**SPECTRAL EVOLUTION OF WIND GENERATED SURFACE GRAVITY
WAVES IN A DISPERSED ICE FIELD**

by

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Abstract

The Marginal Ice Zone includes wide areas covered by dispersed ice floes in which wave conditions are significantly affected by the ice. When the wind blows from the solid ice pack, towards the open sea, growing waves are scattered by the floes, their spectral characteristics being modified. To further understand this problem, a model for the evolution of wind waves in a sparse field of ice floes was developed. The sea state is described by a two-dimensional discrete spectrum. Time-limited wave growth is obtained by numerical integration of the energy balance equation using the exact nonlinear transfer integral. Wave scattering by a single floe is represented in terms of far-field expressions of the diffracted and forced potentials obtained numerically by the Green's function method. The combined effect of a homogeneous field of floes on the wave spectrum is expressed in terms of the Foldy-Twersky integral equations under the assumption of single scattering. The results show a strong dependence of the spectrum amplitude and directional properties on the ratio of the ice floe diameter to the wavelength. For a certain range of this parameter, the ice cover appears to be very effective in dispersing the energy; the wave spectrum rapidly tends to isotropy, limiting its growth both for the energy content and the peak frequency. It is therefore unlikely that an offshore wind blowing over the Marginal Ice Zone would generate a significant wave field.

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Chapter 1

Introduction

The development of a wind wave spectrum and the nature of the energy input and transfer terms involved are now understood well enough, in the case of an open ocean, to be able to provide fairly good wave forecasting (e.g. Janssen et al., 1984; Golding, 1983). As an extension to the existing theory, it is interesting to examine the situation where waves are generated in the presence of a partial ice cover as encountered in the Marginal Ice Zone (MIZ).

The MIZ is defined as that part of the seasonal ice cover which is close enough to the open ocean boundary to be affected by its presence. Among other areas, all cold drift currents (e.g. the Labrador Current) can be considered as part of the MIZ, as well as the edge of the Arctic pack and the outer 200 km of the Antarctic ice cover. The MIZ constitutes an active region of dynamic exchanges between sea ice, water and the atmosphere. Recently, man's activities (offshore oil exploration and production, naval operations) have considerably increased in that region, stressing the need for a better understanding of the physical processes involved in those exchanges.

One aspect of the air-sea-ice interactions has not yet been properly investigated: wave generation by a wind blowing over the ice field towards the open sea. The

presence of scattered ice floes partially covering the sea surface certainly affects the nature of the generated waves, but to what extent? Does the shape of the directional spectrum present major modifications compared with the more familiar situation of an open ocean? Is the generation process inside the MIZ simply irrelevant or does it lead to waves of significant amplitude by the time the open, ice-free sea is reached? Must wave generation in the MIZ be included in wave climate studies and forecasts? These considerations are particularly important for offshore operations in Canadian eastern and Arctic waters.

Most of the previous efforts to study wave-ice interactions in the MIZ have been concentrated on the propagation and attenuation of sea waves and swell entering the ice field (e.g. Wadhams, 1973, 1978; Wadhams et al., 1986). For this purpose the theory of flexural-gravity waves has been used where the energy of the waves is coupled between an ice layer and the water below it (e.g. Wadhams, 1986; Squire, 1984). The propagation of these waves and their dispersive behaviour was first analysed for the simplest case of a uniform semi-infinite floating ice sheet. The transmission coefficient at the ice edge increases with the wavelength, in agreement with the presence of long period waves observed at large distance within the ice field (e.g. Robin, 1963).

To study the wave propagation in a field of discrete floes, an approximate solution based on the same approach has been considered (Wadhams, 1975). The problem is reduced to a two-dimensional one by representing the floes as successive bands of infinite lateral dimension. Each row transmits and reflects a certain fraction of the incident energy dictated by the transmission properties of the flexural-gravity waves on the edge of the band. Although this theory leads to acceptable attenuation rates in most studied cases, it is inadequate. Firstly, as suggested by

Robin (1963), the analysis should be different when the horizontal dimension of the ice floes is small compared to the wavelength, in which case, the bending of the floes can be neglected and the floes visualized as rigid floating plates. Also, apart from assuming that typical irregular floe scatters energy only directly backwards, it also ignores diffraction and the wave-making effect of the floes. A more realistic model must describe the ice cover as discrete floes of typical size and shape and attempt to determine the entire scattered wave field due to such floes. To this end, work has been done to obtain detailed measurements of the motion of realistic ice floes in ocean waves (Squire, 1983). Using two-dimensional numerical methods developed in naval architecture, he obtained estimations of amplitudes of motion in the different modes of oscillation for individual floes of various shapes. These results can be used to improve the ice-band model previously described but there still remains a need for a more accurate three-dimensional representation in which the scattered wave field could be fully analyzed.

All the above mentioned studies have concentrated on the attenuation of ocean waves entering the ice field. However, in this thesis, I attempt to investigate the unresolved problem of wave generation in the MIZ itself by suggesting a way in which the partial ice cover may affect the evolution of the wave spectrum, and to answer the question: "To what extent are the time-rate of change and the energy content of the spectrum affected by the presence of ice floes?" The scattering of the waves is expected to increase the directional spread about the mean direction. The influence of this new energy distribution on the complex nonlinear energy exchange mechanism among waves is a fundamentally interesting aspect of the proposed work. From a more practical point of view, a better knowledge of the wave generation in the MIZ would improve wave and ice conditions forecasting in

a region where these are major environmental hazards, by helping to determine a more accurate value for the effective fetch. The results could also be useful in the prediction of the ice edge position and the degree of inhomogeneity in the ice cover.

The next chapter gives a detailed description of the model and of the numerical simulation method, including a complete derivation of the wave-ice parameterization. The results are presented in Chapter 3, together with a discussion on their implications on wind wave evolution inside the MIZ. Chapter 4 presents conclusions followed by suggestions for future work.

Chapter 2

Methodology

2.1 Introduction

A sea state is commonly represented in terms of its two-dimensional wavenumber spectrum, $F(\vec{k})$, which is the Fourier transform of the covariance of the surface displacement η at points separated by a distance $\vec{r}$:

$$F(\vec{k}) = (2\pi)^{-2} \iint \overline{\eta(\vec{x})\eta(\vec{x} + \vec{r})} e^{-i\vec{k} \cdot \vec{r}} d\vec{r}. \quad (2.1)$$

From the inverse of the Fourier transform,

$$\overline{\eta^2} = \iint F(\vec{k}) d\vec{k}. \quad (2.2)$$

Consequently, $F(\vec{k})$ can be considered to represent the density of contributions to the variance, $\overline{\eta^2}$, per unit area of the wavenumber space. Using the linear dispersion relation,

$$(2\pi f)^2 = gk \tanh(kh) \quad (2.3)$$

(with f the frequency, g the gravitational constant and h the water depth), the corresponding directional frequency spectrum, $F(f, \theta)$, can be simply related to

the wavenumber spectrum by

$$F(f, \theta) = \left[k F(\vec{k}) \frac{\partial k}{\partial f} \right]_{(2\pi f)^2 = gk \tanh(kh)}, \quad (2.4)$$

which is such that

$$\overline{\eta^2} = \int_0^\infty \int_0^{2\pi} F(f, \theta) d\theta df. \quad (2.5)$$

This second spectral form gives the distribution of energy among waves with different frequency, f , and direction of propagation, θ .

For an ice-free situation, in deep water and in the absence of currents, the spectrum evolves in time t and space $\vec{x}$ according to the energy balance equation:

$$\frac{\partial F(f, \theta; \vec{x}, t)}{\partial t} + \vec{C}_g(f, \theta) \cdot \vec{\nabla} F(f, \theta; \vec{x}, t) = S_{in} + S_{ds} + S_{nl} \quad (2.6)$$

where $\vec{C}_g(f, \theta)$ is the group velocity, S_{in} the rate of energy input from the atmosphere, S_{ds} the dissipation rate, mostly through wave breaking, and S_{nl} the energy transfer due to nonlinear interactions among spectral components (Hasselmann et al., 1973). In the presence of ice, this energy balance equation will have to be modified to take account of the effects of ice floes on the wave field.

For wave generation by the wind to be of significant importance, the fraction of the area covered by ice, f_i , has to be relatively small ($\leq 25\%$, say; this study will provide a better quantitative estimate). Thus, the region of interest for this study is restricted to the outer portion of the ice pack where the ice field is usually composed of randomly distributed small floes with no preferred shape or orientation (e.g. Wadhams, 1986).

In the model, the sea surface is assumed to be covered uniformly and sparsely by a random distribution of rigid ice floes. Each mass of floating ice endeavours to

follow the displacement of the supporting water surface within limits imposed by its rigidity and inertia. It re-radiates incident wave energy, slightly diminished by dissipative processes in the water and within the ice itself. This scattering effect tends to decrease the energy content of the wave field and, more importantly, to cause a spectral redistribution, tending to spread out the energy over a broader range of directions. The object of this work is to quantify the modifications due to the partial ice cover on the growing spectrum i.e. to modify the energy terms of Eq.(2.6) for the effect of the ice and to add a new term, S_{ice} , responsible for the extra dissipation and redistribution of energy. In this chapter, a detailed description of the method of integration of Eq.(2.6) is given including the derivation of this new energy term.

2.2 Energy balance equation

Different models have been developed to obtain the wave growth for a full directional wave spectrum by numerical integration of Eq.(2.6), or some approximation thereof. They differ primarily in the form assumed for the source functions (S_{in} , S_{ds} and S_{nl}) for which a variety of formulations exist, based on theoretical and observational grounds (e.g. SWAMP Group, 1985). The specific model used in this study is that described by Hasselmann and Hasselmann (1985a,b).

The input source function, S_{in} , takes the form proposed by Snyder et al. (1981) on the basis of direct measurements of the work done by the atmospheric fluctuations on the waves,

$$S_{in}(f, \theta) = \begin{cases} 0. & \text{if } (\frac{U_s \cos \theta}{c} - 1) < 0. \\ 0.25 \frac{\rho_a}{\rho_w} (\frac{U_s \cos \theta}{c} - 1) \omega F(f, \theta) & \text{otherwise} \end{cases} \quad (2.7)$$

where ρ_a and ρ_w are the densities of air and water, respectively, U_5 the windspeed at 5 m., θ the angle between the wind vector and the wave propagation direction, $\omega = 2\pi f$ the wave angular frequency and c the phase velocity. In its range of validity ($1 \leq \frac{U_5 \cos \theta}{c} \leq 3$), Eq.(2.7) is in reasonable agreement with the theory of Miles (1959), provided it is redefined in terms of u_*/c , where the friction velocity $u_* = \sqrt{\frac{\tau_a}{\rho_a}}$ is determined by the wind shear stress τ_a . To this end, the wind speed U_5 dependence was replaced by a similar dependence on the friction velocity with $u_* = U_5/28$ (Komen et al., 1984).

The less well-known dissipation term, S_{ds} , follows the general form suggested by Hasselmann (1974) for the dissipation due to small-scale white-capping processes. He showed that S_{ds} was to be quasi-linear in the wave spectrum and proportional to the square of the frequency, with a coefficient which depends only on integral spectral parameters:

$$S_{ds}(f, \theta) = -C\bar{\omega}\left(\frac{\omega}{\bar{\omega}}\right)^2\left(\frac{\hat{\alpha}}{\hat{\alpha}_{PM}}\right)^m F(f, \theta). \quad (2.8)$$

Here $\hat{\alpha}$ is the integral wave-steepness parameter,

$$\hat{\alpha} = \frac{E\bar{\omega}^4}{g^2}$$

where

$$\bar{\omega} = E^{-1} \int_0^\infty \int_0^{2\pi} F(f, \theta) \omega df d\theta$$

and

$$E = \int_0^\infty \int_0^{2\pi} F(f, \theta) df d\theta.$$

The parameter $\hat{\alpha}_{PM} = 4.57 \times 10^{-3}$ is the theoretical value of $\hat{\alpha}$ for a Pierson-Moskowitz spectrum. The constant C , determining the overall level of dissipation, has been taken from Komen et al. (1984), with a value of 2 for the power m . They considered the energy transfer equation for well-developed ocean waves and studied the conditions for the existence of an equilibrium solution. The best agreement with the Pierson-Moskowitz spectrum was obtained with $C = 3.33 \times 10^{-5}$.

The nonlinear term, S_{nl} , represents the energy exchange between different wave components interacting weakly among themselves, first demonstrated by Phillips (1960). Although these energy transfers are described as “very weak”, this process plays an important role in the evolution of the spectrum. Wave-wave interactions occur among a set of four spectral components when the resonant conditions are satisfied, namely:

$$\vec{k}_1 + \vec{k}_2 = \vec{k}_3 + \vec{k}_4 \quad (2.9)$$

and

$$\omega_1 + \omega_2 = \omega_3 + \omega_4, \quad (2.10)$$

where the wavenumber, in deep water, is related to the angular frequency by the dispersion relation $k_i = \omega_i^2/g$. These two conditions determine a family of curves in the wavenumber plane specifying the sets of wavenumbers for surface waves capable of undergoing resonant interactions (Fig. (2.1)).

K. Hasselmann (1962) derived an expression for the transfer of energy due to nonlinear interactions by carrying a perturbation analysis to fifth order in a small parameter, the wave slope. From his results, the net rate of change of energy for one spectral component at wavenumber $\vec{k}_4$ can be computed via the Boltzmann-integral

Figure 2.1: Series of curves in the wavenumber plane defined by the resonance conditions (2.9) and (2.10), in deep water. If C and C' are any pair of points on one of the curves, the vectors AC , CB , AC' and $C'B$ form a resonant quartet (from Phillips, 1980).

expression

$$S_{nl}(\vec{k}_4) = \omega_4 \int \cdots \int \sigma(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4) \delta(\vec{k}_1 + \vec{k}_2 - \vec{k}_3 - \vec{k}_4) \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4) \\ [n_1 n_2 (n_3 + n_4) - n_3 n_4 (n_1 + n_2)] d\vec{k}_1 d\vec{k}_2 d\vec{k}_3 \quad (2.11)$$

where $n_i(\vec{k}_i) = F(\vec{k}_i)/\omega_i$ is the action density and the coupling coefficient σ a complicated sixth-order homogeneous function of the wavenumbers involved. Eq.(2.11) describes the net energy transfer rate at a given wave component $\vec{k}_4$ due to interactions with all combinations of four waves satisfying the resonance conditions (2.9) and (2.10) represented by the two Dirac δ functions. Because of the extensive computing time required to evaluate the multiple integral of Eq.(2.11), various approximations to the exact expression are commonly used in operational wave models (e.g. SWAMP Group, 1985). Nevertheless, since those parameterizations of S_{nl} have been fitted to standard directional distributions, they exhibit basic restrictions in their treatment of the nonlinear transfer, and cannot be expected

to provide adequate transfers for unusual spectral shape like the one anticipated when waves develop in the presence of ice floes. Thus, in this study, the S_{nl} term is computed using the full Boltzmann transfer integral (2.11).

An accurate and fast technique, developed recently by Hasselmann and Hasselmann (1981; 1985a), enables a large number of computations to be carried out, as required for this work. The method, based on a symmetrical treatment of the interactions, is one to two orders of magnitude more efficient than previous ones. This is achieved by exploiting the invariance of the coupling coefficient σ with respect to permutations of the wavenumbers, and the principle of detailed balance by which the computation of the change in action density for one wavenumber gives also the identical action changes (except for a simple sign rule) for the other three components participating in a given interaction. Furthermore, another important reduction in computing time is attained by filtering out the regions of the interaction phase space where the contributions to the integral are not significant for a given type of spectrum (where σ is smaller than some preset lower limit). Once the computation of S_{nl} has been performed for a particular spectrum, the nonlinear interactions for another spectrum, presenting only minor differences with the first one, can be obtained by scaling the new spectrum to the reference spectrum with certain spectral parameters, then computing the S_{nl} term for this spectrum with the previously computed filtered grid, and, finally, scaling back the results to obtain the appropriate nonlinear energy transfer. Also, the introduction of stretched variables provide higher resolution in important regions of the interaction space.

Now that the three energy terms of Eq.(2.6) have been satisfactorily defined, the effect of a partial ice cover has to be introduced. First, because waves can neither be generated by the wind nor dissipated by the usual breaking mechanisms

in that fraction f_i of the sea surface covered by ice, both S_{in} and S_{ds} are reduced by a factor $(1 - f_i)$ from their ice-free values given above. There remains the determination of the new term, S_{ice} , which parameterizes the scattering and extra dissipation of energy by the ice.

2.3 Modelling the partial ice cover

The determination of the additional source term, S_{ice} , consists of analyzing the scattering of a random wave field described by its energy spectrum, $F(f, \theta)$, incident on a random array of ice floes. Assuming that each discrete component of the spectrum interacts independently with the floes, two aspects of the problem may be distinguished. First, the ability of a single floe to scatter a certain fraction of the incident energy associated with a single spectral component is considered in Section 2.3.1. Secondly, the average wave field due to scattering of this wave by the whole array of ice floes is computed. The directional scattered spectrum, for each frequency, is then obtained by linear superposition of the contributions from the incident and scattered waves of all directions (Section 2.3.2). Finally, the dissipation of energy due to the presence of the partial ice cover is evaluated (Section 2.3.3).

2.3.1 Scattering of a plane wave by a single floe

There are two distinct approaches to solving the hydrodynamic problem of a floating body in waves: Morison's equation and potential flow theory (e.g. Garrison, 1978). These two procedures have their own limits of applicability. The former considers the flow past a body with velocity and acceleration such as would occur at its centre if it were not present and, therefore, is valid for objects which are

small in relation to the wavelength. In such a situation, drag effects are important, and cause flow separation whose importance can be estimated by the ratio of the wave height to the body diameter, H/D . On the other hand, when the body size to wavelength ratio, D/L , is sufficiently large, the incident wave undergoes important scattering (or diffraction) and the more explicit potential theory is required to describe the flow. However, when both H/D and D/L are small, inertial forces dominate the fluid motion and, if viscous effects are disregarded, the results of diffraction theory approach those based on the inertia term in the Morison equation as the diameter to wavelength ratio tends to 0. Finally, both approaches are limited by the value of the wave steepness parameter, H/L , steep waves being affected by nonlinear effects (with $(\frac{H}{L})_{max} \approx 0.14$ in deep water) (Fig. (2.2)).

In the Morison equation, the force is made up of two components: one due to the drag, as in the case of a real fluid steady flow, and an inertia force due to uniform acceleration of an ideal fluid (e.g. Sarpkaya and Isaacson, 1981). For example, in the case of a vertical cylinder of diameter D , the Morison equation gives the force per unit length as

$$F' = \frac{1}{2}\rho_w D C_d U |U| + \frac{1}{4}\rho_w \pi D^2 C_m \frac{dU}{dt} \quad (2.12)$$

where U is the incident (undisturbed) flow velocity at the centre, C_d the drag coefficient and C_m the inertia coefficient. This is a semi-intuitive approach in that the inertia term is exactly of the form which results from inviscid flow theory while the drag term is of the form used in steady flow. Eq. (2.12) implicitly assumes that the wave slope and the associated pressure gradient are constant across the body and that the wave scattering is negligible. When the body spans a significant

Figure 2.2: Regions of validity for the Morison equation and the potential theory (linear diffraction theory). Here H is the waveheight, L the wavelength and D the body diameter. The dashed lines delimit a region where nonlinear effects are important, and the double-crossed pattern a region where both theories agree, neglecting viscous effects (after Garrison, 1978).

fraction of a wavelength, this is no longer true, and an alternative approach is required.

In the region studied, the outer part of the MIZ, both ocean and wind act on the ice cover to form heavily rafted and ridged small floes (e.g. Bauer and Martin, 1980). These are represented, in the model, by truncated cylinders of diameter $15 \leq D \leq 50\text{m}$ and draft $1 \leq d \leq 4\text{m}$. The geometry of one individual floe can not be fully described by such a simple shape. Sharp corners, for example, would cause flow separation, increasing form drag. However, the overall effect of the ice field on the energy distribution of the growing waves should be properly derived using this simple representation. Within that given range of floe diameters and for the expected values of wavelength in the early stage of this short fetch situation ($L \leq 60\text{m}$), the value of the parameter D/L remains in the region of validity of the potential flow theory. Therefore, in this work, the potential flow theory is used rather than the Morison equation.

Assuming the fluid incompressible and the flow irrotational, the problem reduces to the determination of a velocity potential, Φ , which satisfies the Laplace equation, $\nabla^2\Phi = 0$. The function Φ is defined here such that $\vec{u} = \nabla\Phi$, with $\vec{u}$ the fluid velocity vector. The analysis is based on the assumption that the amplitude of the wave is small. Through linearization, the scattering process can be decomposed and treated as the sum of two distinct mechanisms: the fluid motion produced by a body forced to oscillate in otherwise still water and the interaction of a regular wave with a restrained body (e.g. Sarpkaya and Isaacson, 1981). Because of its symmetry relative to the vertical axis, a floating cylinder will be forced by an incident plane wave to oscillate in three degrees of freedom only: surge, back and forth in the direction of the incident wavenumber $\vec{k}$; heave, up and down; and

Figure 2.3: Cylindrical floe of radius $a = D/2$ and draft d in water of depth h . The three modes of motion have amplitudes of ξ_k with subscript: 1 for surge, 2 for heave and 3 for pitch. Here, (x, y, z) form a Cartesian coordinate system and (r, θ, z) the corresponding cylindrical coordinate system.

pitch, about an axis parallel to the wave crests. For each mode, the resulting small periodic motion, of angular frequency ω , may be expressed in the form $\xi_k e^{-i\omega t}$ where ξ_k is the complex amplitude for the mode of motion k , with $k=1, 2$ and 3 corresponding to surge, heave and pitch respectively (Fig. (2.3)). In linearizing the problem, the velocity potential, also harmonic, can be obtained by superposition of 5 components:

$$\Phi = \text{Re}([\phi_0 + \phi_4 + \sum_{k=1}^3 \xi_k \phi_k] e^{-i\omega t}) \quad (2.13)$$

where ϕ_0 is the undisturbed part of the incident wave potential, ϕ_4 the diffracted potential, ϕ_k , for $k = 1 \dots 3$, the forced potentials and $\text{Re}()$ indicates the real part.

The potential associated with an incident plane wave, of amplitude $H/2$, in water of finite depth, is usually given as

$$\phi_0 = -\frac{igH}{2\omega} \frac{\cosh[k(z+h)]}{\cosh(kh)} e^{ikx} \quad (2.14)$$

where x is measured in the direction of wave propagation and z vertically upwards from the still water level.

It is more convenient, here, to use a cylindrical coordinate system with r measured radially from the z axis and θ counterclockwise from the positive x axis (Fig. (2.3)). The incident potential then takes the form

$$\phi_0 = -\frac{igH}{2\omega} \frac{\cosh[k(z+h)]}{\cosh(kh)} \{\cos(kr \cos \theta) + i \sin(kr \cos \theta)\} \quad (2.15)$$

or, expressing the terms contained in $\{ \}$ as series of Bessel functions (e.g. Abramowitz and Stegun, 1965),

$$\phi_0 = -\frac{igH}{2\omega} \frac{\cosh[k(z+h)]}{\cosh(kh)} \sum_{\ell=0}^{\infty} \beta_{\ell} J_{\ell}(kr) \cos(\ell\theta) \quad (2.16)$$

where

$$\beta_{\ell} = \begin{cases} 1 & \text{if } \ell = 0 \\ 2i^{\ell} & \text{otherwise} \end{cases}$$

and J_{ℓ} is the Bessel function of the first kind of order ℓ .

The complex potentials of Eq.(2.13) must satisfy the Laplace equation and the following boundary conditions:

vanishing of the vertical velocity component on the ocean floor,

$$\frac{\partial \phi_k}{\partial z} = 0 \quad \text{at } z = -h; \quad (2.17)$$

the linearized kinematic and dynamic boundary conditions , at the free surface, combined to give

$$\frac{\partial \phi_k}{\partial z} - \frac{\omega^2}{g} \phi_k = 0 \quad \text{at } z = 0; \quad (2.18)$$

the radiation condition, to insure that the scattered potentials correspond to outgoing waves,

$$\lim_{r \rightarrow \infty} \sqrt{r} \left[\frac{\partial \phi_k}{\partial r} - ik \phi_k \right] = 0 \quad \text{for } k = 1 \dots 4; \quad (2.19)$$

and the condition to be applied on the equilibrium surface of the floating body, S_0 ,

$$\begin{aligned} \frac{\partial(\phi_0 + \phi_4 + \sum_{k=1}^3 \xi_k \phi_k)}{\partial \vec{n}} &= v_n \\ &= \sum_{k=1}^3 -i\omega \xi_k n_k \quad \text{on } S_0 \end{aligned} \quad (2.20)$$

where $\vec{n}$ is the unit vector in the direction normal to S_0 , directed outward from the body and v_n the magnitude of the velocity of the surface in that direction, given by $V_n = \text{Re}(v_n e^{-i\omega t})$. V_n is the sum of the surge, heave and pitch components with $n_1 = n_x$, $n_2 = n_z$ and $n_3 = zn_x - xn_z$ where n_x , n_y and n_z are the direction cosines of $\vec{n}$ with respect to the x , y and z directions respectively.

This boundary-value problem is solved with the Green's function method. In the usual way (e.g. Wehausen and Laitone, 1960), the unknown potentials, $\phi_k(\vec{x})$, are expressed in terms of a surface distribution of sources

$$\phi_k(\vec{x}) = \frac{1}{4\pi} \int_{S_0} f_k(\vec{X}) G(\vec{x}, \vec{X}) dS \quad \text{for } k = 1 \dots 4 \quad (2.21)$$

where $f_k(\vec{X})$ is a source distribution function at $\vec{X} = (X, Y, Z)$, a point on S_0 , and $G(\vec{x}, \vec{X})$ a Green's function for the general point $\vec{x}$ due to a source of unit strength

at $\vec{X}$. The Green's function must satisfy the Laplace equation and the boundary conditions (2.17), (2.18) and (2.19). It remains for $f_k(\vec{X})$ to be chosen so that the body surface boundary condition (2.20) is satisfied.

The source strength distribution functions, $f_k(\vec{X})$, and the amplitudes of motion, ξ_k , for each mode k , are obtained using a method described by Isaacson (1982) and of which the broad outlines are given here. Because of the body's axisymmetry, various functions may be expressed as Fourier series in the angle about the body's vertical axis, θ , reducing considerably the computational effort required from that for bodies of arbitrary shape. The appropriate Green's function, first derived by John (1950), was given in a symmetrical form by Fenton (1978) who used it to compute the diffraction of waves on a fixed body, in finite depth:

$$G = \sum_{\ell=0}^{\infty} \left(\sum_{m=0}^{\infty} G_{\ell m} \right) (2 - \delta_{\ell 0}) \cos[\ell(\theta - \Theta)] \quad (2.22)$$

where δ is the Kronecker delta,

$$G_{\ell m} = 4C_m \cos[\mu_m(Z + h)] \cos[\mu_m(z + h)] K_{\ell} \left(\frac{\mu_m r}{\mu_m R} \right) I_{\ell} \left(\frac{\mu_m R}{\mu_m r} \right),$$

$$C_m = \frac{\mu_m^2 + \nu^2}{(\mu_m^2 + \nu^2)h - \nu}$$

and the cylindrical coordinates (R, Θ, Z) correspond to the point (X, Y, Z) on the surface S_0 . I_{ℓ} and K_{ℓ} are the modified Bessel functions of order ℓ of the first and second kinds, respectively, with the upper of the alternative arguments used if $r \geq R$ and the lower otherwise. Also, $\nu = \omega^2/g$ and the μ_m 's are the roots of

$$\mu_m h \tan(\mu_m h) = -\nu h \quad (2.23)$$

with μ_0 the imaginary root ($\mu_0 = -ik$), and, for $m \geq 1$, μ_m the positive real roots in ascending order.

The source strength distribution function of Eq.(2.21), for each mode of oscillation k , is expanded as:

$$f_k(\vec{X}) = \sum_{\ell=0}^{\infty} f_{k\ell}(s) \cos(\ell\Theta) \quad (2.24)$$

where $s(r, z)$ is the distance measured along the body contour in a vertical plane. The coefficients $f_{k\ell}(s)$ are determined by the boundary condition (2.20) applied on s_0 , the equilibrium body contour. This contour is discretized into a finite number N of short segments with $f_{k\ell}(s)$ assumed uniform over each segment. Hence, Eq. (2.20) leads to a series of N matrix equations that can be solved numerically to give the $f_{k\ell}(s_j)$ for each segment j . The various results, presented in the next chapter, were calculated using $15 \leq N \leq 30$.

The body motion amplitudes, ξ_k , are obtained from the equation of motion for a freely floating axisymmetric body:

$$\sum_{k=1}^3 [-\omega^2(m_{ik} + a_{ik}) - i\omega b_{ik} + c_{ik}] \xi_k = F_i^{(e)} \quad \text{for } i = 1, 2, 3 \quad (2.25)$$

where m_{ik} is the mass matrix, c_{ik} the hydrostatic stiffness coefficient and $F_i^{(e)}$ the exciting forces associated with $(\phi_0 + \phi_4)$ and identical to the wave force components for a fixed body. The added mass coefficient, a_{ik} , and the damping coefficient, b_{ik} , are defined in terms of the force components associated with the forced potentials, $F_{ik}^{(f)}$:

$$a_{ik} = \frac{1}{\omega^2} \text{Re}(F_{ik}^{(f)}) \quad (2.26)$$

$$b_{ik} = \frac{1}{\omega} \text{Im}(F_{ik}^{(f)}) \quad (2.27)$$

where $\text{Im}(\)$ indicates the imaginary part. The source strength distribution functions, $f_{k\ell}(s)$, and the body motion amplitudes, ξ_k , had to be computed for each particular floe shape used in the model, and the details are given in Chapter 3.

Once the source strength distribution functions and the amplitudes of motion, for each floe configuration, have been solved for by the method described above (see Isaacson (1982) for more details on the procedure), an expression for the potential Φ at any point in the fluid can be derived. Substituting Eq.(2.22) and Eq.(2.24) into Eq.(2.21), each scattered (forced and diffracted) potential can be written as

$$\phi_k(r, \theta, z) = \frac{1}{4\pi} \int_{S_0} \sum_{\ell=0}^{\infty} f_{k\ell}(s) \cos(\ell\Theta) \left(\sum_{m=0}^{\infty} G_{\ell m} \right) (2 - \delta_{\ell 0}) \cos[\ell(\theta - \Theta)] dS \quad (2.28)$$

Expressing dS as $Rd\Theta ds$, Eq.(2.28) becomes

$$\phi_k(r, \theta, z) = \frac{1}{4\pi} \sum_{\ell=0}^{\infty} \int_{s_0} f_{k\ell}(s) \left(\sum_{m=0}^{\infty} G_{\ell m} \right) (2 - \delta_{\ell 0}) \int_0^{2\pi} \cos(\ell\Theta) \cos[\ell(\theta - \Theta)] d\Theta R ds \quad (2.29)$$

or, integrating with respect to Θ ,

$$\phi_k(r, \theta, z) = \frac{1}{2} \sum_{\ell=0}^{\infty} \int_{s_0} f_{k\ell}(s) \left(\sum_{m=0}^{\infty} G_{\ell m} \right) \cos(\ell\theta) R ds. \quad (2.30)$$

Introducing in Eq.(2.30) the discretization of s_0 previously mentioned, the integral over s_0 is replaced by a finite sum over N segments of length L_j :

$$\phi_k(r, \theta, z) = \frac{1}{2} \sum_{\ell=0}^{\infty} \sum_{j=1}^N f_{k\ell}(s_j) \sum_{m=0}^{\infty} G_{\ell m}(r, z, R_j, Z_j) \cos(\ell\theta) R_j L_j. \quad (2.31)$$

For large r , the Bessel function, $K_\ell(\mu_m r)$, of Eq.(2.22) takes the asymptotic form (e.g. Abramowitz and Stegun, 1965)

$$\lim_{r \rightarrow \infty} K_\ell(\mu_m r) = \sqrt{\frac{\pi}{2\mu_m r}} e^{-\mu_m r} + \mathcal{O}(r^{-\frac{3}{2}}). \quad (2.32)$$

It follows that the potential (2.31), at large distance from the body, is dominated by the $m = 0$ term, the terms for which $m \geq 1$ dying out exponentially (the roots $(\mu_m)_{m \geq 1}$ being real). The far field expression for Eq.(2.31) is then

$$\lim_{r \rightarrow \infty} \phi_k(r, \theta, z) = \frac{1}{2} \sum_{\ell=0}^{\infty} \sum_{j=1}^N f_{k\ell}(s_j) G_{\ell 0}(r, z, R_j, Z_j) \cos(\ell\theta) R_j L_j + \mathcal{O}(r^{-\frac{3}{2}}) \quad (2.33)$$

with

$$G_{\ell 0}(r, z, R_j, Z_j) = \left\{ 2\sqrt{\frac{2\pi}{k}} i C_0 \cosh[k(Z_j + h)] \cosh[k(z + h)] e^{\frac{-i\pi}{4} - \frac{i\pi}{2}\ell} J_\ell(kR_j) \right\} \frac{1}{\sqrt{r}} e^{ikr}$$

and

$$C_0 = \frac{\nu^2 - k^2}{(\nu^2 - k^2)h - \nu}.$$

In Eq.(2.33), the term corresponding to the so-called local waves decay as $r^{-\frac{3}{2}}$. Finally, the surface displacement, for each mode k , at large distance from the floe, is obtained from the velocity potential of Eq.(2.33), using the linearized dynamic boundary condition at $z = 0$

$$\eta_k(r, \theta, t) = \begin{cases} Re\left(-\frac{\xi_k}{g} \frac{\partial \phi_k(r, \theta, 0)}{\partial t} e^{-i\omega t}\right) & \text{for } k = 1 \dots 3 \\ Re\left(-\frac{1}{g} \frac{\partial \phi_k(r, \theta, 0)}{\partial t} e^{-i\omega t}\right) & \text{for } k = 4 \end{cases}$$

or

$$\eta_k(r, \theta, t) \approx Re\left(R \frac{H}{2} D_k(\theta) \frac{1}{\sqrt{r}} e^{i(kr - \omega t)}\right) \quad (2.34)$$

with, for $k = 1 \dots 3$,

$$D_k(\theta) = -\frac{\omega}{g} \frac{\xi_k}{\frac{H}{2}} \sqrt{\frac{2\pi}{k}} C_0 \cosh(kh) e^{\frac{-i\pi}{4}} \sum_{j=1}^N \sum_{\ell=0}^{\infty} f_{k\ell}(s_j) \cosh[k(Z_j + h)] J_\ell(kR_j) e^{-\frac{i\pi}{2}\ell} R_j L_j \cos(\ell\theta)$$

and, for $k=4$,

$$D_k(\theta) = -\frac{\omega}{g} \frac{1}{\frac{H}{2}} \sqrt{\frac{2\pi}{k}} C_0 \cosh(kh) e^{\frac{-i\pi}{4}} \sum_{j=1}^N \sum_{\ell=0}^{\infty} f_{k\ell}(s_j) \cosh[k(Z_j + h)] J_\ell(kR_j) e^{-\frac{i\pi}{2}\ell} R_j L_j \cos(\ell\theta)$$

and R is a factor introduced to conserve energy, as explained below. In Eq.(2.34), the surface displacements describe cylindrical waves radiating away from the floe with an amplitude of $(\frac{RH}{2} D_k(\theta) \frac{1}{\sqrt{r}})$ as a function of the direction of propagation θ and distance r .

In Appendix A, the infinite depth expression for the scattering amplitude, $D_k(\theta)$, is derived. In the case of large depth, the Green's function of Eq.(2.22) takes a different form, first obtained by John (1950). As one would expect, the finite depth $D_k(\theta)$ of Eq. (2.34) tends to the one obtained in deep water (Eq.(A.12)) by letting $kh \rightarrow \infty$. The details need not be given here, however.

The scattering amplitude of Eq.(2.34) (or Eq.(A.12)) gives the amplitude and the directional properties of the four scattered waves. Also, the total scattering cross-section of a floe, defined as

$$\sigma_s = \sum_{k=1}^4 \int_0^{2\pi} |D_k(\theta)|^2 d\theta, \quad (2.35)$$

determines the fraction of the incident energy being scattered through diffraction and the forced motions. As will be seen in the next Section, the ability of a

particular ice field to disperse energy over all directions strongly depends on the effectiveness of these two processes.

2.3.2 Scattering by the entire ice field

So far, the problem of a plane wave incident on a single floe has been investigated. However, the required wave field is determined by the scattering of each spectral component on the whole array of randomly distributed floes. To obtain this wave field, a statistical average of the scattered waves is done over the entire domain using the theory of wave propagation and scattering in random media.

Since the floes are randomly distributed, the scattered field is not constant and its amplitude and phase fluctuate in a random manner. The total wave field can be divided into the average field, $\langle \eta(\vec{r}, t) \rangle$, also called coherent field, and the fluctuating field, $\eta_f(\vec{r}, t)$, called the incoherent field

$$\eta(\vec{r}, t) = \langle \eta(\vec{r}, t) \rangle + \eta_f(\vec{r}, t). \quad (2.36)$$

At a point $\vec{r}_a$ on the ocean surface, the surface displacement is the sum of the incident wave and the contributions from all the scatterers located at $\vec{r}_s$. The spatially averaged (ergodically equivalent to the ensemble averaged) wave at $\vec{r}_a$, $\langle \eta(\vec{r}_a, t) \rangle$, is given by the Foldy-Twersky integral equation (e.g. Ishimaru, 1978)

$$\langle \eta(\vec{r}_a, t) \rangle = \eta_0(\vec{r}_a, t) + \iint u_s^a \langle \eta(\vec{r}_s, t) \rangle \rho(\vec{r}_s) d\vec{r}_s \quad (2.37)$$

where $\eta_0(\vec{r}_a, t)$ is the incident wave at $\vec{r}_a$ (determined by the potential of Eq.(2.14)), $u_s^a \langle \eta(\vec{r}_s, t) \rangle$ a symbolic notation to indicate the wave at $\vec{r}_a$ due to the scattering of the coherent wave, $\langle \eta(\vec{r}_s, t) \rangle$, incident on a scatterer located at $\vec{r}_s$, and $\rho(\vec{r}_s)$

Figure 2.4: Simple hexagonal model of a homogeneous ice cover. D_{av} is the average distance between floes of radius a .

the number of floes per unit area, also called “number” density.

Because of the finite size of the floes and to prevent divergence of the integral in Eq.(2.37), a decay of the number density with distance from $\vec{r}_a$ must be included to account for shading of remote scatterers by nearby floes. Without shading, the homogeneous field of ice floes can be represented by a simple hexagonal model with its sides equal to the average distance, D_{av} , between floes (Fig. (2.4)). The polygon includes a total of three floes and covers a surface $S = 3\sqrt{3}D_{av}^2/2$. Thus, without shading, the uniform number density is

$$\rho_0 = \frac{3}{S} = \frac{2}{\sqrt{3}D_{av}^2}. \quad (2.38)$$

In this case, the fraction of the area covered by ice, mentioned in Section 2.1, can be written as

$$f_i = \frac{3(\pi a^2)}{S} = \frac{2\pi a^2}{\sqrt{3}D_{av}^2}. \quad (2.39)$$

The shading is introduced by considering the effect of a ring of average radius r

and of small width $dr = 2a$. The fraction of the perimeter of this ring taken by ice floes is

$$f' = \frac{\rho_0(2\pi r 2a)2a}{2\pi r} = \frac{8a^2}{\sqrt{3}D_{av}^2} \quad (2.40)$$

and is independent of r . The transmittivity of each ring of width $2a$, assuming opaque floes, is then

$$T_r = (1 - f') = \left(1 - \frac{8a^2}{\sqrt{3}D_{av}^2}\right). \quad (2.41)$$

On a distance $r = |\vec{r}_s - \vec{r}_a|$ from the centre of a floe located at $\vec{r}_a$, waves scattered at $\vec{r}_s$ will travel through $(r - a)/2a$ rings before reaching $\vec{r}_a$. Therefore, under the single scattering approximation where the amplitude of a wave scattered more than once is assumed to be negligible, the number of floes per unit area effectively radiating waves to $\vec{r}_a$, without being shaded by other floes, is given by

$$\rho_e(r) = \rho_0(T_r)^{\frac{(r-a)}{2a}} = \frac{2}{\sqrt{3}D_{av}^2 \sqrt{1 - \frac{8a^2}{\sqrt{3}D_{av}^2}}} \left(1 - \frac{8a^2}{\sqrt{3}D_{av}^2}\right)^{\frac{r}{2a}} \quad (2.42)$$

or, using Eq.(2.39),

$$\rho_e(r) = \frac{f_i}{\pi a^2 \sqrt{1 - \frac{4f_i}{\pi}}} \left(1 - \frac{4f_i}{\pi}\right)^{\frac{r}{2a}}. \quad (2.43)$$

This “effective” number density, $\rho_e(r)$, decreases with distance at a rate which is a function of the degree of ice cover, f_i , with the importance of the contributions from nearby floes in the integral of Eq.(2.37) increasing with f_i (Fig.(2.5)).

To obtain an expression for the average wave field, Eq.(2.37) is solved by first neglecting all the multiple scattering with

$$u_s^a < \eta(\vec{r}_s, t) > \cong u_s^a \eta_0(\vec{r}_s, t). \quad (2.44)$$

Figure 2.5: Effective number density, $\rho_e(r)$, for different degree of ice cover, f_i . In this example, the floes have a radius $a = 10\text{m}$.

Secondly, since the waves travel through a sparse distribution of scatterers, the far field approximation obtained in Eq.(2.34) is used to give

$$u_s^a \eta_0(\vec{r}_s, t) \cong \text{Re} \left(R \frac{H}{2} \sum_{k=1}^4 D_k(\theta) \frac{1}{\sqrt{r}} e^{i(kr - \omega t)} \right) \quad (2.45)$$

where θ is the angle between the incident direction and $\vec{r} = \vec{r}_a - \vec{r}_s$. Eq.(2.45) is valid only when $\langle \eta(\vec{r}_s, t) \rangle$ can be approximated by $\eta_0(\vec{r}_s, t)$, and when $\vec{r}_a$ is in the far zone of the floe at $\vec{r}_s$. These two assumptions may seem restrictive but, in the present model of low ice concentration, it appears reasonable to assume that the amplitude of a wave scattered more than once is negligible compared to the ones included in the simple scattering theory, and to neglect the local waves. About this last assumption, it is interesting to examine the relative amplitude of the neglected waves. In Eq.(2.33), or Eq.(A.10), the local waves decay as $(\frac{1}{\sqrt{r}} r^{-1})$. As previously mentioned, the importance of the contributions from floes located at small r increases with f_i . Then, referring to the value of the effective number density, $\rho_e(r)$, with distance r (Fig.(2.5)), in the worst situation where $f_i = 0.25$, one has to go as far as $r \cong 21m$ to have the equivalent of one complete floe effectively radiating waves;

$$\int_0^{2\pi} \int_0^R \rho_e(r) r \, dr \, d\theta = 2\pi \int_0^R \rho_e(r) r \, dr \approx 1 \quad \text{for } R = 21m.$$

At this distance, the local wave amplitude is less than 5% ($\frac{1}{21}$) of the far field wave amplitude. For larger ice floes ($a > 10m$), R has an even larger values (see Eq.(2.43)), resulting in a smaller relative amplitude ($< 5\%$) of the neglected local waves. Thus, in this model, for an ice concentration $f_i \leq 0.25$, the contribution to the scattered energy from the waves neglected by the far field approximation is small and can be safely neglected.

To solve Eq.(2.37) it is more convenient to use the cartesian coordinates (x, y) with x in the direction of the incident wave. Since the geometry of the ice field and the incident wave are independent of y , the coherent field must also be independent of y and should behave as plane waves propagating along the x axis. Waves scattered in other directions are then included in the incoherent field. Using the approximation (2.45), Eq. (2.37) becomes

$$\langle \eta(x_a, t) \rangle = \eta_0(x_a, t) + Re(R \frac{H}{2} [\underbrace{\int_{-\infty}^{+\infty} D(\theta) \frac{1}{\sqrt{r}} e^{ikr} \rho_e(r) dy_s dx_s}_{\Pi}] e^{-i\omega t}) \quad (2.46)$$

with

$$r = \sqrt{(x_a - x_s)^2 + (y_a - y_s)^2}$$

and

$$D(\theta) = \sum_{k=1}^4 D_k(\theta).$$

Then, the integral over y_s of Eq.(2.46) is evaluated by the method of stationary phase. This integral can be written as

$$\Pi = \int_{-\infty}^{+\infty} A(x_s, y_s) e^{if(x_s, y_s)} dy_s \quad (2.47)$$

with

$$A(x_s, y_s) = \frac{D(\theta)}{\sqrt{\sqrt{(x_a - x_s)^2 + (y_a - y_s)^2}}} \rho_e(\sqrt{(x_a - x_s)^2 + (y_a - y_s)^2})$$

and

$$f(x_s, y_s) = k\sqrt{(x_a - x_s)^2 + (y_a - y_s)^2}.$$

The stationary phase point, y_0 , is given by

$$\frac{\partial}{\partial y_s} \sqrt{(x_a - x_s)^2 + (y_a - y_s)^2} = 0 \quad (2.48)$$

which yields $y_0 = y_a$. From the general form of the solution obtained by the method of stationary phase (e.g. Ishimaru, 1978, Appendix 14B), Eq. (2.47) takes the form

$$\Pi = A(x_s, y_a) e^{if(x_s, y_a)} \frac{\sqrt{2\pi} e^{i\frac{\pi}{4}}}{\sqrt{\Delta}} \quad (2.49)$$

where

$$\Delta = \left. \frac{\partial^2 f(x_s, y_s)}{\partial y_s^2} \right|_{y_s=y_a} = \frac{k}{(x_a - x_s)}.$$

Substituting the appropriate values for $A(x_s, y_a)$, Δ and $f(x_s, y_a)$, Eq.(2.49) becomes

$$\Pi = \begin{cases} \sqrt{\frac{2\pi}{k}} e^{i\frac{\pi}{4}} D(0) e^{ik(x_a - x_s)} \rho_e(x_a - x_s) & \text{if } x_s < x_a \\ \sqrt{\frac{2\pi}{k}} e^{i\frac{\pi}{4}} D(\pi) e^{ik(x_s - x_a)} \rho_e(x_s - x_a) & \text{if } x_a < x_s. \end{cases} \quad (2.50)$$

Using Eq.(2.50), the coherent field of Eq.(2.46) can now be written as

$$\begin{aligned} \langle \eta(x_a, t) \rangle &= \eta_0(x_a, t) + Re(R \frac{H}{2} [\int_{-\infty}^{x_a} \sqrt{\frac{2\pi}{k}} e^{i\frac{\pi}{4}} D(0) e^{ik(x_a - x_s)} \rho_e(x_a - x_s) dx_s \\ &\quad + \int_{x_a}^{\infty} \sqrt{\frac{2\pi}{k}} e^{i\frac{\pi}{4}} D(\pi) e^{ik(x_s - x_a)} \rho_e(x_s - x_a) dx_s] e^{-i\omega t}). \end{aligned} \quad (2.51)$$

or

$$\begin{aligned} \langle \eta(x_a, t) \rangle &= \eta_0(x_a, t) + Re(R \frac{H}{2} \sqrt{\frac{2\pi}{k}} e^{i\frac{\pi}{4}} [\{ \int_{-\infty}^{x_a} e^{-ikx_s} \rho_e(x_a - x_s) dx_s \} D(0) e^{ikx_a} \\ &\quad + \{ \int_{x_a}^{\infty} e^{ikx_s} \rho_e(x_s - x_a) dx_s \} D(\pi) e^{-ikx_a}] e^{-i\omega t}). \end{aligned} \quad (2.52)$$

This last expression for the coherent field, valid for any x_a , can be simplified by choosing a centred observation point, with $x_a = 0$. Using the effective number density of Eq.(2.43), Eq.(2.52) becomes

$$\langle \eta(x_a = 0, t) \rangle = \eta_0(x_a = 0, t) + \text{Re}\left(R \frac{H}{2} \alpha_c [D(0)e^{ik(0)} + D(\pi)e^{-ik(0)}] e^{-i\omega t}\right) \quad (2.53)$$

with the “coherent scattering” coefficient, α_c , given as

$$\alpha_c = \sqrt{\frac{2\pi}{k}} e^{\frac{i\pi}{4}} \frac{2}{\sqrt{3}D_{av}^2 \sqrt{1 - \frac{8a^2}{\sqrt{3}D_{av}^2}}} \int_0^\infty e^{ikx_s} \left(1 - \frac{8a^2}{\sqrt{3}D_{av}^2}\right)^{\frac{x_s}{2a}} dx_s.$$

In Eq.(2.53), the resultant average field is made up of three components: the incident wave, $\eta_0(x_a, t)$, to which has been added a wave travelling in the same direction and of amplitude proportional to $D(0)$, and a second one travelling in the opposite direction, of amplitude proportional to the backward scattering amplitude, $D(\pi)$. In other words, the only scattered waves interfering in a “coherent” way to be a component of the average field are the ones generated in the $\pm$ incident direction. The square of the magnitude of the coherent scattering coefficient, $|\alpha_c|^2$, which determines the amount of energy associated with these waves, decreases with the parameter ka to reach very small values of $\mathcal{O}(10^{-5})m^{-1}$ (Fig.(2.6)).

Because of the randomness of the ice floe distribution, the phases of waves scattered by different floes have no correlation among themselves. Therefore, their contributions to the total energy of the scattered wave field can be simply added regardless of their relative phases. The total intensity is then the average of the square of the magnitude of the total field, and is the sum of the coherent intensity, $\langle |\eta(x_a, t)|^2 \rangle$, associated with the coherent field, and the incoherent intensity, $\langle |\eta_f(\vec{r}_a, t)|^2 \rangle$, the energy of the fluctuating field

Figure 2.6: The square of the modulus of the coherent scattering coefficient, $|\alpha_c|^2$, as a function of the parameter ka . Results are given for floes of radius $a = 10$ m and ice concentration $f_i = 0.1$ and 0.2 .

$$\begin{aligned}
\langle |\eta(\vec{r}_a, t)|^2 \rangle &= \langle |\langle \eta(x_a, t) \rangle + \eta_f(\vec{r}_a, t)|^2 \rangle \\
&= \langle |\eta(x_a, t)|^2 \rangle + \langle |\eta_f(\vec{r}_a, t)|^2 \rangle. \quad (2.54)
\end{aligned}$$

Accordingly, the energy distribution from the scattering of a single component of the incident spectrum is obtained from Twersky's integral equation for the total intensity, similar to the Foldy-Twersky integral equation (2.37) for the coherent field,

$$\langle |\eta(\vec{r}_a, t)|^2 \rangle = \langle |\eta(x_a, t)|^2 \rangle + \iint |v_s^a|^2 \langle |\eta(\vec{r}_s, t)|^2 \rangle \rho_e(\vec{r}_s) d\vec{r}_s \quad (2.55)$$

where $|v_s^a|^2 \langle |\eta(\vec{r}_s, t)|^2 \rangle$ represents the scattered wave energy at $\vec{r}_a$ due to a scatterer located at $\vec{r}_s$. The incoherent intensity is represented by the integral on the right hand side of this equation. From the expression for the coherent field of Eq.(2.53), and using the far field approximation in the case of single scattering (Eq.(2.45)) to evaluate the energy contribution from the scattered waves, Eq.(2.55) can be written as

$$\langle |\eta(x_a, t)|^2 \rangle = (R \frac{H}{2})^2 [1 + |\alpha_c D(0)|^2 + |\alpha_c D(\pi)|^2 + \iint \frac{|D(\theta)|^2}{r_s} \rho_e(\vec{r}_s) d\vec{r}_s]. \quad (2.56)$$

Expressing $d\vec{r}_s$ as $r_s dr_s d\theta$, Eq.(2.56) becomes

$$\langle |\eta(x_a, t)|^2 \rangle = (R \frac{H}{2})^2 [1 + |\alpha_c D(0)|^2 + |\alpha_c D(\pi)|^2 + \beta \int_0^{2\pi} |D(\theta)|^2 d\theta] \quad (2.57)$$

where

$$\beta = \lim_{r \rightarrow \infty} \int_0^r \rho_e(r_s) dr_s. \quad (2.58)$$

Allowing the energy contributions to come from scatterers located at distances of up to infinity (as in Eq.(2.58)), β tends to its asymptotic value of one floe per

Figure 2.7: Energy factor, $\beta = \int_0^{R_{max}} \rho_e(r) dr$, as a function of the distance of integration R_{max} . Results are given for floes of radius $a = 10m$ and ice concentration, f_i , of 0.05, 0.1 and 0.2.

diameter, $1/2a$, whatever the ice concentration, f_i , is. If the scattering process is limited, in time for example, the upper limit of the integral in Eq.(2.58) takes a finite value, R_{max} , which is the maximum distance travelled by the waves over a certain time interval Δt . In this situation, the influence of the degree of ice cover on β will be noticeable, β increasing with f_i (Fig.(2.7)).

In Eq.(2.57), the incident wave energy, after scattering, is partitioned into: 1- a component identical to the unscattered incident wave, but of reduced (by the factor R) amplitude, $RH/2$; 2- energy associated with the two waves of the coherent field travelling in the $\pm$ incident direction; 3- directional scattering contributions with intensity apportioned as $|D(\theta)|^2$. When a spectrum, $F(f, \theta)$, is in the presence of

an ice field, the energy of each spectral component is spread out over all directions, according to Eq.(2.57). The scattered spectrum is then obtained by summing, for each component, the contributions coming from waves of same frequency and incident from all directions. Thus, for each frequency f_n , the spectrum after scattering, $F(f_n, \theta)$, is obtained by the product of the incident spectrum, $F^*(f_n, \theta)$, and a transfer function $[T]_{f_n}$

$$F(f_n, \theta) = F^*(f_n, \theta) [T]_{f_n} \quad (2.59)$$

or, in tensor notation,

$$F(f_n, \theta_i) = F^*(f_n, \theta_j) (T_{ij})_{f_n}. \quad (2.60)$$

The matrix $[T]_{f_n}$ is symmetric with each element $(T_{ij})_{f_n}$ of the form

$$(T_{ij})_{f_n} = R^2 \{ \beta |D(\theta_{ij})|^2 \Delta\theta + \delta(\theta_{ij}) (1 + |\alpha_c D(0)|^2) + \delta(\pi - \theta_{ij}) |\alpha_c D(\pi)|^2 \}, \quad (2.61)$$

where $\theta_{ij} = |\theta_i - \theta_j|$, $\Delta\theta$ is the angular interval of the spectrum, $\delta()$ the Dirac function, and the parameters β , $|D(\theta_{ij})|^2$ and α are evaluated for the frequency $f = f_n$.

2.3.3 Energy dissipation by the ice

In Section 2.2, a dissipation term, S_{ds} , has been defined for the ice-free situation where energy is lost mainly through wave breaking. When an ice cover is present, some energy is also lost within the ice field by various dissipative processes (ice deformation, wave breaking on the floes, etc...). By analogy to Eq.(2.35), where a scattering cross-section, σ_s , was defined, an absorption cross-section, σ_a , is introduced here to account for this extra dissipation.

Figure 2.8: Percentage of sea surface covered by ice along laser track, June 6, 1972 (from Wadhams, 1975).

The total energy associated with an incident wave of amplitude $RH/2$ travelling through scatterers on a distance x , can be written in terms of this cross-section (e.g. Ishimaru, 1978):

$$\langle |\eta(x, t)|^2 \rangle = \left(R \frac{H}{2}\right)^2 e^{-\rho_o \sigma_a x} \quad (2.62)$$

where ρ_o is the number density without shading of Eq.(2.38). In order to evaluate the parameter σ_a , data collected by P. Wadhams (1975) off the east coast of Newfoundland have been examined. He measured, along a 90 km long line, the “apparent” energy density for different frequency bands of swell entering the MIZ. Details of the degree of ice cover along the transect are given in Fig.(2.8) and the results are plotted in Fig.(2.9).

Figure 2.9: Energy density plotted against distance of penetration inside the ice field. Central period (s) for the bands: band 1, > 11.6; band 2, 11.6; band 3, 10.53; band 4, 9.35; band 5, 8.50; band 6, 7.85; band 7, 7.34. (from Wadhams, 1975)

The transformation of the measured wavenumber spectrum into a frequency spectrum depends on the directional properties of the waves. In this case, the measurements have been made along the major direction of the swell vector which was assumed to have no directional spread. If the energy of the swell is noticeably spread out over direction by the floes, the computed energy at a certain frequency becomes “contaminated” by shorter waves travelling at an angle relative to the incident direction. Therefore, the spectral densities plotted in Fig.(2.9) are associated with the incident, non scattered swell alone only in regions where the scattering by the ice floes is negligible, the incident swell remaining quasi-unidirectional, and can be used to estimate σ_a through Eq.(2.62).

In the outer 20 km, where a high fraction of the surface is covered by smaller floes, the assumption of unidirectionality appears reasonable, the parameter ka remaining small (see Section 3.2). Thus, in this region, the measured loss of energy can be mainly attributed to the absorption cross-section, and Eq.(2.62) should well describe, for each spectral component, the energy attenuation with distance. However, as one goes deeper into the ice cover, the proportion of larger floes gradually builds up and the scattering increases with ka . This causes an apparent increase of energy for a given band due to loss of unidirectionality at higher frequency (as seen in Fig. (2.8) for most bands at penetrations in excess of 20km). Table (2.1) gives the computed absorption cross-sections for the first 20 km where $f_i \approx 0.15$ and $a \approx 7\text{m}$. The value of σ_a rapidly increases to reach a maximum value of about 0.15m. The data being limited, these results constitute a crude estimation of the absorption cross-section but provide a reasonable quantitative description that will be used in the model.

band	T (s)	ka	$\sigma_a(\text{m})$
1	?	?	0.070
2	11.6	0.21	0.012
3	10.53	0.25	0.033
4	9.35	0.32	0.088
5	8.50	0.39	0.148
6	7.85	0.46	0.149
7	7.34	0.53	0.146

Table 2.1: Absorption cross-sections computed for the energy decay of Fig.(2.9). Each band has a central period T .

2.4 Numerical integration procedure

Due to the complexity of the different source functions in the energy balance equation (including the additional S_{ice} term), an important simplification is introduced by assuming the medium unbounded and uniform. This reduces the problem to the evolution of purely time-limited waves owing to the elimination of the advective term, $\vec{C}_g \cdot \vec{\nabla} F$, of Eq.(2.6) which now takes the form

$$\frac{dF(f, \theta; t)}{dt} = (S_{in} + S_{ds})(1 - f_i) + S_{nl} + S_{ice} \quad (2.63)$$

The numerical integration of Eq.(2.63) proceeds following a simple first-order forward difference method. At first, time steps are set to very small values ($\Delta t \approx 10\text{s}$) to account for the rapid initial change in spectral shape (reported in Section 3.3.2). Once this process is stabilized, time steps are dynamically adjusted, starting with $(\Delta t)_n = t_{n-1}/5$ and reducing it by a factor 2 in case of a too large $\Delta F/F$. A simple first-order integration scheme is considered adequate since Δt is determined by the rapidly responding high frequency region of the spectrum, leading to very small ΔF in the energetically relevant region of the spectral peak (Komen et al., 1984).

Furthermore, the choice of a Δt small enough to keep the solution stable insures that the method of integration is consistent and convergent (e.g. Gear, 1971). The spectrum is specified for frequencies distributed according to $f_n = f_0(1.1)^{n-1}$, with $f_0 = 0.07\text{Hz}$ being the lowest frequency, up to $2.5 * f_{peak}$ (where f_{peak} represents the frequency at the maximum of the spectrum). Beyond this value, a high frequency f^{-5} decay is assumed, adjusted independently for each direction. There is an ongoing debate on the exact nature of the high frequency spectral region, with strong evidence for a f^{-4} dependence (e.g. Donelan, 1985; Phillips, 1985). But, the choice of a f^{-5} shape in the short wave region of the spectrum is not, here, considered determining. The angular resolution of the spectrum is of 30° .

The initial wave field specified is a JONSWAP (empirically derived) spectrum (Hasselmann et al., 1973) of the form

$$F(f, \theta) = \frac{\alpha g^2}{(2\pi)^4 f^5} \exp\left[-\frac{5}{4}\left(\frac{f}{f_{peak}}\right)^{-4}\right] \gamma^{\exp\left[-\frac{(f-f_{peak})^2}{2\sigma^2 f_{peak}^2}\right]} G(f, \theta) \quad (2.64)$$

where

$$\sigma = \begin{cases} 0.07 & \text{for } f \leq f_{peak} \\ 0.09 & \text{for } f > f_{peak}. \end{cases}$$

Here the parameter α is equivalent to the usual Phillips's constant but with a time dependence, $\gamma = 3.3$ is the peak enhancement factor and $G(f, \theta)$ a directional spreading function. For a certain initial time $t_0(s)$, the values of α and f_{peak} are obtained from the results of the parametric model of Hasselmann et al. (1976):

$$f_{peak} = \frac{16.8g}{U_{10}} \left(\frac{gt_0}{U_{10}}\right)^{-\frac{3}{7}} \quad (2.65)$$

and

$$\alpha = 0.033 \left(\frac{f_{peak} U_{10}}{g} \right)^{\frac{2}{3}} \quad (2.66)$$

where U_{10} is the wind at 10m.

The spreading function, in Eq.(2.64), is of the cosine-power type and takes the form used by Hasselmann and Hasselmann (1985b)

$$G(f, \theta) = I(p) \cos^{2p} \left(\frac{\theta}{2} \right) \quad (2.67)$$

where

$$p = 10^{0.99} \left(\frac{f}{f_{peak}} \right)^{\kappa}$$

and

$$\kappa = \begin{cases} 4.06 & \text{for } f < f_{peak} \\ -2.34 & \text{for } f \geq f_{peak}. \end{cases}$$

The normalization factor is given by $I(p) = \frac{2^{2p-1}}{\pi} \frac{\Gamma^2(p+1)}{\Gamma(2p+1)}$ with Γ the Gamma function. The angular distribution of Eq.(2.67) is narrower near the peak frequency, in agreement with experiments (e.g. Hasselmann et al., 1980).

At each time step, S_{in} , S_{ds} and S_{nl} are computed to obtain a new spectrum which is then modified, for each frequency f_n , by the scattering transfer function of Eq.(2.59):

$$F(f_n, \theta; t + \Delta t) = [F(f_n, \theta; t) + ((S_{in} + S_{ds})(1 - f_i) + S_{nl})\Delta t][T]_{f_n} \quad (2.68)$$

where, in $[T]_{f_n}$, the value of β is determined by setting the upper limit of integration $R_{max} = C_g \Delta t$ in Eq.(2.58).

Figure 2.10: Fraction of energy lost, f_d , for a time step $\Delta t = 5\text{min}$ and ice concentration, $f_i = 0.1, 0.2$.

The energy factor R , introduced in Eq.(2.34), is obtained, for each frequency, from energy conservation applied to Eq.(2.57), taking the loss of energy through the absorption cross-section, σ_a , into account:

$$R = (1 + |\alpha_c D(0)|^2 + |\alpha_c D(\pi)|^2 + \beta \int_0^{2\pi} |D(\theta)|^2 d\theta + f_d)^{-\frac{1}{2}} \quad (2.69)$$

where f_d , the fraction of energy lost in the scattering process, is given by

$$f_d = (e^{\rho_a \sigma_a C_g \Delta t} - 1). \quad (2.70)$$

Fig. (2.10) presents the variation of the fraction f_d with ka for a time step $\Delta t = 5\text{ min}$ and using absorption cross-sections estimated from values given in Table (2.1). The energy loss rapidly increases to reach a maximum near $ka = 0.5$ and then gradually decreases as the group velocity, C_g , diminishes. The integration procedure is repeated for a series of time steps until dominant trends can be clearly

established.

Chapter 3

Results and discussion

Now that the theoretical development of the model has been given in detail, various results are presented. First, the wave-induced motions of the ice are examined for floes of different dimensions and shapes (Section 3.1). In Section 3.2, the scattered (diffracted and forced) waves are described. Section 3.3 presents the evolution of wave spectra obtained by numerical integration of the energy balance equation under different conditions (including the ice-free situation). Finally, the initial wave generation problem is examined in Section 3.4.

3.1 Floe response amplitudes

The wave-induced motion of a floating body is usually presented in the form of nondimensional response amplitude operators (RAO) defined, here, as $\xi_k/\frac{H}{2}$ for heave and surge, and $L\xi_3/180H$ for pitch (with ξ_3 in degrees). The response varies with the incident wavelength, the water depth, as well as the geometry and dimensions of the floe. In the latter, the position of the centre of gravity and the radius of gyration (in pitch) are important factors in the determination of these RAO's. In order to compute these two parameters, a simplified density structure of the floe was assumed. The density of sea ice is not uniform through its thickness

and varies due to desalination as the floe ages. The floes present in the region described by the model are assumed to be composed of relatively young ice and the density is assumed uniform. Furthermore, due the low thickness of the floes ($\leq 4\text{m}$), the freeboard is expected to be small ($\leq 0.55\text{m}$) (e.g. Tucker et al., 1987). The floes are thus considered, in the estimation of the position of the centre of gravity and the radius of gyration, to be completely submerged, with a uniform ice density equal to 1.

In Fig.(3.1), the computed RAO's are given as a function of ka for a floe of radius $a = 10\text{m}$ and draft $d = 3\text{m}$. As indicated in the figure, the effect of water depth on the response is rather minor except for the low frequency surge response. When the water depth changes from 30m to 100m, the heave and pitch peaks slightly vary in position and magnitude, but the response curves, for these two modes of motion, are almost unchanged. For the surge motion, the two curves are also closely related except for low values of ka for which the shallow water values are much larger. This can be explained by looking at the water particle path changes with depth. When the diameter of the floe is small compared to the wavelength, the floe tends to behave essentially as a fluid particle (e.g. Lever et al., 1984). For deep water waves, the water particle travels along closed circular orbits of radius $H/2$, resulting in a surge RAO of 1. As the water depth to wavelength ratio decreases, the orbits become flat ellipses and the ratio $\xi_1/\frac{H}{2}$ increases.

The typical heave response is shown in Fig. (3.2) where the RAO's are given for three floes of different thickness. For all three floes, the response is perfect for long waves ($ka \rightarrow 0$) and show very small movement in the short wave region ($ka \gg 1$). In the intermediate range of ka , the floe geometry is critical in determining the importance of a resonant peak, with a maximum at ka slightly higher than 1. The

Figure 3.1: Response amplitude operators (RAO) for a floe of radius $a = 10\text{m}$ and draft $d = 3\text{m}$. The RAO's are given for a depth of 30m (solid lines) and a depth of 100m (dashed lines).

Figure 3.2: Response amplitude operators (RAO) in the heave motion for floes of radius $a = 10\text{m}$ and draft $d = 0.5, 2, 3\text{m}$, in a depth of 30m .

Figure 3.3: Response amplitude operators (RAO) in the surge motion for floes of radius $a = 10\text{m}$ and draft $d = 0.5, 2, 3\text{m}$, in a depth of 30m .

peak, which is absent for the thin floe, increases in magnitude, becomes narrower and slowly shifts to lower values of ka , as the floe thickens.

As in the heave motion, the surge RAO indicates a perfect response (with the $\text{RAO} \neq 1$ because of scaling, as explained above) for long waves and very small displacement for short waves (Fig. (3.3)). For intermediate values of ka , the response is generally larger for thinner floes which are moved by faster upper layer flow. Also, when the radius-to-draft ratio decreases, a peak and a minimum develop. This feature is attributed in part to coupling between the pitch and surge motions for which a 180° phase shift occurs simultaneously with the appearance of the peak (e.g. Wehausen, 1971).

Again, the pitch response goes from near perfect response, following the wave

Figure 3.4: Response amplitude operators (RAO) in the pitch motion for floes of radius $a = 10\text{m}$ and draft $d = 0.5, 2, 3\text{m}$, in a depth of 30m .

slope, to negligible motion as ka increases (Fig. (3.4)). As in the heave motion (but for larger ka), a resonant peak appears when the thickness of the floe increases. When the radius-to-draft ratio decreases, it narrows and its magnitude becomes more important, with its position shifting to lower ka values. No attempt has been made to include viscous pitch damping in the simulation. Its effect would be to decrease the pitch and surge resonant peaks but, because the described floes have a rather high radius-to-draft ratio, the results predict small resonant peaks and, therefore, the viscous damping should remain minor.

Finally, the effect of the presence of a keel is examined. Real floes often present deformation features such as ridges and keels, due to intense wave activity or internal stress in the ice pack. The response of a cylindrical floe, to which has been

Figure 3.5: Response amplitude operators of a cylindrical floe with (dashed lines) and without (solid lines) a keel. The floe has a radius $a = 10\text{m}$ and draft $d = 5\text{m}$ with a keel of 3m in depth (water depth of 30m.).

appended a conical keel of 3m in depth, is presented in Fig. (3.5). Although the keel modifies the motion by creating the effect of a slight increase in thickness, the overall characteristics of the response curves remain unchanged.

3.2 Scattering amplitudes and cross-sections

As seen in Section 2.3.2, the nature of the scattering amplitude, $D_k(\theta)$, determines the ability of a given icefield to disperse incident wave energy over all directions. For the three forced waves ($k=1,2,3$), the summation in ℓ of the general expression for $D_k(\theta)$ (Eq. (2.34) or (A.12)) leads to one single non zero term: the $\ell = 0$ term for the heave motion corresponding to isotropic (uniform in all directions) waves,

Figure 3.6: Response amplitude operators, $(RAO)_k$, and scattering cross-sections, Q_k , for the three modes of motion (the thick lines are for the cross-sections). The floe has a radius $a = 10m$ and draft $d = 2m$ in a water depth $h=30m$.

and the $\ell = 1$ term for surge and pitch with a simple $\cos \theta$ dependence of the scattered waves. Also, the magnitude of the scattering amplitude, for each mode, is a combination of the dependence on the response amplitude operator, and of an increase, with the incident wavenumber, of the efficiency of the forced motions to generate waves. Fig. (3.6) gives the RAO and the scattering cross-section, $Q_k = \int_0^{2\pi} |D_k(\theta)|^2 d\theta$, for a typical floe in surge, heave and pitch. In the low ka region, where the floe tends to follow the water particle motion, there is very little energy associated with the forced waves. For intermediate values of ka , a marked increase of energy, accentuated by the presence of a resonant peak in heave and pitch, corresponds to scattered waves of considerable amplitude. Finally, in the

Figure 3.7: Pitch scattering cross-section, Q_3 , for floes of radius $a = 10\text{m}$ and draft $d = 0.5, 2, 3\text{m}$ in a water depth $h=30\text{m}$.

short wave region, the forced waves are of small height due to the vanishing forced motions. When the radius-to-draft ratio diminishes, the increased narrowness and the shift of the peak towards lower ka values result in an increasingly narrow peak for the cross-section, Q , for which the amplitude may even decrease (Fig. (3.7)). Although the heave and pitch RAO peaks are more developed for thicker floes, the wavenumber dependence of the cross-section minimizes their effect on the scattering ability of the floe.

For the diffraction scattering amplitude, $D_4(\theta)$, the summation in ℓ has to be extended to a few terms ($\ell \approx 12$) such that the omitted terms do not contribute noticeably to the results. This produces a more complex directional distribution of the diffracted energy (Fig. (3.8)). At the long wave limit, the small amplitude

Figure 3.8: Diffraction differential scattering cross-section, $|D_4(\theta)|^2$, for $ka = 0.5, 3.75$. The floe has a radius $a = 10\text{m}$ and draft $d = 2\text{m}$ in a water depth $h=30\text{m}$.

diffracted wave is almost perfectly isotropic. When ka increases, the outgoing wave separates into two parts: the shadow-forming wave interfering with the incident wave to reduce the intensity behind the floe, and the rest radiating out in other directions to form the reflected wave. For very large ka , the asymptotic value of the scattering cross-section (also called the effective width) is that for a circular cylinder, namely (e.g. Morse and Feshbach, 1953: p.1381)

$$\lim_{ka \rightarrow \infty} Q_4 = 4a. \quad (3.1)$$

For $ka \gg 1$, the object casts a geometrical shadow so that no energy is lost: half of the diffracted wave must cancel the incident wave for a width $2a$ behind the cylinder and the other half must be the reflected wave, with an effective width of $2a$. A convenient way to compute the diffraction cross-section, independent of the angular resolution of the model, is to use the Forward-Scattering theorem which, in this case, takes the form (Miles, 1971)

$$Q_4 = -\sqrt{\frac{8\pi}{k}} \operatorname{Re}(e^{i\pi/4} D_4(0)). \quad (3.2)$$

The forward-scattering ratio, FSR , defined as

$$FSR = \frac{2\pi |D_4(0)|^2}{Q_4}, \quad (3.3)$$

reduces to unity for isotropic scattering and increases with ka as the shadow-forming energy concentrates into the incident direction (Fig. (3.9)).

The total scattering cross-section, $\sigma_s = \sum_{k=1}^4 Q_k$, from which has been subtracted the incident direction contribution, $|D(0)|^2 \Delta\theta$, measures the efficiency of the ice, for a given ka , to scatter energy away from the incident direction. By

Figure 3.9: Forward-scattering ratio, FSR , for floes of radius $a = 10\text{m}$ and draft $d = 0.5, 2, 3\text{m}$ in a water depth $h=30\text{m}$.

Figure 3.10: Total scattering cross-section corrected for the incident direction contribution, $(\sigma_s - |D(0)|^2 \Delta\theta)$, for floes of radius $a = 10\text{m}$ and draft $d = 0.5, 2, 3\text{m}$ in a water depth $h=30\text{m}$.

looking at the distribution of this corrected σ_s vs ka , three distinct regimes can be identified: the “small floe” regime for $ka \leq 1$.; the “efficient scattering” regime in the intermediate range of ka ; and, finally, the “backscattering” regime in the short wave region (Fig. (3.10)).

For small ka ($ka \leq 1$.), the main contributions to the scattered waves come from the diffracted wave and from the wave forced by the heave motion. In that “small floe” region, those two components are nearly isotropic and of small amplitude. Therefore, when the radius of the floe is small compared to the incident wavelength, only a small fraction of the incident energy is scattered (equally) in all directions.

For all floes, the ability of the ice cover to disperse energy increases considerably

in the intermediate range of ka , but the extent and the nature of this “efficient scattering” regime is a strong function of the geometry of the floes. Thin floes are “good” scatterers over a broad range of ka ($1. \leq ka \leq 6.$), with a maximum in the region where the incident wavelength is comparable to the floe diameter. In this case, the floes are good scatterers over a wide range of ka due to relatively large values of the pitch and heave RAO’s where the forced motions are more efficacious in generating waves. As the radius-to-draft ratio decreases, the resonant heave and pitch RAO peaks shift to lower ka and the extent of this region, dominated by these two maxima, decreases ($1. \leq ka \leq 4.$).

When ka is large, only the diffraction contributes significantly to the scattered wave. In this “backscattering” regime, as one approaches the limit $ka \rightarrow \infty$, about half of the diffracted energy is reflected backward, forming the backscattered wave, and the other half still travels in (or close to) the incident direction (resulting in a corrected σ_s of $2a$ m).

Although this is not the problem addressed here, it is interesting to analyse, with the help of these results, certain measurements of directional wave spectra entering the MIZ. Wadhams et al. (1986), during the MIZEX-84 experiment in the Greenland Sea, observed that the directionality of spectral components broadens significantly more rapidly for the high frequencies than for the swell frequencies. Knowing that the floe size gradually increases away from the ice edge, shorter waves are expected to be the first ones to enter the “efficient scattering” regime and, consequently, their directional character is more rapidly affected. On the other hand, near the ice edge, the swell components propagate through small floes with negligible scattering. Further in the ice pack, as they encounter larger floes, these long waves eventually reach the “efficient scattering” regime with their directionality

seriously affected.

More recently, during an experiment in the Weddel Sea, Squire et al. (1986) observed a marked change in the directional properties of the incident spectra which broadened to become isotropic at a point in the ice pack where the ice floes reached a size comparable to the wavelength of the spectral peak. Referring to Fig. (3.10), this would correspond to a maximum value for the corrected total cross-section near $ka = 3$ (appropriate for thin floes, $d = 0.5m$).

3.3 Time integration of wave spectra

3.3.1 Ice-free situation

The numerical integration of Eq.(2.63) was first performed, for calibration purposes, for the more common case of an ice-free ocean surface ($f_i = 0$), in a water depth $h = 30m$. A wind of $U_{10} = 10 m/s$ was assumed to have been blowing for about 3.25 hours which, according to Eq.(2.65), corresponds to a JONSWAP spectrum with an initial peak frequency $f_{peak} = 0.3$ Hz (Fig.(3.11)). Fig.(3.12) shows the initial one-dimensional (integrated over direction θ) energy balance. The input and dissipation terms are maximum at the peak, according to their linear dependence on the spectrum. The more complex nonlinear term, S_{nl} , has its typical three-lobed distribution; the energy is transferred from the central region of the spectrum to both shorter and longer wave components.

The nonlinear transfer is the principal source of energy on the low frequency forward face of the spectrum. It, rather than direct wind forcing, is thus responsible for the rapid growth rates at frequencies $f < f_{peak}$. S_{nl} controls the shape of the spectrum including the position and development of the peak itself, and generates the sharply peaked JONSWAP shape, characteristic of a growing wind-sea

Figure 3.11: Two-dimensional JONSWAP spectrum, $F(f, \theta)$, with a peak frequency $f_{peak} = 0.3$ Hz.

Figure 3.12: One-dimensional energy balance for a JONSWAP spectrum with $f_{peak} = 0.3$ Hz. The wind blows at $U_{10} = 10$ m/s on an ice-free ocean (F , frequency spectrum; S_{in} , wind input; S_{ds} , dissipation; S_{nl} , nonlinear interactions).

spectrum(e.g. SWAMP, 1985). Intuitively, one would expect the nonlinear term to change the spectrum in the direction of a uniform energy distribution rather than produce a sharp spectral peak. But, although very low energy spectral components always gain energy, as the dominant term of Eq.(2.11), $n_1 n_2 n_3$, is positive, the converse is not true for a sharply peaked spectrum since, in this case, only two of the three dominant terms are negative (Hasselmann, 1963). Webb (1978), among others, examined the transfer mechanism of energy due to nonlinear interactions and showed that the apparent creation of order due to the nonlinear enhancement of the spectral peak, is, in fact, the by-product of the production of a great amount of disorder at high wavenumbers.

The two-dimensional nonlinear term, $S_{nl}(f, \theta)$, has a broader directional distribution in its high frequency lobe than in the two other lobes, and the low frequency lobe is confined to a narrow frequency band and directional distribution (Fig.(3.13)). There are also, near the peak frequency, two relative maxima (a, a') in directions at an angle to the wind, as previously discussed by several authors (Webb, 1978; Fox, 1976; Longuet-Higgins, 1976). This last feature is responsible for the horseshoe-shape of the theoretical fully developed spectrum derived by Komen et al. (1984).

The integration was performed over a short duration of about 1.4 hours. Throughout the integration, the angular distribution of the spectrum can be satisfactorily described by the initial spreading function of Eq.(2.67). The evolution of the total energy, E , and the peak frequency, f_{peak} , is shown in Fig.(3.14) together with the predictions of the parametric wave model of Hasselmann et al. (1976). In the latter, which is based on measurements of fetch-limited wave spectra from various sources (including JONSWAP results), the evolution of f_{peak} and E is obtained

Figure 3.14: Time-limited growth curves for the total energy, E , and the peak frequency, f_{peak} . The wind blows at 10 m/s over an ice-free ocean (model predictions, solid lines; Hasselmann's parametric model, dashed lines).

from Eq.(2.65) and Eq.(2.66) using the relation

$$\frac{Ef_{peak}^4}{g^2\alpha} = 1.6 \times 10^{-4}. \quad (3.4)$$

The decrease of the peak frequency predicted by the model closely follows the parametric model predictions. The total energy increases at about the same rate for the two models, but with an initial disagreement on the absolute value of the energy, E . This difference, although not important, can be explained by the discrepancy between Eq.(3.4), used in the parametric model to relate the total energy of the spectrum to the parameter α , and the relationship obtained from numerical integration of a JONSWAP spectrum (like the one used here to describe the initial sea state), namely (Carter, 1982):

$$\frac{Ef_{peak}^4}{g^2\alpha} = 1.957 \times 10^{-4}. \quad (3.5)$$

3.3.2 Effect of a partial ice cover

The evolution of the spectrum, used in the previous section to describe the initial sea state ($t_0 = 11700s$), is now examined under the action of a partial ice cover. The integration was performed for two different ice concentrations (10% and 20%) with floes of radius $a = 10m$ and draft $d = 2m$, and a wind $U_{10} = 10 m/s$. The resulting spectra, $F(f, \theta)$, are contoured in Fig.(3.15) and Fig.(3.16), and the directional distributions at the peak presented in Fig.(3.17) and Fig.(3.18), for the first 12 minutes of the integration.

In both cases, the ice cover appears to be very efficient in spreading out the wave energy over all directions. The results of Section (3.2) indicate that most of the energy of the initial spectrum is contained in the efficient scattering regime

Figure 3.15: Time evolution of the spectrum, $F(f, \theta)$ ($10^{-3} m^2 Hz^{-1} rad^{-1}$), in the presence of a partial ice cover of 10% with ice floes of radius $a = 10m$ and draft $d = 2m$, $U_{10} = 10m/s$ and $f_{peak}(t_0) = 0.3Hz$.

Figure 3.16: Time evolution of the spectrum, $F(f, \theta)$ ($10^{-3} m^2 Hz^{-1} rad^{-1}$), in the presence of a partial ice cover of 20% with ice floes of radius $a = 10m$ and draft $d = 2m$, $U_{10} = 10m/s$ and $f_{peak}(t_0) = 0.3Hz$.

Figure 3.17: Time evolution of the spectrum at the peak, $F(f_{peak}, \theta)$, in the presence of a partial ice cover of 10% with ice floes of radius $a = 10\text{m}$ and draft $d = 2\text{m}$, $U_{10} = 10\text{m/s}$ and $f_{peak}(t_0) = 0.3\text{Hz}$. The labels indicate the duration of the integration (s), with $t_0 = 11700\text{s}$.

Figure 3.18: Time evolution of the spectrum at the peak, $F(f_{peak}, \theta)$, in the presence of a partial ice cover of 20% with ice floes of radius $a = 10\text{m}$ and draft $d = 2\text{m}$, $U_{10} = 10\text{m/s}$ and $f_{peak}(t_0) = 0.3\text{Hz}$. The labels indicate the duration of the integration (s), with $t_0 = 11700\text{s}$.

region ($1. \leq ka \leq 4.$), with the high frequency components extending into the backscattering regime ($ka \geq 4.$). Accordingly, on the spectral forward face and at the peak, the spectrum becomes almost instantaneously isotropic, and, in the less energetic high frequency region, the energy is scattered more slowly and preferably in the backward direction, resulting in the gradual build-up of a secondary maximum near $\theta = \pm 180^\circ$.

As the spectrum rapidly tends to isotropy, the one-dimensional energy balance is drastically modified. Fig.(3.19) shows the new energy balance obtained after only 12 minutes, with an ice cover of 20%. Here, S_{ice} represents the energy loss due to the ice, through the fraction f_d of Eq.(2.69). The extra dissipation caused by the ice is now an important term in the energy balance with, in this case, a magnitude of about twice that of the S_{ds} term at the peak. Furthermore, the wind input term, S_{in} , has considerably decreased from its ice-free value (see Fig.(3.12)). Although this term is only reduced to 80% of its previous value by the $(1 - f_i)$ factor in Eq.(2.63), the calculated decrease is mainly due to the actual nature of the parameterization of the wind input. In Eq.(2.7), the $(\frac{U_s \cos \theta}{c} - 1)$ term insures that the energy is transferred from the atmosphere to the waves in proportion with the phase velocity component in the direction of the wind, and at a rate proportional to the energy which they already have. As the wave energy is scattered by the floes away from the wind direction, the energy input rapidly diminishes. This reduction in the input function, combined with the extra dissipation of energy by the ice, severely limits the growth of the spectral energy content. The rate at which the energy is supplied by the wind rapidly becomes insufficient to overcome the energy lost through the different dissipation processes, and the energy decays (Fig.(3.20)).

Figure 3.19: One-dimensional energy balance at $t \approx (t_0 + 12)$ min, with an ice concentration $f_i = 0.2$ and $f_{peak}(t_0) = 0.3\text{Hz}$.

Figure 3.20: Time evolution of the total energy, E , of an initial JONSWAP spectrum ($f_{peak} = 0.3$ Hz) in the presence of an ice cover of 0% (from the Hasselmann parametric model), 10% and 20%, with a wind $U_{10} = 10\text{m/s}$.

The nonlinear transfer is also seriously affected as the spectral shape changes. It still has its typical three-lobed shape but with considerably reduced amplitude and larger directional spread (Fig.(3.21)). This known dependence of the nonlinear exchange mechanism on the angular spectral distribution (e.g. Hasselmann and Hasselmann, 1981, 1985a) was first discussed by Hasselmann (1963). He found that, since most of the energy flux is due to interactions in regions of high energy density, an increase in angular spread explains the associated decrease in energy transfer. In our problem, where the energy is being scattered away from the mean direction, the energy transfer effectively decreases and becomes relatively inefficient to shift the peak towards lower frequencies. Thus, a partial ice cover limits the shift of the spectral peak towards longer waves (Fig.(3.22)).

The ice concentration, f_i , does not appear to be a critical parameter in the scattering process. The energy decay rate is noticeably affected by a change in f_i due to the factor $(1 - f_i)$ added in the energy balance equation and to the variation of the attenuation rate in Eq.(2.62). However, as f_i varies from 0.05 to 0.25, the scattering ability of the ice cover remains relatively unchanged, the parameter β of Eq.(2.57) presenting only small variations (see Fig.(2.7)). Therefore, a small change in ice concentration of a noticeable ice cover ($f_i \geq 0.05$), within the range of validity of the model ($f_i \leq 0.25$), does not affect significantly the spectral evolution obtained, the characteristic time scale of the scattering mechanism (time needed for an incident spectrum to become isotropic) being so small relative to the other energy terms modifying the spectrum.

The evolution of a spectrum with the same peak frequency ($f_{peak} = 0.3\text{Hz}$), but generated by a stronger wind, $U_{10} = 20\text{m/s}$, is then examined with an ice cover, $f_i = 0.2$. Eq.(2.65) and Eq.(2.66) gives an initial time $t_o \approx 1.3$ hours and a more

Figure 3.21: Two-dimensional nonlinear term, $S_{nl}(f, \theta)$, at $t \approx (t_0 + 12)$ min, with an ice concentration $f_i = 0.2$ (see Fig.(3.13) for the $f_i = 0$ situation).

Figure 3.22: Time evolution of the peak frequency, f_{peak} , of an initial JONSWAP spectrum ($f_{peak} = 0.3$ Hz) in the presence of a partial ice cover of 0% (from the Hasselmann parametric model), 10% and 20%.

Figure 3.23: Time evolution of the peak frequency, f_{peak} , and total energy, E , for an initial JONSWAP spectrum ($f_{\text{peak}} = 0.3$ Hz) in the presence of a partial ice cover of 20% with a wind $U_{10} = 20\text{m/s}$.

sharply peaked initial spectrum, with $\alpha = 0.024$. As for the lower wind velocity, the spectrum rapidly tends to isotropy, the characteristic time scale of the scattering process remaining relatively small ($\approx 15\text{min}$). Because of the higher wind speed, the dissipation takes longer to overcome the input from the atmosphere but, as the spectrum becomes isotropic, the energy balance rapidly modifies to limit the wave growth both in length and height (Fig.(3.23)).

The integration is then done from an initial JONSWAP spectrum with $f_{\text{peak}} = 0.4\text{Hz}$, describing a younger sea state ($t_0 \approx 1.7\text{hours}$ for $U_{10} = 10\text{m/s}$). In this case, only the low frequency spectral region, near the forward face of the spectrum, is contained in the efficient scattering regime, the rest of the frequencies extending over the backscattering region. Consequently, the spectrum undergoes differential scattering, with the low frequencies becoming rapidly isotropic and the short wave

tail being nearly unaffected (except for the formation of the backscattered “bump”) (Fig.(3.24)). Since an important portion of the energetic region of the spectrum is scattered away from the mean direction, the wind input function and the nonlinear energy transfer are reduced by the scattering, but not as much as in the previous case with the smaller f_{peak} . Thus, the spectrum is, at first, allowed to grow, but at a rate smaller than in the ice-free situation. However, as the peak slowly shifts towards lower frequencies, the energy transferred to the spectral forward face is rapidly spread out over all directions, resulting in the same growth limited situation as before.

So far, in all cases studied, where an important fraction of the energy was spread out by the efficient scattering regime, the ice cover caused a very rapid tendency towards spectral isotropy, leading to a decay of the wave energy. This raises an important question: “Is there a minimum wind speed for which waves can still grow in the presence of a certain ice cover?” The evolution of a spectrum, with $f_{peak} = 0.3\text{Hz}$ and $\alpha = 0.014$ (as in the $U_{10} = 10\text{m/s}$ case), initially isotropic, is examined under different wind conditions. There is a minimum wind speed for which the initial energy balance of such a spectrum results in an increase of the total energy ($(U_{10})_{min} \approx 16\text{m/s}$ for $f_i = 0.2$; $(U_{10})_{min} \approx 13\text{m/s}$ for $f_i = 0.1$). But, because of the important reduction of the nonlinear energy transfer predicted by the model, these waves no longer grow in length. Thus, at first, as the wind transfers energy to the wave field, these short waves rapidly reach the maximum steepness at which they loose their energy through breaking. In other words, the spectral growth of an isotropic spectrum is limited by the S_{ds} term quadratic dependence on the wave-steepness parameter, $\hat{\alpha}$ (see Eq.(2.8)). The total energy of the spectrum rapidly reaches a saturation level at which the three active energy terms, S_{in} , S_{dis}

Figure 3.24: Time evolution of the spectrum, $F(f, \theta)$ ($10^{-3} m^2 Hz^{-1} rad^{-1}$), in the presence of a partial ice cover of 20% with ice floes of radius $a = 10m$ and draft $d = 2m$, $U_{10} = 10m/s$ and $f_{peak}(t_0) = 0.4Hz$.

Figure 3.25: Time evolution of the total energy, E , of an initially isotropic spectrum with $f_{peak}(t_0) = 0.3\text{Hz}$, $f_i = 0.2$, and $U_{10} = 16\text{m/s}$.

and S_{ice} , balance each other. Since the fraction of energy lost due to the ice, in a time step Δt , depends on the wave frequency (through c_g of Eq.(2.70)), each spectral component is attenuated by the ice at a different rate; the longer waves decaying faster. Consequently, the average frequency of the spectrum increases with time. Although this is beneficial to the wind input term, through a decrease of the average wave speed c , the associated higher value of the wave-steepness parameter $\hat{\alpha}$, which varies as $\bar{\omega}^4$, causes a more important increase of the S_{dis} term. Therefore, from the saturation level, the energy of the wave field diminishes with time, as the average frequency increases under the action of the selective S_{ice} term (Fig.(3.25)). Further in the integration, since the parameter $\hat{\alpha}$ also depends on the energy content E , the wind input could again balance the dissipation, but at very low energy levels (for $E < 0.0124 m^2$, in the problem of Fig.(3.25)) corresponding

to waves of negligible amplitude. Thus, in this time-limited situation, even with a wind speed high enough to cause an initial wave growth, the spectral energy rapidly decreases to very low values.

3.4 Initial wave generation problem

All the mentioned integrations have been done with a certain non zero initial wave field suddenly in presence of an ice cover, and, obviously, do not describe the problem of initial wave generation from a calm ocean surface. However, the results presented above can be used to better understand the problem of wave generation by an offshore wind starting to blow over the MIZ. As soon as the ice cover becomes sparser, the wind begins to generate very short fetch limited waves for which a high value of ka locates them in the backscattering region. Although some of the energy is lost in the backward direction, the waves are allowed to grow, but at a rate smaller than in the ice-free situation. However, as those waves travel further along the fetch, their wavenumbers decrease and, likewise, the floes gradually become smaller. Thus, the parameter ka associated with these waves diminishes as they approach the ice edge. In the most favorable conditions (strong wind blowing over a long fetch covered by a low ice concentration of large floes), waves of relatively important wavelength could eventually develop. But, as they would approach the ice edge, these waves would encounter smaller floes ($a \leq 10\text{m}$), and the resulting ka ($((k_{peak}a) \leq 6$, for $L_{peak} \geq 10\text{m}$) would cause, through scattering, a rapid inhibition of the wave growth. If the wind speed is larger than the $(U_{10})_{min}$ required to initially sustain wave growth, the spectral energy would quickly reach a saturation level where the dissipation would balance the wind input to the waves, and the total energy of the spectrum would be restricted to relatively low values. Otherwise, the

energy of the wave field would rapidly decay to a level at which the reduced energy input to these short waves would balance the dissipation terms. In both situations, the energy of the spectrum would be restrained to low values by the partial ice cover. Therefore, according to the model results, an offshore wind blowing over the outer part of the MIZ, can not generate a substantial wave field, the partial ice cover limiting the wave growth.

Chapter 4

Conclusions

The effect of a partial ice cover on the spectral evolution of a wave field is governed by the nature of wave scattering by individual floes. The ice floes, modelled as floating truncated cylinders, diffract the different spectral components and likewise oscillate in three modes of motion: heave, surge and pitch. The response, for each mode, is mainly determined by the floe radius-to-wavelength ratio, ka , going from almost perfect response for very small ka , to negligible movement at large ka ($ka \geq 4$). The geometry of the floes determines the position and magnitude of a heave and surge resonant peak, at intermediate ka , which increases and shift towards lower ka when the radius-to-draft ratio decreases. The incident waves are diffracted on each floe, with the character of the diffracted waves also function of the parameter ka : a small isotropic wave at the long wave limit; and, for very large ka , a shadow-forming wave and a backscattered wave, both of equal energy.

The nature of the scattered (forced and diffracted) waves determines the ability of a given icefield to disperse energy over all directions. Three different regimes have been identified, delimited by the magnitude of the total scattering cross-section (from which has been subtracted the incident direction contribution): a small floe regime ($ka \leq 1$.) where only a small fraction of the incident energy is

scattered (equally) in all directions; an efficient scattering regime, extending over a range of ka varying with the geometry of the floes, in which the energy is rapidly scattered away from the incident direction; and a backscattering regime, for large ka , where about half of the diffracted energy is reflected to form the backscattered wave.

An incident spectrum, for which an important fraction of the energy is contained in the efficient scattering region, becomes almost instantaneously isotropic. This new spectral shape seriously reduces the energy input from the atmosphere and the nonlinear transfer of energy towards lower frequencies. These two effects, combined with an important extra energy loss due to the ice, modifies the energy balance to cause a rapid decay of the spectral energy content. The growth of these waves is then limited both in length and height. The initial wave generation problem in the MIZ was then analysed by noting that the waves, generated along the fetch by a wind blowing towards the open sea, rapidly reach values of ka for which their energy is scattered to become isotropic and, consequently, no longer grow due to the continued dissipation and to the reduced input from the wind.

This model, within its limits of applicability, predicts that a wind blowing seawards over a uniform ice cover, is unlikely to generate an important wave field. The surface waves should remain short, of small amplitude and with a large directional spread. From these results, one should then, in an operational wave forecasting system, safely measure the effective fetch from the point where no noticeable ($f_i < 5\%$) ice cover is present; the energy of the waves formed inside the ice cover being low and spread over all directions. In the same practical point of view, it could be interesting to include an ice factor in a global operational wave forecasting program, in regions where an ice cover is present. This could be done by

assuming the scattering process local in space and time, and by applying, at each time step, the appropriate transfer function to the spectrum, for each grid point of the studied region. This last suggestion strongly depends on the flexibility of the nonlinear transfer parameterization which should be valid for the unusual spectral shape caused by the presence of the ice.

In examining the model results, it is important to keep in mind its limitations. For example, the assumption of a uniform ice cover does not properly describe all situations in the MIZ. It is very common to encounter large polynyas in which regular short fetch wave generation is possible. As suggested by Wadhams (1983), these waves could play, through wave radiation pressure, an important role in the formation of ice edge bands frequently observed, especially in the Bering Sea.

Although it provides satisfying answers to the questions raised at the beginning of this thesis, the model could certainly be improved, particularly in the parameterization of the different dissipative processes. Also, a similar three dimensional approach, to analyse the scattered potentials, could be applied, with appropriate modifications, to situations where the fraction of the surface covered by ice is larger, and, therefore, where the far-field approximation is no longer valid, and where the floe-floe interactions are important. It could also be used in the better studied case of waves and swell entering the icefield, where the estimation of the energy attenuation rate could certainly be improved by a more complete analysis of the scattered waves.

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Appendix A

Scattering amplitudes in infinite depth

In deep water, the appropriate form for the Green's function of Eq.(2.21) is (John, 1950)¹:

$$G(\vec{x}, \vec{X}) = \frac{1}{R'} - \int_0^\infty \frac{\nu + \mu}{\nu - \mu} e^{(z+Z)\mu} J_0(\mu q) d\mu, \quad (\text{A.1})$$

where

$$R' = \sqrt{(x - X)^2 + (y - Y)^2 + (z - Z)^2},$$

$$q = \sqrt{(x - X)^2 + (y - Y)^2},$$

$\nu = \omega^2/g$, and the path of integration runs below the root ν of the denominator.

At large distance from the floe, John (1950) showed that this equation reduces to

$$\lim_{q \rightarrow \infty} G(\vec{x}, \vec{X}) = 2\pi i k e^{k(z+Z)} H_0^{(1)}(kq) + O\left(\frac{1}{q^3}\right) \quad (\text{A.2})$$

where $H_0^{(1)}$ is the Hankel function of the first kind of order 0. It is convenient to convert Eq.(A.2) to cylindrical coordinates with

¹with appropriate corrections on signs, in agreement with Wehausen and Laitone, 1960.

$$r^2 = x^2 + y^2, \quad R^2 = X^2 + Y^2, \quad \tan \theta = \frac{y}{x}, \quad \tan \Theta = \frac{Y}{X};$$

then

$$q^2 = R^2 + r^2 - 2rR \cos(\theta - \Theta). \quad (\text{A.3})$$

Eq.(A.2) can now be transformed in a symmetrical form, in the same way that Fenton (1978) obtained Eq.(2.22) in the finite depth case. The use of Graf's addition theorem (e.g. Abramowitz and Stegun, 1965) gives

$$H_0^{(1)}(kq) = \sum_{\ell=-\infty}^{\infty} H_{\ell}^{(1)}(kr) J_{\ell}(kR) \cos[\ell(\theta - \Theta)]. \quad (\text{A.4})$$

In this series, the $-\ell^{th}$ term is equal to the ℓ^{th} term. Thus, introducing a Kronecker delta, Eq.(A.4) can be written as

$$H_0^{(1)}(kq) = \sum_{\ell=0}^{\infty} (2 - \delta_{\ell 0}) H_{\ell}^{(1)}(kr) J_{\ell}(kR) \cos[\ell(\theta - \Theta)]. \quad (\text{A.5})$$

Substituting this new form for the Hankel function into Eq.(A.2), the deep water Green's function, for large r , is given by

$$\lim_{r \rightarrow \infty} G = 2\pi i k e^{k(z+Z)} \sum_{\ell=0}^{\infty} (2 - \delta_{\ell 0}) \cos[\ell(\theta - \Theta)] J_{\ell}(kR) H_{\ell}^{(1)}(kr) + O(r^{-3}). \quad (\text{A.6})$$

Substituting Eq.(A.6) and (2.24) in (2.21), the scattered far field potentials, for $k = 1, \dots, 4$, take the form:

$$\phi_k(r, \theta, z) \cong \frac{1}{4\pi} \int_{S_0} \sum_{\ell=0}^{\infty} f_{k\ell}(s) \cos(\ell\Theta) G_{\ell} (2 - \delta_{\ell 0}) \cos[\ell(\theta - \Theta)] dS \quad (\text{A.7})$$

with

$$G_\ell = 2\pi i k e^{k(z+Z)} J_\ell(kR) H_\ell^{(1)}(kr).$$

As in the finite depth case, dS is expressed as $Rd\Theta ds$ and the equilibrium contour, s_0 , discretized into N segments of length L_j :

$$\begin{aligned} \phi_k(r, \theta, z) &\cong \frac{1}{4\pi} \int_{s_0} \sum_{\ell=0}^{\infty} f_{k\ell}(s) G_\ell \int_0^{2\pi} (2 - \delta_{\ell 0}) \cos(\Theta) \cos[\ell(\theta - \Theta)] d\Theta R ds \\ &\cong \frac{1}{4\pi} \sum_{j=1}^N \sum_{\ell=0}^{\infty} f_{k\ell}(s_j) G_\ell(r, R_j, z, Z_j) \int_0^{2\pi} (2 - \delta_{\ell 0}) \cos(\ell\Theta) \\ &\quad \cos[\ell(\theta - \Theta)] d\Theta R_j L_j. \end{aligned} \quad (\text{A.8})$$

Integrating over Θ , Eq.(A.8) reduces to

$$\phi_k(r, \theta, z) \cong \frac{1}{2} \sum_{j=1}^N \sum_{\ell=0}^{\infty} f_{k\ell}(s_j) G_\ell(r, R_j, z, Z_j) \cos(\ell\theta) R_j L_j. \quad (\text{A.9})$$

For large r , $H_\ell^{(1)}(kr)$ contained in G_ℓ takes the asymptotic form (e.g. Abramovitz and Stegun, 1965)

$$\lim_{r \rightarrow \infty} H_\ell^{(1)}(kr) = \sqrt{\frac{2}{\pi kr}} e^{-i(\frac{\pi}{2}\ell + \frac{\pi}{4})} e^{ikr} + \mathcal{O}(r^{-\frac{3}{2}}). \quad (\text{A.10})$$

Substituting Eq.(A.10) in Eq.(A.9), the far field potential can be written as

$$\begin{aligned} \phi_k(r, \theta, z) &= \left\{ \sqrt{2\pi k} i e^{-i\frac{\pi}{4}} \sum_{j=1}^N \sum_{\ell=0}^{\infty} f_{k\ell}(s_j) e^{k(z+Z_j)} J_\ell(kR_j) \right. \\ &\quad \left. e^{-i\frac{\pi}{2}\ell} R_j L_j \cos(\ell\theta) \right\} \frac{1}{\sqrt{r}} e^{ikr} + \mathcal{O}(r^{-\frac{3}{2}}). \end{aligned} \quad (\text{A.11})$$

Finally, the surface displacement, far enough from the floe, is obtained from the far field potential at the surface ($z = 0$), using Eq.(2.34). As in the case of finite depth, it takes the form of outgoing cylindrical waves:

$$\eta_k(r, \theta, t) = Re \left(R \frac{H}{2} D'_k(\theta) \frac{1}{\sqrt{r}} e^{i(kr - \omega t)} \right) \quad (\text{A.12})$$

with, in this case,

$$D'_k(\theta) = -\frac{\omega}{g} \frac{\xi_k}{\frac{H}{2}} \sqrt{2\pi k} e^{-\frac{i\pi}{4}} \sum_{j=1}^N \sum_{\ell=0}^{\infty} f_{k\ell}(s_j) e^{kZ_j} J_{\ell}(kR_j) e^{-\frac{i\pi}{2}\ell} R_j L_j \cos(\ell\theta)$$

for $k = 1, 2, 3$, and

$$D'_k(\theta) = -\frac{\omega}{g} \frac{1}{\frac{H}{2}} \sqrt{2\pi k} e^{-\frac{i\pi}{4}} \sum_{j=1}^N \sum_{\ell=0}^{\infty} f_{k\ell}(s_j) e^{kZ_j} J_{\ell}(kR_j) e^{-\frac{i\pi}{2}\ell} R_j L_j \cos(\ell\theta)$$

for $k = 4$.

Appendix B

List of Symbols

k : wavenumber

$F(\vec{k})$: two-dimensional wavenumber spectrum

η : surface displacement

$\bar{\eta}^2$: surface displacement variance

f : wave frequency

g : gravitational constant

h : water depth

$F(f, \theta)$: directional frequency spectrum

c_g : group velocity

S_{in} : rate of energy input from the wind

S_{ds} : dissipation rate (wave breaking)

S_{nl} : nonlinear energy transfer

f_i : fraction of the area covered by ice

S_{ice} : ice energy term

ρ_a : air density

ρ_w : water density

U_5 : wind speed at 5 m

ω : angular frequency

c : phase velocity

u_* : friction velocity
 τ_a : wind shear stress
 $\hat{\alpha}$: integral wave-steepness parameter
 $\bar{\omega}$: average angular frequency
 E : total energy of the spectrum
 C : dissipation constant
 n_i : action density for wavenumber k_i
 σ : nonlinear coupling coefficient
 $\delta()$: Dirac function
 H : wave height
 D : floe diameter
 L : wavelenght
 F' : force per unit lenght on a vertical cylinder
 U : flow velocity
 C_d : drag coefficient
 C_m : inertia coefficient
 d : floe draft
 Φ : velocity potential with $\Phi = \phi e^{-i\omega t}$
 a : floe radius
 ξ_k : floe motion amplitudes (ξ_1 , surge; ξ_2 , heave; ξ_3 , pitch)
 (x, y, z) : Cartesian coordinate system (see Fig.(2.3))
 (r, θ, z) : cylindrical coordinate system (see Fig.(2.3))
 $Re()$: real part
 β_ℓ : $=1$ for $\ell = 0$, $=2i^\ell$ for $\ell \geq 1$
 J_ℓ : Bessel function of the first kind of order ℓ
 S : immersed floe surface
 S_o : equilibrium immersed surface of the floe
 $\vec{n}$: unit vector normal to S_o , directed outward

V_n : velocity of the surface in direction $\vec{n}$
 v_n : magnitude of V_n
 $n_{x,y,z}$: direction cosines of $\vec{n}$
 $\vec{X}$: point on S_o , (X, Y, Z)
 $f_k(\vec{X})$: source strenght distribution function
 $G(\vec{x}, \vec{X})$: Green's function
 $\delta_{\ell 0}$: Kronecker delta
 (R, Θ, Z) : cylindrical coordinates corresponding to (X, Y, Z)
 I_ℓ, K_ℓ : modified Bessel functions of order ℓ of the first and second kinds
 $\nu : = \omega_2/g$
 s : distance measured along the body contour in a vertical plane
 s_o : equilibrium body contour
 N : number of segments along s
 m_{ik} : mass matrix components
 c_{ik} : stiffness matrix components
 a_{ik} : added mass coefficient
 b_{ik} : damping coefficient
 $F_i^{(e)}$: exciting force components
 $F_{ik}^{(f)}$: force components associated with the forced motions
 $Im()$: imaginary part
 L_j : segment lenght
 R : factor of wave amplitude reduction
 $D_k(\theta)$: scattering amplitude
 $H_\ell^{(1)}$: Hankel function of the first kind of order ℓ
 σ_s : total scattering cross-section
 D_{av} : average distance between floes
 ρ_o : number density, without shading

f' : fraction of a perimeter taken by ice floes
 T_r : transmittivity of a ring
 ρ_e : effective number density
 $D(\theta) : = \sum_{k=1}^4 D_k(\theta)$
 α_c : coherent scattering coefficient
 β : energy spreading factor
 σ_a : absorption cross-section
 Δt : time step
 f_{peak} : frequency at the maximum of the spectrum
 α : Phillips's constant
 γ : peak enhancement factor
 $G(f, \theta)$: directional spreading function
 t_o : initial time of the integration
 U_{10} : wind speed at 10 m
 Γ : Gamma function
 f_d : fraction of the energy lost in the scattering process
 Q_k : scattering cross-section