WAVE-GUIDE PROPAGATION OF ACOUSTIC-GRAVITY WAVES IN AN ISOTHERMAL LAYER MODEL OF THE STRATOSPHERE

by

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ABSTRACT

It is suggested that the stratosphere may act as a wave-guide for certain types of acoustic-gravity waves. An isothermal layer model is proposed which introduces gravity terms into the equations governing wave propagation. An expression for the phase change on reflection at a boundary of the layer is derived. Numerical solutions to the equation for wave-guide modes of propagation are obtained by the use of a digital computer.

Results are given in the form of dispersion curves. Cutoff is found to occur at a frequency well above the Brunt resonant frequency for the stratosphere. The model stratosphere proposed in this study does not behave as a lossless wave-guide for gravity coupled acoustic waves. This is shown to be consistent with the results of a more complete study by Press and Harkrider.

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CHAPTER I

INTRODUCTION

One of the striking features of the temperature profile of the earth's atmosphere (Figure 2) is the prominent low temperature region which defines the stratosphere. It is characterized by quite steep temperature gradients at its upper and lower boundaries at approximately 30 and 10 km altitude respectively, and by a relatively uniform temperature of about 220 $^{\circ}$K throughout the intervening 20 km thickness.

The air in such a layer can be expected to exhibit resonance at some particular frequency of excitation, and the layer may be expected to act as a duct or waveguide for acoustic waves of certain frequencies. If the resonant frequency of the air in the layer is one of those frequencies which may be guided by the layer, then it is to be expected that the corresponding frequency might predominate.

The natural frequency of oscillation of a parcel of air in a stable atmosphere is called the Brunt-Väisälä frequency. It is given by the expression

$$\omega_B = g/c\ (\gamma - 1)^{1/2}$$

where g is the gravitational acceleration, c the speed of sound and γ the ratio of specific heats, c_p/c_v. For the stratosphere ω_B turns out to be approximately 0.021 rad/sec

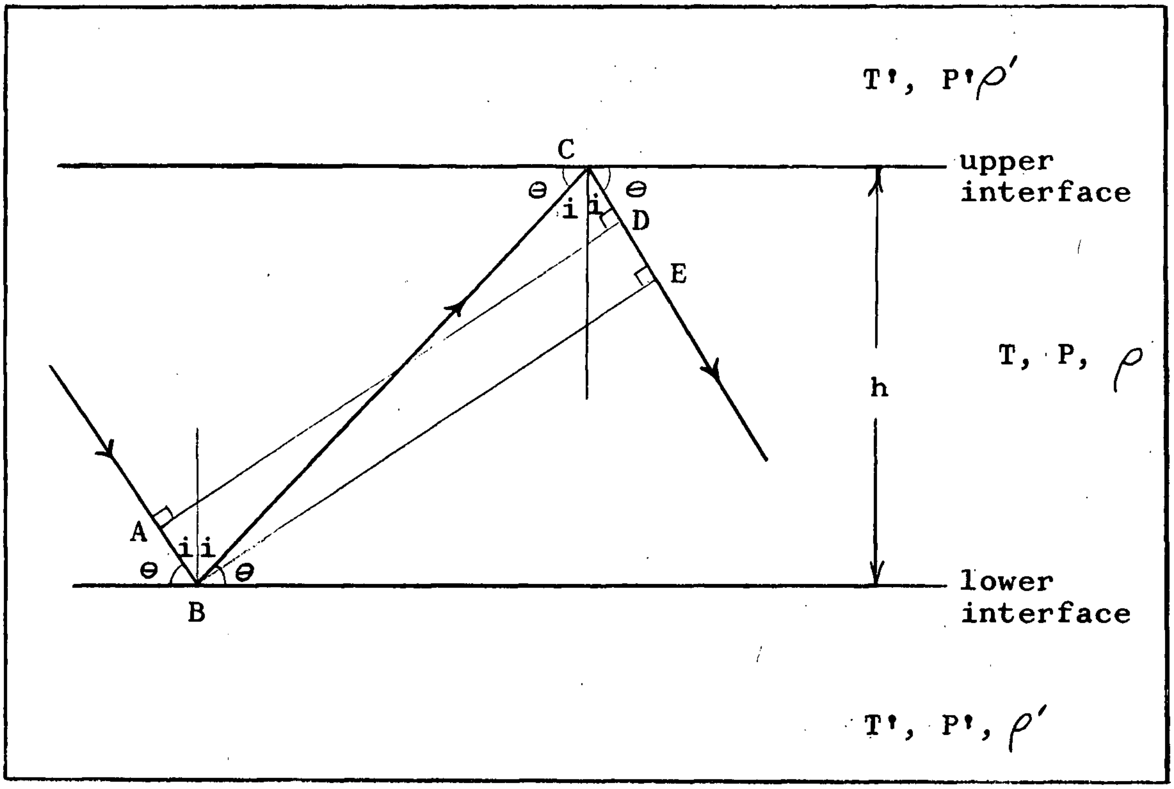

FIGURE 1. Ray path diagram for wave-guide modes.

giving a characteristic period of approximately 5.2 minutes.

The investigation of wave-guide modes of propagation is not a simple matter, but it is possible to use ray theory to demonstrate that such modes are possible, at least for simple acoustic waves. Consider a single layer of air with thickness h and acoustic velocity c_o, surrounded by air with acoustic velocity c_o'. To have unattenuated waveguide transmission within the layer of air, only those modes are permissible which have constant phase in planes perpendicular to the ray paths.

In Figure 1, the effective path length $\overline{ABCD}$ must equal path length $\overline{BCDE}$.

$$s = \frac{h}{\sin \theta} + \frac{h}{\sin \theta} \cos (2i)$$

$$= \frac{h}{\sin \theta} \; (1 + \cos (180° - 2\theta))$$

$$= 2h \sin \theta$$

The condition for constructive interference requires that the phase change from, say, A to D must be an integral multiple of 2π. That is, the change of phase due to the distance travelled by the wave, together with the total phase change on reflection, must be equal to $2\pi m$, where $m = 1, 2, 3, \ldots$.

$$2h(\omega/c_o)\sin \theta + \varphi = 2m\pi.$$

Here, the total phase change produced by the two reflections is represented by φ . At the critical angle of incidence $\theta = \theta_c$ and $\varphi = 0$, leaving simply

$$\omega/c_o = \frac{m\pi}{h \sin \theta_c} \qquad m = 1, 2, \ldots.$$

In the diagram, θ_c is the complement of the critical angle of incidence. The critical angle of incidence is given by

$$\sin\,(90^\circ - \theta_c) = \frac{c_o}{c_o'}$$

That is,

$$\sin\,\theta_c = (1 - (\frac{c_o}{c_o'})^2)^{1/2}$$

Evidently, wave-guide modes are possible for acoustic waves having frequencies greater than

$$\omega_c = \frac{\pi m}{h} \quad \frac{1}{\sqrt{(\frac{1}{c_o})^2 - (\frac{1}{c_o'})^2}}$$

Considering the fundamental mode ($m = 1$), and temporarily assuming

$h = 20$ km

$c_o = 297$ m/sec

$c_o' = 329$ m/sec

one finds that this cutoff will occur for ω_c equal to 0.108 rad/sec. Clearly, this excludes the Brunt frequency ($\omega_B = 0.02$ rad/sec) which is well below the acoustic cutoff.

In constructing this simple model, however, we have ignored completely the effect of gravity, and assumed that the boundaries to the layer were effectively static. We shall see how the inclusion of gravity terms affects the phase change. In subsequent sections we will construct a model which takes these factors into account in the form of more sophisticated boundary conditions, and pays rather more attention to the assumptions involved.

CHAPTER II

THE MODEL ATMOSPHERE

In the derivations to follow, the low velocity region corresponding to the stratosphere is approximated by an isolated layer of air at a constant temperature of T degrees Kelvin. The surrounding layers, above and below, are represented by two infinitely thick layers both at a constant absolute temperature of T'. The thickness h of the low velocity layer is taken to be 20 kilometers, with T and T' at 220°K and 270°K respectively. This is shown, along with the ARDC standard 1959 atmosphere, in Figure 2. The ambient pressure is assumed to be a unique function of density, which, along with temperature, is a function only of the altitude z.

For waves having periods such as will be encountered in this research, we are justified in neglecting the curvature of the earth and in treating the interfaces between the atmospheric layers as infinite planes. Further, we shall assume that we are sufficiently far from the source of an observed disturbance that the waves propagating in the positive x direction are plane and infinite in the y direction. Their amplitude is assumed to be small. Although waves in an isothermal atmosphere are not refracted by density change (i.e. rays are straight lines), the amplitudes of waves do depend on altitude because ρ_o varies with z. In this sense the waves are not strictly plane waves.

The air itself is considered to be a perfect gas with the ordinary sound velocity given by

$$c_o = \left(\frac{\gamma RT}{m}\right)^{1/2}$$

where the constants take the following values:

Ratio of specific heats	$\gamma = \frac{c_p}{c_v} = 1.40$
Universal gas constant R	$= 8314.9$ MKS units
Molecular weight m	$= 29.$

Thus for the enclosed layer, the ordinary sound velocity will be 297 m/sec and for the surrounding air 329 m/sec.

Although we are keenly interested in the effect of gravity on wave propagation in the layer, it is possible to make certain simplifying assumptions about the nature of the gravitational field. Taking as the radius of the earth the figure 6371 km, and 9.80 m/sec^2 as a typical value of the acceleration due to gravity at the earth's surface, a rough calculation can be made of the magnitude of g at the upper and lower boundaries of the isolated layer. At the lower boundary (altitude of 10 km) we find g approximately 9.76 m/sec^2; at the upper (30 km) boundary, 9.70 m/sec^2. It is not unreasonable then, to treat g as being approximately constant at 9.7 m/sec^2 within the layer of interest.

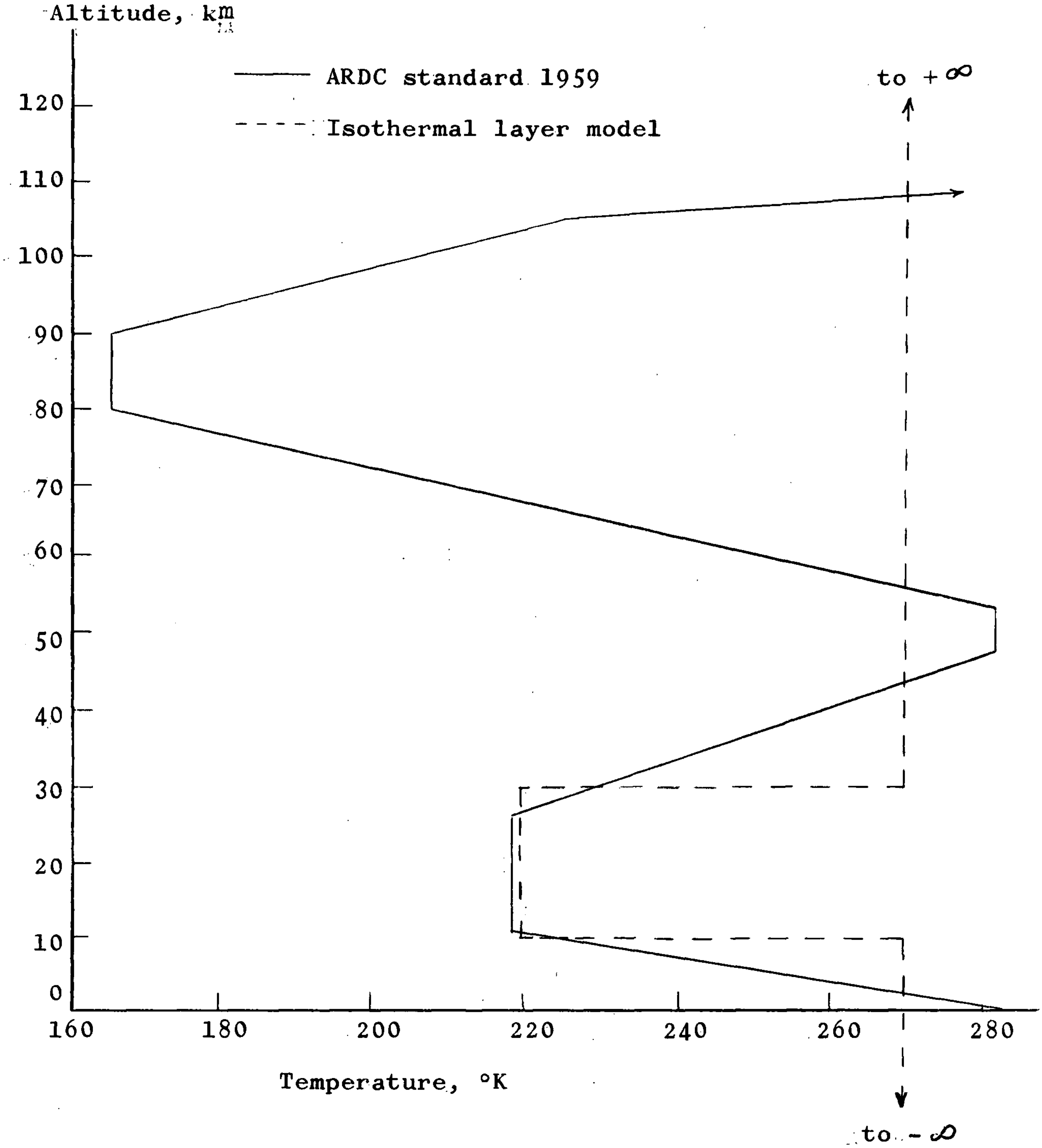

FIGURE 2. Temperature profiles of the atmosphere. ARDC standard 1959 Atmosphere after Wares et al is shown with the isothermal layer representation of the stratosphere.

CHAPTER III

THEORY

By expansion of rigorous first order equations describing the behaviour of a perfect fluid where the pressure is everywhere a unique function of the density, Bergmann has derived non-static (time dependent) relationships. In the following equations $\bar{u}$ is the velocity vector, ρ the density, p the pressure, B the bulk modulus, V the gravitational potential, and t the time. The subscripts refer to consecutive orders of expansion with the 0 subscript referring to the static solution. The first-order equations as given by Bergman are:

<u>Equation of continuity</u>

$$\frac{\partial \rho_1}{\partial t} + \nabla(\rho_o \bar{u}_1) = 0 \qquad \ldots\ldots \qquad (1)$$

<u>Equation of motion</u>

$$\rho_o \frac{\partial \bar{u}_1}{\partial t} + \nabla p_1 + \rho_1 \nabla V = 0 \qquad \ldots\ldots \qquad (2)$$

<u>Equation of state</u>

$$\frac{\partial p_1}{\partial t} + \bar{u}_1 \cdot \nabla p_o = \frac{B_o}{\rho_o} \left(\frac{\partial \rho_1}{\partial t} + \bar{u}_1 \cdot \nabla \rho_o\right) \qquad (3)$$

Equation (1) can be written

$$\frac{\partial \rho_1}{\partial t} + \bar{u}_1 \cdot \nabla \rho_o + \rho_o \nabla \bar{u}_1 = 0 \qquad \ldots\ldots \qquad (1')$$

The first two terms are the same as the expression in the brackets in equation (3). Substitution gives

$$\frac{\partial p_1}{\partial t} + \bar{u}_1 \cdot \nabla p_o + B_o \nabla \bar{u}_1 = 0 \qquad \ldots\ldots \qquad (4)$$

Time dependence of the form e^{iwt} is assumed.

$$i\omega \rho_1 + \bar{u}_1 \cdot \nabla \rho_o + \rho_o \nabla \bar{u}_1 = 0 \qquad \ldots \qquad (1')$$

$$i\omega \rho_o \bar{u}_1 + \nabla p_1 + \rho_1 \nabla V = 0 \qquad \ldots\ldots \qquad (2')$$

$$i\omega p_1 + \bar{u}_1 \cdot \nabla p_o + B_o \nabla \bar{u}_1 = 0 \qquad \ldots\ldots \qquad (4')$$

From equation (1')

$$\nabla \cdot \bar{u}_1 = - \frac{1}{\rho_o} (i\omega \rho_1 + \bar{u}_1 \cdot \nabla \rho_o)$$

This is substituted for $\nabla \bar{u}_1$ in equation (4) to get

$$i\omega p_1 + \bar{u}_1 \cdot \nabla p_o - \frac{B_o}{\rho_o} (i\omega p_1 + \bar{u}_1 \cdot \nabla \rho_o) = 0$$

By rearranging terms, we get

$$\bar{u}_1 \cdot (\nabla p_o - \frac{B_o}{\rho_o} \nabla \rho_o) + i\omega p_1 - \frac{B_o}{\rho_o} i\omega \rho_1 = 0$$

$$\ldots\ldots \qquad (5)$$

Equation (2) can be written

$$\bar{u}_1 = \frac{i}{\omega \rho_o} (\nabla p_1 + \rho_1 \nabla V) \qquad \ldots\ldots \qquad (6)$$

In the model atmosphere we have chosen V = g z and so

$\nabla V = g\bar{k}$. Thus (6) becomes

$$\bar{u}_1 = \frac{1}{\omega\rho_o} \left(\nabla p_1 + \rho_1 g \bar{k} \right) \qquad \ldots\ldots \qquad (6')$$

The right hand expression can be substituted for $\bar{u}_1$ in equation (5), giving

$$\frac{i}{\omega\rho_o} (\nabla p_1 + \rho_1 g \bar{k}).(\nabla p_o - \frac{B_o}{\rho_o} \nabla\rho_o) + i\omega p_1 - \frac{B_o}{\rho_o} i\omega\rho_1 = 0$$

or, more simply,

$$(\nabla p_1 + \rho_1 g \bar{k}).(\nabla p_o - \frac{B_o}{\rho_o} \nabla\rho_o) + \omega^2 p_1 \rho_o - \omega^2 B_o \rho_1 = 0$$

Collecting terms in ρ_1 we have

$$\rho_1 \left\{ g\bar{k}.(\nabla p_o - \frac{B_o}{\rho_o} \nabla\rho_o) - B_o\omega^2 \right\} = -\left\{ \nabla p_1 . (\nabla p_o - \frac{B_o}{\rho_o} \nabla\rho_o) + \omega^2 \rho_o p_1 \right\} .$$

Divide both sides by B_o, rearrange and solve for ρ_1:

$$\rho_1 = - \frac{\left\{ \nabla p_1 . \bar{G} - \dfrac{\omega^2 \rho_o}{B_o} p_1 \right\}}{\left\{ g\bar{k}.\bar{G} + \omega^2 \right\}} \qquad \ldots\ldots \qquad (7)$$

where we have defined a vector

$$\bar{G} = \frac{\nabla\rho_o}{\rho_o} - \frac{\nabla p_o}{B_o}$$

$\bar{G}$ depends only on the ambient pressure and temperature, which in turn are functions only of altitude.

$$\bar{G} = \left\{ \frac{\bar{k}\,\frac{\partial \rho_o}{\partial z}}{\rho_o} - \frac{\bar{k}\,\frac{\partial \rho_o}{\partial z}}{B_o} \right\} = \left\{ \frac{1}{\rho_o}\frac{\partial \rho_o}{\partial z} - \frac{1}{B_o}\frac{\partial \rho_o}{\partial z} \right\} \bar{k}$$

For any fluid in a uniform gravitational field,

$$\frac{\partial p_o}{\partial z} = - \rho_o g$$

Also, since $c_o^2 = \dfrac{B_o}{\rho_o}$

we can write $G = \left\{ \dfrac{1}{\rho_o} \dfrac{\partial \rho_o}{\partial z} + \dfrac{g}{c_o^2} \right\} \bar{k}$

For a perfect gas

$$\frac{\partial p_o}{\partial z} = \frac{\partial \rho_o}{\partial z} \cdot \frac{RT}{m}$$

$$- \rho_o g = \frac{\partial \rho_o}{\partial z} \cdot \frac{RT}{m}$$

$$\frac{1}{\rho_o} \frac{\partial \rho_o}{\partial z} = - \frac{mg}{RT}$$

Now since $c_o^2 = \dfrac{\gamma p_o}{\rho_o} = \dfrac{\gamma RT}{m}$

then $\dfrac{1}{\rho_o} \dfrac{\partial \rho_o}{\partial z} = - \dfrac{\gamma g}{c_o^2}$

Combining these results

$$\bar{G} = \left\{ - \frac{\gamma g}{c_o^2} + \frac{g}{c_o^2} \right\} \bar{k} = - \left\{ (\gamma - 1) \frac{g}{c_o^2} \right\} \bar{k}$$

Since $\gamma > 1$, $\bar{G}$ is intrinsically negative for the atmosphere.

Equation (7) can now be written

$$\rho_1 = - \frac{\left\{ \frac{\partial p_1}{\partial z} G - (\omega/c_o)^2 p_1 \right\}}{\left\{ g G + \omega^2 \right\}} \qquad (7')$$

where $G = -(\gamma - 1) g/{c_o}^2$

By substituting the above value of ρ_1 (equation (7')) into equation (6') we have

$$\bar{u}_1 = \frac{i}{\rho_o \omega} \left\{ \nabla p_1 + g\bar{k}\left(\frac{(\omega/c_o)^2 p_1 - \frac{\partial p_1}{\partial z} G}{\omega^2 + g G} \right) \right\}$$

The components of velocity are:

$$u = \frac{i}{\rho_o \omega} \frac{\partial p_1}{\partial x}$$

$$v = \frac{i}{\rho_o \omega} \frac{\partial p_1}{\partial y}$$

and

$$w = \frac{i}{\rho_o \omega} \left\{ \frac{\partial p_1}{\partial z} + g \left(\frac{(\omega/c_o)^2 p_1 - \frac{\partial p_1}{\partial z} G}{\omega^2 + g G} \right) \right\}$$

$$= \frac{i \omega}{\rho_o(\omega^2 + gG)} \left(\frac{\partial p_1}{\partial z} + \frac{g}{{c_o}^2} p_1 \right) \qquad (8)$$

Examination of Boundary Conditions

Consider the case of a plane wave incident upon an interface from above, as in Figure 3.

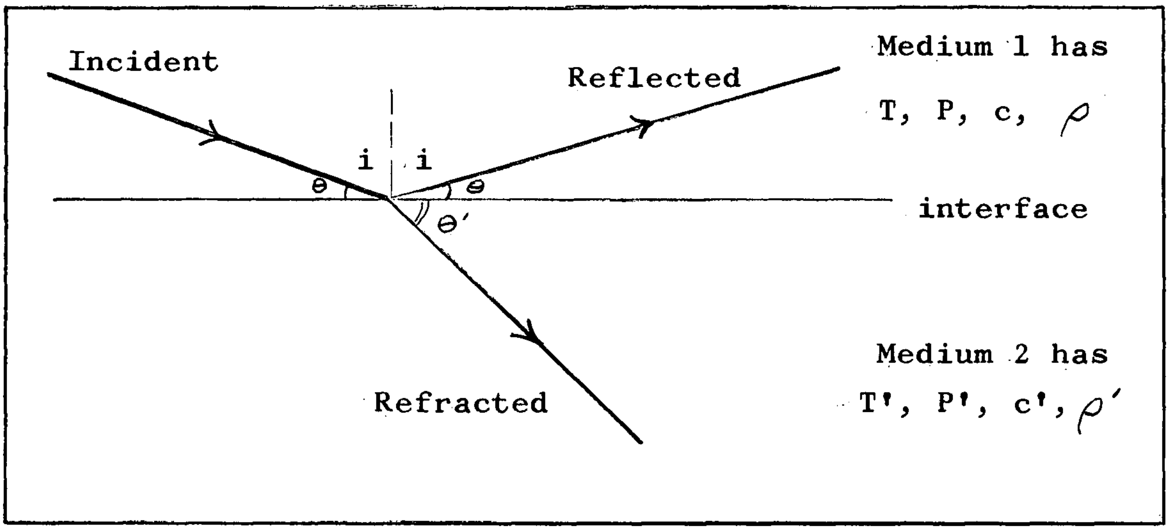

FIGURE 3. Reflection and refraction at an interface.

The general solutions can be written:

$$p_i = A\ e^{i[\omega t - xk\cos\theta + zk\sin\theta]}$$

$$p_r = B\ e^{i[\omega t - xk\cos\theta - zk\sin\theta]}$$

$$p_r' = D\ e^{i[\omega t - xk'\cos\theta' + zk'\sin\theta']}$$

where the primed quantitites refer to the lower region and the unprimed quantities refer to the upper.

Since we will later be calculating the reflection coefficient, it is convenient to assign unit amplitude to the incident wave so that A = 1, B/A = R, the amplitude of the reflected wave, and D/A = C, the amplitude of the refracted wave.

(a) Pressure

At an interface the pressure must be equal on both sides ($p = p'$). Taking terms up to 1st order only, we have

$$p_o + p_1 = p_o' + p_1'$$

The unperturbed pressure may be written $p_o = \rho_o g \zeta$ where terms of order ζ^2 have been neglected. Thus,

$$\rho_o g \zeta + p_1 = \rho_o' g \zeta + p_1'$$

where $z = \zeta$ is the equation of the interface. It is true for all time, thus

$$\frac{D}{Dt}\left(p_1 - p_1' + (\rho_o - \rho_o')g\zeta\right) = 0$$

Approximately $i\omega(p_1 - p_1') + (\rho_o - \rho_o')g\ w' = 0$.

$$w' = \frac{i\omega}{(\rho_o - \rho_o')g}(p_1 - p_1') \qquad \ldots\ldots \quad (9)$$

(b) Velocity

The vertical component of velocity must be continuous across an interface

$$w = w' \qquad \ldots\ldots \quad (10)$$

Application of Boundary Conditions at z = 0

(a) Combining equations (9) and (9) we have

$$\frac{i\,\omega}{\rho_o'(\omega^2+gG')}\left(\frac{\partial p_1'}{\partial z}+\frac{g}{c_o'^2}p_1'\right)=\frac{i\,\omega}{(\rho_o-\rho_o')g}(p_1-p_1')$$

$$\left(\frac{\partial p_1'}{\partial z}+\frac{g}{c_o'^2}p_1'\right)=\frac{\rho_o'(\omega^2+gG')}{(\rho_o-\rho_o')g}(p_1-p_1')$$

$$C\left(ik'\sin\theta'+\frac{g}{c_o'^2}+\frac{\rho_o'}{(\rho_o-\rho_o')}\frac{(\omega^2+gG)}{g}\right)=\frac{\rho_o'}{(\rho_o-\rho_o')}\left(\frac{\omega^2+gG'}{g}\right)(1+R)$$

all at z = 0

$$C\left(ik'\sin\theta'+\frac{g}{c_o'^2}+\left(\frac{\rho_o}{\rho_o-\rho_o'}\right)\left(\frac{\omega^2+gG}{g}\right)\right)=\frac{\rho_o'}{\rho_o-\rho_o'}\left(\frac{\omega^2+gG'}{g}\right)(1+R) \qquad \ldots\ldots (11)$$

(b) Combining equations (8) and (10) we have

$$\frac{i\,\omega}{\rho_o(\omega^2+gG)}\left(\frac{\partial p_1}{\partial z}+\frac{g}{c_o^2}p_1\right)=\frac{i\,\omega}{\rho_o'(\omega^2+gG')}\left(\frac{\partial p_1'}{\partial z}+\frac{g}{c_o'^2}p_1'\right)$$

$$\frac{\rho_o'(\omega^2+gG')}{\rho_o(\omega^2+gG)}\left\{ik\sin\theta(1-R)+\frac{g}{c_o^2}(1+R)\right\}=\left\{ik'\sin\theta+\frac{g}{c_o'^2}\right\}C \qquad \ldots\ldots (12)$$

Reflection Coefficient at z = 0

The order to eliminate C and find R, the reflection coefficient, multiply equation (11) by $(ik'\sin\theta'+g/c_o'^2)$ and equation (12) by $\left(ik'\sin\theta'+g/c_o'^2+\frac{\rho_o'}{(\rho_o-\rho_o')}\left(\frac{\omega^2+gG'}{g}\right)\right)$ and subtract to get

$$\frac{\rho_o'}{(\rho_o-\rho_o')}\frac{(\omega^2+gG')}{g}(ik'\sin\theta' + g/c_o'^2)(1+R)$$

$$= \left\{(ik'\sin\theta' + g/c_o'^2 + \frac{\rho_o}{(\rho_o-\rho_o')}\frac{(\omega^2+gG')}{g}\right\}\left\{\frac{(\rho_o'(\omega^2+gG')}{\rho_o(\omega^2+gG)}\right\}$$

$$\left\{(ik\sin\theta)(1-R) + \frac{g}{c_o^2}(1+R)\right\}$$

$$R = -\frac{\left\{\frac{\rho_o}{(\rho_o-\rho_o')}\frac{(\omega^2+gG)}{g}\frac{(ik'\sin\theta' + \frac{g}{c_o'^2})}{(ik'\sin\theta' + \frac{g}{c_o'^2} + \frac{\rho_o'}{(\rho_o-\rho_o')}\frac{(\omega^2+gG)}{g}} - ik\sin\theta - \frac{g}{c_o^2}\right\}}{\left\{\frac{\rho_o}{(\rho_o-\rho_o')}\frac{(\omega^2+gG)}{g}\frac{(ik'\sin\theta' + \frac{g}{c_o'^2})}{(ik'\sin\theta' + \frac{g}{c_o'^2} + \frac{\rho_o'}{(\rho_o-\rho_o')}\frac{(\omega^2+gG)}{g}} + ik\sin\theta - \frac{g}{c_o^2}\right\}}$$

Note that if g is set to zero

$$R_{g=0} = -\frac{\rho_o ik'\sin\theta' - \rho_o' ik\sin\theta}{\rho_o ik'\sin\theta' + \rho_o' ik\sin\theta}$$

$$= -\frac{\rho_o c\sin\theta' - \rho_o' c'\sin\theta}{\rho_o c\sin\theta' + \rho_o' c'\sin\theta}$$

which is the simple acoustical expression for the reflection coefficient.

Referring again to Figure 3 we note that by Snell's law

$$c' \cos \theta = c \cos \theta'$$

$$\cos \theta' = \frac{c'}{c} \cos \theta \ .$$

$$\sin \theta' = (1 - \cos^2 \theta')^{1/2} = (1 - (\frac{c'}{c})^2 \cos^2 \theta)^{1/2}$$

subject to the condition that $\cos \theta > c/c'$

$$\sin \theta' = - i(\frac{c'^2}{c^2} \cos^2 \theta - 1)^{1/2}$$

Since c' and c are both real, $\sin \theta'$ is pure imaginary for angles of incidence greater than critical.

The reflection coefficient then becomes:

$$R = - \frac{\left\{ \frac{\rho_o(\omega^2+gG)}{(\rho_o-\rho_o')g} \, \frac{(-k'\sqrt{(\frac{c'}{c})^2\cos^2\theta-1} + \frac{g}{c_o'^2})}{(-k'\sqrt{(\frac{c'}{c})^2\cos^2\theta-1} + \frac{g}{c_o'^2} + \frac{\rho_o(\omega^2+gG')}{(\rho_o-\rho_o')g})} - ik\sin\theta - \frac{g}{c_o^2} \right\}}{\left\{ \frac{\rho_o(\omega^2+gG)}{(\rho_o-\rho_o')g} \, \frac{(-k'\sqrt{(\frac{c'}{c})^2\cos^2\theta-1} + \frac{g}{c_o'^2}}{(-k'\sqrt{(\frac{c'}{c})^2\cos^2\theta-1} + \frac{g}{c_o'^2} + \frac{\rho_o(\omega+gG')}{(\rho_o-\rho_o')g})} - \frac{g}{c_o'^2} + ik\sin\theta \right\}} \qquad \ldots\ldots \quad (13)$$

Note that this expression is of the form

$$R = - \frac{a - ik \sin \theta}{a + ik \sin \theta}$$

where a is pure real. Therefore $|R| = 1$.

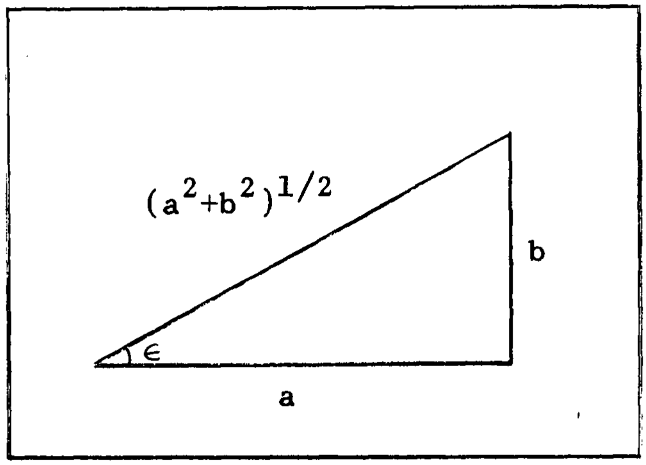

FIGURE 4. Phase angle ϵ

Phase shift on reflection

Let us write for convenience $R = -\frac{a - ib}{a + ib}$

Let us also define a triangle containing an angle ϵ as in Figure 4.

Divide top and bottom by $(a^2 + b^2)^{1/2}$

$$R = -\frac{\left\{\frac{a}{\sqrt{a^2 + b^2}} - i\frac{b}{\sqrt{a^2 + b^2}}\right\}}{\left\{\frac{a}{\sqrt{a^2 + b^2}} + i\frac{b}{\sqrt{a^2 + b^2}}\right\}}$$

$$= -\frac{(\cos \epsilon - i \sin \epsilon)}{(\cos \epsilon + i \sin \epsilon)}$$

Finally, $R = -e^{2i\epsilon}$

where 2ϵ is interpreted in the phase change on reflection.

From the triangle in Figure 4

$$\epsilon = \arctan b/a$$

$$\epsilon = \arctan \left\{ \frac{k \sin \theta}{\left(\frac{\rho_o(\omega^2+gG)}{(\rho_o-\rho_o')g} \frac{\left(-k'\sqrt{(\frac{c'}{c})^2\cos^2\theta-1} + \frac{g}{c_o'^2}\right)}{\left(-k'\sqrt{(\frac{c'}{c})^2\cos^2\theta-1} + \frac{\rho_o(\omega^2+gG)}{(\rho_o-\rho_o')g} + \frac{g}{c_o'^2}\right)} - \frac{g}{c_o'^2} \right)} \right\}$$

. . . . . (14)

Reflection at z = h

Obviously, since the reflection coefficient does not depend on the choice of a coordinate system, ϵ must have the same algebraic form for any similar reflection. We will show that ϵ has the same numerical value for the two interfaces which exist in the model atmosphere of this problem.

First, replace k by (ω/c_o) and $\cos^2\theta$ by $(1 - \sin^2\theta)$ in equation (14) to get

$$\tan\epsilon = \frac{(\omega/c_o)\sin\theta}{\left\{ \frac{\rho_o(\omega^2+gG)}{(\rho_o-\rho_o')g} \frac{\left(-\sqrt{\omega^2(\frac{1}{c_o^2}-\frac{1}{c_o'^2})-(\frac{\omega}{c_o})^2\sin^2\theta} + \frac{g}{c_o'^2}\right)}{\left(-\sqrt{\omega^2(\frac{1}{c_o^2}-\frac{1}{c_o'^2})-(\frac{\omega}{c_o})^2\sin^2\theta} + \frac{g}{c_o'^2}\ \frac{\rho_o'(\omega^2+gG')}{(\rho_o-\rho_o')g}\right)} - \frac{g}{c_o'^2} \right\}}$$

Recall that at $z = \zeta$ we can write

$$p = p'$$

$$p/\rho = \frac{RT}{m}$$

$$p'/\rho' = \frac{RT'}{m}$$

Combining these, $\rho T = \rho' T'$. This is true for steady state, thus $\rho_o/\rho_o' = \frac{T'}{T}$ and from (7') $G = -(\gamma - 1)g/c_o^2$

$$\text{Now } \tan \epsilon = \frac{(\omega/c_o) \sin \theta}{\left\{ \dfrac{\left(\omega^2 - \dfrac{g^2(\gamma-1)}{c_o^2}\right)\left(-\sqrt{\omega^2\left(\frac{1}{c_o^2} - \frac{1}{c_o'^2}\right) - \left(\frac{\omega}{c_o}\right)^2 \sin^2\theta} + \dfrac{g}{c_o'^2}\right)}{g(1 - T/T')\left(-\sqrt{\omega^2\left(\frac{1}{c_o^2}\ \frac{1}{c_o'^2}\right) - \left(\frac{\omega}{c_o}\right)^2 \sin^2\theta} + \dfrac{g}{c_o'^2} + \dfrac{\left(\omega^2 - g^2\left(\frac{\gamma-1}{c_o'^2}\right)\right)}{(T'/T - 1)g}\right)} - \dfrac{g}{c_o'^2} \right\}}$$

. . . . . (15A)

Since g is assumed constant, and since $c_o^2 = \frac{\gamma RT}{m}$, for a given ω and angle of incidence the phase change on reflection depends only on the temperatures of the layers.

This demonstrates that the phase change is the same for both reflections in this problem, since the temperatures of the layers are the same. Altitude in itself does not change ϵ. This is rather remarkable since the air densities are vastly different at the two interfaces.

Secular Equation

Recalling that the equation describing the condition required for wave-guide propagation was

$$2h\ (w/c_o)\ \sin\theta + \varphi = 2m\pi \qquad m = 1, 2, 3, \ldots$$

We have shown the phase change on reflection at both the upper and lower boundaries was 2ϵ, thus the total phase change φ is 4ϵ, and it is now possible to write

$$2h\ (w/c_o)\ \sin\theta + 4\epsilon = 2m\pi \qquad \ldots\ldots \qquad (15)$$

$$m = 1, 2, 3, \ldots.$$

where ϵ is defined in equation (15A).

CHAPTER IV

NUMERICAL ANALYSIS

In order to deduce some physically observable variables from the preceding theory, it is necessary to determine the relationship between ω and θ. This is accomplished by solving equation (15) which may be written

$$\frac{1}{2\pi}\left\{2h\left(\frac{\omega}{c_o}\right)\sin\theta + 4\epsilon\right\} = m$$

where m is an integer constant associated with the mode of propagation.

It is convenient to make the following substitution:

$$(\omega/c_o)\sin\theta = x$$

This gives

$$f(x) = \frac{1}{2\pi}\left(2hx + 4\quad\right) = m \qquad \ldots\ldots \quad (16)$$

$$m = 1, 2, \ldots$$

where

$$\epsilon = \arctan\left\{\frac{x}{\dfrac{\left(\omega^2 - \dfrac{g^2(\gamma-1)}{c_o^2}\right)\left(-\sqrt{\omega^2\left(\dfrac{1}{c_o^2} - \dfrac{1}{c_o'^2}\right) - x^2} + \dfrac{g}{c_o'^2}\right)}{g\left(1-\dfrac{T}{T'}\right)\left(-\sqrt{\omega^2\left(\dfrac{1}{c_o^2}-\dfrac{1}{c_o'^2}\right)-x^2} + \dfrac{g}{c_o'^2} + \dfrac{\left(\omega^2 - \dfrac{g^2(\gamma-1)}{c_o'^2}\right)}{(T'/T-1)\,g}\right)} - \dfrac{g}{c_o'^2}}\right\}$$

Equation (16) was solved numerically for x for many different values of ω.

In order to describe the techniques used for solution, we introduce arbitrarily some temporary notation which simplifies the expression for the phase change ϵ .

Let
$$a_1 = \left(\frac{1}{c_o^2} - \frac{1}{c_o'^2}\right)$$
$$a_2 = g/c_o^2$$
$$a_3 = g/c_o'^2$$
$$a_4 = h$$
$$a_5 = (\gamma - 1)g$$
$$a_6 = (1 - T/T')g$$
$$a_7 = (T'/T - 1)g$$

The constants a_1 through a_7 depend only on the physical constants of the model atmosphere; they are all independent of ω and x. Using this simplified notation, the expression for f(x) becomes

$$f(x) = \frac{1}{2\pi}\left\{a_4 x + 4 \arctan\left[\frac{x}{\dfrac{(\omega^2 - a_3 a_5)(-\sqrt{\omega^2 a_1 - x^2} + a_3)}{a_6\left(-\sqrt{\omega^2 a_1 - x^2} + a_3 + \dfrac{(\omega^2 - a_3 a_5)}{a_7}\right)} - a_3}\right]\right\}$$
$$= m \qquad \ldots\ldots \quad (17)$$

Certain critical values of ω and x can readily be seen on inspection of the argument of the arctangent function.

For instance, when $\omega^2 = a_3 a_5$, that is when $\omega = g/c_o(\gamma-1)^{\frac{1}{2}}$ the complicated term in the denominator vanishes. This the Brunt frequence ω_B, and we must watch our results carefully in the neighbourhood of $\omega = \omega_B$.

Also, it can be seen that when x is greater than $\omega^2 a_1$ the denominator becomes imaginary. If we are to search for real roots (a necessary condition for the wave-guide modes we seek) x must be constrained to be less than or equal to $\omega^2 a_1$. This condition that x^2 must be less than $\omega^2 (\frac{1}{c_o^2} - \frac{1}{c_o'^2})$ in order to keep f(x) real is merely a restatement of the requirement that the angle of incidence must be less than the critical angle. This can be shown as follows:

$$x \leq \omega^2 (\frac{1}{c_o^2} - \frac{1}{c_o'^2})$$

$$\omega^2/c_o^2 \ (\sin\theta)^2 \leq \omega^2 (\frac{1}{c_o^2} - \frac{1}{c_o'^2})$$

$$(\sin\theta)^2 \leq 1 - (\frac{c_o}{c_o'})^2$$

$$(\cos\theta)^2 \geq (\frac{c_o}{c_o'})^2$$

The angle θ is the complement of the angle of incidence; hence

$$\sin i \geq c_o/c_o' \qquad \text{for a real root.}$$

As usual $\sin i_c = c_o/c_o'$ defines the critical angle.

Computation

In order to gain insight into the behaviour of f(x), a FORTRAN program was written for the IBM 7094-II at the University of Toronto. For a particular value of ω the program computed and plotted values of f(x) and its first derivative f'(x) for a wide range of values of x subject to the limitation $x \leq \omega^2 a_1$.

The expression used for the derivative was

$$f'(x) = \frac{\partial f(x)}{\partial x} = \frac{1}{\pi}\left\{ a_4 - 2\left(\frac{1}{(1 + \frac{S^2}{R^2})}\left[\frac{S'R - R'S}{R^2}\right]\right)\right\}$$

where $S = x\, a_6\left(a_3 - \sqrt{\omega^2 a_1 - x^2} + \dfrac{(\omega^2 - a_3 a_5)}{a_7}\right)$

$$S' = \frac{\partial S}{\partial x} = a_6\left\{a_3 - \sqrt{\omega^2 a_1 - x^2} + \frac{(\omega^2 - a_3 a_5)}{a_7}\right\} + x a_6\left\{\frac{x}{\sqrt{\omega^2 a_1 - x^2}}\right\}$$

$$R = (\omega^2 - a_2 a_5 - a_3 a_6)\left[a_3 - \sqrt{\omega^2 a_1 - x^2} - a_3 a_6\left(\frac{\omega^2 - a_3 a_5}{a_7}\right)\right]$$

$$R' = \frac{\partial R}{\partial x} = (\omega^2 - a_2 a_5 - a_3 a_6)\left(\frac{x}{\sqrt{\omega^2 a_1 - x_2}}\right)$$

Several exploratory runs of this first program were made, using different values of ω with $m = 1$. In each case tested, $f(x)$ either had no real root at all, or else it had a real root and both $f(x)$ and $f'(x)$ were continuous near the root. On the strength of these results, a second FORTRAN program was written which used Newton's formula

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

in an automatic iterative procedure.

Unfortunately, for values of ω which approached the cutoff frequency ω_c, the derivative $f'(x)$ became unstable and discontinuous, and it was not possible to use methods which depend on continuity of the function and its derivative. Instead, it became necessary to revert to a program of the first type, and to study $f(x)$ in detail for each value of ω. Wherever a real root of $f(x) = m$ was found to exist, it was calculated by a short program which performed a simple geometrical interpolation between the closest points on the graph.

Values of ω were chosen in such a way as to ensure good coverage of the range between ordinary acoustic frequencies (periods of a few seconds) and the cutoff frequency ω_c. A careful study was made of the behaviour of $f(x)$ at frequencies below ω_c to make sure that no additional branches of the $x(\omega)$ curve could exist undetected. Particular attention was paid to frequencies near $\omega_B = g/c_o(\gamma-1)^{1/2}$, the Brunt Frequency.

Having calculated $x(\omega) = (\omega/c_o)\sin\theta$, it is a simple matter to compute certain physically observable parameters which are useful in analysing the behaviour of the model.

The horizontal component of the propagation constant (wave number) is simply

$$k_x(\omega) = \left(\frac{\omega}{c_o}\right)\cos\theta = \left(\frac{w^2}{c_o^{\,2}} - x^2\right)^{1/2}$$

From this equation the phase velocity $\frac{\omega}{k_x}$ and the group velocity $\frac{\omega}{k_x}$ may be derived.

The foregoing describes the method of computation. The results are presented and discussed in subsequent sections.

CHAPTER V

QUANTITATIVE RESULTS

The detailed results of the computations for the fundamental mode in the gravity-acoustic model of the stratosphere are presented in Table 1. It shows the angular frequency at which the calculation was made, the calculated value of the variable x, and the corresponding value of the horizontal wave number k_x.

Cutoff occurs at ω_c = 0.21 rad/sec for the fundamental mode. Below that frequency, the function f(x) has no real roots. A careful study was made of the behaviour of f(x) in the neighbourhood of the Brunt frequency, and it can be conclusively stated that there is no branch of the $k_x(\omega)$ curve containing ω_B.

Table 2 is derived from Table 1, and contains lists of group and phase velocities and the corresponding angular frequencies and periods. Data from Table 1 is plotted in Figure 5. Data from Table 2 is plotted in Figure 6 and the results of Press and Harkrider in Figure 7.

Similar calculations were made for other modes (m = 2, 3, 4) but they have been excluded as uninteresting, since, as would be expected, the cutoff frequencies were much higher. No anomalous behaviour was discovered for those modes.

TABLE 1 - COMPUTED RESULTS FOR m = 1

ω	x (x 10^3)	k_x (x 10^2)
2.210	0.16140621	0.74353431
2.110	0.16120411	0.70987358
2.010	0.16162911	0.67621192
1.910	0.16187618	0.64254919
1.810	0.16215156	0.60888520
1.710	0.16246045	0.57521971
1.610	0.16280932	0.54155242
1.510	0.16320650	0.50788293
1.410	0.16366328	0.47421076
1.310	0.16419543	0.44053522
1.210	0.16482115	0.40685540
1.110	0.16556674	0.37317001
0.810	0.16902441	0.27205659
0.710	0.17104694	0.23831610
0.610	0.17350604	0.20454258
0.510	0.17733682	0.17070651
0.410	0.18360871	0.13674602
0.310	0.19601491	0.10246311
0.300	0.19801848	0.09899493
0.290	0.20023009	0.09551460
0.270	0.20710156	0.09192143
0.260	0.20565351	0.08850240
0.250	0.21302164	0.08138840
0.240	0.21789300	0.07777002
0.235	0.22077025	0.07593811
0.230	0.22405705	0.07408563
0.225	0.22794032	0.07220453
0.220	0.23252976	0.07028790
0.215	0.23828189	0.06831543
CUTOFF	——	——

TABLE 2 - ANGULAR FREQUENCY, PHASE VELOCITY, GROUP VELOCITY AND PERIOD.

ω rad/sec	Period, Minutes	Phase Velocity m/sec	Group Velocity m/sec
2.210	.04738	297.23	
2.160	.04848		297.08
2.110	.04963	297.24	
2.060	.05983		297.07
2.010	.05210	297.24	
1.960	.05343		297.06
1.910	.05483	297.25	
1.860	.05630		297.05
1.810	.05786	297.26	
1.760	.05950		297.04
1.710	.06124	297.28	
1.660	.06308		297.02
1.610	.06504	297.29	
1.560	.06713		297.00
1.510	.06935	297.31	
1.460	.07173		296.98
1.410	.07427	297.34	
1.360	.07700		296.95
1.310	.07994	297.37	
1.260	.08311		296.91
1.210	.08655	297.40	
1.160	.09028		296.86
1.110	.09434	297.45	
0.960	.10908		296.70
0.810	.12928	297.73	
0.760	.13778		296.38
0.710	.14749	297.92	
0.660	.15867		296.09
0.610	.17167	298.23	
0.560	.18700		295.54
0.510	.20533	298.76	
0.460	.22765		294.46
0.410	.25541	299.83	
0.360	.29089		291.69
0.310	.33781	302.55	
0.305	.34334		288.42
0.300	.34907	303.04	
0.295	.35499		287.33
0.290	.36110	303.62	
0.280	.37400		285.22
0.270	.38785	305.08	

Table 2, Cont'd

ω rad/sec	Period, Minutes	Phase Velocity m/sec	Group Velocity m/sec
0.265	.39517		282.45
0.260	.40277	306.02	
0.255	.41067		279.83
0.250	.41888	307.17	
0.245	.42743		276.37
0.240	.43633	308.60	
0.2375	.44093		272.94
0.2350	.44562	309.46	
0.2325	.45041		269.91
0.2300	.45530	310.45	
0.2275	.46031		265.80
0.2250	.46542	311.61	
0.2225	.47065		260.88
0.2200	.47600	313.00	
0.2175	.48147		253.49
0.2150	.48707	314.72	
CUTOFF			

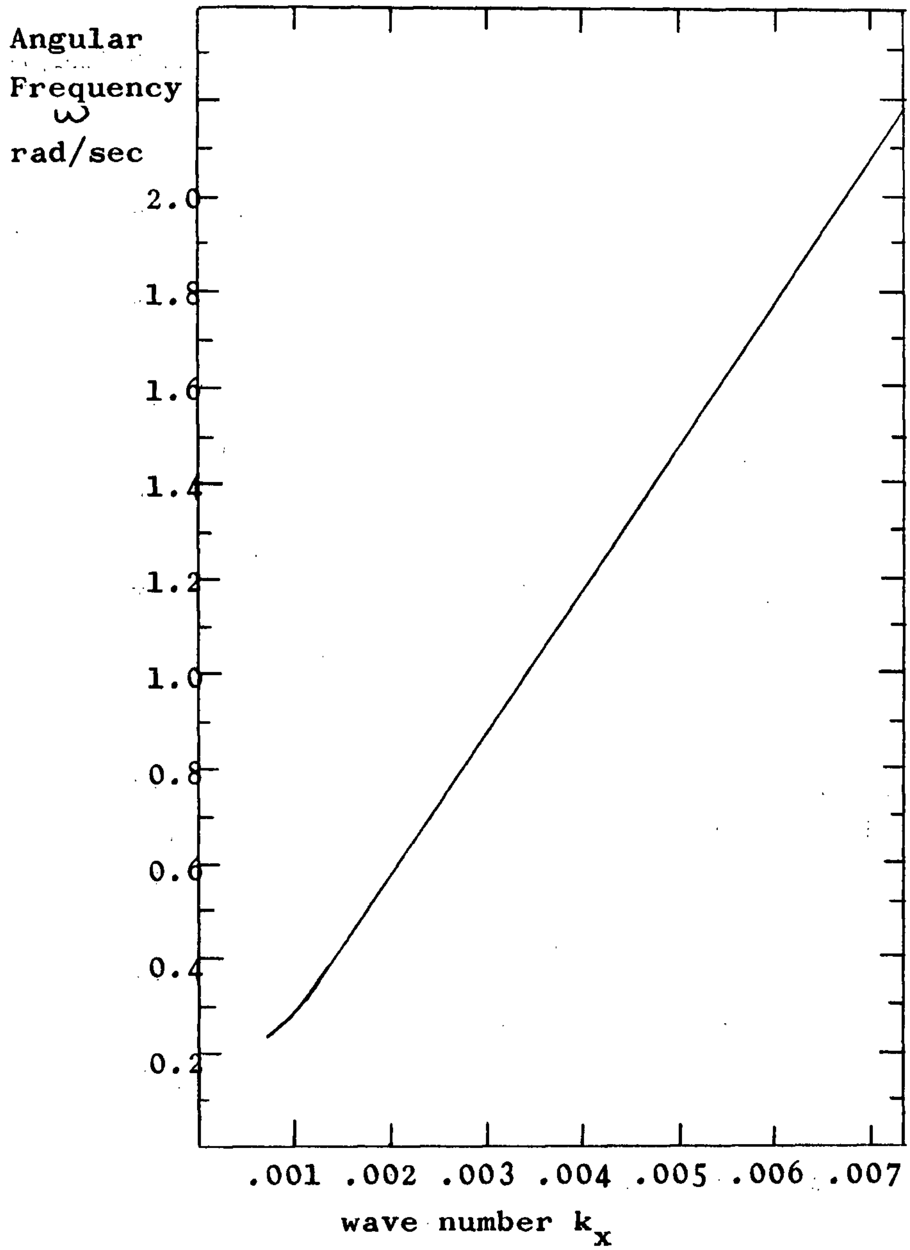

FIGURE 5. Graph of frequency against wave number for the isothermal layer model of the stratosphere.

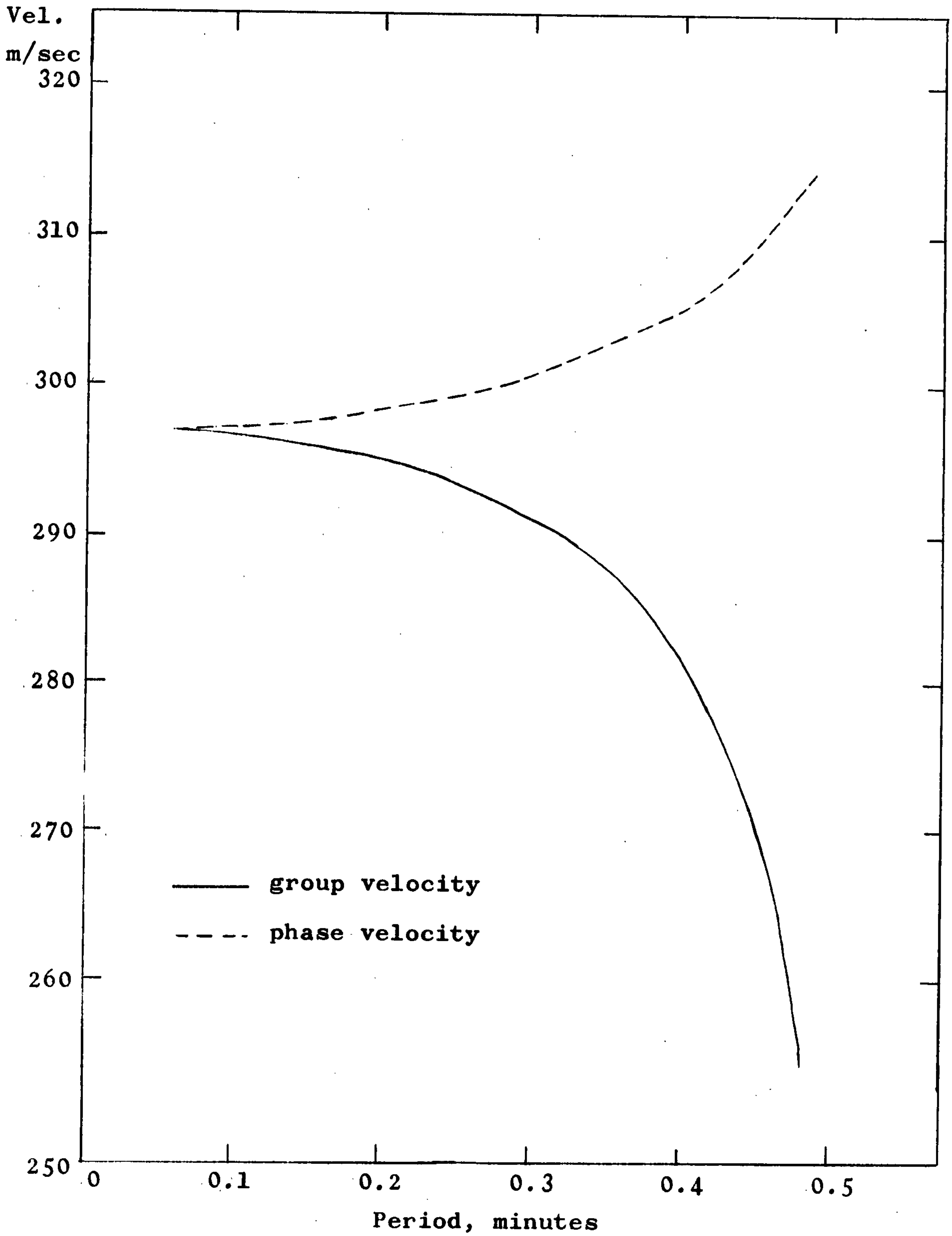

FIGURE 6. Group velocity and phase velocity for waveguide modes in the isothermal layer model.

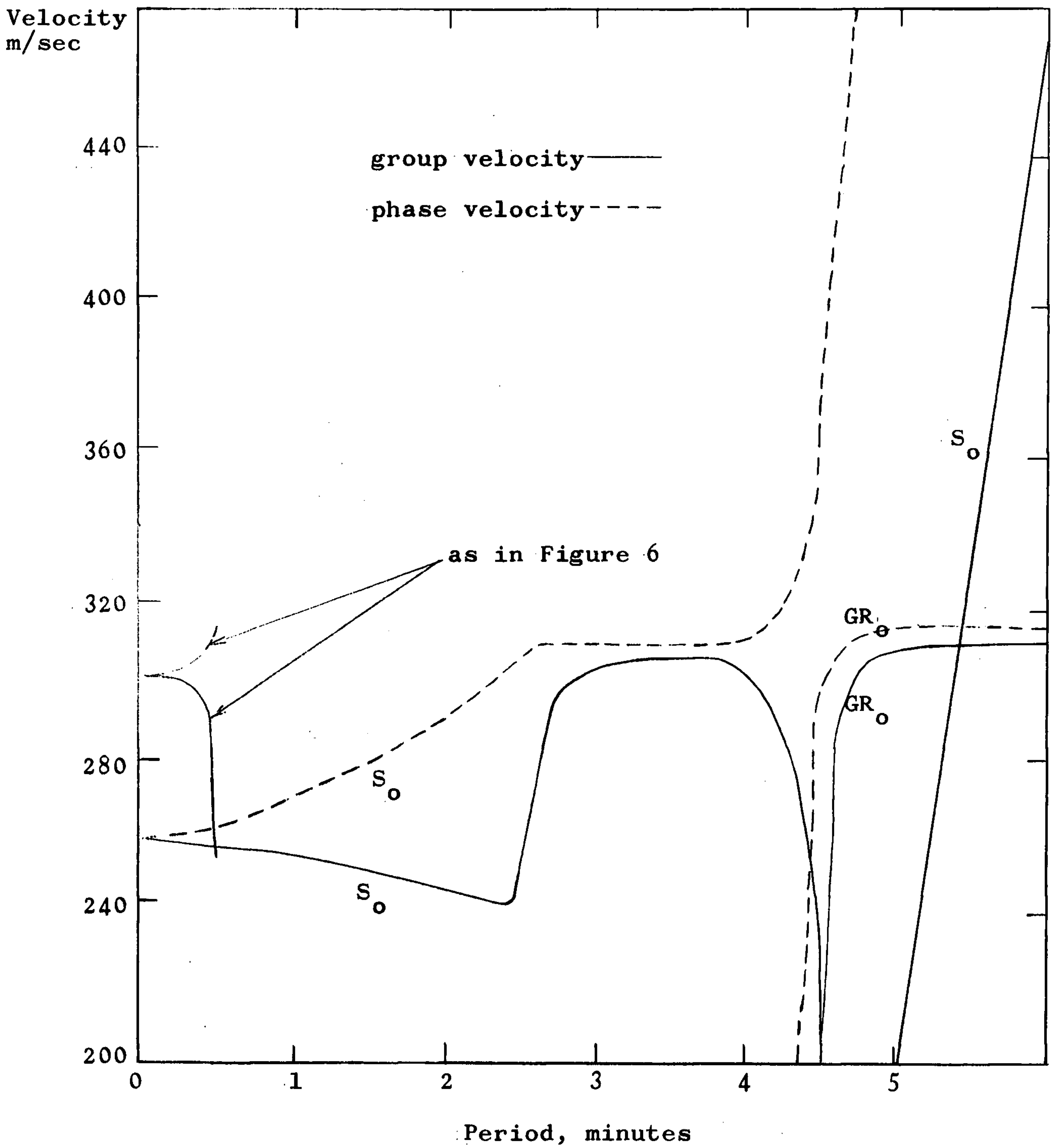

FIGURE 7. Dispersion curves of Press and Harkrider (sketched from their Figure 5) compared with the dispersion curves for the gravitating stratosphere model.

CHAPTER VI

DISCUSSION

The behaviour of the gravity-acoustic model can be summarized quite simply.

Waves having frequencies greater than 0.21 rad/sec (period of less than 0.5 minutes) can be trapped in the layer and propagate entirely within it, being reflected repeatedly from the top and bottom interfaces. The condition for constructive interference determines the eigenvalues of the angle of incidence for each value of ω, and consequently the horizontal component of wave number k_x. Values of k_x are presented in Table 1.

Waves having periods greater than 0.5 minutes cannot satisfy the condition of constructive interference for angles of incidence greater than the critical angle, and thus leak out of the guide into the surrounding media.

The model is not capable of, and was not intended to be capable of, representing the behaviour of the whole atmosphere. It is useful only for examining the waveguide effect of the stratosphere, since the model does not accurately portray those parts of the atmosphere which lie outside the layer. Our model atmosphere as a whole behaves as an infinitely deep gravitating atmosphere, because, unlike the true atmosphere, the model extends infinitely in both directions. As a consequence the natural gravity waves have infinite velocities and are not indicated in our solution. Moreover, only those waves which satisfy the secular equation of the layer (Equation 16) are portrayed accurately, and no conclusions can be drawn with respect to the subsequent paths of acoustic waves which have leaked out of the layer.

One of the most interesting features of the results is the fact that the cutoff frequency for the gravity-acoustic formulation is only 0.21 rad/sec, which is a considerably higher frequency than the cutoff for the simple acoustic case (0.11 rad/sec). Thus it would appear that by taking account of gravity in the theory we have modified the boundary conditions in a manner which permits waves which would be trapped in the ordinary acoustic theory to escape from the layer. It is instructive to re-examine the mathematics to see how such a phenomenon can occur.

<u>Frequency dependence of phase change at the critical angle</u>

Unlike the simple acoustic model, the gravity-acoustic model has a phase change which does not always approach zero as the angle of incidence approaches the critical angle. Let us denote by x_m the value of $(\omega/c_o)\sin\theta$ where $(90° - \theta) = i_c$. Of course, x_m need not be a root of the secular equation, but it is always equal to or greater than the root, if a root exists, because x_m is the largest value that x can take while ensuring that f(x) is pure real. Actually, x_m approaches the value of the root as ω approaches ω_c, the cutoff frequency.

At the critical angle, the expression for the phase change on each reflection becomes

$$2\epsilon = 2\arctan\left\{\frac{\omega\sqrt{\frac{1}{c_o^2}-\frac{1}{c_o'^2}}}{\frac{(\omega^2-\omega_B^2)}{g(1-T/T')}\cdot\frac{g/c_o'^2}{\left(\frac{(\omega^2-\omega_B^2)}{g(1-T/T')}+\frac{g}{c_o'^2}\right)}-\frac{g}{c_o'^2}}\right\}$$

where we have recognized $gG = -\omega_B^2$, the square of the Brunt frequency.

On simplification this becomes

$$2\epsilon = 2\arctan\left\{-\left[\frac{\omega}{g}\sqrt{\left(\frac{c_o'}{c_o}\right)^2-1}\left(1+\frac{c_o'^2(\omega^2-\omega_B^2)}{g^2(1-T/T')}\right)\right]\right\}$$

With the set of physical parameters adopted in this model, every term in the square brackets is always positive except $(\omega^2-\omega_B^2)$. For values of $\omega < \omega_B$ the term $(\omega^2-\omega_B^2)$ becomes negative. Indeed the sign of the whole argument of the arctangent changes when

$$\frac{c_o'^2}{g^2}\ \frac{(\omega^2-\omega_B^2)}{(1-T/T')} < -1$$

Using physical values for the gravity-acoustic model, this is found to occur when ω is less than 0.0163 rad/sec.

It should be emphasized that this calculation is for the particular value of x corresponding to the critical angle; the secular equation is in general satisfied by a different value of x. But the fact remains that the expression for the phase change on reflection is negative for acoustic frequencies and becomes positive at some frequency below the Brunt frequency.

This means that at acoustic frequencies, up to and including the acoustic cutoff frequency, the phase change on reflection has a negative sign and thus tends to effectively shorten the acoustic path length between subsequent reflections. This has the same effect as a decrease in h, the layer thickness, in that it raises the cutoff frequency.

It is interesting to note that if the cutoff frequency had been lower, that is if $\omega_c \approx \omega_B$ the effect of the change of sign of the phase change would have been to extend the cutoff to lower frequencies or even to introduce a stopping band and a new low frequency branch to the dispersion curve.

Such was not the case with this model, however, and although a careful check was made, no such branch was found to exist. The cutoff frequency was so much higher than the frequency at which ϵ changed sign that no interaction was possible. The f(x) curve did undergo an abrupt change of shape, however, when the sign of ϵ changed.

Comparison

The dispersion curves produced by the analysis of this model can be compared with those derived by Press and Harkrider for a multi-layered approximation to the Standard ARDC atmosphere. Figure 7 is a sketch showing the results for the fundamental modes for the model described in this thesis and for a model studied by Press and Harkrider having a half-space beginning at 220 km (their Figure 5). In both cases, the high frequency limit of both the phase and group velocities is the sound velocity of the channel with the lowest velocity. In their case that channel is the second low velocity layer at about 85 km, while in ours, naturally, it is the isothermal layer representing the stratosphere.

It is clear that in the complete model the wave guided in the stratosphere would leak through to the low velocity channel in the mesosphere. Thus the stratosphere would not be the perfect guide that we have proposed in this study. Nevertheless, for the wavelengths studied here the stratosphere would be a sufficiently good guide that it must be considered for many problems. In this respect our approximate treatment is superior to the more complete treatment of Press and Harkrider. The waves we have studied here would be included as slowly attenuated modes in a complete theory.

Press and Harkrider tried several variations of their model to determine which parts of the atmosphere govern the various segments and branches of the dispersion curves. One such version had the lower (20 km) temperature minimum entirely absent. Its removal left the short period portion of the curve relatively unchanged, but with the long period plateaus displaced. This was indicative of dependence of the long period plateaus on the structure of the lower 50 km of the atmosphere.

Another version had the second (85 km) low temperature zone removed. This resulted in dispersion curves having the same high velocity plateaus at long periods (over 5 minutes) as in the Standard version, but with the very short period waves having velocities corresponding to the 20 km channel. The velocity minimum at 2.5 minutes was eliminated. This version demonstrates that the short period portions of the dispersion curves depend heavily on the structure of the atmosphere above 50 km, and in particular on the low velocity channel at 85 km.

With this background it is possible to discuss the detailed behaviour of the Press and Harkrider curves and to compare the results of our more simple model based on a gravitating stratosphere.

Waves having periods below about 2.5 minutes are apparently guided by the upper (85 km) low velocity channel in much the same way as we have postulated for the lower channel in our model. The group and phase velocity curves are quite similar in shape for the two models. The reason that our curve cuts off at 0.5 minutes while the other is unbroken to 2.5 minutes is attributed to the much greater thickness and temperature contrasts which exist in the case of the upper channel. The absence of branches corresponding to the lower channel for very small periods indicates that the two low velocity channels are coupled, and that energy from the lower leaks into the upper.

In the Press and Harkrider model, energy begins to leak out of the upper channel at periods about 2.5 minutes. Although the group velocity is close to the sound velocity of the lower channel, we know from the results of our own model that the branch from 2.5 minutes to 4.5 minutes does not correspond to waves guided simply by the lower channel. Instead it must represent a more complicated mode involving at least the lower 50 km of the atmosphere.

At periods greater than 4.5 minutes, acoustic-gravity waves are no longer guided by any of the structures of the lower atmosphere, but appear to represent the gravity branch. They, too, appear to be quite dependent on the structure of the atmosphere below 50 km.

Conclusions

1. An isolated isothermal layer such as that proposed to represent the stratosphere does not behave as a lossless wave-guide for frequencies in the order of the Brunt frequency and lower.

2. The effect of gravity on the boundary conditions is to increase the cutoff frequency by modifying the expression for the phase change on reflection. The effect is sufficient to change the cutoff by a factor of two.

3. A wave-guide which conducts waves of periods greater than 0.5 minutes requires increased layer thickness, increased temperature contrasts, a rigid earth's surface, or all three.

4. A simple isothermal layer model of the stratosphere is not sufficient to account for the gravity-acoustic wave transmission properties of the atmosphere over any frequency range. If, however, we had modeled the mesosphere channel instead, our results would have been valid for periods up to about 1 minute.

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