NONLINEAR STABILITY AND STATISTICAL EQUILIBRIUM
OF FORCED AND DISSIPATED FLOW

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ABSTRACT

A global analysis for the hydrodynamical system defined for a homogeneous, incompressible layer of fluid on the $\beta$-plane is performed in both infinite and finite function space. Its application to global stability has yielded an algorithm for characterizing flows based on the existence of initially growing perturbations as opposed to the normal mode analysis; its application to the search for optimal initial perturbations has led to the least upper bound of energy growth rate; its application to multiple equilibria has given rise to a necessary condition for their existence; its application to the study of the relationship of modal to nonmodal growth rates has uncovered the cause underlying many aspects of the limitation of the modal stability analysis including the failure to predict transient growth of disturbances in stable flows and the underestimation of the intensity of initial development of instability in unstable flows. Numerical illustrations made for some specific flows have strengthened the general results, suggesting that a stability analysis of a hydrodynamical system without a global analysis is likely to be limited in many important aspects.

The local analysis of asymptotic behavior of nonmodal disturbances to hyperbolic equilibria of the system have established: a) for any subcritical flow outside of
monotonic, global stability regime, there exists a finite neighborhood around the origin of $\mathbb{R}^M$ such that a disturbance initialized in this neighborhood will ultimately decay to zero after it exhibits Orr's temporal amplification; b) for any supercritical flow, there exists a finite neighborhood adjacent to the origin of $\mathbb{R}^M$ such that a disturbance initialized in this neighborhood will persist as $t \to \infty$; and c) the nature of the persistent disturbances is related to the nature of the nonhyperbolic point in parameter space of interest. The numerical experiments are seen to confirm these predictions.

Closure modeling of forced-dissipated statistical equilibrium of perturbed flows arising from initially uniform zonal flows over random topography is done with special regard to the correlation between disturbance and underlying topography and the resulting stress. Such an exercise has led to, on one hand, the numerical results for topographical stress suggesting clearly the significance of this force in overall momentum budget of large scale ocean circulations. On the other hand, it has led to an appreciation that the detailed conservation of energy and potential enstrophy, which holds regardless of the presence of dissipation in the system, provides a means for systematic investigation of nonlinear transfer of the these quantities among interacting triads, an area not accessible to other approaches.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABSTRACT</td>
<td>i</td>
</tr>
<tr>
<td>TABLE OF CONTENTS</td>
<td>iii</td>
</tr>
<tr>
<td>LIST OF FIGURES</td>
<td>vii</td>
</tr>
<tr>
<td>ACKNOWLEDGEMENTS</td>
<td>xi</td>
</tr>
</tbody>
</table>

## CHAPTER 1

**INTRODUCTION**

1.1 Overview .................................................................. 1  
1.2 Objective .................................................................. 5  
1.3 Preview of subsequent chapters .................................. 6

## CHAPTER 2

**GLOBAL ANALYSIS I: INFINITE DIMENSIONAL SYSTEM,**  
with application to: global stability, optimal perturbation, multiple equilibria and relation of initial modal to nonmodal growth rates

2.1 Introduction ................................................................ 8  
2.2 IBVP For Stability and Predictability ......................... 9  
2.3 Symmetrized Energy (Error) Equation .......................... 13  
2.4 Generalized Rayleigh Quotient and its properties ......... 14  
2.5 An Optimal Problem for the Generalized Rayleigh Quotient ........................................................................ 18  
2.6 Application I: global stability and optimal perturbation ................................................................. 22  
2.7 Application II: multiple equilibria ............................ 29
2.8 Application III: initial modal
   vs. nonmodal growth rate .................. 35
2.9 Concluding Remarks .......................... 43
Appendix 2A A variational approximation method for the
   optimal value of generalized Rayleigh quotient $\rho$. 45

CHAPTER 3

GLOBAL ANALYSIS II: FINITE DIMENSIONAL SYSTEM,
   with application to: global stability and optimal perturbation

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1</td>
<td>Introduction</td>
<td>46</td>
</tr>
<tr>
<td>3.2</td>
<td>Governing Equations in $\mathbb{R}^M$</td>
<td>48</td>
</tr>
<tr>
<td>3.3</td>
<td>Global analysis in $\mathbb{R}^M$</td>
<td>49</td>
</tr>
<tr>
<td>3.3.1</td>
<td>Disturbance equation</td>
<td>49</td>
</tr>
<tr>
<td>3.3.2</td>
<td>Symmetrized energy equation</td>
<td>51</td>
</tr>
<tr>
<td>3.3.3</td>
<td>Generalized Rayleigh Principle</td>
<td>54</td>
</tr>
<tr>
<td>3.4</td>
<td>Application to: global stability and optimal perturbation</td>
<td>56</td>
</tr>
<tr>
<td>3.5</td>
<td>Numerical illustrations</td>
<td>58</td>
</tr>
<tr>
<td>3.5.1</td>
<td>Set-up of the numerical experiments</td>
<td>58</td>
</tr>
<tr>
<td>3.5.2</td>
<td>Numerical results</td>
<td>59</td>
</tr>
<tr>
<td>3.6</td>
<td>Concluding Remarks</td>
<td>63</td>
</tr>
<tr>
<td>Appendix 3A</td>
<td>Algebraic properties of the set ${s_i}$</td>
<td>64</td>
</tr>
<tr>
<td>Appendix 3B</td>
<td>Physical properties of the set ${s_i}$</td>
<td>65</td>
</tr>
</tbody>
</table>
CHAPTER 4

FINITE AMPLITUDE NONMODAL DISTURBANCE I: INITIAL BEHAVIOR, its relation to initial intense development of disturbances

4.1 Introduction ........................................... 74
4.2 Nonmodal disturbance over transient period .......... 75
   4.2.1 Modal vs. nonmodal growth rate
      at initial instant .................................... 75
   4.2.2 Explosive development of nonmodal disturbance .. 77
4.3 Numerical illustrations .................................. 78
4.4 Concluding Remarks ........................................ 82
Appendix 4A Modal growth rate expressed in terms of \( \rho \) .... 83
Appendix 4B Fundamental properties of nonmodal disturbance 84

CHAPTER 5

FINITE AMPLITUDE NONMODAL DISTURBANCE II: ASYMPTOTIC BEHAVIOR, its relation to bifurcation and multiple equilibria

5.1 Introduction ............................................... 95
5.2 Asymptotic decay as \( t \to \infty \) in subcritical flow ....... 96
5.3 Persistence as \( t \to \infty \) in supercritical flow ........... 99
5.4 Persistence, criticality and supercritical bifurcation 104
5.5 Numerical illustrations ..................................... 108
5.6 Concluding remarks ........................................ 113
Appendix 5A The direct method of Liapunov .................. 115
Appendix 5B Auxiliary lemmas .................................. 116
Appendix 5C Real canonical theory of linear operator ....... 119
Appendix 5D Liapunov instability function .................... 121
CHAPTER 6

CLOSURE MODELING: FORCED-DISSIPATED STATISTICAL EQUILIBRIUM OF LARGE SCALE QUASI-GEOSTROPHIC FLOWS OVER RANDOM TOPOGRAPHY

6.1 Introduction ........................................... 134

6.2 Closure Formulation ................................. 135

6.2.1 A self-consistent model .......................... 135

6.2.2 Moment Equations ................................. 136

6.2.3 Closure Hypothesis and Master Equations ...... 139

6.3 Numerical results and comparison with DNS .......... 144

6.3.1 Solution method .................................. 144

6.3.2 Model parameters ................................. 146

6.3.3 Numerical results ................................ 147

6.4 Concluding remarks .................................. 155

Appendix 6A Expressions for $T_{1,k}, T_{2,k}, S_{2,k}$ and $S_{3,k}$ .... 157

Appendix 6B Conservation properties of the closure model .. 158

CHAPTER 7

CONCLUSIONS .............................................. 169

REFERENCES ................................................. 174
3.1 Streamfunctions of three representative equilibrium states for $U^* = 22 \text{ m/s}$ and $h_i = 500$ m. (a) $1/r = 29$ days; (b) $1/r = 11$ days; (c) $1/r = 5.5$ days. The rest of the parameters are given in the text.

3.2 Stability regime diagram. The solid line for the finite stability measure $r_N(\Psi)$. The dash lines $r = r^* = 0.17/\text{day}$ for the MGS boundary, $r = r_L = 0.075/\text{day}$ for the linear stability boundary and $r = r$ is for the diagonal dash line which is used for testing the condition $r_N(\Psi) - r > 0$.

3.3 An example of MGS. The equilibrium state $\Psi$ is (c) in Fig 3.1. The initial perturbations $\delta$ are randomly generated with the ratio $\delta$ of their initial kinetic energy to the energy in $\Psi$ ranging from 0.2 to 1.0.

3.4 Existence of initially growing nonmodal $s_m$ to equilibria $\Psi(r)$ in subcritical regime, i.e., region (II) in Fig 3.2.

3.5 The spatial configurations of five initially growing nonmodal perturbations $s_i$ to the basic state (b) in Fig 3.1.

3.6 The energy time series for five disturbances initialized from the $s_i$ shown in Fig 3.5.

3.7 The growth rates of disturbances initialized from nonmodal perturbations $s_i$ over an initial growing period. The basic state $\Psi$ is Fig 3.1 (a).

viii
4.1 Maximum initial modal versus nonmodal growth rate, i.e., $r_{L}(\tilde{V}) - r$ vs. $r_{N}(\tilde{V}) - r$, for the set of equilibria $\{\tilde{V}(U)\}$ over zonal wavenumber-1 topography of height 1000 m. The Ekman damping coefficient $r = 1/(15\text{days})$ is indicated by the horizontal dash line. The three vertical dash lines are for the MGS boundary $U = \bar{U} = 6.6$ m/s, and for linear stability boundaries $U^* = \bar{U}_L^* = 10.5$ m/s and $U^* = \bar{U}_N^* = 26.3$ m/s, where $\bar{U}_N$ and $\bar{U}_L$, $\bar{U}_N^*$ and $\bar{U}_L^*$ are obtained as roots to the equations, $r_N(\tilde{V}(U^*)) - r = 0$ and $r_L(\tilde{V}(U^*)) - r = 0$, respectively. ..............

4.2 (a) Topography contours; (b)-(f) streamfunctions of several representative states from the set of $\{\tilde{V}(U)\}$ used in Fig 4.1, corresponding to $U = 6.20$ m/s in (I), $9.12$ m/s in (II), $11.33$ and $17.85$ m/s in (III) and $40.13$ m/s in (IV), respectively. ...

4.3 The same as Fig 4.1 except that the set of equilibria $\{\tilde{V}(U)\}$ used here corresponds to the zonal wavenumber-2 topography. ..................

4.4 The same as Fig 4.1 except that the set of equilibria $\{\tilde{V}(U)\}$ used here corresponds to $1/r = 30$ days. ....................

4.5 Basic states, nonmodal and modal initial perturbations for the numerical experiments. (a) for streamfunction of the equilibrium state taken from Fig 4.4 corresponding to $U = 16.3$ m/s, (b) and (c) for its fast growing initial nonmodal and modal perturbations, respectively. ................

4.6 Evolution of the growth rate with time. The optimal nonmodal initial perturbation $s_M$ (see Fig 4.5(b)) has growth rate $\sigma_N(s_M;\tilde{V}) = 1/(5.29 \text{days})$, whereas the fast growing modal perturbation (FGMP) (see Fig 4.5(c)) has $\Re(\sigma) = 1/(9.52 \text{days})$. The initial perturbations are indicated in Fig 4.5 (a) and (b) .................

4.7 Evolution of disturbance energy with time for the case shown in Fig 4.6. The initial perturbation energy is 20% of the energy in the equilibrium states to ensure finite amplitude for the initial perturbations. ..................
5.1 Stability regime diagram for a family of equilibria \( \{ \dot{\mathbf{v}}(r) \} \) with \( 1/r \) ranging from 29.5 days to 3.5 days, \( U = 22.0 \) m/s and topography being of zonal wavenumber-1 and of height 500 m. The top curve and the curve \( abcd \) are its \( r_N(\mathbf{v}) \) and \( r_L(\mathbf{v}) \), respectively, with \( r=\overline{r}_N=1/(5.71 \text{ days}) \) and \( r=\overline{r}_L=1/(13.33 \text{ days}) \) as its MGS boundary and linear stability boundary. The curve \( efgb \) is \( r_L(\mathbf{v}) \) for the set of equilibria \( \{ \dot{\mathbf{v}}(r) \} \) bifurcating at criticality \( r=\overline{r}_L \) from the primary branch \( \{ \mathbf{v}(r) \} \). \( r=\overline{r}_L'=1/(18.5 \text{ days}) \) is the linear stability boundary for the set \( \{ \dot{\mathbf{v}}'(r) \} \).

5.2 Asymptotic nonvanishing steady states of nonmodal disturbances. The underlying equilibrium states are those located on the part \( (f \rightarrow b) \) of the primary branch (cf. Fig 5.1) for values of \( r \) given in the figure.

5.3 Streamfunctions for bifurcation of an equilibrium state \( (a) \) into a new steady flow \( (f) \). The snapshots \( (c)-(f) \) are from the experiment for \( 1/r=17.1 \text{ days} \) (cf. the thick solid line in Fig 5.2).

5.4 Local uniqueness of asymptotic steady state of nonmodal disturbances. The underlying equilibrium state is the same as one in Fig 5.2 for experiment of \( 1/r=16.4 \text{ days} \). The \( s_M' \) is obtained from scaling \( s_M \) such that \( \mathbf{v}(t;s_M') \) at \( t=0 \) has 10% of the basic state energy.

5.5 Periodic limiting states of nonmodal disturbances, with periods 46.3 days for solid line and 85.8 days for dash line, respectively. The underlying equilibria are located on the unstable section of the stationary bifurcation branch (cf. Fig 5.1), with the values of \( r \) as indicated.

5.6 Streamfunctions for bifurcation of an equilibrium state \( (a) \) into a periodic flow. The snapshots \( (b)-(f) \) are taken from the experiment for \( 1/r=24.8 \text{ days} \) (cf. the solid line in Fig 5.5) over a cycle of oscillation, with \( t'=336.9 \text{ days} \) and \( T=49.3 \text{ days} \).
5.7 Repeated supercritical bifurcation for the primary branch of equilibria (cf. Fig 5.1). The point marked by $x$ on the primary branch is a stationary bifurcation point whereas the symbol + indicates the Hopf bifurcation point. The lines drawn with dash corresponds to unstable equilibrium states. .......................... 132

5.8 Nonmodal versus modal initial perturbations in transition to a periodic state. The basic state is from the stationary bifurcation branch $\{\tilde{\psi}(r)\}$ with $1/r=18.8$ days (cf. Fig 5.1), located near the secondary bifurcation point $r=r'$. The initial growth rate of $\dot{\psi}(t;\mathbf{s})$ and $\dot{\phi}(t;\text{Re}(\mathbf{z}))$ are $1/(5.08$ days) and $1/(400.0$ days), respectively. ............ 133

6.1 Topographic stress $\tau$ as a function of $U$. The parameters are $(h_{\text{max}}, \beta, r) = (4.0, 0.8, 0.12)$. The solid line is for the closure results and symbols for the dns data. The resonant point corresponds to $U_r = 1.0$. .................. 163

6.2 Streamfunction ($\psi = -Uy+\phi$) for the two representative flows at $t=20$ (or $t=230$ days), with parameters as the same as in Fig 6.1. (a): the subresonant flow with $U = 0.25$; (b): the superresonant flow with $U = 2.75$. The dash contours are for negative values. .................. 164

6.3 Enstrophy (a), topographic stress (b) and vorticity-topography correlation (c) spectra (solid lines) for the subresonant flow case $U = 0.25$, with parameters the same as in Fig 6.1. The symbols are for the five dns ensemble data. 165

6.4 Vorticity (a) and Topography (b) for the superresonant flow shown in Fig 6.2(b). .......... 166

6.5 The same as in Fig 6.3 but for $U = 2.75$. ........ 167

6.6 Topographic stress $\tau$ as a function of $r$. The parameters are $(h_{\text{max}}, \beta, U) = (6.2.0, 0.8, 3.0)$. The solid line is for the closure results and symbols for the dns data. .................. 168
I would like to thank Dr. William Hsieh, my supervisor, for giving me the opportunity to roam far and wide in search of the ultimate theme of the thesis work as well as for his unfailing support and constant encouragement along the way. I would also like to acknowledge Dr. Greg Holloway for his long term coaching of my study of turbulence theory and for his supervision of my work on closure modelling, and Dr. John Fyfe for his enlightening suggestions in my stability studies and for his kindly providing me the β-plane channel model which has yielded many vivid illustrations.

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CHAPTER 1 INTRODUCTION

1.1 Overview

Determining the temporal evolution of disturbances in a given hydrodynamical system has been of much interest to oceanographers and meteorologists, since it is generally believed that many observed phenomena including the development of mesoscale eddies in the oceans and cyclone waves in the westerlies can be traced to some kind of instability. Mathematically, the task amounts to finding a solution to an initial boundary value problem (IBVP) posed for the disturbance. In practice, the IBVP is often linearized around the given flow and its solution often sought in a special form (i.e., modal form). This approach, referred to as modal analysis hereafter for simplicity, has occupied a prominent position in the literature. Much of the present understanding of hydrodynamical stability owes its origin to this approach. It has long been associated with the theory of Charney (1947) and Eady (1949) on baroclinic instability of the westerly wind.

However, some aspects of the temporal evolution of disturbances fall outside the scope of modal analysis. This is even true over an initial period of evolution of small disturbances, which is generally considered adequate for
linear theory. For example, it has been demonstrated using linear initial value problems that (a) properly configured perturbations exhibit large transient growth in some subcritical flows despite the absence of growing modal perturbations (cf. Orr; 1907; Rosen, 1971; Farrell, 1982; Boyd, 1983; O'Brien, 1990); and (b) the instantaneous growth rate is often considerably larger than the maximum modal growth rate over the initial period (cf. Farrell, 1982; 1988; 1989a; Boyd, 1983; O'Brien, 1990).

Further limitations of the modal analysis are noticed in applying the modal growth rate to account for the development of mesoscale eddies in the oceans and synoptic scale disturbances in the atmosphere. For example, the growth rates of mesoscale oceanic eddies from modal analysis are of the order of one year (Schulman, 1967) whereas the observation data in Crease (1962), Swallow (1971) and Koshlyakov and Grachev (1973) suggest that it is of order of a few months. Another example is found in the study of cyclonegenesis. It is well known that the typical period of deepening of observed cyclones is between 12 and 48 hours (cf. Roebber, 1984; Sanders, 1986), clearly larger than any found in modal analysis of baroclinic instability, say, 133 hours in Valdes and Hoskins (1988).

While there is little doubt that the study based on the
linear IBVP (e.g., Case, 1960; Rosen, 1971) yields more accurate description of the onset and early evolution of infinitesimal disturbances than modal analysis by allowing for a more general type of initial perturbation, the asymptotic behavior of disturbances as $t \to \infty$ is somewhat beyond the capacity of the linear IBVP approach, especially when the flow under concern is subject to growing modal disturbances. In this case, arises the problem of nonlinear saturation which is concerned with the maximum amplitude attained by growing disturbances (Shepherd, 1988, 1989); or the problem of nonlinear equilibration which deals with the mechanisms responsible for the arrest of exponentially growth (Pedlosky, 1970, 1981; Salmon, 1980; Mak, 1985). Consideration of asymptotic behavior of disturbances also arises when one wonders what happens to the initially growing disturbances after transient growth (cf. Rosen, 1971; Boyd, 1983; Shepherd, 1985); or how a transition from one equilibrium state to another takes place in the presence of multiple equilibria (Charney & Devore, 1979; Vickroy & Dutton, 1979; Wiin-Nielsen, 1979a; Källén, 1981; Legras & Gill, 1983; Proefschrift, 1989).

The asymptotic behavior of nonmodal disturbances is not only of theoretical interest but also of practical value. Viewing the synoptic scale disturbances as transients superposed on the planetary scale westerlies, long range
forecast could benefit from the basic understanding of the long term behavior of nonmodal disturbances in model atmosphere flows such as those considered in Charney and Devore (1979). Taking the view that the climate itself is a question of distribution among all possible equilibrium states reachable by the atmosphere-oceans system and that climate change is a matter of the redistribution under the influence of a changing boundary (Charney and Devore, 1979), climate modelers are essentially facing the same problem as considered here (e.g., Marotzke, 1989).

Situations may arise in which the $t \to \infty$ behavior of persistent disturbances (or the resulting flows) can perhaps be usefully characterized only in a statistical sense. This will be the case when specific forms of initial perturbations to the hydrodynamical system under concern (or, equivalently when initial conditions for the perturbed system) are not precisely known in a deterministic sense but rather given in terms of some probabilistic measure. Statistical description of spectral behavior for the conservative quasi-geostrophic system on $\beta$-plane is obtained using equilibrium statistical mechanics by Salmon, Holloway and Hendershott (1976), on a rotating sphere by Frederiksen and Sawford (1980), for dissipated-forced flows on an $f$-plane using closure theory by Herring (1976) and Holloway (1978), and on a $\beta$-plane with large scale zonal flow component by Holloway (1987).
1.2 Objective

This thesis is to establish some basic properties regarding the temporal evolution of disturbances in a relatively simple nonlinear hydrodynamical system, taking the view that the analysis for such a system would yield some insight into the temporal behavior of disturbances in more realistic systems such as those modelling the Gulf Stream in the Atlantic Ocean and the westerlies at midlatitudes. The specific system considered here is the one defined for a homogeneous, incompressible layer of fluid on a $\beta$-plane, i.e.,

\[
(\partial/\partial t)\nabla^2 \psi + J(\psi, \nabla^2 \psi + \beta y + (h/H) f_0) = -r(\nabla^2 \psi - \nabla^2 \psi^*),
\]

subject to

\[
\begin{align*}
\psi(x+1,y,t) &= \psi(x,y,t), \\
\psi(x+1,y,t) &= \psi(x,y+d,t), \\
\psi(x,y,t) &= \psi(x,y,t), \\
\psi(x,y,t) &= \psi_0(x,y),
\end{align*}
\]

where $\psi$ is the streamfunction; $\nabla^2$ the two-dimensional Laplacian; $J$ the Jacobian; $\beta$ the beta parameter; $f_0$ the Coriolis parameter at reference latitude $\theta_0$; $h(x,y)$ the bottom topography; $H$ the mean depth of the fluid, $r$ the
Ekman damping coefficient and $\psi^*$ an externally prescribed forcing function; $l$ the periodic length in east-west direction (x-axis); $d$ in (1.2.2a) the channel width, or in (1.2.2b) the periodic length in north-south direction (y-axis). Specific steps taken to achieve this objective is detailed in subsequent preview of the thesis.

1.3 Preview of subsequent chapters

In chapter 2, we develop a global analysis for the system (1.2.1)-(1.2.3) in a Hilbert space consisting of all \textit{kinematically admissible functions} (defined as those satisfying B.C.(1.2.2a) or (1.2.2b)), with no assumption made on the nature of the flow except that it is governed by the system. The results are applied to a number of geophysical fluid dynamical problems ranging from global stability to the relation of modal to nonmodal initial growth rates.

In chapter 3, a finite dimensional version of the global analysis is made for flows subject to the channel condition (1.2.2a) and forced by an external zonal momentum source $\psi^* = -U^*y$. The restriction to this specific type of forcing is by no means essential but rather for the convenience of numerical experiments. Numerical calculations are made to illustrate the basic notions put forward in this and previous chapters.
Extending the study of the relation of initial nonmodal to modal perturbations in the Hilbert space, we give a finite dimensional account for the subject in chapter 4. Specifically, we extend the conclusions strictly true only at initial instant from the global analysis to an initial period of time by establishing some fundamental properties of nonmodal disturbances.

In chapter 5, a local analysis in state space is made on the asymptotic behavior of nonmodal disturbances as \( t \to \infty \), and on its relation to bifurcation of one equilibrium flow into another. The results from both numerical experiments and bifurcation analysis confirm the theoretical predictions.

In chapter 6, we use a finite dimensional version of (1.2.1), (1.2.2b) to study the stationary statistics of asymptotic states of perturbed flows arising from initially uniform zonal flows over random topography, with focus on the correlation between topography and vorticity and the resulting topographic stress. The work is based on a closure theory (Holloway, 1987) and direct numerical simulation of flows. The numerical results are obtained for parameters relevant to the midocean environment. Some brief concluding remarks on the results of the present study and possible extensions from the present work are given in chapter 7.
CHAPTER 2

GLOBAL ANALYSIS PART I: INFINITE DIMENSIONAL SYSTEM,
with application to: global stability, optimal perturbation,
multiple equilibria and relation of initial modal to nonmodal
growth rates

2.1 Introduction

Since Charney's striking theory on the instability of
westerly winds in the atmosphere (cf. Charney, 1947), the
normal modes stability analysis (hereafter referred to as
modal analysis for simplicity) has been playing, and will
continue to play, a vital role in geophysical fluid dynamics.
However, some shortcomings of the analysis have come into
light in various contexts of hydrodynamics (cf. Rosen, 1971;

In this chapter, we present an analysis which circumvents
some of these shortcomings and, together with the modal
analysis, will yield a fuller description of the stability
properties of a hydrodynamical system. For the reason that
will become apparent, the analysis is hereafter referred to
as global analysis as opposed to the modal analysis. The
global analysis performed here makes no reference to any
specific flow except that it is governed by the continuum
system (1.2.1)-(1.2.3). It is thus expected that the global
analysis and the resulting conclusions apply to any flow
The chapter is organized as follows. The global analysis is constructed step by step from § 2.2 to § 2.5, with § 2.2 devoted to the formulation of an initial boundary value problem (IBVP) for the stability (or predictability) of a flow $\psi(x,t;\psi_0)$ governed by (1.2.1)-(1.2.3); § 2.3 to derivation of a symmetrized kinetic energy (or error) equation for disturbance (or error); § 2.4 to introduction of the generalized Rayleigh quotient into the kinetic energy (or error) equation, and to its basic properties; § 2.5 to construction of an algorithm for finding the extreme value of the generalized Rayleigh quotient in some relevant function space $W$. In the remainder of the chapter, the global analysis is applied to monotonic, global stability (MGS), to a search for optimal initial perturbations in § 2.6; to multiple equilibria and predictability in § 2.7; to relation of initial modal to nonmodal growth rate in § 2.8, followed by concluding remarks in § 2.9.

2.2 IBVP for stability and predictability

The first step in the global analysis is to formulate an IBVP for disturbances (or errors) in a given flow. Let $\psi(x,t;\psi_0)$ denote a flow governed by (1.2.1)-(1.2.3) and initialized from $\psi_0(x,y)$. Consider another flow $\psi(x,t;\psi_0+\phi_0)$ which is realized under the same conditions as those for
\( \psi(x,t; \psi_0) \) except for its initial field: \( \psi_0 + \phi_0 \). To address the question if and under what condition the two flows stay together (roughly speaking, stable or predictable) or stay apart (unstable or unpredictable), we consider the difference field \( \phi \), given by

\[
\phi(x,t) = \psi(x,t; \psi_0 + \phi_0) - \psi(x,t; \psi_0).
\]  

The system describing the dynamics of \( \phi \) is obtained after subtracting the governing equations for \( \psi(x,t; \psi_0 + \phi_0) \) from those for \( \psi(x,t; \psi_0) \)

\[
(\partial/\partial t) \nabla^2 \phi + \mathcal{L}[\phi; \psi] + N[\phi] = 0,
\]  

\[
B.C. (III): \begin{cases}
\partial \phi/\partial x = 0, \text{for } y = 0, d \text{ and } 0 \leq x \leq 1, \\
\phi(x,y,t) = \phi(x+1,y,t), \text{for } 0 \leq y \leq d,
\end{cases}
\]  

or \n
\[
B.C. (IV): \begin{cases}
\phi(x,y,t) = \phi(x+1,y,t), \text{for } 0 \leq y \leq d, \\
\partial \phi/\partial x = 0, \text{for } y = 0, d \text{ and } 0 \leq x \leq 1,
\end{cases}
\]  

\[
I.C. \quad \phi(x,t_0) = \phi_0(x),
\]  

where \( \mathcal{L} \) and \( N : \mathbb{V} \rightarrow \mathbb{V} \) are respectively linear and nonlinear differential operators, and are given by (in indicial notation with summation convention applied to the repeated indices)

\[
\mathcal{L}[\phi; \psi] = \varepsilon_{ijk} \partial_j \psi \partial^i_m \partial^j_n \phi + \varepsilon_{ijk} \partial_j \phi \partial^i_m \partial^j_n \psi + r \partial_j \partial^j \phi,
\]  

\[
N[\phi] = \varepsilon_{ijk} \partial_j \phi \partial^i_m \partial^j_n \psi.
\]
\[ \frac{\partial a}{\partial x_1} \equiv (x_1, x_2, x_3) \equiv (x, y, z), \quad \text{(2.2.7a)} \]
\[ (z_1, z_2, z_3) = (0, 0, 1), \quad \text{(2.2.7b)} \]
\[ \epsilon_{ijk} = \text{alternating tensor}, \quad \text{(2.2.7c)} \]
\[ Q = \frac{\partial a}{\partial x_1} \psi + \beta y + (h/H) f_0 = \text{potential vorticity in } \psi, \quad \text{(2.2.8)} \]

with \( \mathcal{H} \) being the set of all kinematically admissible functions for which the evolution problems (1.2.1)-(1.2.3) and (2.2.2)-(2.2.4) are well defined. In particular, the elements from \( \mathcal{H} \) satisfy the following requirements:

(i) B.C. (III) (or B.C. (IV)) \quad \text{(2.2.9)}

(ii) smoothness sufficient to assure that the spatial derivatives in (1.2.1)-(1.2.3) (or (2.2.2)-(2.2.4) are well defined. \quad \text{(2.2.10)}

From now on we assume that \( \mathcal{H} \) is a Hilbert Space. This means, among the other things, that it is a vector space over \( \mathbb{R} \) with an inner product

\[ \langle a, b \rangle = \int_\Omega \overline{ab}; \|a\| = \langle a, a \rangle^{1/2}, \quad \forall \ a \text{ and } b \in \mathcal{H}, \quad \text{(2.2.11)} \]

where the overbar denotes the complex conjugate which is irrelevant for the elements from \( \mathcal{H} \) but is needed for the elements from \( \overline{\mathcal{H}} \). The latter is the counterpart of \( \mathcal{H} \) over \( \mathbb{C} \). The symbol \( \Omega \) denotes the area enclosed by the periodic channel (cf. (2.2.3a)) or the double periodic cell (cf. (2.2.3b) depending on the model geometry.

**Remark 2.2.1** The system (2.2.2)-(2.2.4) admits two
interpretations. If $\phi_0$ is taken as an initial perturbation to the flow $\psi(x,t;\psi_0)$ at $t = t_0$, then the system governs the subsequent evolution of the initial perturbation $\phi_0$, thereby leading to the statement on stability of the flow $\psi(x,t;\psi_0)$. On the other hand, if $\phi_0$ is viewed as an error introduced in assigning initial state $\psi_0$ to the flow $\psi(x,t;\psi_0)$, the system (2.2.2)-(2.2.4) then describes growth or decay of the initial error $\phi_0$, thus yielding information on predictability of the flow $\psi(x,t;\psi_0)$. In what follows, no assumption on the nature of $\phi_0$ is made unless specified explicitly, thereby unifying the treatment of stability and predictability within one framework.

It is obvious that if $\phi_0 = \text{constant}$, then $\phi = 0$ $\forall$ $t > 0$ (cf. (2.2.2)). It is not difficult to see that $\phi = 0$ is a null solution of (2.2.2)-(2.2.4) subject to $\phi_0 = \text{constant}$ for all $x \in \Omega$. Moreover, the stability of $\phi = 0$ implies the stability of the flow $\psi(x,t;\psi_0)$ governed by (1.2.1)-(1.2.3) as seen from (2.2.1). This correspondence leads us to study the stability problem for flow $\psi(x,t;\psi_0)$ via the one for $\phi = 0$. The latter is defined by (2.2.2)-(2.2.4) with $\phi_0 = \text{constant}$ for $x \in \Omega$. With this note in mind, we state the results below only in terms of the flow $\psi(x,t;\psi_0)$ for the sake of space. Also note that the restriction of $\phi_0$ to a non-constant field is not essential here but is physically motivated, since $\phi_0 = \text{constant}$ corresponds to a perturbation with kinetic energy.
equal to zero, or an error with variance being zero.

2.3 Symmetrized energy (error) equation

The second step in the global analysis is to symmetrize the linear operator $\mathcal{L}$ (cf. (2.2.5)), i.e., to remove the asymmetric constituents in $\mathcal{L}$ (e.g., the second term in $\mathcal{L}$) such that the time tendency in the disturbance energy equation (or error equation) is expressed in terms of a self-adjoint operator (cf., Reddy, 1986).

We start with taking the inner product of (2.2.2) with $\phi \in \mathcal{H}$, followed by directly evaluating individual terms and using the adjoint operator $\mathcal{L}^*$ of $\mathcal{L}$ defined by $<\mathcal{L}a,b>=<a,\mathcal{L}^*b>$, $\forall \ a,b \in \mathcal{H}$ (cf., chapter 8 in Reddy, 1986). Omitting the algebraic details involved, we give the resulting equation below

\[
\frac{d}{dt} <\delta_i^T, \delta_i^T> = <\phi, (\mathcal{L}+\mathcal{L}^*) \phi>,
\]  

(2.3.1)

\[
\mathcal{L}^* [\phi; \psi] = - \epsilon_{ijk} z_j \theta_i^T \theta_i \phi + \epsilon_{ijk} z_j \theta_i^T \theta_i \phi + r \delta_i \delta_i^T \phi,
\]  

(2.3.2)

\[
(\mathcal{L}+\mathcal{L}^*) [\phi; \psi] = - \delta_j (\delta_j \mathcal{U} \delta_i \phi) - \delta_j \mathcal{U} \delta_i \delta_j \phi + 2r \delta_i \delta_i \phi,
\]  

(2.3.3)

\[
\mathcal{U} = \epsilon_{ijk} z_j \theta_i^T \theta_i \psi,
\]  

(2.3.4)

where all the symbols here are the same as those in (2.2.2)-(2.2.8). Several remarks on (2.3.1)-(2.3.4) are in order.
Remark 2.3.1 The self-adjointness of $\mathcal{L} + \mathcal{L}^*$ can be easily verified by noting that $\langle a, (\mathcal{L} + \mathcal{L}^*) b \rangle = \langle (\mathcal{L} + \mathcal{L}^*) a, b \rangle$, $\forall a, b \in \mathcal{H}$. This mathematical property will be essential to our approach.

Remark 2.3.2 In adding the adjoint $\mathcal{L}^*$ to $\mathcal{L}$ to form the operator $\mathcal{L} + \mathcal{L}^*$, the physical processes corresponding to the non self-adjoint constituents in $\mathcal{L}$ are annihilated from (2.3.1) as desired. For example, the advection of potential vorticity $Q$ in flow $\psi$ by $\phi$, as represented by the skew-symmetric operator $\mathcal{L}^* [\phi; \psi] = \epsilon_{ijk} \partial_j \phi \partial_k \psi Q$, is eliminated from $\mathcal{L} + \mathcal{L}^*$.

Remark 2.3.3 In the symmetrization leading to (2.3.1)-(2.3.4), the nonlinear term $N[\phi]$ vanishes exactly without invoking linearization hypothesis. Thus, any statements which follow from (2.3.1) hold for arbitrary $\phi \in \mathcal{H}$ regardless of its size.

2.4 Generalized Rayleigh Quotient and its properties

The next two crucial steps in the global analysis are: to introduce a functional such that it is bounded in $\mathcal{H}$, and to construct an algorithm for obtaining the extreme value of the functional in $\mathcal{H}$. The objective of this section is to accomplish the former. We do this by a further manipulation of the operator $\mathcal{L} + \mathcal{L}^*$. 

14
While the mathematical manipulation involved is elementary, we must note the following two crucial physical facts. First, the velocity gradient \( \partial_j U_i \) in the flow \( \psi(x,t;\psi_0) \) (cf. (2.3.3)) can be decomposed into the rate of strain tensor \([d_{ij}]\) and the vorticity tensor \([v_{ij}]\), i.e.,

\[
[d_{ij}] = [d_{ij}] + [v_{ij}],
\]

\[
d_{ij} = (1/2) (\partial_i U_j + \partial_j U_i), \tag{2.4.1a}
\]

\[
v_{ij} = (1/2) (\partial_i U_j - \partial_j U_i). \tag{2.4.1b}
\]

Second, the spinning of \( \phi \) by \( \psi \), associated with \([v_{ij}]\), is conservative as far as the kinetic energy of \( \phi \) is concerned, which can be readily shown by noting the asymmetric nature of the tensor \([v_{ij}]\). Omitting the algebraic details, we give the resulting form

\[
\left( \frac{d}{dt} \right) (1/2) \langle \partial_1 \phi, \partial_1 \phi \rangle = \langle \partial_1 \phi, \partial_1 \phi \rangle (\rho(\phi;\psi) - r), \tag{2.4.2}
\]

\[
\rho(\phi;\psi) = \langle \phi, d\phi \rangle / \langle \phi, B\phi \rangle, \tag{2.4.3}
\]

\[
A[\phi;\psi] = -\partial_i d_{ij} \partial_j \phi, \tag{2.4.4}
\]

\[
B[\phi] = -\partial_i \partial_j \phi, \tag{2.4.5}
\]

where \( \rho(\phi;\psi): \mathbb{H} \to \mathbb{R} \) is the generalized Rayleigh quotient, \( A \) and \( B: \mathbb{H} \to \mathbb{H} \) are differential operators.

**Remark 2.4.1** Kinematically, \( \langle \phi, d\phi \rangle \) represents the straining of \( \phi \) by \( \psi \) via the presence of rate of strain tensor \([d_{ij}]\) of
\( \psi \) in (2.4.4). In terms of energetics, it serves as an energy source for the development of \( \phi \), or more precisely as an energy conversion between \( \psi \) and \( \phi \). On the other hand, \( \langle \phi, B\phi \rangle \) is the total kinetic energy in \( \phi \). Note that if \( r \neq 0 \), the total Ekman damping on \( \phi \) is \( r \langle \phi, B\phi \rangle \). Thus, the generalized Rayleigh Quotient \( \rho(\phi; \psi) \) can be physically interpreted as the ratio of energy conversion between \( \psi \) and \( \phi \) to the energy in \( \phi \), or to the Ekman dissipation if \( r \neq 0 \) (or the ratio of the error generation to the error).

The following properties of operators \( A \) and \( B \) are used throughout the chapter.

**Lemma 2.1** Let \( A \) and \( B \) be given by (2.4.4) and (2.4.5). Then \( A \) and \( B \) are self-adjoint on \( \mathcal{H} \), viz.

(i) \( A^* = A \), \hspace{1cm} (2.4.6)

(ii) \( B^* = B \), \( \langle a, Ba \rangle > 0 \), \( \forall a \in \mathcal{H} \) with \( a \neq \) constant. \hspace{1cm} (2.4.7)

**Proof:** (i) for any \( a, b \in \mathcal{H} \), we have

\[
\langle Aa, b \rangle = \int_\Omega b (-d_i d_{ij} \delta_j a) = - \int_\Omega \left\{ \partial_i (bd_{ij} \delta_j a) - \partial_i bd_{ij} \delta_j a \right\}
\]

\[
= \int_\Omega d_j bd_{ji} \delta_i a \quad (i \leftrightarrow j)
\]

\[
= \int_\Omega d_j bd_{ij} \delta_i a \quad (\text{since } d_{ij} = d_{ji})
\]

\[
= \int_\Omega \left\{ \partial_i (ad_{ij} \delta_j b) - a\partial_i d_{ij} \delta_j b \right\} = \langle a, Ab \rangle .
\]
On the other hand, we have $<\Delta a, b> = <a, \Delta^* b>$ by definition (cf., Reddy, 1986), which together with the last line in the above calculation of $<\Delta a, b>$ establishes the self-adjointness of $\Delta$.

(ii) follows similarly.

\textbf{Lemma 2.2} Let $\rho(\phi; \psi)$ be defined by (2.4.3). Then:

$$|\rho(\phi; \psi)| \leq \max_{x \in \Omega} \max (\lambda_1, \lambda_2), \ \forall \ \phi \in \mathcal{M},$$

where $\lambda_i$ ($i=1, 2$) are principal values of the rate of strain tensor $[d_{ij}]$ at $x \in \Omega$.

\textbf{Proof:} First, we can put $\rho(\phi; \psi)$ in the form

$$\rho(\phi; \psi) = \int_{\Omega} \delta_1 \phi d_{ij} \delta_j \phi / \int_{\Omega} \delta_1 \phi \delta_1 \phi, \ \phi \in \mathcal{M},$$

(2.4.8)

with $d_{ij}$ defined by (2.4.1b).

Since the rate of strain tensor $[d_{ij}]$ is real and symmetric, there thus exists a set of principal axes at each point $x \in \Omega$ with respect to which $[d_{ij}]$ has a diagonal form. In fact, such a diagonal form is achieved locally (i.e., at any $x \in \Omega$) by performing an orthogonal similarity transformation

$$[c_{ij}]^T [d_{ij}] [c_{ij}] = \text{diag} (\lambda_1, \lambda_2),$$

(2.4.10)

where $\lambda_i$ ($i=1, 2$) are the eigenvalues of

$$\det ([d_{ij}] - \lambda I) = 0, \ I = \text{a unit matrix} \in \mathbb{R}^{2x2},$$

(2.4.11)

and hence are the principal values of $[d_{ij}]$ at $x$, and the
column vectors of \( [c_{ij}] \in \mathbb{R}^{2 \times 2} \) are the corresponding eigenvectors and thus constitute a set of principal axes at \( \mathbf{x} \) for the rate of strain tensor \( [d_{ij}] \). It is clear that \( [c_{ij}] \) constructed with the normalized eigenvectors of (2.4.11) enjoys the following orthogonal property

\[
[c_{ij}] [c_{ij}]^T = [c_{ij}]^T [c_{ij}] = I \quad ; \quad [c_{ij}]^{-1} = [c_{ij}]^T.
\] (2.4.12)

An estimate of the numerator in (2.4.9) thus follows from (2.4.10) and (2.4.12)

\[
\left| \int_{\Omega} \partial_i \phi d_{ij} \partial_j \phi \right| \leq \int_{\Omega} \left| \partial_i \phi \right| \left( [c_{ij}]^T \right)^{-1} [c_{ij}]^T [d_{ij}] [c_{ij}]^T \partial_j \phi \left| \partial_i \xi \right| \quad \text{with} \quad \partial_i \xi = [c_{ij}]^{-1} \partial_j \phi
\]

\[
\leq \max_{\mathbf{x} \in \Omega} (\lambda_1, \lambda_2) \int_{\Omega} \partial_i \xi \partial_i \xi
\]

\[
= \max_{\mathbf{x} \in \Omega} (\lambda_1, \lambda_2) \int_{\Omega} \partial_i \phi \partial_i \phi
\]

from which (2.4.8) is obtained.

2.5 An optimal problem for the generalized Rayleigh quotient

As the very final step in the global analysis, we demonstrate that the extreme value of \( \rho \) in \( H \) can be obtained via a boundary eigenvalue problem (BEP).

**Lemma 2.3** If \( \bar{\phi} \in H \) maximizes the generalized Rayleigh Quotient \( \rho(\phi; \phi) \), then \( \bar{\phi} \) solves the boundary eigenvalue problem:

\[18\]
\[ \Delta \phi = \lambda \psi, \quad (2.5.1) \]

\[ \text{\phi satisfies the B.C. (III) (or B.C. (IV)),} \quad (2.5.2) \]

\[ \text{with } \lambda = \lambda[\psi] = \rho(\tilde{\phi};\psi) = \max_{\psi \in \mathcal{H}} \rho(\phi;\psi), \quad (2.5.3) \]

and \( A, S \) given by (2.4.4) and (2.4.5), respectively.

**Proof:** First, we associate with the domain \( \mathcal{H} \) of the Rayleigh quotient \( \rho(\phi;\psi) \), a set \( \mathcal{M} \) of comparison functions such that for any \( \phi \in \mathcal{H} \) and any \( \xi \in \mathcal{M} \), \( \phi + \varepsilon \xi \in \mathcal{H} \) for any \( \varepsilon \in \mathbb{R} \). This is done by choosing \( \mathcal{M} = \mathcal{H} \).

Now, define linear functionals \( \mathcal{F}, \mathcal{G} : \mathcal{H} \to \mathbb{R} \) by

\[ \mathcal{F}[\phi] = \langle \phi, \Delta \phi \rangle, \quad (2.5.4) \]
\[ \mathcal{G}[\phi] = \langle \phi, B\phi \rangle, \quad (2.5.5) \]

respectively. Then, for \( \rho(\phi) = \mathcal{F}[\phi]/\mathcal{G}[\phi] \) (where for simplicity of notation, \( \psi \) as an argument is suppressed),

\[ \delta \rho(\phi;\xi) = (1/\mathcal{F}[\phi]^2) \{ \mathcal{G}[\phi] \delta \mathcal{F}[\phi;\xi] - \delta \mathcal{G}[\phi;\xi] \mathcal{F}[\phi] \}. \]

If \( \tilde{\phi} \) in \( \mathcal{H} \) maximizes \( \rho \), then \( \delta \rho(\tilde{\phi};\xi) = 0 \), or

\[ \mathcal{G}[\tilde{\phi}] \delta \mathcal{F}[\tilde{\phi};\xi] - \delta \mathcal{G}[\tilde{\phi};\xi] \mathcal{F}[\tilde{\phi}] = 0 \quad \text{for all } \xi \in \mathcal{M}. \quad (2.5.6) \]

It is found from lemma 2.1 that

\[ \delta \mathcal{F}[\tilde{\phi};\xi] = 2\langle \xi, \Delta \tilde{\phi} \rangle, \quad (2.5.7a) \]
\[ \delta \mathcal{G}[\tilde{\phi};\xi] = 2\langle \xi, B\tilde{\phi} \rangle. \quad (2.5.7b) \]

Introducing (2.5.7) into (2.5.6) yields

\[ \langle \xi, (A-\lambda B)\tilde{\phi} \rangle = 0, \quad \forall \text{ any } \xi \in \mathcal{M}. \]
with $\lambda = \tilde{\lambda} = \mathcal{F}[\hat{\phi}] / \mathcal{G}[\hat{\phi}] = \max_{\phi \in \mathcal{H}} \mathcal{F}[\phi] / \mathcal{G}[\phi]$, which implies

$$(d - \lambda B) \hat{\phi} = 0,$$

thus completing the proof.

While lemma 2.3 provides a systematic means of solving the optimal problem $\max \rho$ in $\mathcal{H}$, its practical implementation may not turn out to be simple in the case of zonally asymmetrical flow $\psi$ which in generally leads to nonseparable coefficients in (2.5.1). However, the lemma is crucial to our subsequent development of the theory, as we will see soon. In appendix 2A, we present a Rayleigh-Ritz procedure for the numerical solution of the problem: $\max \rho$ in $\mathcal{H}$.

**Remark 2.5.1** As seen in writing $\tilde{\lambda}[\psi]$, $\tilde{\lambda}$ depends on the state of the flow $\psi(x,t;\psi_0)$ and hence is a function of time $t$ if $\psi_0 \in \{\psi\}$, where $\{\psi\}$ denotes the set of equilibrium states of (1.2.1)-(1.2.3). However, it is clear that

$$\tilde{\lambda}[\psi] = \tilde{\lambda}[\psi_0] = a \text{ constant if } \psi_0 \in \{\psi\}.$$  \hspace{1cm} (2.5.8)

For practical concerns, the following lemma assures us some desired properties of the eigensystem (2.5.1)-(2.5.2).

**Lemma 2.4** For the eigensystem (2.5.1)-(2.5.2), it holds that
(i) all eigenvalues $\lambda$ are real;

(ii) $\lambda$ is bounded;

(iii) if $\lambda_1 \neq \lambda_2$, then the corresponding eigenfunctions $\phi_1, \phi_2$ are orthogonal in the sense that $\langle \delta_1 \phi_1, \delta_1 \phi_2 \rangle = 0$.

Proof: (i) Let $(\lambda, \phi)$ be determined from (2.5.1). Consider

$$\langle A \phi, \phi \rangle = \langle \phi, A \phi \rangle = \langle \phi, A \phi \rangle \quad (\text{by Lemma 2.1(i)})$$  \hspace{1cm} (2.5.9)

It thus follows from (2.5.1) and (2.5.9) that

$$\langle \lambda B \phi, \phi \rangle = \langle \phi, \lambda B \phi \rangle \Rightarrow$$

$$\lambda \langle B \phi, \phi \rangle = \bar{\lambda} \langle \phi, B \phi \rangle = \bar{\lambda} \langle \phi, B \phi \rangle \quad (\text{by Lemma 2.1(ii)})$$

which holds only if $\bar{\lambda} = \lambda$, i.e., $\lambda$ is real.

(ii) The boundedness of $\lambda$ follows from lemma 2.2.

(iii) let $\lambda_1$ and $\phi_1$ (i=1,2) be stated above. Now, consider

$$\langle \phi_1, A \phi_2 \rangle = \langle A \phi_1, \phi_2 \rangle \Rightarrow$$

$$\langle \phi_1, \lambda_2 B \phi_2 \rangle = \langle \lambda_1 B \phi_1, \phi_2 \rangle \Rightarrow$$

$$(\lambda_1 - \lambda_2) \langle \delta_1 \phi_1, \delta_1 \phi_2 \rangle = 0$$

which leads to $\langle \delta_1 \phi_1, \delta_1 \phi_2 \rangle = 0$ if $\lambda_1 \neq \lambda_2$. \hfill $$

Remark 2.5.2 The physical interpretation of (iii) in lemma 2.4 is that the velocity fields associated with different $\lambda$ are orthogonal w.r.t the inner product (2.2.11).

To summarize, the global analysis includes: 1) the symmetrized energy (or error) equation (2.4.2); 2) an
algorithm for optimizing the generalized Rayleigh quotient $\rho$, i.e., lemma 2.3.

In view of the fact that a bounded functional is introduced in § 2.4 and a variational argument is invoked to relate the extreme problem to an Euler-type boundary eigenvalue problem, the global analysis developed here may fall into the category of variational energy method (Serrin, J., 1959a, 1959b; Joseph, D.D., 1966, 1976).

In the remainder of the chapter, we shall explore the results of the global analysis (i.e., lemma 2.1-2.4) in various contexts. First, we note from remark 2.2.1 that our results can be stated in terms of the stability of the flow $\psi(x,t;\psi_0)$, or equally well in terms of its predictability, depending on our interpretation of the initial data $\phi_0$. Except for a few occasions in the subsequent developments, however, much of argument is presented only in terms of the stability for the sake of space.

2.6 Application I: Global stability and optimal perturbation

First, we give the definition of global stability for the system (1.2.1)-(1.2.3)

**Definition 2.1** (global stability) Let $\phi(x,t;\phi_0)$ satisfy (2.2.2)-(2.2.4). Then:
(i) a flow $\psi(x,t;\psi_0)$ governed by (1.2.1)-(1.2.3) is said to be asymptotically stable if there exists a finite number $\delta$ such that $\forall \phi_0 \in H$

$$\langle \delta_1 \phi_0, \delta_1 \phi_0 \rangle \leq \delta \Rightarrow \langle \delta_1 \phi, \delta_1 \phi \rangle / \langle \delta_1 \phi_0, \delta_1 \phi_0 \rangle \to 0 \text{ as } t \to \infty.$$  

(2.6.1)

(ii) a flow $\psi(x,t;\psi_0)$ is said to be globally stable if the limit (2.6.1) holds when $\delta \to \infty$.

(iii) a flow $\psi(x,t;\psi_0)$ is said to be monotonically, globally stable if it is globally stable and, in addition, $(d/dt)\langle \delta_1 \phi, \delta_1 \phi \rangle \leq 0$, for $t>0$.

Remark 2.6.1 The $\delta$ here is referred to as an attracting radius of the flow $\psi(x,t;\psi_0)$ in $H$ since $\delta$ defines a subset of $H$ in which the acclaimed properties of $\psi(x,t;\psi_0)$ hold. In the case of global stability, $\delta = \infty$ and hence $\psi(x,t;\psi_0)$ has an infinite large attracting domain in $H$.

Definition 2.2 The finite stability measure of a flow $\psi(x,t;\psi_0)$, denoted by $r_N(\psi)$, is defined as

$$r_N(\psi) = \sup_{t > t_0} \lambda,$$  

(2.6.2)

where $\lambda$ is given by (2.5.3).

Theorem 2.1 A flow $\psi(x,t;\psi_0)$ is monotonic, globally stable if and only if

$$r_N(\psi) - r < 0.$$  

(2.6.3)
Proof: (i) sufficiency.
From (2.4.2) and (2.6.2)

\[ (d/dt')(1/2)\langle \partial_i \phi, \partial_i \phi \rangle \leq \langle \partial_i \phi, \partial_i \phi \rangle (r_N(\psi)-r). \]  

(2.6.4)

Integrating (2.6.4) with respect to \( t' \) from \( t_0 \) to \( t \) gives

\[ \langle \partial_i \phi, \partial_i \phi \rangle \leq \langle \partial_i \phi_0, \partial_i \phi_0 \rangle \exp \{2(r_N(\psi)-r)(t-t_0)\}, \forall t \geq t_0 \]  

(2.6.5)

where \( \phi_0 \) is initial perturbation. Taking the limit \( t \to \infty \) in (2.6.5) under the condition (2.6.3), we establish the sufficiency.

(ii) the converse (proof by contradiction). Suppose that \( \psi \) is MGS but \( r_N(\psi) - r \geq 0 \).

Then, take \( \tilde{\phi} \) (cf. lemma 2.3) as an initial perturbation, i.e.,

\[ \phi(x, t; \tilde{\phi}) = \hat{\phi}, \text{ at } t = t_0, \]

and evaluate the initial tendency of kinetic energy of \( \phi(x, t; \tilde{\phi}) \)

\[ (d/dt)(1/2)\langle \partial_i \phi, \partial_i \phi \rangle|_{t=t_0} = \langle \partial_i \hat{\phi}, \partial_i \hat{\phi} \rangle (\rho(\hat{\phi}; \psi) - r) \]  

(by 2.4.2)

\[ = \langle \partial_i \hat{\phi}, \partial_i \hat{\phi} \rangle (r_N(\psi) - r) \]

\[ \geq 0, \text{ (by 2.6.6)} \]

a statement contradicting to MGS of \( \psi(x, t; \psi_0) \). This completes the proof. \( \blacksquare \)

The concept of global stability occupies a prominent position in the global theory of stability of the Navier-Stokes equation (cf. Serrin, J., 1959a, 1959b; Joseph,
D.D., 1976) and the Boussinesq equation (cf. Joseph, D.D., 1966). However, it has barely appeared in geophysical fluid dynamics (GFD). The explicit account of global stability in GFD were taken in Vickroy and Dutton (1979) and Källén and Wiin-Nielsen (1980) for a three dimensional system. The paper by Crisciani and Mosetti (1990) essentially deals with the aspect of global stability for wind-driven ocean circulation. No optimization is made in these earlier papers. The lack of interest in this aspect of stability may arise from the consideration that any geophysically relevant flows can hardly be globally stable. Given some element of truthfulness in the above consideration, we will see that the global analysis originated from the study of global stability has applications in a number of important problems in GFD such as multiple equilibria, bounds on the growth rate of disturbances in supercritical flows and transient growth in subcritical flows. It is clear from the above proof that theorem 2.1 can also be stated in terms of the existence of initially growing perturbations (or errors) to a given flow \( \psi(x, t; \psi_0) \) at instant \( t \). Thus, we have

**Corollary 2.1** At any instant \( t' ( \geq t_0 ), \) there exists at least one initially growing perturbation (error) to the state \( \psi(x, t'; \psi_0) \) if and only if \( \lambda(\psi(x, t'; \psi_0)) - r > 0, \) where \( \lambda \) is given by (2.5.3).
Corollary 2.1 covers an important special case where $t' = t_0$ and the initial state $\psi_0 \in \{\psi\}$. In this special case, we have

**Corollary 2.2** Let $\psi_0 \in \{\psi\}$. Then:

$\lambda(\psi_0) - r > 0 \iff$ existence of initially growing $\phi_0$ to $\psi_0$, or

$r(\psi_0) - r > 0 \iff$ existence of initially growing $\phi_0$ to $\psi_0$.

Identifying initially growing perturbations to a given flow, a subject with a long history starting from Orr's work on plane Couette flow (1907), has been the focus of several recent papers on inviscid constant shear flows (Boyd 1983; Shepherd 1985 and Farrell 1987), on a specific zonally asymmetric flow (Farrell 1989c), on the Charney and Eady models with Ekman damping (Farrell 1982, 1984 and 1989a) and on the Green model (O'Brien, 1990). Directing attention to such perturbations is natural to synoptic meteorologists since forecast rules often emphasize the roles of these properly configured perturbations (Palmen and Newton, 1969). For zonal shear flows, it has been known since Orr (1907) that a perturbation with its phase lines oriented against the mean shear exhibits temporal amplification.

The forgoing results (i.e., Corollary 2.1 and 2.2) may be regarded as a continued effort of the above cited works in the context of time-dependent, nonlinear and zonal asymmetric
flows. Indeed, the corollaries together with lemma 2.3 constitute an algorithm for a systematical search of initially growing perturbations to a given flow. As opposed to the common notion of "lean against shear", it is not difficult to see from (2.4.2) and (2.4.3) that for instantaneous growth of a perturbation \( \phi(x) \) to a given flow \( \psi(x,t;\psi_0) \), it suffices to require that the ratio of the energy conversion between \( \phi \) and \( \psi \) to the energy in \( \phi \) at that instant be larger than the Ekman damping \( r \). It is significant that in the case of zonally asymmetric flows, what matters for a perturbation to grow is not simply the energy conversion rate between \( \psi \) and \( \phi \) but rather the generalized Rayleigh quotient (i.e., the energy conversion rate per unit disturbance energy), a scenario not obvious to our intuition.

To give the notion of optimal initial perturbation (or worst error) a quantitative measure, we introduce

**Definition 2.3** The instantaneous growth rate \( \sigma_N(\phi;\psi) \) of a disturbance (or error) \( \phi(x,t;\phi_0) \) in a flow \( \psi(x,t;\psi_0) \) is defined as

\[
\sigma_N(\phi;\psi) = \frac{1}{2} \left( \frac{d}{dt} \right) \ln \langle \delta J_1 \phi, \delta J_1 \phi \rangle.
\]  

(2.6.7)

**Theorem 2.2** At any instant \( t \), among all possible perturbations (or errors) \( \phi \) in \( \mathbb{H} \) to a flow \( \psi(x,t;\psi_0) \), \( \phi \)
(cf. lemma 2.3) is the optimal one (or the worst one) in the sense that \( \sigma_N(\phi;\psi) \geq \sigma_N(\phi;\psi), \forall \phi \in H. \)

Proof: from (2.4.2) and (2.6.7)
\[
\sigma_N(\phi;\psi) = \rho(\phi;\psi) - r \quad (\text{by lemma 2.3})
\]
\[
\leq \rho(\phi;\psi) - r \quad (\text{by (2.5.3)})
\]
\[
= \lambda(\psi) - r \quad (\text{by (2.5.3)})
\]
\[
= \sigma_N(\phi;\psi),
\]

where \( \lambda(\psi) \) is given by (2.5.3), which completes the proof. \( \blacksquare \)

The idea of optimal perturbation emerged in the search for favorably configured perturbation and its growth rate to explain the explosive growth observed in cyclonegenesis (Roebber, 1984; Sanders, 1986) or in model studies of initial value problems (Farrell, 1988, 1989a; O'Brien 1990). The optimal perturbations obtained in early studies have found other applications. For example, those determined for barotropic constant shear flow are used in optimal excitation of neutral Rossby waves (Farrell, 1988).

The optimal perturbations found in this study differ from those in Farrell (1988, 1989b) and O'Brien (1990) in several aspects. First, our search is carried out within a fully nonlinear context whereas the others were made using linear models. Second, the functional to be optimized is the
generalized Rayleigh quotient in our case but is the energy norm or the $L^2$ norm in other studies. Third, the optimization in this study is taken with respect to all possible kinematically admissible functions whereas the same process in previous studies is carried out conditionally, i.e., subject to some constraints which are physically sound but essentially subjective.

2.7 Application II: multiple equilibria

The notion of multiple equilibria was fairly recently introduced into geophysical fluid dynamics in an effort to account for the vacillation between the low index ("blocking pattern") and high index flows in the atmosphere (Charney & Devore, 1979) and for nonlinear effects on predictability (Vickroy & Dutton, 1979). The concept has since been extended to increasingly sophisticated physical models, yielding much insight into the variability in large scale atmospheric flows (Källén & Wiin-Nielsen, 1980; Källén; 1981; Legras & Gill, 1983; Rambaldi & Mo, 1984; Tung & Rosenthal, 1985), recurrence of weather systems (Proefschrift, 1989) and climate changes (Marotzke, 1989).

The difficulty with nonlinearity has led the early investigators to low order spectral models (Charney and Devore, 1979; Proefschrift, 1989) or numerical models (Källén, 1982, 1985). In the former case, the question
naturally arising is: are multiple equilibrium states simply the consequence of severe truncation to the original infinite dimensional systems? or do multiple equilibria exist in the original continuum models? In the latter case, given an equilibrium state obtained from numerical integration of the model equations, one wants to know if there exists another equilibrium state.

In this section, we address the above questions in the context of system (1.2.1)-(1.2.3) based on the global analysis. We do so by considering the problem of uniqueness of the equilibrium state. It is important to distinguish this uniqueness problem from the one for IBVP. In this problem, one seeks multiplicity of equilibrium states for a given set of external and internal model parameters. In contrast, the one for IBVP is concerned with how many flows can evolve from a given initial state. We shall show below that the uniqueness of IBVP holds unconditionally whereas the uniqueness of equilibrium state may break down for some values of model parameters, thus implying the possibility of multiple equilibria. First, we have

**Theorem 2.3** (uniqueness of IBVP (1.2.1)-(1.2.3)) The Velocity field of a flow \( \psi(x,t;\phi_0) \) is continuous w.r.t its initial condition and hence there can be only one flow evolving from a given initial velocity field.
Proof: Consider another flow $\psi(x,t;\psi_0+\phi_0)$ of (1.2.1)-(1.2.3) realized under the same conditions as those for $\psi(x,t;\psi_0)$ except its initial state $\psi_0+\phi_0$. It thus follows from (2.4.2) and (2.6.2) that the kinetic energy of the difference field $\phi(x,t;\phi_0)$ satisfies

$$(d/dt)<\partial_x \phi, \partial_x \phi> \leq 2<\partial_x \phi, \partial_x \phi>(r_N(\psi)-r)$$

which implies that for $t_0 \leq t$

$$0 \geq \int_{t_0}^{t} \left\{ (d/dt')<\partial_x \phi, \partial_x \phi>-2(r_N(\psi)-r)<\partial_x \phi, \partial_x \phi> \right\} e^{-2(r_N(\psi)-r)t'} dt'$$

$$= \left. <\partial_x \phi, \partial_x \phi>e^{-2(r_N(\psi)-r)t'} \right|_{t_0}^{t}$$

(2.7.1a)

Now, let the difference between the initial velocity fields, as measured by $\|\nabla \phi_0\|$, approach to zero. Then, we have from (2.7.1a) that at any fixed $t=t_0$

$$0 \leq c^2 \|\nabla \phi\|^2 = c^2 <\partial_x \phi, \partial_x \phi> \leq <\partial_x \phi_0, \partial_x \phi_0> = \|\nabla \phi_0\|^2 \to 0,$$  

(2.7.1b)

with $c^2 = e^{-2(r_N(\psi)-r)\Delta t}$, $\Delta t = t-t_0$,

which leads to $\|\nabla \phi\| \to 0$ as $\|\nabla \phi_0\| \to 0$ and hence the continuity of velocity field w.r.t its initial data. The uniqueness of the IBVP follows from (2.7.1b) after setting $\|\nabla \phi_0\| = 0$ in (2.7.1b), which establishes the desired results.

Viewing the prediction of motions in the atmosphere or oceans as an initial value problem, we arrive at the notion,
on the basis of theorem 2.3, that the flow \( \psi(x,t;\psi_0) \) governed by (1.2.1)-(1.2.3) is perfectly predictable in the sense that its initial state \( \psi_0 \) uniquely determines the subsequent evolution of the flow evolving from the given \( \psi_0 \). In practice, this point of view has difficulties, as the "perfect prediction" mentioned above depends on the perfect knowledge of boundary conditions (1.2.2) and initial condition (1.2.3), which is not humanly attainable. This, together with the fact that under certain conditions (cf. corollary 2.1), initial errors in B.C. or I.C. tend to amplify as time advances, implies that predictability problems remain despite the uniqueness of IBVP.

Taking the point of view that the atmosphere or oceans act as dynamic systems vacillating among the possible equilibrium states, limit cycles and strange attractors, it appears that the uniqueness problem of the equilibrium state is more relevant to the prediction problems. This point of view may be traced back to Lorenz's work (1963, 1969). In connection to climate changes, it is reflected in the remark that climate itself is a question of distribution among the possible equilibrium states, and climate variation a matter of how the changed boundary alters the distribution (Charney and Devore, 1979). It is clear that for such an approach to be successful, it is important to have an accurate count of multiple equilibria available to systems for given
conditions. Unfortunately, no general algorithm for accomplishing this task has come into light. The difficulties in determining whether a system has other equilibrium states other than the known one arise frequently in practice. The following result provides a limited solution to the problem.

**Theorem 2.4** (necessary condition for multiple equilibria)

Let $\psi_0 \in \{\Psi\}$. If the system has an equilibrium state other than $\psi_0$, then $r_N(\psi_0) - r \geq 0$.  

(2.7.2)

*Proof:* (by contradiction) Suppose that the system has an equilibrium state other than $\psi_0$ but $r_N(\psi_0) - r < 0$.  

(2.7.3)

Let $\psi'_0$ denote the other equilibrium state, $\phi_0$ the difference field between $\psi_0$ and $\psi'_0$, i.e., $\phi_0 = \psi_0 - \psi'_0$. Now, consider two realizations of (1.2.1)-(1.2.3), $\psi(x,t;\psi_0)$ and $\psi(x,t;\psi'_0)$, initialized from $\psi_0$ and $\psi'_0$, respectively, and the difference field $\phi(x,t;\phi_0)$. By the assumption that $\psi_0$ and $\psi'_0$ are two distinct equilibria of (1.2.1)-(1.2.2) and the uniqueness of the two realizations (cf. theorem 2.3), we have

\[ ||\nabla \phi||_t = ||\nabla \phi_0||, \quad \forall \ t \geq t_0. \]  

(2.7.4)

On the other hand, we apply the energy inequality (2.6.5) over the interval $[t,t_0]$ to have

\[ ||\nabla \phi|| \leq ||\nabla \phi_0|| \exp\{2(r_N(\psi) - r) (t - t_0)\}, \quad \forall \ t \geq t_0. \]
< \| \nabla \phi \|_0 \quad (by \ (2.7.3))

contradicting (2.7.4). This completes the proof.

To the extent that the system (1.2.1)-(1.2.3) governs the motions in the atmosphere or the oceans, the forgoing results indicate that the phenomenon of multiple equilibrium states as observed in Charney and Devore (1979) and in many subsequent works (e.g., Källén, 1981; Proefschrift, 1989) may not arise from the truncations invoked in these studies but rather occurs under certain external and internal conditions for which (2.7.2) or some corresponding form of (2.7.2) is met. We will provide numerical evidence to strengthen this point in the next chapter. By now, there appears to be little doubt on the existence of multiple equilibria at mathematical level. The remaining concern is how relevant the parameter values for which multiple equilibria are observed are to the real atmosphere and climate system (Tung & Rosenthal, 1985).

For the practical concerns mentioned at the onset of the section, the condition (2.7.2) with \( r_N(\psi_0) \) determined from the global analysis (i.e. from (2.5.1) and (2.5.2)) provides a quick check on the possibility of multiple equilibria. It is important to note that when (2.7.2) as a necessary condition is satisfied a direct calculation is needed before concluding the existence of multiple equilibria. However, a
violation of (2.7.2) clearly rules out such a possibility.

2.8 Application III: initial modal vs. nonmodal growth rate

The limitations of the modal growth rate in accounting for some observed oceanic and atmospheric phenomena have been known for some times. For example, the growth rates of mesoscale oceanic eddies from modal analysis are of the order of one year (Schulman, 1967) whereas the observation data in Crease (1962), Swallow (1971), and Koshlyakov and Grachev (1973) suggest that it is of order of a few months. This situation has led to optimization of the modal growth rate expression w.r.t. model parameters such as \( \beta \) and topographic height in order to match the observed population of eddies in the oceans (Cf. Robinson and Mcwilliams, 1974). Another example is found in the study of cyclonegenesis. It is well known that the typical period of deepening of observed cyclones is between 12 and 48 hours (cf. Roebber, 1984; Sanders, 1986), clearly larger than any found in modal analysis of baroclinic instability, e.g., 133 hours in Valdes and Hoskins (1988). Further examples may be found in stability studies of hydrodynamical flows via initial value problems. It is observed that the maximum modal growth rate is often considerably lower than the instantaneous growth rate during the initial period, and this happens even when the flow supports strong modal instability (cf. Farrell, 1982; Boyd, 1983; O'Brien, 1990).
Despite the increasing evidence for this defect and the suggestion that the neglect of the continuous spectrum in the modal analysis is responsible for the flaw, the following questions remain:

1) does the modal growth rate inevitably underestimate the growth rate of disturbances to a given flow?
2) if so, what is the underlying cause?
3) is there any systematic procedure to overcome the defect?

Our objective of this section is to provide explicit answers to the above questions in the context of barotropic, nonlinear and zonally varying flows. We do this by an explicit comparison of BEP (2.5.1)–(2.5.2) arising in the global analysis with the BEP arising from the modal analysis.

The BEP arising in the modal analysis is obtained after assuming that the solution to the linearized version of (2.2.2)–(2.2.4) has the modal form $e^{\sigma t} \xi(x)$, where $\sigma \in \mathbb{C}$ and $\xi(x) \in \mathcal{H}$ (cf. section 2.2), and introducing it into (2.2.2) and (2.2.3),

$$\sigma \nabla^2 \xi + L[\xi; \psi] = 0, \quad (2.8.1)$$

where $\xi$ satisfies B.C. (III) or (IV), with $L$ given by (2.2.5).

To make the subsequent discussions precise, we introduce
the following definitions

Definition 2.4 A solution \( \phi(x,t;\phi_0) \) to (2.2.2)-(2.2.4) is referred to as a finite amplitude nonmodal disturbance (or simply a disturbance); a solution \( \varphi(x,t;\varphi_0) \) to the linearized version of (2.2.2)-(2.2.4) as a linear disturbance; a linear disturbance in the form \( \varphi(x,t;\xi,\sigma) = \xi e^{\sigma t} \), with \((\xi,\sigma)\) determined from (2.8.1), as a modal disturbance. Furthermore, \( \phi_0 \) is said to be a modal initial perturbation if \( \phi_0 = \text{Re}(\xi) \) or \( \text{Im}(\xi) \) with \( \xi \in \) the eigenspace \( \{\xi\} \) of (2.8.1), satisfying (2.8.1), otherwise as a nonmodal initial perturbation. A growth rate is said to be modal growth rate if it is obtained as the real part of the eigenvalue of (2.8.1) (i.e., \( \text{Re}(\sigma) \)), otherwise nonmodal growth rate.

To setup a stage for comparison of the two BEPs we first establish

**Lemma 2.5** Let \( \psi \) be an equilibrium state of (1.2.1)-(1.2.3). Then, its modal disturbance \( e^{\sigma t}\xi(x) \) has the growth rate

\[
\text{Re}(\sigma) = \rho(\xi;\psi) - r, \tag{2.8.2a}
\]

\[
\rho(\xi;\psi) = \frac{\langle \xi, d\xi \rangle}{\langle \xi, B\xi \rangle}, \tag{2.8.2b}
\]

where \( A \) and \( B \) are defined by (2.4.4) and (2.4.5).

**Proof:** let \( \xi \in \{\xi\} \). Taking the inner product of (2.8.1) with \( \xi \) and adding it to its complex conjugate, we have
where $\mathcal{L}^*$ is the adjoint of $\mathcal{L}$ given by (2.3.2). Since $\mathcal{B}$ is a self-adjoint on $\mathcal{V} \ominus \{\xi\}$ (cf. lemma 2.1 (ii)), $\langle \mathcal{B} \xi, \xi \rangle$ is thus real on $\{\xi\}$. It hence follows from (2.8.3) that

\[ -2\text{Re}(\sigma) \langle \xi, \mathcal{B} \xi \rangle + \langle \xi, (\mathcal{L} + \mathcal{L}^*) \xi \rangle = 0, \quad (\text{by self-adjointness of } \mathcal{L} + \mathcal{L}^*) \]

\[ -2\text{Re}(\sigma) \langle \xi, \mathcal{B} \xi \rangle + 2 \langle \xi, (\mathcal{A} - r \mathcal{B}) \xi \rangle = 0, \quad (\text{by } (2.3.3)) \]

which immediately leads to (2.8.2).

Next, we introduce the notion of linear stability measure of an equilibrium in

**Definition 2.5** The linear stability measure of an equilibrium state $\psi$, denoted by $r_L(\psi)$, is defined as

\[ r_L(\psi) = \max_{\xi \in \{\xi\}} \rho(\xi; \psi). \quad (2.8.4) \]

It then follows from (2.8.2) and (2.8.4) that

\[ \max \text{Re}(\sigma) = r_L(\psi) - r, \quad (2.8.5a) \]

as opposed to the maximum nonmodal growth rate (cf. (2.6.8))

\[ \max \sigma_N(\phi; \psi) = r_N(\psi) - r, \quad (2.8.5b) \]

where $r_N(\psi)$ is defined by (2.6.2).

The answer to the question 1) raised above lies in the relative size of $r_N(\psi)$ to $r_L(\psi)$, which is settled in
Lemma 2.6 (comparison of stability measures) For an equilibrium state $\psi$ of (1.2.1)-(1.2.3), it holds that
$$r_L(\psi) \leq r_N(\psi).$$ (2.8.6)

To prove the lemma, we need the following inequality:

Let $\{x_n\}$ and $\{y_n\}$ be two real finite sequences and $y_n > 0$ for any $n$. Then
$$\sum_n x_n \sum_n y_n \leq \max_n \left(\frac{x_n}{y_n}\right).$$ (2.8.7)

To see (2.8.7), it suffices to note that
$$\sum_n x_n \sum_n y_n = \sum_n \frac{(x_n \cdot y_n)}{y_n} \sum_n y_n$$

which immediately gives (2.8.7). Now, we turn to

Proof: (i) Let $\xi \in \{\xi\}$ and be written $\xi = \xi_r + i\xi_i$. Expressing $\rho(\xi;\psi)$ in terms of $(\xi_r, \xi_i)$, we have
$$\rho(\xi;\psi) = \left\{ \sum_{n=1}^{\infty} \langle \xi_n, d\xi_n \rangle \right\} \left\{ \sum_{n=1}^{\infty} \langle \xi_n, d\xi_n \rangle \right\}.$$ (2.8.8)

As an eigenstructure, $\xi(x) \neq 0$ for $x \in \Omega$. Furthermore, let $\xi(x)$ be constant (since kinetic energy of a normal mode with constant structure is zero, thus corresponding to a case of little interest physically). The latter constraint implies two cases: (i) both $\xi_r$ and $\xi_i$ are not constant and (ii) one of them is constant. For case (i), either term in the
denominator of (2.8.8) is strictly positive (cf. (2.4.7)) and hence (2.8.7) is applicable to (2.8.8), which yields

\[ \rho(\xi; \psi) = \max_{n=1, r} \left\{ \frac{\langle \xi_n, A\xi_n \rangle}{\langle \xi_n, B\xi_n \rangle} \right\} \]

\[ \leq \max_{\phi \in \mathcal{H}} \left\{ \frac{\langle \phi, A\phi \rangle}{\langle \phi, B\phi \rangle} \right\} = r_N(\psi). \]  

(2.8.9)

For case (ii), w.l.o.g., suppose that \( \xi_r \neq 0 \) and \( \xi_i = 0 \). Then, only the term for \( \xi_r \) appears in (2.8.8). Thus, we have (2.8.9) without appealing to (2.8.7). To this end, taking the maximum on the two side of (2.8.9) over \( \{\xi\} \) proves the assertion.

The fundamental property (2.8.6) in conjunction with (2.8.5) yields an explicit affirmative answer to question 1) raised above. To see this, we note that for any \((\sigma, \xi)\) determined from (2.8.1), it holds that

\[ \text{Re}(\sigma) = \rho(\xi; \psi) - r, \]  

(by (2.8.2))

\[ \leq r_L(\psi) - r, \]  

(by (2.8.4))

\[ = r_N(\psi) - r, \]  

(by lemma 2.3)

\[ = \sigma_N(\phi; \psi) \]  

(by theorem 2.2)

which allows for the conclusion:

**Theorem 2.5 (initial nonmodal and modal growth rate)** Let \( \psi \) be an equilibrium state of (1.2.1)-(1.2.3). Then, it has at
least one nonmodal initial perturbation which has a growth rate not less than the maximum modal growth rate.

Moreover, the interpretation of the property (2.8.9) reveals the answer to question 2): the ratio of energy conversion between a basic flow and a disturbance to disturbance energy in the eigenspace \( \xi \in \mathcal{N} \) is always less than or equal to the extreme value of the same ratio in the space of all real kinematically admissible disturbances (i.e., in \( \mathcal{W} \)).

The global analysis (i.e., lemma 2.1-2.3) together with the comparison of the stability measures (2.8.6) constitutes a general procedure for circumventing the underestimation of growth rate of disturbances by the modal analysis, thus closing question 3) above.

Further applications of the global analysis and the property (2.8.6) can be found, for example, in explaining why there exists transient growth of disturbances in subcritical flows, or equivalently why the modal analysis inevitably fail to predict the temporal amplification phenomenon in stable flows. To see this, consider a subcritical flow \( \tilde{\psi} \) of (1.2.1)-(1.2.3). It is evident from (2.8.5a) that \( r_L(\tilde{\psi})-r<0 \) for the \( \tilde{\psi} \). However, by the comparison of stability measure (2.8.6), it is still possible to have \( r_N(\psi)-r>0 \) if the \( \psi \) is
outside of the MGS regime, and hence to have an initial growing nonmodal perturbation. On the other hand, if \( r_N(\Psi) < 0 \), then the \( \Psi \) has no initial growing nonmodal disturbance. Moreover, by the comparison of the stability measures, it has no growing modal disturbance either. To summarize:

Theorem 2.6 (initially growing nonmodal perturbation in subcritical flows) If an equilibrium state \( \Psi \) of (1.2.1)-(1.2.3) has no initially growing nonmodal perturbation, then it does not have any growing modal perturbation; the converse is false.

The phenomenon of transient growth of disturbances in flows supporting no growing modal disturbances was originally reported by Orr (1907) in his work on plane Couette flow. It has been documented in several recent works on initial value problems of the Charney and Eady models (Farrell, 1985), of barotropic constant shear flows on \( \beta \)-plane (Boyd, 1983; Farrell, 1987), and of the Green model (O'Brien, 1990). It is frequently found in these studies that for a realistic value of the Ekman damping the model flows have disturbances on the synoptic cyclone spatial scales which exhibit large transient growth despite the absence of, or presence of only weakly unstable, modal perturbations. This observation has led to the argument that the modal instability cannot in general
serve to explain the occurrence of cyclone scale disturbances in midlatitude but rather they arise perhaps predominantly from the release of mean flow potential energy by properly configured initial nonmodal perturbations (Farrell, 1985).

Given the possibility that cyclone scale disturbances are initiated by fairly large amplitude perturbations as emphasized in synoptic experiences (cf. Petterssen, 1955; Palmen and Newton, 1969), we believe that a successful inclusion of baroclinicity into the present formalism may result in a framework which allows for finite amplitude perturbations and hence more suitable for the study of cyclonegenesis.

2.9 Concluding remarks

We have performed a global analysis of the system (1.2.1)-(1.2.3), with results applied to a number of GFD problems. When applied to the global stability, the analysis has yielded a systematic procedure for characterizing stability of flows on the basis of whether there exists an initially growing perturbation regardless of its amplitude and spatial structure, as opposed to the modal analysis which characterizes stability according to whether there exists an exponentially growing perturbation. The same procedure can also be used to identify an optimal initial nonmodal perturbation to a given flow. When applied to the study of
relation of initial modal to nonmodal growth rates, the global analysis, together with the fundamental comparison property (cf., lemma 2.6), has uncovered the cause underlying the limitation of modal growth rate as an indicator of potential development of disturbances, and the cause underlying phenomena such as the well known transient growth in subcritical flows and explosive development in weakly unstable flows. The global analysis has also found application in the study of multiple equilibrium states, resulting in a necessary condition for (or a sufficient condition for ruling out) the existence of such phenomenon. It is expected that these results hold for all flows governed by (1.2.1)-(1.2.3).

In the following two chapters, we will present analyses which are parallel to, and extend, those in this chapter, but are performed in a finite dimensional function space (or equivalently in $\mathbb{R}^n$) and for a more specific physical system. Moreover, we will complement the analyses there with numerical experiments.
Appendix 2A A variational approximation method for the optimal value of generalized Rayleigh quotient $\rho$

In this appendix, we present a Rayleigh-Ritz procedure (cf. Reddy, 1986) for obtaining the approximate solution $\hat{\phi}$ to the optimal problem $\max \rho(\phi;\psi)$ for $\phi \in \mathcal{H}$. Let $u_1, u_2, \ldots, u_M$ denote $M$ linearly independent functions in $\mathcal{H}$ and write

$$\hat{\phi} = c_i u_i$$  \hspace{1cm} (2A.1)

where the summation convention is applied to the repeated indices, with $c_i$ ($i=1,2,\ldots,M$) being constants to be determined. Evaluating $\rho(\phi;\psi)$ at $\hat{\phi}$, we have

$$\Pi(c_1, c_2, \ldots, c_M) = \rho(\hat{\phi}) = C(c_1, c_2, \ldots, c_M) / D(c_1, c_2, \ldots, c_M),$$  \hspace{1cm} (2A.2)

$$C(c_1, c_2, \ldots, c_M) = \langle u_i, d u_j \rangle c_i c_j,$$  \hspace{1cm} (2A.3)

$$D(c_1, c_2, \ldots, c_M) = \langle u_i, B u_j \rangle c_i c_j,$$  \hspace{1cm} (2A.4)

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathcal{H}$ (cf. (2.2.11)), with the operators $d$ and $B$ given by ((2.4.4) and (2.4.5), respectively. The optimizing constants $c=(c_1, c_2, \ldots, c_M)$ in (2A.1) then must satisfy $\partial \Pi / \partial c_i = 0$ ($i=1,2,\ldots,M$), which leads to

$$\partial C / \partial c_i = \Pi \partial D / \partial c_i , i=1,2,\ldots,M$$  \hspace{1cm} (2A.5)

or in terms of matrix eigenvalue form

$$[\langle u_i, d u_j \rangle] c = \mu [\langle u_i, B u_j \rangle] c,$$  \hspace{1cm} (2A.6)

where $\mu = \Pi(c)$. The largest eigenvalue of (2A.6) is then the Rayleigh-Ritz approximation to the optimal value of $\rho(\phi;\psi)$ in $\mathcal{H}$, and its corresponding eigenvector $c$ will yield an optimal disturbance $\hat{\phi}$ via (2A.1).
CHAPTER 3
GLOBAL ANALYSIS II: FINITE DIMENSIONAL SYSTEM,
with application to: global stability and optimal perturbation

3.1 Introduction

No general theory of global analysis for any hydrodynamical system in finite dimensional function space (or equivalently in $\mathbb{R}^M$) has been known, except for a few attempts made to low order systems (cf. Vickroy and Dutton, 1979; Källén and Wiin-Nielsen, 1980). The objective of this chapter is to perform such an analysis.

To facilitate the subsequent numerical experiments, the analysis is done for a system defined by (1.2.1), (1.2.2a) (periodic channel geometry) and (1.2.3) with the external forcing $\psi^*$ acting as uniform zonal momentum source, i.e.,

$$\psi^* = -U^* y$$

(3.1.1)

where $U^*$ is a parameter describing the forcing strength. We note that the restriction of $\psi^*$ to the form (3.1.1) is not essential to the analysis.

Before concluding the introduction, let us make a simple physical observation (e.g., Gravel, 1988) that the streamfunction $\psi$ can be decomposed into two parts: one arising directly from the uniform zonal momentum forcing
(3.1.1) and one representing the deviation, i.e.,

\[ \psi(x,y,t) = -U^* y + \phi(x,y,t). \]  

(3.1.2)

Thus, it follows from introducing (3.1.2) into (1.2.1), (1.2.2a) and (1.2.3) that

\[ (\partial/\partial t) \nabla^2 \phi + J(-U^* y + \phi, \nabla^2 \phi + \beta y + (h/H) f_0) = -r \nabla^2 \phi, \]  

(3.1.3)

with

\[ \phi(0,y,t) = \phi(0,y,t), \quad \text{at } 0 \leq y \leq 1, \]  

(3.1.4a)

and

\[ \phi(x,y,0) = \phi_0(x,y), \]  

(3.1.5)

where \( \phi_0(x,y) \) is such that \( \psi(x,y,0) = -U^* y + \phi_0(x,y) \). A solution to the system (3.1.3)-(3.1.5) thus determines the response of the original system to the external forcing (3.1.1).

Furthermore, since the uniform zonal part \(-U^* y\) of \( \psi \) (cf. (3.1.2) is time independent, the study of the stability of \( \psi \) is then effectively reduced to that for \( \phi \).

The chapter is organized as follows. The system (3.1.3)-(3.1.5) is cast into a spectral form in § 3.2. A general procedure for the global analysis in \( \mathbb{R}^M \) is presented in § 3.3, with results from its application to global stability and search for initial growing perturbations and optimal initial nonmodal perturbations presented in § 3.4. Numerical illustrations are made using a set of equilibria depending on \( r \) in § 3.5, with a concluding remark in § 3.6.
3.2 Governing equations in $\mathbb{R}^M$

By virtue of the channel boundary condition (3.1.4), we, following Mitchell and Derome (1983), introduce a set of orthonormal functions $\mathcal{G} = \{G_i\}$, with $G_i$ given by

$$G_i = \begin{cases} 
2^{1/2} \cos(\pi ny/d) & \text{if } 1 \leq i \leq I \\
2 \sin(2 \pi mx/l) \sin(\pi ny/d), & \text{if } I+1 \leq i \leq I+I^*J \\
2 \cos(2 \pi mx/l) \sin(\pi ny/d), & \text{if } I+I^*J+1 \leq i \leq I+2I^*J \\
\end{cases}$$

where $m=1,2,\ldots,J; \; n=1,2,\ldots,I; \; M=I+2I^*J.$

(3.2.1)

where the index $i$ for $G_i$ increases based on the columnwise convention for the two-dimensional array $G(m,n)$. Taking (3.2.1) as a basis for $\mathbb{H}$ (cf. § 2.2), we expand all dependent variables in the form

$$\Phi = \sum_{i=1}^{M} (\Phi_i, h_i) G_i$$

and insert them into (3.1.3) to obtain the spectral version of (3.1.3)-(3.1.5) below

$$\frac{d}{dt} \Phi = F(\Phi; \alpha), \quad (3.2.3)$$

$$\Phi(0) = \Phi_0, \quad (3.2.4)$$

where $\Phi, \Phi_0$ and $\Phi$ are column vectors in $\mathbb{R}^M$, e.g.,

$$\Phi = [\Phi_1, \Phi_2, \ldots, \Phi_M]^T$$

and $F: \mathbb{R}^M \to \mathbb{R}^M$ is an $M$-dimensional nonlinear vector field, with the $i$th component $F_i$ given by

$$F_i(\Phi; \alpha) = (1/a_i^2) \left\{ \sum_j b_{ij} \left[ (f_0/H) h_j - a_j^2 \Phi_j \right] \right\}$$

48
with $a_i^2$, $b_{ij}$, $c_{ijk}$ given by

$$a_i^2 = a(m, n) = \begin{cases} 
(\pi n/d)^2, & \text{if } 1 \leq i \leq I \\
(2\pi m/l)^2 + (\pi n/d)^2, & \text{if } I+1 \leq i \leq I+I*J \\
(2\pi m/l)^2 + (\pi n/d)^2, & \text{if } I+1+I*J \leq i \leq I+2I*J 
\end{cases} \tag{3.2.6}$$

$$c_{ijk} = \langle G_i (G_j G_k) \rangle = c_{jki} = c_{kij} = -c_{jik}. \tag{3.2.7}$$

$$b_{ij} = \langle G_i \partial G_j / \partial x \rangle = -b_{ji}. \tag{3.2.8}$$

Finally, $\alpha$ in (3.2.3) represents any model parameter whose effect on the stability of equilibrium states of (3.2.3) is under concern.

### 3.3 Global analysis in $\mathbb{R}^M$

While the details of the global analysis in $\mathbb{R}^M$ is noticeably different from its counterpart in $\mathbb{R}$, the basic ideas are essentially the same. In light of this, description of the motivations behind the subsequent mathematical manipulations is considerably simplified or neglected (see chapter 2 for details).

#### 3.3.1 Disturbance equation

Let $\mathbf{\phi}(t; \mathbf{\phi}_0)$ denote the solution of (3.2.3) initialized
from $\delta_0$. Consider another realization $\delta(t;\delta_0 + \delta_0)$ of (3.2.3) made under the same condition as those for $\delta(t;\delta_0)$ except its I.B.: $\delta_0 + \delta_0$. As in the section 2.2, the stability problem for $\delta(t;\delta_0)$ can be investigated by considering the temporal evolution of the difference

$$\delta = \delta(t;\delta_0 + \delta_0) - \delta(t;\delta_0),$$

(3.3.1)

which is governed by the initial value problem

$$(d/dt)\delta = f(\delta,\dot{\delta},\alpha),$$

(3.3.2a)

$$\delta(t_0) = \delta_0,$$

(3.3.2b)

where $f: \mathbb{R}^M \to \mathbb{R}^M$ is an $M$-dimensional vector field arising from the expansion of the nonlinear vector field $F$ as a Taylor series around the motion $\delta$, defined by

$$F(\delta;\dot{\delta},\alpha) = F_0(\delta|\delta) + (1/2)F^{\delta\delta}(\delta|\delta) + 0(|\delta|^2),$$

(3.3.3a)

$$F_0(\delta|\delta) = \delta \delta(\delta + \delta_1 + \delta_2 + \alpha)/\delta \delta_1 \delta_2|_{\delta_1 = \delta_2 = 0},$$

(3.3.3b)

$$F^{\delta\delta}(\delta|\delta) = \delta^2 F(\delta + \delta_1 + \delta_2 + \alpha)/\delta \delta_1 \delta \delta_2|_{\delta_1 = \delta_2 = 0},$$

(3.3.3c)

A direct evaluation of (3.3.3b) and (3.3.3c) yields

$$F_0(\delta;\dot{\delta}) = A\delta,$$

(3.3.4a)

$$F^{\delta\delta}(\delta;\dot{\delta}) = 2B\delta\delta,$$

(3.3.4b)

or in component form

$$\{F_0(\delta;\dot{\delta})\}_i = \sum_j A_{ij}\dot{\phi}_j,$$

(3.3.4c)

$$\{F^{\delta\delta}(\delta;\dot{\delta})\}_i = 2\sum_{jk} B_{ijk}\dot{\phi}_j\dot{\phi}_k.$$
with

\[ A_{ij} = [A_{ij}], \]

\[ A_{ij} = \left( \frac{1}{a_i^2} \right) \left\{ U^* b_{ij} (-a_j^2) + \beta b_{ij} - ra_i^2 \delta_{ij} \right. \]

\[ \left. + \sum_k \left[ c_{ikj} \phi_k (-a_j^2) + c_{ijk} (f_0 h_k / H - a_i^2 \phi_k) \right] \right\}, \quad (3.3.4e) \]

\[ B = [B_{ijk}], \quad B_{ijk} = (1/a_i^2) c_{ijk} (-a_j^2). \quad (3.3.4f) \]

Note that the linear operator \( A : \mathbb{R}^M \to \mathbb{R}^M \) and bilinear operator \( B : \mathbb{R}^M \times \mathbb{R}^M \to \mathbb{R}^M \) are the counterparts of \( \$ \) and \( \mathcal{N} \) (cf. 2.2.5) and (2.2.6) in \( \mathbb{R}^M \), respectively. Also, note that the higher order terms in (3.3.3) vanish identically since \( F \) is a polynomial of degree 2 in \( \phi \) (cf. (3.2.5)). To this end, we have a closed form for \( f \)

\[ f = \sum_i A_{ij} \phi_j + \sum_j \sum_k B_{ijk} \phi_j \phi_k, \quad (3.3.5a) \]

or

\[ f = A^2 \phi + B \phi^2. \quad (3.3.5b) \]

### 3.3.2 Symmetrized energy equation

The next step is to put the disturbance energy equation into a form such that its time tendency can be expressed in terms of a self-adjoint operator in \( \mathbb{R}^M \) and hence in terms of a bounded functional. To accomplish this, we first state a result from matrix algebra.
Lemma 3.1 Let $\mathbf{P} = [P_{ij}] \in \mathbb{R}^{N \times N}$ be a symmetric matrix and $\mathbf{Q} = [Q_{ij}] \in \mathbb{R}^{N \times N}$ be a skew-symmetric matrix. Then

$$\mathbf{PQ} = \sum_{i} \sum_{j} P_{ij} Q_{ij} = 0. \quad (3.3.6)$$

With (3.3.5), (3.3.2) can be written in component form

$$(d/dt)\phi_i = \sum_{j} A_{ij} \phi_j + \sum_{j} B_{ijk} \phi_j \phi_k. \quad (3.3.7)$$

Now, we multiply (3.3.7) with $a_i^2 \phi_i$ and sum over $i$ to have

$$(d/dt) \left\{ \sum_{i} (1/2) a_i^2 \phi_i^2 \right\} = \sum_{i} \sum_{j} a_i^2 A_{ij} \phi_i \phi_j + \sum_{i} \sum_{j} \sum_{k} a_i^2 B_{ijk} \phi_i \phi_j \phi_k. \quad (3.3.8)$$

Note that the bilinear operator $a_i^2 B_{ijk}$ corresponds to a skew-symmetric matrix with respect to indices $i, j$ by virtue of (3.2.7) and (3.3.4f) whereas $\phi_i \phi_j \phi_k$ is obviously symmetric w.r.t $i, j$. It thus follows from lemma 3.1 that the second summation in (3.3.8) vanishes exactly. Similarly, the symmetric properties (3.2.7) and (3.2.8) demand that the terms invoking $b_{ij}$ and $c_{ijk}$ in the linear operator $a_i^2 A_{ij}$ (cf. (3.3.4e)) make zero contribution to the first summation in (3.3.8). With these observations, (3.3.8) is reduced to (in terms of inner product on $\mathbb{R}^N$)

$$(d/dt) (1/2) \langle \phi, D\phi \rangle = \langle \phi, \hat{A} \phi \rangle - r \langle \phi, D\phi \rangle, \quad (3.3.9)$$

$$\hat{A} = [\hat{A}_{ij}], \quad \hat{A}_{ij} = \sum_{k} c_{ik} \phi_j \phi_k (-a_j^2), \quad (3.3.10)$$

52
\[ \mathbf{D} = [D_{ij}] = \mathbf{D}_{ij}^T, \quad D_{ij} = \delta_{ij}^2 \delta_{ij}. \]  

(3.3.11)

It is clear from (3.3.10) that (3.3.9) has not yet achieved the desired form due to the non self-adjointness of the operator \( \mathbf{A} \). To remove the non self-adjoint constituent in \( \mathbf{A} \), we observe that

**Lemma 3.2** Let \( \mathbf{P} \in \mathbb{R}^{M \times M} \). Then, there exists a decomposition

\[ \mathbf{P} = \mathbf{P}^+ + \mathbf{P}^- \],

with

\[ \mathbf{P}^T = \mathbf{P}^+ = (1/2) (\mathbf{P} + \mathbf{P}^T), \quad \mathbf{P}^- = -\mathbf{P}^- = (1/2) (\mathbf{P} - \mathbf{P}^T), \]

such that \( \langle \phi, \mathbf{P} \phi \rangle = \langle \phi, \mathbf{P}^+ \phi \rangle \), \( \forall \phi \in \mathbb{R}^M \).

The lemma follows from lemma 3.1 after noting \( \mathbf{P}^T = -\mathbf{P}^- \).

Applying this observation to (3.3.9), we obtain

\[ (d/dt) (1/2) \langle \phi, \mathbf{D} \phi \rangle = \langle \phi, \mathbf{P}^+ \phi \rangle - r \langle \phi, \mathbf{D} \phi \rangle, \]

or in terms of the Generalized Raleigh Quotient \( \rho(\phi; \phi) \)

\[ (d/dt) (1/2) \langle \phi, \mathbf{D} \phi \rangle = \langle \phi, \mathbf{P}^+ \phi \rangle (\rho(\phi; \phi) - r), \]

(3.3.12a)

(3.3.12b)

where \( \mathbf{A}^+ \) is given by

\[ \mathbf{A}^+ = (1/2) (\mathbf{A}^T + \mathbf{A}), \]

(3.3.13)

\[ \rho(\phi; \phi) = \langle \phi, \mathbf{A}^+ \phi \rangle / \langle \phi, \mathbf{D} \phi \rangle. \]

(3.3.14)

**Remark 3.3.1** It follows from (3.3.11) and (3.3.13) that the operators \( \mathbf{A}^+ \) and \( \mathbf{D} \) are self-adjoint in \( \mathbb{R}^M \), i.e.,

(i) \( \mathbf{A}^+ = (\mathbf{A}^+)^T \),

(ii) \( \mathbf{D} = \mathbf{D}^* \) and \( \mathbf{D} \) is positive definite,
where $()^*$ denotes the adjoint of $()$. The above two properties are the reminiscent of those for the differential operators $\mathcal{A}$ and $\mathcal{B}$ in $\mathcal{H}$ (cf. (2.4.6) and (2.4.7)).

### 3.3.3 Generalized Rayleigh Principle

As in the case of the continuum system, the crucial steps in the global analysis are to introduce a bounded functional and to construct an algorithm for optimizing the functional. As suggested by (3.3.12b), we consider the extreme values of $\rho$ (cf. (3.3.14)) in $\mathbb{R}^M$. For this, it is important to recognize that $\rho(\hat{\psi}; \delta) = \langle \hat{\psi}, \mathbf{A}^+ \delta \rangle / \langle \delta, \mathbf{D} \delta \rangle$ in (3.3.12b) would reduce to the standard Rayleigh quotient if $\mathbf{D}$ was a unit operator $\mathbf{I}$ of order $M$, and that in that case $\rho$ would be bounded above by the largest eigenvalue of $\mathbf{A}^+$ and below by the smallest according to Rayleigh principle (cf. Kreyszig, 1978).

Motivated by this observation, we shall show below that given $\rho$ defined by (3.3.14) with $\mathbf{A}^+$ and $\mathbf{D}$ satisfying the properties (3.3.15) and (3.3.16), it is possible to generalize the standard Rayleigh principle such that $\rho$ is bounded in $\mathbb{R}^M$ and the disturbance $\hat{\delta}$ in $\mathbb{R}^M$ realizing the extreme values of $\rho$ can be obtained systematically. This is made possible by establishing lemma 3.3 (see Appendix 3A).

**Lemma 3.3** (Algebraic properties of the $\{s_i\}$)

Let $\mathbf{A}^+$ and $\mathbf{D}$ be given by (3.3.13) and (3.3.11), respectively.
Then, the eigensystem

$$\mathbf{A}^+ \mathbf{s} = \lambda \mathbf{ds},$$  

(3.3.17)

has the following properties

(i) all $\lambda_i$ are real,  

(3.3.18)

(ii) (3.3.17) has $M$ linearly independent eigenvectors $\mathbf{s}_i$,  

(iii) let $\mathbf{S}$ and $\Lambda$ be given by

$$\mathbf{S} = [\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_M],$$  

(3.3.19)

$$\Lambda = \text{diag}[\lambda_1], \text{ with } \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_M,$$  

(3.3.20)

then

$$\mathbf{S}^T \mathbf{A}^+ \mathbf{S} = \Lambda, \quad \tag{3.3.21a}$$

$$\mathbf{S}^T \mathbf{D} \mathbf{S} = \mathbf{I}, \quad \tag{3.3.21b}$$

where $\mathbf{I}$ is a unit matrix of order $M$.

As an aside, we observe the striking resemblance of the matrix eigenvalue problem (MEP) (3.3.17) to the BEP (2.5.1)-(2.5.2). Based on lemma 3.3., we can show that

Lemma 3.4 (Generalized Rayleigh principle) Let $\rho(\hat{\phi}; \hat{\psi})$ be given by (3.3.14). Then:

(i) $\lambda_1 \leq \rho(\hat{\phi}; \hat{\psi}) \leq \lambda_M$, $\forall \hat{\phi} \in \mathbb{R}^M$,  

(3.3.22)

(ii) $\min_{\hat{\phi} \in \mathbb{R}^M} \rho(\hat{\phi}; \hat{\psi}) = \rho(\mathbf{s}_1; \hat{\psi}) = \lambda_1(\hat{\psi})$,  

(3.3.23)

(iii) $\max_{\hat{\phi} \in \mathbb{R}^M} \rho(\hat{\phi}; \hat{\psi}) = \rho(\mathbf{s}_M; \hat{\psi}) = \lambda_M(\hat{\psi}) = r_N(\hat{\psi})$,  

(3.3.24)

where $\mathbf{s}_1, \mathbf{s}_M, \lambda_1$ and $\lambda_M$ are determined from (3.3.17).
Proof: We use the nonsingularity of $S$ to define a linear mapping $S: \mathbb{R}^M \to \mathbb{R}^M$

$\tilde{\phi} = S\tilde{\phi}$.

It thus follows from (3.3.20) and (3.3.21) that in terms of $\tilde{\phi}$

$p(\tilde{\phi}; \tilde{\psi}) = p(S\tilde{\phi}; S\tilde{\psi}) = \langle \tilde{\phi}, S^{-1} \tilde{\psi} \rangle$

and hence that $\forall \tilde{\phi} \in \mathbb{R}^M$

$p(\tilde{\phi}; \tilde{\psi}) - \lambda = \left\{ \sum_{i} (\lambda_i - \lambda) \tilde{\phi}_i \tilde{\psi}_i \right\}/\sum_{i} \tilde{\phi}_i \tilde{\phi}_i \leq 0, \text{ (by (3.3.20))}$

which proves that $p$ is bounded above by $\lambda$. A similar argument establishes the left side of (3.3.22). Using the fact that $s_i$ (i=1,M) satisfy (3.3.17) and that $\lambda_i$ are real (cf. (3.3.18)), we evaluate $p(\tilde{\phi}; \tilde{\psi})$ at $s_M$ and $s_1$ to obtain (3.3.23) and (3.3.24), which completes the generalization.

In summary, the global analysis in $\mathbb{R}^M$ for the $M$-dimensional system (3.2.3) includes: 1) symmetrized disturbance energy equation (3.3.12); 2) the eigensystem (3.3.17); and 3) the generalized Rayleigh principle.

3.4 Application to: global stability and optimal perturbation

By arguments parallel to those in § 2.6, we obtain the following results with the proof omitted (see § 2.6 for parallelism)
Theorem 3.1 (monotonic, global stability) Let $r_N(\hat{\phi})$ denote the finite stability measure of $\hat{\phi}$ and be defined by

$$r_N(\hat{\phi}) = \max_{\phi \in \mathbb{R}^M} \rho(\hat{\phi}; \phi) = \lambda_M$$

with $\lambda_M$ given by (3.3.24). Then, an equilibrium state $\hat{\psi}$ of (3.2.3) is MGS iff $r_N(\hat{\psi}) - r < 0$. (3.4.2)

Corollary 3.1 There exists at least one initially growing disturbance to an equilibrium state $\hat{\psi}$ of (3.2.3) if and only if $r_N(\hat{\psi}) - r > 0$ (3.4.3)

Theorem 3.2 (optimal initial perturbations) Let the growth rate $\sigma_N(\phi; \hat{\psi})$ of a disturbance $\phi$ to a given flow $\hat{\psi}$ be given by

$$\sigma_N(\phi; \hat{\psi}) = \frac{1}{2} \left( \frac{d}{dt} \ln \langle \phi, D\phi \rangle \right).$$

Then, among all the possible initial perturbations $\phi_0 \in \mathbb{R}^M$ to a given equilibrium state $\hat{\psi}$, $s_M$ is the optimal one in the sense that

(i) the growth rate of $\phi(t; s_M)$ at $t=t_0$ is the largest in $\mathbb{R}^M$

$$\sigma_N(s_M; \hat{\psi}) \geq \sigma_N(\phi_0; \hat{\psi}), \forall \phi_0 \in \mathbb{R}^M,$$ (3.4.5)

(ii) the value $\sigma_N(s_M; \hat{\psi})$ is the least upper bound on the growth rate of all $\phi(t; \phi_0) \forall \phi_0 \in \mathbb{R}^M$ for $t \geq t_0$. Further properties of the set $\{s_I\}$ determined from (3.3.17) as nonmodal initial perturbations are given in Appendix 3B.
3.5 Numerical illustrations

The objective of this section is to illustrate some of concepts, put forward here and in chapter 2, regarding global stability, initially growing perturbations and optimal perturbations by applying them to planetary scale atmospheric flows.

3.5.1 Set-up of the numerical experiments

The geometry of the flow field is the same as in Fyfe and Derome (1987), i.e., the $\beta$-plane re-entry channel centered around the midlatitude, characterized by

$$ l = 2\pi R \cos(\theta_0), \quad d = 4 \times 10^6 m, \quad H = 1 \times 10^4 m, \quad (3.5.1a) $$

$$ f_0 = 2\Omega \sin(\theta_0), \quad \beta = 2\Omega \cos(\theta_0)/R, \quad (3.5.1b) $$

where $\Omega$, $R$ and $\theta_0$ are the rotational rate of the earth, the radius of the earth and the reference latitude, respectively, taking the values of $7.292 \times 10^{-5}/s$, $6.37 \times 10^6 m$ and $45^0N$, respectively. The lower boundary of the geometry is assumed to have a sinusoidal form

$$ h(x,y) = h_1 \sin(2\pi mx/l) \sin(nny/d), \quad (3.5.2) $$

with $m = 1$, $n = 1$ and $h_1 = 500 m$. The physical parameters of the system (3.2.3) are the uniform zonal momentum forcing strength $U^*$ and the Ekman damping coefficient $r$. For illustration purpose, we set $U^* = 22 m/s$ based on the
previous studies (e.g., Tung and Rosenthal, 1985), and take $r$ as the controlling parameter, i.e., set $a = r$ (cf. section 3.2), with $r$ ranging from $1/(1 \text{ day})$ to $1/(35 \text{ days})$.

The numerical experiments are performed via the initial value problems defined by the system (3.2.3)-(3.2.4). The system is solved numerically with the nonlinear terms calculated via FFT and the time stepping taken via leap-frog scheme in conjunction with the Robert filter (see Fyfe, 1989 for more details). We emphasize that while theorem 3.1 and 3.2 hold regardless of truncation, a sufficient resolution is required to obtain convergent numerical solutions. In what follows, a relatively low truncation at $M=15$ with $I=2$ and $J=3$ (cf. (3.2.1)) is made. It must be stressed that while some changes are observed in runs (not shown) where $M$ is increased from $M=15$ to $M=210$, qualitatively speaking, the numerical results are unchanged.

### 3.5.2 Numerical results

#### a) equilibrium states

A family of equilibrium states of (3.2.3), denoted by $\{\hat{\Psi}(r)\}$, is obtained numerically using the continuation algorithm (cf. Keller, 1978). In Fig 3.1, we present three representative equilibria corresponding to (a) weak dissipation ($r = 1/29$ days), (b) moderate dissipation ($r =$
1/11 days) and (c) strong dissipation ($r = 1/5.5$ days), respectively. It is seen, as expected, that strong wavy motion has developed as the Ekman damping decreases.

b) optimal MGS boundary and MGS

To obtain the optimal MGS boundary in $r$-space, we solve the generalized matrix eigenvalue problem (3.3.17) for each $\hat{\psi}$ $\in \{\hat{\psi}(r)\}$ to obtain $r_N(\hat{\psi})$ using the Eispack routines (Garbow, 1974), with results shown in Fig 3.2. It is seen that $r_N(\hat{\psi})$ monotonically decreases as a function of $r$ and intersects with the dashed line $r=r_N$ at $r=r_N$. It is clear from theorem 3.1 that the line $r=r_N$ defines the MGS boundary. Also shown in Fig 3.2 is the normal mode stability boundary $r=r_L$, where $r_L$ is numerically determined from the usual modal stability analysis of the set $\{\hat{\psi}(r)\}$. It is evident from Fig 3.2 that the two critical values of Ekman damping, i.e., $r_N$ and $r_L$, divide the whole stability regime of the family $\{\hat{\psi}(r)\}$ into three distinct subregimes: supercritical regime (I), subcritical regime (II) with initially growing perturbation and MGS regime (III).

To illustrate MGS of the equilibria $\hat{\psi}$, we performed five numerical experiments corresponding to five different initial conditions of the form $\hat{\psi}(0) = \hat{\phi}_0 + \hat{\psi}_0$, where $\hat{\psi}$ is the equilibrium state shown in Fig 3.1 (c) and $\hat{\phi}_0$ in each experiment is a randomly generated initial perturbation, with
the ratio $\hat{\delta}$ of initial energy in $\Phi_0$ to the energy in $\Psi$ varying from 0.2 to 1 in these experiments. Fig 3.3 shows the time series of disturbance kinetic energy (scaled by $\rho_a H d l$, with $\rho_a$ the density of air and $H$, $d$ and $l$ the spatial scales defined before). It is seen in Fig 3.3 that even for $\Phi_0$ as large as $\Psi$ (i.e., with $\hat{\delta} = 1$), the predicted monotonic decay to zero is observed.

c) Initially transient growth in subcritical flows

The existence of initially growing nonmodal perturbations to those $\Psi$ in regime (II) (cf. Fig 3.2) is illustrated in Fig 3.4-6. In each of the six experiments reported in Fig 3.4, we set $\tilde{\Phi}(0)$ to be $\Phi_0 = \Phi + s_m$, where $s_m$ is obtained from solving (3.3.17) using $\Psi$ in regime (II). The existence of such perturbations is clearly demonstrated even for the equilibrium state $\Psi(r)$ with $1/r=6$ days, which is just across the MGS boundary. The growth rates of the six initial nonmodal perturbation are found to be 0.207/day, 0.204/day, 0.2/day, 0.197/day and 0.187/day, respectively. It is seen that these disturbances initialized from these perturbations undergo transient growth over a period as long as 5 days, saturate at an energy level as high as roughly 60% more of their initial values, then asymptotically decay to zero.

The existence of more than one such perturbation to a given subcritical flows is illustrated for the equilibrium
state \( \mathbf{s} \) shown in Fig 3.1(b). In Fig 3.5, we show the spatial configuration of five such nonmodal perturbations \( s_i \), with Fig 3.6 showing the time series of kinetic energy of the disturbances started from them. It is interesting to note in Fig 3.5 that with a decrease in their initial growth rate (from the top panel to the bottom one), \( s_i \) appear to be more elongated in the zonal direction. As for the temporal evolution of these \( \mathbf{\phi}(t; s_i) \), much of the observations for Fig 3.4 can be carried over to Fig 3.6.

The transient growth has been reported in a variety of model flows (cf. Orr, 1907; Rosen, 1971; Boyd, 1983; etc). Its relevance to cyclonegenesis has been strongly suggested in Farrell (1985; 1989a). As a cautionary note, we stress that the asymptotic decay to zero revealed in Fig 3.5 and Fig 3.6 does not constitute a proof for establishing this as a generic phenomenon in subcritical flows. We will return to this aspect in chapter 5.

d) optimal nonmodal perturbations in supercritical regime

In terms of growth rate, the superior status of \( \mathbf{s}_m \) to all initial disturbances \( \mathbf{\phi}_0 \in \mathbb{R}^m \) is guaranteed by theorem 3.2. Here, we report the results from five numerical experiments designed to illustrate the conclusion. Each of the experiments starts from one of the following initial states 
\[
\mathbf{\phi}(0) = \mathbf{\phi}_0 = \mathbf{\phi} + \mathbf{s}_{m-1}, \quad i=0,1,\ldots,4,
\]
where $\mathbf{\hat{v}}$ is the equilibrium state shown in Fig 3.1(a), and $s_i$ are its initially growing nonmodal perturbations determined from (3.3.17).

In Fig 3.7, we show the time series of growth rate $\sigma_N(\mathbf{\hat{\phi}}; \mathbf{\hat{v}})$ over the first 15 days. The initial growth rate of $\mathbf{\hat{\phi}}(t; s_{M-1})$ ($i=0,1,\ldots,4$) are found to be 0.185/day, 0.182/day, 0.144/day, 0.113/day and 0.028/day, respectively, in contrast to the maximum modal growth rate 0.070/day, which clearly demonstrate the optimal status of $s_M$ as an initial perturbation to $\mathbf{\hat{v}}$.

We note that identifying favorably configured perturbations in models other than ours has been a subject of several recent works (Farrell, 1988, 1989a; O'Brien, 1990). The plausible role of these perturbations in explosive development of cyclone waves in midlatitude has been suggested, for example, in Farrell (1985).

3.6 Concluding remarks

A reduction in dimension of the system has made the global analysis in $\mathbb{R}^M$ a procedure amenable to standard matrix eigenvalue calculation package such as Eispack (cf. Garbow, 1974). The numerical examples given in § 3.5 clearly indicate that a stability analysis of a hydrodynamical system without a global analysis is likely to be limited.
Appendix 3A  Algebraic properties of the set \( \{s_i\} \)

Proof: The reality of \( \lambda \) follows from the self-adjointness of \( \tilde{A}^+ \) and \( D \) (cf. (3.3.15)-(3.3.16)). To see the existence of \( M \) linearly independent \( s_i \), we consider the positive square root \( D^{1/2} \) of the operator \( D \) and the linear mapping defined by 
\[
\tilde{s} = D^{1/2}s
\]
with which, (3.3.17) yields 
\[
\tilde{A}\tilde{s} = \lambda\tilde{s}, \text{ with } \tilde{A} \text{ given by } \quad (3A.2)
\]
\[
\tilde{A} = (D^{1/2})^{-1}\tilde{A}^+ (D^{1/2})^{-1},
\]
where the existence of the inverse of \( D^{1/2} \) is assured by \( D \) being positive definite. From (3.3.16), it is not difficult to see that \( D^{1/2} \) is diagonal and hence \( \tilde{A} \) is self-adjoint. It follows then that the eigenvalue problem (3A.2) has an orthonormal set of \( M \) eigenvectors \( \tilde{s}_i \), i.e.,
\[
\tilde{A}\tilde{s}_i = \lambda_i\tilde{s}_i, \quad \forall \ i=1,2,\ldots,M \quad (3A.3)
\]
\[
\tilde{s}_i^\top \tilde{s}_j = \delta_{ij}, \quad \forall \ i,j=1,2,\ldots,M \quad (3A.4)
\]
which yield, via the linear mapping (3A.1),
\[
\tilde{A}^+ s_i = \lambda_i Ds_i, \quad \forall \ i = 1, 2, \ldots, M. \quad (3A.5)
\]

To see that \( S \) and \( \Lambda \) constructed according to (3.3.20) satisfy (3.3.21), it suffices to note that (3A.1) and (3A.4) yield (3.3.21b), and that summing up (3A.5) over \( i \) gives
\[
\mathbf{A}^+[\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_n] = \mathbf{D}[\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_n] \Lambda
\]
\[
\Rightarrow S^T \mathbf{A}^+ S = S^T \mathbf{D} \Lambda = \Lambda
\]
as desired. \hfill \blacksquare

\textbf{Appendix 3B} Physical Properties of the set \( \{ \mathbf{s}_i \} \)

\textbf{Corollary 3.1} Let \( \mathbf{s}_i \) be determined by (3.3.17). Then, \( \mathbf{s}_i \) as a nonmodal initial perturbation has the properties:

(i) the initial growth rate of \( \hat{\phi}(t; \mathbf{s}_i) \) is independent of the size of \( \mathbf{s}_i \), i.e.,

\[
\sigma_n (c \mathbf{s}_i; \bar{\mathbf{v}}) = \sigma_n (\mathbf{s}_i; \bar{\mathbf{v}}), \quad \forall \ c \in \mathbb{R}
\]  \hspace{1cm} (3B.1)

(ii) the velocity fields associated with different \( \lambda_i \) are orthogonal in sense that

\[
\langle \mathbf{v}(\mathbf{s}_i), \mathbf{v}(\mathbf{s}_j) \rangle = 0 \text{ if } \lambda_i - \lambda_j \neq 0,
\]  \hspace{1cm} (3B.2)

where \( \mathbf{v}(\mathbf{s}_i) \) and \( \mathbf{v}(\mathbf{s}_j) \) are velocity fields of \( \mathbf{s}_i \) and \( \mathbf{s}_j \), respectively, and \( \langle ., . \rangle \) is the inner product in \( \mathbb{H} \) (cf. (2.2.11)).

\textbf{Proof:} (i) The proof for (i) follows from (3.3.12b), (3.3.17) and (3.4.4).

(ii) First note that

\[
\langle \mathbf{s}_i, \mathbf{D} \mathbf{s}_j \rangle (\lambda_i - \lambda_j) = 0. \quad \text{(by (3.3.16) and (3.3.17))}
\]  \hspace{1cm} (3B.3)

Next, let \( \hat{\phi}(\mathbf{s}_i) \) and \( \hat{\phi}(\mathbf{s}_j) \) denote the streamfunctions generated using \( \mathbf{s}_i \) and \( \mathbf{s}_j \) as expansion coefficients according to (3.2.2). It thus follows that the inner product of \( \mathbf{v}(\mathbf{s}_i) \) and \( \mathbf{v}(\mathbf{s}_j) \) w.r.t the (2.2.11)
\[ \langle \mathbf{v}(s_i), \mathbf{v}(s_j) \rangle = \langle \mathbf{\nabla} \phi(s_i), \mathbf{\nabla} \phi(s_j) \rangle = \sum_{\alpha} s_{\alpha i} s_{\alpha j}^* \]  

(3B.4)

which is identical to \( \langle \mathbf{s}_i, \mathbf{D} \mathbf{s}_j \rangle \) if the inner product in \( \mathbb{R}^M \) is used to represent the summation in (3B.4), where \( s_{\alpha i} \) and \( s_{\alpha j} \) (\( \alpha = 1, 2, \ldots, M \)) are the the components of \( \mathbf{s}_i \) and \( \mathbf{s}_j \). The orthogonality of \( \mathbf{v}(s_i) \) and \( \mathbf{v}(s_j) \) then follows from (3B.3) and (3B.4). \( \blacksquare \)
Fig 3.1 Streamfunctions of three representative equilibrium states for $U^* = 22.\text{m/s}$ and $h_i = 500 \text{ m}$. (a) $1/r=29$ days; (b) $1/r=11$ days; (c) $1/r=5.5$ days. The rest of the parameters are given in the text.
Fig 3.2 Stability regime diagram. The solid line for the finite stability measure $r_N(\Psi)$. The dashed lines $r=r_N^-=0.17$/day for the MGS boundary, $r=r_L^-=0.075$/day for the linear stability boundary and $r=r$ is for the diagonal dashed line which is used for testing the condition $r_N(\Psi) - r > 0$. 
Fig 3.3 An example of MGS. The equilibrium state $\psi$ is (c) in Fig 3.1. The initial perturbations $\delta$ are randomly generated with the ratio $\delta$ of their initial kinetic energy to the energy in $\psi$ ranging from 0.2 to 1.0.
Fig 3.4 Existence of initially growing nonmodal \( s_m \) to equilibria \( \hat{\psi}(r) \) in subcritical regime, i.e., region (II) in Fig 3.2.
Fig 3.5 The spatial configurations of five initially growing nonmodal perturbations $s_i$ to the basic state (b) in Fig 3.1.
Fig 3.6 The energy time series for five disturbances initialized from the $s_i$ shown in Fig 3.5.
Fig 3.7 The growth rates of disturbances initialized from nonmodal perturbations $s_i$ over an initial growing period. The basic state $\psi$ is Fig 3.1 (a).
4.1 Introduction

The limitation of the growth rate of the most unstable (or the least damped) normal mode as an indicator of potential development of disturbances to a given flow has led one to search for favorably configured perturbation of nonmodal structure as alternatives (cf., Farrell, 1988; O’Brien, 1990; or § 2.6 and § 3.4 in this thesis). Given that such a perturbation is found by some means, e.g., by the global analysis, it remains necessary to establish that there exists an initial period over which the amplification triggered by this perturbation proceeds more intensely than that due to any modal perturbation in order to ascribe (or relate) the explosive development observed, for example, in cyclonegenesis (cf. Roebber, 1984; Sanders, 1986) to the properly configured nonmodal perturbations. A common approach to demonstrate this is based on numerical solutions of initial value problems, as done in § 3.5. While the results from such a approach (e.g., Farrell, 1989b, or Fig 3.7 in the previous chapter) suggest affirmative answer to the above concern, no direct proof has been known to date.
The objective of this chapter is to show that this is indeed the case in § 4.2. This is done in part by extending some results in § 2.8 to the finite dimensional system (3.2.3), and in part by establishing some basic properties of finite amplitude nonmodal disturbances (here defined as solutions to the nonlinear system (3.2.3)-(3.2.4), hereafter referred to as nonmodal disturbances for simplicity) in Appendix 4B. Numerical examples are given in § 4.3, followed by concluding remarks in § 4.4.

4.2 Nonmodal disturbance over transient period

The approach taken here is to compare the initial behavior of nonmodal disturbances starting from optimal nonmodal perturbations with the initial behavior of those initialized from any other perturbation, including those of modal structure.

4.2.1 modal vs. nonmodal growth rate at initial instant

First, we establish the optimal status for the nonmodal perturbations obtained as solutions to the MEP (3.3.17) w.r.t the most unstable normal mode at initial instant $t_0$. For the sake of space, only the outline of the treatment is given here (see § 2.8 for the counterpart in $\mathcal{N}$).

To obtain the MEP for the modal analysis of (3.2.3), we consider the linearized version of (3.3.2)
\( \frac{d}{dt} \hat{\psi} = \Lambda \hat{\psi}, \)  
\( \hat{\psi}(t_0) = \hat{\psi}_0 \)

where \( \Lambda : \mathbb{C}^M \rightarrow \mathbb{C}^M \) is given by (3.3.4e), with \( \hat{\psi} \) and \( \hat{\psi}_0 \in \mathbb{C}^M \).

Introducing a modal form solution

\[ \hat{\psi}(t; \xi, \sigma) = \xi e^{\sigma t}, \text{ for } \xi \in \mathbb{C}^M \text{ and } \sigma \in \mathbb{C} \]

into (4.2.1), we have the standard matrix eigenvalue problem

\[ (\Lambda - \sigma I) \xi = 0, \]

where \( I \) is the unit operator of order \( M \). With the derivation directed to Appendix 4A, we state the result in

**Lemma 4.1** For any modal disturbance \( \hat{\psi}(t; \xi, \sigma) = \xi e^{\sigma t} \) to a given equilibrium state \( \hat{\psi} \) of (3.2.3), with \( \xi \) and \( \sigma \) determined from (4.2.4), its growth rate is given by

\[ \text{Re}(\sigma(\hat{\psi}; \hat{\psi})) = \rho(\xi; \hat{\psi}) - r \]

where \( \rho \) is generalized Rayleigh Quotient as given by (3.3.14).

Defining the linear finite stability measure by

\[ r_L(\hat{\psi}) = \max_{\xi \in \{\hat{\psi}\}} \rho(\xi; \hat{\psi}) \quad (4.2.6) \]

with \( \rho \) being the generalized Rayleigh quotient and \( \{\hat{\psi}\} \subset \mathbb{C}^M \) denoting the set of eigenvectors of (4.2.4), we can show, using the properties (3.3.15) and (3.3.16), that

**Lemma 4.2** (comparison of stability measures) Let \( r_N(\hat{\psi}) \) and \( r_L(\hat{\psi}) \) be given by (3.3.24) and (4.2.6), respectively. Then:
In terms of \( r_L (\tilde{\psi}) \) and \( r_N (\tilde{\psi}) \), we can write
\[
\max \Re (\sigma) = r_L (\tilde{\psi}) - r \quad (4.2.8)
\]
as opposed to the maximum nonmodal growth rate
\[
\max \sigma_N (\phi; \tilde{\psi}) = r_N (\tilde{\psi}) - r \quad (4.2.9)
\]
where we have used (3.3.12b), (3.3.24) and (3.4.4) to get (4.2.9). From (4.2.7), we then have (cf. theorem 2.5 in H)

**Theorem 4.1** For any equilibrium state \( \tilde{\psi} \) of (3.2.3), it has at least one nonmodal initial perturbation which has a growth rate not less than its maximum modal growth rate.

### 4.2.2 Explosive development of nonmodal disturbance

To demonstrate the desired assertion from theorem 4.1, strictly speaking, we need the continuity of solution \( \phi(t; \phi_0) \) of (3.3.2) in time \( t \), and the continuity of the instantaneous growth rate \( \sigma_N (\phi; \tilde{\psi}) \) in \( \phi \) for \( \phi \) in some open set of \( \mathbb{R}^n \), both of which are established in Appendix 4B. Given these auxiliary properties of nonmodal disturbances, we proceed by contradiction, viz., supposing that for a sufficiently small initial period \( [t_0, t_0 + T) \) and \( \forall \ t = t_0 + t' \) with \( t' \leq T \),
\[
\sigma_N (\phi(t_0 + t' ; s_N) ; \tilde{\psi}) < \sigma_N (\phi(t_0 ; s_N) ; \tilde{\psi}), \tag{4.2.10}
\]
where \( s_0 = \Re (\tilde{\xi}) \) or \( = \Im (\tilde{\xi}) \), with \( \tilde{\xi} \) the fastest growing modal perturbation (FGMP) determined from (4.2.4). Taking \( \lim t' \to 0^+ \)
in (4.2.10) and using the continuity of $\omega_N(\phi;\psi)$ in $\phi$ and of 
$\hat{\phi}(t; \hat{\phi}_0)$ in $t$ (cf. Appendix 4B), we have

$$\sigma_N(\phi(t_0+0+, \mathbf{s}_M); \psi) \leq \sigma_N(\phi(t_0+0+, \hat{\phi}_0); \psi).$$ (4.2.11)

Again, by continuity of $\hat{\phi}(t; \mathbf{s}_M)$ and $\hat{\phi}(t; \hat{\phi}_0)$ at $t_0$, we 
identify $\phi(t_0+0+, \mathbf{s}_M)$ in (4.2.11) with $\phi(t_0; \mathbf{s}_M)$, and similarly 
$\phi(t_0+0+, \hat{\phi}_0)$ with $\phi(t_0; \hat{\phi}_0)$, thus leading to a statement 
contradicting the established fact that $\sigma_N(\mathbf{s}_M; \psi) \geq \sigma_N(\hat{\phi}_0; \psi)$
for $\hat{\phi}_0 \in \mathbb{R}^M$ (cf. (3.4.5)). Similarly, it can be shown that 
$\sigma_N(\phi(t; \mathbf{s}_M); \psi)$ is not less than $\max \text{Re}(\sigma) \forall \ t=t_0+t' \in
[t_0, t_0+T)$ based on theorem 4.1. To summarize, we have

**Theorem 4.2** There exists an initial period $[t_0, t_0+T)$ over 
which the initial development of at least one nonmodal initial perturbation to a given equilibrium state of (3.2.3) 
will proceed at a rate not less than the maximum modal growth rate or not less than that of nonmodal disturbances 
initialized from modal initial perturbations.

**4.3 Numerical illustrations**

The purpose of this section is twofold: first to 
supplement some numerical calculations to the discussion of 
the relation of initial modal to nonmodal growth rates (cf. §
2.8 and § 4.2.1), and second to provide further numerical 
illustrations for theorem 4.2 as supplementary to Fig 3.7.
We do so by considering the problem of planetary scale atmospheric flow over topography, with the model geometry being the same as in § 3.5. Three sets of equilibria of (3.2.3) are used here, each of which is obtained numerically using the continuation algorithm (Keller, 1978) with $U'$ as the controlling parameter ranging from 5m/s to 50m/s, a range covering the typical wind velocity in the upper part of troposphere, with the rest of model parameters specified as needed.

(a) Initial modal vs. nonmodal growth rates

From (4.2.8) and (4.2.9), it suffices to compare $r_N(\hat{\psi})$ with $r_L(\hat{\psi})$ in order to see the underestimation of potential amplification of disturbances by the modal analysis (cf., theorem 4.1). Here, $r_N(\hat{\psi})$ is calculated as in § 3.5.2(b), with $r_L(\hat{\psi})$ obtained from max Re($\sigma$) via (4.2.8) after solving the standard eigenvalue problem $(A - \sigma I)\hat{\xi} = 0$.

Fig 4.1 shows the results from one case of comparison. It is seen that the curve for $r_N(\hat{\psi})$ is always above the one for $r_L(\hat{\psi})$ over the whole range of $U'$, thus confirming the predicted potentially more intense initial development of nonmodal disturbances. The model parameters for the set of equilibria $\{\hat{\psi}(U')\}$ are given in the caption, with some representative states from the set $\{\hat{\psi}(U')\}$ shown in Fig 4.2.
Viewing the stability of the set \( \{ \hat{\psi}(U^*) \} \) as a function of the external forcing \( U^* \), it is seen that in the direction of increasing \( U^* \), the \( \{ \hat{\psi}(U^*) \} \) first loses monotonic, global stability (MGS) at \( U^* = \tilde{U}_N = 6.6 \) m/s, then loses modal stability once across \( U^* = \tilde{U}_L = 10.5 \) m/s from below, and recovers the modal stability for \( U^* \) larger than \( \tilde{U}_L^2 = 26.3 \) m/s (cf. Fig 4.1). Corresponding to the exchange of stability of \( \{ \hat{\psi}(U^*) \} \) is the loss and regain of zonality in equilibrium flows in physical space (cf. Fig 4.2). The physical mechanism underlying the supercritical regime (III) is frequently associated with the so-called topographic instability, which has received much attention in the effort to explain the formation, maintenance and break down of "blocking patterns" in atmospheric flows (e.g., Charney and Devore, 1979).

The consequence of the underestimation varies from one stability regime to another. In the monotonic global stability regime (i.e., region (I) in Fig 4.1), it is physically insignificant since no disturbance is allowed to grow by (4.2.8) and (4.2.9) anyway; in the subcritical regimes (i.e., regions (II) and (IV) in Fig 4.1), it leads to the loss of temporal amplification phenomenon; in the supercritical regime (i.e., region (III), it implies potential explosive intensification of nonmodal disturbances. The consequence is more dramatic when the basic states \( \{ \hat{\psi}(U^*) \} \) are only weakly unstable or stable, as shown in Fig
4.3. It is seen that over the entire range of $U^*$, the potential initial amplifications of disturbances to $\hat{\psi}(U^*)$ solely arise from properly configured nonmodal perturbations since modal growth rate is negative everywhere.

Consider the change of physical parameters in the direction of promoting the modal growth rate to see how this will affect the relation of initial modal to nonmodal growth rate. This is done for the set of $\{\hat{\psi}(U^*)\}$ determined using the same set of parameters as in Fig 4.1 except the reduced Ekman damping time scale $1/r^* = 30$ days. In Fig 4.4, while the portion of $U^*$-parameter space supporting the modal growth is enlarged compared to region (III) in Fig 4.1, the concern for the underestimation of potential amplification of disturbances using the modal growth rate remains as before.

(b) Intense development of instability over an initial period

To see the development of nonmodal disturbances at a rate larger than would be expected from the maximum modal growth rate over some initial period, it suffices to consider if it happens when the underlying equilibrium state supports strongly unstable modal perturbations. Numerical experiments based on the initial value problem (3.2.3) are performed as in § 3.5. The equilibrium state $\hat{\psi}$ used here is shown in Fig 4.5(a), corresponding to the peak on the $r^*(\hat{\psi})$ curve in Fig 4.4. The instantaneous growth rates from three experiments
are shown in Fig 4.6, with the corresponding energetics given in Fig 4.7. It is seen that the response of $\hat{\psi}$ to the nonmodal initial perturbations $s_m$ and $s_{m-1}$ is noticeably more intense than to the fastest growing modal initial perturbation $\zeta$. Specifically, the initial growth rates of $\hat{\phi}(t;s_{m-1})$ ($i=0,1$) are nearly 50% larger than the maximum modal growth rate and are noticeably higher than that of $\hat{\phi}(t;\phi_0)$ with $\phi_0 = \text{Re}(\zeta)$ over a period of two days. In terms of energetics, the energy level of $\hat{\phi}(t;s_{m-1})$ ($i=0,1$) is considerably higher than that of $\hat{\phi}(t;\text{Re}(\zeta))$ for more than five days (cf. Fig 4.7).

4.4 Concluding remarks

The global analysis (cf. § 2.2-2.5 and § 3.3) provides a systematical algorithm for the search of favorably configured nonmodal perturbations. The study of the relation of initial modal to nonmodal growth rate reveals the cause responsible for the limitation of modal analysis and provides the rationale for phenomena such as the transient growth in subcritical flows. The investigation of initial behavior of nonmodal disturbances initialized from those favorably configured nonmodal perturbations is a key to understanding the rapid development of disturbances observed, for example, in the atmosphere (Sanders, 1986).
Appendix 4A  Modal growth rate expressed in terms of $\rho$

Proof: First, we put the standard eigenvalue problem (4.2.4) into an equivalent form

$$\hat{A} \hat{r} = \sigma D \hat{r}; \hat{A} = DA,$$  \hfill (4A.1)

where $\hat{A}$ and $D$ are given by (3.3.4e) and (3.3.11), respectively. The equivalence follows from the fact that $D$ is positive definite in $\mathbb{R}^N$ (cf. (3.3.11)) and hence invertible.

At this point, it is crucial to note from (3.3.4e), (3.3.11) and (4A.1) that $\hat{A}$ can be decomposed as

$$\hat{A} = \hat{A}^+ + \hat{A}^-,$$  \hfill (4A.2)

with

$$\hat{A}^+ = \hat{A} - \sigma D, \quad (4A.3a)$$

$$\hat{A}_{ij} = \sum_k c_{ikj} \psi_k (-a_j^2), \quad (4A.3b)$$

$$\hat{A}^- = [\hat{A}^-]_i^j = -\hat{A}^{-T}, \quad (4A.4a)$$

$$\hat{A}^-_{ij} = \beta b_{ij} + \sum_k c_{ijk} (f_0 \psi_k^2 / H - a_k^2 \psi_k). \quad (4A.4b)$$

Taking inner product of (4A.1) with respect to an arbitrary eigenvector $\xi$ of (4.2.4) gives

$$\langle \xi, \hat{A} \xi \rangle = \sigma \langle \xi, D \xi \rangle,$$  \hfill (4A.5)

where the overbar denotes the complex conjugate. Adding the complex conjugate of (4A.5) to itself, we obtain, with the use of the definitions of $\hat{A}^+$ and $\hat{A}^-$,

$$2 \text{Re} (\sigma) \langle \xi, D \xi \rangle = \langle \xi, \hat{A}^+ \xi \rangle + \langle \xi, \hat{A}^- \xi \rangle = \langle \xi, \hat{A}^+ \xi \rangle + \langle \xi, \hat{A}^+ \xi \rangle.$$
where the contribution from $\hat{A}^-$ to the RHS of (4A.6) cancels out by virtue of the fact (4A.4a), with $\hat{A}^+$ defined by (3.3.13). The realness of the RHS of (4A.6) is ensured by the self-adjointness of $\hat{A}^+$ and $D$ (cf. (3.3.13) and (3.3.11)). To this end, introducing the definition of $\rho$ into (4A.6) leads to the assertion.

**Appendix 4B** Fundamental properties of nonmodal disturbance

**B-1 Existence, uniqueness and continuity of $\hat{\phi}(t;\hat{\phi}_0)$**

The continuity of nonmodal disturbance $\hat{\phi}(t;\hat{\phi}_0)$ in time $t$ as well as other fundamental properties such as existence and uniqueness of solutions to (3.3.2) can be easily obtained via a simple application of the Picard-Lindelöf theorem (cf. theorem 3.1 in Hale, 1963). The essential condition assuring these properties is that the vector field $f(\hat{\phi};\hat{\psi},\alpha)$ given by (3.3.3) is locally Lipschitzian in $\hat{\phi}$, as stated in

**Lipschitzianism of the vector field $f(\hat{\phi};\hat{\psi},\alpha)$**

Let $f(\hat{\phi};\hat{\psi},\alpha)$ be defined by (3.3.3) and let $\mathcal{U}$ be any bounded, closed set in $\mathbb{R}^M$. Then, $f(\hat{\phi};\hat{\psi},\alpha)$ is locally Lipschitzian in $\hat{\phi} \in \mathcal{U}$.

**Proof:** The boundedness and closedness of set $\mathcal{U}$ implies the existence of a closed ball $\mathbb{B}(0,d)$ of radius $d$ centered at $0$, the origin of $\mathbb{R}^M$, such that $\mathcal{U} \subseteq \mathbb{B}(0,d)$. Now, take any $\hat{\phi}, \tilde{\phi}$
ε U and consider
\[ \| f(\tilde{\phi}; \psi, \alpha) - f(\tilde{\phi}; \tilde{\psi}, \alpha) \| \leq \| A \| \| \tilde{\phi} - \tilde{\phi} \|
+ \| B(\tilde{\phi}; \tilde{\psi}) \| + \| B(\tilde{\phi}; \tilde{\psi}) \|, \]
where we have used (3.3.5) for f, with A and B given by (3.3.4e) and (3.3.4f), respectively, and \( \| \| \) being the \( L^2 \) norm. Note that every linear operator in \( \mathcal{L}(\mathbb{R}^n) \), a vector space of linear operators on \( \mathbb{R}^n \), is bounded (cf. theorem 2.7-9 in Kreyszig, 1978). In fact, by the Schwarz inequality, we obtain
\[ \| A \| \leq \left( \sum_{i,j} A_{ij}^2 \right)^{1/2}, \]
where \( A_{ij} \) is entry of A (cf. (3.3.4e)). For the two terms involving B in (4B.1), we have
\[ \| B(\tilde{\phi}; \tilde{\psi}) \| \leq d \left( \sum_{i,j,k} \sum B_{ijk}^2 \right)^{1/2} \| \tilde{\phi} - \tilde{\phi} \|, \]
\[ \| B(\tilde{\phi}; \tilde{\psi}) \tilde{\phi} \| \leq d \left( \sum_{i,j,k} \sum B_{ijk}^2 \right)^{1/2} \| \tilde{\phi} - \tilde{\phi} \|. \]
Introducing (4B.2)-(4B.4) into (4B.1), we get
\[ \| f(\tilde{\phi}; \psi, \alpha) - f(\tilde{\phi}; \tilde{\psi}, \alpha) \| \leq L(U) \| \tilde{\phi} - \tilde{\phi} \|, \]
where \( L(U) \) is Lipschitzian constant of set \( U \)
\[ L(U) = \max \left\{ \left( \sum_{i,j} A_{ij}^2 \right)^{1/2}, 2d \left( \sum_{i,j,k} \sum B_{ijk}^2 \right)^{1/2} \right\}. \]
This completes the proof.
By virtue of the Picard-Lindelöf theorem, we have

Existence, uniqueness and continuity Let $U$ be any bounded, closed set in $\mathbb{R}^n$ and let $E \subset \mathbb{R} \times \mathbb{R}^n$ denote the domain of definition for the solution $\phi(t;\phi_0)$ of (3.3.2)

$$E = \left\{ (t, \phi_0) \mid a(\phi_0) \leq t \leq b(\phi_0); \phi_0 \in U \subset \mathbb{R}^n \right\},$$  \quad (4B.5)

with $(a(\phi_0), b(\phi_0)) \subset \mathbb{R}$ being the maximum interval of existence of $\phi(t;\phi_0)$. Then, for any $\phi_0 \in U$, the system (3.3.2) has a unique solution $\phi(t;\phi_0)$ through $\phi_0$. Moreover, $\phi(t;\phi_0)$ is continuous in $E$.

B-2: Continuity of $\sigma_n(\phi;\psi)$ in $\phi$

The continuity of $\sigma_n(\phi;\psi)$ in $\phi$ for $\phi$ in some open set of $\mathbb{R}^n$ is given by:

Continuity of $\sigma_n(\phi;\psi)$ in $\phi$ Let the growth rate $\sigma_n(\phi;\psi)$ of $\phi(t;\phi_0)$ be defined by (3.4.4), and let $U$ be any bounded, closed set in $\mathbb{R}^n$. Then, $\sigma_n: U \to \mathbb{R}$ is continuous.

Proof: From (3.3.12b), (3.3.24) and (3.4.4), it suffices to show that $\rho: U \to \mathbb{R}$ is continuous. Recall that $\rho$ is the generalized Rayleigh Quotient defined by (3.3.14). For this, we observe that for any $\phi', \phi \in U \subset \bar{B}(0, d)$

$$|<\phi', A^+\phi> - <\phi, A^+\phi>| \leq 2d\|A^+\phi' - \phi\|,$$ \quad (4B.6)

86
\[ |\langle \phi', D\phi' \rangle - \langle \phi, D\phi \rangle| \leq 2d \|D\| \|\phi' - \phi\|, \]  

where \( d \) is radius of the closed ball \( \bar{B}(0, d) \) chosen to enclose \( U \), with \( \mathbf{A}^+ \) and \( D \) given by (3.3.13) and (3.3.11), respectively. It is evident from (4B.6)-(4B.7) that both \( \langle \phi, \mathbf{A}^+\phi \rangle \) and \( \langle \phi, D\phi \rangle \) are continuous in a neighborhood of \( \phi \) defined by the open ball \( B(\phi; \delta) \), with \( \delta \) given by \( \min \left\{ \frac{1}{2d \|\mathbf{A}^+\|}, \frac{1}{2d \|D\|} \right\} \), from which we conclude that \( \rho(\phi; \psi) \) (\( = \langle \phi, \mathbf{A}^+\phi \rangle/\langle \phi, D\phi \rangle \)) and hence \( \sigma_n(\phi; \psi) \) are continuous in some neighborhood of \( \phi \) not larger than \( B(\phi; \delta) \).
Fig 4.1 Maximum initial modal versus nonmodal growth rate, i.e., $r_L(\Psi) - r$ vs. $r_N(\Psi) - r$, for the set of equilibria $\{\Psi(U)\}$ over zonal wavenumber-1 topography of height 1000 m. The Ekman damping coefficient $r = 1/(15 \text{days})$ is indicated by the horizontal dashed line. The three vertical dashed lines are for the MGS boundary $U^* = \bar{U}_N = 6.6 \text{ m/s}$, and for linear stability boundaries $U^* = \bar{U}_L = 10.5 \text{ m/s}$ and $U^* = \bar{U}_L^2 = 26.3 \text{ m/s}$, where $\bar{U}_N$ and $\bar{U}_L^1$, $\bar{U}_L^2$ are obtained as roots to the equations, $r_N(\Psi(U^*)) - r = 0$ and $r_L(\Psi(U^*)) - r = 0$, respectively.
Fig 4.2 (a) Topography contours; (b)-(f) streamfunctions of several representative states from the set of $\{\tilde{V}(U')\}$ used in Fig 4.1, corresponding to $U'=6.20$ m/s in (I), 9.12 m/s in (II), 11.33 and 17.85 m/s in (III) and 40.13 m/s in (IV), respectively.
Fig 4.3 The same as Fig 4.1 except that the set of equilibria \( \{ \Phi(u') \} \) used here corresponds to the zonal wavenumber-2 topography.
Fig 4.4 The same as Fig 4.1 except that the set of equilibria ($\mathcal{E}(U^*)$) used here corresponds to $1/r = 30$ days.
Fig 4.5 Basic states, nonmodal and modal initial perturbations for the numerical experiments. (a) for streamfunction of the equilibrium state taken from Fig 4.4 corresponding to $U=16.3$ m/s, (b) and (c) for its fast growing initial nonmodal and modal perturbations, respectively.
Fig 4.6  Evolution of the growth rate with time. The optimal nonmodal initial perturbation $s_m$ (see Fig 4.5(b)) has growth rate $\sigma_N(s_m; \Psi) = 1/(5.29 \text{ days})$, whereas the fast growing modal perturbation (FGMP) (see Fig 4.5(c)) has $\text{Re}(\sigma) = 1/(9.52 \text{ days})$. The initial perturbations are indicated in the Fig 4.5 (a) and (b).
Fig 4.7 Evolution of disturbance energy with time for the case shown in Fig 4.6. The initial perturbation energy is 20% of the energy in the equilibrium states to ensure the finite amplitude for the initial perturbations.
CHAPTER 5

FINITE AMPLITUDE NONMODAL DISTURBANCE II: ASYMPTOTIC BEHAVIOR
its relation to bifurcation and multiple equilibria

5.1 Introduction

Consideration of the asymptotic behavior of disturbances arises in a variety of circumstances. When a modal analysis reveals the existence of exponentially growing perturbations to a given flow, one wants to determine the maximum amplitude that can be attained by the growing disturbances, a problem of nonlinear saturation (Shepherd, 1988, 1989); or to inquire about the arrest of the exponential growth, a problem of nonlinear equilibration (Pedlosky, 1970, Salmon, 1980; Mak, 1985). On the other hand, the existence of initially growing nonmodal perturbations uncovered by such analysis as optimization procedures (Farrell, 1988; O'Brien, 1990; chapter 2 of this thesis) raises the question: what happens to these disturbances after the initial transient growth. Further, identifying multiple equilibria from a bifurcation analysis (Vickroy and Dutton, 1979) leads one to wonder how a transition from one equilibrium state to another takes place. The answers to these problems or questions lie in the study of the asymptotic behavior of finite amplitude nonmodal disturbances.

The objective of this chapter is to establish some basic
facts regarding the $t \to \infty$ behavior for the system (3.2.3)-(3.2.4). The discussion of asymptotic decay in subcritical flows is given in § 5.2, with the treatment for persistent disturbances in supercritical flows in § 5.3. The \textit{a priori} determination of the nature of the asymptotic states of nonmodal disturbances is addressed in § 5.4. Numerical illustrations are given in § 5.5, followed by concluding remarks.

5.2 Asymptotic decay as $t \to \infty$ in subcritical flow

In this and next sections, we confine the discussion to a generic case where the underlying equilibrium state $\tilde{\psi}$ of (3.2.3) is \textit{hyperbolic}, i.e., $\text{Re}(\sigma_i(A(\tilde{\psi}))) \neq 0$, $i = 1,2,\ldots,M$, where $A(\tilde{\psi})$ is the linear part of the vector field $F$ (or $f$) (cf. (3.2.3) or (3.3.3a)) evaluated at $\tilde{\psi}$ (or at $\phi = 0$). The reason for the restriction to hyperbolicity is more than for convenience. In fact, it is motivated by the fact that hyperbolicity is a generic property of linear operators on $\mathbb{R}^M$. More precisely, the set of hyperbolic operators is dense and open in $\mathcal{L}(\mathbb{R}^M)$, which means that any operator $P \in \mathcal{L}(\mathbb{R}^M)$ with $\text{Re}(\sigma_i(P)) = 0$ for some $i$ can be approximated arbitrarily closely by some $Q \in \mathcal{L}(\mathbb{R}^M)$ satisfying $\text{Re}(\sigma_i(Q)) \neq 0$ for any $i$ (cf. theorem 3, pp.157 in Hirsh and Smale, 1976). However, violation of hyperbolicity often occurs at a critical equilibrium state of (3.2.3) which in turn corresponds to a bifurcation point in some underlying physical parameter
space, which is the subject of § 5.4.

The main result of this section is theorem 5.1. The analysis leading to this result is based on the direct method of Liapunov (Hahn, 1967; Hale, 1969; Hirsh and Smale, 1976; Verhulst, 1985). The basic ideas of the method are given in Appendix 5A for reference. For a better focus on the main development, auxiliary lemmas are directed to Appendix 5B.

**Theorem 5.1** Suppose an equilibrium state \( \hat{\psi} \) of (3.2.3) is hyperbolic for which the following hold: (i) \( r_L(\hat{\psi}) - r < 0 \); (ii) \( r_N(\hat{\psi}) - r > 0 \), where \( r_L(\hat{\psi}) \) and \( r_N(\hat{\psi}) \) are given by (3.3.24) and (4.2.6), respectively. Then, there exists a neighbourhood \( U \) of the null solution \( \hat{\psi} = 0 \) of (3.3.2) such that any nonmodal disturbance \( \hat{\phi}(t;\hat{\phi}_0) \) to \( \hat{\psi} \) approaches 0 as \( t \to \infty \) whenever \( \hat{\phi}_0 \in U \).

Proof: Hyperbolicity of \( \hat{\psi} \) and the condition (i) implies that \( \text{Re}(\sigma(A)) < 0 \). It thus follows from lemma 5.2 (cf. Appendix 5B) that there exists a positive definite \( Q \in \mathbb{R}^{M \times M} \) as a solution to Liapunov matrix equation

\[
A^TQ + QA = -I, \quad I \text{ a unit matrix of order } M \tag{5.2.1}
\]

where we have set \( R \) in (5B.3) to \(-I\). Put \( V(\hat{\phi}) = \langle \hat{\phi}, Q\hat{\phi} \rangle \). We assert that such a chosen \( V \) is a Liapunov function of the nonlinear system (3.3.2) at \( \hat{\phi} = 0 \) (cf. Appendix 5A). It is
obvious that

\[ V(0) = 0 \text{ and } V(\varphi) = \langle \dot{\varphi}, Q \dot{\varphi} \rangle > 0 \text{ in } \mathbb{R}^n - \{0\} \]  
\hspace{1cm} (5.2.2)

with the latter property due to the positive definiteness of \(Q\). Now, consider the orbital derivative of \(V\) along the trajectory of any disturbance \(\varphi(t; \varphi_0)\) to \(\varphi\)

\[ \dot{V} = \langle (d/dt)\varphi, Q\varphi \rangle + \langle \varphi, Q(d/dt)\varphi \rangle \]

\[ = \langle A\varphi, Q\varphi \rangle + \langle \varphi, QA\varphi \rangle + \langle B\varphi, Q\varphi \rangle + \langle \varphi, QB\varphi \rangle \]  
\hspace{1cm} (by (3.3.5b))

\[ \equiv V_2(\varphi) + w(\varphi) \]  
\hspace{1cm} (5.2.3)

where \(V_2(\varphi)\) corresponds to the first two terms in (5.2.3), and \(w(\varphi)\) to the last two ones. Moreover,

\[ V_2(\varphi) = \langle \dot{\varphi}, (A^TQ + QA)\varphi \rangle = -||\varphi||^2 \]  
\hspace{1cm} (by (5.2.1))

and hence is negative definite in \(\mathbb{R}^n - \{0\}\). For \(w(\varphi)\), we have the following estimate

\[ |w(\varphi)| \leq |\langle B\varphi, Q\varphi \rangle| + |\langle \varphi, QB\varphi \rangle| \]

\[ \leq \||B\varphi||Q\varphi\| + ||\varphi||Q||B\varphi\| \]  
\hspace{1cm} (by Schwarz inequality)

\[ \leq 2c||Q||^3 ||\varphi||^3 \]  
\hspace{1cm} (by lemma 5.3 (cf. Appendix 5B))

It thus follows from lemma 5.1 (cf. Appendix 5B) that there exists a neighbourhood \(\mathcal{U}\) of the null solution \(\varphi = 0\) of (3.3.2) such that \(V\) satisfies (5A.4). This last property of \(V\) together with (5.2.2) proves that \(V\) is a Liapunov function for the null solution, which yields the desired result by Liapunov's criterion for asymptotic stability.
Remark 5.2.1 It follows that $\psi(t; s_i) \to 0$ after an initial transient growth if $s_i \in \mathbb{U}$, where $s_i$ is an initially growing nonmodal perturbation determined from (3.3.17) and assured by the condition (ii). This result provides a theoretical basis for the asymptotic decaying to zero seen in Fig 3.4 and Fig 4.6.

It has well been known for linear model flows (Orr, 1907, Rosen, 1971; Boyd, 1983; Farrell, 1987) that following the transient growth associated with the continuous spectrum is an asymptotic decay to zero. By imposing hyperbolicity and conditions (i)-(ii) on $\Psi$, we have essentially managed to demonstrate that this is also true for the nonlinear system (3.2.3). Note that this extension remains local as long as the neighborhood $\mathbb{U}$ is finite.

5.3 Persistence as $t \to \infty$ in supercritical flow

To conduct the subsequent discussion in a most transparent manner, we assume that the $\Psi$ under concern satisfies

(a) $\Psi$ is hyperbolic, 
(b) $\text{Re}(\sigma_i(\mathbf{A})) < 0$, $i = 1, 2, \ldots, p$, 
(c) $\text{Re}(\sigma_i(\mathbf{A})) > 0$, $i = p+1, p+2, \ldots, s, s \neq p$

where $\mathbf{A}$ is given by (3.3.4e) evaluated at $\Psi$, with the $\sigma_i(\mathbf{A}) = \alpha_i + i\beta_i$ ($\beta_i \geq 0$), $i=1,2,\ldots,s$, being its distinct eigenvalues of algebraic multiplicity $n_i$. The condition $s \neq p$
is meant to exclude the case discussed in § 5.2.

It is clear that this $\hat{\psi}$ has initially growing modal perturbations by (5.3.1c) and hence nonmodal perturbations by (4.2.7)-(4.2.9). From earlier works though for other model flows (e.g., Stuart, 1960; Pedlosky, 1970, 1981; Mak, 1985), it may be expected that the disturbances started from unstable normal modes would be ultimately equilibrated via mechanisms such as nonlinear cascade (Salmon, 1980). It remains unclear to those disturbances arising from perturbations of nonmodal structure. This concern is strengthened by the argument that the observed rapid deepening of synoptic scale disturbances perhaps predominantly arises from the release of the mean flow potential energy by favorably configured nonmodal perturbations (Farrell, 1985).

The main result of this section (i.e., theorem 5.2) may be regarded as an effort to address the concern. Essentially, we first find a new basis for the space of all the kinematically admissible disturbances (i.e., for $\tilde{R}$) according to the real canonical theory of linear operator (Hirsh & Smale, 1974), and then divide the disturbances into two subset $\tilde{R}^+$ and $\tilde{R}^-$ (cf. lemma 5.4 in Appendix 5C). Next, we show that under the conditions (5.3.1) there exists a Liapunov instability function $V^+(\phi)$ in the neighbourhood $\hat{\phi}=0$ whose
properties allow us to detect the $t \to \infty$ asymptotic behavior of any disturbance initialized in $U^+ \cap R^+ \setminus \{0\}$ (cf. lemma 5.5 in Appendix 5D for properties of $V^+$). Now, we state

Theorem 5.2 Let the equilibrium state $\hat{\psi}$ of (3.2.3) satisfy (5.3.1). Then, any nonmodal disturbance $\hat{\phi}(t; \hat{\phi}_0)$ to $\hat{\psi}$ will persist as $t \to \infty$ if $\hat{\phi}_0 \in U^+ \cap R^+ \setminus \{0\}$, where $R^+$ and $U^+$ are the same as in lemma 5.4 and lemma 5.5, respectively. This is particularly true for $\hat{\phi}(t; s_i)$ when $s_i$ is contained in $U^+ \cap R^+ \setminus \{0\}$, $s_i$ the nonmodal initial perturbation from (3.3.17).

Proof: Under the hypotheses (5.3.1) on $\hat{\psi}$, the property (a) of function $V^+$ holds (cf. Appendix 5D). Thus, for a given constant $a>0$, there exists a closed ball $B(0, b)$ such that

$$|V^+(\phi)| \leq a \text{ for } \phi \in B(0, b). \quad (5.3.2)$$

Moreover, by continuity of $V^+$ at 0, it is possible to choose $b$ to ensure the closed ball in (5.3.2) to satisfy

$$B(0, b) \subseteq U^+. \quad (5.3.3)$$

We assert that any nonmodal disturbance $\hat{\phi}(t; \hat{\phi}_0)$ to $\hat{\psi}$ will leave the ball in finite time if $\hat{\phi}_0 \in B(0, b) \cap U^+ \cap R^+ \setminus \{0\}$. We prove this by contradiction, assuming that

$$\hat{\phi}(t; \hat{\phi}_0) \subseteq B(0, b) \text{ for } t \geq t_0 \text{ if } \hat{\phi}_0 \in B(0, b) \cap U^+ \cap R^+ \setminus \{0\}. \quad (5.3.4)$$

From property (c) of $V^+$ in lemma 5.5, we choose $\hat{\phi}_0$ such that
$\mathcal{V}_0 \in \mathbb{B}(0, b) \cap \mathbb{R}^+ - \{0\}$, i.e.,

$$V^+(\mathcal{V}_0) > 0.$$  \hfill (5.3.5)

Since for $t \geq t_0$

$$V^+(\mathcal{V}(t; \mathcal{V}_0)) - V^+(\mathcal{V}_0) = \int_{t_0}^{t} \dot{V}^+ > 0 \quad \hfill (5.3.6)$$

$V^+(\mathcal{V}(t; \mathcal{V}_0))$ is nondecreasing along the orbit $\mathcal{V}(t; \mathcal{V}_0)$, where the positive sign in (5.3.6) follows from (b) in lemma 5.5, (5.3.3) and (5.3.5). Put

$$\mathcal{V} = \left\{ \mathcal{V} \mid V^+(\mathcal{V}(t; \mathcal{V}_0)) > V^+(\mathcal{V}_0); \mathcal{V}(t; \mathcal{V}_0) \in \mathbb{B}(0, b) \right\}.$$ 

It is obvious from (5.3.5), and (5.3.6) that the set $\mathcal{V} \neq 0$, and hence from (b) of lemma 5.5 that $\mu = \inf_{\mathcal{V}} \dot{V}^+ > 0$. Continuing (5.3.6) leads to

$$V^+(\mathcal{V}(t; \mathcal{V}_0)) - V^+(\mathcal{V}_0) \geq \mu (t - t_0) \to \infty \text{ as } t \to \infty,$$

a statement contradicting (5.3.2), thus completing the proof.

**Remark 5.3.2** It is clear from lemma 5.4 and 5.5 (cf. Appendices 5C and 5D) that the set $U^+ \cap \mathbb{R}^+ - \{0\}$ is not empty under the condition (5.3.1), implying that there always exist perturbations leading to persistent disturbances to a flow $\tilde{\mathcal{V}}$ for which (5.3.1) holds, and that a transition from the given $\tilde{\mathcal{V}}$ to a new state is potentially inevitable.

We consider the foregoing result useful primarily for its capacity to cope with perturbations not restricted to any
special type and basic flows not necessarily zonally sheared, and for its assurance of the existence of a set of perturbations capable of triggering a transition from a given flow to another distinct state, but not for its prediction of details of limiting status. In fact, it yields no information on the nature of persistent disturbances (i.e., steady or periodic or chaotic), as opposed to weakly nonlinear theory (e.g., Stuart, 1960; Watson, 1960; Pedlosky, 1970, 1981). Further, it is silent about the mechanism responsible for the equilibration which the persistent evolving disturbances are expected to experience ultimately (Pedlosky, 1970; Salmon, 1980), and about the saturation level (Shepherd, 1988, 1989).

It is important to note that a perturbation superimposed onto a $\hat{\psi}$ for which (5.3.1) is satisfied may not survive in time, i.e., it may decay to 0 as $t \to \infty$, despite the supercritical nature of $\hat{\psi}$. Indeed, by a similar argument, we can have

**Remark 5.3.2** Under the hypothesis on $\hat{\psi}$ in (5.3.1), there exists a function $\tilde{V}: \mathcal{U} \to \mathbb{R}$ ($\mathcal{U}$ a neighbourhood of $\hat{\phi} = 0$) defined by

$$\tilde{V} = \langle \tilde{\phi}, Q^- \tilde{\phi} \rangle, \quad Q^- = \text{diag}[Q_1, -Q_2]$$

satisfying

(a) $\tilde{V}(0) = 0$ 

(5.3.8a)
(b) \( \dot{\mathbf{v}}^- < 0 \) in \( \mathcal{U}^R - \{0\} \),  

(5.3.8b)  

c) \( \dot{\mathbf{v}}^- (\tilde{\phi}) > 0 \) \( \forall \tilde{\phi} \in \mathcal{R}^- - \{0\} \)  

(5.3.8c)  

where \( \tilde{\phi} \in \mathbb{R}^n \) is written w.r.t the basis \( \mathcal{B} \) (5C.7), \( \mathcal{Q}_1 \) and \( \mathcal{Q}_2 \) the solutions to (5D.5) and (5D.6), respectively. Furthermore, It follows from (5.3.8) and lemma 5.4 (cf. Appendix 5C) that any nonmodal disturbance \( \dot{\phi}(t;\tilde{\phi}_0) \) to \( \tilde{\psi} \) will approach to 0 as \( t \to \infty \) if \( \tilde{\phi}_0 \in \mathcal{U}^- \cap \mathcal{R}^- - \{0\} \), especially if \( \tilde{\phi}_0 = s_i \in \mathcal{U}^- \cap \mathcal{R}^- - \{0\} \), where \( s_i \) is determined from (3.3.17). The set \( \mathcal{U}^- \cap \mathcal{R}^- - \{0\} \) and set \( \mathcal{U}^+ \cap \mathcal{R}^+ - \{0\} \) are reminiscences of the stable and unstable manifolds of \( \tilde{\phi} = 0 \) associated with (3.3.2), respectively (cf. Kelley, 1967; Hale, 1969).

5.4 Persistence, criticality and supercritical bifurcation

Given the existence of persistent \( \dot{\phi}(t;\tilde{\phi}_0) \) to a supercritical \( \tilde{\psi} \) (cf. theorem 5.2), it remains to determine the precise nature of the persistent nonmodal disturbances (i.e., stationary, periodic or chaotic). From the bifurcation theory of Navier-Stokes equation (Joseph and Sattinger, 1972; Joseph, 1976), it would be expected that this nature is related to the properties of principal eigenvalue \( \tilde{\sigma} \) (defined as the one with the largest real part) of the linear part of \( \mathcal{F}(\tilde{\phi};\tilde{\psi}(\alpha);\alpha) \) at some critical values of \( \alpha \). The goal of this section is to verify this expectation for the system (3.3.2).

It can be argued that the linear stability boundary \( \alpha = \bar{\alpha} \).
for the set equilibria \( \{ \tilde{\psi}(\alpha) \} \) of (3.2.3) in \( \alpha \)-parameter space is a criticality (recall that \( \tilde{r}_L \) is the root to the equation, \( r_L (\tilde{\psi}(\alpha)) - r = 0 \)). To see this, we first note that \( \text{Re}(\tilde{\sigma})|_{\alpha=\tilde{r}_L} = r_L (\tilde{\psi}(\tilde{r}_L)) - r = 0 \) by (4.2.8), which implies that \( \tilde{\psi}(\tilde{r}_L) \) is nonhyperbolic. Next, under the hyperbolicity condition, the asymptotic behavior of \( \tilde{\phi}(t;\tilde{\phi}_0) \) undergoes qualitative changes upon crossing \( \alpha = \tilde{r}_L \) (cf. theorem 5.1 and theorem 5.2). Further, the existence of persistent \( \tilde{\psi}(\alpha) \) with \( \alpha \) in some neighborhood of \( \alpha = \tilde{r}_L \) indicates that \( \alpha = \tilde{r}_L \) is a supercritical bifurcation point of (3.3.2) since in the supercritical side of \( \alpha = \tilde{r}_L \), (3.3.2) also has \( \tilde{\phi} = 0 \) as its solution.

To see the link between the property of \( \tilde{\sigma}(\tilde{r}_L) \) and the nature of persistent \( \tilde{\phi}(t;\tilde{\phi}_0) \), we shall focus on some neighborhood of \( \alpha = \tilde{r}_L \) and restrict discussion to the case in which the following holds:

(a) \( \tilde{\sigma}(\tilde{r}_L) \) (\( = i\omega \)) is a simple eigenvalue of \( \tilde{f}_\phi \) at \( (\tilde{\phi},\tilde{\alpha}) = (0,\tilde{r}_L) \); furthermore, \( \tilde{f}_\phi \) at \( (0,\tilde{r}_L) \) has no eigenvalue of the form \( im\omega \) with \( m \in \mathbb{Z}\{1,-1\} \); \hfill (5.4.1)

(b) \( \tilde{f}_\alpha \) at \( (0,\tilde{r}_L) \) \( \in \) range \( \tilde{f}_\phi \) at \( (0,\tilde{r}_L) \) \hfill (5.4.2)

(c) the loss of stability of \( \{ \tilde{\psi}(\alpha) \} \) at \( \alpha = \tilde{r}_L \) is strict in the sense that \( \left( \frac{d}{dt}\text{Re}(\tilde{\sigma}) \right)|_{\alpha=\tilde{r}_L} > 0 \). \hfill (5.4.3)

It is of interest to note that while the above assumptions are made for simplicity, it turns out that these conditions
are well satisfied in numerical calculations such as those reported in the next section.

We first consider the case of $\bar{\sigma}(\bar{r}_L)=0$ (i.e. $\omega = 0$ by (5.4.1)). It is crucial to note that the stationary solutions of (3.3.2) trace out a curve in state-parameter space (i.e., in $\mathbb{R}^M \times \mathbb{R}$) as the parameter $\alpha$ varies in some physically relevant range. We shall show next that under conditions (a) and (b), there are two such curves passing through the point $(0, \bar{r}_L)$, one for the null solution $\dot{\phi}=0$ and one for the stationary persistent state of $\dot{\phi}(t;\phi_0)$. We begin with the observation that

$$\dim \{\text{range } \mathbf{f}(0;\mathbf{v}(\bar{r}_L), \bar{r}_L)\} = \dim \{\text{range } \mathbf{A}(\mathbf{v}(\bar{r}_L))\} = M-1 \quad \text{(by (3.3.5))}$$

Now, define $Df : \mathbb{R}^{M+1} \rightarrow \mathbb{R}^M$ by $Df = [f_{\phi}, f_\alpha]$. Then, condition (a) and (b) together with (5.4.4) imply that

$$\dim \{\text{null } Df\}|_{\alpha=\bar{r}_L} = (\dim \{\text{domain } Df\} - \dim \{\text{range } Df\})|_{\alpha=\bar{r}_L} = 2. \quad (5.4.5)$$

Next, set $(d/dt)\dot{\phi} = 0$ for any stationary limiting state $\dot{\phi}$ of (3.3.2) to have

$$\mathbf{f}(\phi; \mathbf{v}(\alpha); \alpha) = 0. \quad (5.4.6)$$

For tangent $(d\phi, d\alpha)$ at point $(\phi, \alpha)$ on any solution curve of (5.4.6), we take the differential on both sides of (5.4.6) to get
\[ f_\phi (\phi; \psi (\alpha); \alpha) \, d\phi + f_\alpha (\phi; \psi (\alpha); \alpha) \, d\alpha = 0, \text{ or} \]
\[ Df[d\phi, d\alpha]^T = 0. \quad (5.4.7) \]

It then follows from (5.4.5) that the solution space of (5.4.7) is of dimension 2, and hence that there exists two distinct tangents at \((0, \bar{r}_L)\).

For the case \(\sigma(\bar{r}_L) = i\omega\) with \(\omega \neq 0\), we appeal to the well known Hopf theorem (cf. Marsden & McCracken, 1976) and note that the conditions (5.4.1)-(5.4.3) coincide with those leading to Hopf theorem. Thus, without further effort, we summarize the discussion for both \(\omega = 0\) and \(\omega \neq 0\):

**Theorem 5.3** (Asymptotic state and criticality) Let \(f(\phi; \psi (\alpha); \alpha)\) be given by (3.3.5), and satisfy the conditions (5.4.1)-(5.4.3) at \((\phi_0, \alpha) = (0, \bar{r}_L)\). Then, generically, for an equilibrium state \(\bar{\psi}(\alpha)\) of (3.2.3) for which \(\alpha\) is in a small neighborhood of \(\bar{r}_L\), its nonmodal disturbance \(\bar{\phi}(t; \phi_0)\) approaches an unique stationary state as \(t \to \infty\) if \(\sigma(\bar{r}_L) = 0\), or a periodic limiting state if \(\sigma(\bar{r}_L) = i\omega\) with \(\omega \neq 0\).

**Remark 5.5.1** The foregoing results yield no information on the direction in which the expected bifurcation branches off. A determination of the bifurcation direction requires, in general, higher order derivatives of the vector field \(f\) (cf. Guckenheimer & Holmes, 1983).
Remark 5.5.2 By definition (3.3.1), $\hat{\phi}(t; \hat{\phi}_0)$ is the difference between the perturbed flow and the original equilibrium state. It then follows that under the conditions of theorem 5.2 and theorem 5.3, supercritical bifurcation occurs at criticality $\alpha = \bar{r}_L$. More precisely, out of instability of a given equilibrium state $\hat{\psi}(\alpha)$ to an initial perturbation $\hat{\phi}_0$, a new steady motion $\hat{\psi}(\alpha) + \hat{\phi}(t; \hat{\phi}_0)$ of (3.2.3) emerges if $\bar{\sigma}(\bar{r}_L) = 0$, or a periodic flow bifurcates from $\hat{\psi}(\alpha)$ if $\bar{\sigma}(\bar{r}_L) = i\omega$ with $\omega \neq 0$.

Remark 5.5.3 Let the conditions (5.4.1)-(5.4.3) be satisfied. A necessary condition for subcritical bifurcation branching off at $\alpha = \bar{r}_L$ is violation of hyperbolicity for $\alpha$ in the subcritical side of some sufficiently small neighborhood of $\alpha = \bar{r}_L$. Note that the above results essentially solve the existence problem. Actually obtaining these limiting states of nonmodal disturbances requires, in general, further numerical bifurcation analysis (see Appendix 5E for details).

5.5 Numerical illustrations  

a) Primary equilibrium branch and its stability  
To illustrate the main results of this chapter, we consider planetary scale atmospheric flows over topography. In Fig 5.1, we show the modal and global stability analysis results for a set of such flows $\{\hat{\psi}(r)\}$ obtained as equilibrium states
of (3.2.3) (see the caption for model parameters). The top curve and the curve \( \text{abcd} \) are its linear and nonlinear stability measure \( r^N_\text{L} \) and \( r^L_\text{L} \), respectively. It is not difficult to see from (4.2.8)-(4.2.9) that \( r=r^\text{N}_\text{L} \) and \( r=r^\text{L}_\text{L} \) serve as the monotonic global stability boundary and the linear stability boundary of the set \( \{\hat{V}(r)\} \), respectively. The curve \( \text{efgb} \) is \( r^L_\text{L}(\hat{V}) \) for the set of equilibria \( \{\hat{V}(r)\} \) which bifurcates at the criticality \( r=r^\text{L}_\text{L} \) from the primary branch \( \{\hat{V}(r)\} \) and is obtained via a numerical bifurcation analysis (see Appendix 5E for details), with \( r=r^\text{L}_\text{L}' \) as its linear stability boundary. For illustration of the main results, we will focus on regimes (I) and (II). See Fig 3.3-3.6 for asymptotic decay to zero in the subcritical regime.

b) Steady limiting states and static bifurcation  Moving along the primary branch of equilibria \( \{\hat{V}(r)\} \) in the direction of decreasing \( r \), we cross the linear stability boundary \( r=r^L_\text{L}=1/(13.33 \text{ day}) \) and step into the supercritical regime. It is expected from theorem 5.2 that \( \hat{V}(r) \) with \( r \) in region (I) and (II) has persistent nonmodal disturbances \( \hat{\phi}(t;\hat{\phi}_0) \). Further, the nature of the nonvanishing limiting states of \( \hat{\phi}(t;\hat{\phi}_0) \) as \( t \to \infty \) is related to the properties of \( f^\hat{\phi} \) at \( (0,\hat{r}_\text{L}) \) when the underlying \( \hat{V}(r) \) has \( r \) close to \( \hat{r}_\text{L} \). The direct calculations indicate that the conditions listed in (5.4.1)-(5.4.3) are satisfied at the criticality \( \hat{r}_\text{L}=1/(13.33 \text{ day}) \).
day) with \( \omega = 0 \). Thus, the long term behavior of \( \hat{\phi}(t; \hat{\phi}_0) \) to \( \hat{\psi}(r) \) for \( r \) in the supercritical side of a small neighborhood of \( \bar{r}_L \) is expected to be stationary and unique (cf. theorem 5.3).

To verify these expectations, numerical experiments based on (3.3.2) are performed, with the results shown in Fig 5.2. The underlying equilibria are taken from the part \( (f' \rightarrow b) \) on the primary branch to ensure that the four \( \hat{\psi}(r) \) are close enough to the criticality \( \bar{r}_L \) (= 13.33 days). It is clearly seen that after the transient growth, the disturbances settle down to some steady states. As an example, we show in Fig 5.3 the streamfunctions for bifurcation of an equilibrium state Fig 5.3(a) into a new steady flow Fig 5.3(f). The snapshots (c)-(f) are taken from the experiment for \( 1/r = 17.1 \) days (cf. the thick solid line in Fig 5.2). Fig 5.4 indicates that the limiting state of \( \hat{\phi}(t; \hat{\phi}_0) \) is independent of initial conditions. This local uniqueness is the consequence of the fact that \( \{\text{Null } D\phi\}_{r=\bar{r}_L} = 2 \) (cf. (5.4.5)) in the present case. The experiment initialized from \( s' \) suggests that the region over which the uniqueness holds is fairly large.

**c) periodic states and Hopf bifurcation** We shall use the set \( \{\hat{\psi}'(r)\} \) bifurcating from the primary branch at \( r = \bar{r}_L = 1/(13.33 \text{ day}) \) for illustration of the second part of theorem 5.3. It
is seen from $r_L(\dot{\psi}')$ (cf. curve efg in Fig 5.1) that the exchange of stability for the set $\{\dot{\psi}'(r)\}$ occurs at $r=r'_L=1/(18.5 \text{ days})$. Moreover, a direct calculation reveals that $r=r'_L$ is a Hopf bifurcation point, i.e., the conditions (5.4.1)-(5.4.3) hold with $\omega \neq 0$. It is thus anticipated that nonmodal disturbances $\dot{\phi}(t;\phi_0)$ to those $\dot{\psi}'(r)$ with $r$ in the supercritical side of some neighborhood of $r=r'_L$ will tend to asymptotic periodic states. A set of numerical experiments confirms this anticipation. The results from two of these experiments are shown in Fig 5.5 and Fig 5.6, with the former displaying the time series of disturbance energy and the latter for the streamfunctions of bifurcation of a given steady flow shown in Fig 5.6 (a) (corresponding to the thick curve in Fig 5.4) into a periodic one shown in Fig 5.6 (b)-(f).

d) Repeated supercritical bifurcation In agreement with the forgoing numerical experiments, it is found from the bifurcation analysis (cf. Appendix 5E) that two stationary solution curves of (3.3.2) emanates from $(0,r'_L)$ supercritically. The one for steady limiting state of $\dot{\phi}(t;\phi_0)$ plus the underlying primary equilibria $\{(r)\}$ yields the stationary bifurcation branch as seen in Fig 5.7. A modal stability analysis of this branch $\{\psi'(r)\}$ indicates that when the difference between $\dot{\psi}'(r)$ and $\dot{\psi}(r)$ as measured by the amplitude of the corresponding nonmodal disturbance $\dot{\phi}(t;\phi_0)$
is small $\bar{v}'(r)$ is linearly stable but exchanges the stability when the bifurcating flows deviate further away from the original ones (cf. Fig 5.1), a result virtually true to all supercritical bifurcation (e.g., Joseph, 1976). Moreover, the criticality on the stationary bifurcation $\{\bar{v}'(r)\}$ is a Hopf bifurcation point from which a periodic branch of $\hat{\phi}(t;\hat{\phi}_0)$ bifurcates and is numerically obtained (cf. Appendix 5E). The latter plus the underlying $\{\bar{v}'(r)\}$ gives rise to the periodic branch $\{\bar{v}'(r)\}$ as shown in Fig 5.7. It is remarkable that each time out of instability of a given flow to persistent nonmodal disturbances, a new flow emerges with a higher level of energy norm, which manifests as an increase in the complexity level of temporal behavior (e.g., from steady to periodic) and spatial structure (e.g., from predominant zonal to wavy motion). The repeated supercritical bifurcation such as the one seen in Fig 5.7 has been a prominent topic in the subject of transition to turbulence since Landau’s conjecture (1944) on this subject (cf., Joseph, 1976). The present treatment of the repeated bifurcation could be carried out further by inquiring the stability of the periodic branch $\{\bar{v}'(r)\}$ via Floquet theory (cf. Arnol’d, 1983; Guckenheimer & Holmes, 1983), which is certainly necessary to study transition to turbulence or chaotic behavior in the deterministic system (3.2.3).
e) Transition due to modal vs. nonmodal initial perturbations

Given the increasing evidence for a plausible role of nonmodal initial perturbations in explosive development of disturbances observed in the atmosphere (e.g., Sanders, 1986) and in model studies (e.g., Farrell, 1985; Fig 3.7 and Fig 4.6 and 4.7 in this thesis), it is of interest to see the role played by the two types of perturbations in transition from one state of flow to another. Fig 5.8 displays the results from two experiments, one triggered by the optimal nonmodal perturbation $s_{\text{M}}$ and one by the fast growing normal model $\Re(\xi)$. It is seen clearly that the former is much more effective as a triggering agent for transition than the latter.

5.6 Concluding remarks

In this chapter, we have analyzed the asymptotic behavior of nonmodal disturbances $\phi(t;\phi_0)$ for a generic case where the perturbed state is hyperbolic using the direct method of Liapunov and the like. Specifically, we have shown that under certain conditions (cf. theorem 5.1) the transient growth associated with finite amplitude nonmodal disturbances will ultimately diminish as $t \to \infty$, a phenomenon which has long been known for some linear models (e.g., Pedlosky, 1964). We have also established that a transition from a given flow to another distinct state is inevitable when the flow under concern has growing modal perturbations (cf., theorem 5.2).
Further, we have demonstrated that the nature of asymptotic behavior of persistent disturbances is related to the nature of the neighboring nonhyperbolic point in some underlying parameter space. As a final note, the results here are local in the state space $\mathbb{R}^M$ or in the underlying parameter space.
Appendix 5A The direct method of Liapunov

The method is widely available in textbooks on dynamical systems (see, e.g., Hirsh and Smale, 1976; Verhulst, 1985, for more extensive treatment). To state the basic features of the subject, we consider the system

\[
\frac{d}{dt}x = g(x), \quad x(t_0) = x_0
\]  

(5A.1)

where \( g: D \to \mathbb{R}^m \) is a \( C^1 \) vector field on an open set \( D \subset \mathbb{R}^m \). Let \( x_e \) be an equilibrium of (5A.1). Let \( V: U \to \mathbb{R} \) be a differentiable function defined in a neighbourhood \( U \subset D \) of \( x_e \). An orbital derivative of function \( V \) along the solution \( x(t,x_0) \) of (5A.1), denoted by \( \dot{V} \), is defined by \( \dot{V} = Vg \), where \( V \) the gradient operator in \( \mathbb{R}^m \). \( V \) is said to be a Liapunov function if it satisfies:

(a) \( V(x_e) = 0 \) and \( V(x) > 0 \) in \( U - \{x_e\} \),

(5A.2)

(b) \( \dot{V} \leq 0 \) in \( U - \{x_e\} \).

(5A.3)

If \( V \) meets the condition (a) and

(c) \( \dot{V} < 0 \) in \( U - \{x_e\} \),

(5A.4)

it is called a strict Liapunov function.

Theorem (Liapunov's criterion for stability). An equilibrium state \( x_e \) of (5A.1) is stable if there exists a Liapunov function \( V(x) \) in a neighbourhood \( U \subset D \) of \( x_e \); It is asymptotically stable if there exists a strict Liapunov function \( V \) in \( U \).
Appendix 5B  Auxiliary lemmas

The following lemmas are developed to facilitate the proof of the main results in § 5.2 and § 5.3 and included here for convenience of reference. Some variants of lemma 5.1 and 5.2 may be found in Hale (1969) or in Hirsh and Smale (1974).

**Lemma 5.1** Let $f: \mathbb{R}^M \to \mathbb{R}$ be written as $f(x) = f_p(x) + w(x)$ such that (i) $f_p(x)$ is a negative (positive) definite homogeneous polynomial of degree $p$; (ii) $w(x) = o(\|x\|^p)$ as $\|x\| \to 0$. Then, $f(x)$ is negative (positive) definite in $U\setminus\{0\}$, $U$ a neighbourhood of $\phi = 0$.

**Proof:** (for negative definite case) By (i), we put

$$\max_{\|x\|=1} f_p(x) = -a < 0 \quad (5B.1)$$

and hence have, for $x \in \mathbb{R}^M \setminus \{0\}$

$$f_p(x) = \|x\|^p f_p(x/\|x\|^p) \leq -a \|x\|^p. \quad (\text{by (5B.1) and (i)})$$

Note that condition (ii) implies that for a given $\varepsilon > 0$, there exists a closed ball $\bar{B}(0,\delta)$ such that

$$|w(x)| \leq \varepsilon \|x\|^p \quad \text{for} \quad x \in \bar{B}(0,\delta). \quad (5B.2)$$

Now, setting $\varepsilon = (1/2)a$ in (5B.2) and then combining (5B.1) and (5B.2) yield

$$f(x) \leq -(a/2) \|x\|^p < 0, \text{ for } x \in \bar{B}(0,\delta) \setminus \{0\},$$

a desired result. Identifying $\bar{B}(0,\delta)$ with $U$ completes the proof. \[116\]
Lemma 5.2 Let $P, Q, R \in \mathcal{L}(\mathbb{R}^n)$. The Liapunov matrix equation

$$P^T Q + QP = R$$

(5B.3)

has a positive definite $Q$ as a solution for any negative definite $R$ if and only if $\text{Re}(\sigma(P)) < 0$.

Proof: We prove the converse of lemma 5.2, which is used subsequently. For any given negative definite $R$, we construct a positive definite $Q$ according to

$$Q = \int_0^\infty \exp(P^T t) (-R) \exp(P t) \, dt$$

(5B.4)

where $\exp(P t) = \sum_{i=1}^\infty (P t)^i/i!$ is the exponential of operator $P t$. The convergence of the integrant in (5B.4) follows from the fact that there exist positive constants $c_1$ and $\alpha_i$, $i=1,2$, such that

$$\|\exp(P t)\| \leq c_1 \exp(-\alpha_1 t),$$

(5B.5a)

$$\|\exp(P^T t)\| \leq c_2 \exp(-\alpha_2 t)$$

(5B.5b)

if $\text{Re}(\sigma(P)) < 0$. Thus, $Q$ is well defined. To see $Q$ satisfies (5B.3), consider

$$\left(\frac{d}{dt}\right) \{ \exp(P^T t) (-R) \exp(P t) \}$$

$$= P^T \exp(P^T t) (-R) \exp(P t) + \exp(P^T t) (-R) \exp(P t) P$$

(5B.6)

where the fact that $P \exp(P t) P^{-1} = \exp(P t)$ is used. Integrating (5B.6) from 0 to $\infty$, we obtain (5B.3) with the use of the estimate

$$0 \leq \| \exp(P^T t) (-R) \exp(P t) \|$$

$$\leq \| \exp(P^T t) \| \| (-R) \| \| \exp(P t) \|$$
\[ \|(-R)\|c_1\exp(-\alpha_1 t)c_2\exp(-\alpha_2 t) \to 0 \text{ as } t \to \infty. \text{ (by } (5B.5)\)\]

and the corresponding limit
\[ \exp(P^*t)(-R)\exp(Pt) \to 0 \text{ as } t \to \infty. \]

This establishes the converse of lemma 5.2.

**Lemma 5.3** The bilinear operator \( B \) defined by (3.3.4f) is bounded such that
\[
|B_{xy}| \leq c|x||y|, \text{ for any } x \text{ and } y \in \mathbb{R}^n, \tag{5B.7a}
\]
where \( c \) is given by
\[
c^2 = c_1^4 \sum_{i \neq j \neq k} \left( \max_{j \neq k} |B_{ijk}| \right)^2 \tag{5B.7b}
\]
with \( c_1 \) a positive constant.

**Proof:** First, by norm equivalence in \( \mathbb{R}^n \) (cf. Kreyszig, pp.96), there exists a constant \( c_1 > 0 \) that
\[
\|x\|_1 \leq c_1 \|x\|, \text{ for any } x \in \mathbb{R}^n \tag{5B.8}
\]
where norm \( \| \cdot \|_1 \) is defined as the sum of absolute values of all components of \( x \). Now, for any \( x \) and \( y \) in \( \mathbb{R}^n \), consider
\[
|B_{xy}|^2 = \sum_{i} \left\{ \sum_{j} \sum_{k} B_{ijk}x_j y_k \right\}^2 \\
\leq \sum_{i} \left\{ \sum_{j} \sum_{k} |B_{ijk}x_j y_k| \right\}^2 \\
= \sum_{i} \left( \max_{j \neq k} |B_{ijk}| \right)^2 \left( \sum_{j} \sum_{k} |x_j| |y_k| \right)^2 \\
= \sum_{i} \left( \max_{j \neq k} |B_{ijk}| \right)^2 \|x\|_1^2 \|y\|_1^2
\]
\[ sc^2 |x|^2 \|y\|^2 \] (by (5B.8))

which leads to (5B.7a) immediately, with c a constant given by (5B.7b).

**Appendix 5C** Real canonical theory of linear operator

We apply the real canonical theory of linear operators on \( \mathbb{R}^M \) (e.g. Hirsh and Smale, 1974) to find a new basis for \( \mathbb{R}^M \) w.r.t which the linear part \( A \) (cf. (3.3.4a) of the vector field \( f \) (3.3.3a) takes a specific form.

**Lemma 5.4** Let the equilibrium state \( \bar{y} \) of (3.2.3) satisfy (5.3.1). Then, \( \mathbb{R}^M \) has a basis \( \mathcal{B} \) such that

\[ (i) \mathbb{R}^M = \mathbb{R}^- \oplus \mathbb{R}^+, \quad \dim \mathbb{R}^- = M-k \text{ and } \dim \mathbb{R}^+ = k, \]

with \( k \) equal to the sum of algebraic multiplicity for eigenvalues with positive real parts;

\[ (ii) A \text{ in the basis } \mathcal{B} \text{ has a matrix of form } \]

\[ A_{\mathcal{B}} = \text{diag}[A_-, A_+] \]

where \( A_- \in \mathcal{L}(\mathbb{R}^-) \) and \( A_+ \in \mathcal{L}(\mathbb{R}^+) \), satisfying

\[ \text{Re}(\sigma(A_-)) < 0 \text{ and } \text{Re}(\sigma(A_+)) > 0, \]

respectively.

**Proof:** Let \( E_i(\sigma_i) \subset \mathbb{C}^M \) denote the generalized eigenspace of \( A \) belonging to \( \sigma_i(A) \), i.e., \( E_i = \text{Null}(A - \sigma_i(A)) \), where \( n_i \) is algebraic multiplicity of \( \sigma_i \). Put \( \sigma_i = \alpha_i + i\beta_i \). If \( \sigma_i(A) \) is real (i.e., \( \beta_i = 0 \)), by the S-N decomposition (cf. theorem 2, pp.129 in Hirsh and Smale, 1974), \( E_i \) has a basis \( \mathcal{B}_i \) giving
the restriction of \( A \) to \( E_i \) (i.e., \( A|E_i \)) a matrix in the real canonical form

\[
(A_i)_{\delta_1} = \text{diag } [A_i^{(1)}, A_i^{(2)}, \ldots, A_i^{(\mu_i)}]
\]

where each of diagonal blocks in (5C.4) has a form

\[
\begin{bmatrix}
\alpha_i \\
\vdots \\
\alpha_i
\end{bmatrix},
\]

with \( \mu_i \) the geometric multiplicity of \( \sigma_i \) (\( \equiv \text{dim Null}(A - \sigma_i I) \)). If \( \sigma_i(A) \) is complex with \( \beta_i > 0 \), \( \bar{\sigma}_i(A) \) is also an eigenvalue of \( A \) as \( A \in \mathcal{L}(\mathbb{R}^n) \). Moreover, taking the union of real parts and imaginary parts of the basis elements of \( E_i(\sigma_i) \) yields a basis \( \delta_i \) for \( E_{i_1} \equiv E_i(\sigma_i) \oplus E_i(\bar{\sigma}_i) \) which has a decomplexification \( E_{i_1} \equiv E_{i_1} \cap \mathbb{R}^n \) of dimension \( 2n_i \).

Again by the S-N decomposition, \( A|E_{i_1} \) under the basis \( \delta_i \) has a matrix of form (5C.4) but with each blocks characterized by the form

\[
\begin{bmatrix}
I_{2i} & D_i \\
I_{2i} & I_{2i}
\end{bmatrix}; \quad D_i = \begin{bmatrix} \alpha_i & -\beta_i \\ \beta_i & \alpha_i \end{bmatrix}; \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

Note that the diagonal blocks of form (5C.5) or (5C.6) in (5C.4) appear as many times as \( \mu_i \), whereas \( \sigma_i \) or \( D_i \) appears in (5C.4) as often as \( n_i \). It is clear that these bases \( \{\delta_i\} \) form a linearly independent set and hence the union of \( \delta_i \) provides a basis \( \delta \) for \( \mathbb{R}^n \)
\[ \mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \ldots \cup \mathcal{A}_p \cup \mathcal{A}_{p+1} \cup \ldots \cup \mathcal{A}_s. \]  

(5C.7)

The direct sum decomposition (5C.1) of \( \mathbb{R}^M \) follows immediately from subdividing the basis \( \mathcal{A} \) (5C.7) into two parts, one consisting of the bases \( \mathcal{A}_i \) \((i=1,2,\ldots,p)\) of \( \mathbb{E}_i(\sigma_i) \) with \( \text{Re}(\sigma_i(\mathcal{A})) < 0 \) and one composed of the remaining \( \mathcal{A}_i \) of \( \mathbb{E}_i(\sigma_i) \) with \( \text{Re}(\sigma_i(\mathcal{A})) > 0 \). Since \( \forall \mathbf{x} \in \mathbb{E}_i \) (or \( \mathbb{E}_i^R \)), it holds that

\[ (\mathcal{A}_i-\sigma_i(\mathcal{A})) \mathbf{x} = 0, \mathbb{E}_i \) (or \( \mathbb{E}_i^R \)) is thus invariant under \( \mathcal{A} \).

Hence, \( \mathcal{A} \) under the basis \( \mathcal{A} \) has a matrix representation of form

\[ \mathbf{A}_\mathcal{A} = \text{diag} \left\{ (\mathcal{A}_1)_\mathbb{E}_1, (\mathcal{A}_2)_\mathbb{E}_2, \ldots, (\mathcal{A}_s)_\mathbb{E}_s \right\} \]  

(5C.8)

which yields the asserted form (5C.2) after grouping the first \( p \) blocks in (5C.8) into \( \mathbf{A}_+ \) and the reminder into \( \mathbf{A}_- \).

The property (5C.3) of \( \mathbf{A}_+ \) and \( \mathbf{A}_- \) follows from the fact that the eigenvalues \( \sigma_i \) of \( \mathcal{A} \) in \( \mathbb{E}_i \) (or \( \mathbb{E}_i^R \)) are as the same as those of \( (\mathcal{A}_i)_\mathbb{E}_i \) by (5C.4)-(5C.6). This completes the proof.  

**Appendix 5D** Liapunov instability function

Existence of a function \( \mathcal{V}^+ \) satisfying the properties (a)-(c) is the key step to establish theorem 5.2. The construction of such \( \mathcal{V}^+ \) for the system (3.3.2) is motivated by the treatment in Hale (1969).

**Lemma 5.5** Let the equilibrium state \( \hat{\mathbf{x}} \) of (3.2.3) satisfy (5.3.1). Then, there exists a \( C^1 \) function \( \mathcal{V}^+: \mathbb{U}^+ \rightarrow \mathbb{R}, \mathcal{V}^+ \)
the neighborhoods of the null solution $\dot{\phi} = 0$ of (3.3.2), such that

(a) $V^+(\phi) = 0$ as $\phi \to 0$,

(b) $\dot{V}^+ > 0$ in $U^+ - \{0\}$,

(c) $V^+(\phi_n) > 0$, $\forall \{\phi_n\} \subset R^+$ with $\phi_n \to 0$ as $n \to \infty$,

where $R^+$ is determined in Lemma 5.4.

Proof: We prove the lemma by constructing a function $V^+$ which meets the conditions listed in (5D.1)-(5D.3). First, we note that matrices representing a given operator in different bases are similar. It then follows that there exists a nonsingular matrix $P \in \mathcal{L}(R^n)$ such that

$$P^{-1}AP = A_\& = \text{diag}[A_-,A_+]$$  \hspace{1cm} (5D.1)

where $A$ is the linear part of the vector field $f$ (cf., (3.3.3)-(3.3.5)) and $A_\&$ is its counterpart w.r.t. the basis & (cf., lemma 5.4). In fact, $P$ itself defines a map from $R^M$ with the basis & (cf. (5C.7)) to $R^m$ with the standard basis. Substituting

$$\dot{\phi} = P\hat{\phi}.$$  \hspace{1cm} (5D.2)

into (3.3.2), we have the governing equation written w.r.t the basis &

$$\frac{d}{dt}\hat{\phi} = \text{diag}[A_-,A_+]\hat{\phi} + BP\hat{\phi}$$  \hspace{1cm} (5D.3)

where $B$ is the same as before, given by (3.3.4f). Now, define the function $V^+$ by
\[ V^+ = \langle \tilde{\phi}, Q^+ \tilde{\phi} \rangle, \quad Q^+ = \text{diag} \{ -Q_1, Q_2 \} \]  
(5D.4)

where \( Q_1 \in \mathcal{L}(\mathbb{R}^{M-k}) \) and \( Q_2 \in \mathcal{L}(\mathbb{R}^k) \) are the solutions to

\[ A^+_1 Q_1 + Q_1 A_- = -I_{M-k}, \]  
(5D.5)

\[ (-A^+_2)Q_2 + Q_2 (-A_+) = -I_k, \]  
(5D.6)

\( I_{M-k} \) and \( I_k \) the unit operators of order \( M-k, k, \) respectively. The existence of \( Q_1 \) and \( Q_2 \) are guaranteed by lemma 5.2. The property (a) follows immediately from the definition (5D.4).

To see (c), we use the fact (5C.1) to write \( \tilde{\phi} \) in the form \( \tilde{\phi} = [u,v]^T, u \in \mathbb{R}^{-} \) and \( v \in \mathbb{R}^{+} \). Then, for any \( \{ \tilde{\phi}_n \} \subset \mathbb{R}^{+} \) with \( \tilde{\phi}_n \to 0 \) as \( n \to \infty \), (5D.4) yields

\[ V^+ (\tilde{\phi}_n) = \langle v_n, Q_2 v_n \rangle > 0. \]  
(5D.7)

The positiveness in (5D.7) results in from the same property of \( Q_2 \). We turn to (b). A direct evaluation of \( \dot{V}^+ \) along the orbit of \( \dot{\phi}(t; \tilde{\phi}_0) \) of (5D.3) gives,

\[ \dot{V}^+ = \langle \dot{\tilde{\phi}}, \text{diag}[-A^T_1 Q_1 - Q_2 A_-, A^T_2 Q_2 + Q_2 A_+ \tilde{\phi}] \rangle + o(\|\tilde{\phi}\|^2) 
= \|\dot{\tilde{\phi}}\|^2 + o(\|\tilde{\phi}\|^2). \]  
(5D.8)

where we have used an argument similar to lemma 5.3 to get the estimate of the high order terms in (5D.8). Applying lemma 5.1 results in (b), which completes the proof.

**Appendix 5E Algorithm for bifurcation analysis**

Here, we only briefly outline the numerical procedures for finding the stationary or periodic limiting states of
nonmodal disturbances governed by (3.3.2a) (see the references given below for a detailed account). For steady states, the following several steps are involved: 1) to locate the stationary bifurcation point; 2) to find the tangents on the bifurcating branches emanating from \( (0, \bar{r}_L) \) (which are the solutions to (5.4.7) and then use them as a predictor to obtain an approximate steady solution to (3.3.2a); and 3) to apply the Newton-Raphson algorithm (cf. Parker and Chua, 1989) as a corrector to improve it to get the true steady solutions of (3.3.2a); 4) to use the continuation algorithm (Keller, 1978) to trace out the remaining branch. For 1), \( \det (M) \) is introduced as a test function to locate the stationary bifurcation point, with \( M \) defined by

\[
M = \begin{bmatrix} f_\phi & f_\alpha \\ \frac{\partial f}{\partial \phi} & \frac{\partial f}{\partial \alpha} \end{bmatrix}
\] (5E.1)

where \( f \) is given by (3.3.3a). It can be shown that a sign change in \( \det (M) \) during the process of continuation w.r.t. parameter \( \alpha \) indicates that a bifurcation point is passed over (cf. Parker and Chua, 1989). An application of bisection algorithm to \( \det (M) \) yields the bifurcation point. For step 2, (5.4.7) is solved for the null space of \( Df \) using the single value decomposition (SVD) (cf. Nobel and Daniel, 1988). The basis elements of the null space of \( Df \) are then used as approximate tangents. More sophisticated branching switch techniques are available but increase the amount of
computation significantly (cf. Kubicek and Marek, 1983).

For periodic asymptotic states, we convert the initial value problem (3.3.2) into a two point boundary value problem

\[
\frac{d\dot{\phi}}{dT} = \left[ f\left(\dot{\phi}, \dot{\psi}, \alpha\right) \right]
\]

subject to the boundary conditions

\[
\begin{bmatrix}
    \dot{\phi}(0) - \dot{\phi}(T) \\
    f_k(\dot{\phi}(0), \dot{\psi}, \alpha)
\end{bmatrix} = 0
\]

where \( T \) is the period and the \( f_k(\dot{\phi}(0), \dot{\psi}, \alpha) = 0 \) is introduced into (5E.3) as a phase condition, with \( k \) arbitrarily fixed to between 1 and \( M \) (cf. Seydel, 1988). Other types of phase conditions are also possible (Seydel, 1988). The standard shooting method (cf. Ascher, et al, 1988) is used for solutions of (5E.2)-(5E.3).
Fig 5.1 Stability regime diagram for a family of equilibria \( \{ \tilde{\psi}(r) \} \) with \( 1/r \) ranging from 29.5 days to 3.5 days, \( U = 22.0 \) m/s and topography being of zonal wavenumber-1 and of height 500 m. The top curve and the curve abcd are its \( r_n(\tilde{\psi}) \) and \( r_L(\tilde{\psi}) \), respectively, with \( r=r_n=1/(5.71 \) days) and \( r=r_L=1/(13.33 \) days) as its MGS boundary and linear stability boundary. The curve efgb is \( r_L(\tilde{\psi}') \) for the set of equilibria \( \{ \tilde{\psi}'(r) \} \) bifurcating at criticality \( r=r_L' \) from the primary branch \( \{ \tilde{\psi}(r) \} \). \( r=r_L'=1/(18.5 \) days) is the linear stability boundary for the set \( \{ \tilde{\psi}'(r) \} \).
Fig 5.2 Asymptotic nonvanishing steady states of nonmodal disturbances. The underlying equilibrium states are those located on the part \((f \to b)\) of the primary branch (cf. Fig 5.1) for values of \(r\) given in the figure.
Fig 5.3 Streamfunctions for bifurcation of an equilibrium state (a) into a new steady flow (f). The snapshots (c)-(f) are from the experiment for $1/r=17.1$ days (cf. the thick solid line in Fig 5.2).
Fig 5.4 Local uniqueness of asymptotic steady state of nonmodal disturbances. The underlying equilibrium state is the same as one in Fig 5.2 for the experiment of $l/r=16.4$ days. The $s'_M$ is obtained from scaling $s_M$ such that $\phi(t;s'_M)$ at $t=0$ has 10% of the basic state energy.
Fig 5.5 Periodic limiting states of nonmodal disturbances, with periods 46.3 days for the solid line and 85.8 days for dashed line, respectively. The underlying equilibria are located on the unstable section of the stationary bifurcation branch (cf. Fig 5.1), with the values of $r$ as indicated.
Fig 5.6 Streamfunctions for bifurcation of an equilibrium state (a) into a periodic flow. The snapshots (b)-(f) are taken from the experiment for $l/r=24.8$ days (cf. the solid line in Fig 5.5) over a cycle of oscillation, with $t'=336.9$ days and $T=49.3$ days.
Fig 5.7 Repeated supercritical bifurcation for the primary branch of equilibria (cf. Fig 5.1). The point marked by $\times$ on the primary branch is a stationary bifurcation point whereas the symbol + indicates the Hopf bifurcation point. The lines drawn with dashed corresponds to unstable equilibrium states.
Fig 5.8 Nonmodal versus modal initial perturbations in
transition to a periodic state. The basic state is from the
stationary bifurcation branch ($\Psi'(r)$) with $1/r=18.8$ days
(cf. Fig 5.1), located near the secondary bifurcation point
$r=r'$. The initial growth rate of $\delta(t; S_M)$ and $\delta(t; \text{Re}(\xi))$ are
$1/(5.08 \text{ days})$ and $1/(400.0 \text{ days})$, respectively.
6.1 Introduction

The statistical equilibrium achieved by quasi-geostrophic flows over a random topography has been investigated by many authors. Salmon, Holloway and Hendershott (1976; hereafter referred to as SHH) obtained the inviscid-unforced statistical equilibrium (also referred to as the absolute equilibrium) for one- and two-layer flows using the method of classical statistical mechanics. They found that at absolute equilibrium, the lower-layer flow is anti-cyclonic circulation around hills. Furthermore, for scales larger than the internal deformation radius, the flow correlation with topography was shown to extend throughout the depth of fluid.

While the absolute equilibrium provides valuable insights into the dynamics of a problem, omission of forcing and dissipation leads to the results which has little to do with viscous flows in many aspects of dynamics. To cope with these aspects, Herring (1977) and Holloway (1978) developed statistical theories for forced-dissipated flows on the $f$-plane using the direct interaction approximation (Kraichnan, 1967) and the test field model (TFM) (Kraichnan, 1971). Recently, some effort has been made to extend the
f-plane formulation to the β-plane case with a nonvanishing domain averaged zonal velocity component (Holloway, 1987; hereafter referred to H87)

The objective of this work is to study the statistical equilibrium established when a flow is forced over a random topography by an external uniform zonal momentum source. The study is based on a closure model (H87) as well as on direct numerical simulation (dns). One aspect of the equilibria is singled out for consideration, i.e., the vorticity-topography correlation and the resulting topographic stress acting on the equilibrium flows.

In § 6.2, the closure formulation is outlined and two invariants of motion for the physical system under concern are briefly discussed. In § 6.3, we solve numerically the closure equations for parameters relevant to midocean environment, with results compared to those from ensemble dns. A brief summary and discussion of the results are presented in § 6.4.

6.2 Closure formulation

6.2.1 A self-consistent model

Consider a flow governed by (1.2.1), forced by an external zonal momentum source (i.e., $\psi^*=-U^*y$ in (1.2.1)) and bounded by a double periodic cell (i.e., satisfying B.C.(1.2.2b)).
Further, we write the streamfunction as (cf. H87)
\[ \psi(x,y,t) = -U(t)y + \Phi(x,y,t), \]  
(6.2.1)
then a self-consistent system is obtained (cf. H87)
\[ \delta_t \nabla^2 \Phi + J(\Phi - U y, \nabla^2 \Phi + \beta y + h) = -D \nabla^2 \Phi, \]  
(6.2.2)
\[ \frac{d}{dt} U = r(U^* - U) + \frac{\overline{h}\Phi}{\overline{\partial x}}, \]  
(6.2.3)
\[ \Phi(x+1,y,t) = \Phi(x,y,t), \Phi(x,y+1,t) = \Phi(x,y,t), \]  
(6.2.4)
where \((h/H) f_0\) in our earlier notation (cf. (1.2.1)) is simply written as \(h\) to be more consistent with the notation commonly used in closure formalism, and \(D\) in (6.2.2) denotes the dissipation operator including the Ekman and biharmonic dissipations. The tendency equation (6.2.3) may be obtained from considering the \(x\)-directed momentum equation (e.g., Hart, 1979). The term in (6.2.3) with overbar is referred to as the topographic stress \(\tau\) arising from the flow-topography interaction, where the overbar denotes the domain average.

In the next section, the problem defined by (6.2.2)-(6.2.4) is projected onto a Fourier space, giving a set of coupled nonlinear ordinary differential equations for which the corresponding statistical problem is formulated via moment equations.

### 6.2.2 Moment equations

By virtue of (6.2.4), we choose the set \(G = \{ e^{ik.x} \}\) as the basis functions, where \(x = (x,y)\) and \(k \in K\) defined by
\[ k = \{ k = (m,n) \frac{2\pi}{L} \mid (m,n) = 0, \pm 1, \pm 2, \ldots, 1 \leq |k| \leq k_{\text{max}} \}, \]

and expand \( \phi, h \) in the form

\[ (\psi, h) = \sum_{k} (\phi_{k}, h_{k}) e^{ik \cdot x}, \quad (6.2.5) \]

where the summation is over all \( k \in \mathbb{K} \). After introducing (6.2.5) into (6.2.2) and (6.2.3), and using the orthonormal property of set \( \mathbb{G} \), we obtain the spectral version of (6.2.2)-(6.2.3), in terms of \( \zeta_{k} = -k^{2}\phi_{k} \)

\[ \frac{d}{dt} \zeta_{k} = -ik \cdot (U \cdot \zeta_{k}) - (i\omega_{k} + \nu_{k}) \zeta_{k} - \sum_{\Delta_{pq}} A_{kpq}(\zeta_{p}^{*} \zeta_{q} + \zeta_{p} \zeta_{q}^{*}), \quad (6.2.6) \]

\[ \frac{d}{dt} U = r(U^{*} - U) - \text{Im} \left\{ \sum_{k} k \cdot \zeta_{k}^{*} h_{k}/k^{2} \right\}, \quad (6.2.7) \]

\[ \omega_{k} = (U - \beta/k^{2}) k, \quad (6.2.8) \]

\[ A_{kpq} = \hat{z} \cdot (k \times p)/p^{2} = \hat{z} \cdot (p \times q)/p^{2} = \hat{z} \cdot (q \times k)/p^{2}, \quad (6.2.9) \]

where \( (\cdot)^{*} \) denotes the complex conjugate of quantity \( (\cdot) \); the symbol \( \sum \) represents the summation over wavevectors \( p, q \) such that \( k + p + q = 0 \); \( \omega_{k} \) is the linear Rossby wave frequency for mode \( k \) and \( A_{kpq} \) is the interaction coefficient for the triad \( (k,p,q) \); \( \nu_{k} \) is the dissipation operator in the spectra space; finally, \( \hat{z} \) is the unit vector along the z-axis.

Consider a phase space \( \Gamma \) spanned by the real and imaginary part of all \( \zeta_{k} \) and \( h_{k} \) plus \( U \) (i.e., \( \text{Re}(\zeta_{k}), \text{Im}(\zeta_{k}), \text{Re}(h_{k}), \text{Im}(h_{k}), U \)), where the symbols \( \text{Im}, \text{Re} \) denote the imaginary and real part of the quantity). Then,
one single realization of (6.2.6) and (6.2.7) yields a trajectory in $\Gamma$, and an ensemble of such realizations corresponds to the evolution of a cluster of phase points $\{\text{Re}(\zeta_k), \text{Im}(\zeta_k), \text{Re}(h_k), \text{Im}(h_k), U\}$ in $\Gamma$. A complete statistical description of the ensemble requires the knowledge of probability distribution function $P(\text{Re}(\zeta_k), \text{Im}(\zeta_k), \text{Re}(h_k), \text{Im}(h_k), U, t)$ which exists in principle as the solution to the Liouville equation implied by (6.2.6) and (6.2.7), or equivalently, requires the joint moments of all orders between the phase points $\{\text{Re}(\zeta_k), \text{Im}(\zeta_k), \text{Re}(h_k), \text{Im}(h_k), U\}$.

However, the determination of $P$ can rarely be carried out in the presence of external forcing $U^*$ and dissipation $D$, while it is possible to seek time independent $P$ as a functional of the invariants of (6.2.6) and (6.2.7) for the case where $U^* = D = 0$ (e.g., SHH; Carnevale & Frederiksen, 1987). In this study, we restrict ourself to those states of the ensemble for which the ensemble statistics is time-independent. We refer to those states as forced-dissipative statistical equilibrium as opposed to the absolute equilibrium. Furthermore, we are only interested in two second order moments of the equilibria, i.e., $Z_k = \langle \zeta_k \zeta_k^* \rangle$ and $C_k = \langle \zeta_k h_k \rangle$, where the angle bracket denotes the ensemble average. Physically, the first represents the modal spectral for the vorticity and the second for the modal spectral of the vorticity-topography correlation.
In what follows, we outline the closure procedure at the level $Z_k$ and $C_k$. From (6.2.6), we have

$$
(d/dt)Z_k = 2k U \text{Im}(C_k) - 2v_k Z_k
$$

$$
- \sum_{\Delta} 2A_{kpq} \left\{ \text{Re}\langle \zeta_p \zeta_q \zeta_k \rangle + \text{Re}\langle \zeta_p h_k \zeta_k \rangle \right\},
$$

(6.2.10)

$$
(d/dt)C_k = ik U H_k + (i\omega_k - \nu_k) C_k
$$

$$
- \sum_{\Delta} A_{kpq} \left\{ \langle \zeta_p \zeta_q h_k \rangle + \langle \zeta_p h_k h_k \rangle \right\},
$$

(6.2.11)

where $H_k = \langle h_k h_k \rangle$. It is clear that the solutions of (6.2.10) and (6.2.11) require the knowledge of the third order moments of the forms $\langle \zeta \zeta \zeta \rangle$, $\langle \zeta \zeta h \rangle$ and $\langle h \zeta h \rangle$ which are in turn depend on the fourth order moments, and so on, leading to a unclosed hierarchy of moment equations. At this stage, some closure hypothesis on the relations among different order moments are necessary in order to close the hierarchy at the level of $Z_k$ and $C_k$.

6.2.3 Closure hypothesis and master equations

For this purpose, we first assume that the ensemble statistics of topography $\{h_k\}$ is Gaussian. The direct consequence of this assumption is the vanishing of odd moments of the form $\langle hhh \rangle$. Next, we assume: (i) the role of the fourth order cumulants, introduced in expressing fourth order moments in terms of products of the second order
moments $Z_k$ and $H_k$, is to provide damping action leading to the saturation of third order moments; (ii) the characteristic relaxation time for third order moments is shorter than those for the second order moments (i.e., $Z_k$).

Viewed simply, the hypothesis (i) permits replacement of the fourth order cumulants in the third order moment equations by a linear damping term with characteristic eddy-damping rate $\mu_{kpq}$ for the triad $(k,p,q)$. The hypothesis (ii) allows us to obtain the quasi-stationary triple correlations from the third order moment equations

$$<C_{C,C_k}> = -\theta_{kpq}^{(1)} \left\{ \hat{A}_{pqk} \frac{Z^2}{k} + \hat{A}_{qpk} \frac{Z^2}{p} + \hat{A}_{kpq} \frac{Z^2}{q} ight\},$$

$$<C_{C_k,C_p}> = -\theta_{kpq}^{(2)} \left\{ \hat{A}_{pqk} \frac{Z^2}{k} + \hat{A}_{qpk} \frac{Z^2}{p} + \hat{A}_{kpq} \frac{Z^2}{q} - \hat{A}_{qpk} \frac{Z^2}{k} - \hat{A}_{kpq} \frac{Z^2}{p} - \hat{A}_{qpk} \frac{Z^2}{q} \right\},$$

$$<C_{C_k,C_p,C_q}> = -\theta_{kpq}^{(3)} \left\{ \hat{A}_{pqk} \frac{C}{k} + \hat{A}_{qpk} \frac{C}{p} + \hat{A}_{kpq} \frac{C}{q} - \hat{A}_{qpk} \frac{C}{k} - \hat{A}_{kpq} \frac{C}{p} - \hat{A}_{qpk} \frac{C}{q} \right\},$$

where $\theta_{kpq}$ are characteristic of relaxation (towards...
quasi-equilibrium) by the nonlinear transfer and external
dissipation. Specification of the three arrays $\theta_{kpq}^{(i)}$ (i=1,2,3)
are subject to various constraints resulted from the
consideration of conservation requirements on the closure
model and the statistical realizability (i.e., $Z_k \geq 0 \forall k \in K$
and $t \geq 0$) (cf., H78), among which are the following

(i) $\theta_{kpq}^{(i)} = \theta_{kpq} = 0$, for $i = 1, 2, 3$ and $k,p,q \in K$, (6.2.13a)
(ii) $\theta_{kpq}$ is invariant to permutation to $k,p,q$, (6.2.13b)

where $\theta_{kpq}$ is a real array to be specified below.

The actual elements of $\theta_{kpq}$ may be obtained via various
theories (e.g., TFM and EDQNM (cf. Lesieur, 1987)). In TFM,
$\theta_{kpq}$ is calculated from an auxiliary problem in which $\theta_{kpq}$ is
related to the rate at which advection of a test field due to
the turbulence field $\zeta$ introduces the exchange between the
solenoidal and compressive parts of the test field
(Kraichnan, 1971). The case of 2-D turbulence with Rossby
waves but without topography has been examined by Holloway
and Hendershott (1977; hereafter referred to as HH77).
Generalizing the result in HH77 to the present study, we have

$$\theta_{kpq} = \mu_{kpq} / (\omega_{kpq}^2 + \mu_{kpq}^2),$$
(6.2.13c)

$$\mu_{kpq} = |k|^2 + |q|^2 + \mu_{kqp},$$
(6.2.13d)

$$\omega_{kpq} = \omega_k + \omega_P + \omega_q,$$
(6.2.13e)
where \( g \) is a phenomenological constant of order unity. With (6.2.12)-(6.2.13), the moment equations (6.2.10)-(6.2.11) and zonal flow tendency equation (6.2.7) are closed at the level \( Z_k \) and \( C_k' \), leading to

\[
\begin{align*}
\frac{d}{dt} Z_k &= 2k_x U_k - 2v_k Z_k + T_{1,k} + T_{2,k} \\
\frac{d}{dt} C_k &= i k_x U_k + (\mathbf{\omega}_k - \nu_k) C_k + S_{2,k} + S_{3,k} \\
\frac{d}{dt} U &= r(U - U) - \sum_{k} \frac{k_x I_k'}{k^2},
\end{align*}
\] (6.2.14) (6.2.15) (6.2.16)

where \( I_k \) is the imaginary part of \( C_k \), and \( T_{1,k} \) and \( T_{2,k} \) (\( i = 1,2 \)) are the enstrophy transfer and the correlation production, respectively, with the subscripts corresponding to the three distinct third order moments of form \( \langle \zeta \zeta \zeta \rangle \), \( \langle \zeta \zeta \rangle \) and \( \langle \zeta h \rangle \) in (6.2.10) and (6.2.11). The detailed expressions for \( T \) and \( S \) in terms of \( Z_k \) and \( C_k \) are obtained using (6.2.12) and listed in the Appendix 6A.

In the absence of nonconservative effects such as Ekman damping \( r \), the continuous system (6.2.1)-(6.2.4) has two invariants of the motion, i.e., the total kinetic energy and total potential enstrophy.
\[ E(\psi) = \frac{1}{1/L} \int_{\Omega} (1/2) |\nabla \psi|^2 d\Omega \]
\[ = \frac{1}{1/L} \int_{\Omega} (1/2) |\nabla \psi|^2 d\Omega + (1/2) U^2, \]  
\[ \text{(6.2.17)} \]

\[ Q(\psi) = \frac{1}{1/L} \int_{\Omega} (1/2) (\nabla^2 \psi + h)^2 d\Omega + \beta U, \]  
\[ \text{(6.2.18)} \]

where \( \psi \) is given by (6.2.1) and \( \Omega \) is the double periodic cell of length 1. The existence of the two invariants can be readily obtained from (6.2.2) and (6.2.3) by applying the divergence theorem and the boundary condition (6.2.4) (cf., Carnevale and Frederiksen, 1987). For the truncated system (6.2.6) and (6.2.7), it can be shown that the system has the truncated version of (6.2.17) and (6.2.18), denoted by \( E^{(N)} \) and \( Q^{(N)} \), as its invariants. Moreover, the closure model (6.2.13)-(6.2.16) conserves \( E^{(N)} \) and \( Q^{(N)} \) in the statistical sense, i.e., it conserves \( \langle E \rangle \) and \( \langle Q \rangle \) (see Appendix 6B for proof)

\[ \langle E \rangle \equiv \langle E^{(N)} \rangle = \sum_{k} (1/2) Z_k/k^2 + (1/2) U^2, \]  
\[ \text{(6.2.19)} \]

\[ \langle Q \rangle \equiv \langle Q^{(N)} \rangle = \sum_{k} (1/2) (Z_k + 2R_k) + \beta U, \]  
\[ \text{(6.2.20)} \]

It is noted in establishing (6.2.19) and (6.2.20) that the invariants, \( \langle E \rangle \) and \( \langle Q \rangle \), result from some delicate cancellation of terms in the enstrophy transfer (i.e., \( T_{1,k}, T_{2,k} \)) and the correlation production (i.e., \( S_{2,k}, S_{3,k} \)) (cf. Appendix 6A). It is thus expected that neglecting some terms in the transfer functions while
retaining others may result in the loss of $\langle E \rangle$ and $\langle Q \rangle$, as is found in H87.

6.3 Numerical results and comparison with DNS

With these conservation properties aside, we proceed to use the closure model to study the forced-dissipative statistical equilibrium. For this objective, it suffices to solve the set (6.2.14)-(6.2.16) for its stationary solutions, though the set is appropriate to the study of quasi-stationary statistics of the ensemble realizations of (6.2.6)-(6.2.7). We start with a brief description of the solution method and the physical parameters, followed by a presentation of numerical results.

6.3.1 solution method

At the forced-dissipative statistical equilibrium, the time tendency for $Z_k$, $C_k$ and $U$ in (6.2.14)-(6.2.16) vanishes, thus leaving (6.2.14)-(6.2.16) as a set of coupled nonlinear algebraic equations for the modal spectra $Z_k$, $C_k$ and for $U$. The task of finding the stationary values for $Z_k$, $C_k$ and $U$ is then equivalent to the one of finding the roots to that set of equations. There exist at least two ways to accomplish this task: one in which (6.2.14)-(6.2.16) are simultaneously solved for $\{Z_k, C_k, U\}$ with a given set of $U^*$ and $r$; and one in which $Z_k$ and $C_k$ are first obtained by solving
(6.2.14)-(6.2.15) for some prescribed values of $U$ and $r$, with the value of $U^*$ necessary to achieve such stationary $Z_k(C_k$ and $U$ then found from (6.2.16). In this study we adopt the second approach mainly because of the consideration that the physical basis for assigning values to $U^*$ is unclear whereas the range of mean velocity for geophysical relevant flows is better documented in the literature (for oceans, e.g., Crease, 1962; Swallow, 1971). The actual numerical scheme used here is the one used by Bartello and Holloway (1991) (hereafter referred to as BH) in their study of passive scalar transport in $\beta$-plane turbulence. The algorithm is iterative, with the dissipation terms treated pseudo-analytically (for details see the Appendix in BH).

Based on the same consideration, the direct numerical simulation of (6.2.6) is carried out with the uniform zonal flow $U$ set to some prescribed value. As in the closure case, the external momentum forcing $U^*$ necessary for maintaining the forced-dissipative equilibrium may be found from (6.2.7). The dns is done in a spectral domain truncated isotropically at $k_{\text{max}}$ with the interaction terms among triads $(k,p,q)$ in (6.2.6) calculated by the dealiased pseudo spectral method (Orszag, 1971), the dissipation terms evaluated analytically, and the time derivative approximated by the leapfrog scheme with Robert filter (for details see Ramsden, et al, 1985).
6.3.2 Model parameters

All the numerical results reported below are expressed in terms of the model units

\[ U_0 = 0.05 \text{m/s}; \quad L_0 = 1/2\pi; \quad T_0 = 10^6 \text{s} \]

where \( l \), the length of the periodic cell, is set to 320 km. The choice of the model units, to some extent, is arbitrary, and is made mainly for the convenience of presenting numerical results.

The model parameters (e.g., \( \beta \), \( r \) and \( h_{\text{rms}} \)) are set to those representative of the midlatitude deep ocean environment. In particular, we set \( \beta = 1.6 \times 10^{-11} \text{m}^{-1} \text{s}^{-1} \), a typical value for planetary \( \beta \) at midlatitude. We take \( v_k \approx r + v_4 k^4 \), where \( v_4 \) is introduced to remove the enstrophy piled up near \( k_{\text{max}} \) due to finite truncation. For \( k_{\text{max}} = 15 \), we set \( v_4 = 6 \times 10^{-5} \) to have proper damping at higher \( k \). As for \( r \), we take \( r = 0.12 \) (or \( 1/(100 \text{ days}) \)), a value corresponding to weak damping for most of our numerical computations.

The topography \( \{h_k\} \) for dns and \( \{H_k\} \) for the closure model are generated according to the isotropic topography variance spectra (H87)

\[ H(k) = h_0 / (3. + k)^{2.5} \]

with the phase of \( \{h_k\} \) generated randomly satisfying uniform distribution over \((0, 2\pi)\). We chose \( h_0 \) in (6.3.1) such that
\( h_{\text{rms}} \) is equal to 4.0 (or \( 4.0 \times 10^{-6} \) /s) corresponding to 200m bumps in a 5000m deep ocean. Such roughness of our "ocean floors" may be considered as representative of real ocean floors. As for \( U \), we consider the equilibrium for which \( U \) varies from 0 to 5 (or from 0 to 0.25m/s), a range around the typical mean velocity for flows below the main thermocline: 0.05 - 0.1 m/s (Crease, 1962).

To obtain a sense of how the numerical results depend on the model parameters, we consider cases where the Ekman damping coefficient \( r \) varies from 0.12 (1/(100 days)) to 1.16 (1/(10 days)), with \( h_{\text{max}} = 2.0 \) and \( \beta = 0.8 \).

A test of convergence of the solution to the closure model (6.2.15)-(6.2.16) with respect to \( k_{\text{max}} \) is made for \( k_{\text{max}} = 5, 15 \) and 30. The degrees of freedom retained in these truncations are 96, 748 and 2932, respectively. The test results show that relative improvement from \( k_{\text{max}} = 15 \) to \( k_{\text{max}} = 30 \) is insignificant for the range of \( U \) and the parameters mentioned above. For example, at \( U = 0.25 \), the relative change of topographic stress is less than 1 % but the cpu time increases an order of magnitude. Thus, the truncation \( k_{\text{max}} = 15 \) is retained for the subsequent calculations.

6.3.3 Numerical results

In the following presentation of our numerical results, we
first make the observations based on the closure results and then make remarks on the degree to which these observations agree with dns ensemble data. The model parameters which are used both in dns and in closure calculations described in (a) and (b) are set to

\[ r = 0.12, \quad h_{\text{rms}} = 4.0 \quad \text{and} \quad \beta = 0.8 \]

with the values for \( U \) specified as needed.

(a) Topographic stress as a function of \( U \) and \( r \)

The first set of calculations involved a family of forced-dissipative statistical equilibria for which \( U \) varies from 0.0 or 5.0. The stationary moments \( Z_k \) and \( C_k \) of these equilibria are obtained from (6.2.14)-(6.2.15) using the method described above. The topographic stress \( \tau \) acting on the equilibrium flows is then calculated according to the second term on the right hand side of (6.2.16) using the stationary \( C_k \), with the results shown in Fig 6.1 in a solid line. It is seen that \( \tau \) as a function of \( U \) first rapidly increases when \( U \) moves away from zero until the resonant \( U_r \) is reached (for the parameters used, \( U_r = 1.0 \)). Then, \( \tau \) monotonically deceases as \( U \) approaches large values.

The resonant behavior may be anticipated from the modal spectral of the topographic stress \( \tau \) (see (16)-(18) in H87 for the complete form)
\( \tau_k \propto H_k / (\omega_k^2 + \nu_k^2) \) \tag{6.3.2}

with \( \nu_k^2 \) a term which may not concern us for the present discussion. First recall that \( \omega_k = (U-\beta/k^2)k \). It is thus expected that \( \tau_k \) reaches a maximum for \((U,k)\) such that \( U - \beta/k^2 = 0 \). Next, note that for the topographic stress spectrum \((6.3.1)\) and the parameters used here, the dominant contribution to \( \tau \) is found from those \( \tau_k \) with \(|k| (=k) = 1 \) (cf. Fig 6.3 (b) and Fig 6.5 (b)). It then follows from \((6.3.2)\) and \( \beta = 0.8 \) that \( \tau \) reaches its maximum around \( U = 1.0 \), in agreement with the numerical observation \( U_r = 1.0 \).

For the identical parameters, five ensemble dns runs are performed for some representative values of \( U = 0.25, 1.0, 1.75, 2.75 \) and \( 3.75 \), each of which consists of 5 experiments corresponding to five random realizations of \( \{h_k\} \). The quantities of interest are then calculated in each experiment as time-averages over statistical stationary period. The data points for the topographic stress in Fig 6.1 are obtained from evaluation of the second term on RHS of \((6.2.7)\).

It is seen in Fig 6.1 that the agreement between the theoretical values and dns data is better for those equilibria with \( U \) away from the resonant \( U_r (=1.0) \). It is noted that over the range of \( U \) where the considerable discrepancy occurs, dns data exhibit noticeable variations.
from one realization of (6.2.6) to another, which implies that $\tau$ is sensitive to the details of arrangement of these little bumps on the "ocean" floors.

Given that the parameters used in this study are representative of midlatitude deep ocean environment, and given that the present status of ocean currents is at statistical equilibrium, it then follows from Fig 6.1 that the current with typical mean $U = 0.25$ (or 1.25 cm/s) is subject to $|\tau| = 0.233$ (or 0.06 N/m$^2$), a value comparable with the mean wind stress of order $O(10^{-1} \text{N/m}^2)$, where the ocean depth $H_o = 5000$ m and seawater density $\rho = 10^3 \text{kg/m}^3$ are assumed. Even for currents with $U$ away from the resonant region, our study indicates that $\tau$ is not a negligible factor in the momentum budget.

From the analysis of the stationary $Z_k$ and $C_k$ and the ensemble dns for $U \in (0., 5.0)$, it is noted that the statistical equilibria for which $U < U_r$ share much of characters of those with $U$ near $U_r$. This may be anticipated from Fig 6.1 since the subresonant region is very narrow and also close to the $U_r$. Moreover, these equilibria differ in some aspects of spectral behavior from those with $U$ located in the superresonant region ($U > U_r$) but away from $U_r$. It thus appears to be instructive to consider two representative cases, separately.

150
(b) Two representative cases

(i) Case 1: subresonant flow \((U = 0.25)\)

First, we present Fig 6.2 (a) to show a representative subresonant flow in physical space before discussion in spectral domain. The flow field shown in Fig 6.2 (a) is obtained from one of the random realization of (6.3.1). It is seen in (a) that at \(t = 20.0\) (or at \(t = 230\) days) which is well into the statistical stationary period, the flow has developed a strong meridional circulation on the west side of the domain but a concentrated zonal jet in the central part with pronounced wavy motions on the scale \(k < 2\) located to its south and a large stagnant region to its north.

Three spectra of the equilibria are singled out for discussion. They are enstrophy variance \(Z(k)\), topographic stress \(\tau(k)\) and vorticity-topography correlation \(R(k)\), shown in Fig 6.3 in (a), (b) and (c), respectively. Here, \(Z(k)\), \(\tau(k)\) and \(R(k)\) are band-averaged spectra obtained from corresponding modal spectra: \(Z_k, \tau_k = -k_x I_k/k^2\) and \(R_k = \text{Re} \{C_k\}\) according to

\[
\langle (\kappa) \rangle (k) = (2\pi k/N(k)) \sum_{|k| - 1/2 \leq |\kappa| \leq k + 1/2} \langle \kappa \rangle_k
\]

where the summation is taken over all the modes \(\kappa\) contained within the band defined by \(k-1/2 \leq |\kappa| \leq k + 1/2\), with \(N(k)\) being
the number of modes in the $k$th band. The corresponding dns spectra represented by the symbols are obtained in a 5-dns ensemble in the same manner according to the spectral versions of $\overline{\zeta \zeta}, \overline{\delta \Phi / \delta x}$ and $\overline{\zeta}$, where the overbar denotes the domain average as before.

It is seen in Fig 6.3 (a) that $Z(k)$ has a broad peak around $k = 3$, which may account for the dns observation (e.g., Fig 6.2(a)) that the dominant eddy activity in subresonant flows is on the scale $k = 2$. It is also noted from the solutions to (6.2.14)-(6.2.15) for $U \leq U_r$ that the peak will shift to some higher wave number and become sharper when $U$ moves away from $U_r$ towards $U = 0$, which is characteristic of the equilibria in the subresonant regime. As for spectral decomposition of the topographic stress, it is noted in (b) that $\tau$ will mainly be felt by the flow on the $k \leq 3$. Practically, the contributions to $\tau$ from higher wavenumbers are negligible. The correlation $R(k)$ is seen to have a broad peak around $k = 2$ and 3 and diminish as $k$ approaches $k_{\max}$. Note that the V-shaped correlation spectra characterizes all the subresonant flows.

The above observations are seen to be confirmed by the dns data in Fig 6.3, at least qualitatively, though the extent to which the two calculations agree with each other varies from one quantity to another and from one scale to
another. Comparison of the solid curves with data points indicates that the closure predictions tend to overestimate at low wave numbers, and that the discrepancy becomes smaller as $k$ approaches $k_{\text{max}}$ for $\tau(k)$ and $R(k)$ but persists for $Z(k)$. It is important to note that the discrepancy varies from one realization to another, as seen in Fig 6.3. To assess the statistical significance of the comparison, statistical analysis of the ensemble data at individual $k$ is performed, with results indicating that the worst discrepancy occurs around the $k$ where the standard deviation of dns data reaches its maximum. For example, it is noted from the analysis for $R(k)$ that the standard deviation take its maximum at $k = 3$ whereas the difference between closure and dns reaches maximum at $k = 2$.

In relation to previous work, our results for $\tau$ appear to support the observation (Treguier, 1989) that topographic stress is dominated by its component at the largest available scale, though for subresonant flows the present study suggests that $\tau$ is also felt by the motions on the scales slightly smaller than the largest one (e.g., Fig 6.3).

(ii) Case 2: superesonant flows ($U = 2.75$)

As in the previous case, we first present a representative superesonant flow in the physical domain. Fig 6.2 (b) displays its streamfunction, with its vorticity in Fig 6.4(b)
and the topography randomly realized in this experiment in Fig 6.4(a). As seen in Fig 6.2(b), the superresonant flow exhibits eddy activity on the largest resolved scale \( k=1 \) in contrast to the previous case where noticeable eddy scale is around \( k = 2 \) or even 3 (cf, Fig 6.3(a)). Indeed, it is found from the dns ensemble mentioned in (a) that with a further increase in \( U \), the streamfunction fields at the statistical equilibrium are characterized by a strong uniform zonal component \(-Uy\) and a relative weak eddy motion \( \phi \). As for the vorticity field (cf.Fig 6.4 (a) and (b)), the mirror image relation between \( h \) and \( \zeta \) in this case is most clearly seen at the largest resolved topographic scale.

Fig 6.5 shows the spectra \( Z(k) \), \( \tau(k) \) and \( R(k) \) for this case. Instead of going to make detailed comments on the spectral decomposition, we simply point out the difference between the current case and the previous case. First, \( Z(k) \) is seen to peak at \( k = 1 \) (cf, (a)), which is consistent with the dns observation based on Fig 6.2(b). Second, the contribution to \( \tau \) essentially comes from one single scale, that is, \( k = 1 \), in a sharp contrast to the near resonant case where \( \tau \) acts also significantly on the intermediate scale motions, say, \( k = 2 \) (cf, Fig 6.3 (b)). Third, instead of a V-shaped form for \( R(k) \) as seen in the subresonant flows (e.g., Fig 6.3 (c)), \( R(k) \) exhibits monotonic behavior (see (c)) with changes in \( k \). In particular, the largest
correlation takes place at $k = 1$ in agreement with dns observation (cf. Fig 6.4). As for the agreement between closure and dns at individual $k$, much of the remarks for the previous case applies to the current case.

(c) **Topographic stress as a function of $r$**

To have a sense of dependence of the above numerical results on the model parameters, we have taken the Ekman coefficient $r$ as an example and considered the case where $r$ varies from $0.12$ ($1/(100\, \text{days})$) to $1.16(1/10 \, \text{days})$, with $U = 3.0$, $h_{\text{max}} = 2.0$ and $\beta$ as the same as above. The results from these calculations are presented for the topographic stress in Fig 6.6. It is seen that the stress increases with increased $r$ and the closure prediction is in a good agreement with the ensemble mean of the experiment data. Note that the increase of $\tau$ with $r$ as seen in Fig 6.6 is not always the case. It is noted in our test runs (not shown here) that for those statistical equilibrium flows with $U$ in the subresonant regime, the dependence of $\tau$ on $r$ will be opposite to that seen in Fig 6.6

6.4 Concluding remarks

The closure model is used, together with dns, to study the forced-dissipated statistical equilibrium approached by the momentum-driven topographic flows. A particular attention is directed to the topographic stress which acts on the uniform
zonal component of the equilibrium flows. The results indicate that the stress exhibits resonant behavior for those equilibria with $U$ near the resonant value $U_r$. (cf., Fig 6.1). It is found that the spectral dynamics of the equilibria undergoes some qualitative changes upon crossing $U_r$ (cf. Fig 6.3 and Fig 6.5).

Recall that the periodic cell length $l$ is set to 320 km, a size corresponding to the grid size in a coarse-resolution ocean circulation model, and that $U$ corresponds to the domain averaged mean. Thus, the topographic stress $\tau$ may be viewed as the effect of topographic features of subgrid scale on the $U$ defined at coarse grids. With this point of view, our calculations seem to strengthen the concern, which has been recognized for some time, about the adequacy of resolution in large scale ocean modeling.
Appendix 6A: Expressions for $T_{1,k}, T_{2,k}, S_{2,k}$ and $S_{3,k}$

Note that the energy and potential enstrophy transfer $T_{1,k}$ and $S_{1,k}$ are the closure approximations to the four triad summations in (6.2.10) and (6.2.11), respectively. It thus follows using (6.2.12) that

$$T_{1,k} = \sum_{\Delta} 2A_{kpq}^2 k_{pqr} \left\{ \hat{A}_{pqk} Z_{q} Z_{k} + \hat{A}_{pqk} Z_{p} Z_{k} + \hat{A}_{kpq} Z_{p} Z_{q} \right\}$$

$$+ A_{pqk} Z_{q} R_{k} + A_{kpq} Z_{k} R_{q} + A_{qpk} Z_{p} R_{k} + A_{qkp} Z_{k} R_{p}$$

$$+ A_{kpq} Z_{p} R_{q} + A_{kqp} R_{q} R_{p} \right\},$$

and similarly,

$$T_{2,k} = \sum_{\Delta} 2A_{kpq}^2 k_{pqr} \left\{ \hat{A}_{pqk} Z_{q} R_{k} + \hat{A}_{pqk} Z_{p} R_{q} + A_{kpq} Z_{k} H_{q} + A_{kpq} Z_{p} H_{q} \right\}$$

$$+ A_{pqk} (R_{q} R_{k} + I_{q} I_{k}) + A_{kqp} (R_{p} R_{q} + I_{p} I_{q}) \},$$

where the interaction coefficients $A_{kpq}$ and $\hat{A}_{kpq}$ are given by (6.2.9) and (6.2.12d) in the text.
Appendix 6B Conservation properties of the closure model

The proof of (6.2.19) and (6.2.20) lies on somewhat more stringent conservation properties of the model, viz., the detailed conservation of the energy and potential enstrophy, which are summarized in

**Detailed conservation** Let \( k, p, q \in K \) form a triad \((k, p, q)\) for which \( k + p + q = 0 \). Let \( T_{e}(k|p, q) \) and \( T_{q}(k|p, q) \) be the symmetrized energy and potential enstrophy transfer densities at wavenumber \( k \), defined by

\[
T_{e,k} = \frac{(T_{1,k} + T_{2,k}^{'})/k^2}{\Delta} = \sum_{\Delta} T_{e}(k|p, q), \quad (6B.1)
\]

\[
T_{q,k} = (T_{1,k} + T_{2,k}^{'}) + 2\text{Re} \ (S_{2,k} + S_{3,k}) = \sum_{\Delta} T_{q}(k|p, q), \quad (6B.2)
\]

where \( T_{1,k} \), \( T_{2,k}^{'}, S_{2,k} \) and \( S_{3,k} \) are given by (6A.1)-(6A.4), respectively, with the symbol \( \sum \) defined as in (6.2.6). Then, \( T_{e}(k|p, q) \) and \( T_{q}(k|p, q) \) for the closure model (6.2.13)-(6.2.16) satisfy

\[
T_{e}(k|p, q) + T_{e}(p|q, k) + T_{e}(q|k, p) = 0, \quad (6B.3)
\]

\[
T_{q}(k|p, q) + T_{q}(p|q, k) + T_{q}(q|k, p) = 0. \quad (6B.4)
\]

**Proof:** Since the method used to prove (6B.4) parallels to the one for establishing (6B.3), the proof given here is thus
made explicitly for the energy argument (6B.3). First, we symmetrize the terms in \( T_e(k) \) (cf. (6B.1)) using (6A.1) and (6A.2) w.r.t. \( p, q \) to get \( T_e(k|p,q) \)

\[
T_e(k|p,q) = T_e^{(2Z)}(k|p,q) + T_e^{(2R)}(k|p,q)
+ T_e^{(2H)}(k|p,q) + T_e^{(RR+II)}(k|p,q),
\]

(6B.5)

\[
T_e^{(2Z)}(k|p,q) = \theta_{kpq} \left\{ \hat{A}_{kpq} \hat{A}_{p} \hat{q} \hat{k} - \hat{A}_{kqp} \hat{q} \hat{k} \hat{p} \right\} \frac{1}{k^2},
\]

(6B.6)

\[
T_e^{(2R)}(k|p,q) = \theta_{kpq} \left\{ A_{kpq} \hat{A}_{p} \hat{k} \hat{q} R + A_{kqp} \hat{A}_{q} \hat{k} \hat{p} R \right\}
+ A_{k} \hat{A}_{q} \hat{k} \hat{p} \hat{q},
\]

(6B.7)

\[
T_e^{(2H)}(k|p,q) = \theta_{kpq} \left\{ A_{k} \hat{A}_{q} \hat{k} \hat{p} R + A_{k} \hat{A}_{q} \hat{k} \hat{p} R \right\}
+ A_{k} \hat{A}_{q} \hat{k} \hat{p} \hat{q},
\]

(6B.8)

\[
T_e^{(RR+II)}(k|p,q) = \theta_{kpq} \left\{ A_{k} \hat{A}_{q} \hat{k} \hat{p} R R + A_{k} \hat{A}_{q} \hat{k} \hat{p} R R \right\}
+ I_R I_q I_k
\]

159
where $\theta_{kpq}$, $A_{kpq}$, and $\tilde{A}_{kpq}$, etc, are given by (6.2.9), (6.2.12d) and (6.2.13), respectively. Next, note that to establish (6B.3), it suffices to show that (6B.3) holds for any one of the components in $T_e(k|p,q)$. To demonstrate that the latter is true, we take $T_e^{(ZH)}(k|p,q)$ as an example and show that

$$T_e^{(ZH)}(k|p,q) + T_e^{(ZH)}(p|q,k) + T_e^{(ZH)}(q|k,p) = 0. \quad (6B.10)$$

Interchanging $k$ with $p$ in (6B.8) yields $T_e^{(ZH)}(p|q,k)$. Similarly, the operation: $k \leftrightarrow q$ in (6B.8) gives $T_e^{(ZH)}(q|k,p)$. Omitting the lengthy algebra, we simply point out that (6B.10) follows after introducing the two into (6B.10) and canceling the terms pairwise with the use of the symmetric properties of $A_{kpq}$ and $\tilde{A}_{kpq}$, i.e.,

$$A_{kpq} = -A_{qpk}; \quad \tilde{A}_{kpq} = \tilde{A}_{qpk}. \quad (6B.11)$$

and the symmetric property of $\theta_{kpq}$ (cf. (6.2.13)). With the arguments leading to (6B.10), it thus follows that the detailed conservation of total energy (6B.3) holds for the closure model (6.2.13)-(6.2.16).
Now, we show that the two invariants \( <E> \) and \( <Q> \) given by (6.2.19) and (6.2.20) are simply the consequence of the detailed conservation properties (6B.3) and (6B.4). Specifically, we have

**Invariants of the motion.** Let \( <E> \) and \( <Q> \) given by (6.2.19)-(6.2.20). Then \( <E> \) and \( <Q> \) are conserved by the nonlinear transfer in (6.2.13)-(6.2.16) in the sense

\[
\sum_{k} \frac{(T_{1,k} + T_{2,k})}{k^2} = 0,
\]

\[
\sum_{k} \left( T_{1,k} + T_{2,k} + 2 \Re(S_{2,k} + S_{3,k}) \right) = 0,
\]

Moreover, \( <E> \) and \( <Q> \) are the invariants of the model in the absence of the dissipation.

**Proof:** The closure versions of the total kinetic energy and potential enstrophy equations follow from (6.2.14)-(6.2.16)

\[
\frac{d}{dt} <E> = r(U^* - U)U
\]

\[
- \sum_{k} \nu_k Z_k/k^2 + (1/2) \sum_{k} \left( T_{1,k} + T_{2,k} \right)/k^2,
\]

(6B.14)

\[
\frac{d}{dt} <Q> = - \sum_{k} \nu_k (Z_k + R_k)
\]

\[
+ (1/2) \sum_{k} \left\{ (T_{1,k} + T_{2,k}) + 2 \Re(S_{2,k} + S_{3,k}) \right\}.
\]

(6B.15)

In terms of \( T_e(k|p,q) \) and \( T_q(k|p,q) \), (6B.14) and (6B.15) read

161
\[
(d/dt)<E> = r(U^* - U)U - \sum_{k} v_k \frac{Z_k}{kk} + \frac{1}{2} \sum_{p} \sum_{q} T_e(k|p,q) \delta(k+p+q),
\]

\[
(6B.16)
\]

\[
(d/dt)<Q> = - \sum_{k} v_k (Z_k + R_k) + \frac{1}{2} \sum_{p} \sum_{q} T_q(k|p,q) \delta(k+p+q),
\]

\[
(6B.17)
\]

where the summations are over all \( k, p, q \in K \) and \( \delta(k+p+q) = 1 \) when \( k+p+q = 0 \) otherwise it vanishes. Interchanging the dummy indices in (6B.14)-(6B.15): \( k \leftrightarrow p \) and \( k \leftrightarrow q \), and adding the resulting equations to (6B.16) and (6B.17), respectively, yield

\[
(d/dt)<E> = r(U^* - U)U - \sum_{k} v_k \frac{Z_k}{k} + \frac{1}{6} \sum_{p} \sum_{q} \left( T_e(k|p,q) + T_e(p|q,k) + T_e(q|k,p) \right) \delta(k+p+q),
\]

\[
(6B.18)
\]

\[
(d/dt)<Q> = - \sum_{k} v_k (Z_k + R_k) + \frac{1}{6} \sum_{p} \sum_{q} \left( T_q(k|p,q) + T_q(p|q,k) + T_q(q|k,p) \right) \delta(k+p+q),
\]

\[
(6B.19)
\]

The conditions (6B.12) and (6B.13) immediately follow after comparing (6B.18) with (6B.14) and (6B.19) with (6B.15) and then invoking (6B.3) and (6B.4). Introducing (6B.12) and (6B.13) into (6B.14) and (6B.15) shows that \( <E> \) and \( <Q> \) are the invariants of the motion when no dissipation is in the system, which completes the proof.
Fig 6.1  Topographic stress $\tau$ as a function of $U$. The parameters are $(h_{\text{max}}, \beta, \tau) = (4.0, 0.8, 0.12)$. The solid line is for the closure results and symbols for the dns data. The resonant point corresponds to $U_r = 1.0$. 

163
Fig 6.2 Streamfunction ($\psi = -Uy+\theta$) for the two representative flows at $t=20$ (or $t=230$ days), with parameters as the same as in Fig 6.1. (a): the subresonant flow with $U = 0.25$; (b): the superresonant flow with $U = 2.75$. The dash contours are for negative values.
Fig 6.3: Enstrophy (a), topographic stress (b) and vorticity-topography correlation (c) spectra (solid lines) for the subresonant flow case $U = 0.25$, with parameters as the same as in Fig 6.1. The symbols are for the five dns ensemble data.
Fig 6.4 Vorticity (a) and Topography (b) for the superresonant flow shown in Fig 6.2(b)
Fig 6.5 The same as in Fig 6.3 but for $U = 2.75$. 
Fig 6.6 Topographic stress $\tau$ as a function of $r$. The parameters are $(h_{\text{max}}, \beta, U) = (6.2.0, 0.8, 3.0)$. The solid line is for the closure results and symbols for the dns data.
We have analyzed the problem of temporal evolution of perturbations to the hydrodynamical system (1.2.1)-(1.2.3) in both infinite and finite dimensional function spaces, and in both deterministic and probabilistic sense.

In doing so, we have developed an algorithm for the global analysis of the system (1.2.1)-(1.2.3) as opposed to the usual modal analysis. The global feature arises from the fact that the analysis is conducted in a space of all kinematically admissible disturbances, as opposed to the modal analysis which takes care of only infinitesimal disturbances with the modal form, or as opposed to weakly nonlinear theory which is restricted to a subset, consisting of finite amplitude disturbances, of the above space. This global nature of the analysis allows us to overcome several difficulties found in the usual modal analysis.

Its application to global stability has yielded a systematic algorithm for separating flows with initial growing perturbations from those without (cf. theorem 2.1 and 3.1). The same algorithm has also been used to find an optimal nonmodal perturbation to a given flow (cf. theorem 2.2 and 3.2). The growth rate of such a perturbation is shown
to be the least upper bound on the growth rate. Its application to multiple equilibria has led to a necessary condition for the existence of multiple equilibria. Its application to the study of relation of modal to nonmodal growth rate, in conjunction with the comparison of the stability measures, has uncovered the cause underlying many shortcomings of modal analysis, or equivalently have resolved such paradoxes as growth of disturbances in subcritical flows and explosive development of instability in weakly supercritical flows.

Numerical illustrations made for some specific flows have strengthened the general results, suggesting that a stability analysis of a hydrodynamical system without a global analysis such as the one developed here is limited in many important aspects. Also, these numerical examples have clearly demonstrated how the global analysis can be systematically implemented to specific flow problems.

In our local analysis of asymptotic behavior of nonmodal disturbances to hyperbolic equilibria of the system, we have established: a) for any subcritical flow outside of MGS regime, there exists a finite, though perhaps small, neighborhood around the origin of $\mathbb{R}^n$ such that a nonmodal disturbance initialized in this neighborhood will ultimately decay to zero after exhibiting Orr's temporal amplification
(theorem 5.1); b) for any supercritical flow, there exists a finite, though perhaps small, neighborhood adjacent to the origin of $\mathbb{R}^n$ such that a nonmodal disturbance initialized in this neighborhood will persists as $t \to \infty$ (cf. theorem 5.2); and c) the nature of the persistent nonmodal disturbances is related to the nature of nonhyperbolic point in parameter space of interest (cf. theorem 5.3). The numerical experiments are seen to confirm these predictions.

Our probabilistic study of forced-dissipated statistical equilibrium of perturbed flows arising from initial uniform zonal flows over random topography is done with special regard to the correlation between disturbance and underlying topography and the resulting force. While many questions remain to be answered, such an exercise has led to an appreciation about the method itself as well as about the problem of flow-topography interaction. For the former, we note that the detailed conservation of energy and potential enstrophy, which holds regardless of the presence of dissipation in the system (cf. Appendix 6B), provides a means for systematical investigation of nonlinear transfer of the these quantities among interacting triads, an area not accessible to other approaches. For the latter, numerical results for topographical stress clearly indicate the significance of this force in the overall momentum budget of large-scale ocean circulation and strengthen the need to
parameterize it in general circulation models.

There are several possible extensions from the present work. First, the obvious limitation of the system (1.2.1)-(1.2.3) as a model to study geophysically relevant flows is the neglect of baroclinicity, which has been known to be a primary source of instability in the atmosphere and oceans since Charney (1947) and Eady (1949). It is clear that a global analysis for a system with this effect included allows one to draw conclusions on the origin, development and decay of disturbances from model studies in closer connection to such phenomena as synoptical scale disturbances in the westerly winds and mesoscale eddies in the oceans. It is expected from the work on the Boussinesq equation (cf. Joseph, 1966) that the physical principle (cf. lemma 2.2) which allows for the present analysis will hold while it will take a more general form involving the ratio of the conversion between the total energy (i.e., kinetic energy plus potential energy) in a basic flow and in a disturbance to the total dissipation. It thus follows that an extension from the present barotropic system to a baroclinic system is ensured in principle while the analysis in the baroclinic case will be far more involved than the present case.

Second, an orbital stability analysis via Floquet theory (Arnol'd 1983; Guckenheimer & Holmes, 1983) is a necessary
next step to study the transition from periodic flows to quasi-periodic ones and eventually to chaotic or turbulent flows. This would allow us to complete the repeated supercritical bifurcation shown in Fig 5.7.

Finally, as the topographic stress appears significant from the rough comparison between it and the wind stress, an important task is to bring it into general ocean circulation models to see if it enhances their performance w.r.t. some specific failures of these models, e.g., latitude overshoot of the Gulf Stream. This effort, in conjunction with Holloway's proposal for Unprejudiced ocean circulation (Holloway, 1991), may lead to some practical scheme to parameterize the effect of subgrid-scale topography on the grid-scale motions.
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