding of cardinality to match the written numerals to their corresponding pictorial displays.

[52] This response is labelled a mapped response because the child uses his or her understanding of cardinality to match the written numerals to their corresponding pictorial displays.

numerals has become linked to his or her knowledge of cardinality (Griffin, 2005; Resnick et al., 1973).

Instances of the Behaviour

When asked to match the written numerals from 1 to 10 to their pictorial displays, the child can read the written numerals, subitize pictorial displays of up to 4 or 5 objects, and match the written numerals to their corresponding pictorial displays.

Example: *Researcher: "What's this one, Anne? What's that one?" (The researcher holds up the card showing the number 5.) Anne: "5." (Anne looks at the card and moves toward the researcher.) Researcher: "Good. Which one is 5?" (Anne kneels in front of the researcher and looks at the card.) Anne: "There." (Anne picks up the card showing 5 dots and shows it to the researcher.)*

When asked to match the written numerals from 1 to 10 to their pictorial displays, the child can read the written numerals, count the objects in the pictorial displays, and match the written numerals to their corresponding pictorial displays.

Example: *There are no examples of this type of response in the study.*

When asked to match the written numerals from 1 to 10 to their pictorial displays, the child can read the written numerals, count the objects in the pictorial displays, repeat or emphasize the last number word said, and match the written numerals to their corresponding pictorial displays.

Example: *There are no examples of this type of response in the study.*

When asked to match the written numerals from 1 to 10 to their pictorial displays, the child can read the written numerals, count the objects in the pictorial displays, and match the written numerals to their corresponding pictorial displays.

Example: *Researcher: " And this one?" (The researcher holds up the card showing the number 8.) Kevin: "8." (Kevin looks at the card as he says 8. Kevin looks at the cards on the floor.) Kevin: "1, 2, 3, 4, 5, 6, 7, 8." (Kevin touches each dot on the card showing 8 dots with his finger as he counts from 1 to 8.) Researcher: "Okay" (Kevin picks the card up from the floor and takes the card showing the number 8 from the researcher.)*

IV. (PW) Putting the Written Numerals From 1 to 10 in the Correct Order on the Number Line

A. (DI) Differentiated Response[53]

General Definition

The child recognizes that the written numerals follow the same order as the number words in the number word sequence. The child uses his or her knowledge of the number words from 1 to 10 to order the written numerals from 1 to 10 (Fuson, 1988). The child's knowledge of the written numerals has become linked to his or her knowledge of the number sequence from 1 to 10 (Griffin, 2005).

Instances of the Behaviour

When asked to put the numerals from 1 to 10 in the correct order, the child can count up from 1 and put the numeral in the correct position.

***Example**: Researcher: "And where would you put the number 6?" (Kevin turns, picks up the number 6, turns back to the number line and looks at the number line.) Kevin: "1. 1, 2, 3, 4, 5, 6. 6." (Kevin points to the first space on the horizontal line as he says 1, points to this space again as he says 1 the second time, points to the second, third, fourth and fifth spaces as he counts from 2 to 5 and puts the number 6 in the sixth space as he says 6.)*

B. (LA) Linked Across Response[54]

General Definition

The child demonstrates some awareness that the written numerals are ordered in terms of their numerosity (Baroody, 1989). The child recognizes that numbers have magnitude and that numbers that come later in the number sequence have greater magnitude (are bigger) than numbers that come earlier in the number sequence (Baroody, 1989). The child uses his or her knowledge of the after/before relations of the number sequence (uses a global comparison strategy) to order the written numerals from 1 to 10 (Fuson, 1988).

---

[53] This response is labelled a differentiated response because the child uses his or her understanding of the position of the number words in the number word sequence to put the written numerals in the correct order.

[54] This response is labelled a linked across response because the child uses his or her incomplete understanding of cardinal order relations on the number words to put the written numerals in the correct order.

The child's knowledge of the written numerals has become linked to his or her incomplete knowledge of the cardinal order relations on the number words.

Instances of the Behaviour

When asked to put the numerals from 1 to 10 in the correct order, the child can put the lowest numeral in the first position.

***Example**: Researcher: "Where does 1 go? Good." (Anne picks up the number 1 and puts it on the first space on the horizontal line.)*

When asked to put the numerals from 1 to 10 in the correct order, the child can put the highest numeral in the last position.

***Example**: Researcher: "Where does 10 go? Good girl. Okay." (Anne puts the number 10 on the number 10 space.)*

When asked to put the numerals from 1 to 10 in the correct order, the child can put lower numerals toward the beginning of the number sequence.

***Example**: Researcher: "Where does it (2) go?" Kevin: "2." (Kevin looks at the horizontal line, points to the second space on line and puts the number 2 in that space. Kevin has not yet placed the numbers 1 or 3 on the line.)*

When asked to put the numerals from 1 to 10 in the correct order, the child can put higher numerals toward the end of the number sequence.

***Example**: Researcher: "Where does 9 go? Good. Okay." (Kevin picks up the number 9 and puts the number 9 in the ninth space on the horizontal line. Kevin has not yet placed the numbers 8 or 10 on the line.)*

C. (MA) Mapped Response[55]

General Definition

The child understands that the written numerals are ordered in terms of their numerosity (Resnick et al., 1973). The child recognizes that numbers have magnitude and that the number that comes just after each number in the number sequence has a greater relative magnitude than the number that precedes it (Baroody, 1989). The child uses his or her

[55] This response is labelled a mapped response because the child uses his or her understanding of the cardinal order relations on the number words to put the number words in the correct order.

knowledge of the just after/just before relations of the number sequence (uses a fine-comparison strategy) to order the written numerals from 1 to 10 (Fuson, 1988). The child's knowledge of the written numerals has become linked to his or her knowledge of the cardinal order relations of the number words (Fuson, 1988; Resnick et al., 1973).

Instances of the Behaviour

When asked to put the written numerals from 1 to 10 in the correct order, the child can put two written numerals in the correct order, put a third written numeral in the correct order with respect to the other two and continue to use this method of comparing the written numerals until all the written numerals have been put in the correct order.

***Example**: Researcher: "And where would your number 7 go?" (Kevin looks at the number 7 space and puts the number 7 on the number line in the number 7 space between the number 6 and the number 8.)*[56]

When given the numerals from 1 to 10 and asked to put the numerals in the correct order, the child can put the lowest (or highest) written numeral in the first position, the next lowest (or highest) written numeral in the second position, the next lowest written numeral (or highest) in the third position and so on until all the numerals have been placed in the correct order.

***Example**: There are no examples of this type of response in the study.*

When given the numerals from 1 to 10 and asked to put the numerals in the correct order, the child can recognize an incorrectly ordered series.

***Example**: Researcher: "Now. Where would you put 4?" (Sarah turns, looks at the number 4 on the floor, picks up the number 4 and looks at the number line.) Researcher: "Okay." (Sarah takes the number 5 off the horizontal and puts the number 4 where the number 5 was, in the seventh space on the line.) Researcher: "Good for you." (Sarah takes the number 8 which is two spaces to the left of the number 4 and moves it one space to the left.) Researcher: "Okay. Good." (Sarah takes the number 7 which is one space to the left of the number 4 and moves it one space to the left. Sarah then puts the number 5*

[56] The children in the study did not continue to use this method until all the numerals were put in the correct order. The children only put a third written numeral in the correct order with respect to two other written numerals if the two other written numerals were already in the correct position.

in the empty space just to the left of the number 4. Sarah has ordered the written numerals from 1 to 10 from right to left.)[57]

Interobserver Agreement

Interobserver agreement is demonstrated when two independent observers "agree on how big a block of codable data is" and "use roughly the same codes for the same blocks of data" (Miles & Huberman, 1994, p. 64). Interobserver agreement was assessed to determine the accuracy of the observations and to refine and clarify the code definitions (Miles & Huberman, 1994).

The data were coded by two independent observers: the researcher and a graduate student in the Department of Educational and Counselling Psychology and Special Education at the University of British Columbia who was familiar with Case's (1991c; 1996a; 1998a) theoretical perspective. The researcher coded the entire data set. The graduate student then recoded 6 of the 29 instructional sessions (sessions 1, 6, 12, 18, 24 and 29). The percentage of interobserver agreement was calculated. An interobserver agreement of 97% was initially achieved. Differences were discussed and resolved. An interobserver agreement of 100% was achieved after differences were discussed and resolved.

[57] Occasionally, young children order the written numerals from right to left.

Chapter 5:
Results

This study explored how individual, 4- to 5-year-old children, who displayed average to above-average mathematical ability for their age, responded to an instructional program that was designed to facilitate the construction of the mental counting line. Intra-individual and inter-individual variability in the rate and the pattern of construction of the mental counting line and the transfer of the understanding represented in the mental counting to new tasks and problem situations were observed and described.

The microgenetic approach advocated by Catán (1986b) was used to collect and analyze the data. An instructional program was used to stimulate and accelerate the construction of the mental counting line (Catán, 1986b; Inhelder et al., 1974; Lee & Karmiloff-Smith, 2002; McKeough & Sanderson, 1996). Changes in the children's cognitive structures as they constructed the mental counting line and the learning processes hypothesized to cause these changes to occur were inferred from the children's responses to the instructional program (Catán, 1986a; Lee & Karmiloff-Smith, 2002; Siegler & Crowley, 1991).

Five research questions guided the exploration of how the children responded to the instructional program. The research questions were:

1. When and how will the children elaborate and differentiate the content of their schemas? (When and how will the children come to know that adding one unit to the number 4 gives you the number 5 in a counting sequence?)
2. When and how will the children make linkages between the content of their schemas? (When and how will the children come to know that the number 5 involves a greater quantity than the number 4?)
3. When and how will the children map together the content of their schemas in order to abstract a new higher-order numerical principle? (When and how will the children come to know that adding one unit to a set of objects will result in a number that is one step further along in a string of numerals and that taking away one unit from a set of objects will result in a number that is one step behind in a string of numerals?)
4. When and how will the children apply the mental counting line to new tasks and problem situations?

5. When and how will the children acquire the written symbols used to represent the elements of the mental counting line?

The research questions were based on the "requirements that would have to be met" (Case, 1998a, p. 778) to construct the mental counting line. These requirements describe the developmental sequence that leads to the construction of the mental counting line and specify the changes that occur in children's cognitive structures (elaboration, differentiation, linking across, mapping, consolidation, and acquisition of written numerals) as children construct the mental counting line.

A hierarchical coding scheme was developed to detect individual differences in the rate and the pattern of construction of the mental counting line as the children progressed through the instructional program. The coding scheme was based on the requirements described above and the mathematical understandings defined in the instructional units of the instructional program.

Table 5 shows the relationship between the research questions, the units in the instructional program, the changes that occur children's cognitive structures as children construct the mental counting line, and the related levels of mathematical understanding.

Table 5. Research Questions, Instructional Units, Changes that Occur in Children's Cognitive Structures, and Related Levels of Mathematical understanding

Research Questions	Instructional Units	Changes in Cognitive Structure	Related Levels of Mathematical understanding
Question # 1	Unit 1	Elaboration	Cannot recite number words from 1 to 10 in the correct order
		Differentiation	Can recite number words from 1 to 10 in the correct order
Question # 1	Unit 2	Elaboration	Cannot recite number words from 10 to 1 in the correct order
		Differentiation	Can recite number words from 10 to 1 in the correct order
Question # 2	Unit 3	Linking Across	Cannot maintain one-to-one correspondence between numbers and objects when counting
Question # 3		Mapping	Can maintain one-to-one correspondence between numbers and objects when counting
Question # 2	Unit 4	Linking Across	Cannot count out sets of a specified size Counts sets, but does not repeat the last number word said
Question # 3		Mapping	Counts out sets of a specified size Counts sets and repeats the last number word said
Question # 2	Unit 5	Linking Across	Does not count up from larger number to obtain the answer
Question # 3		Mapping	Counts up from larger number to obtain the answer
Question # 2	Unit 6	Linking Across	Does not count down from larger number to obtain the answer
Question # 3		Mapping	Counts down from larger number to obtain the answer
Question # 2	Unit 7	Linking Across	Can make gross numerical comparisons between sets
Question # 3		Mapping	Can make fine numerical comparisons between sets
Question # 4	Unit 8	Non-consolidation	Uses a visual comparison strategy to determine which has more
		Consolidation	Counts and compares the resulting numbers to determine which has more
Question # 5	Unit 9	Elaboration	Cannot identify all the written numerals from 1 to 10 Uses a count up from 1 strategy to order the written numerals from 1 to 10
		Differentiation	Can read all the written numerals from 1 to 10
		Linking Across	Cannot match all the written numerals from 1 to 10 to their corresponding set size Uses a global numerical estimation strategy to order the written numerals from 1 to 10
		Mapping	Can match all the written numerals from 1 to 10 to their corresponding set size Uses a fine numerical comparison strategy to order the written numerals from 1 to 10 Uses a lowest/highest, next lowest/next highest strategy to order the written numerals from 1 to 10 Can recognize an incorrectly ordered series

Presentation of the Results

The results are presented in three parts. The qualitative analysis of the children's scores on the pretest and posttest measures is presented in Part 1. The results of this analysis provided evidence of the children's initial level of mathematical understanding, the children's progress in the construction of the mental counting line, and the relationship between the children's initial level of mathematical understanding and the children's post-instructional level of mathematical understanding.

An overview of each instructional unit and the results of the descriptive microgenetic quantitative and qualitative analyses of the children's responses to the instructional program are presented in Part 2. The results of these analyses addressed the research questions and provided evidence of individual pathways in the construction of each of the separate mathematical understandings represented in the mental counting line.

The results of the trend analysis of the children's performance across the instructional units is presented in Part 3. The results of this analysis provided a more comprehensive description of the construction process and evidence of individual pathways in the construction of the mental counting line across the instructional units.

Part 1: Qualitative Analysis of the Children's Pretest and Posttest Scores

The results are presented in terms of the children's performance on the screening measure and the measures of conceptual understanding.

Pretest Scores on the SB5

Table 6 shows the children's pretest scores on the Quantitative Reasoning factor index of the SB5. The Quantitative Reasoning factor index score is a measure of children's mathematical ability (Roid, 2003a). Although lower level items on the quantitative reasoning subtests of the SB5 assess mathematical ability rather than mathematical skills acquired through instruction, environmental experience significantly influences children's levels of performance on these tests (Roid, 2003a).

Table 6. Pretest Scores on the Quantitative Reasoning Factor Index of the SB5

Participant	SB5 Quantitative Reasoning Factor Index		
	Standard Score	Percentile Rank	Score Range
Anne	108	70	Average
Kevin	114	82	Average
Sarah	116	86	Above-average
Wendy	125	95	Above-average

Table 7 shows the children's pretest scores on the Verbal Quantitative Reasoning and Nonverbal Quantitative Reasoning subtests of the SB5. For 3 children there was no significant difference between the nonverbal and verbal Quantitative Reasoning subtest scores; for one child there was a significant difference between these subtest scores.

Table 7. Pretest Scores on the Verbal Quantitative Reasoning and Nonverbal Quantitative Reasoning Subtests of the SB5

Participant	SB5 Nonverbal Quantitative Reasoning Subtest			SB5 Verbal Quantitative Reasoning Subtest		
	Scaled Score[58]	Percentile Rank	Score Range	Scaled Score	Percentile Rank	Score Range
Anne	11	63	Average	12	75	Average
Kevin	15	95	Above-average	10	50	Average
Sarah	13	84	Above-average	13	84	Above-average
Wendy	15	95	Above-average	14	91	Above-average

Pretest and Posttest Scores on the Measures of Conceptual Understanding

Table 8 shows the children's pretest and posttest scores on the measures of conceptual understanding. As can be seen, all of the children progressed to the intermediate level or above on the Number Knowledge test. None of the children progressed to the unidimensional level on this test. This indicated that the children were still in the process of integrating the mental counting line. Differences in the children's posttest scores indicated that some of the children

[58] Scaled scores are normalized raw scores with a mean of 10 and a standard deviation of 3.

were further along in the integration process than others. Relative to where they started, Kevin progressed the most on this test, followed by Anne and Sarah, and Wendy.

Although the children did not progress to the unidimensional level on the Number Knowledge test, movement from the pretest to the posttest was significant for all of the children in the study. The older average ability children (Anne aged 4.8; Kevin age 4.11) progressed to a level that was appropriate for their age. The younger above-average ability children (Sarah aged 4.0; Wendy age 4.1) progressed to a level that was somewhat advanced for their age, but not more than two years in advance of their age[59].

Table 8. Pretest and Posttest Scores on the Measures of Conceptual Understanding[60]

Participant	Number Knowledge Test		Balance Beam Task		Money Knowledge Task		Birthday Party Task	
	Pretest	Posttest	Pretest	Posttest	Pretest	Posttest	Pretest	Posttest
Anne	1.1	1.7	1.0	2.0	1.0	1.0	1.0	1.5
Kevin	1.1	1.8	1.0	1.5	1.2	1.4	1.5	2.0
Sarah	1.0	1.6	1.0	1.5	1.0	1.0	1.0	0.5
Wendy	1.3	1.8	2.0	2.0	0.8	1.0	1.0	2.0

Anne's performance on the remaining conceptual measures (transfer tasks) at pretest was similar to her performance on the number Knowledge test. With the exception of the Balance Beam task, her performance on these measures at posttest was lower than her performance on the Number Knowledge test. At posttest, generalization of the understanding represented in the mental counting line was faster to the Balance Beam task (2.0) and slower to the Birthday Party task (1.5). There was no generalization to the Money Knowledge task (1.0).

With the exception the Balance Beam task, Kevin's performance on the remaining conceptual measures at pretest was higher than his performance on the Number Knowledge test. With the exception the Birthday Party task, his performance on these measures at posttest was lower than his performance on the Number Knowledge test. At posttest, generalization of the

[59] The CA (chronological age) equivalents of the following scores on the Number Knowledge test are: 1.0, 3-4 years; 1.5, 4-5 years; and 2.0, 5-6 years (Griffin & Case, 1997).

[60] The scores on the measures of conceptual understanding ranged from 1.0 (Level 1 or predimensional level) to 2.0 (Level 2 or dimensional level). A score of 1.5 represents the intermediate level.

understanding represented in the mental counting line was faster to the Birthday Party task (2.0) and slower to the Money Knowledge task (1.4) and the Balance Beam task (1.5).

Sarah's performance on the remaining conceptual measures at pretest was similar to her performance on the number Knowledge test. Her performance on these measures at posttest was lower than her performance on the Number Knowledge test. At posttest, generalization of the understanding represented in the mental counting line was faster to the Balance Beam task (1.5). There was no generalization to the Money Knowledge task (1.0) and the Birthday Party task (0.5).

With the exception of the Balance Beam task (Wendy scored 2.0 on the Balance Beam task at pretest and at posttest), Wendy's performance on the remaining conceptual measures at pretest was lower than her performance on the Number Knowledge test. With the exception of the Money Knowledge task, her performance on these measures at posttest was higher than her performance on the Number Knowledge test. At posttest, generalization of the understanding represented in the mental counting line was faster to the Birthday Party task (2.0). There was no generalization to the Money Knowledge task (1.0).

All of the children demonstrated generalization of the understanding represented in the mental counting line to at least one of the remaining conceptual measures at posttest. However, the breadth and the extent of transfer varied from child to child. This indicated all of the children were beginning to integrate the numerical and spatial components of the mental counting line and at least some of the children were beginning to transfer the understanding represented in the mental counting line to the remaining conceptual measures. However, generalization appeared to be more rapid for children who achieved the highest scores on the Number Knowledge test. Kevin was the only child who showed transfer to all of the remaining conceptual measures at posttest. Wendy was the only child who scored at the unidimensional level on two of the three remaining conceptual measures at posttest.

Relationship Between Pretest Scores on the SB5 and Posttest Scores on the Number Knowledge Test

The comparison of the children's pretest scores on the Quantitative Reasoning factor index of the SB5 and the children's posttest scores on Number Knowledge test showed that the children's pretest scores on the quantitative reasoning component of the SB5 were related to their posttest scores on the Number Knowledge test. Although all of the children progressed from the pretest to the posttest on the Number Knowledge test, progress appeared to be more significant for the younger, above-average ability children than for the older, average ability children. The older, average ability children (Anne aged 4.8; Kevin aged 4.11) progressed to a level that was appropriate for their age. The younger, above-average ability children (Sarah age 4.0; Wendy aged 4.1), progressed to a level that was somewhat advanced for their age.[61] This indicated the younger, above-average ability children, who had better developed mathematical understandings prior to instruction, showed earlier integration of the numerical and spatial components of the mental counting line for their age.

Part 2: Descriptive Microgenetic Quantitative and Qualitative Analyses of the Children's Responses

The results are presented in terms of the research questions and the units in the instructional program, both of which follow the developmental sequence that leads to the integration and consolidation of the mental counting line and the acquisition of the written symbols that represent the elements of the mental counting line. The research questions and the related instructional units are presented in the order in which the developmental sequence unfolds.

Research question 1 is presented first, followed by the results of the descriptive microgenetic quantitative and qualitative analyses of the children's responses to the tasks presented in Units 1 and 2. Research questions 2 and 3 are presented next, followed by the results of the descriptive microgenetic quantitative and qualitative analyses of the children's responses to the tasks presented in Units 3 to 7. Research question 4 is presented next, followed by the results of the descriptive microgenetic quantitative and qualitative analyses of the children's responses to the tasks presented in Unit 8. Research question 5 is presented last, followed by the results of the descriptive microgenetic quantitative and qualitative analyses of the children's responses to the tasks presented in Unit 9.

One advantage of the microgenetic approach was that it provided a detailed and precise description of the children's responses to the instructional program. This approach allowed the researcher to view the children's first independent differentiated, mapped, or consolidated responses and the responses prior to and subsequent to these responses. Investigating the children's responses in detail made it possible to identify patterns of change in the children's responses.

---

[61] The CA (chronological age) equivalents of the following scores on the Number Knowledge test are: 1.0, 3-4 years; 1.5, 4-5 years; and 2.0, 5-6 years (Griffin & Case, 1997).

To provide access to the data at a microgenetic level, the children's coded response were plotted on graphs (see Figure 1). Level of understanding was indicated along the vertical axis of each graph (the levels of understanding represent a hierarchically organized scale). Individual responses were plotted along the horizontal axis of each graph (the individual responses represent responses over time). The children's responses were identified by unit and session. Each point on the graph represented an opportunity to respond.

Individual responses were displayed within sessions for each unit (each unit represented a separate understanding). For selected understandings, individual responses were displayed by session across the instructional sequence. Responses displayed by session across the instructional sequence were numbered according to the session in which they occurred (Unit 4, Session 9, 11; Unit 5, Session 12, 13, 14 , 15, etc.).

In the majority of sessions across the instructional sequence, the children were given one opportunity to respond. In some sessions the children were given more than one opportunity to respond. In these instances, session numbers were listed more than once to indicate that the children were given more than one opportunity to respond (Unit 8, Session 24, 25, 26, 26).

When and How the Children's Schemas Were Elaborated and Differentiated

Unit 1 (Sessions 1-2)

Unit 1 was designed to teach the children the number words from 1 to 10. In Session 1 the children were asked to recite the number words up from 1 and stop at a designated number word. The highest number counted to increased as the children moved through Session 1. At the end of Session 1 the children counted from 1 up to 10. In Session 2 the children were asked to give the next number word up in the number word sequence, count on from that number word, and stop at a designated number word (Griffin et al., 1992). A vertical number line showing the numerals from 1 to 10 was used in Session 2 to help the children learn the number words and become aware of the position of each number word in the number word sequence. The children counted from 1 up to 10 across the instructional sequence.

Descriptive Microgenetic Quantitative Analysis[62]

Individual Pathways Within Unit 1

Figure 1 shows when the children first generated an independent differentiated response for the number words from 1 to 10 in Unit 1 and how the children responded prior to and subsequent to this response.[63] As can be seen, no abrupt change from lack of understanding to understanding was evident in any of the children's responses. Overall, the children's pattern of response indicated they knew the number words from 1 to 10 and understood the correct position of each number word in the number sequence from 1 to 10.

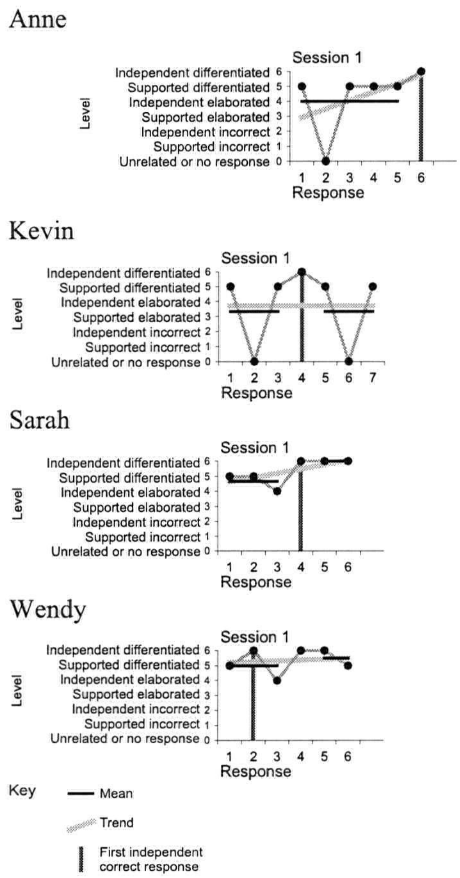


Figure 1. When the children first generated an independent differentiated response for the number words from 1 to 10 in Unit 1 and how the children responded prior to and subsequent to this response.

---

[62] The analyses were based on less complex (LC) questions such as "I want you to count from 1 to 6.", "Let's count from 1 to 10"., "I start first, 1, 2, 3 and then …" and "I'm winking at you so you stop.". More complex (MC) questions were not asked in this unit.

[63] The "Pointing and Winking Task", presented in Session 2, Unit 1, was not included in this analysis. This task required the children to count up from a specified number word and stop at a specified number word while counting. The children found this task difficult and were only able to complete this task with researcher support. The children's performance on this task is described in the qualitative analysis.

Individual Pathways Across Units 3 to 9

Figure 2 shows the responses the children generated across Units 3 to 9 for the number words from 1 to 10. As can be seen, although Anne's level of response was variable in Units 7 to 9 and Kevin's level of response was variable in Units 6 and 7, their overall level of response across Units 3 to 9 indicated they knew the number words from 1 to 10 and understood the correct position of each number word in the number sequence from 1 to 10. Sarah's and Wendy's patterns of response across Units 3 to 9 clearly indicated they knew the number words from 1 to 10 and understood the correct position of each number word in the number sequence from 1 to 10.

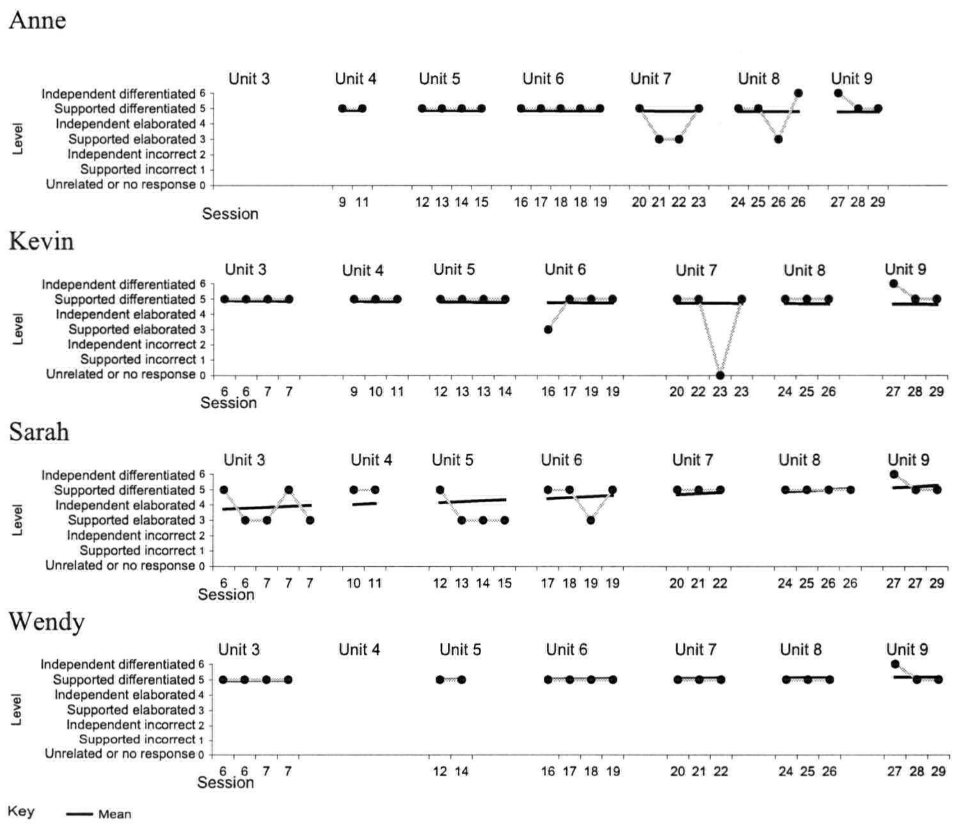


Figure 2. The responses the children generated across Units 3 to 9 for the number words from 1 to 10.

Cross-Case Comparison

All of the children first generated independent differentiated responses for the number words from 1 to 10 in Unit 1. Sarah was the only child who generated only independent differentiated responses subsequent to her first independent response in Session 1.[64]

[64] The "Pointing and Winking Task", presented in Session 2, Unit 1 was not included in this analysis.

None of the children showed an abrupt change in their level of response prior to and subsequent to their first independent differentiated response in Session 1. All of the children showed a relatively high level of response from the beginning of Session 1. This indicated the children learned to say the number words prior to instruction and that progress was related to the expansion of this mathematical understanding rather than to the construction of a new mathematical understanding.

Overall, Sarah progressed the most in Session 1, followed by Wendy, Kevin, and Anne.[65] Although Sarah generated her first independent differentiated response less quickly than Wendy, she generated only independent differentiated responses subsequent to her first independent differentiated response whereas Wendy did not consistently generate independent differentiated responses subsequent to her first independent differentiated response. The children who progressed the most in Session 1 had pathways characterized by less variability in their level of response prior to and subsequent to their first independent differentiated response and higher frequency of independent differentiated responses subsequent to their first independent differentiated response. The children who progressed the least in Session 1 had pathways characterized by greater variability in their level of response prior to and subsequent to their first independent differentiated response and lower frequency of independent mapped responses subsequent to their first independent differentiated response.

The children's most frequent responses across Units 3 to 9 were supported differentiated responses.[66] Although the frequency of researcher support was high, it consisted primarily of the researcher giving the first number word in the number word sequence, "1" or saying, "Start with 1". Overall, the results of the descriptive microgenetic quantitative analysis indicated the children knew the number words from 1 to 10 and understood the correct position of the number words from 1 to 10.

Descriptive Microgenetic Qualitative Analysis

By the end of Session 1, all of the children could say the number words separately and distinctly when reciting the number words from 1 to 10. All of the children could say the number

[65] Progress was based on when the first independent differentiated, mapped, or consolidated response occurred and the frequency (number of occurrences) of differentiated, mapped, or consolidated responses generated subsequent to the first independent differentiated, mapped, or consolidated response.

[66] Although researcher support was high, it consisted of the instructor giving the first number word in the number word sequence, "1." or saying, "Start with 1.".

words correctly up from 1 and stop at a specified number word with and without researcher support in Unit 1, with three exceptions. When asked to say the number words from 1 to 7 in Session 1, Unit 1, Sarah and Wendy said the number words up from 1, failed to stop at 7 and continued to 8 (Sarah also failed to stop at 10 on several occasions across Units 3 to 9). When asked to say the number words from 1 to 5 in Session 2, Unit 1, Kevin said the number words up from 1, failed to stop at 5 and continued to 9.

All of the children could enter the number word sequence at a specified number word, give the next number word up in the number word sequence, count on from that number word, and stop at a specified number word, with researcher support in Session 2, Unit 1. On at least one occasion Anne, Wendy, and Sarah counted on from 1 when asked to count on from the next number word up in the number word sequence. The following protocol illustrates a supported elaborated response for the number words from 1 to 10 in Session 2, Unit 1.

> *Researcher: "Okay, let's start. 1, 2, 3." Wendy: "1". Researcher: "No. You start where I left off, 1, 2." (The researcher points to Wendy.) Wendy: "1." Researcher: "I go 1, 2, 3. Then, you go ...". (The researcher points to Wendy again.) Wendy: "1" . Researcher: "No. 4, 5. (Wendy says 5 along with the researcher. The researcher points to Wendy as she says 5) "Then you stop cause I'm winking at you. And then I go 6, then I point at you and you go ..." (The researcher points at Wendy.) "What comes after 7?" Wendy: "8."*

Unit 2 (Sessions 3-5)

Unit 2 was designed to teach the children the number words backward from 10 to 1. In Sessions 3 and 4 the children were asked to recite the number words down from a designated number word. The number the children counted down from increased as the children moved through Sessions 3 and 4. At the end of Session 4 all of the children counted down from 10 to 1. In Session 5 the children were asked to give the next number word down in the number word sequence, count down from that number word, and stop at a designated number word (Griffin et al., 1992). A vertical number line showing the numerals from 1 to 10 was used in Session 5 to help the children say the number words backward from 10 to 1 and become aware of the position of each number word in the number word sequence. The children counted from 10 to 1 across the instructional sequence.

Descriptive Microgenetic Quantitative Analysis[67]

Individual Pathways Within Unit 2

Figure 3 shows the responses the children generated for the number words from 10 to 1 in Unit 2.[68] As can be seen, Anne's, Kevin's, and Wendy's patterns of response indicated they were just beginning to understand next before relationships for the numbers words from 10 to 1. They had difficulty generating the next number word back for many of the numbers from 10 to 1. Sarah's pattern of response indicated she had not yet begun to understand next before relationships for the number words from 10 to 1. She had difficulty generating the next number word back for all of the number words from 10 to 1.

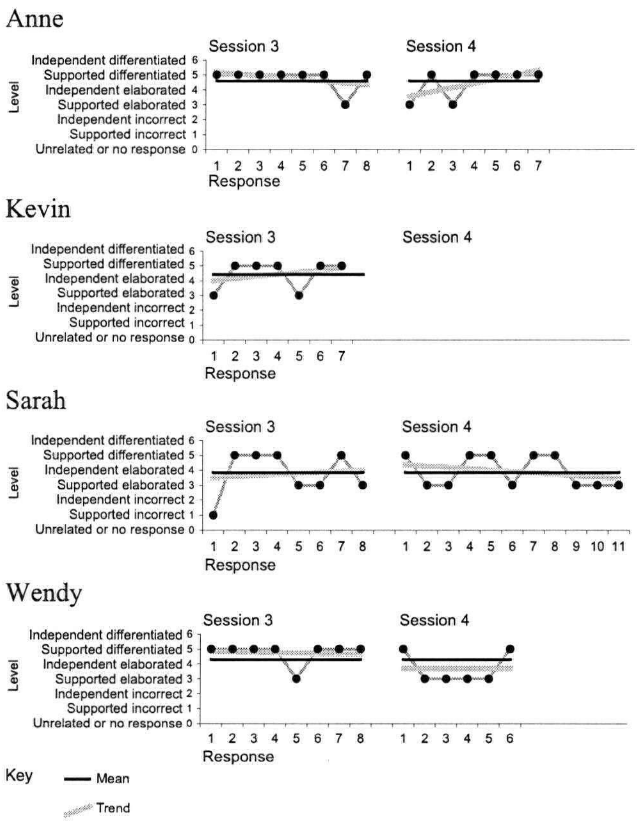


Figure 3. The responses the children generated for the number words from 10 to 1 in unit 2.

[67] The analyses were based on less complex (LC) questions such as "Let's start with 5. 5.", "Let's start with 4 and go down.", "Shall we do it from 5 again?", "I'm going to start counting and I'm going to count 10, 9. And then I point to you and go …". More complex (MC) questions were not asked in this unit.

[68] The "Pointing and Winking Task", presented in Session 5, Unit 2, was not included in this analysis. This task required the children to count down from a specified number word and stop at a specified number word while counting. The children found this task difficult and were only able to complete this task with researcher support. The children's performance on this task is described in the qualitative analysis.

Individual Pathways Across Units 3 to 9

Figure 4 shows when the children first generated an independent differentiated response for the number words from 10 to 1 across Units 3 to 9 and how the children responded prior to and subsequent to this response. As can be seen, no abrupt change from lack of understanding to understanding was evident in Anne's, Kevin's, and Wendy's response. Overall, their patterns of response indicated their understanding of next before relationships for the number words from 10 to 1 increased as they moved across Units 3 to 9. Sarah's pattern of response indicated her understanding of next before relationships for the number words from 10 to 1 did not increase as she moved across Units 3 to 9. She failed to generate an independent differentiated response for the number words from 10 to 1 across Units 3 to 9 and her data show a variable level of response with a gradually decreasing trend.

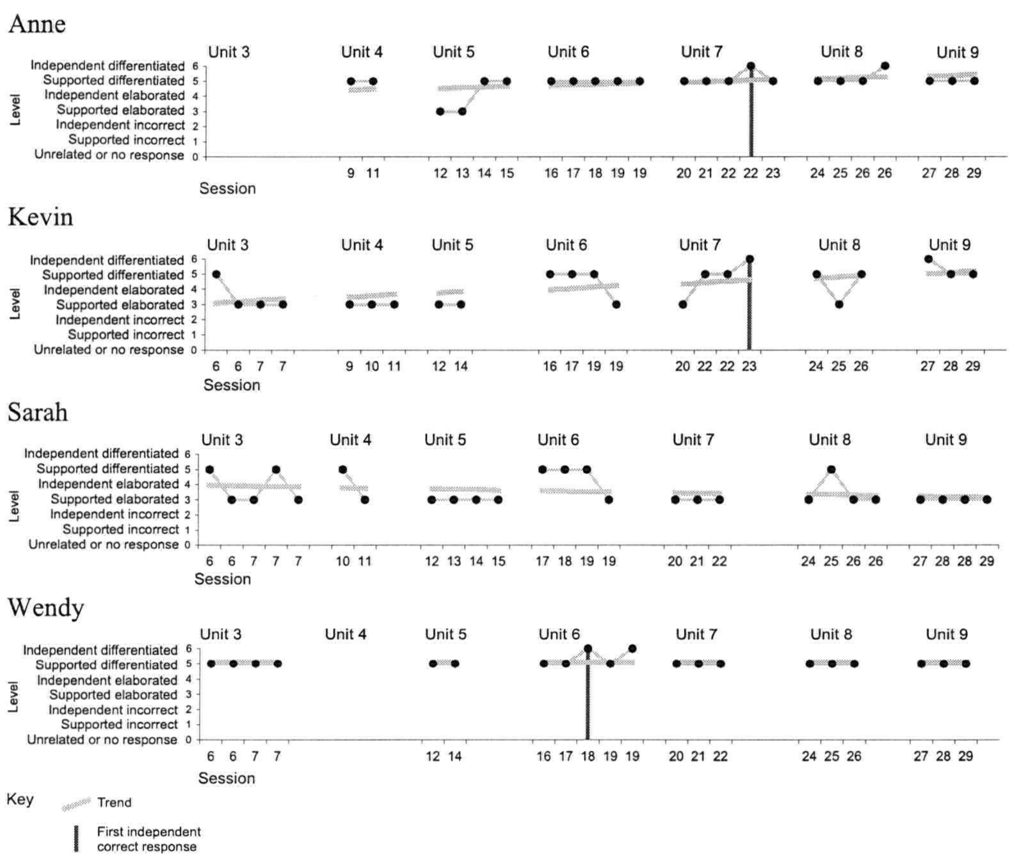


Figure 4. When the children first generated an independent differentiated response for the number words from 10 to 1 across Units 3 to 9 and how the children responded prior to and subsequent to this response.

Cross-Case Comparison

None of the children generated independent differentiated responses for the number words from 10 to 1 in Unit 2. The tasks presented in Unit 2 required greater differentiation of the number word sequence than the tasks presented in Unit 1. Children typically learn to say the number words from 10 to 1 later in the developmental sequence. Ann, Kevin, and Wendy each generated two independent differentiated responses for the number words from 10 to 1 across Units 3 to 9. However, Wendy generated these responses toward the middle of the instructional sequence. Kevin and Anne generated these responses toward the end of the instructional sequence. Sarah was the only child who failed to generate an independent differentiated response for the number words from 10 to 1 across Units 3 to 9. Anne and Wendy showed stable levels of response and Kevin and Wendy showed variable levels of response across Units 3 to 9. With the exception of Sarah, all of the children's most frequent responses subsequent to their first independent differentiated response were supported differentiated responses. None of the children showed an abrupt change in their level of response prior to and subsequent to their first independent differentiated response across Units 3 to 9.

Overall, Wendy progressed the most across Units 3 to 9, followed by Anne, Kevin, and Sarah. The children who progressed the most across Units 3 to 9 had pathways characterized by earlier attainment of their first independent response (Wendy), less variability in their level of response prior to and subsequent to their first independent differentiated response, and higher frequency of independent and supported differentiated responses subsequent to their first independent differentiated response. The children who progressed the least across Units 3 to 9 had pathways characterized by greater variability in their level of response and higher frequency of supported elaborated and supported differentiated responses. Overall, the results of the descriptive microgenetic quantitative analysis indicated the children were just beginning to or had not yet begun to establish next back relationships for the number words from 10 to 1 in Unit 2 and with the exception of Wendy, continued to acquire this mathematical understanding across Units 3 to 9.

Descriptive Microgenetic Qualitative Analysis

All of the children could say the number words from 10 to 1 in Unit 2. However, the level of researcher support was high and consisted of the researcher giving the next number word down in the number word sequence each time or allowing the children to look at a number line showing the numerals from 1 to 10 while saying the number words down to 1. The following protocol illustrates a supported differentiated response for the number words from 10 to 1 in Unit 2.

Researcher: "Okay. We'll start with 10." Wendy: "10." (Wendy remains standing.)

Researcher: "10. Okay, 9." (Wendy says "9" along with the researcher. Wendy and the

researcher crouch as they say 9.) Researcher: "8." Wendy: "8."(Wendy and the researcher crouch down each time they say a number word.) Researcher: "7." (The researcher crouches down.) Wendy: "7." (Wendy crouches right down.) Researcher: "6, 5, 4, 3, 2, 1." (Wendy says the number words along with the researcher. (The researcher crouches right down and Wendy and the researcher move their hands down as they say the number words from 5 to 2. Wendy and the researcher touch the floor with both hands as they say "1." Wendy and the researcher jump up as they say "Blast-off".)

All of the children could say the number words correctly down to 1 from a specified number word with researcher support in Unit 2. The level of researcher support was high and consisted of giving the children the next number word down in the number word sequence or saying the number words while looking at a vertical number line showing the numerals from 1 to 10. The exception was Anne, who said the number words correctly from 5 to 1 with minimal researcher support on one occasion in Session 3, Unit 2 and from 6 to 1 with minimal researcher support on one occasion in Session 4, Unit 2. All of the children had difficulty entering the number word sequence at a specified number word, giving the next number word down in the number word sequence, counting down from this number word, and stopping at a specified number word without considerable researcher support in Session 4, Unit 2.

Although the children were unable to say the number words correctly from 10 to 1 without researcher support in Unit 2, Anne, Kevin, and Wendy said the number words correctly from 10 to 1 without researcher support further along in the instructional sequence. The exception was Sarah who continued to have difficulty saying the number words from 10 to 1 throughout the instructional sequence. The following protocol illustrates a supported elaborated response for the number words from 10 to 1 across Units 3 to 9.

Researcher: "Good. Now let's go backwards." (The researcher holds her hand in position.) Sarah: "10, 9" (The researcher moves her hand down as Sarah says 9). Researcher: "Good." (The researcher moves her hand down). Sarah: "8." Researcher: "Good." (The researcher moves her hand down). Sarah: "7." Researcher: "Good." (The researcher moves her hand down). Sarah: "7." Researcher: "6." Sarah: "6, 3." (The researcher moves her hand down as Sarah says 6.) Researcher: "5." Sarah: "5, 4, 3, 2, 1." (The researcher moves her hand down as Sarah says 4, 3, 2, 1.)

When and How the Children's Schemas Were Linked Across and Mapped

Unit 3 (Sessions 6-8)

Unit 3 was designed to teach the children the principle of one-to-one-correspondence (Griffin et al., 1992). In Sessions 6 to 8 the children were asked to touch each block once each time they said a number word as they built a tower of blocks up to a specified number word (counted forward). The children were also asked to touch each block once each time they said a number word as they took the tower down (counted backward). The highest number counted to increased as the children moved through Sessions 6 and 7. The children started counting with five blocks at the beginning of Sessions 6 and 7, counted with 7 blocks at the end of Session 6, and counted with 10 blocks at the end of Session 7. In Session 8, the number of blocks counted varied depending upon the number of dots shown on the dot-set cards.[69]

<u>Descriptive Microgenetic Quantitative Analysis</u>[70]

<u>Individual Pathways Within Unit 3</u>

Figure 5 shows when the children first generated an independent mapped response for object counting using one-to-one correspondence on forward and backward counts in Unit 3 and how the children responded prior to and subsequent to this response. As can be seen, no abrupt change from lack of understanding to understanding was evident in any of the children's responses. Overall, Anne's pattern of response indicated she understood the principle of one-to-one correspondence and could touch each object once and only once when counting on the majority of responses. Kevin's pattern of response indicated he understood the principle of one-to-one correspondence. However, his performance across the second half of Unit 3 was inconsistent, due largely to errors on backward counts (he tended to start backward counts with 1). Sarah's pattern of response indicated she understood the principle of one-to-one correspondence. However, her performance across Unit 3 was inconsistent due to failure to tag

[69] In Session 8 dot-set cards were presented showing dots from 1 to 10. On each opportunity to respond, the children were asked to count the dots on the card and build the tower up the same number of blocks as there were dots on the card. Counting dots on the dot-set cards was not included in this analysis. This task was coded as counting objects in sets and saying that the last number word said is the number of objects in the set. The children's performance on this task is described in the qualitative analysis.

[70] The analyses were based on less complex (LC) questions such as "Let's build it up. Let's go up to 8 this time.", "Okay. Let's go back down.", "Let's go up to 10 this time.", and "Now lets go back down." More complex (MC) questions were not asked in this unit.

each item properly when counting and errors on backward counts. She also appeared to lose interest in the task at the end of Session 8. Wendy's pattern of response indicated she understood the principle of one-to-one correspondence and could touch each object once and only once when counting on the majority of responses.

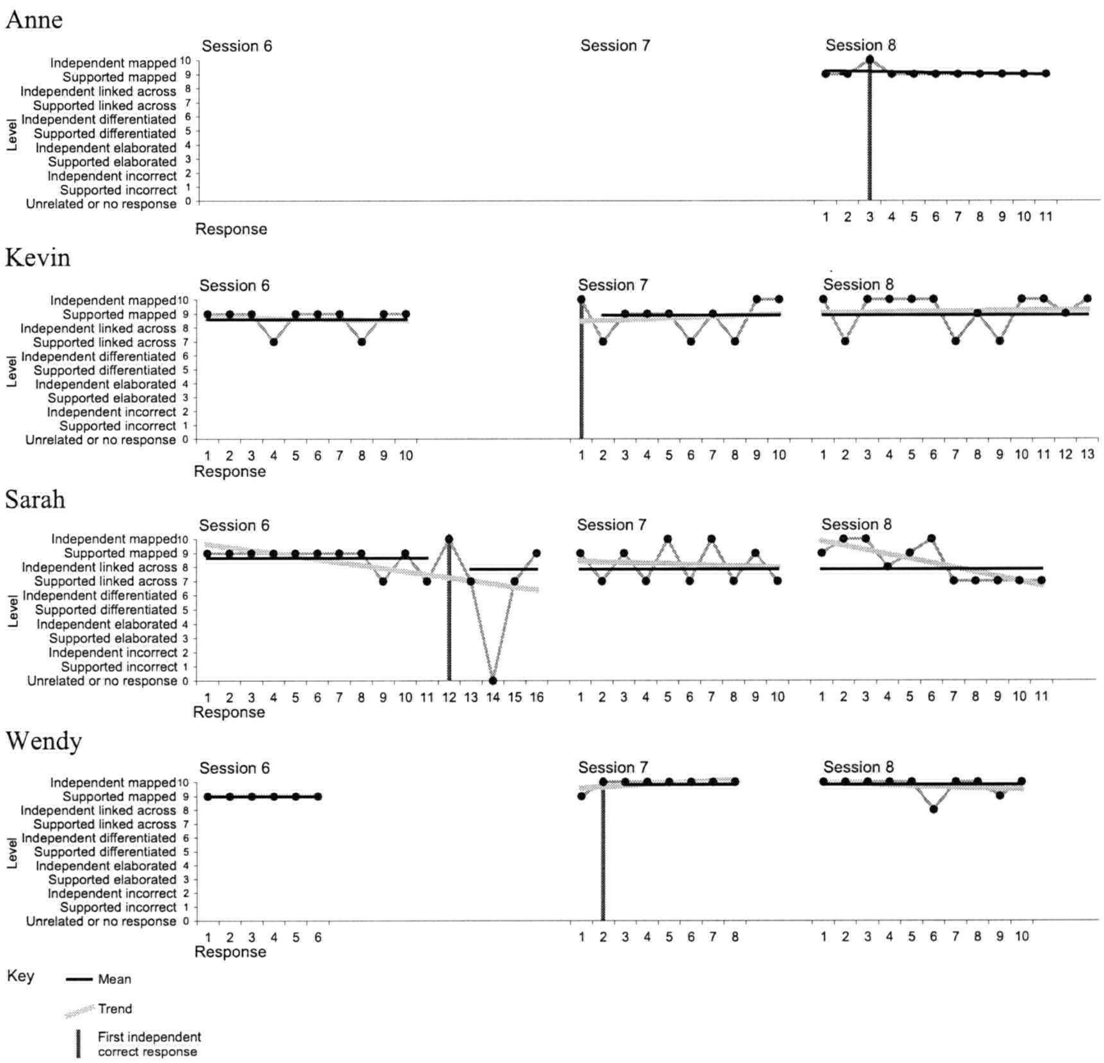


Figure 5. When the children first generated an independent mapped response for object counting using one-to-one correspondence on forward or backward counts in Unit 3 and how the children responded prior to and subsequent to this response.

Cross-Case Comparison

All of the children first generated independent mapped responses for object counting using one-to-one correspondence on forward and backward counts in Unit 3. Overall, Wendy progressed the most, followed by Anne, Kevin and Sarah. Although Wendy generated her first independent mapped response less quickly than Anne, she generated almost all independent mapped responses subsequent to her first independent mapped response. Anne generated only supported mapped responses subsequent to her first independent mapped response. However, the presence of researcher support may have underestimated Anne's level of performance. Although the level of researcher support was frequent, it consisted primarily of the researcher pointing to or touching the first block or dot in a pictorial display, pointing to or touching subsequent blocks or dots in a pictorial display, or saying "Start with 6.", "Start with 1.", "What comes after 7?", "What comes on top of 2?".

The children who progressed the most in Unit 3 had pathways characterized by less variability in their level of response prior to and subsequent to their first independent mapped response and higher frequency of independent mapped responses subsequent to their first independent mapped response (Anne was an exception for the reason mentioned above). The children who progressed the least in Unit 3 had pathways characterized by greater variability in their level of response prior and subsequent to their first independent mapped response and higher frequency of supported linked across responses prior to and subsequent to their first independent mapped response.

None of the children showed an abrupt change in their level of response prior to and subsequent to their first independent mapped response. Kevin, Sarah, and Wendy showed high levels of response from the beginning of Session 6. Anne showed a high level of response from the beginning of Session 8. This pattern of response indicated the children may have learned the principle of one-to-one correspondence prior to instruction and that progress was related to the expansion of this mathematical understanding rather than to the construction of a new mathematical understanding. The children's knowledge of the number sequence from 1 to 10 had already become integrated with their ability to touch each object once when counting (Griffin & Case, 1996; Griffin et al., 1992). Overall, the results of the descriptive microgenetic quantitative analysis indicated the children understood the principle of one-to-one correspondence.

Descriptive Microgenetic Qualitative Analysis

All of the children generated supported and independent mapped responses on the majority of forward counts in Unit 3. Touching errors occurred most frequently when the children counted dots in the pictorial displays. Errors consisted of saying a number word and running a finger over more than one dot, touching a dot twice when saying a number word, or failing to touch a dot while saying a number word.

Counting errors occurred most frequently on backward counts when the children took the towers down. Errors consisted of failing to give the next number word down in the number word sequence or starting backward counts with 1. Researcher support was frequent and consisted of saying the next number down before and after errors occurred. The following protocol illustrates a supported linked across response on a backward count for object counting using one-to-one correspondence in Unit 3.

> ***Researcher "Good. Now, take them down, Sarah. What do you start with?" (The researcher puts her hands on the blocks to steady the tower. The researcher keeps her hand on the eight block.) Sarah: "8." (Sarah takes the eight block down and places it on the floor.) Researcher: "Good." Sarah: "8." (Sarah takes the seventh block down and places it on the floor.) Researcher: "7." Sarah: "7, 4." (Sarah takes the sixth block.) Researcher: "6." (Sarah puts the sixth block down on the floor.) Sarah: "6, 3." (Sarah takes the fifth block as she says 3.) Researcher: "5." Sarah: "5." Sarah: "4." (Sarah takes the fourth block down and puts it on the floor.) Researcher: "Good." Sarah: "3." (Sarah takes the third block down and puts it on a block the floor.) Researcher: "Good." Sarah: "2, 1" (Sarah takes the second block down and puts it on a block on the floor as she says 2. Sarah takes the first block and puts it on a block on the floor after she says 1.)***

Counting errors occurred infrequently on forward counts. However, Kevin and Sarah occasionally made counting errors on forward counts. Errors consisted of starting the counts with a number other than 1 or skipping number words in the middle of the counts.

On several occasions Wendy (and on one occasion Anne) looked at or pointed to the first block and touched subsequent blocks when building the tower up and taking the tower down. The following protocol illustrates this type of independent mapped response on a forward count for object counting using one-to-one correspondence in Unit 3.

> ***Researcher: "Good. Okay. Let's build the tower up to 7 this time. Okay. Start with 1." (The researcher adds an extra block as she says "Okay. Let's build the tower up to 7 this time.") Wendy: "1". (Wendy points to the first block on the floor.) Researcher: "Uhuh." Wendy: "2, 1, 2, 3, 4, 5." (Wendy puts a second block on the first block as she says 2. Wendy takes the second block off the first block and places it on the floor. Wendy puts her hand on the first block on the floor as she says 1. Wendy puts a second block on the first block as she says 2. Wendy puts the third block on the second block and the fourth block on the third block as she says 3, 4. The researcher steadies the blocks with her hands. Wendy puts the fifth block on the fourth block as she says 5.) Researcher: "Good." (The researcher steadies the blocks with her hand.) Wendy: "6." (Wendy puts the sixth block on the fifth block.) Researcher: "Good." (The researcher***

steadies the blocks with her hand.) Wendy: "7." (Wendy puts the seventh block on the sixth block.)

Unit 4 (Sessions 9-11)

Unit 4 was designed to teach the children the cardinality principle (Griffin et al., 1992). In Sessions 9 to 11 the children were asked to say how many animals were in a set and were asked to count out sets of a specified size. The number of animals the children counted and put into the fields increased from 2 to 5 as the children moved through Session 9, from 6 to 9 as the children moved through Session 10, and from 8 to 10 as the children moved through Session 11. The children were also asked supplemental questions about the relative size of the sets (e.g., "Do you have more pigs or more cows? How do you know?"). Responses to these questions were analyzed in Unit 7.

Descriptive Microgenetic Quantitative Analysis[71]

Two mathematical understandings were included in Unit 4; an understanding of how many objects are in a set and an understanding of how to produce sets of a specified size. Both mathematical understandings make up the cardinality principle. Counting out sets of a specified size is more difficult than saying how many objects are in a set. To count out sets of a specified size children must remember the number that represents the size of the set and must stop counting at that number (Resnick & Ford, 1981 as cited in Baroody, 1989).

All of the children were able to produce sets of a specified size in Unit 4. However, the children did not spontaneously repeat the last number word said when asked how many objects were in a set (repeating the last number word said has been used as a marker to indicate that children understand the last number word represents the number of objects in the set) (Baroody, 1989). As a result the children's responses to the "How many?" questions presented in Unit 4 were coded at a lower level of understanding (coded independent linked across rather than independent mapped).

The children did, however, repeat the last number word said when the question "How many?" was asked immediately after the count (e.g., Researcher: "And how many pigs are in that group?" Wendy: "1, 2, 3, 4, 5. The same as the horses." Researcher: "Okay. How many?"

---

[71] The analyses were based on less complex (LC) questions such as "Can you put 7 cows in that field for me, please?". The analyses of more complex (MC) questions such as "And how do you know you have 2 cows in the field?", "And how do you know you've got 4?" and "How do you know?, How can you tell?" are described in the qualitative analysis.

Wendy: "5."). This indicated the children understood the last number word said represented the number of objects in the set and the level of understanding assigned to the "How many?" questions presented in Unit 4 underestimated the children's level of understanding.

For the above reason, only responses to questions asking the children to count out sets of specified size were analyzed in Unit 4. These responses represent a higher level of understanding and hence provided a better description of the children's understanding of cardinality. The sequence of the children's responses to both questions are presented in Session 9 to provide a sense of the temporal pattern of how the children responded in Unit 4.

Figure 6 shows the children's responses for counting to determine quantity in Unit 4.

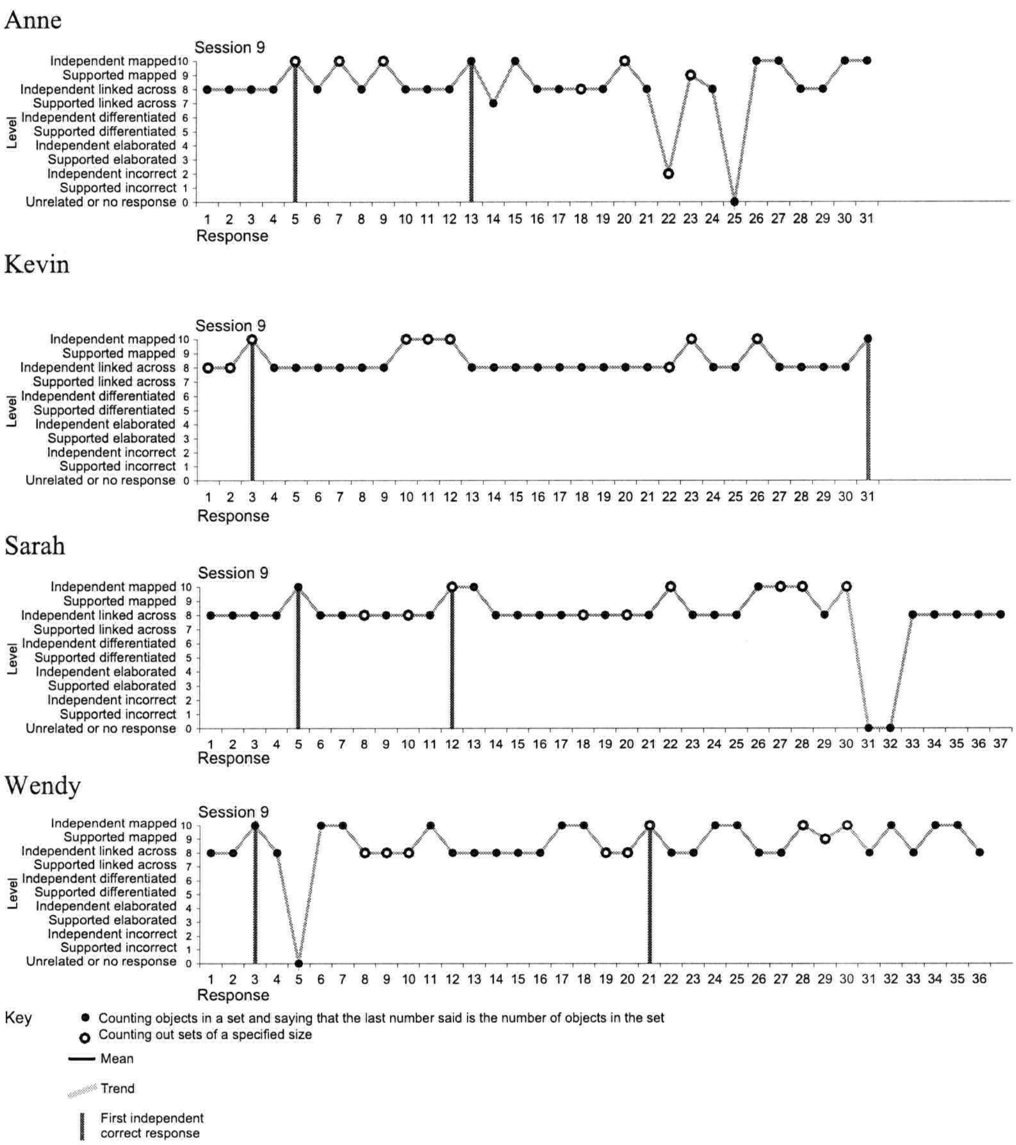


Figure 6. The children's responses for counting to determine quantity in Unit 4.

Each point in Figure 6 represents an opportunity to count objects in a set and say either the last number word said is the number of objects in the set (CS) or count out sets of a specified size (CO). As can be seen, all of the children generated independent linked across responses for CS from the beginning of Session 9. On these responses the children were able to say how many objects were in the set when the "How many?" question was asked immediately after the count. This indicated that at least for small sets of objects all of the children understood the last number word said indicated the number of objects in the set from the beginning of Session 9.

Individual Pathways for CO Within Unit 4

Figure 7 shows when the children first generated an independent mapped response for counting out sets of a specified size (CO) in Unit 4 and how the children responded prior to and subsequent to this response. As can be seen, no abrupt change from lack of understanding to

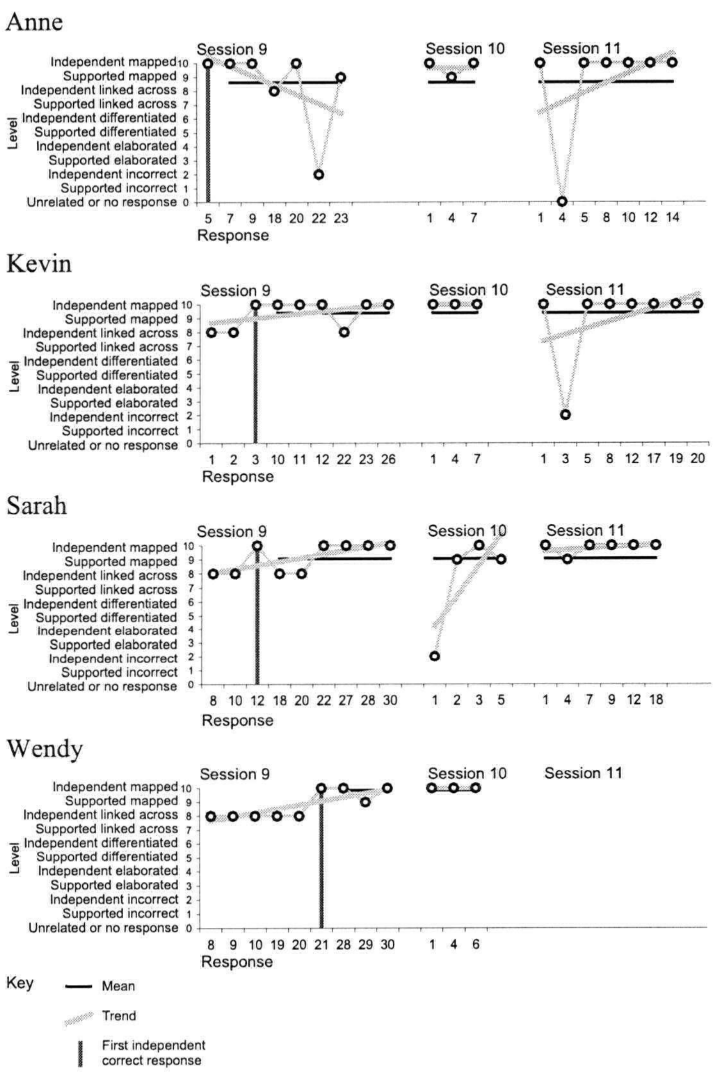


Figure 7. When the children first generated an independent mapped response for counting out sets of a specified size (CO) in Unit 4 and how the children responded prior to and subsequent to this response.

understanding was evident in Anne's response. Overall, Anne's pattern of response indicated she could count out sets of a specified size. This suggested Anne understood the cardinality principle. Although Kevin, Sarah, and Wendy generated independent linked across responses prior to their first independent mapped response, a change of two levels of response, they all subitized small sets of objects on these responses. Consequently, abrupt change was not evident in their patterns of response. Overall, Kevin's, Sarah's, and Wendy's patterns of response indicated they could count out sets of a specified size. This suggested they understood the cardinality principle.

Cross-Case Comparison

All of the children first generated independent mapped responses for CO in Unit 4. None of the children showed an abrupt change in their level of response prior to and subsequent to their first independent mapped response. Although Kevin, Sarah, and Wendy generated only independent linked responses prior to their first independent mapped response, these responses were due to subitizing small sets of objects rather than to an inability to understand the requirements of the task. Although Anne, Kevin, and Sarah generated a small number of independent linked across responses and independent incorrect responses subsequent to their first independent mapped response, these responses were due counting out sets incorrectly and subitizing small sets of objects rather than to an inability to understand the requirements of the task. Overall, all of the children showed a high level of response from the beginning of Unit 4. This indicated the children understood the principle of cardinality prior to instruction and that progress was related to the expansion of this mathematical understanding rather than to the construction of a new mathematical understanding. The children's counting skills had already become integrated with their knowledge of the cardinal values of sets (Griffin & Case, 1996; Griffin et al., 1992).

Descriptive Microgenetic Qualitative Analysis

All of the children could subitize sets of up to 5 objects. All of the children could count out sets of a specified size. Errors were infrequent and tended to occur when the children counted out larger sets objects, were distracted, or counted the sets too quickly. Touching errors occurred more frequently than counting errors. The following protocol illustrates this type of supported linked across response for saying how many objects are in a set (CS) in Unit 4.

Researcher: "Okay. Touch each one as you count it. Touch each one. 1." (The researcher points to the horses. Sarah touches the first horse as the researcher says 1.)

Sarah: "2" (Sarah whispers and touches the second horse.) Researcher: "No. Count out loud." Sarah: "1, 2, 3, 4, 5, 6, 7, 8, 9, 10." (Sarah touches each horse as she counts from 1 to 5, but does not match touches and horses as she counts from 6 to 10).

On one occasion Anne spontaneously repeated the last number word immediately after counting the set. The following protocol illustrates this type of independent mapped response for saying how many objects are in a set (CS) in Unit 4.

Researcher: "How many pigs have you got left in your field?" Anne: "1, 2, 3, 4. 4.". (Anne touches the first, second, third and fourth pigs as she says 1, 2, 3, 4, looks at the researcher and repeats the last number word said.)

When asked to count out sets of a specified size all of the children counted out sets of up to 10 animals and subitized sets of up to 5 animals without researcher support. All of the children were able to count out smaller sets from larger sets. Errors were infrequent and tended to occur when the children counted out larger sets of 6 to 10 objects. Errors were touching errors rather than counting errors, and consisted of touching an object twice when saying a number word, failing to touch an object when saying a number word, or touching more than one object when saying a number word. Although the level of researcher support was high, it consisted primarily of pointing to a particular animal or set of animals or moving a hand or finger over a set of animals.

When asked how they knew a particular number of objects were in a set, the majority of the children's responses indicated they understood counting or subitizing could be used to determine how many objects were in a set. Anne, Kevin and Sarah's responses to this question were "Cause I see it", "I counted them", "Because I know", "Because there's 3", "Because I counted". Wendy's response to this question was to recount the set.

Unit 5 (Sessions 12-15)

Unit 5 was designed to teach the children the increment rule (Griffin et al., 1992). In Sessions 12 to 15 the children were asked to say how many objects there were when 1 object was added to sets of 1 to 9, 2 objects were added to sets of 2 to 5, and 3 objects were added to sets of 2 to 5. The size of the larger addend increased as the children moved from Sessions 12 to 15. Sets of 1 to 4 objects were presented in Session 12 and sets of 5 to 9 objects were presented in Sessions 13 to 15. One object was added each time in Sessions 12 and 13. The following variations were also introduced in Sessions 14 and 15. Two or 3 objects were added to sets of 2 to 5 objects in Session 14 and 2 objects were added to sets of 2 to 5 objects in Session 15.

Descriptive Microgenetic Quantitative Analysis[72]

Individual Pathways Within Unit 5

Figure 8 shows when the children first generated an independent mapped response for incrementing sets in Unit 5 and how the children responded prior to and subsequent to this response. As can be seen, no abrupt change from lack of understanding to understanding was evident in any of the children's responses. Although Anne's, Kevin's, and Wendy's most frequent responses subsequent to their first independent mapped response were independent mapped

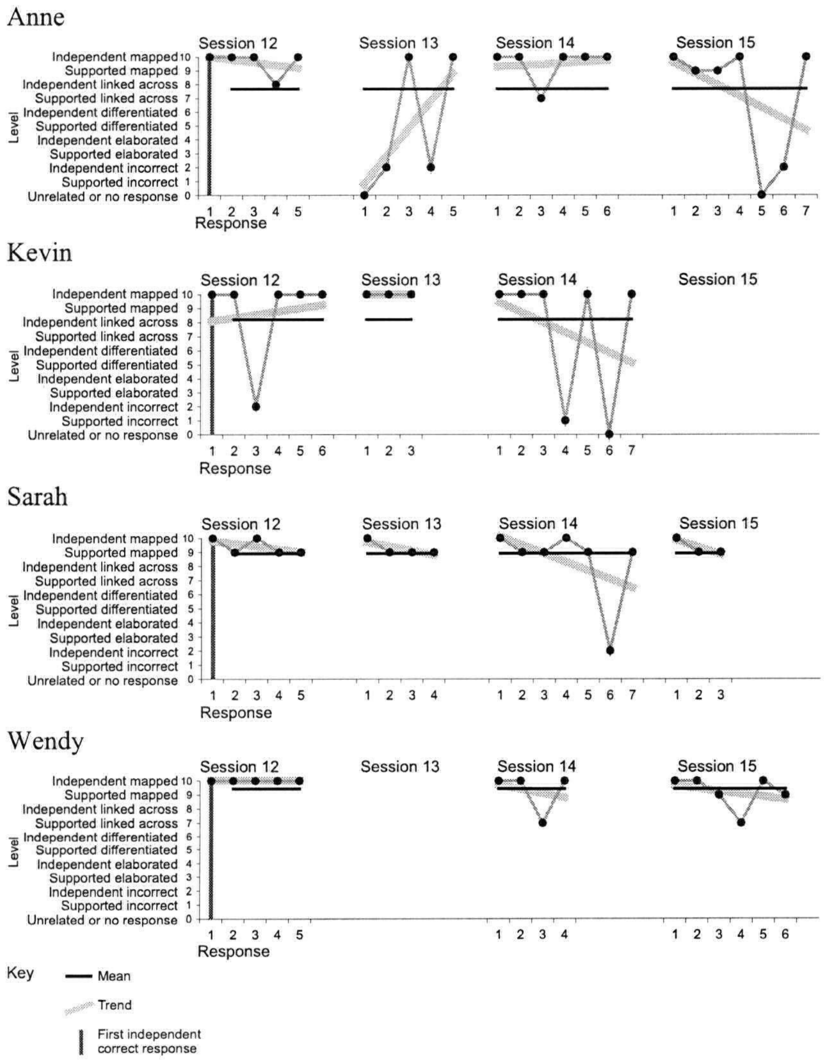


Figure 8. When the children first generated an independent mapped response for incrementing sets in Unit 5 and how the children responded prior to and subsequent to this response.

[72] The analyses were based on less complex (LC) questions such as "How many cookies do you have now?", How many cookies are in the cookies bag now?" and "And how many do you have now?". The analyses of more complex (MC) questions such as "And how do you know the Good Fairy put another cookie in?", "How do you know there are 3 cookies in the bag?", "So what did the Good Fairy do?" and "And how do you know you have 10 for sure?" are described in the qualitative analysis.

responses, they counted up from 1 on these responses. Although Sarah's most frequent responses subsequent to her first independent mapped response were supported mapped responses, she subitized small sets of objects and counted up from one these responses. Anne and Wendy began to count on from the larger addend for small numbers in Sessions 14 and 15 and Kevin began to count on from the larger addend for small numbers in Sessions 13 and 14. Sarah counted on from the larger addend (2+1) on only one occasion in Session 15. Variability in Anne's and Sarah's levels of response appeared to be due to inattention and the introduction of larger addends. Variability in Kevin's level of response appeared to be due to inattention and variability in Wendy's level of response appeared to be due to difficulty counting on from the larger addend when more than one object was added to the set. However, despite this, the children's patterns of response indicated they understood the number sequence could be used to find the answer to addition problems such as N+1, N+2, and N+3. However, they were just beginning to understand the incrementing rule.

Individual Pathways Across Units 6 to 9

Figure 9 shows the responses the children generated across Units 6 to 9 for incrementing sets. The size of the larger addend varied from 2 to 10 and the number of objects added to each set varied from 1 to 3 as the children moved across Units 6 to 9 (1 to 3 objects were added to each set depending on the capability of the child). As can be seen, Anne's and Wendy's most frequent responses across Units 6 to 9 were supported mapped responses and Kevin's most frequent responses were supported mapped and supported linked across responses. This indicated that despite the number of unrelated or no responses and independent and supported incorrect responses, their understanding of the incrementing rule gradually improved as they moved across Units 6 to 9. Although Sarah's most frequent responses were supported mapped responses, she generated a large proportion of unrelated or no responses. This indicated she was beginning to understand the incrementing rule as she moved across Units 6 to 9.

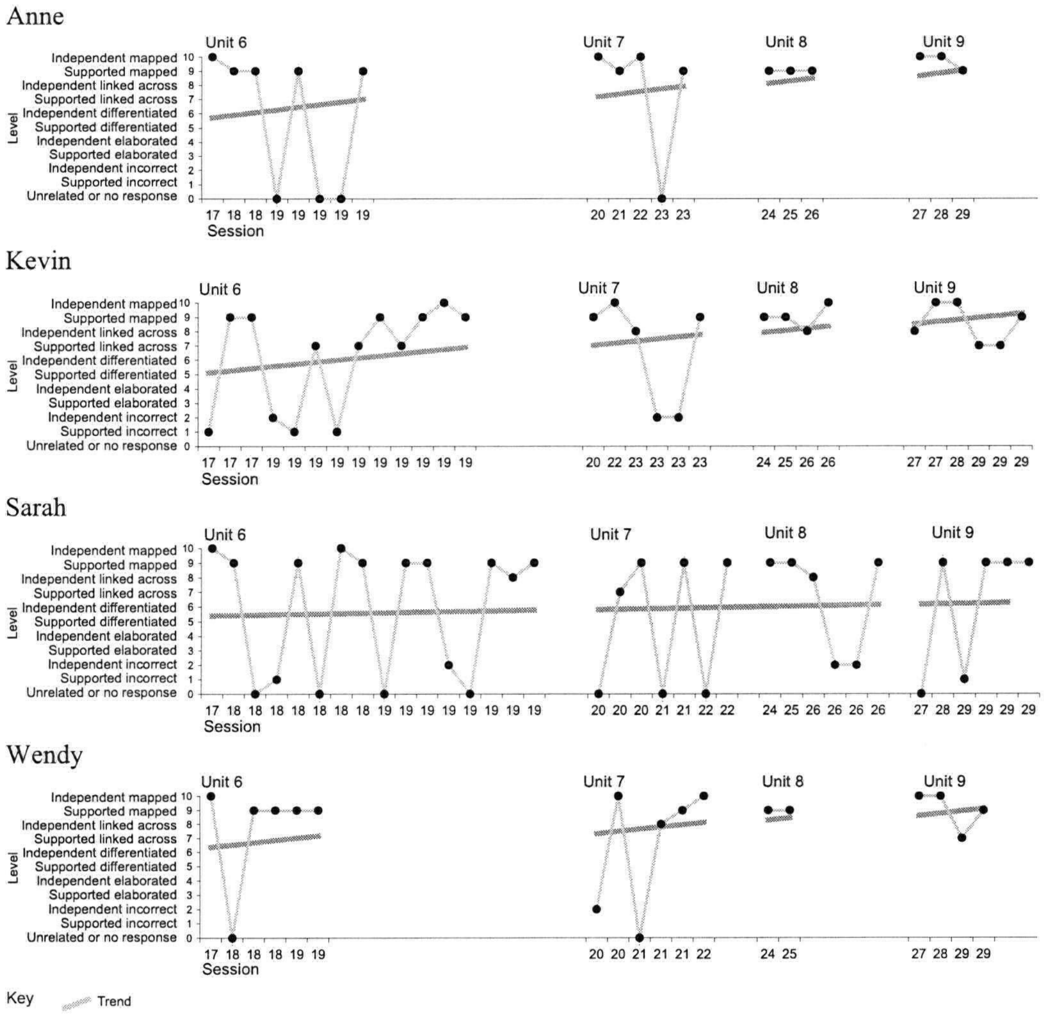


Figure 9. The responses the children generated across Units 6 to 9 for incrementing sets.

<u>Cross-Case Comparison</u>

All of the children first generated independent mapped responses for incrementing sets on their first opportunity to respond in Unit 5. However, the set sizes were small, only one object was added to each set, the level of instructor support was high, and the children frequently counted up from 1 rather than on from the larger addend to find the answer.

None of the children showed an abrupt change in their level of response prior to and subsequent to their first independent mapped response in Unit 5. All of the children showed some degree of variability in their level of response. This indicated the children's understanding of the incrementing rule was developing slowly throughout Unit 5 (however, some children were developing this understanding more quickly than others) and that progress was related to construction of a this mathematical understanding rather than to the expansion of an existing numerical understanding. The children appeared in an intermediate phase between lack of understanding and understanding, showing progress, but with regressions in between (Case, 1998a).

Overall, Wendy progressed the most in Unit 5, followed by Kevin, Anne, and Sarah. Although Kevin and Anne generated more independent mapped responses than Wendy, they generated a higher frequency of supported and independent linked across responses, supported and independent incorrect responses, or unrelated or no responses. The children who progressed the most in Unit 5 had pathways characterized by more consistent generation of independent mapped responses (generated independent mapped responses consecutively) subsequent to their first independent mapped response. The children who progressed the least in Unit 5 had pathways characterized by greater variability in their level of response (less consistent generation of independent mapped responses) subsequent to their first independent mapped response.

Although Anne, Kevin, and Wendy showed variability in their levels of response across Units 6 to 9, their understanding of the increment rule gradually improved. Sarah's understanding, however, appeared to develop more slowly than the other children's understanding. She continued to generate a large number of unrelated or no responses and very few independent mapped responses. Overall, the results of the descriptive microgenetic analysis indicated the children were just beginning to understand the incrementing rule in Unit 5. With the exception of Sarah, the children gradually improved their understanding of the increment rule across Unit 6 to 9.

Descriptive Microgenetic Qualitative Analysis

When asked how many N+1 were, all of the children counted up from 1 or subitized (up to 5) the number of cookies in the bag on the majority of opportunities to respond, with and without researcher support, in Unit 5. Anne, Wendy, and Kevin attempted to count on from the larger addend when the researcher held the top of the bag closed (so they could not see the cookies inside). However, they were unable to give the correct answer, without researcher support, until the end of Unit 5, when they immediately gave the correct answer for questions such as 2+1 and 3+1. Anne could immediately give a correct answer for questions such as 5+1 and 8+1 without researcher support in Unit 5. However, all 3 children counted up from 1 on at least one occasion when more then one object was added to the larger addend, the size of the larger addend was increased, or the cookies were available to count. Sarah resisted counting on from the larger addend and consistently counted up from 1. She required considerable researcher support throughout Unit 5.

When asked how many N+2 and N+3 were, all of the children counted up from 1 on all opportunities to respond, with and without researcher support, with three exceptions. On one occasion, Anne spontaneously gave the correct answer for 2+2 and on another occasion, she spontaneously gave the correct answer for 3+3. On one occasion, Wendy spontaneously gave the correct answer for 5+2. When asked how she knew there were 7, she counted up from 1 to make sure she was correct. The following protocol illustrates this type of independent mapped response for incrementing sets in Unit 5.

Researcher: "I'm going to put 5 cookies in the bag. Good fairy comes along and puts 2 more in. How many do you have now?" (The researcher puts 5 cookies in the bag as she says "I'm going to put 5 cookies in the bag.". Researcher drops 2 more cookies in the bag as she says "Cookie fairy comes along and puts 2 more in.".) Wendy: "7." (Wendy takes the bag and looks inside.) Researcher: "Okay. You check. How do you know there are 7 in there?" (Wendy dumps the cookies out of the bag.) Wendy: "Maybe not 7." (Wendy spreads the cookies out on the floor and looks at them.) Researcher: "Oops. Maybe one got stuck. Did we put 5 and 1 ... Yep". Wendy: "1, 2, 3, 4, 5, 6, 7". (Wendy touches each cookie as she counts from 1 to 7..

Errors occurred less frequently on N+1 questions when the largest addend was a small. Errors occurred more frequently on N+1 questions when the largest addend was large and on N+2 and N+3 questions. On at least one occasion, all of the children forgot how many cookies the Good Fairy initially put in the bag or how many cookies the Good Fairy added to the bag. This occurred most frequently for numbers from 6 to 10. The following protocol illustrates this type of supported linked across response for incrementing sets in Unit 5.

Researcher: "There are 8 cookies in the bag. How many cookies are in the bag, Anne?" (The research puts 8 cookies in the bag and puts the bag on the floor in front of Anne.) Anne: "8". Researcher: "Good. Cookie Fairy comes along and puts 1 more in." (The researcher puts 1 more cookie in the bag.) "How many cookies are in the bag now? How many cookies are in the bag now?" Anne: "Uhmmmmm 8".

All of the children understood the number of cookies in the bag increased when the Good Fairy added 1, 2, or 3 cookies to the cookies already in the bag. Anne, Wendy, and Kevin appeared to understand the answer could be obtained by counting up from the larger addend. When asked how many 4+1 was, Kevin said "4, 5." and then said "she added 1 more to make 5". When asked how many 6+2 was, Wendy moved her finger from 6 up to 8 on the number line.

When asked how many 8+2 was, Anne moved her finger up from 8 to 10 on the number line. Sarah did not appear to understand the answer could be obtained by counting up from the larger addend. She rarely used this strategy without considerable researcher support.

When asked how they knew there were N+1, N+2, or N+3 cookies in the bag after 1, 2, or 3 cookies had been added to the bag, the majority of the children's responses indicated they understood addition problems such as N+1, N+2, or N+3 could be solved by counting on from the larger addend, even though they did not consistently count on from the larger addend to find the answer. The children's responses were "Added 1 more", "She added 1 more to make 8", "1 more to make 10", "Cause I counted", "Cause there's, because there is 1 more", "Count", and "Cause there was after 6, there's 7."

The children's ability to count on from the larger addend gradually improved as they moved across Units 6 to 9. However, the level of researcher support was high (as it was in Unit 5) and consisted of counting the cookies before and after cookies had been added to the bag, counting the researcher's fingers, and counting up one, two, or three units on a number line (a vertical number line showing the numerals from 1 to 10). Wendy required the most researcher support in Unit 5 and across the instructional sequence. However, Anne, Kevin, and Wendy required a great deal of researcher support as well. Even at the end of the instructional sequence, the children counted up from 1 when the cookies were available to count.

Unit 6 (Sessions 16-19)

Unit 6 was designed to teach the children the decrement rule (Griffin et al., 1992). In Sessions 16 to 19 the children were asked to say how many objects were when 1 object was subtracted from sets of 2 to 10 and 2 objects were subtracted from sets of 3 to 5. The size of the minuend (larger number) varied from 2 to 10 objects throughout Sessions 16 to 19 (2 to 5 objects were presented for less capable children and 6 to 10 objects were presented for more capable children). A vertical number line showing the numerals from 1 to 10 was used to help the children acquire this mathematical understanding.

Descriptive Microgenetic Quantitative Analysis[73]

Individual Pathways Within Unit 6

Figure 10 shows when the children first generated an independent mapped response for decrementing sets in Unit 6 and how the children responded prior to and subsequent to this response. As can be seen, no abrupt change from lack of understanding to understanding was evident in Anne's and Kevin's levels of response. Although a change of more than one level was evident in Sarah's pattern of response at the point where she generated her first independent

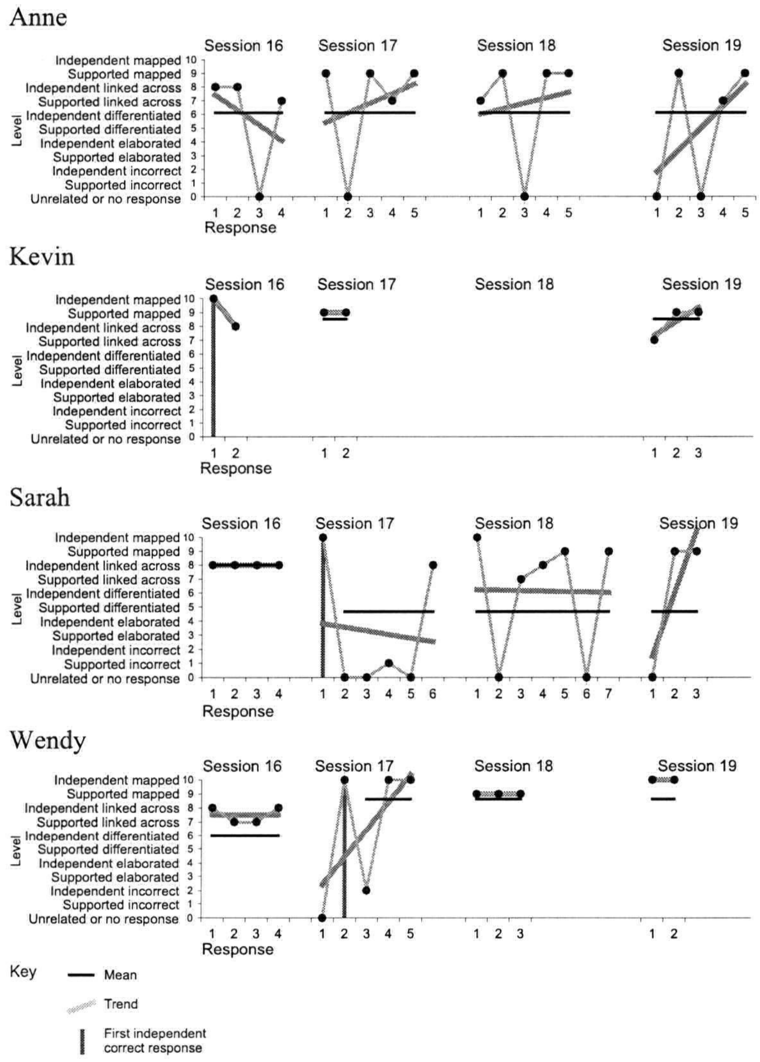


Figure 10. When the children first generated an independent mapped response for decrementing sets in Unit 6 and how the children responded prior to and subsequent to this response.

[73] The analyses were based on less complex (LC) questions such as "How many cookies do you have in the bag now?" and "How many cookies are in the bag now?". The analyses of more complex (MC) questions such as "How do you know you've got 2?" and "How do you know there are 4 cookies in the bag now?" are described in the qualitative analysis.

mapped response, her responses subsequent to her first independent mapped response were highly variable. Although Wendy's level of response was variable in Session 17 and her overall number of responses was small, a change of more than one level was evident in her pattern of response prior to and subsequent to her first independent mapped response. Also, her most frequent responses subsequent to her first independent mapped response were independent mapped responses. Overall, Anne's and Kevin's patterns of response indicated they were beginning to understand the decrementing rule. Sarah's pattern of response indicated she was slowly beginning to understand the decrementing rule and Wendy's pattern of response indicated she understood the decrementing rule.

Individual Pathways Across Units 7 to 9

Figure 11 shows the responses the children generated across Units 7 to 9 for decrementing sets. The size of the minuend varied from 2 to 10 and the number of objects

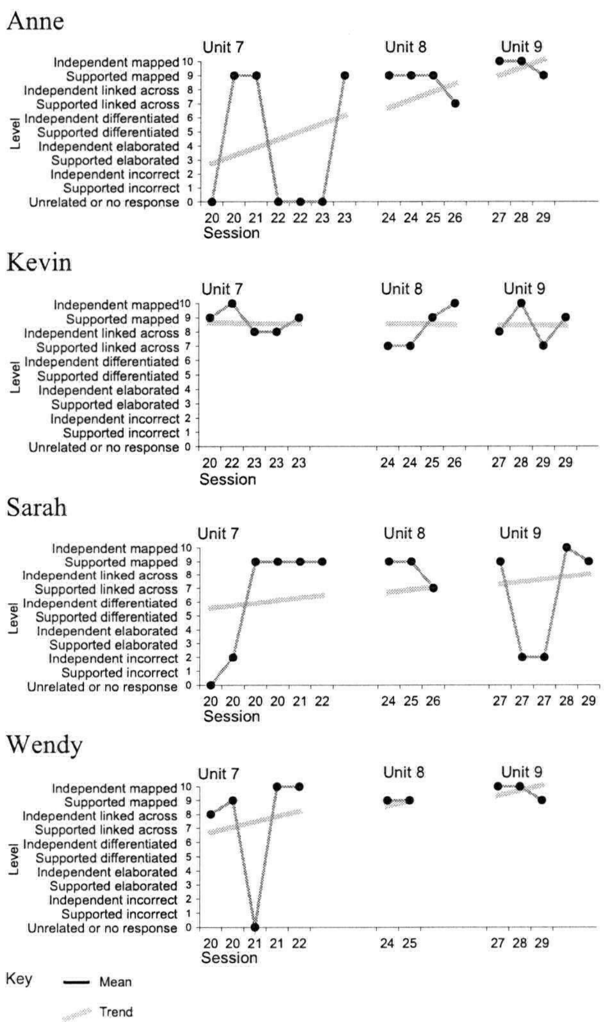


Figure 11. The responses the children generated across Units 7 to 9 for decrementing sets.

subtracted from each set varied from 1 to 2 as the children moved across Units 7 to 9 (2 to 5 objects were presented for less capable children and 6 to 10 objects were presented for more capable children; one to 2 objects were subtracted from each set depending on the capability of the child). As can be seen, Anne's level of response was highly variable in Unit 7, but gradually increased as she moved across Units 8 and 9. Although Kevin generated three independent mapped responses, almost half of his responses were supported and independent linked across responses. This indicated that although Anne's and Kevin's understanding of the decrementing rule gradually improved, they had difficulty counting back from the minuend when the size of the minuend increased and more than one object was subtracted from the set. Sarah's level of response was initially low in Unit 7 and then high and variable in Units 8 and 9. This indicated that although she was beginning to understand the decrementing rule, she had difficulty counting back from the minuend to find the answer. Although Wendy's level of response dropped in the middle of Unit 7, it remained high and stable in Units 8 and 9. This indicated that although she understood the decrementing rule, she had difficulty counting back from the minuend when the size of the minuend increased and more than one object was subtracted from the set.

Cross-Case Comparison

With the exception of Anne, all of the children first generated independent mapped responses for decrementing sets in Unit 6. Anne first generated an independent mapped response for decrementing sets in Unit 9. With the exception of Wendy, the set sizes the children were presented with were small and only one object was subtracted from each set.

Overall, Wendy progressed the most in Unit 6, followed by Kevin, Anne, and Sarah. Although Sarah generated one independent mapped response subsequent to her first independent mapped response, she also generated six unrelated or no responses. Although Kevin's number of responses was small, he generated his first independent mapped response on his first opportunity to respond.

The child who progressed the most in Unit 6 had a pathway characterized by a clear difference in her level of response prior to and subsequent to her first independent mapped response and a relatively stable level of response subsequent to her first independent mapped response. The children who progressed the least in Unit 6 had pathways characterized by greater variability in their levels of response subsequent to their first independent mapped response and a lower frequency of independent mapped responses subsequent to their first independent mapped response.

With the exception of Wendy, none of the children showed an abrupt change in their level of response prior to and subsequent to their first independent mapped response in Unit 6. Anne's, Kevin's, and Sarah's patterns of response indicated their understanding of the incrementing rule was developing slowly throughout Unit 6 and that progress was related to the construction of a this mathematical understanding rather than to the expansion of an existing numerical understanding (however, Kevin and Anne appeared to be developing this understanding faster than Sarah). Anne, Kevin, and Sarah were in an intermediate phase between lack of understanding and understanding, showing progress, but with regressions in between (Case, 1998a). Although Wendy's number of responses was small, she was the only child who appeared to progress directly from lack of understanding to understanding without passing through an intermediate phase for this numerical understanding (Case, 1998a).

All of the children showed variability in their level of response across Units 7 to 9. However, Anne's, Wendy's, and Sarah's levels of response showed increasing trends. Overall, the results of the descriptive microgenetic analysis indicated, with the exception of Wendy, the children were just beginning to understand the decrementing rule in Unit 6 and Anne and Sarah gradually improved their understanding of the increment rule across Unit 7 to 9.

Descriptive Microgenetic Qualitative Analysis

When asked how many N-1 or N-2 were, all of the children counted or subitized (up to 5) the number of cookies left in the bag (when they were allowed to look in the bag), on the majority of opportunities to respond in Unit 6. Errors occurred most frequently on N-1 and N-2 questions when the children were not allowed to look at the number of cookies left in the bag. Errors consisted most frequently of giving numbers smaller or larger than the minuend. The following protocol illustrates of this type of supported linked across response for decrementing in Unit 6.

> ***Researcher: "I put 5 cookies in the cookie bag. Close your eyes." (The researcher puts 5 cookies in the bag and holds the top of the bag closed.) "The cookie monster comes and takes 2 cookies away." (The researcher takes 2 cookies out of the bag.) "How many cookies are in the cookie bag now?" Anne: "Can I take the counting board?" (Anne takes the number line in her hand and looks at the number line.) "Uhmmmm 2".***

Kevin, Sarah, and Wendy were able to count down from the minuend for smaller sets such as 2-1, 3-1, and 4-1 without researcher support in Unit 6. However, Sarah said "I guessed" on her opportunity to respond. On at least one occasion, all of supp the children forgot how many cookies the Cookie Monster initially put in the bag (particularly for numbers from 6 to 10) or how many cookies the Cookie Monster took out of the bag. The following protocol illustrates an independent mapped response for decrementing sets in Unit 6.

> ***Researcher: "How many cookies are in the bag?" Wendy: "5". Researcher: "Okay. Do you want to check or can you remember?" Wendy: "I can remember." Researcher: "Okay. 5 cookies in the bag." (The researcher holds the opening of the bag closed with her hand.) Researcher: "Close your eyes. Here comes Cookie, Cookie Monster and takes 1 cookie out of the bag." (The researcher takes 1 cookie out of the bag.) "How many cookies are in the bag now?" Wendy: "4".***

All of the children required researcher support (verbal prompting, a number line showing the numerals from 1 to 10 and the researcher's finger pointing to the numbers) to count down from the minuend for numbers from 6 to 10 in Unit 6, with two exceptions. On one occasion, Wendy immediately gave the correct answer for 7-1 and on another occasion she immediately gave the correct answer for 6-1, without researcher support. On at least one occasion, all of the children moved their fingers down from the minuend on the vertical number line to find the answer to questions such as 4-1, 5-1, 6-1 or 7-1.

When asked how they knew there were N-1 or N-2 cookies in the bag after 1 or 2 cookies had been removed from the bag, the majority of the children's responses indicated the children understood the number sequence (just before relations) could be used to solve subtraction problems such as N-1 or N-2 by the end of Unit 6. The children's responses were "Cause there's 1 gone away", "1 tooked away", "Took 1 out", and "Because he took 1 away". However, the level of researcher support was high and Sarah may have been repeating the researcher's justification for the answer rather than generating her own in Unit 6.

The children's ability to count down from the minuend improved as they moved across Units 7 to 9. However, all of the children, with the exception of Wendy, continued to required a great deal of researcher support (a vertical number line showing the numerals from 1 to 10 and the instructor's fingers). All of the children counted the remainder when the cookies were available to count.

Unit 7 (Sessions 20-23)

Unit 7 was designed to teach the children the fine-comparison rule for the number sequence from 1 to 10 (Griffin et al., 1992). In Sessions 20 to 23 the children were asked to map sets from 1 to 10 onto a vertical number line showing the numerals from 1 to 10 and use the numerals on the number line to make gross and fine numerical comparisons between the sets (both vertical and horizontal number lines showing the numerals from 1 to 10 were used). The magnitude of the numbers increased as the children moved from Session 20 to Session 21. Sets of 1 to 5 were compared in Session 20 and sets of 6 to 10 were compared in Session 21. Sets of 1 to 10 were compared in Session 22 and Session 23.

Two mathematical understandings, saying one set has more than another set and saying one set has less than another set, were included in Unit 7. Saying one set has less than another set is more difficult than saying one set has more than another set ("the word more is used more frequently and is more readily understood by children") (Baroody, 1989, p. 101). However, the two understandings were analyzed to provide a sense of the temporal pattern of the children's understanding of next number relationships (just before relations and just after relations).

Descriptive Microgenetic Quantitative Analysis[74]

Individual Pathways Within Unit 7

Figure 12 shows the responses the children generated for using numbers to compare quantities in Unit 7.[75] Each point in Figure 6 represents an opportunity to say one set has more than another set (SM) or one set has less than another set (SL). Throughout Sessions 20 to 23 the children's level of response for SM and SL was determined by whether the children were asked to make gross or fine numerical comparisons between sets.[76] As can be seen, all of the children generated supported mapped responses for SM across Sessions 20 to 23. Anne and Kevin generated a small number of supported mapped responses for SL across Sessions 20 to 23. These responses, combined with the low number of supported incorrect and unrelated or no responses indicated all of the children could make fine numerical comparisons between sets for

---

[74] The analyses were based on less complex (LC) questions such as "Which number is smaller 6 or 10?", Which number is bigger 4 or 5?", Which animal do we have the littlest amount of?", Which animal do we have the biggest amount of?" and "Which animal do we have the next biggest amount of?". The analyses of more complex (MC) questions such as "And how do you know that 4 is bigger than 3?", " and "And how do we know we have more cats?" are described in the qualitative analysis.

[75] Independent mapped responses could not be scored in Unit 7 because the task included a number line. Number lines are a type of physical support. Therefore, the most sophisticated response a child could generate was a supported mapped response.

[76] Supported linked across responses indicated the children could make gross numerical comparisons or fine visual comparisons between sets with researcher support. Supported mapped responses indicated the children could make fine numerical comparisons between sets with researcher support.

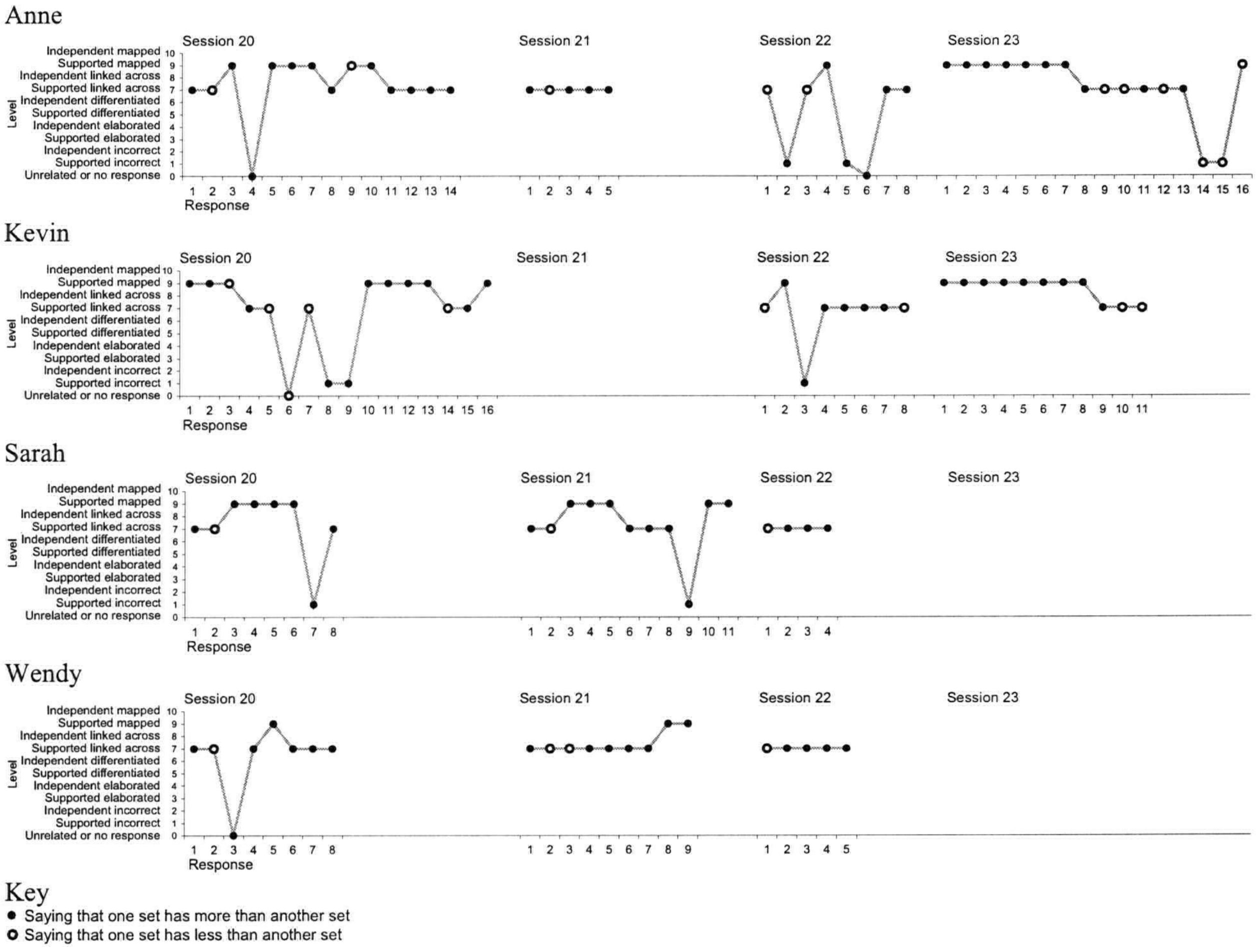


Figure 12. The responses the children generated for using numbers to compare quantities in Unit 7.

SM (at least for selected number pairs) and Anne and Kevin could make fine numerical comparisons for SL with researcher support in Unit 7. Overall, Anne's and Kevin's patterns of response suggested they understood the fine-comparison rule for SM and SL (for selected number pairs) and Sarah's and Wendy's patterns of response suggested they understood the fine-comparison rule for SM (for selected number pairs).[77]

<u>Individual Pathways Across Units 4, 5, and 8</u>

Figure 13 shows when the children first generated independent mapped responses for using numbers to compare quantities across Units 4, 5, and 8 and how the children responded prior to and subsequent to these responses. Throughout Units 4, 5, and 8 the children's level of response for SM and SL was determined by whether the children were asked to make

[77] For a full understanding the children must understand "each" next number up in the number sequence represents a set that has been incremented by 1 unit.

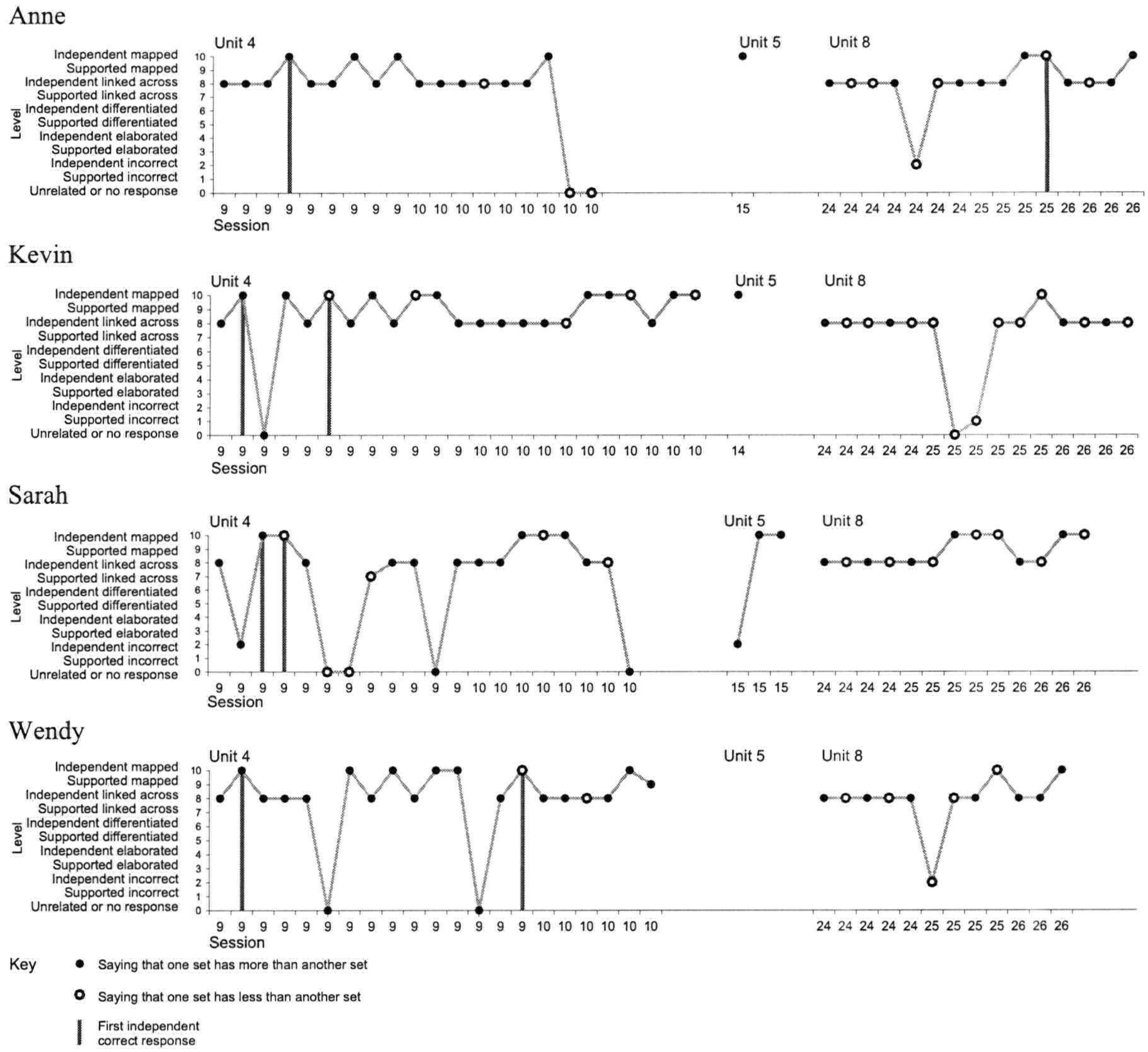


Figure 13. When the children first generated independent mapped responses for using numbers to compare quantities across Units 4, 5, and 8 and how the children responded prior to and subsequent to these responses.

gross or fine numerical comparisons between sets.[78] As can be seen, all of the children first generated an independent mapped response for SM in Session 9, Unit 4 (compared 2 vs. 3). Kevin, Sarah, and Wendy first generated an independent mapped response for SL in Session 9, Unit 4 (compared 2 vs. 3 or 3 vs. 4). Anne first generated an independent mapped response for SL in Session 25, Unit 8 (compared 5 vs. 6). With the exception of Anne, all of the children continued to generate independent mapped responses for SM and SL across Units 4, 5, and 8.

[78] Independent linked across responses indicated the children could make gross numerical comparisons or fine visual comparisons between sets without researcher support. Independent mapped responses indicated the children could make fine numerical comparisons between sets without researcher support. Unit 7 was not included in this analysis because independent mapped responses could not be scored in Unit 7.

Overall, with the exception of Anne, the children's patterns of response suggested they understood the fine-comparison rule for SM and SL in Unit 4 (for selected number pairs).

Cross-Case Comparison

All of the children could make fine numerical comparisons between sets for SM with researcher support in Unit 7. Anne and Kevin could make fine numerical comparisons between sets for SL with researcher support in Unit 7. However, Wendy and Sarah did not have an opportunity to make fine numerical comparisons for SL in Unit 7. All of the children could make fine numerical comparisons between sets for SM without researcher support from the beginning of Unit 4. With the exception of Anne, all of the children could make fine numerical comparisons for SL without researcher support in Unit 4. However, the number of opportunities to make fine numerical comparisons for SM and SL was small and the children were not asked to make fine-numerical comparisons for all number pairs from 1 to 10.

None of the children showed an abrupt change in their level of response prior to and subsequent to their first independent mapped response. All of the children's pathways for SM and SL in Unit 7 and across Units 4, 5, and 9 were characterized by mapped responses (supported in Unit 7, independent across Units 4, 5, and 9) and low variability in their level of response (at least for a selected number of number pairs). This indicated the children understood the fine-comparison rule for SM and SL (for selected number pairs) prior to instruction and that progress was related to the expansion of this mathematical understanding rather than to the construction of a new mathematical understanding.

Descriptive Microgenetic Qualitative Analysis

All of the children could judge which animal they had the littlest amount of, the next biggest amount of, and the biggest amount of in Unit 7. All of the children used the numbers on the vertical and horizontal number lines to make gross (3 vs. 5, 5 vs. 2, 9 vs7, 4 vs. 2, 8 vs. 4 and 3 vs. 7) and fine (2 vs. 3, 5 vs. 6, 5 vs. 4, 7 vs. 6, 4 vs. 3) numerical comparisons between sets (sets from 1 to 10 were mapped to the numbers on the vertical number and horizontal number lines). All of the children understood the terms bigger and smaller and more and less. However, Kevin initially had some difficulty with the words "next biggest". When asked which animal he had the next biggest amount of after the number 1, Kevin looked at the number line and pointed to the hippo which was at the number 3.

On at least one occasion Wendy, Kevin, and Sarah based their judgements on a visual comparison of how high the animal was up on the number line (sets from 1 to 10 were mapped to the numbers on a vertical number line.). When asked how they knew they had a larger or smaller amount of a particular animal, Wendy said "Cause it's higher", Sarah said "I can see it", and Kevin said "Because it's bigger". The following protocol illustrates this type of supported linked across response for SM in Unit 7.

Researcher: "And which animal do we have the biggest amount of?" Wendy: "The turtle." (Wendy points to the picture of the turtle.) Researcher: "Good for you. Okay. And you, how did you know? How did you know that that was the littlest amount and that was the biggest amount?".(The researcher points to the cat and then to the turtle.) Wendy: "Cause higher. Way higher than any of them.".

All of the children could make gross and fine numerical comparisons between sets in Units 4, 5 and 8. However, the sets sizes and the range of sets compared were small. The following protocol illustrates an independent mapped response for SM in Unit 4.

Researcher: "Do you have more pigs or more horses?" Kevin: "More horses." (Kevin looks at the horses.) "And how do you know you have more horses?" Kevin: "Because these guys are 3 and these guys are 1 and 2." (Kevin picks up a pig in each hand and looks at them as he says 1 and 2) Researcher: "Good. And which is the bigger number, 3 or 2?" Kevin: "3".

When asked how they knew which animal they had the biggest or littlest amount of in Unit 7, the majority of the children's responses indicated they understood numbers and the number sequence could be used to make gross and fine numerical comparisons between sets. When asked how they knew they had more or less owls, rabbits or hippos etc., Anne said "I counted", Wendy said "Cause it's 5" or "Cause there's 2", Sarah said "I looked at the number" and Kevin said "Because they had 9 chickens".

Throughout Unit 7 the level of instructor support was high. Vertical and horizontal number lines showing the numerals from 1 to 10 were used to help the children make gross and fine numerical comparisons between the sets. However, number lines were not used to help the children make numerical comparisons between sets in Units 4 and 8.

When and How the Children Applied the Mental Counting Line to New Tasks and Problem Situations

Unit 8 (Sessions 24-26)

Unit 8 was designed to teach the children to use the mental counting line to make relative quantity judgements in more than one quantitative dimension (Griffin et al., 1992). In Sessions 24 and 25 the children were presented with two jars filled with different quantities of blocks and were asked to say which jar contained the larger or smaller amount. In Session 26 the children were presented with two chains of unequal length (folded into piles so they appeared to be the same size) and were asked to say which chain was longer or shorter. Sets with large numerical differences (8,2; 9,4; 5,10) were presented in Session 24. Sets with small numerical differences (7, 9; 5,6; 8,7) were presented in Session 25. Sets with large and small numerical differences (4,5; 4,9; 2,7; 8,9; 5,6) were presented in Session 26.

Descriptive Microgenetic Quantitative Analysis[79]

Individual Pathways Within Unit 8

Figure 14 shows when the children first generated an independent consolidated response for comparing sets in two different quantitative dimensions in Unit 8 and how the children responded prior to and subsequent to this response. As can be seen, no abrupt change from lack of understanding to understanding was evident in Anne's or Sarah's level of response. Anne also failed to generate an independent consolidated response. Although Kevin generated only one independent non-consolidated response prior to his first independent consolidated response and his overall number of responses was small, a change of more than one level was evident in his pattern of response prior to and subsequent to his first independent consolidated response. Although Wendy generated only three independent non-consolidated responses prior to her first independent consolidated response and her overall number of responses was small, a change of more than one level was evident in her level of response prior to and subsequent to her first independent consolidated response. Overall, Anne's and Sarah's patterns of response indicated

[79] The analyses were based on less complex (LC) questions such as "Which jar has more?", "Which one has more?", "Which is the longest chain?", or "Which is the shortest chain?". The analyses of more complex (MC) questions such as "And how do you know that one has more?", "How can you be more certain that jar has more?", "How can you find out?", or "How do you know?, How can you find out without stretching the chains out?" are described in the qualitative analysis.

they had not yet begun to consolidate the mental counting line and Kevin's and Sarah's patterns of response suggested they were beginning to consolidate the mental counting line.

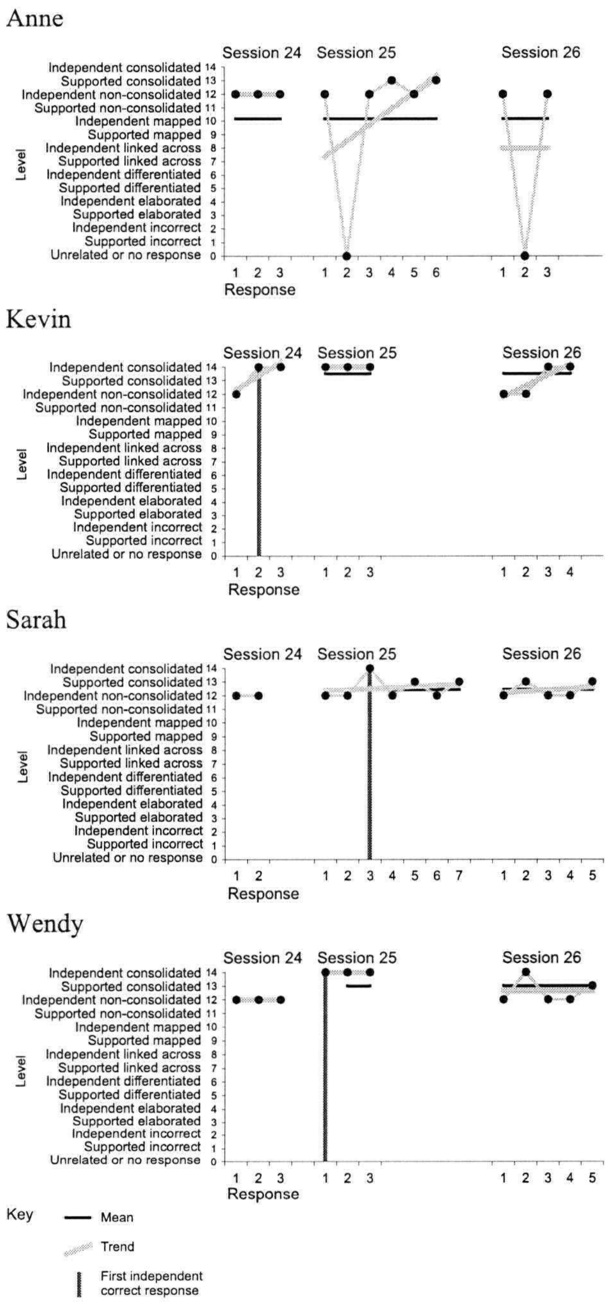


Figure 14. When the children first generated an independent consolidated response for comparing sets in two different quantitative dimensions in Unit 8 and how the children responded prior to and subsequent to this response.

Cross-Case Comparison

With the exception of Anne, all of the children first generated an independent non-consolidated response for comparing sets in two different quantitative dimensions in Unit 8. Overall, Kevin progressed the most in Unit 8, followed by Wendy, Anne, and Sarah. Although

Sarah generated her first independent consolidated response earlier than Anne, she did not generate an independent consolidated response subsequent to her first independent consolidated response.

The pathways of the children who progressed the most were characterized by a change of more than one level prior to and subsequent to their first independent consolidated response, less variability in their level of response, and higher frequency of independent consolidated responses subsequent to their first independent consolidated response. Their patterns of response suggested they more rapidly applied the understanding represented in the mental counting line to new tasks and problem situations. The pathways of the children who progressed the least were characterized by greater variability in their level of response prior to and subsequent to their first independent consolidated response, higher frequency of independent non-consolidated, supported incorrect, or unrelated or no responses, and lower frequency of independent consolidated responses subsequent to their first independent consolidated response. Their patterns of response indicated they more slowly applied the understanding represented in the mental counting line to new tasks and problem situations. Overall, the results of the descriptive microgenetic quantitative analysis indicated two of the children were beginning to consolidate the mental counting line and two of the children were not.

Descriptive Microgenetic Qualitative Analysis

All of the children based their initial relative quantity judgements (in the dimension of number) on visual comparisons of the sets, with and without researcher support. However, the task presented in Session 24 strongly influenced the manner in which the children responded. There were large numerical differences between all of the sets presented in Session 24.

Kevin and Wendy began to use the count and compare strategy in Sessions 25 and 26 when there were small numerical differences between the sets. On at least two occasions they immediately counted each set and compared the resulting numbers, without instructor support, when there were small numerical differences between the sets. The following protocol illustrates an independent consolidated response for using numbers to say which of two sets has more or less in the dimensions of number and the dimension of weight in Unit 8.

Researcher: "Which one has more?" (Wendy closes her eyes as Researcher says "Close your eyes.". Researcher puts different quantities of blocks in the 2 jars. Researcher puts the jar to Wendy's right to Wendy's left and the jar to Wendy's left to Wendy's right as she says "Okay. Which one has more?".) Wendy: "Mmm. 1, 2. 1, 2, 3,

4, 5, 6, 7. 1, 2, 3, 4, 5, 6, 7, 8." (Wendy looks in the jar to her right, puts her hand in the jar and touches each block as she says 1, 2. Wendy touches each block in the jar to her right as she says 1, 2, 3, 4, 5, 6, 7. Wendy looks in the jar to her left and touches each block as she says 1, 2, 3, 4, 5, 6, 7, 8.) Researcher: "Good. Okay. Which one has more? " Wendy: "That one? Wendy puts her hand over the jar to her left.)

However, Anne and Sarah continued to base their relative quantity judgements on visual comparisons of the sets. Both Anne and Wendy required a great deal of prompting to count each set and compare the resulting numbers. Kevin and Wendy less frequently based their relative quantity judgements on visual comparisons of the sets. They also required less prompting to count each set and compare the resulting numbers. The following protocol illustrates an independent non-consolidated response for using numbers to say which of two sets has more or less in the dimensions of number and the dimension of weight in Unit 8.

Researcher: "Let's do one more. Close your eyes tight. Okay. Okay. Let's do one more." (The researcher puts 6 blocks in one jar and 5 blocks in the other jar.) "Which one is bigger?" Anne: "That one." (Anne taps the rim of the jar containing 6 blocks.)

When asked how they knew which of two jars had more or less or which of two chains was longer or shorter, Anne's and Sarah's responses indicated they based their relative quantity judgements on global, visual comparisons of the sets. Their responses were "Because I see more blocks in it", "Cause it's big", "Because it has more", or "Because it's the longest". Kevin's and Wendy's responses indicated they based their relative quantity judgements numerical comparisons of the sets. Their responses were "Because this one has 4 chains and this one has 3 chains", "Because it has 8 and that one has 7", or "There's 3 on this one and not on this one".

When and How the Children Acquired Knowledge of the Written Numerals

Unit 9 Sessions (27-29)

Unit 9 was designed to teach the children to identify the written numerals from 1 to 10, read the written numerals from 1 to 10, match the written numerals from 1 to 10 to their corresponding pictorial displays, and put the written numerals from 1 to 10 in the correct order (Griffin et al., 1992).

Descriptive Microgenetic Quantitative Analysis[80]

Individual Pathways Within Unit 9

Figure 15 shows the responses the children generated for reading the written numerals from 1 to 10 in Unit 9. As can be seen, with the exception of Anne, all of the children's responses indicated they could read the written numerals from 1 to 10. Independent elaborated responses occurred for Anne when she confused written numerals with similar defining characteristics (6, 8, and 9).

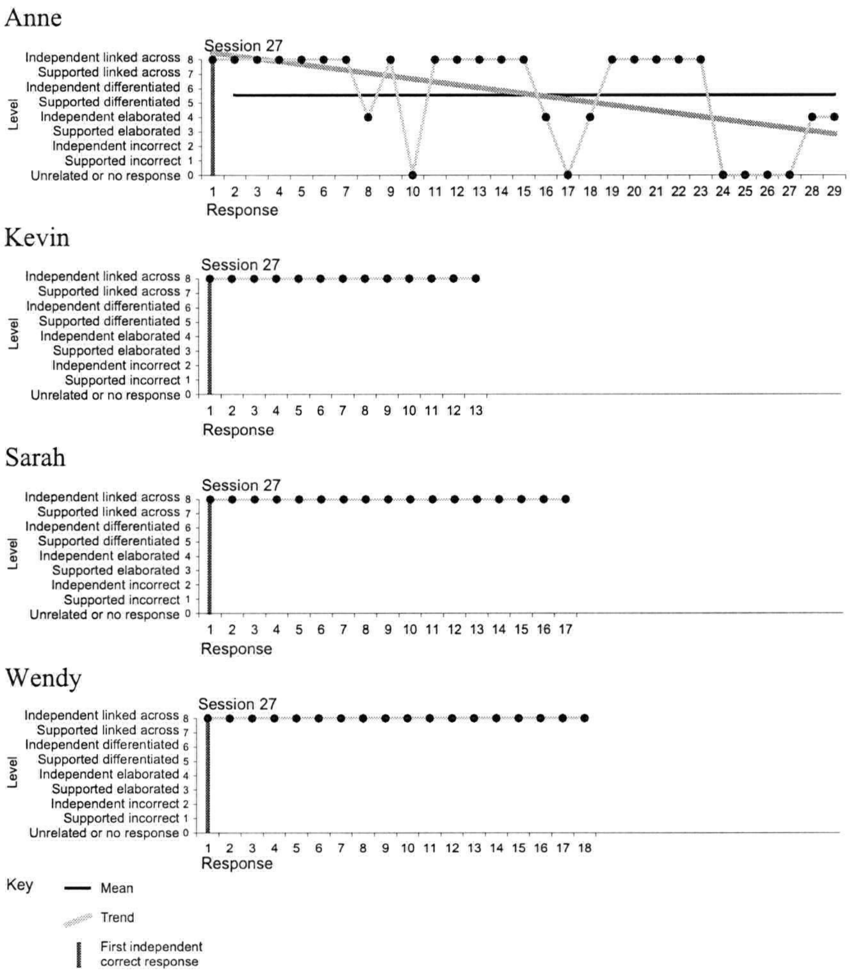


Figure 15. The responses the children generated for reading the written numerals from 1 to 10 in Unit 9.[81]

---

[80] The analyses of the mathematical understandings presented in Unit 9 are based on less complex (LC) questions such as "Can you point to each card and tell me the number?, What's that one?" (reading written numerals), "Find the 6. Now find the 10. Now find 4." (identifying written numerals), "Now you find the card, the dot card that matches that one (8)." (matching written numerals to their corresponding set size) and "Where do you think that goes on the number line?" "Where do you think 5 should go?" (putting written numerals in the correct order on a number line). More complex questions were included in this unit.

[81] There are three sessions in Unit 9. However, only sessions relevant to the analysis are shown.

Figure 16 shows the responses the children generated for matching the written numerals from 1 to 10 to their corresponding pictorial displays in Unit 9. As can be seen, all of the children's responses indicated they could match the written numerals from 1 to 10 to their corresponding set sizes. The unrelated or no response occurred for Anne because she was not paying attention, the supported mapped responses occurred when she was reminded to count the dots on the dot-set cards.

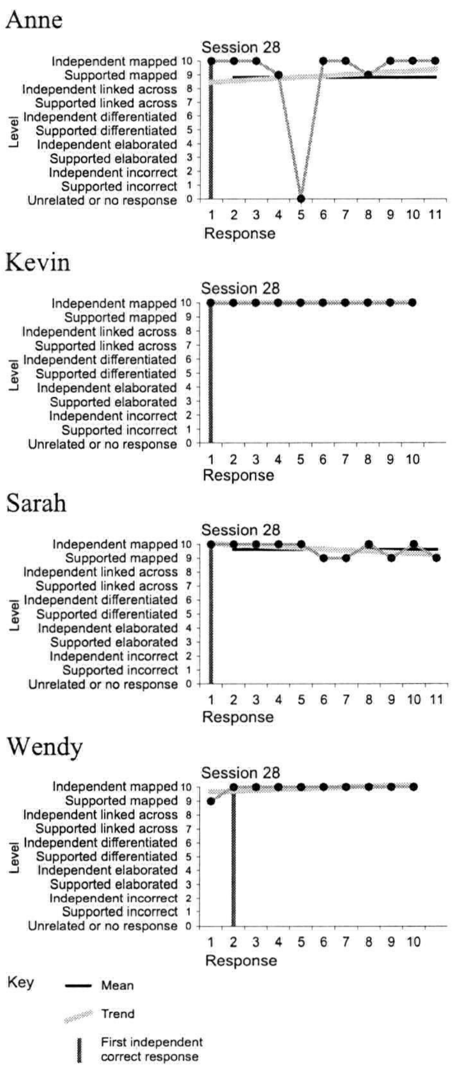


Figure 16. The responses the children generated for matching the written numerals from 1 to 10 to their corresponding pictorial displays in Unit 9.

Figure 17 shows the responses the children generated for identifying the written numerals from 1 to 10 in Unit 9.[82] As can be seen, the children's patterns of response in Session 29

[82] The written numerals were placed randomly on the floor in front of the children. The children picked up the written numerals when the names of the written numerals were called out.

indicated they could identify all of the written numerals from 1 to 10. The independent elaborated response occurred for Anne when she confused written numerals with similar defining characteristics (8 and 9).

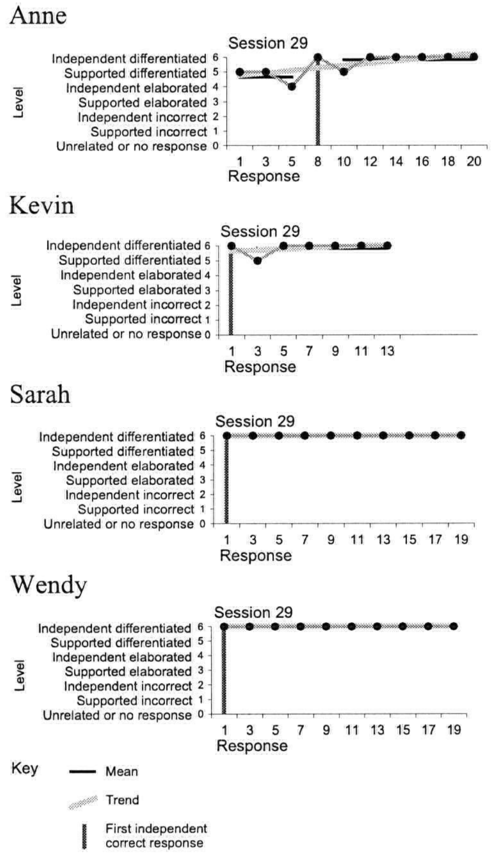


Figure 17. The responses the children generated for identifying the written numerals from 1 to 10 in Unit 9.

Figure 18 shows the responses the children generated for putting the written numerals from 1 to 10 in the correct order in Unit 9 (placed numeral 5 in the correct position).[83] As can be seen, Anne's pattern of response indicated she used the count up from one strategy and the fine-comparison strategy almost as frequently as the global comparison strategy to put the written numerals in the correct order. Kevin's pattern of response indicated that although he most frequently used the count up from one strategy to put the written numerals in the correct order, he used the global comparison strategy and the fine-comparison strategy as well. Sarah's pattern of response indicated that although she most frequently used the fine-comparison strategy to put the written numerals in the correct order, she used the count up from one strategy and the global

[83] In Unit 9 the spaces on the number line did not have accompanying numbers. The number line was part of the task rather than a type of support.

comparison strategy as well. Wendy's pattern of response indicated that although she most frequently used the fine-comparison strategy to put the written numerals in the correct order, she used the global comparison strategy as well.

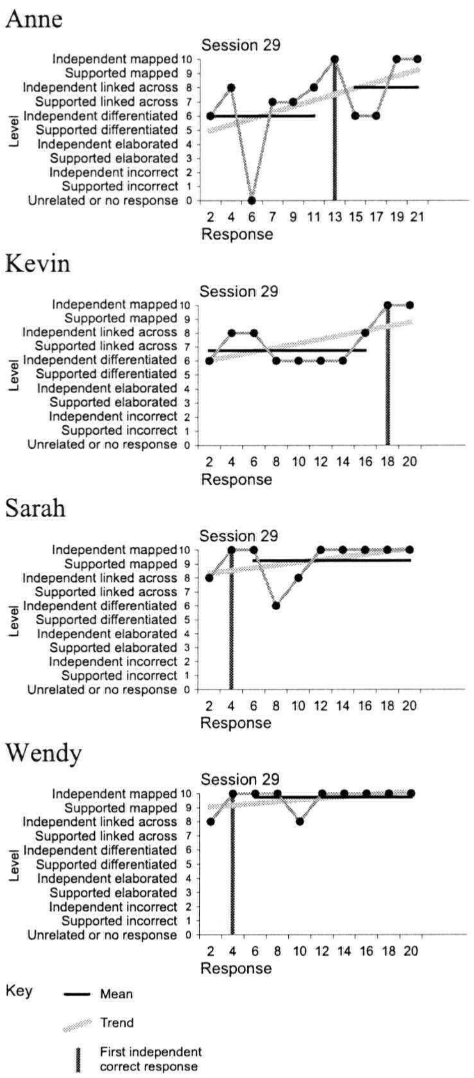


Figure 18. The responses the children generated for putting the written numerals from 1 to 10 in the correct order in Unit 9.

Cross-Case Comparison

All of the children could identify the written numerals, read the written numerals, and match the written numerals to their appropriate set size from the beginning of Sessions 26, 27, and 28. Anne was the only child who confused written numerals with similar defining characteristics (6, 8, and 9). The children who progressed the most had pathways characterized by early attainment of their first independent differentiated, linked across, or mapped response, little or no variability in their level of response, and higher frequency of differentiated, linked across, or mapped responses subsequent to their first independent differentiated, linked across, or mapped response. The children who progressed the least had pathways characterized by greater variability and lower frequency of differentiated, linked across, or mapped responses subsequent to their first independent differentiated, linked across, or mapped response. This indicated the children learned to identify the written numerals, read the written numerals, and match the written numerals to their corresponding set size prior to instruction.

All of the children first generated an independent mapped response for ordering the written numerals from 1 to 10 in Session 29. However, Kevin most frequently used the count up from one strategy and Sarah and Wendy most frequently used the fine-comparison strategy. Anne used the count up from one strategy, the global comparison strategy, and the fine-comparison strategy with almost equal frequency. Overall, Wendy progressed the most in Session 29 for ordering the written numerals, followed by Sarah, Anne, and Kevin. The children who progressed the most had pathways characterized by earlier attainment of their first independent mapped response, a clear difference in their level of response prior to and subsequent to their first independent mapped response, less variability in their level of response, and higher frequency of independent mapped responses subsequent to their first independent mapped response. The children who progressed the least had pathways characterized by later attainment of their first independent mapped response, greater variability in their level of response prior to and subsequent to their first independent mapped response, and lower frequency of independent mapped responses subsequent to their first independent mapped response.

Descriptive Microgenetic Qualitative Analysis

Kevin and Anne ordered the written numerals conventionally from left to right. Sarah and Wendy ordered the written numerals from right to left. All of the children used a combination of strategies (count up from one strategy, a global comparison strategy, and a fine-comparison strategy) to put the written numerals in the correct order. However, Kevin most frequently used the count up from one strategy, Sarah and Wendy most frequently use the fine-comparison strategy, and Anne used all three strategies with approximately equal frequency. The following protocol illustrates an independent differentiated response (counting up from one strategy) for putting the written numerals in the correct order in Session 29.

Researcher: "Now. You put the 5 on the number line where you think it should go."

Kevin: "1, 2, 3, 4, 5." (Kevin puts his hand on each of the first five spaces on the horizontal line as he counts from 1 to 5 and then puts the numeral 5 in the fifth space.)

Part 3: Trend Analysis of the Children's Performance

Figures 19 to 22 show the children's performance across the instructional sequence. Units, sessions, and opportunities to respond are shown on the horizontal axes of the graphs. Levels for each mathematical understanding represented in the mental counting line are shown on the vertical axes of the graphs (details of the children's performance are shown in Figures 1 to 18).The points in the figures represent the children's responses to the instructional program. The broken black line represents the child's individual developmental pathway. The broken grey line represents a break in the developmental pathway (where a child failed to generate a first independent differentiated, mapped, or consolidated response within the relevant instructional unit). The black horizontal line between Units 8 and 9 represents the division between knowledge of written numerals and the "more fundamental" (Case, 1998a, p. 765) components of the mental counting line. The grey zones represent Units 1 to 9 in the instructional program. The "preferred developmental pathway" (Case, 1996d, p. 211) moves across the top left corner of each step in the instructional sequence. When the children's responses were examined within and across the instructional units, the following trends were identified for each child.

Anne

Figure 19 shows Anne's developmental pathway across the instructional sequence. As can be seen, Anne generated her first independent differentiated response for saying the number words from 1 to 10 at the end of Session 1, Unit 1. Despite this, her level of response in Units 1, 3, and 4 was high. Variability in her level of response was low. This indicated Anne acquired the mathematical understandings taught in Units 1, 3, and 4 prior to instruction and progress was related to the expansion of these mathematical understandings. Her performance in Units 1, 3, and 4 was consistent with her performance on the Quantitative Reasoning subtests of the SB5 and her pretest performance on the Number knowledge test.

Anne's level of response in Unit 2 indicated she was just beginning to establish next before relationships for the number words from 10 to 1. Although her performance improved as she moved across Units 3 to 9, she did not generate a first independent differentiated response for this mathematical understanding until late in the instructional sequence. Her performance, however, was consistent with research on young children's mathematical understandings (Baroody, 1987, 1989; Fuson, 1988; Griffin & Case, 1996). Knowledge of the number word sequence from 10 to 1 is not usually acquired until later in the developmental sequence (Baroody, 1987, 1989; Fuson, 1988; Griffin & Case, 1996).

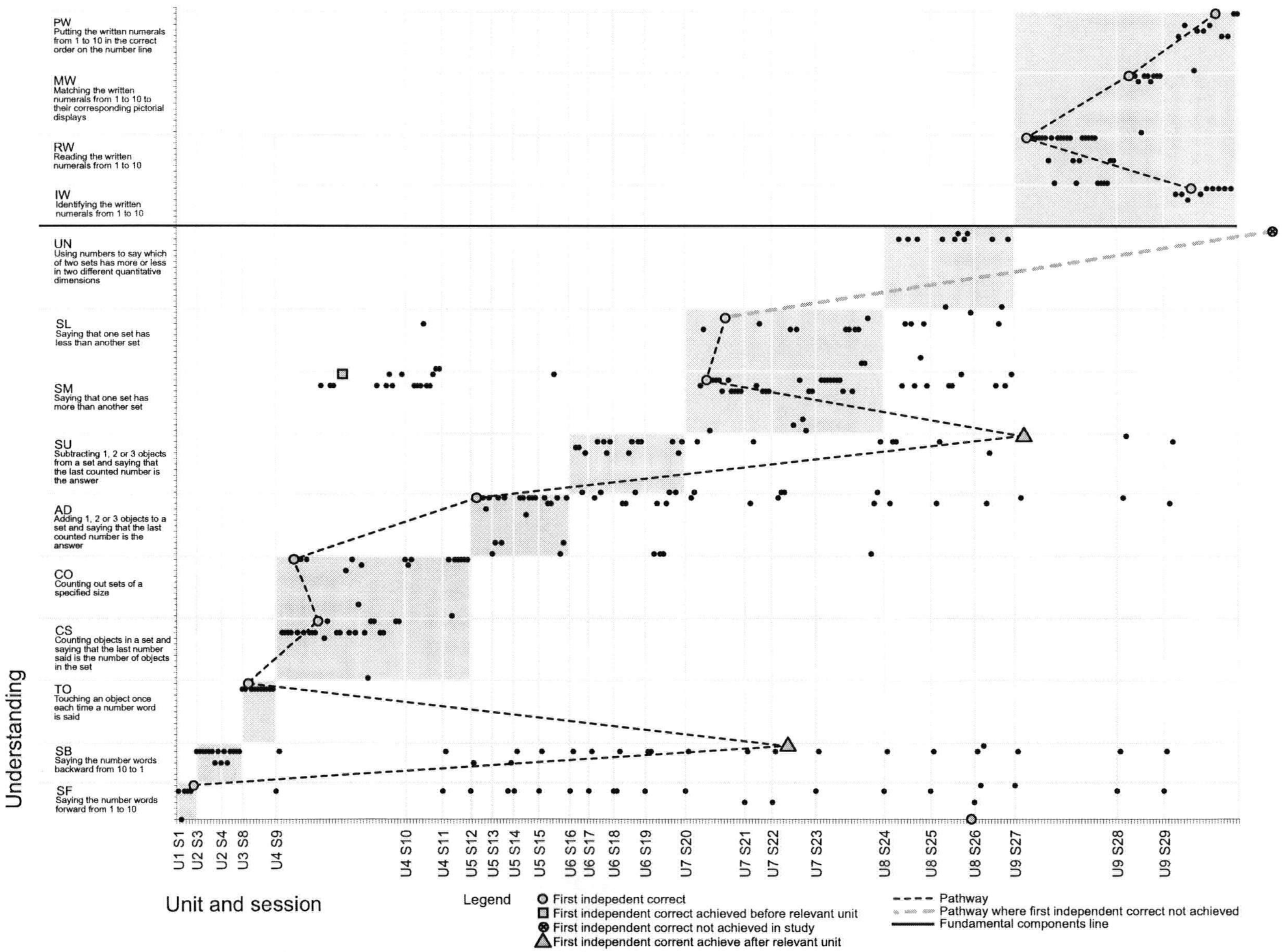


Figure 19. Anne's developmental pathway across the instructional sequence.

Note: IW is shown before RW because IW is typically acquired before RW in the developmental sequence.

Anne's level of response in Units 5 and 6 was high, but variable. This indicated she was just beginning to understand the incrementing and decrementing rules and progress was related to further construction of these mathematical understandings (Griffin & Case, 1996). Slower construction of the decrementing rule appeared to be related to slower development of her ability to give the next number word back for the number words from 10 to 1 (Fuson, 1988).

Anne's level of response in Unit 7 and across Units 4, 5, and 8 for SM and SL was high, and somewhat variable. This indicated she understood the fine-comparison rule and could make fine-numerical comparisons for SM and SL for selected number pairs without instructor support as early as Unit 4. Variability in her level of response was related primarily to whether she was asked to make gross or fine numerical comparisons between sets. However, Anne's level of response in Unit 8 showed she was not yet able to apply the understanding represented in the mental counting line to new tasks and problem situations. This indicated integration of the numerical and spatial components of the mental counting line was not complete and the bootstrapping process described in Case's (1996a; 1996b; 1998a) model of the process of structural change may not have occurred.

Anne's level of response in Unit 8 was consistent with her posttest performance on the Number Knowledge test. Her performance on this test indicated she was still in the process of integrating the numerical and spatial components of the mental counting line. However, her level of response in Unit 8 was not consistent with her posttest performance on the remaining conceptual measures (transfer tasks). Her performance on these measures showed generalization of the understanding represented in the mental counting line to two of the three measures. However, her performance on these measures may have been due to specific experience with the tasks (in the testing situation) rather than to transfer of the understanding represented in the mental counting line. Case (1996d) suggested the understanding represented in central conceptual structures may be acquired in specific problem-solving situations as well as via the operation of the hierarchical learning loop.

Anne's level of response for identifying, reading, and matching the written numerals in Unit 9 was high and somewhat variable. Variability occurred when she confused written numerals with similar defining characteristics. Despite this, her performance indicated she acquired knowledge of the written numerals prior to instruction and connected this knowledge to the more fundamental components of the mental counting line. Anne's level of response for ordering the written numerals showed she relied on less sophisticated strategies, such as the count up from one strategy and the global comparison to carry out this task. Her performance however, was consistent with her performance in Units 5 to 8 and her posttest performance on the Number Knowledge test and the remaining conceptual measures which indicated she was still in the process of integrating the mental counting line.

Overall, Anne moved directly from elaboration and differentiation of forward counting to one-to-one correspondence and cardinality. She then moved to an understanding of the incrementing and decrementing rules and to an understanding of the fine-comparison rule. However, she did not move directly from elaboration and differentiation of forward counting to elaboration and differentiation of backward counting. Her understanding of the fine-comparison rule also appeared to be developing earlier in the instructional sequence. Despite this, further integration of the numerical and spatial components of the mental counting line was slow and consolidation of the mental counting line did not appear to occur. Although Anne followed the developmental pathway that was expected, deviation from this developmental pathway resulted in a slower rate of learning and weak integration and lack of consolidation of the mental counting line.

Kevin

Figure 20 shows Kevin's developmental pathway across the instructional sequence. As can be seen, Kevin's level of response in Units 1, 3, and 4 was high. Variability in his level of response was low. This indicated he acquired the mathematical understandings taught in these instructional units prior to instruction and progress was related to the expansion of these mathematical understandings. Kevin's performance in Units 1, 3, and 4 was consistent his performance on the Quantitative Reasoning subtests of the SB5 and his pretest performance on the Number knowledge test.

Kevin's level of response in Unit 2 indicated he was just beginning to establish next before relationships for the number words from 10 to 1. Although his performance improved as he moved across Units 3 to 9, he did not generate a first independent differentiated response for this mathematical understanding until late in the instructional sequence. His performance, however, was consistent with research on young children's mathematical understandings (Baroody, 1987, 1989; Fuson, 1988; Griffin & Case, 1996). Knowledge of the number word sequence from 10 to 1 is not usually acquired until later in the developmental sequence (Baroody, 1987, 1989; Fuson, 1988; Griffin & Case, 1996).

Kevin's level of response in Units 5 and 6 was high and somewhat variable. This indicated he was just beginning to understand the incrementing and decrementing rules and

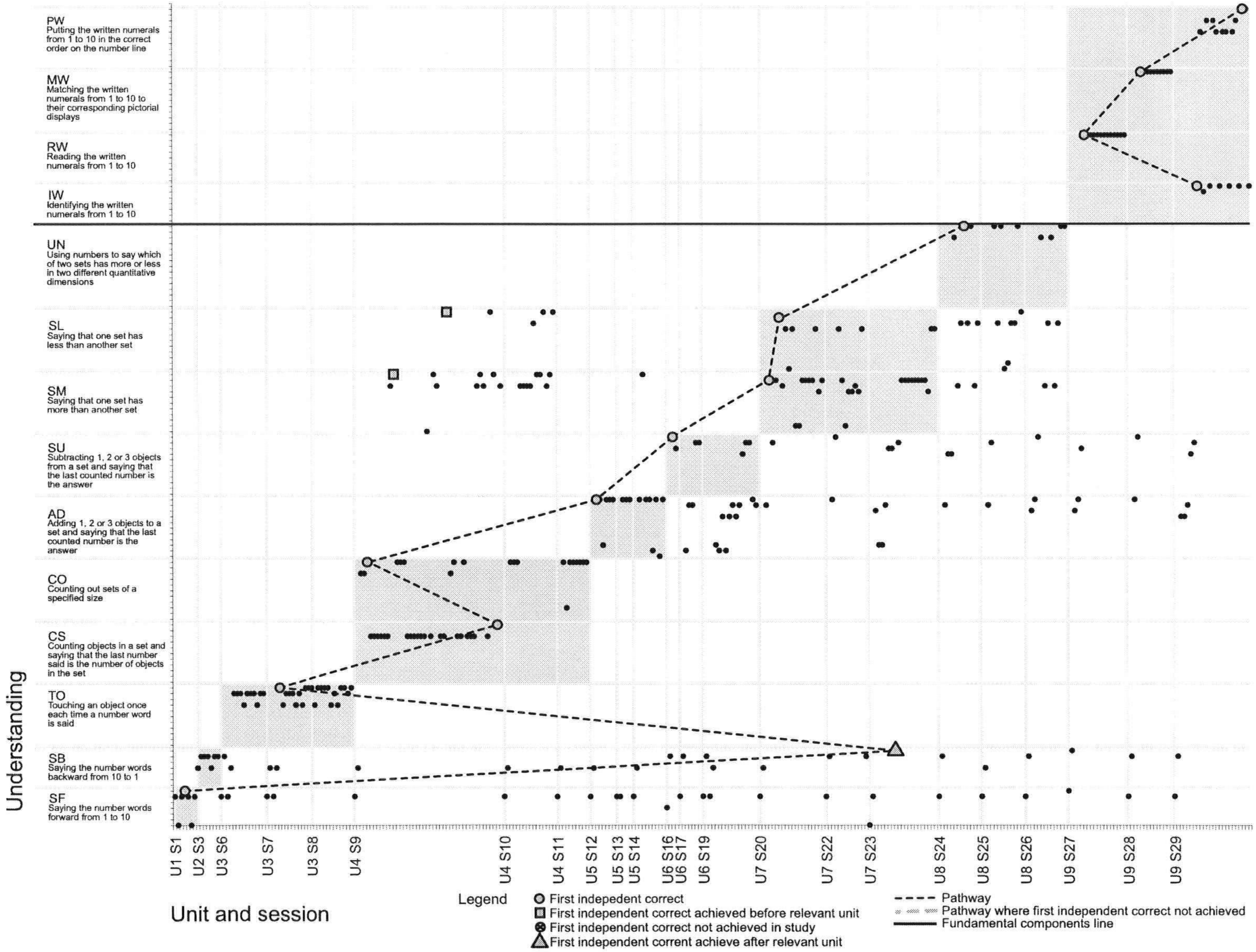


Figure 20. Kevin's developmental pathway across the instructional sequence.

Note: IW is shown before RW because IW is typically acquired before RW in the developmental sequence.

progress was related to the construction of these mathematical understandings (Griffin & Case, 1996).Slower construction of the decrementing rule appeared to be related to slower development of his ability to give the next number word back for the number words from 10 to 1 (Fuson, 1988).

Kevin's level of response in Unit 7 and across Units 4, 5, and 8 for SM and SL was high, and somewhat variable. This indicated he understood the fine-comparison rule and could make fine-numerical comparisons between sets for selected number pairs without instructor support as early as Unit 4. Variability in his level of response was related primarily to whether he was asked to make gross or fine numerical comparisons between sets.

Kevin's level of response in Unit 8 indicated integration of the numerical and spatial components of the mental counting line was almost complete and he was beginning to apply the understanding represented in the mental counting line to new tasks and problem situations. The bootstrapping process described in Case's (1996a; 1996b; 1998a) model of the process of structural change was beginning to occur.

Kevin's level of response in Unit 8 was consistent with his posttest performance on the Number Knowledge test and the remaining conceptual measures (transfer tasks). His performance on the Number Knowledge test indicated integration of the numerical and spatial components of the mental counting line was well under way. His performance on the remaining conceptual measures showed generalization of the understanding represented in the mental counting line to all of these measures (however, generalization was more rapid to some measures than to others). This suggested the understanding represented in the mental counting line was transferred to specific problem-solving situations via the operation of the hierarchical learning loop (Case, 1996d, 1998a).

Kevin's level of response for identifying, reading, and matching the written numerals in Unit 9 was high. There was no variability in his level of response. This indicated he acquired knowledge of the written numerals prior to instruction and connected this knowledge to the more fundamental components of the mental counting line. However, Kevin's level of response for ordering the written numerals showed he relied primarily on the less sophisticated, count up from one strategy to carry out this task. His performance was not consistent with his performance in Units 5 to 8. However, his performance may be consistent with the children's performance in Siegler's (1996a) study on how children discovered and used different addition and subtraction strategies. In this study Siegler found children retained and used a repertoire of more and less sophisticated strategies as their learning progressed (Siegler, 1996a).

Overall, Kevin moved directly from elaboration and differentiation of forward counting to one-to-one correspondence and cardinality. He then moved to incrementing and decrementing small sets of objects and to an understanding of the fine-comparison rule. However, he did not move directly from elaboration and differentiation of forward counting to elaboration and differentiation of backward counting. His understanding of the fine-comparison rule also appeared to be developing earlier in the instructional sequence. Further integration of the numerical and spatial components of the mental counting line continued and consolidation of the mental counting line began to occur. Despite slower development of next before relationships for the number sequence from 1 to 10, Kevin followed the developmental pathway that was expected. Although difficulty with backward counting led to errors when decrementing sets, it did not slow his rate of learning significantly or interfere with his ability to integrate and begin to consolidate the mental counting line.

Sarah

Figure 21 shows Sarah's developmental pathway across the instructional sequence. As can be seen, Sarah's level of response in Units 1, 3, and 4 was high. Variability in her level of response was intermediate. Although Sarah understood the principle of one-to-one correspondence, touching errors and errors in backward counting lowered her level of performance in Unit 3. Despite this, her level of response indicated she acquired the mathematical understandings taught in these instructional units prior to instruction and progress was related to the expansion of these mathematical understandings. Her performance in Units 1, 3, and 4 was consistent with her performance on the Quantitative Reasoning subtests of the SB5 and her pretest performance on the Number knowledge test.

Sarah's level of response in Unit 2 indicated she had not yet begun to establish next before relationships for the number words from 10 to 1. She had difficulty generating the next number word back for all of the number words from 10 to 1. Although her performance slowly improved as she moved across Units 3 to 9, she never generated an independent differentiated response for the number word sequence from 10 to 1. Despite this, her performance was consistent with research on young children's mathematical understandings (Baroody, 1987, 1989; Fuson, 1988; Griffin & Case, 1996). Knowledge of the number word sequence from 10 to 1 is not usually acquired until later in the developmental sequence(Baroody, 1987, 1989; Fuson, 1988; Griffin & Case, 1996).

Sarah's level of response in Units 5 and 6 was high, but variable. This indicated she was just beginning to understand the incrementing and decrementing rules and progress was related

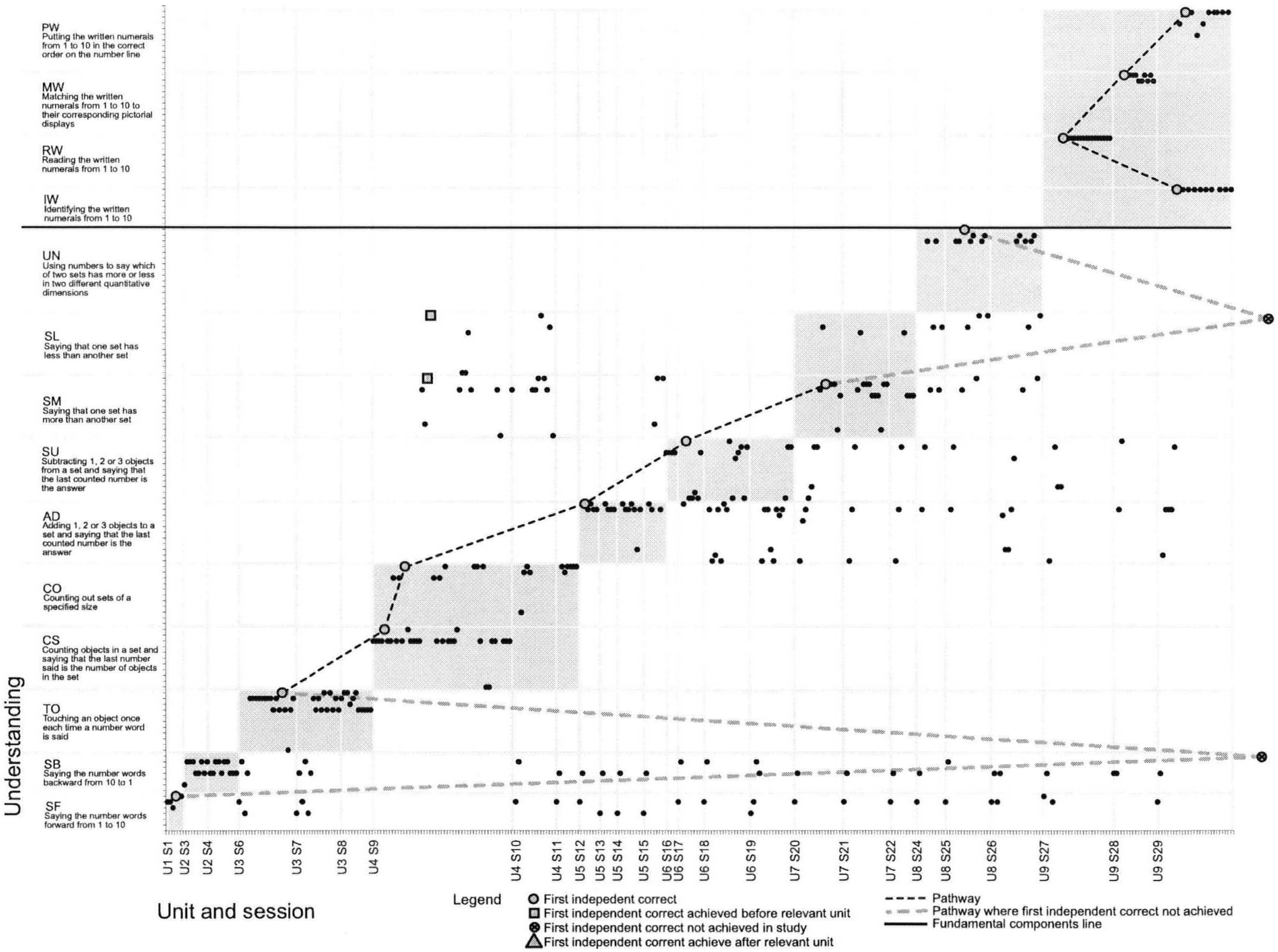


Figure 21. Sarah's developmental pathway across the instructional sequence.

Note: IW is shown before RW because IW is typically acquired before RW in the developmental sequence.

to the construction of these mathematical understandings (Griffin & Case, 1996). However, she found both incrementing and decrementing sets difficult and consistently counted up from one or counted the remainder to find the answer to addition and subtraction problems. Slower construction of these mathematical understandings was influenced by slower development of her understanding of next after and next before relationships for the number words from 1 to 10 (Fuson, 1988).

Sarah's level of response in Unit 7 and across Units 4, 5, and 8 for SM and SL was high, and somewhat variable. This indicated she understood the fine-comparison rule and could make fine numerical comparisons for SM and SL for selected number pairs without instructor support as early as Unit 4. Variability in her level of response was related primarily to whether she was asked to make gross or fine numerical comparisons between sets. However, Sarah's level of response in Unit 8 indicated she could not yet apply the understanding represented in the mental counting line to new tasks and problem situations. This indicated integration of the numerical and spatial components of the mental counting line was only just beginning and the bootstrapping process described in Case's (1996a; 1996b; 1998a) model of the process of structural change may not have occurred.

Sarah's level of response in Unit 8 was consistent with her posttest performance on the Number Knowledge test and the remaining conceptual measures (transfer tasks). Her performance on the Number Knowledge test indicated she was just beginning to integrate the numerical and spatial components of the mental counting line. Her performance on the remaining conceptual measures showed generalization of the understanding represented in the mental counting line to one of the three measures. However, her performance on this measure may have been due to specific experience with the task (in the testing situation) rather than to transfer of the understanding represented in the mental counting line. Case (1996d) suggested the understanding represented in central conceptual structures may be acquired in specific problem-solving situations as well as via the operation of the hierarchical learning loop.

Sarah's level of response for identifying, reading, and matching the written numerals in Unit 9 was high. Variability in her level of response was low. This indicated she acquired knowledge of the written numerals prior to instruction and connected this knowledge to the more fundamental components of the mental counting line. Sarah's level of response for ordering the written numerals showed she most frequently used the fine-comparison strategy to carry out this task. Her performance was not consistent with her performance in Units 5 to 8 which suggested she might use less sophisticated strategies such as the count up from one strategy or the global comparison strategy. Her ordering of the written numerals from left to right rather than right to left also suggested she might use less sophisticated strategies to carry out this task. However, she used the fine-comparison strategy more frequently toward the end of Session 29. This suggested the numerals she initially placed on the number line may have facilitated use of the fine-comparison strategy.

Overall, Sarah moved directly from elaboration and differentiation of forward counting to one-to-one correspondence and cardinality. She then moved to an understanding of the incrementing and decrementing rules and to an understanding of the fine-comparison rule. However, she did not move directly from elaboration and differentiation of forward counting to elaboration and differentiation of backward counting. Her understanding of the fine-comparison rule also appeared to be developing earlier in the instructional sequence. Despite this, further integration of the numerical and spatial components of the mental counting line was slow and consolidation of the mental counting line did not appear to occur. Although Sarah followed the developmental pathway that was expected, slow development of next after and next before relationships for the number sequence from 1 to 10 slowed her rate of learning and resulted in weak integration and lack of consolidation of the mental counting line.

Wendy

Figure 22 shows Wendy's developmental pathway from across the instructional sequence. As can be seen, Wendy's level of response in Units 1, 3, and 4 was high. Variability in her level of response was low. This indicated she acquired the mathematical understandings taught in Units 1, 3, and 4 prior to instruction and progress was related to the expansion of these mathematical understandings. Wendy's performance in Units 1, 3, and 4 was consistent with her performance on the Quantitative Reasoning subtests of the SB5 and her pretest performance on the Number Knowledge test.

Wendy's level of response in Unit 2 indicated she was just beginning to establish next before relationships for the number words from 10 to 1. Although her performance improved as she moved across Units 3 to 9, she did not generate a first independent differentiated response for this mathematical understanding until the middle of the instructional sequence. Her performance, however, was consistent with research on young children's mathematical understandings(Baroody, 1987, 1989; Fuson, 1988; Griffin & Case, 1996). Knowledge of the number word sequence from 10 to 1 is not usually acquired until later in the developmental sequence (Baroody, 1987, 1989; Fuson, 1988; Griffin & Case, 1996).

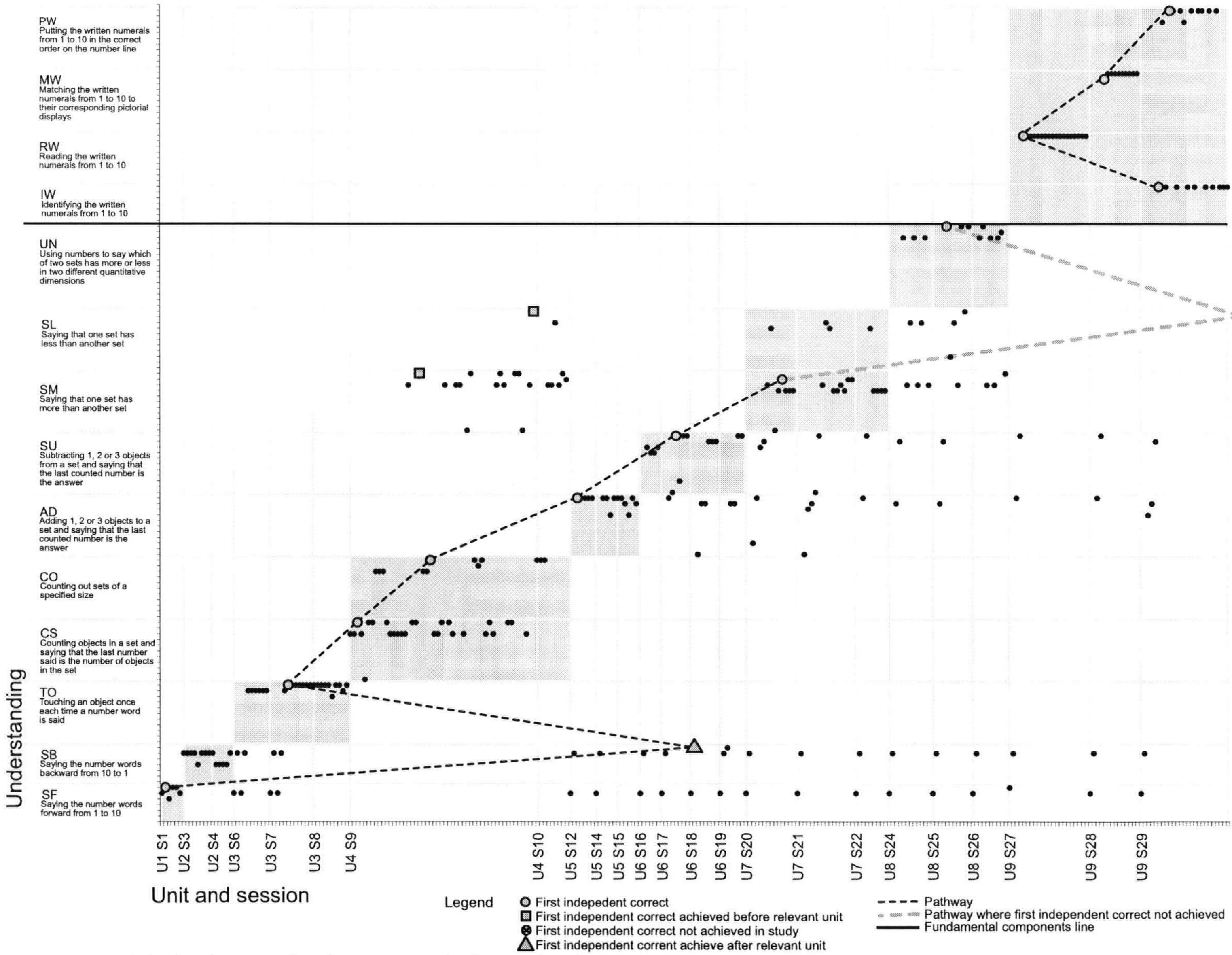


Figure 22. Wendy's developmental pathway across the instructional sequence

Note: IW is shown before RW because IW is typically acquired before RW in the developmental sequence.

Wendy's level of response in Units 5 and 6 was high and relatively stable. Her performance in Units 5 and 6 indicated she understood the decrementing rule and was beginning to understand the incrementing rule (Griffin & Case, 1996). For both mathematical understandings, progress was related to her developing understanding of next after and next before relationships for the number words from 10 to 1 (Fuson, 1988).

Wendy's level of response in Unit 7 and across Units 4, 5, and 8 for SM and SL was high, and somewhat variable. This indicated she understood the fine-comparison rule and could make fine-numerical comparisons for SM and SL for selected number pairs without instructor support as early as Unit 4. Variability in her level of response was related primarily to whether she was asked to make gross or fine-numerical comparisons between sets.

Wendy's level of response in Unit 8 indicated integration and consolidation of the numerical and spatial components of the mental counting line was almost complete and she was beginning to apply the understanding represented in the mental counting line to new tasks and problem situations. The bootstrapping process described in Case's (1996a; 1996b; 1998a) model of the process of structural change was beginning to occur.

Wendy's level of response in Unit 8 was consistent with her posttest performance on the Number Knowledge test and the remaining conceptual measures (transfer tasks). Her performance on the Number Knowledge test indicated integration of the numerical and spatial components of the mental counting line was well under way. Her performance on the remaining conceptual measures showed generalization of the understanding represented in the mental counting line to two of the three measures. This suggested the understanding represented in the mental counting line was transferred to specific problem-solving situations via the operation of the hierarchical learning loop (Case, 1996d, 1998a).

Wendy's level of response for identifying, reading, and matching the written numerals in Unit 9 was high. There was no variability in her level of response. This indicated she acquired knowledge of the written numerals prior to instruction and connected this knowledge to the more fundamental components of the mental counting line. Wendy's level of response for ordering the written numerals showed she most frequently used the fine-comparison strategy to carry out this task. Her performance was consistent with her performance in Units 5 to 8 which suggested she might use a more sophisticated strategy. Despite this, her ordering of the written numerals from left to right rather than right to left suggested she might use a less sophisticated strategy to carry out this task. However, her performance may be consistent with the children's performance in Siegler's (1996a) study on how children discovered and used different addition and subtraction strategies. In this study Siegler found children retained and used a repertoire of more and less sophisticated strategies as their learning progressed (Siegler, 1996a).

Overall, Wendy moved directly from elaboration and differentiation of forward counting to one-to-one correspondence and cardinality. She then moved to an understanding of the incrementing and decrementing rules and to an understanding of the fine-comparison rule. However, she did not move directly from elaboration and differentiation of forward counting to elaboration and differentiation of backward counting. Her understanding of the fine-comparison rule also appeared to be developing earlier in the instructional sequence. Further integration of the numerical and spatial components of the mental counting line continued and consolidation of the mental counting line began to occur. Despite somewhat slower development of next before relationships for the number sequence from 1 to 10, Wendy followed the developmental pathway that was expected. Difficulty with backward counting did not appear to slow her rate of learning or interfere with her ability to integrate and begin to consolidate the mental counting line.

This chapter articulated how changes in the children's conceptual thinking occurred as they negotiated the critical transition from the relational to the dimensional stages of the mental counting line. The following chapter focuses on how the changes that were observed in the children's conceptual thought support Case's (1991c; 1996a; 1998a) neo-Piagetian theory of cognitive development, Case's (1996a; 1996b; 1998a) model of the process of structural change, and Porath's (1988; 1991b) model of the intellectual development of academically advanced children. This chapter also shows how the results of the study are consistent with the results of existing macrodevelopmental studies from Case's (1991c; 1996a; 1998a) neo-Piagetian theoretical perspective, other neo-Piagetian theoretical perspectives (Fischer et al., 1984), and the results of existing macrodevelopmental studies on the development of young children's mathematical understanding (Robinson et al., 1996; Robinson et al., 1997). The contribution of the study, educational implications of the study, limitations of the study, and directions for future research are also presented.

Chapter 6:
Discussion

The major objective of this study was to explore how individual, 4- to 5-year-old children, who displayed average to above-average mathematical ability for their age, responded to an instructional program that was designed to facilitate the construction of the mental counting line. The more specific objective of this study was to observe and describe intra-individual and inter-individual variability in the rate and the pattern of construction of the mental counting line and the transfer of the understanding represented in the mental counting to new tasks and problem situations.

The mental counting line is a central conceptual structure that develops in the domain of whole number understanding as children negotiate the critical transition between the relational and dimensional stages of Case's (1991c; 1996a; 1998a) neo-Piagetian theory. The mental counting line represents "a 'preferred developmental pathway' for a large class of individuals" (Case, 1996d, p. 211). However, individual differences in this developmental pathway may exist (Case, 1996d, 1998a; Case et al., 1993).

The microgenetic approach advocated by Catán (1986b) was used in this study because this approach allowed for the observation and description of the changes that occurred in the children's understanding as the children acquired and integrated the separate numerical and spatial components that make up the mental counting line (Lee & Karmiloff-Smith, 2002; Parziale, 2002; Siegler, 1997). This microgenetic approach also allowed for the observation of aspects of the instructional program that may have contributed to the integration of these components of the mental counting line (Catán, 1986b; Siegler & Crowley, 1991).

Individual Pathways

This study found all of the children followed the developmental sequence Case (1996a; 1998a) suggested leads to the construction of the mental counting line, all of the children began to integrate the numerical and spatial components of the mental counting line, and some of the children began to generalize the understanding represented in the mental counting line to new tasks and problem situations. The older average ability children progressed to a level that was appropriate for their age and the younger above-average ability children progressed to a level that was advanced for their age.

However, the most important finding of this study was that the microgenetic approach made it possible to observe and describe individual differences in the children's developmental pathways as they responded to the instructional program that was designed to facilitate the construction of the mental counting line. Although all of the children progressed through the sequence of developmental steps suggested by Case (1998a), the children showed individual differences in the rate and the pattern of construction of each of the separate mathematical understandings that make up the mental counting line. This resulted in differences in the degree to which the children integrated the numerical and spatial components of the mental counting line and differences in the children's ability to transfer the understanding represented in the mental counting line to new tasks and problem situations. The children who progressed the most showed fewer deviations from the developmental sequence outlined by Case (1996a; 1998a), moved through the separate mathematical understandings at a more optimal level, and showed a more rapid rate of learning through the middle and latter portions of the developmental sequence. The children who progressed the least showed more deviations from this developmental sequence, moved through the separate mathematical understandings at a less optimal level, and showed a slower rate of learning through the middle and latter portions of the developmental sequence.

The differences that were observed suggest that although there was some consistency in the children's performance, the children all followed individual developmental pathways through the expected sequence of developmental steps. When and how the children acquired and integrated each of the separate components of the mental counting line determined how successful they were in negotiating the critical transition between the relational and dimensional stages of Case's (1991c; 1996a; 1998a) theory. Optimal development through this critical transition is essential because the manner in which children negotiate this transition determines the course of their future development (Case, 1996a, 1998a). The differences that were observed in the children's developmental pathways, therefore, have important implications for our understanding of the development of children's mathematical thought and the development of effective instruction.

Rate and Pattern of Construction

All of the children progressed in the construction of the mental counting line as a result of instruction. All of the children began to integrate the numerical and spatial components of the mental counting line. This finding supports Case's contention that the separate mathematical understanding represented in the mental counting line can be taught to children who have not yet acquired these mathematical understandings (Griffin, 2004b; Griffin & Case, 1997; Griffin et al., 1992). This finding is consistent with the results of instructional studies from Case's (1991c; 1996a; 1998a) theoretical perspective (Case & Sandieson, 1991; Griffin, 2004b; Griffin & Case, 1996, 1997; Griffin et al., 1995). These studies have consistently demonstrated that the mathematical understandings represented in the mental counting line can be successfully taught (Griffin & Case, 1996, 1997; Griffin et al., 1995; Griffin et al., 1994). This finding is also consistent with the results of a microgenetic study, and a time series analysis of children's narrative understanding in the domain of social/narrative understanding (McKeough et al., 2005; McKeough & Sanderson, 1996). This study demonstrated that stories with action-intentional plot structures can be taught. It is possible the children's progress may have been due to maturation (the instructional program was taught over a 7-week period) rather than to the instruction the children received. However, the mathematical understandings that were taught are normally acquired over a much longer period of time. Also, the instructional program that was used was designed to compress the learning of these mathematical understandings into a much tighter time frame so that changes in the children's understandings could be observed and described (Catán, 1986a). This suggests the children's progress may be attributed to the instruction they received rather than to maturation.

However, all of the children progressed to different points in the construction of the mental counting line. This finding is consistent with the results of instructional studies from Case's (1991c; 1996a; 1998a) theoretical perspective (Case & Sandieson, 1991; Griffin, 2004b; Griffin & Case, 1996, 1997; Griffin et al., 1995). These studies have consistently found that although the majority of the children in these studies moved to the dimensional stage of the mental counting line, the remaining children also demonstrated some degree of progress in the construction of this central conceptual structure (Griffin & Case, 1996, 1997; Griffin et al., 1995; Griffin et al., 1994). This finding supports Case's (1991c; 1996a) contention that children's progress in the construction of central conceptual structures such as the mental counting line will vary depending on a combination of talent, interest, motivation, and the amount of domain-specific experience children receive.

The older average ability children progressed to a level that was appropriate for their age (Griffin & Case, 1997). This suggests that the children may have moved to a level that growth in their working memory capacity would allow (Case et al., 1993). Movement from one major stage to the next is related to system-wide, age-related increases in children's working memory capacity (Case, 1991c, 1996a, 1996d). Children's working memory capacity sets an upper limit on the number of elements children can pay attention to or hold in their working memory while processing information (Case, 1991c). In a series of studies Griffin (1994, as cited in Case, 1996d) found that children who had the required amount of working memory capacity, but who lacked crucial components of the mental counting line, moved to the level their working memory capacity would allow when instruction was provided.

The younger above-average ability children moved to a level that was somewhat advanced for their age. This suggested that at least for one of the children movement to a higher level of understanding may have been related to automatization of some of the components of the mental counting line. The child who progressed the most appeared to have better developed counting skills. She was also better able to construct the understandings (generative rule, fine-comparison rule; and consolidation of the mental counting line) that were taught from the middle to the end of the instructional sequence. Automatization due to practice, prior learning, or the vertical and horizontal mental counting lines that were used to scaffold the children's understanding may have allowed her to more readily engage in attentionally mediated learning and move more quickly to higher levels of conceptual thought. This finding supports Case's (1996a; 1996b) contention that children from enriched environments experience acceleration of their development in particular domains up to the maximum allowed by their working memory capacity as a result of the bootstrapping process set up by the operation of the hierarchical learning loop. This finding is consistent with predictions of the mathematical version of Case's (1996a; 1996b; 1998a) model of the process of structural change and existing empirical data from Case's (Case, 1991c, 1996a, 1998a) theoretical perspective. This finding supports Porath's contention (1991b) that academically advanced children are similar to children of more average ability in terms of the development of their conceptual understandings, but different from children of more average ability in terms of the rate at which they acquire specific skills in particular domains. In a series of studies, Porath (1991b; 1993; 1996a; 1996b) consistently found the conceptual understandings of academically advanced children did not exceed the conceptual understandings of more average ability children by more than two years. However, specific skills were acquired at a much more rapid rate and the thinking characteristic of each developmental stage was applied in a more elaborate and flexible way. This finding also supports Robinson, Abbott, Berninger, and Busse's (1996) finding that it is possible to identify advanced mathematical capabilities in very young children before they enter school and Robinson, Abbott, Berninger, Busse, and Mukhopadhyay 's (1997) finding that young mathematically precocious children benefit from enrichment in this domain.

The children who progressed more quickly through the instructional sequence showed more optimal levels of response and tended to maintain these levels of response when researcher support was withdrawn. This finding is consistent with Fischer et al.'s (1984) finding that children who progress at different rates show differences in their developmental level or developmental range in different situations and contexts (situations and contexts of high and low support). In a study investigating the developmental pathways of preschool children as they constructed an understanding of social roles, Fischer et al. (1984) consistently found individual differences in developmental range when the children's levels of performance were compared across different situations and contexts (Fischer et al., 1984).

The children who progressed more quickly through the instructional sequence also began to generalize the understanding represented in the mental counting line to new tasks and problem situations. This supports Case's (1991c; 1996a; 1998a) contention that once the mental counting line is integrated it is "consolidated in a fashion that transcends any specific context" (Case, 1998a, p. 791) and the understanding that is represented in the mental counting line is used to make sense of new tasks and problem situations. This also supports Case's (1996a; 1998a) suggestion that the action of the hierarchical learning loop is responsible for this pattern of response. What appeared to be a more uniform rate of growth was apparent in the children's performance on the remaining conceptual measures at posttest. This suggests the understanding represented in the mental counting line was in the process of being transferred to these tasks via the operation of the hierarchical learning loop (Case, 1996b). This finding is consistent with predictions of the mathematical version of Case's (1996a; 1996b; 1998a) model of the process of structural change and existing empirical data from Case's (Case, 1991c, 1996a, 1998a) theoretical perspective. This finding is also consistent with the results of instructional studies from Case's (1991c; 1996a; 1998a) theoretical perspective (Case & Sandieson, 1991; Griffin, 2004b; Griffin & Case, 1996, 1997; Griffin et al., 1995). These studies have consistently shown transfer of the mathematical understandings represented in the mental counting line to tasks that depended on this understanding (Griffin & Case, 1996, 1997; Griffin et al., 1995; Griffin et al., 1994).

All of the children followed the developmental sequence outlined by Case in the "requirements that would have to be met" (Case, 1998a, p. 778) to construct a new higher order central conceptual structure. This indicated that the changes that were observed in the children's thinking were developmental rather than idiosyncratic. This also supports Case's contention that although variability is apparent in children's development as a result of differences in talent, interest, motivation, experience, or the context of specific tasks, "a good deal of consistency" is also present (Case et al., 1993, p. 97). Although acquisition of counting backward did not follow acquisition of counting forward, this was not an unexpected finding since this understanding was moved ahead in the instructional sequence to give the children more practice.

However, a great deal of variability was apparent around the expected developmental pathway, particularly for the children who progressed the least. This finding indicated the children followed individual pathways into and through each of the separate understandings represented in the mental counting line. The tasks presented in each instructional unit provided independent assessments of each of the separate mathematical understandings that make up the mental counting line (Case et al., 1993). None of the children performed at exactly the same level on these tasks. This produced individual pathways or developmental webs around the expected developmental sequence. This finding also showed that differences in the acquisition and integration of the separate components that make up the mental counting line can affect the overall process of integration of the mental counting line and children's progress through this developmental transition (Case, 1996d). The children who progressed the least showed poorer integration of the separate components of the mental counting line. This finding supports Case's suggestion that although the mental counting line represents "a 'preferred developmental pathway' for a large class of individuals" (Case, 1996d, p. 211), children will respond differently to the tasks presented at each step in this developmental sequence (Case et al., 1993). This finding is also consistent with the results of McKeough and Sanderson's (1996) microgenetic study and McKeough, Davis, Forgeron, Marini, and Fung's (2005) time series analysis of children's narrative understanding in the domain of social/narrative understanding. In these studies they found that although all of the children integrated and consolidated the mental story line, individual differences were apparent in the children's use of stereotypic plot sequences, stereotypic elaborative sequences, and original event sequences (McKeough & Sanderson, 1996).

At the beginning of the instructional sequence all of the children showed a pattern of gradual change in their conceptual understanding, similar to the pattern described by Siegler (1996a). However, two of the children began to show a pattern of more rapid, abrupt change in their conceptual understanding from the middle to the end of the instructional sequence. These findings support Case's (1998a) suggestion that the operation of the hierarchical learning loop can lead to both types of learning depending on the specific characteristics of the children and the problem-solving situation. The change in the children's rate of learning became apparent at the point in the instructional sequence where strong number sequence skills were required to carry out the tasks. Knowing the position of each number in the number sequence, being able to quickly enter the number sequence at a specified point, and move backward and forward from that point was particularly important. Good next before and next after skills seemed to be related to more rapid construction of the increment and decrement rules and the fine-comparison rule. Children with less automatized number sequence skills showed a slower rate of learning here. For this reason, the number lines were particularly important at this point in the instructional sequence. The number lines supported the development of the children's number sequence skills and facilitated the use of more sophisticated count on and count back addition and subtraction strategies. Without the number line the children who were learning at a slower pace tended to drop back to less sophisticated strategies such as counting up from one and counting the remainder. As the next before and next after relationships between the numbers in the number sequence became more firmly established, the children began to use the number line as a problem-solving tool. Providing the children with number lines seemed to help them move more quickly to using the mental counting line as a problem-solving tool.

Overall, the results of this study provide evidence for the development of the mental counting line in the domain of whole number understanding, evidence for qualitative changes in the children's thinking, and evidence for individual pathways in the development of the children's thinking as they moved between the steps on "The Mind's Staircase". The results of the study support Case's (1991c; 1996a; 1998a) neo-Piagetian theory of cognitive development, Case's (1996a; 1996b; 1998a) model of the process of structural change, and Porath's (1988; 1991b) model of the intellectual development of academically advanced children. The results of the study are consistent with the results of existing macrodevelopmental studies from Case's (1991c; 1996a; 1998a) neo-Piagetian theoretical perspective, other neo-Piagetian theoretical perspectives (Fischer et al., 1984), and the results of existing macrodevelopmental studies on the development of young children's mathematical understanding (Robinson et al., 1996; Robinson et al., 1997).

Contribution of the Study

This study represents the first microgenetic analysis of the changes that occur in children's thinking as they construct the mental counting line in the domain of whole number understanding. The study contributes to the development of a richer model of children's understanding in this domain. As well, it provides insight into the process of conceptual change for individual children and information about where the instructional program can be changed to accommodate the needs of individual children. The study also links Case's (1991c; 1996a; 1998a) neo-Piagetian theory of cognitive development, Case's (1996a; 1996b; 1998a) model of the process of structural change, and Porath's (1988; 1991b) model of the intellectual development of academically advanced children with behavioural data at the individual level (Case, 1996c; Siegler, 1996b) and the results of existing macrodevelopmental studies (Case, 1996c; Siegler, 1996b). The coding scheme that was developed to analyze the microgenetic data contributes to the field by providing a tool to conduct a finer grained analysis of the "requirements that would have to be met" (Case, 1998a, p. 778) to construct the mental counting line.

This study shows how the development of individual children can inform instruction. The study showed that children who fail to acquire and integrate particular components of the mental counting line or who are slow to acquire and integrate these components will show different developmental pathways through this critical transition. The different developmental pathways children move through as they negotiate this transition must be taken into account if instruction is to be effective. This research will contribute to our understanding of the process of integration and consolidation in the development of the mental counting line and will contribute to the development of better methods of instruction.

Educational Implications

Although manipulatives are appropriate for young children in supporting their learning of mathematics, they should be used in conjunction with mental counting lines if children are to begin to move easily and flexibly backward and forward along their own mental counting lines. The children in this study constructed more sophisticated numerical strategies when they were solving problems using the mental counting lines. The children counted on from the larger addend when using the mental counting line. However, when the mental counting lines were not available the children tended to drop back to the less sophisticated strategy of counting up from one to find the answer.

The children in this study did not spontaneously recognize the utility of using their counting and comparing skills to determine which of two jars had more blocks or which of two chains was longer. The children's patterns of response to these activities suggest that conceptual understandings take a long time to develop. Children need a lot of opportunities to explore the concepts that were taught. They need to revisit these understandings again and again, at different times and in a variety of different situations and problem-solving contexts. Children can be taught specific understandings, but they also need to have experiences where they discover these understandings on their own.

Limitations of the Study

One limitation of the study was the design of the instructional program. The instructional program was highly structured and the level of researcher support was high. Preferred strategies and correct responses were modelled by the instructor. As well, the children were consistently prompted to respond in a particular way throughout the study. Hence, the children were not allowed to explore the materials and come up with their own strategies or solutions. This constrained the range of the children's responses and made it more difficult to observe the extent to which the children differed in their learning patterns.

A second limitation of the study was the difficulty of working with very young children. Young children get tired; they have short attention spans and are not concerned with doing well on exams. These factors may have influenced the children's levels of performance on the pretest and the posttest, particularly on the posttest. The children were posttested at the end of the preschool year. The weather was warm and their families were planning summer vacations. The children were excited about their summer holidays and were not motivated to do well on the tests. The children's behaviour during the posttest may have underestimated what they were able to do.

A third limitation of the study concerned the measures of conceptual understanding. The Money Knowledge task may not have been suitable for young children. The children were unfamiliar with the content of this task and all of the children had difficulty answering the questions on this task.

A fourth limitation of the study was that missed instructional sessions were not made up. It was difficult to schedule additional instructional session within the framework of the preschool program. This differentially reduced the children's exposure to the instructional program and may have influenced their level of performance.

Suggestions for Future Research

This study focused on how numerical capabilities and understanding developed over time as a result of instruction. This study did not look at how specific instructional procedures influenced the children's levels of performance as they progressed through the instructional program. Future studies could look at how specific instructional procedures might contribute to intra-individual and inter-individual differences in the children's performance (McKeough & Sanderson, 1996).

The Number Knowledge test evaluated only a small number of the children's numerical skills and understandings. It would be helpful to include a wider range of skills and understandings at the lower levels of the Number Knowledge test. For example, including addition questions such as N+1 and subtraction questions such as N-1 on the Number Knowledge test would provide more information on children's mathematical understandings and strategy use and more information on individual differences in the development of the mental counting line.

This study did not address how children from different socioeconomic backgrounds would respond to the instructional procedure that was designed to facilitate construction of the mental counting line. All of the children in this study were from middle- to upper-middle class homes. The parents were well educated and were interested in their children's intellectual development. They also provided the children with a range of educational experiences, games and toys. Future studies could look at how children from a variety of other socioeconomic backgrounds responded to the instructional procedure.

Bibliography

Bain, S. K., & Allin, J. D. (2005). Stanford-Binet Intelligence Scales, Fifth Edition. *Journal of Psychoeducational Assessment, 23*(1), 89-95.

Baroody, A. J. (1987). *Children's mathematical thinking: A developmental framework for preschool, primary, and special education teachers*. New York: Teachers College Press.

Baroody, A. J. (1989). *A guide to teaching mathematics in the primary grades*. Needham Heights, MA: Allyn and Bacon.

Baroody, A. J., & Wilkins, J. L. M. (1999). The development of informal counting, number, and arithmetic skills and concepts. In J. V. Copley (Ed.), *Mathematics in the early years* (pp. 48-65). Washington, DC: National Association for the Education of Young Children.

Bidell, T., & Fischer, K. W. (1992). Cognitive development in educational contexts. In A. Demetriou, M. Shayer & A. Efklides (Eds.), *Neo-Piagetian theories of cognitive development*. New York, NY: Routledge.

Biggs, J. B. (1992). Returning to school: Review and discussion. In A. Demetriou, M. Shayer & A. Efklides (Eds.), *Neo-Piagetian theories of cognitive development* (pp. 277-294). New York, NY: Routledge.

Case, R. (1985). *Intellectual development: Birth to adulthood.* Orlando, FL: Academic Press, Inc.

Case, R. (1991a). General and specific views of the mind, its structure and its development. In R. Case (Ed.), *The mind's staircase: Exploring the conceptual underpinnings of children's thought and knowledge* (pp. 3-15). Hillsdale, NJ: Lawrence Erlbaum Associates.

Case, R. (1991b). The mind and its modules: Toward a multi-level view of the development of human intelligence. In R. Case (Ed.), *The mind's staircase: Exploring the conceptual underpinnings of children's thought and knowledge* (pp. 343-376). Hillsdale, NJ: Lawrence Erlbaum Associates.

Case, R. (1991c). A neo-Piagetian approach to the issue of cognitive generality and specificity. In R. Case (Ed.), *The mind's staircase: Exploring the conceptual underpinnings of children's thought and knowledge* (pp. 17-51). Hillsdale, NJ: Lawrence Erlbaum Associates.

Case, R. (1996a). Introduction: Reconceptualizing the nature of children's conceptual structures and their development in middle childhood. In R. Case & Y. Okamoto (Eds.), *The role of central conceptual structures in the development of children's thought* (Vol. 61, pp. 1-26). Chicago, IL: University of Chicago Press.

Case, R. (1996b). Modeling the dynamic interplay between general and specific change in children's conceptual understanding. In R. Case & Y. Okamoto (Eds.), *The role of central conceptual structures in the development of children's thought* (Serial No. 246 ed., Vol. 61, pp. 156-188). Chicago: University of Chicago Press.

Case, R. (1996c). Modeling the process of conceptual change in a continuously evolving hierarchical system. In R. Case & Y. Okamoto (Eds.), *The role of central conceptual structures in the development of children's thought* (Serial No. 246 ed., Vol. 61, pp. 283-297). Chicago: University of Chicago Press.

Case, R. (1996d). Summary and conclusion. In R. Case & Y. Okamoto (Eds.), *The role of central conceptual structures in the development of children's thought* (Serial No. 246 ed., Vol. 61, pp. 189-214). Chicago: University of Chicago Press.

Case, R. (1998a). The development of conceptual structures. In D. Kuhn & R. S. Siegler (Eds.), *Cognition , Perception and Language* (Fifth ed., Vol. 2, pp. 745-800).

Case, R. (1998b). *A psychological model of number sense and its development.* Paper presented at the Annual meeting of the American Education Research Association, San Diego, CA.

Case, R., & Bereiter, C. (1984). From behaviourism to cognitive behaviourism to cognitive development: Steps in the evolution of instructional design. *Instructional Science, 13*, 141-158.

Case, R., & Edelstein, W. (1993). Introduction: Structural approaches to individual differences. In R. Case & W. Edelstein (Eds.), *The new structuralism in cognitive development: Theory and research on individual pathways* (pp. 1-10). New York, NY: Karger.

Case, R., & Griffin, S. (1990). Child cognitive development: The role of central conceptual structures in the development of scientific and social thought. In C.-A. Hauert (Ed.), *Developmental Psychology: Cognitive, Perceptuo-Motor, and Neuropsychological Perspectives* (pp. 193-230). Oxford, England: North-Holland.

Case, R., Griffin, S., & Kelly, W. M. (2001). Socioeconomic differences in children's early cognitive development and their readiness for schooling. In S. N. Golbeck (Ed.), *Psychological perspectives on early childhood education: Reframing dilemmas in research and practice* (pp. 37-63). Mahwah, NJ: Lawrence Erlbaum Associates, Publishers.

Case, R., Griffin, S., McKeough, A., & Okamoto, Y. (1991). Parallels in the development of children's social, numerical, and spatial thought. In R. Case (Ed.), *The mind's staircase: Exploring the conceptual underpinnings of children's thought and knowledge* (pp. 269-284). Hillsdale, NJ: Erlbaum.

Case, R., & Mueller, M. P. (2001). Differentiation, integration, and covariance mapping as fundamental processes in cognitive and neurological growth. In J. L. McClelland & R. S. Siegler (Eds.), *Mechanisms of cognitive development: Behavioral and neural perspectives. Carnegie Mellon symposia on cognition.* (pp. 185-219). Mahwah, NJ, US: Lawrence Erlbaum Associates, Inc.

Case, R., Okamoto, Y., Henderson, B., & McKeough, A. (1993). Individual variability and consistency in cognitive development: New evidence for the existence of central conceptual structures. In R. Case & W. Edelstein (Eds.), *The new structuralism in cognitive development: Theory and research on individual pathways* (pp. 71-100). New York, NY: Karger.

Case, R., Okamoto, Y., Henderson, B., McKeough, A., & Bleiker, C. (1996). Exploring the macrostructure of children's central conceptual structures in the domains of number and narrative. In R. Case & Y. Okamoto (Eds.), *The role of central conceptual structures in the development of children's thought* (Vol. 61, pp. 59-82). Chicago, Illinois: University of Chicago Press.

Case, R., & Sandieson, R. (1991). Testing the presence of a central quantitative structure: Use of the Transfer Paradigm. In R. Case (Ed.), *The mind's staircase: Exploring the conceptual underpinnings of children's thought and knowledge* (pp. 117-132). Hillsdale, NJ: Erlbaum.

Case, R., Stephenson, K. M., Bleiker, C., & Okamoto, Y. (1996). Central spatial structures and their development. In R. Case & Y. Okamoto (Eds.), *The role of central conceptual structures in the development of children's thought* (Vol. 61, pp. 103-130). Chicago, IL: University of Chicago Press.

Catán, L. (1986a). The dynamic display of process: Historical development and contemporary uses of the microgenetic method. *Human Development, 29*, 252-263.

Catán, L. (1986b). *Literacy and cognition: A microgenetic study of the construction and transformation.* Unpublished doctoral dissertation, University of Sussex.

Chen, Z., & Siegler, R. S. (2000). *Across the great divide: Bridging the gap between understanding of toddlers' and older children's thinking* (Vol. 65 (2, Serial No. 261)). Oxford, UK: Blackwell.

Cooper, J., O., Heron, T. E., & Heward, W. L. (2007). *Applied behavior analysis* (2nd ed.). Upper Saddle River, NJ: Pearson Education, Inc.

Crammond, J. (1991). Analyzing the basic cognitive-developmental processes of children with specific types of learning disability. In R. Case (Ed.), *The mind's staircase: Exploring the underpinnings of children's thought and knowledge* (pp. 285-302). Hillsdale, NJ: Lawrence Erlbaum Associates.

Crowley, K., Shrager, J., & Siegler, R. S. (1997). Strategy discovery as a competitive negotiation between metacognitive and associate mechanisms. *Developmental Review, 17*, 462-489.

D'Amato, R. C., & Johnson, J. A. (2004). Review of the Stanford-Binet Intelligence Scales, Fifth Edition. *Mental Measurements Yearbook.*

de Ribaupierre, A. (1993). Structural invariants and individual differences: On the difficulty of dissociating developmental and differential processes. In R. Case & W. Edelstein (Eds.), *The new structuralism in cognitive development: Theory and research on individual pathways* (pp. 11-32). New York, NY: Karger.

de Ribaupierre, A., & Rieben, L. (1995). Individual and situational variability in cognitive development. *Educational Psychologist, 30*(1), 5-14.

Dennis, S. (1991). Stage and structure in the development of children's spatial representations. In R. Case (Ed.), *The mind's staircase: Exploring the underpinnings of children's thought and knowledge* (pp. 285-302). Hillsdale, NJ: Lawrence Erlbaum Associates.

Edelstein, W., Keller, M., & Schroder, E. (1990). Child development and social structure: A longitudinal study of individual differences. In P. B. Baltes, D. L. Featherman & R. M. Lerner (Eds.), *Life-span development and behavior* (Vol. 10, pp. 152-185). Hillsdale, NJ: Erlbaum.

Efklides, A., Demetriou, A., & Gustafsson, J.-E. (1992). Training, cognitive change and individual differences. In A. Demetriou, M. Shayer & A. Efklides (Eds.), *Neo-Piagetian theories of cognitive development* (pp. 122-143). New York, NY: Routledge.

Fischer, K. W. (1980). A theory of cognitive development: The control and construction of hierarchies of skills. *Psychological Review, 87*(6), 477-531.

Fischer, K. W., Ayoub, C., Singh, I., Noam, G., Maraganore, A., & Raya, P. (1997). Psychopathology as adaptive development along distinctive pathways. *Development and Psychopathology, 9*(4), 749-779.

Fischer, K. W., & Canfield, R. L. (1986). The ambiguity of stage and structure in behavior: Person and environment in the development of psychological structures. In I. Levin (Ed.), *Stage and structure: Reopening the debate* (pp. 246-267). Norwood, NJ: Ablex.

Fischer, K. W., Hand, H. H., Watson, M. W., Van Parys, M. M., & Tucker, J. L. (1984). Putting the child into socialization: The development of social categories in preschool children. In L. G. Katz (Ed.), *Current topics in early childhood education* (Vol. V). Norwood, NJ: Ablex Publishing.

Fischer, K. W., & Kennedy, B. P. (1997). Tools for analyzing the many shapes of development: The case of self-in-relationships in Korea. In E. Amsel & K. A. Renninger (Eds.), *Change and development: Issues of theory, method and application*. Mahwah, NJ: Lawrence Erlbaum Associates.

Fischer, K. W., Knight, C. C., & Van Parys, M. (1993). Analyzing diversity in developmental pathways: Methods and concepts. In R. Case & W. Edelstein (Eds.), *The new structuralism in cognitive development: Theory and research on individual pathways* (pp. 33-56). New York, NY: Karger.

Fuson, K. C. (1988). *Children's counting and concepts of number*. New York: Springer-Verlag.

Gelman, R., & Gallistel, C. R. (1978). *The child's understanding of number*. Cambridge, MA: Harvard University Press.

Ginsburg, H. P., Klein, A., & Starkey, P. (1998). The development of children's mathematical thinking: Connecting research with practice. In I. E. Sigel & K. A. Renninger (Eds.), *Child Psychology in Practice* (Fifth ed., Vol. 4, pp. 401-476). New York, NY: John Wiley & Sons, Inc.

Goldin-Meadow, S., & Alibali, M. W. (2002). Looking at the hands through time: A microgenetic perspective on learning and instruction. In N. Granott & J. Parziale (Eds.), *Microdevelopment: Transition processes in development and learning* (pp. 80-105). New York, NY: Cambridge University Press.

Granott, N. (1998). Unit of analysis in transit: From the individual's knowledge to the ensemble process. *Mind, Culture and Activity, 5*(1), 42-66.

Griffin, S. (2004a). Building number sense with Number Worlds: a mathematics program for young children. *Early Childhood Research Quarterly, 19*, 173-180.

Griffin, S. (2004b). Contributions of central conceptual structure theory to education. In A. Demetriou & A. Raftopoulos (Eds.), *Cognitive developmental change: Theories, models and measurement.* Cambridge, UK: Cambridge University Press.

Griffin, S. (2005). Fostering the development of whole-number sense: Teaching mathematics in the primary grades. In M. S. Donovan & J. D. Bransford (Eds.), *How students learn: History, mathematics and science in the classroom* (pp. 257-308). Washington, DC: The National Academies Press.

Griffin, S., & Case, R. (1996). Evaluating the breadth and depth of training effects when central conceptual structures are taught. In R. Case & Y. Okamoto (Eds.), *The role of central conceptual structures in the development of children's thought* (Vol. 61, pp. 83-102). Chicago, Illinois: University of Chicago Press.

Griffin, S., & Case, R. (1997). Rethinking the primary school math curriculum: An approach based on cognitive science. *Issues in Education, 3*, 1-65.

Griffin, S., Case, R., & Capodilupo, A. (1995). Teaching for Understanding: The Importance of the Central Conceptual Structures in the Elementary Mathematics Curriculum. In A. McKeough, J. Lupart & A. Marini (Eds.), *Teaching for Transfer: Fostering Generalization in Learning* (pp. 123-151). Mahwah, New Jersey: Lawrence Erlbaum Associates, Inc.

Griffin, S., Case, R., & Carpenter, P. (1992). *Rightstart: A foundational math program for kindergarten.*

Griffin, S., Case, R., & Sandieson, R. (1991). Synchrony and asynchrony in the acquisition of children's everyday mathematical knowledge. In R. Case (Ed.), *The mind's staircase: Exploring the conceptual underpinnings of children's thought and knowledge.* (pp. 75-97). Hillsdale, NJ: Lawrence Erlbaum Associates, Inc.

Griffin, S., Case, R., & Siegler, R. S. (1994). Rightstart: Providing the central conceptual structures for children at risk of school failure. In K. McGilly (Ed.), *Classroom lessons: Integrating cognitive theory and classroom practice* (pp. 25-49). Cambridge, MA: MIT Press.

Harrison, P. L. (1990). *Early Screening Profiles: Manual.* Circle Pines, MN: American Guidance Service.

Hawkins, R. P. (1982). Developing a behavior code. In D. P. Hartmann (Ed.), *Using observers to study behavior*. San Francisco: Jossey-Bass.

Hollos, M., & Cowan, P. A. (1973). Social isolation and cognitive development: Logical operations and role taking abilities in three Norwegian social settings. *Child Development, 44*, 630-641.

Hoppe-Graff, S. (1993). Individual differences in the emergence of pretend play. In R. Case & W. Edelstein (Eds.), *The new structuralism in cognitive development: Theory and research on individual pathways* (pp. 57-70). New York, NY: Karger.

Horowitz, F. D., & O'Brien, M. (1986). Gifted and talented children. *American Psychologist, 41*(10), 1147-1152.

Inhelder, B. (1943). *The diagnosis of reasoning in the mentally retarded* (W. Stephens, Trans.). New York, NY: The John Day Company, Inc.

Inhelder, B., Sinclair, H., & Bovet, M. (1974). *Learning and the development of cognition* (S. Wedgwood, Trans.). Cambridge, MA: Harvard University Press.

Jackson, N. E. (1993). Case's neo-Piagetian theory: A work in progress, *Contemporary Psychology* (Vol. 38, pp. 902-903).

Kail, R., & Bisanz, J. (1992). The information-processing perspective on cognitive development in childhood and adolescence. In R. J. Sternberg & C. A. Berg (Eds.), *Intellectual development* (pp. 229-260). Cambridge, UK: Cambridge University Press.

Kalchman, M., & Case, R. (1998). Teaching mathematical functions in primary and middle school: An approach based on Neo-Piagetian theory. *Scientia Pedagogica Experimentalis, 35*(1), 7-53.

Kalchman, M., Moss, J., & Case, R. (2001). Psychological models for the development of mathematical understanding: Rational numbers and functions. In S. M. Carver & D. Klahr (Eds.), *Cognition and instruction: Twenty five years of progress* (pp. 1-38). Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Kaufman, A. S., & Kaufman, N. L. (1983). *Kaufman Assessment Battery for Children.* Circle Pines, MN: American Guidance Service.

Keating, D. P. (1991). Central conceptual structures: Seeking developmental integration. In R. Case (Ed.), *The mind's staircase: Exploring the conceptual underpinnings of children's thought and knowledge* (pp. 276-282). Hillsdale, NJ: Lawrence Erlbaum Associates.

Kennedy, C. H. (2005). *Single-Case Designs for Educational Research.* Boston: Allyn & Bacon.

Klahr, D. (1989). Information processing approaches. *Annals of Child Development, 6,* 133-185.

Knight, C. C., & Fischer, K. W. (1992). Learning to read words: Individual differences in developmental sequences. *Journal of Applied Developmental Psychology, 13,* 377-404.

Larivée, S., Normandeau, S., & Parent, S. (2000). The French connection: Some contributions of French-language research in the post-Piagetian era. *Child Development, 71*(4), 823-839.

Lee, K., & Karmiloff-Smith, A. (2002). Macro- and microdevelopmental research: Assumptions, research strategies, constraints, and utilities. In N. Granott & J. Parziale (Eds.), *Microdevelopment: Transition processes in development and learning* (pp. 243-292). New York, NY: Cambridge University Press.

Marini, Z. (1984). *The development of social and physical cognition in childhood and adolescence.* Unpublished doctoral dissertation, University of Toronto (OISE).

Marini, Z. (1991). Synchrony and asynchrony in the development of children's scientific reasoning. In R. Case (Ed.), *The mind's staircase: Exploring the conceptual underpinnings of children's thought and knowledge.* (pp. 55-73). Hillsdale, NJ: Lawrence Erlbaum Associates, Inc.

Masten, A. S., & Coatsworth, D. J. (1998). The development of competence in favorable and unfavorable environments: Lessons from research on successful children. *American Psychologist, 53*(2), 205-220.

McClelland, J. L. (1995). A connectionist perspective on knowledge and development. In T. J. Simon & G. S. Halford (Eds.), *Developing cognitive competence: New approaches to process modeling.* Hillsdale, NJ: Lawrence Erlbaum Associates, Publishers.

McKeough, A. (1991). A neo-structural analysis of children's spatial representations. In R. Case (Ed.), *The mind's staircase: Exploring the conceptual underpinnings of children's thought and knowledge* (pp. 171-188). Hillsdale, NJ: Lawrence Erlbaum.

McKeough, A., Davis, L., Forgeron, N., Marini, A., & Fung, T. (2005). Improving story complexity and cohesion: A developmental approach to teaching story composition. *Narrative Inquiry, 15*(2), 241-266.

McKeough, A., & Sanderson, A. (1996). Teaching storytelling: A microgenetic analysis of developing narrative competency. *Journal of Narrative and Life History, 6*(2), 157-192.

Merriam, S. B. (1998). *Qualitative research and case study applications in education.* San Francisco: Jossey-Bass Publishers.

Miles, M. B., & Huberman, A. M. (1994). *Qualitative data analysis: An expanded sourcebook (2nd ed.).* Thousand Oaks, CA: Sage Publications.

Mix, K. S., Huttenlocher, J., & Levine, S. C. (2002). *Quantitative development in infancy and early childhood.* Oxford, UK: Oxford University Press.

Montague-Smith, A. (1997). *Mathematics in nursery education.* London, UK: David Fulton Publishers.

Moss, J., & Case, R. (1999). Developing children's understanding of the rational numbers: A new model and an experimental curriculum. *Journal for Research in Mathematics Education, 30*(2), 122-147.

O'Donnell, A. M., Reeve, J., & Smith, J. K. (1997). *Educational psychology: Reflection for action.* Hoboken, NJ: John Wiley & Sons.

Okamoto, Y., & Case, R. (1996). Exploring the microstructure of children's central conceptual structures in the domain of number. In R. Case & Y. Okamoto (Eds.), *The role of central conceptual structures in the development of children's thought* (Vol. 61, pp. 27-58).

Okamoto, Y., Case, R., Bleiker, C., & Henderson, B. (1996). Cross-cultural investigations. In R. Case & Y. Okamoto (Eds.), *The role of central conceptual structures in the development of children's thought* (Vol. 61, pp. 131-155). Chicago, Illinois: University of Chicago Press.

Parziale, J. (1997). *Microdevelopment during an activity based science lesson.* Unpublished Doctoral dissertation, Harvard University, Cambridge, MA.

Parziale, J. (2002). Observing the dynamics of construction: Children building bridges and new ideas. In N. Granott & J. Parziale (Eds.), *Microdevelopment: Transition processes in development and learning* (pp. 157-180). New York, NY: Cambridge University Press.

Pascual-Leone, J. (1970). A mathematical model for the transition role in Piaget's developmental stages. *Acta Psychologica, 32*, 301-345.

Pascual-Leone, J. (1989). An organismic process model of Witkin's field-dependence-independence. In T. Globerson & T. Zelniker (Eds.), *Cognitive Style and Cognitive Development, Human Development* (Vol. 3, pp. 36-70). Norwood, NJ: Ablex Publishing Corporation.

Piaget, J. (1952). *The origins of intelligence in children.* New York: W. W. Norton & Company Inc.

Piaget, J. (1964). *Development and learning.* Paper presented at the Piaget rediscovered: A report on the Conference on Cognitive Studies and Curriculum Development, Ithaca, NY.

Piaget, J. (1983). Piaget's theory. In W. Kessen (Ed.), *History, methods, and methods* (Fourth ed., Vol. 1, pp. 103-028). New York, NY: John Wiley and Sons.

Porath, M. (1988). *Cognitive development of gifted children: A neo-Piagetian perspective.* Unpublished doctoral dissertation, University of Toronto (OISE).

Porath, M. (1991a). Educational implications of a developmental perspective on intellectual giftedness. *Exceptionality Education Canada, 1*(4), 61-76.

Porath, M. (1991b). Stage and structure in the development of children with various types of "giftedness". In R. Case (Ed.), *The mind's staircase: Exploring the underpinnings of children's thought and knowledge.* Hillsdale, NJ: Lawrence Erlbaum Associates, Publishers.

Porath, M. (1993). Gifted young artists: Developmental and individual differences. *Roeper Review, 16*, 29-33.

Porath, M. (1996a). Affective and motivational considerations in the assessment of gifted learners. *Roeper Review, 19*(1), 13-17.

Porath, M. (1996b). Narrative performance in verbally gifted children. *Journal for the Education of the Gifted, 19*, 273-292.

Resnick, L. B. (1983). A Developmental Theory of Number Understanding. In H. Ginsburg (Ed.), *The Development of mathematical thinking* (pp. 109-151). New York: Academic Press.

Resnick, L. B. (1989). Developing mathematical knowledge. *American Psychologist, 44*(2), 162-169.

Resnick, L. B., Wang, M. C., & Kaplan, J. (1973). Task analysis in curriculum design: A hierarchically sequenced introductory mathematics curriculum. *Journal of Applied Behavior Analysis, 6*(4), 679-710.

Rittle-Johnson, B., Siegler, R. S., & Alibali, M. W. (2001). Developing conceptual understanding and procedural skill in mathematics: An iterative process. *Journal of Educational Psychology, 96*(2), 346-362.

Robinson, N. M., Abbott, R. D., Berninger, V. W., & Busse, J. (1996). The structure of abilities in math-precocious young children: Gender similarities and differences. *Journal of Educational Psychology, 88*(2), 341-352.

Robinson, N. M., Abbott, R. D., Berninger, V. W., Busse, J., & Mukhopadhyay, S. (1997). Developmental changes in mathematically precocious young children: Longitudinal and gender effects. *Gifted Child Quarterly, 41*, 145-158.

Roid, G. H. (2003a). *Stanford-Binet Intelligence Scales, Fifth Edition, Examiner's Manual.* Itasca, IL: Riverside Publishing.

Roid, G. H. (2003b). *Stanford-Binet Intelligence Scales, Fifth Edition, Technical Manual.* Itasca, IL: Riverside Publishing.

Rusch, F. R., Rose, T., & Greenwood, C. R. (1988). *Introduction to behavior analysis in special education.* Englewood Cliffs, NJ: Prentice-Hall.

Schaeffer, B., Eggleston, V. H., & Scott, J. L. (1974). Number development in young children. *Cognitive Psychology, 6*(3), 357-379.

Shrager, J., & Siegler, R. S. (1998). SCADS: A model of children's strategy choices and strategy discoveries. *Psychological Science, 9*(5), 405-410.

Shultz, T. R., & Schmidt, W. C. (1991). *A cascade-correlation model of balance scale phenomena.* Paper presented at the Thirteenth Annual Conference of the Cognitive Science Society, Chicago, IL.

Siegler, R. S. (1996a). *Emerging minds: The process of change in children's thinking.* New York, NY: Oxford University Press.

Siegler, R. S. (1996b). A grand theory of development. In R. Case & Y. Okamoto (Eds.), *The role of central conceptual structures in the development of children's thought* (Serial No. 246 ed., Vol. 61, pp. 266-275). Chicago: University of Chicago Press.

Siegler, R. S. (1997). Concepts and methods for studying cognitive change. In E. Amsel & K. A. Renninger (Eds.), *Change and development: Issues of theory, method, and application* (pp. 77-97).

Siegler, R. S., & Crowley, K. (1991). The microgenetic method: A direct means for studying cognitive development. *American Psychologist, 46*(6), 606-620.

Siegler, R. S., & Jenkins, E. (1989). *How children discover new strategies*. Hillsdale, NY: Lawrence Erlbaum Associates, Inc.

Siegler, R. S., & Stern, E. (1998). Conscious and unconscious strategy discoveries: A microgenetic analysis. *Journal of Experimental Psychology: General, 127*(4), 377-397.

Smith, L. D., Best, L. A., Stubbs, D. A., Archibald, A. B., & Roberson-Nay, R. (2002). Constructing knowledge: The role of graphs and tables in hard and soft psychology. *American Psychologist, 57*(10), 749-761.

van Geert, P. (1994). *Dynamic systems of development: Change between complexity and chaos*. New York, NY: Harvester.

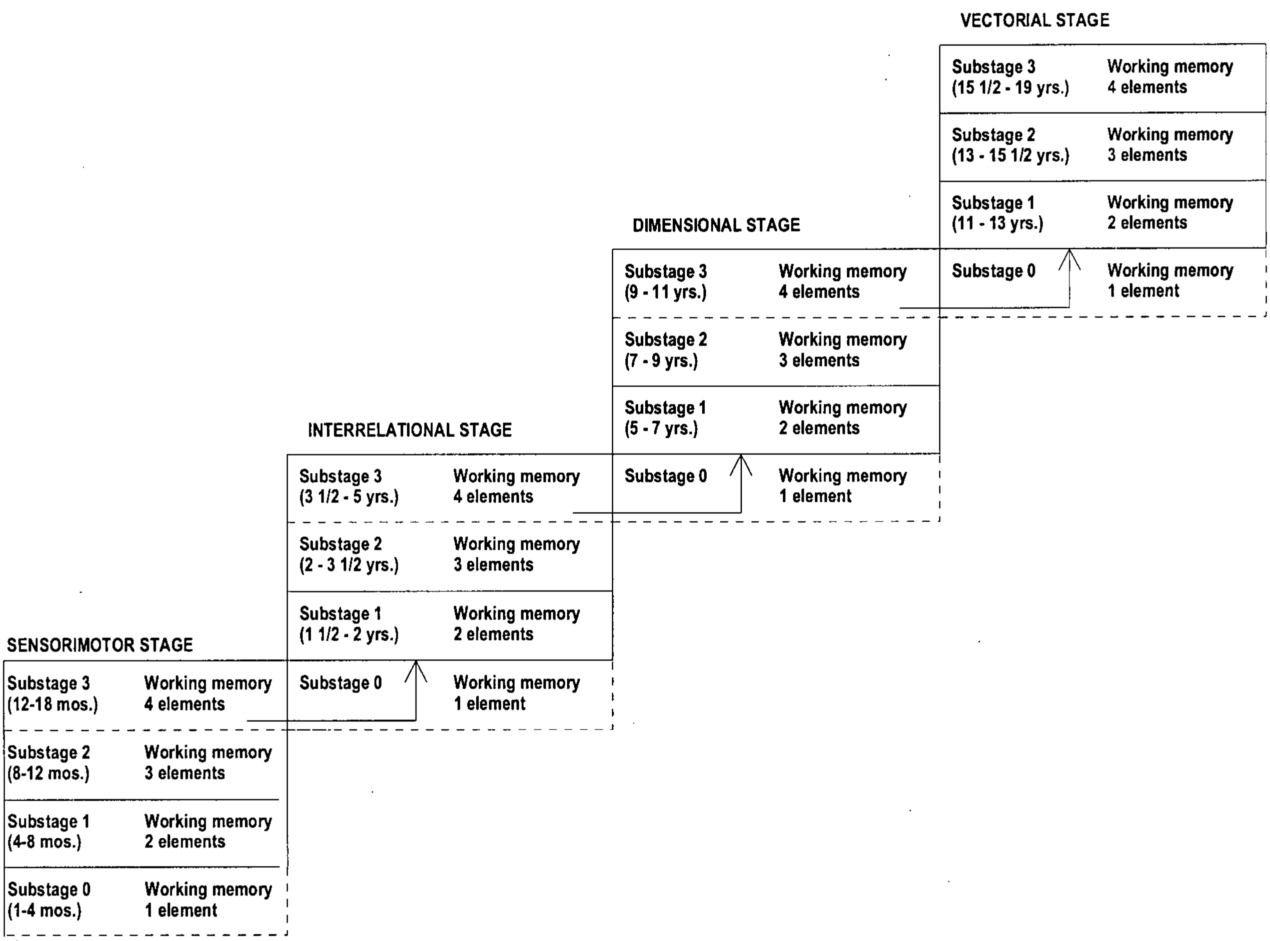
Waxman, B., Robinson, N. M., & Mukhopadhyay, S. (1996). *Teachers nurturing math: Talented young children*. Storrs, CT: National Research Center on the Gifted and Talented.

Wechsler, D. (1989). *Wechsler Preschool and Primary Scale of intelligence - Revised.* San Antonio, TX: The Psychological Corporation.

Yin, R. K. (1994). *Case study research: Design and methods* (Second ed.). Thousand Oaks, CA: Sage Publications Inc.

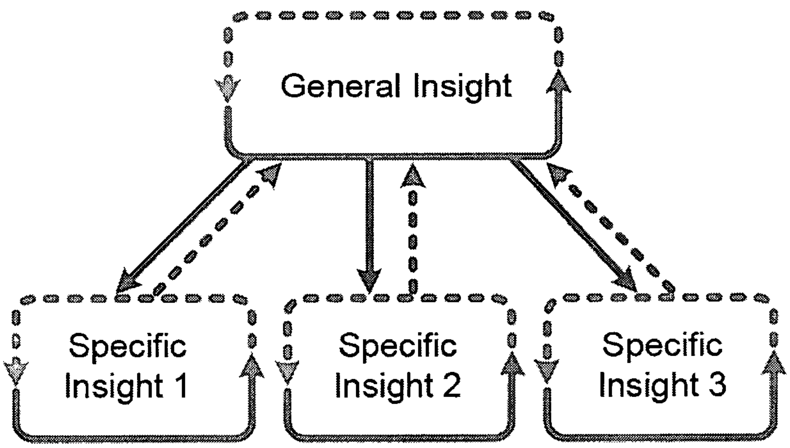
Appendix A
Stages, Substages and Corresponding Working Memory Capacity Described in Case's (1991c; 1996a; 1998a) Neo-Piagetian Theory of Cognitive Development

The diagram below is based on the diagram in Case (1996d, p. 201).

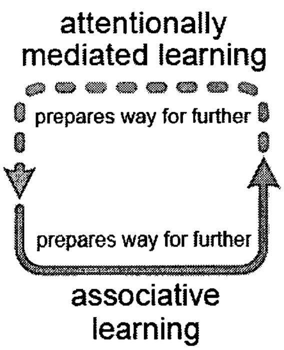


Appendix B
Hierarchical Learning Loop

The diagram below is based on the diagram in Case (1996a, p. 21).



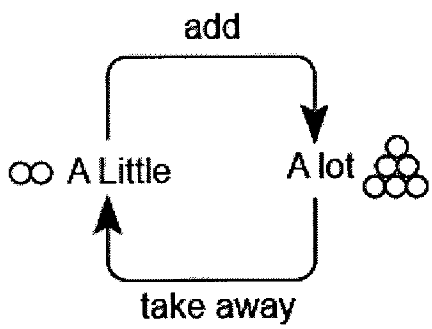
Iterative learning loop



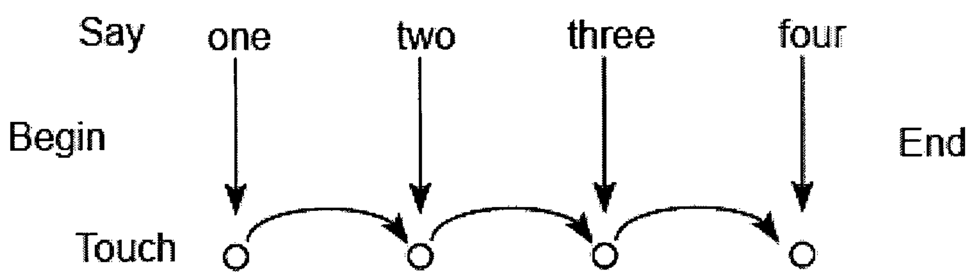
Appendix C
The Counting Schema and Global Quantity Schema

The diagrams below are based on the diagrams in Case (1996a, p. 6).

Global Quantity Schema

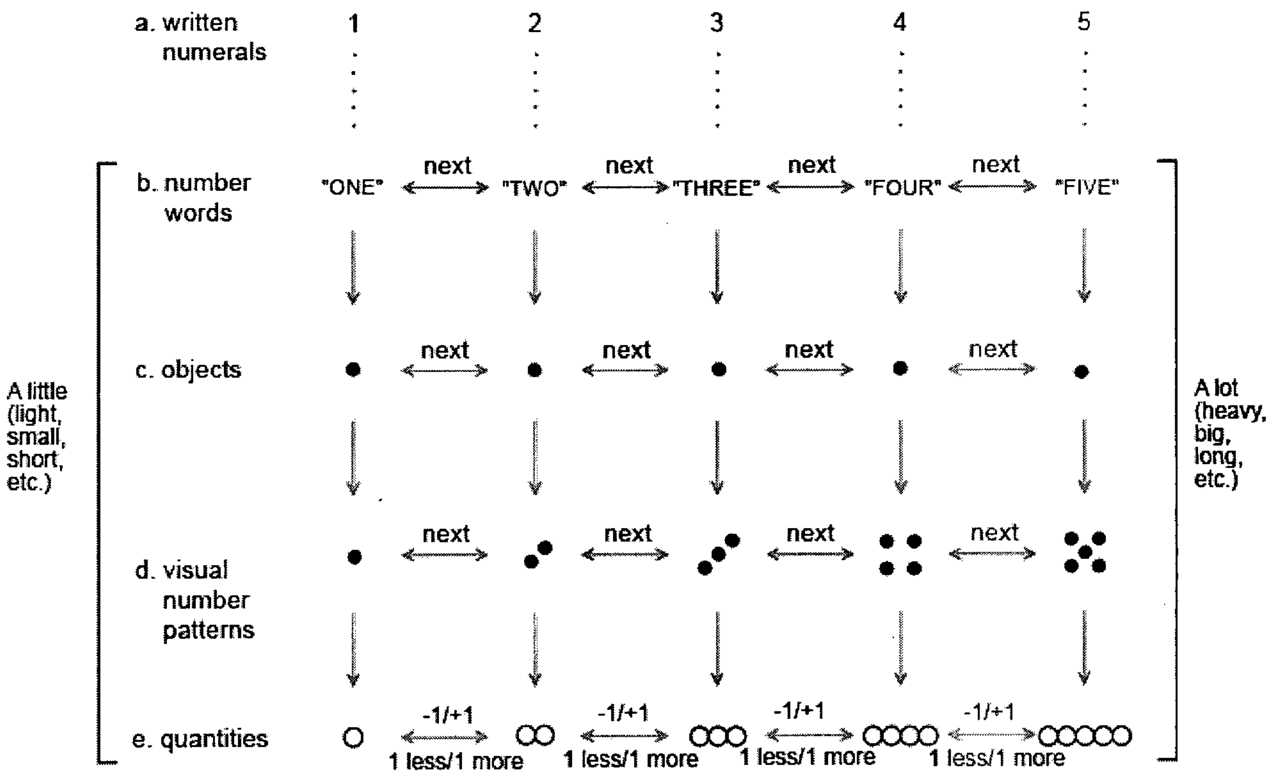


Counting Schema



Appendix D
Mental Counting Line

The diagram below is based on the diagram in Griffin and Case (1996, p. 84).



date the second copy and retain this copy for your own records. Your signature indicates that you have given your consent for your child's participation in this study and that you have received a copy of this consent form for your own records. Thank you for your interest and cooperation.

I, ______________________, parent or guardian of______________________,

☐ do/ ☐ do not consent to allow my child to participate in this study. I acknowledge that I have received a copy of this consent form.

Signature:________________________________ Date: _________________

Appendix F
Script for Children's Assent
Individual Pathways in the Development of Young Children's Mathematical Understanding

I am studying at the University of British Columbia. I am really interested in how children your age learn about numbers. If you would like to, we will play some number games together and I will ask you some questions about numbers.

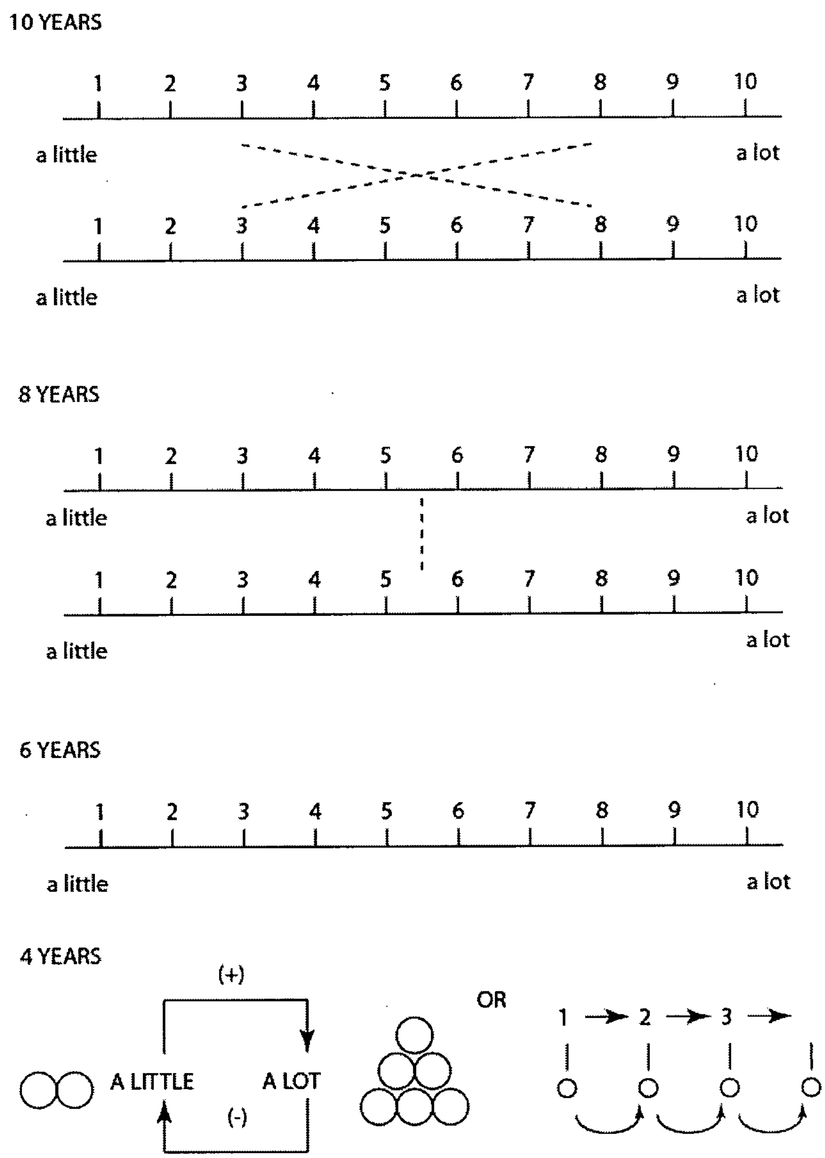
Would you like to be in this study? (Continue if the child says yes.)

You do not have to be in the study if you don't want to. You can stop being in the study any time after we start. No one will get mad at you if you want to stop.

Do you have any questions?

Appendix G
Complexity of Children's Mathematical understanding at Each Level of the Measures of Conceptual Understanding

The diagram below is based on the diagram in Case (1996d, p. 192).



Appendix H
Number Knowledge Test

Number Knowledge Test: Problems

The problems for each of the four levels of the Number Knowledge test (Griffin & Case, 1997).

Dimensional Stage

Preliminary Level

Let's see if you can count from 1 to 10. Go ahead.

Level 1 (4-year-old level)[84]: Go to Level 1 if 3 or more correct

1. (Show 5 unordered chips.) Would you count these for me?
2. I'm going to show you some counting chips (show mixed array of 3 red and 4 blue chips). Count just the blue chips and tell me how many there are.
3. Here are some circles and triangles (show mixed array of 7 circles and 8 triangles). Count just the triangles and tell me how many there are.
4. Pretend I'm going to give you 2 pieces of candy and then I'm going to give you 1 more (do so). How many will you have altogether?

Level 2 (6-year-old level): Go to Level 2 if 5 or more correct

1. If you had 4 chocolates and someone gave you 3 more, how many chocolates would you have altogether?
2. What number comes right after 7?
3. What number comes two numbers after 7? (Accept either 9 or 10)

4a. Which is bigger, 5 or 4?

4b. Which is bigger, 7 or 9?

5a. (This time, I'm going to ask you about smaller numbers.) Which is smaller, 8 or 6?

5b. Which is smaller, 5 or 7?

6a. (Show visual array on 8"x 8" card.) Which number is closer to 5, 6 or 2?

6b. (Show visual array on 8" x 8" card.) Which number is closer to 7, 4 or 9?

[84] The understanding that is described at each level is acquired during a two-year period. The ages assigned to each level represent the midpoint of this two-year period (Griffin, 2005).

7. How much is 2+4? (OK to use fingers for counting)
8. How much is 8 take away 6 (OK to use fingers for counting)

9a. (Show visual array -8 5 2 6- on 8" x 8" card. Ask child to point to and name each numeral.) When you are counting, which of these numbers do you say first?

9b. When you are counting, which of these numbers do you say last?

Level 3 (8-year-old level): Go to Level 3 if 5 or more correct

1. What number comes 5 numbers after 49? (Accept either 54 or 55)
2. What number comes 4 numbers before 60? (Accept either 56 or 55)

3a. Which is bigger, 69 or 71?

3b. Which is bigger, 32 or 28?

4a. (This time I'm going to ask you about smaller numbers.) Which is smaller, 27 or 32?

4b. Which is smaller, 51 or 39?

5a. (Show visual array on 8" x 8" card.) Which number is closer to 21, 25 or 18?

5b. (Show visual array on 8" x 8" card.) Which number is closer to 28, 31 or 24?

6. How many numbers are there in between 2 and 6? (Accept either 3 or 4)
7. How many numbers are there in between 7 and 9? (Accept either 1 or 2)
8. (Show 12 and 54 on 8" x 8" card.) How much is 12+54? (No credit if number increased by one with fingers.)
9. (Show 47 and 21 on 8" x 8" card.) How much is 47 take away 21? (No credit if number decreased by one with fingers.)

Level 4 (10-year-old level): Go to Level 4 if 4 or more correct

1. What number comes 10 numbers after 99?
2. What number comes 9 numbers after 999?

3a. Which difference is bigger, the difference between 9 and 6 or the difference between 8 and 3?

3b. Which difference is bigger, the difference between 6 and 2 or the difference between 8 and 5?

4a. Which difference is smaller, the difference between 99 and 92 or the difference between 25 and 11?

4b. Which difference is smaller, the difference between 48 and 36 or the difference between 84 and 73?

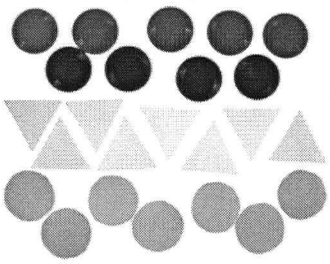
5. (Show 13 and 39 on 8" x 8" card.) How much is 13 + 39?

6. (Show 36 and 18 on 8" x 8" card.) How much is 36 – 18?
7. How much is 301 take away 7?

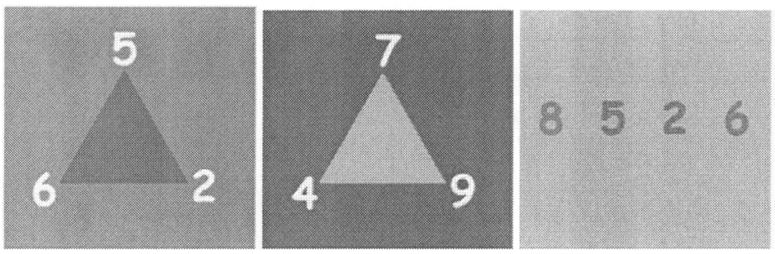
Number Knowledge Test: Materials

The materials for Levels 1, 2, and 3 of the Number Knowledge test (Griffin & Case, 1997).

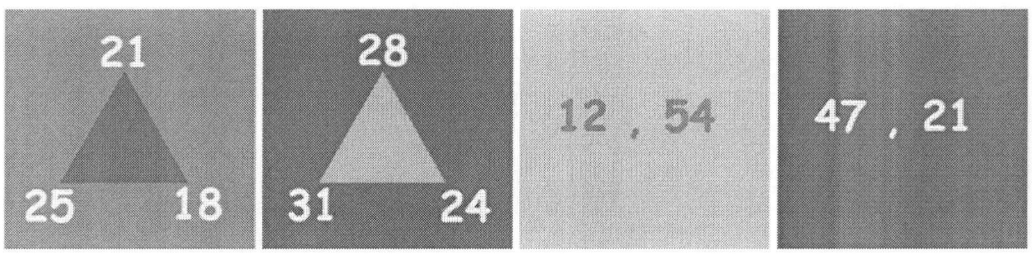
Level 1 (4-year-old level)



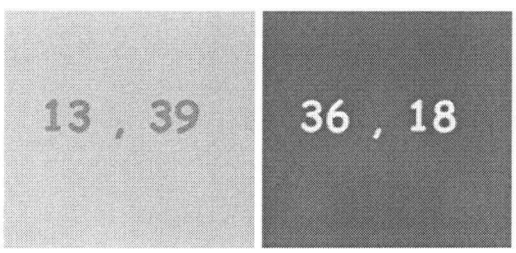
Level 2 (6-year-old level)



Level 3 (8-year-old level)



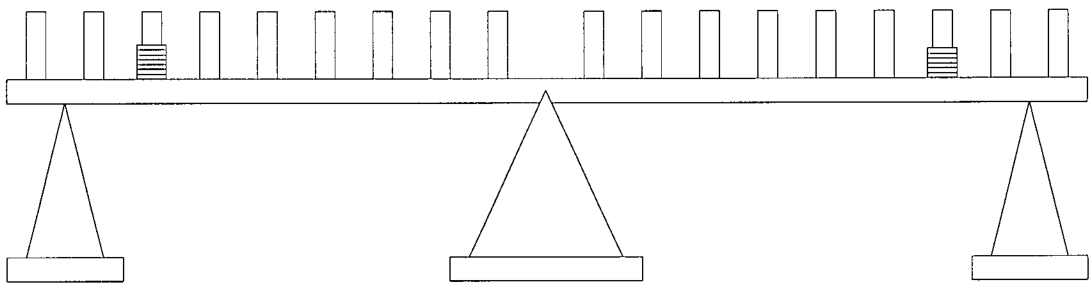
Level 4 (10-year-old level)



Appendix I
Balance Beam Task

Balance Beam Task: Apparatus

The balance beam apparatus (Marini, 1991).



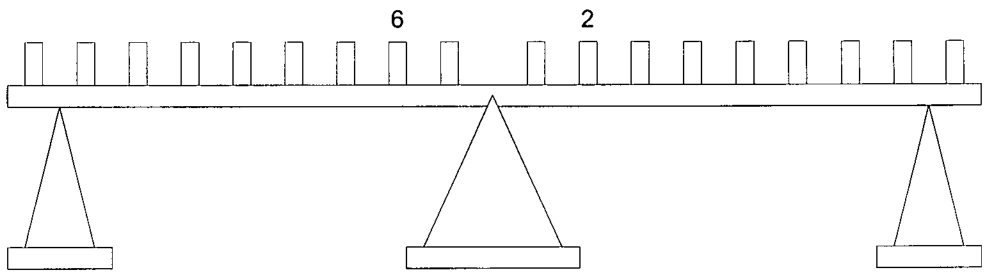
Balance Beam Task: Problems

The problems for each of the four levels of the Balance Beam task (Marini, 1991).

Dimensional Stage

Level 1: Predimensional Level (3½-5 years)

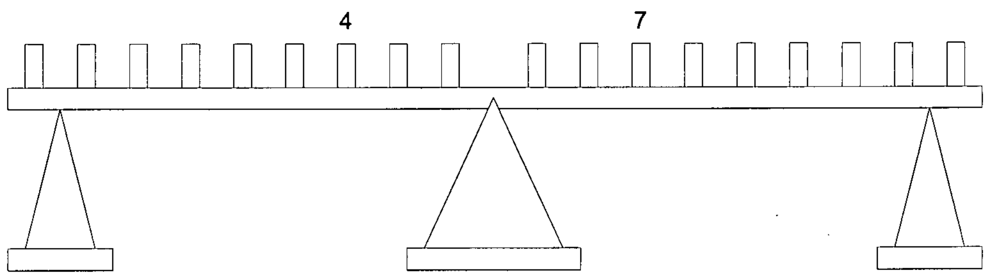
1.



Answer ______________________________

Why? ______________________________

2.

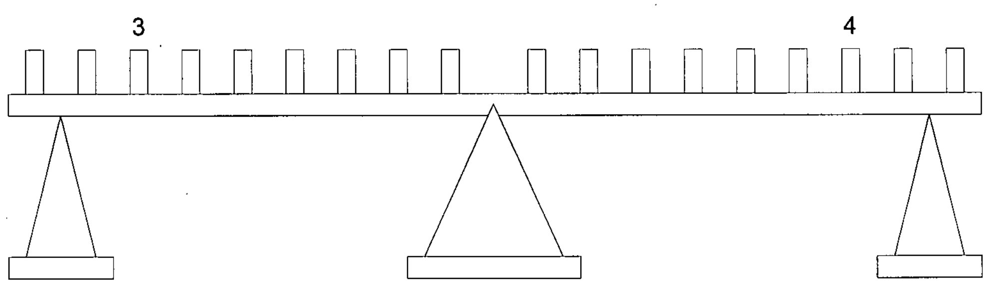


Answer ______________________________

Why? ______________________________

Level 2: Unidimensional Level (5-7 years)

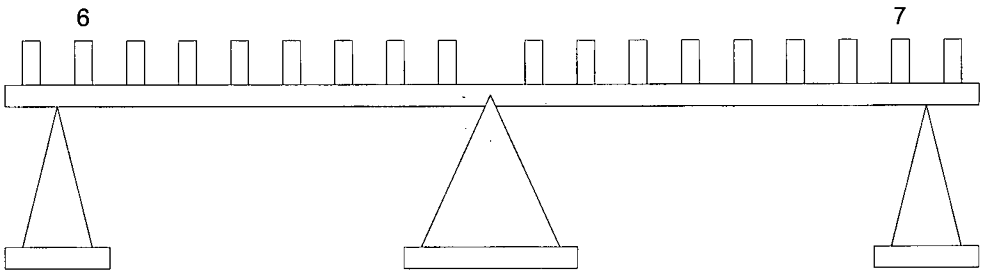
1.



Answer ______________________________

Why? ______________________________

2.

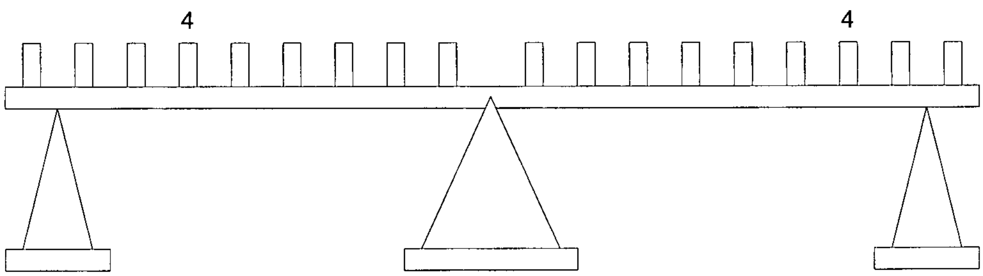


Answer ______________________________

Why? ______________________________

Level 3: Bidimensional Level (7-9 years)

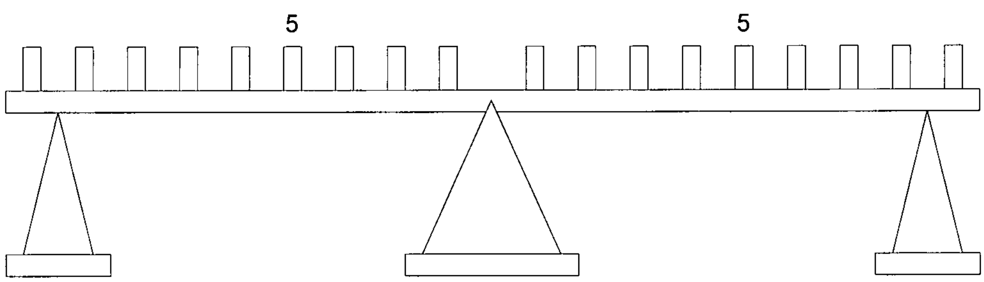
1.



Answer ______________________________

Why? ______________________________

2.

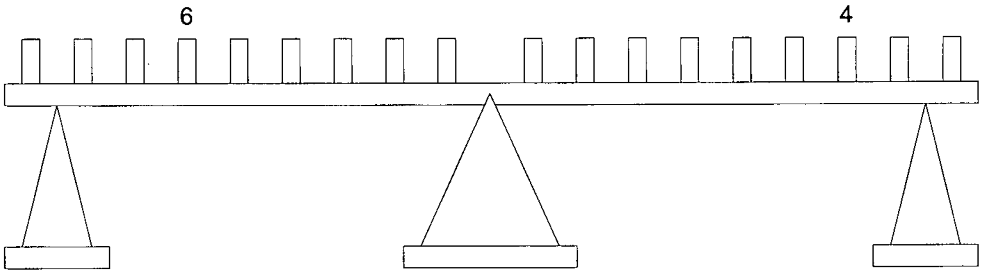


Answer ______________________________

Why? ______________________________

Level 4: Integrated Bidimensional Level (9-11 years)

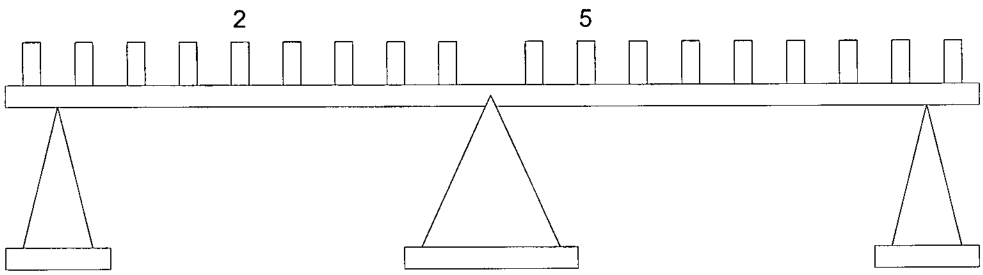
1.



Answer ____________________

Why? ____________________

2.



Answer ____________________

Why? ____________________

Balance Beam Task: Scoring Criteria

The scoring criteria for each of the four levels of the Balance Beam task (Case, Okamoto et al., 1996; Marini, 1991).

Dimensional Stage

Level 1: Predimensional Level (3½-5 years)

The child estimates the size of the stack of weights on each side of the beam (makes a global estimate of the size of each stack of weights), compares the results and says that the side with the bigger or heavier stack will go down (Case, Okamoto et al., 1996; Marini, 1991). For example: "It has more" (Case, Okamoto et al., 1996, p. 217).

Level 2: Unidimensional Level (5-7 years)

The child counts the weights in each stack, compares the resulting numbers and says that the side with the stack having the most weights will go down. The response must show that the child counted the washers in each stack (Case, Okamoto et al., 1996; Marini, 1991). For example: "This side will go down because it has four,"; "Because it has four and the other one has three" (Case, Okamoto et al., 1996, p. 217)

Level 3: Bidimensional Level (7-9 years)

The child counts the weights in each stack, counts the number of spaces each stack is from the fulcrum, compares the results and says that the side with the stack farthest from the fulcrum will go down. The response must show that the child considered both the dimensions of weight and distance (Case, Okamoto et al., 1996; Marini, 1991). For example: "Because these weights are further from the end"; "Because these ones are on the third peg" " (Case, Okamoto et al., 1996, p. 217).

Level 4: Integrated Bidimensional Level (9-11 years)

The child considers both the dimensions of weight and distance and uses an addition or subtraction procedure in order to effect a compensation between weight and distance. The response must show that the child used and addition or subtraction strategy to obtain the answer (Case, Okamoto et al., 1996; Marini, 1991). For example: "This side because it has two more weights than the other side but is only one farther out"; "This side because it is seven out and three up, so it is 10; the other side is four out and five up, which is only nine" (Case, Okamoto et al., 1996, p. 217).

Addition Strategy: The child adds the weight and distance values on the left side of the beam, adds the weight and distance values on the right side of the beam, compares the sums and says that the side with the larger sum will go down.

Subtraction Strategy: The child subtracts the smaller weight from the larger weight, subtracts the smaller distance from the larger distance, compares the differences and says, if the difference between the distances is greater the side with the stack farthest from the fulcrum will go down; or if the difference between the weights is greater the side with the larger stack will go down.

Appendix J
Money Knowledge Task

Money Knowledge Task: Problems

The problems for each of the four levels of the Money Knowledge task (Case, Okamoto et al., 1996).

Dimensional Stage

Level 1: Predimensional Level (3½-5 years)

1a. Which is worth more, a dollar or a penny?

1b. Which is worth less, a dollar or a penny?

2a. Does a car cost a lot or a little?

2b. Does a piece of gum cost a lot or a little?

3. I'm going to give you 1 penny [do so], and then I'm going to give you 3 more [do so]. How many pennies do you have altogether?

4. Here's one bunch of pennies [show 2 pennies], and here's another bunch [show 8 pennies]. Which bunch is worth more?

5. Here's one set of dollars [show 5 loonies], and here's another set [show 2 loonies]. Which is worth more?

Level 2: Unidimensional Level (5-7 years)

1. Now I'm going to show you some more money [show a \$5.00 bill, a loonie and a toonie]. which is worth the most?

2. If I give you this [show a \$5.00 bill] and this [show 2 loonies], how much money did I give you altogether?

3. Suppose you go to a store to buy a candy and you want to buy this candy [show index card with real piece of candy taped to it]. This candy costs 5 cents, but you look in your pocket, and you only have 4 cents. How much more money do you need to buy the candy?

4. This time you want to buy this candy [show index card with real piece of candy taped to it]. This candy costs 7 cents, and you give 10 cents. How much do you get in change?

5a. [show a dime and a nickel]. Which is worth more?

5b. [show a $5.00 bill and 2 loonies]. Which is worth more?

Level 3: Bidimensional Level (7-9 years)

1. If I give you a dime, and then I give you 6 more [show no objects for this item]. How much have I given you altogether?
2. [show a $5.00 bill with 1 cent and a loonie with approximately 20 cents]. Which is worth more?
3. Suppose you want to buy this bike. The price tag shows $45.00 [show bike picture]. You count your money, and you have 3 $20.00 bills. How much money do you get in change?

4a. Which is closer to 8 cents, a nickel or a dime?

4b. Which is closer to 19 cents, a quarter or a dime?

Level 4: Integrated Bidimensional Level (9-11 years)

1. If I give you 2 quarters, and then I give you 4 quarters, how much is it worth altogether? How many cents have I given you?

2a. [show visual array] Which is closer to $25.35, $20.00 or $30.00?

2b. [show visual array] Which is closer to $46.45, $46.00 or $47.00?

3a. [show visual array] Which is closer to $40.00, $29.95 or $61.05?

3b. [show visual array] Which is closer to $15.00, $9.95 or $19.95?

4a. [show 2 groups of coins] Suppose you have a quarter and a dime and I have 4 dimes. Who has more money, you [child] or me [tester]?

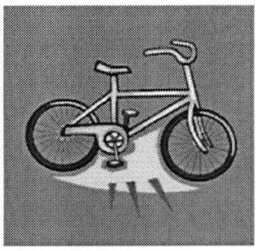
4b. [show 2 groups of coins] Suppose you have 3 quarters and I have 5 dimes and 2 nickels. Who has more money, you [child] or me [tester]?

5. Your hot lunch cost $3.45, and you gave a $20.00 bill and 2 quarters. How much change should you receive?

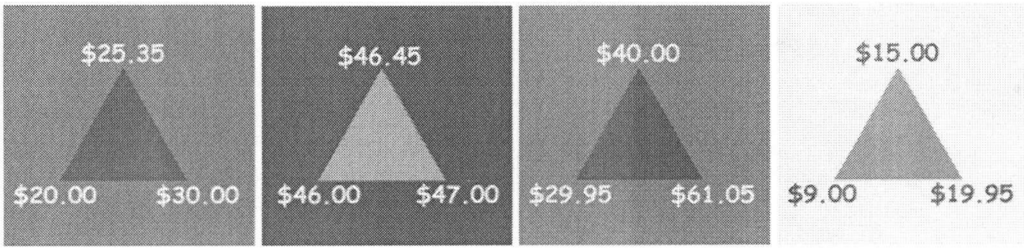
Money Knowledge Task: Materials

The materials for Levels 3 and 4 of the Money Knowledge task.

Level 3: Bidimensional Level (7-9 years)



Level 4: Integrated Bidimensional Level (9-11 years)



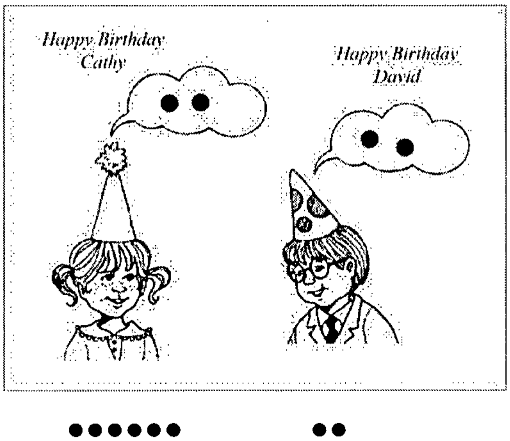
Appendix K
Birthday Party Task

Birthday Party Task: Materials

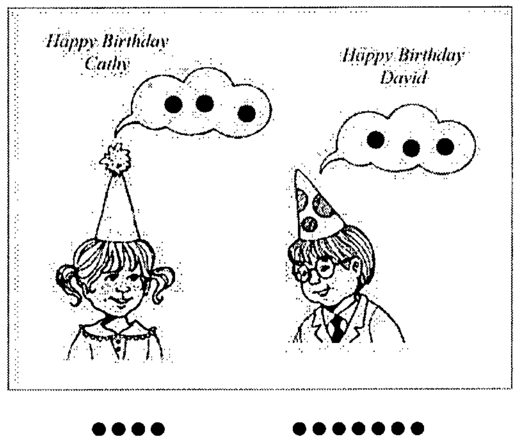
The materials for each of the four levels of the Birthday Party task (Marini, 1984).

Level 1: Predimensional Level (3½-5 years)

1.

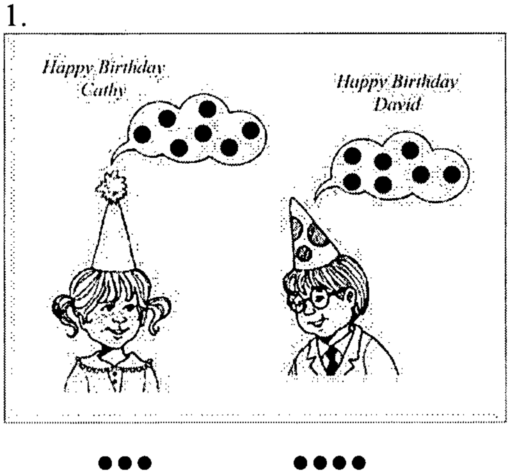


2.

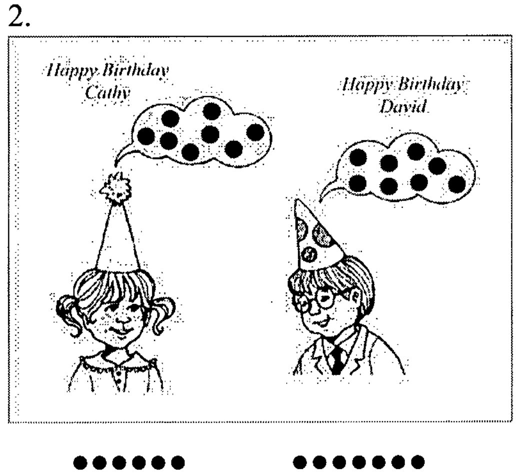


Level 2: Unidimensional Level (5-7 years)

1.

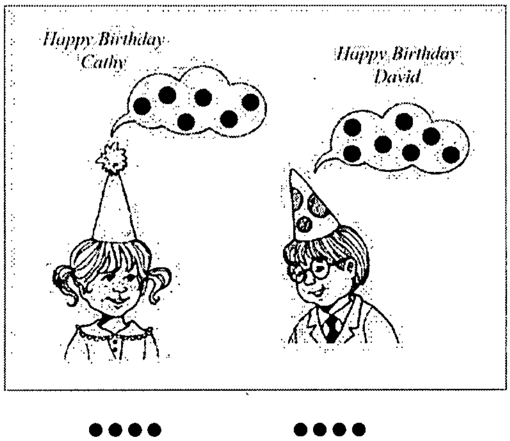


2.

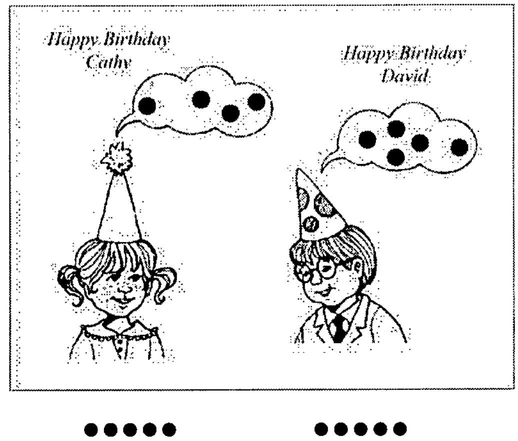


Level 3: Bidimensional Level (7-9 years) - score

1.

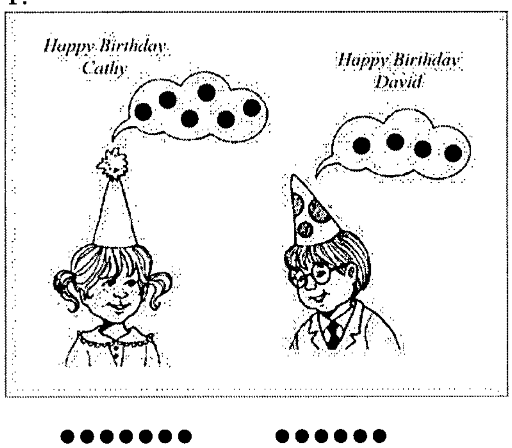


2.

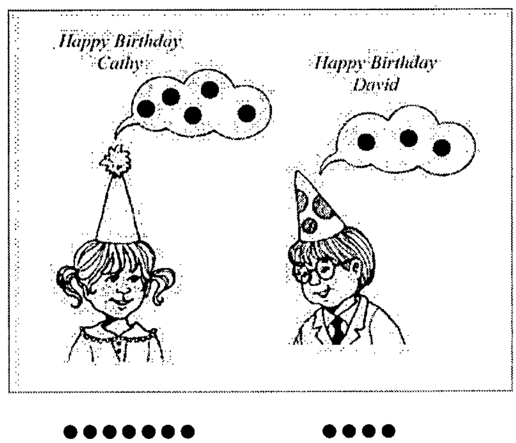


Level 4: Integrated Bidimensional Level (9-11 years)

1.



2.



Birthday Party Task: Problems

The problems for each of the four levels of the Birthday Party task (Marini, 1984).

Dimensional Stage

Level 1: Predimensional Level (3½-5 years)

1.

Cathy		David	
Wants	Gets	Wants	Gets
2	6	2	2

Who is happier?

Cathy is happier. David is happier. Cathy & David are both happy

Why? ______________________________

____

____

2.

Cathy		David	
Wants	Gets	Wants	Gets
3	4	3	7

Who is happier?

Cathy is happier. David is happier. Cathy & David are both happy

Why? ______________________________

____

____

Level 2: Unidimensional Level (5-7 years)

1.

Cathy		David	
Wants	Gets	Wants	Gets
7	3	7	4

Who is happier?

Cathy is happier. David is happier. Cathy & David are both happy

Why? ____

2.

Cathy		David	
Wants	Gets	Wants	Gets
8	6	8	7

Who is happier?

Cathy is happier. David is happier. Cathy & David are both happy

Why? ____

Level 3: Bidimensional Level (7-9 years) - score

1.

Cathy		David	
Wants	Gets	Wants	Gets
6	4	7	4

Who is happier?

Cathy is happier.   David is happier.   Cathy & David are both happy

Why? ____

____

____

2.

Cathy		David	
Wants	Gets	Wants	Gets
4	5	5	5

Who is happier?

Cathy is happier.   David is happier.   Cathy & David are both happy

Why? ____

____

____

Level 4: Integrated Bidimensional Level (9-11 years)

1.

Cathy		David	
Wants	Gets	Wants	Gets
6	7	4	6

Who is happier?

Cathy is happier. David is happier. Cathy & David are both happy

Why? ______________________________

____

____

2.

Cathy		David	
Wants	Gets	Wants	Gets
5	7	3	4

Who is happier?

Cathy is happier. David is happier. Cathy & David are both happy

Why? ______________________________

____

____

Birthday Party Task: Scoring Criteria

The scoring criteria for each of the four levels of the Birthday Party task (Case, Okamoto et al., 1996; Marini, 1984)

Dimensional Stage

Level 1: Predimensional Level (3½-5 years)

The child estimates the number of presents that each child received (makes a global estimate of the size of each set of presents), compares the results and says that the child who received more is happier (happiness is determined by the amount each child received) (Case, Okamoto et al., 1996; Marini, 1984). For example: "Because he has lots"; "Because she has more" (Case, Okamoto et al., 1996, p. 219).

Level 2: Unidimensional Level (5-7 years)

The child counts the number of presents that each child received, compares the resulting numbers and says that the child who received the larger number of presents is happier (happiness is determined by the amount each child received). The response must show that the child counted the presents that each child received (Case, Okamoto et al., 1996; Marini, 1984). For example: "David, because he got 4 and she got 3" (Case, Okamoto et al., 1996, p. 219).

Level 3: Bidimensional Level (7-9 years)

The child counts the number of presents that each child wanted and then counts the number of presents that each child received. The child compares the number of presents that each child wanted to the number of presents that each child received and says that the child who received more presents than he or she wanted is happier or that the child who received the exact number of presents that he or she wanted is happier because he or she received exactly what he or she wanted (happiness is determined by the amount each child wanted and received). The response must show that the child considered what each child wanted and what each child received (Case, Okamoto et al., 1996; Marini, 1984). For example; "They both got 5, but Cathy wanted only 4, so she is happier"; "They both got 5, but David got exactly 5, so he is happier" (Case, Okamoto et al., 1996, p. 219).

Level 4: Integrated Bidimensional Level (9-11 years)

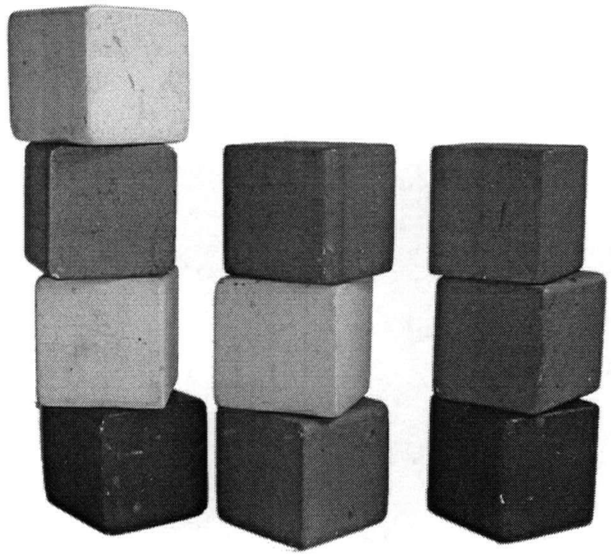
The child counts the number of presents that each child wanted and the number of presents that each child received. The child then computes the difference between what each child wanted and received, compares the two differences and says that the child who received the most is happier (happiness is determined by the amount each child wanted and received). The response must show that the child considered what each child wanted and what each child received (Case, Okamoto et al., 1996; Marini, 1984). For example: "David has 2 more than he wanted, but Cathy got only 1 more than she wanted, so he is happier" (Case, Okamoto et al., 1996, p. 219).

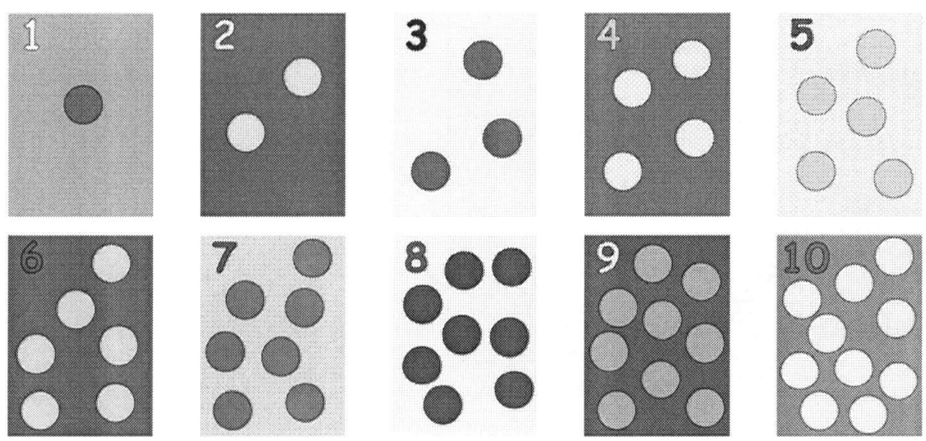
Appendix L
Pointing and Winking Task

The materials used in the Pointing and Winking task.

Appendix M
Build the Tower Task

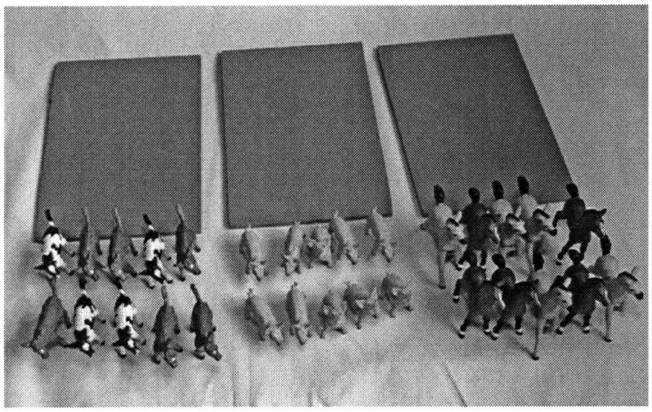
The materials used in the Build the Tower task.

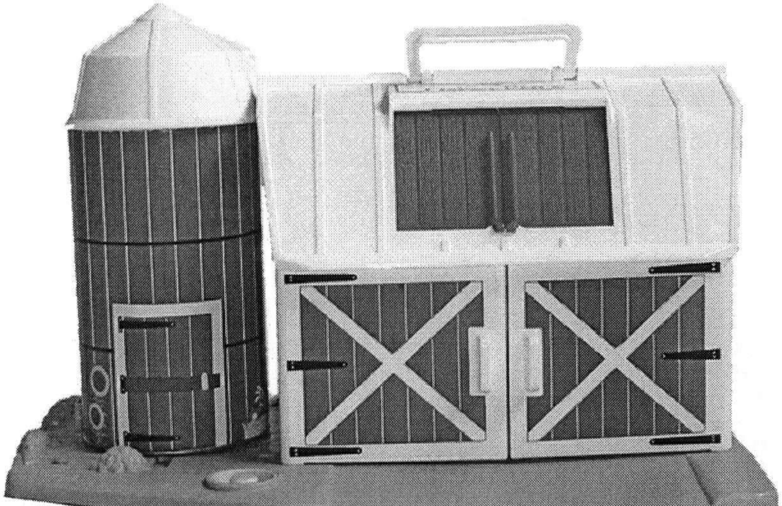




Appendix N
Help the Farmer Task

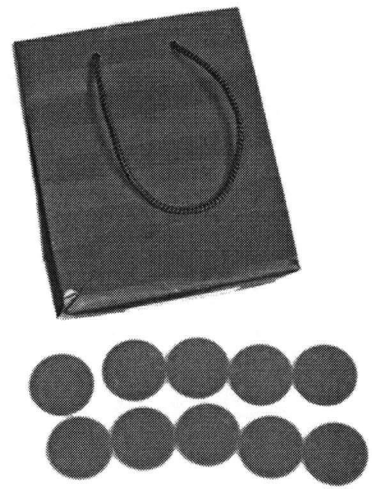
The materials used in the Help the Farmer task.





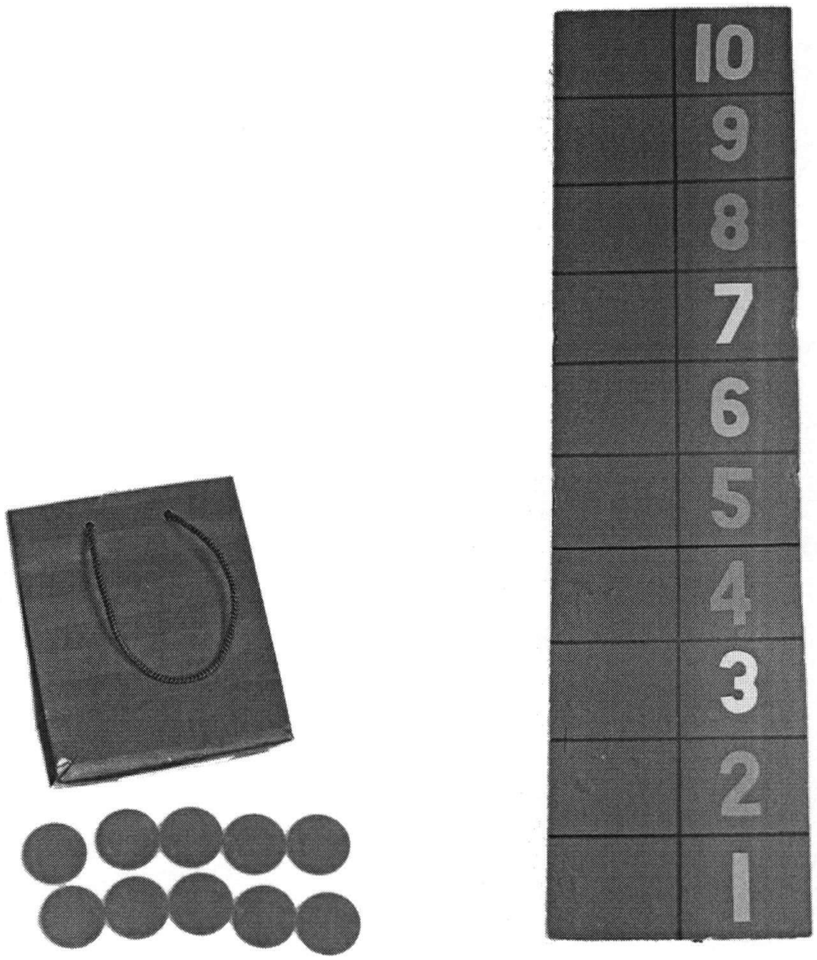
Appendix O
Good Fairy Task

The materials used in the Good Fairy task.



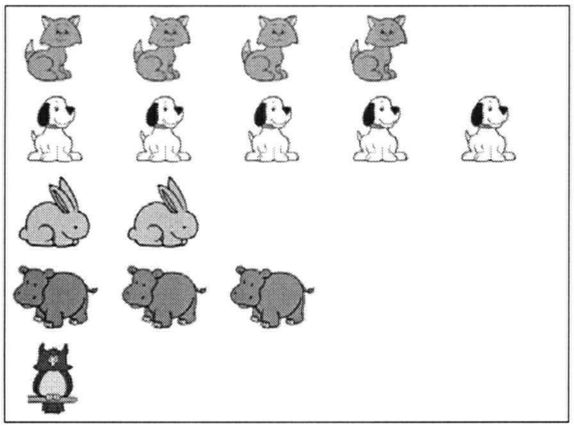
Appendix P
Cookie Monster Task

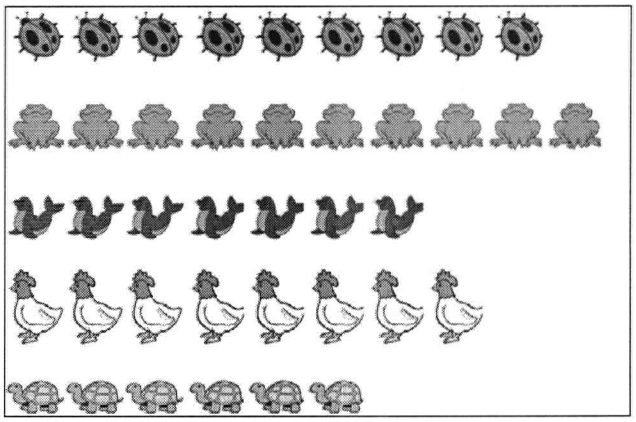
The materials used in the Cookie Monster task.

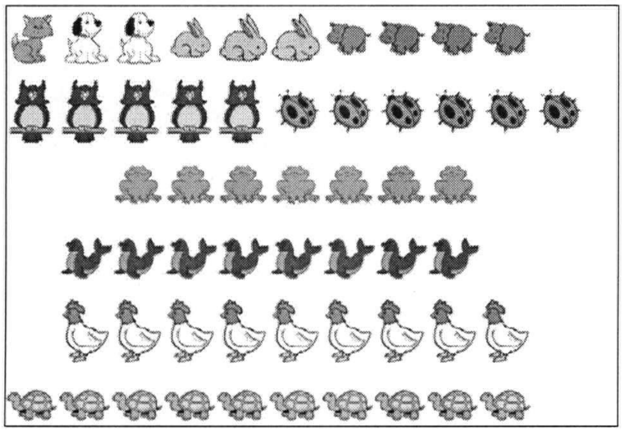


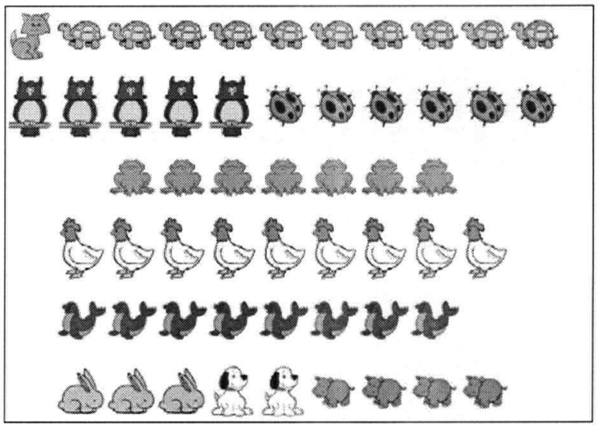
Appendix Q
Animals on the Number Line Task

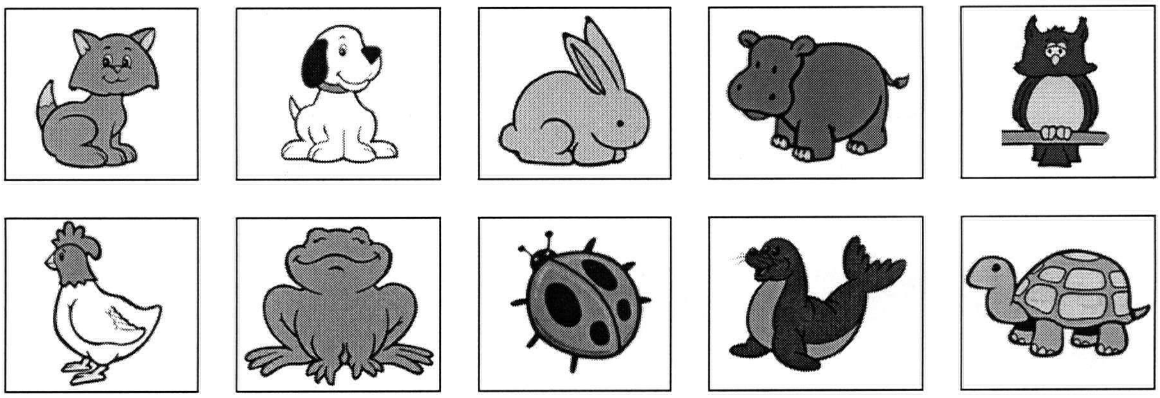
The materials used in the Animals on the Number Line task.

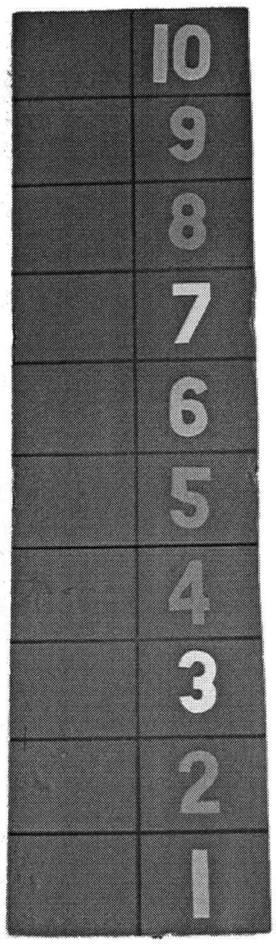






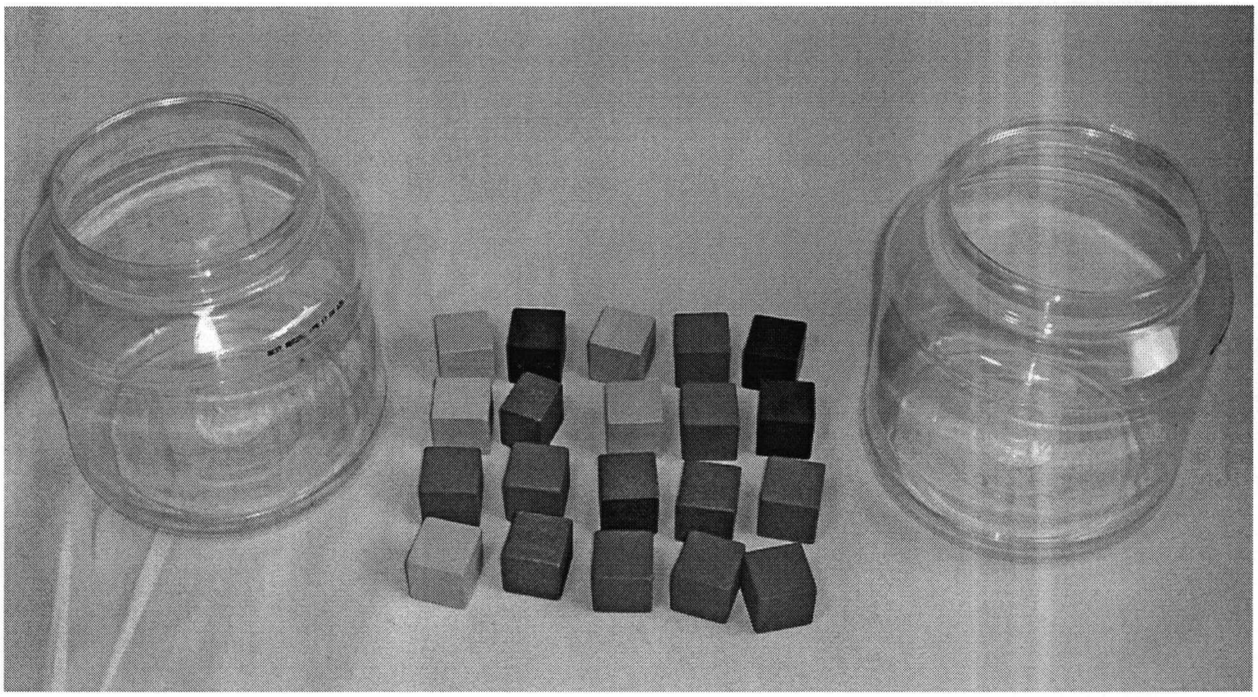





10
9
8
7
6
5
4
3
2
1

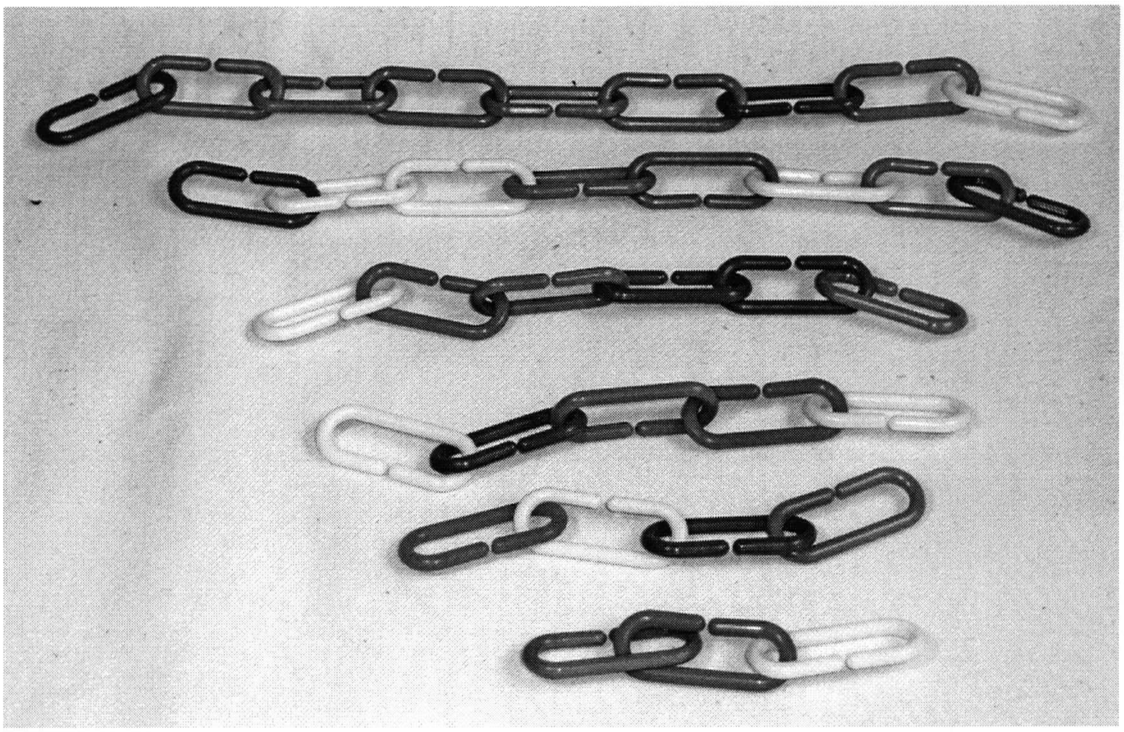
Appendix R
Which Has More Task

The materials used in the Which Has More task.



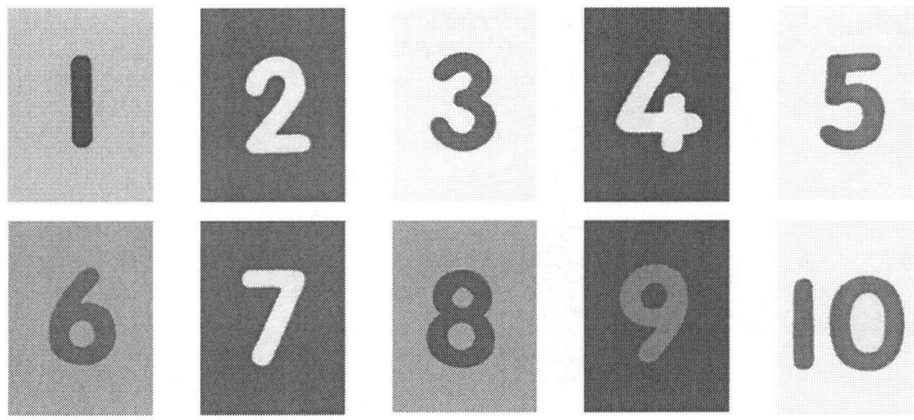
Appendix S
Let's Compare Task

The materials used in the Let's Compare task.



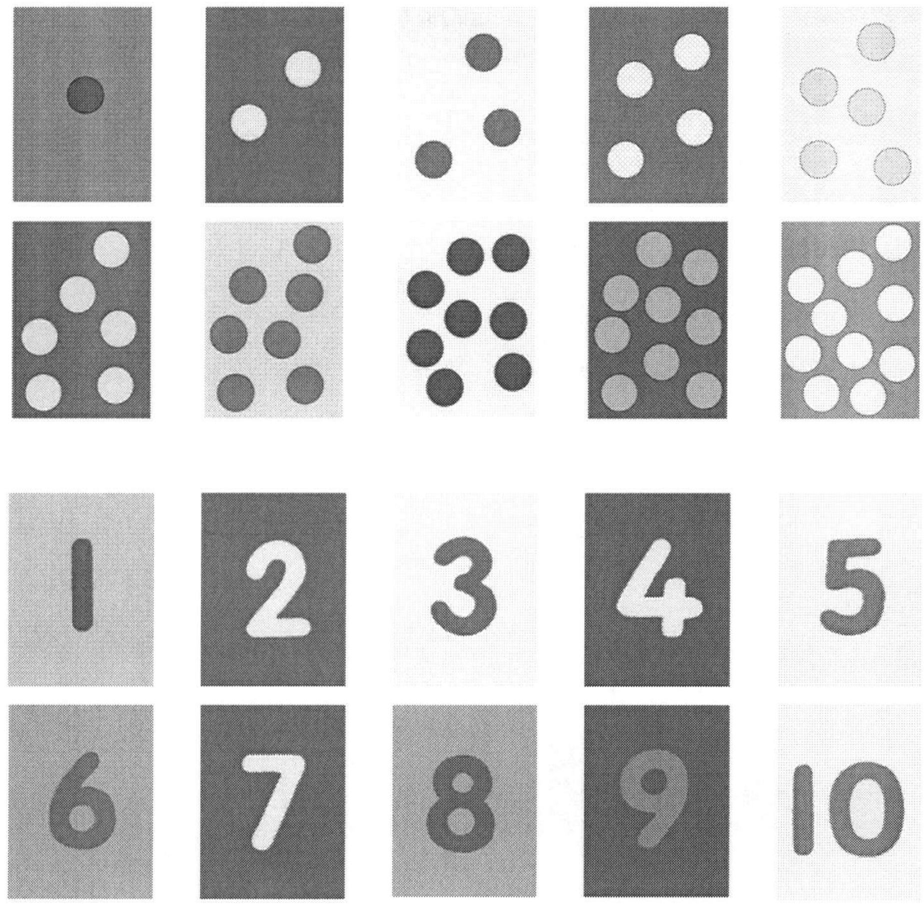
Appendix T
Name That Numeral Task

The materials used in the Name That Numeral task.



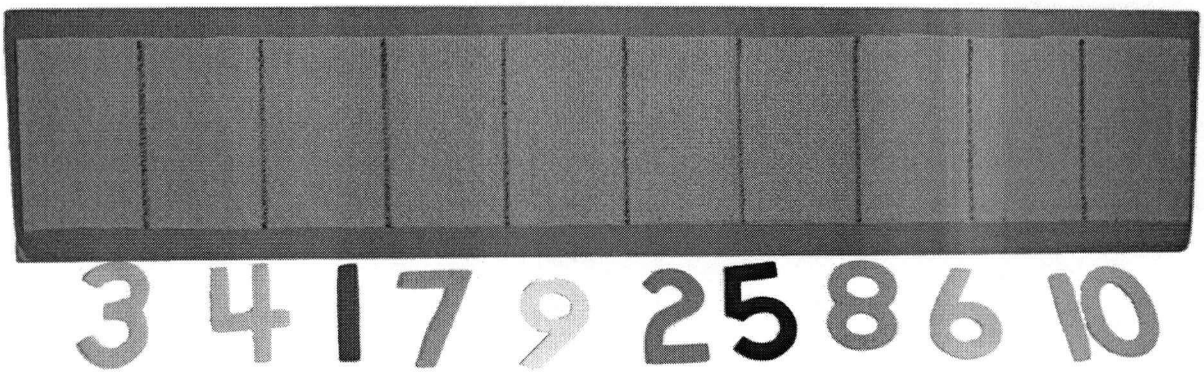
Appendix U
Match Task

The materials used in the Match task.



Appendix V
Numerals on the Number Line Task

The materials used in the Numerals on the Number Line task.



Appendix W
Questions Related to the Children's Responses

First Independent Differentiated, Mapped or Consolidated Response

Within Sessions

1. How many responses were there in the session?
2. On which response, within the session, did the first independent differentiated, mapped or consolidated response (first independent correct response)[85] occur?

Across Sessions

1. How many responses were there across the sessions?
2. On which response, across the sessions, did the first independent differentiated or mapped response (first independent correct response) occur?

Responses Before and After the First Independent Differentiated, Mapped or Consolidated Response

Before

3. How many unrelated or no responses were there before the first independent differentiated, mapped or consolidated response (first independent correct response?)
4. How many supported incorrect responses were there before the first independent differentiated, mapped or consolidated response (first independent correct response?)
5. How many supported elaborated, linked across or non-consolidated responses (supported partially correct responses)[86] were there before the first independent differentiated, mapped or consolidated response (first independent correct response)?
6. How many supported differentiated, mapped or consolidated responses (supported correct responses) were there before the first independent differentiated, mapped or consolidated response (first independent correct response)?

---

[85] Linked across responses for reading the written numerals are included here because they represent a correct response for this mathematical understanding.

[86] Nonconsolidated responses can be classified as correct responses when the difference between two sets is large and the difference can be correctly determined using a visual assessment strategy. Nonconsolidated responses are included here because they represent a lower level response indicating that the mental counting line has not yet been consolidate.

7. How many independent, incorrect responses were there <u>before</u> the first independent differentiated, mapped or consolidated response (first independent correct response)?
8. How many independent elaborated, linked across or non-consolidated responses (independent partially correct responses) were there <u>before</u> the first independent differentiated, mapped or consolidated response (first independent correct response)?

After

9. How many unrelated or no responses were there <u>after</u> the first independent differentiated, mapped or consolidated response (first independent correct response)?
10. How many supported incorrect responses were there <u>after</u> the first independent differentiated, mapped or consolidated response (first independent correct response)?
11. How many supported elaborated, linked across or non-consolidated responses (supported partially correct responses) were there <u>after</u> the first independent differentiated, mapped or consolidated response (first independent correct response)?
12. How many supported differentiated, mapped or consolidated responses (supported correct responses) were there <u>after</u> the first independent differentiated, mapped or consolidated response (first independent correct response)?
13. How many independent incorrect responses were there <u>after</u> the first independent differentiated, mapped or consolidated response (first independent correct response)?
14. How many independent elaborated, linked across or non-consolidated responses (independent partially correct responses) were there <u>after</u> the first independent differentiated, mapped or consolidated response (first independent correct response)?
15. How many independent differentiated, mapped or consolidated responses (independent correct responses) were there <u>after</u> the first independent differentiated, mapped or consolidated response (first independent correct response)?

Appendix X
Coding Sheet

The coding sheet used to code the data.

Coding Sheet

Subject ________________ Coder __________________ Date___________

Researcher's Question						Child's Response										Qualitative Description
Unit	Session	Question Content	Opportunities to Respond	Question Level of Complexity	Instructor Support	Nature of Response	Correct-ness of Response	Justification	Correctness of Justification	Demonstrated Levels of Understanding						
										(EL) Elaborated Response	(DI) Differentiated Response	(LA) Linked Across Response	(MA) Mapped Response	(CO) Consolidated Response	(NC) Non-Consolidated Response	