ELECTRIC AND MAGNETIC FIELDS ASSOCIATED WITH A VERTICAL FAULT

by

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ABSTRACT

Interest in the vertical fault problem for electromagnetic fields has been recently revived by the papers of I. d'Erceville and G. Kunetz (1962) and D. Rankin (1962). In the derivation of his equations Rankin used d'Erceville's theory which contains some fallacious assumptions. These have been pointed out by J.T. Weaver (1962) and also in this thesis.

This thesis follows the lines of mathematical attack first employed by d'Erceville and Kunetz, and later developed by Weaver, in applying the theory of integral transforms to the partial differential equations satisfied by land and sea conductors. The problem of both a vertical fault and also a sloping fault, i.e. $0 < \alpha < 90°$ where α is the angle of dip of the fault are considered.

The results in the general case are inconclusive, no solution has been found and no solution is suggested. The case of $\alpha = 90°$ has proved to be equally indeterminate, but a solution has been suggested, which, although it has not been proved rigourously, does not appear to violate any physical principles and also seems to represent the field equations on the surface of the land and the sea.

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TABLE OF CONTENTS

I	INTRODUCTION	1
II	INDUCTION EQUATIONS	3
III	PHYSICAL INTERPRETATIONS	13
IV	BOUNDARY CONDITIONS	19
V	BOUNDARY CONDITIONS FOR $\alpha = 90°$	24
VI	SOLUTION OF THE EQUATIONS FOR $\alpha = 90°$	28
	H-polarization	28
	E-polarization	30
VII	PRACTICAL IMPLICATIONS	41
VIII	CONCLUSIONS	52
APPENDIX A1	SNELL'S LAW APPLIED TO A NONCONDUCTOR-CONDUCTOR BOUNDARY	54
APPENDIX A2	FRESNEL EQUATIONS	56
APPENDIX A3	PROOF OF DIV $\hat{E} = 0$	58
APPENDIX A4	VALUES OF ψ AND Ψ FOR G(y)	59
APPENDIX A5	SOLUTION OF THE EQUATIONS IN THE GENERAL CASE	60

LIST OF FIGURES

I	Co-ordinate systems at a land-sea-air contact.	3
II	Diagram showing the transformation from one co-ordinate system to another.	8
III	Magnetic field generated from an electric field in a right angled conductor.	15
IV	Magnetic fields generated from electric fields in two right angled conductors; $j_1 > j_3$ so that the vertical magnetic field, H_z , at the discontinuity is directed upwards.	15
V	Incident, transmitted and reflected rays at a boundary.	20
VI	General field vector in nonrectangular co-ordinates.	21
VII	Vertical magnetic anomaly, after J.T. Weaver.	40
VIII	Magnetogram records, adapted from U. Schmucker.	42
IX	Map of Southern California.	43
X	Schematic cross-section at the sea coast for an oversimplified conductivity cliff.	45
XI	Velocity depth curve (shear waves) under the Canadian Shield (Model CANSD) and under the ocean (Model 8099), after J. Brune and J. Dorman.	45
XII(a)	Idealized plot of skin depth vs. period for two conductivities.	47
(b)	Probable variation of H_z with period.	47

INTRODUCTION

Interest in the vertical fault discontinuity problem has been recently stimulated by the papers of I. d'Erceville and G. Kunetz (1962) and D. Rankin (1962). The present work is an extension of a paper by J.T. Weaver (1963) who was looking for a theoretical explanation of an anomalously high vertical magnetic component of geomagnetic micropulsations found at a land-sea contact. In 1961 D.A. Christoffel, J.A. Jacobs and J.A. Shand published a paper comparing the ratios of the vertical to the horizontal components of the magnetic fields at Victoria, B.C., which can be considered to be situated at a land-sea contact at the edge of the Pacific, and Ralston, which is in the Southern part of the plains of Alberta and thus is removed from a land-sea contact. The results showed that there was a definite vertical anomaly in the magnitude of the geomagnetic amplitudes in the micropulsation range (0.001 cps to 3.0 cps) at a land-sea contact. This contradicts an assumption used by d'Erceville in his theoretical treatment of the problem.

In this thesis the initial electromagnetic field is assumed to be vertically incident. Even though this approximation is not exact, it is found (see Appendix 1) that it is equivalent to the condition that the displacement current is negligible in Maxwell's equations, i.e. that $|\sigma| \gg |\omega\epsilon|$. As a result, the angle that the plane of incidence makes with the discontinuity is arbitrary and hence it has been assumed to be at right angles to the discontinuity. This permits simplifying assumptions to be made and the problem becomes two dimensional.

The difficulties arising from variations of conductivity with depth have been circumvented by assuming that the conductivity does not vary for depths less than 200 km. and that the electromagnetic field has been "completely" attenuated at that depth, i.e. for theoretical purposes a depth of 200 km. can be regarded as the depth to infinity. Neither land nor sea have been assumed to be perfect conductors (although both may be considered so when compared with air).

INDUCTION EQUATIONS

For the general case the coordinate axes O (X, Y, Z) have been chosen with the X-axis along the discontinuity junction of land, sea and air, the Y axis along the sea-air contact, and the Z axis along the land-sea contact. In the case of a vertical discontinuity ($\alpha = 90°$), the axes are rectangular and are renamed O (x, y, z). Thus OX and Ox are identical as are OY and Oy (see Figure 1).

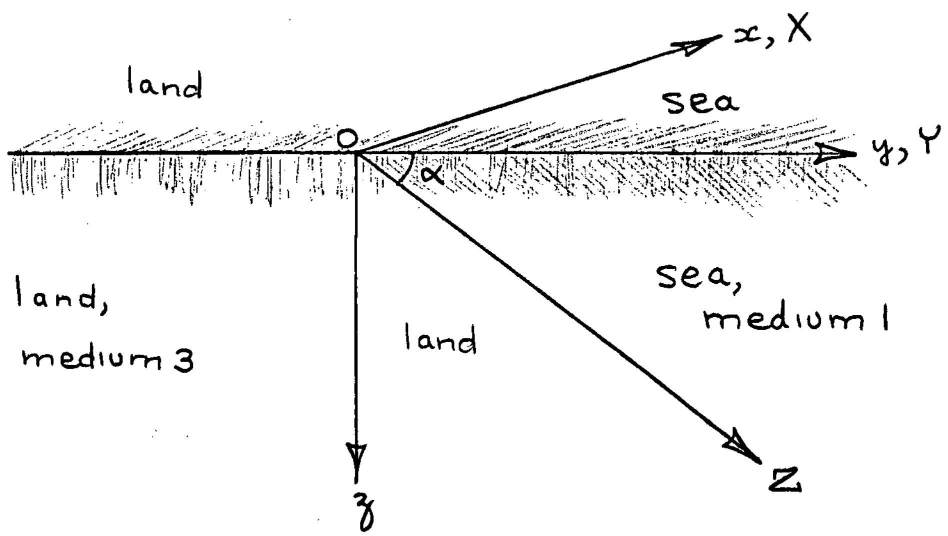


FIGURE 1 Co-ordinate systems at a land-sea-air contact.

It is assumed that the incident electromagnetic field is a wave of the form

$$\vec{H}_L = \vec{H}_o \, e^{i(\omega t - k_2 \hat{n}_o \cdot \vec{r})}$$

where:

H_0 = the constant, finite amplitude of the magnetic field

k_2 = the wave number in air
i.e. $k_2 = \omega\sqrt{\mu_0\epsilon_0}$ where μ_0 and ϵ_0 are the permeability and permittivity in air

$\hat{n}_0$ = the unit vector in the direction of propagation

and ω = the frequency

Upon striking the land-sea conductor the magnetic field is "split up" into two parts, the reflected and transmitted fields. Both these fields are assumed to vary as follows

$$\vec{H}_t = \vec{H}_t(y,z)\, e^{i\omega t}$$

$$\vec{H}_r = \vec{H}_r(y,z)\, e^{i\omega t}$$

Hence in the atmosphere the field is given by

$$\vec{H} = \vec{H}_i + \vec{H}_r = e^{i\omega t}\left[\vec{H}_0\, e^{-i k_2 \hat{n}_0\cdot\vec{r}} + \vec{H}_r(y,z)\right]$$

A similar set of equations holds for the electric field, assuming

$$\vec{E}_i = \vec{E}_0\, e^{i(\omega t - k_2 \hat{n}_0\cdot\vec{r})}$$

It is possible to simplify Maxwell's equations in a homogeneous, isotropic medium.

(a) $\nabla \times \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t}$ (b) $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ (1)

(a) $\nabla \cdot \vec{H} = 0$ (b) $\nabla \cdot \vec{E} = \rho / \epsilon$ (2)

If the fields are varying in a purely periodic manner then for the two dimensional case, equation 2(b) is found to be automatically equal to zero, i.e.

$$\nabla \cdot \vec{E} = 0 \tag{3}$$

The proof of this is given in Appendix A3.

Within the conductors it is necessary to consider the ratio $\left| \frac{\sigma}{\omega \epsilon} \right|$, which is found to be much greater than unity. Table I gives the value of this ratio for land and sea respectively:

Table I

Medium	Conductivity, σ mho/m	Dielectric constant, ϵ Farad/m	Frequency ω cps	$\left\lvert \frac{\sigma}{\omega \epsilon} \right\rvert$
sea	4	80×10^{-10}	3	$\sim 10^{+10}$
land	10^{-5}-10^{-6}	3×10^{-10}-4×10^{-10}	3	$\sim 10^{+4}$

Hence it is possible in both cases to assume that displacement currents may be neglected in the conductors, i.e. that

$$\nabla \times \vec{H} = \vec{j} = \sigma \vec{E} \tag{4}$$

in both land and sea. However there is no justification in ignoring the displacement current in an uncharged atmosphere. Such a justification is only possible after a solution of the electric and magnetic field has been obtained. Further if we treat the fields as purely electrostatic or magnetostatic then there is no justification for using Maxwell's equations to find the remaining field. Thus assuming $\nabla \times \vec{H} = 0$ (for a magnetostatic field), we cannot use $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ to find the electric field. This point has been dealt with more fully in Dr. A. Nishida's Ph.D. Thesis (1962). Hence in the neutral atmosphere

$$\text{(a)} \quad \nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} \qquad \text{(b)} \quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{5}$$

Therefore using equations (1) - (4) it is possible to express the magnetic and electric fields as diffusion equations for the case when $\alpha = 90°$, i.e.

$$\nabla^2 \vec{E} = i\mu\sigma\omega \vec{E} \quad \text{and} \quad \nabla^2 \vec{H} = i\mu\sigma\omega \vec{H}$$

If we substitute $\eta^2 = \mu\sigma\omega$ (6)

then the equations to be solved in each conductor are

$$\nabla^2 \vec{E} = i\eta \vec{E} \tag{7(a)}$$

and
$$\nabla^2 \vec{H} = i\eta^2 \vec{H}. \tag{7(b)}$$

In the atmosphere the equations to be solved can be derived in a similar manner and are

$$\text{(a)} \quad \nabla^2 \vec{E} = -\mu\epsilon\omega^2 \vec{E} \quad \text{and (b)} \quad \nabla^2 \vec{H} = -\mu\epsilon\omega^2 \vec{H} \tag{8}$$

Because of the two dimensional nature of the model all field vectors are independent of x and we can find two solutions of equations (7), one for E-polarization and another for H-polarization. In both cases the polarized field vectors are directed along the x-axis. Hence for H-polarization

$$E_x = 0 \qquad E_y = \frac{1}{\sigma}\frac{\partial H_x}{\partial z} \quad \text{and} \quad E_z = \frac{-1}{\sigma}\frac{\partial H_x}{\partial y} \qquad \alpha = 90^\circ \tag{9}$$

and for E-polarization

$$H_x = 0 \qquad H_y = \frac{i}{\mu\omega}\frac{\partial E_x}{\partial z} \quad \text{and} \quad H_z = \frac{-i}{\omega\mu}\frac{\partial E_x}{\partial y} \qquad \alpha = 90^\circ \tag{10}$$

These two cases are completely independent of each other and the solution of the diffusion equations will be obtained for each case separately.

For the case of H-polarization the equation to be solved for $z \geqslant 0$ is

$$\frac{\partial^2 H_x}{\partial z^2} + \frac{\partial^2 H_x}{\partial y^2} = i\eta^2 H_x \qquad \alpha = 90^\circ \tag{11}$$

and for E -polarization the equation is

$$\frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} = i\eta^2 E_x \qquad \alpha = 90^\circ \tag{12}$$

So far no mention has been made of the diffusion equation when $90^\circ > \alpha > 0^\circ$. For the cases of E- and H -polarization they may be obtained from equations (11) and (12) using the equations of transformation for the two coordinate systems O (X, Y, Z) and O (x, y, z) viz.

$$\left.\begin{aligned} x &= X \\ y &= Y + Z\cos\alpha \\ z &= Z\sin\alpha \end{aligned}\right\} \quad (13) \text{ or } \quad \left.\begin{aligned} X &= x \\ Y &= y - z\cot\alpha \\ Z &= z\operatorname{cosec}\alpha \end{aligned}\right\} \tag{14}$$

See Figure II.

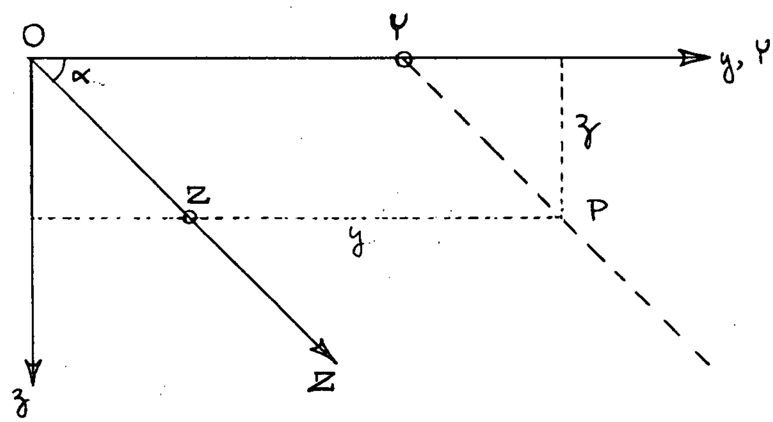


FIGURE II Diagram showing the transformation from one co-ordinate system to another.

Therefore

$$\left.\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial}{\partial X}, \\ \frac{\partial}{\partial y} &= \frac{\partial}{\partial Y} \\ \text{and} \quad \frac{\partial}{\partial z} &= -\cot\alpha \frac{\partial}{\partial Y} + \operatorname{cosec}\alpha \frac{\partial}{\partial Z} \end{aligned}\right\} \tag{15}$$

and hence

$$\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \operatorname{cosec}^2\alpha\left(\frac{\partial^2}{\partial Y^2} - 2\cos\alpha \frac{\partial^2}{\partial Y \partial Z} + \frac{\partial^2}{\partial Z^2}\right)$$

Therefore equations (11) and (12) may be written

$$\frac{\partial^2 F}{\partial Y^2} - 2\cos\alpha \frac{\partial^2 F}{\partial Y \partial Z} + \frac{\partial^2 F}{\partial Z^2} = i\eta^2 \sin^2\alpha \, F \tag{16}$$

where F is equal to either E_x or H_x.

The value of the transmitted magnetic field across z=0 can be expressed as a function of y only, say $G(y)$, and time, t, for H-polarization and the value of the transmitted electric field can be expressed as another function of y for E-polarization, say $C(y)$ and time, t. Hence assuming solutions of equations (8) to be of the form

$$H_x = H(y,z)\, e^{+i\omega t}$$

and

$$E_x = E(y,z)\, e^{+i\omega t}$$

we can express the component H_x in air (omitting the term $e^{+i\omega t}$ as before) as

$$H_x = e^{i(my+nz)}$$

The condition which must be satisfied for a solution of (8) is therefore

$$m^2 + n^2 = \mu\epsilon\omega^2 \tag{17}$$

$$\left.\begin{array}{lll} \text{i.e.} & n = \sqrt{\mu\epsilon\omega^2 - m^2} & \text{for } \omega\sqrt{\mu\epsilon} > |m| \\ \text{and} & n = i\sqrt{m^2 - \mu\epsilon\omega^2} & \text{for } \omega\sqrt{\mu\epsilon} < |m| \end{array}\right\} \tag{18}$$

The general expression for the reflected wave can be expressed by

$$H_x = \int_{-\infty}^{+\infty} A(m)\, e^{i(my+nz)}\, dm \tag{19}$$

where n is given by (18).

Since $H_i = H_o e^{-i(m_2 y + n_2 z)}$

where the subscript 2 refers to air, at z=0 for H-polarization the tangential component of the magnetic field is continuous across the land-sea conductor i.e.

$$\hat{n} \times (\vec{H}_i + \vec{H}_r) = \hat{n} \times \vec{H}_t \tag{20}$$

where $\hat{n}$ is the unit vector normal to the surface.

For H -polarization

$$\vec{H}_i + \vec{H}_r = \vec{H}_t \qquad \text{or, on } z=0,$$

$$H_0 e^{-im_2 y} + \int_{-\infty}^{+\infty} A(m) e^{imy} dm = G(y)$$

$$\therefore \int_{-\infty}^{+\infty} A(m) e^{imy} dm = G(y) - H_0 e^{-im_2 y}$$

Applying a Fourier transform, we can express the function $A(m)$ as a function of m, viz.

$$A(m) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left(G(y) - H_0 e^{-im_2 y} \right) e^{-imy} dy$$

Since

$$\int_{-\infty}^{+\infty} e^{icx} dx = 2\pi \, \delta(c)$$

where $\delta(c)$ is the Dirac delta function,

$$A(m) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} G(y) e^{-imy} dy - H_0 \, \delta(m + m_2) \tag{21}$$

or $$H_x = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dm \, e^{i(my+nz)} \int_{-\infty}^{+\infty} G(\eta) e^{-im\eta} d\eta - H_0 e^{i(-m_2 y + n_2 z)}$$

Hence knowing the function $G(y)$ by solving equations (11) and (12) it is possible to express the magnetic and electric field in the air. A similar function defined by $B(m)$ can be obtained for E-polarization. The proof follows similar lines, and it can easily be shown that

$$B(m) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[C(y) - E_0 e^{-im_2 y} \right] e^{-imy} dy \tag{22}$$

where $C(y)$ is the value of the transmitted electric field, which corresponds to $G(y)$, on z=0. Hence

$$E_x = \int_{-\infty}^{+\infty} B(m)\, e^{i(my + nz)}\, dm \tag{23}$$

is the electric field in the atmosphere ($z < 0$).

For H-polarization, the electric field in the atmosphere is given by:

$$E_x = 0 \qquad E_y = \frac{1}{i\omega\epsilon}\frac{\partial H_x}{\partial z} \quad \text{and} \quad E_z = \frac{i}{\omega\epsilon}\frac{\partial H_x}{\partial y}, \; \alpha = \frac{\pi}{2} \tag{24}$$

and for E-polarization, the magnetic field in the atmosphere is given by:

$$H_x = 0 \qquad H_y = \frac{i}{\mu\omega}\frac{\partial E_x}{\partial z} \quad \text{and} \quad H_z = \frac{-i}{\mu\omega}\frac{\partial E_x}{\partial y}, \; \alpha = \frac{\pi}{2} \tag{25}$$

Equations (9), (10), (24) and (25) may be generalized for the case of $0 < \alpha \leqslant 90°$ by applying equations (15).

PHYSICAL INTERPRETATIONS

When an electromagnetic wave meets a change in conductivity (e.g. at an air-sea contact) the incident wave is propagated through the new conducting medium with a velocity of $\sqrt{\frac{2\omega}{\mu\sigma}}$ where μ and σ are the permeability and conductivity of the medium.

Using Maxwell's equations the magnetic and electric fields can be expressed in the following manner:

$$\nabla^2 \vec{H} = (i\omega\mu\sigma - \omega^2\mu\epsilon)\vec{H}$$

and $$\nabla^2 \vec{E} = (i\omega\mu\sigma - \omega^2\mu\epsilon)\vec{E}$$

However $|\sigma| \gg \omega\epsilon$ so that these equations can be approximated by equations (7). The effect of this on any physical interpretation is to heavily damp the electromagnetic wave in the conductive medium and to permit the current density to take a finite time to build up, which results in the diffusion phenomenon described by equations (7).

The description of the electric and magnetic fields associated with the current densities can be found from the vector potential

$$\vec{A} = \frac{\mu}{4\pi}\int \frac{\vec{j}(y,t)}{r}\, d\tau$$

Since $\vec{B} = \mu \vec{H} = \nabla \times \vec{A}$

and $\nabla \times \vec{H} = (\sigma + i\omega\epsilon) E$, $E = -\dfrac{\partial \vec{A}}{\partial t}$

if the gradient of the scalar potential of the electrostatic field is small compared with $\dfrac{\partial \vec{A}}{\partial t}$. Hence if we take a "wedge" of sea, or to simplify it, take the case when the angle $\alpha = 90°$ then it can be seen that for vertical propagation a current system is set up, decaying exponentially with depth (z). This current system radiates waves in all directions, both in the y-direction and in the negative z-direction (reflected ray). The radiation in the y-direction is not part of the reflected ray but is referred to as a diffraction term. The direction of the magnetic field is given in Figure III, assuming that the electric field in the conductor is directed along the x-axis. It will be noticed that in this case (i.e. E -polarization), the magnetic field at the vertical boundary has become vertical. For the case of H -polarization the corresponding electric field becomes vertical. The problem now arises of what happens when two conductors are placed side by side. If the angle α is still assumed to be 90° then the magnetic field for each conductor, separated as in Figure IV, is generated in the same manner as before but now there is a mutual induction field near y=0, which tends to oppose the existing electric field in both conducting media. Upon joining the two conductors on the plane y=0 then there is an

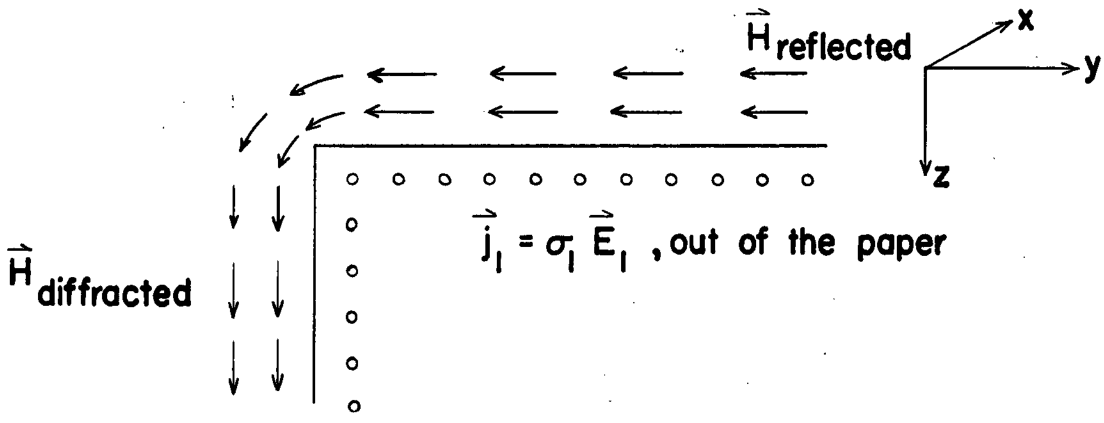


FIGURE III Magnetic field generated from an electric field in a right angled conductor.

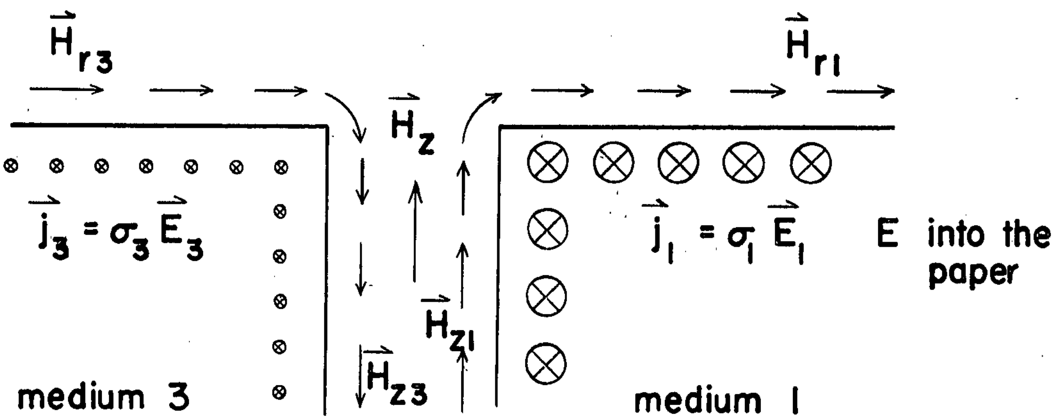


FIGURE IV Magnetic fields generated from electric fields in two right angled conductors; $j_1 > j_3$ so that the vertical magnetic field, H_z , at the discontinuity is directed upwards.

electric field discontinuity between $\vec{E}_3$ and $\vec{E}_1$ and hence there is a diffusion of electric field in the y-direction, in the same manner as the transmitted electric field diffuses, until the value of the electric field reaches a common value on y=0.

The difference in current density gives rise to an anomalous vertical component of the magnetic field, $\vec{H}_z$. There are three cases to consider:

1) $\vec{j}_3 > \vec{j}_1$ when $\vec{H}_z$ is downwards at the discontinuity junction.
2) $\vec{j}_3 = \vec{j}_1$ in which case the two media are identical (and therefore there is no conductivity discontinuity).
3) $\vec{j}_3 < \vec{j}_1$ when $\vec{H}_z$ is upwards (see Figure IV).

Rather than try to compute the diffusion of the electric field from first principles, a reasonable guess is made. In much the same way as the electric field is attenuated with depth, for E -polarization, the diffusion of the electric field can be considered to be "attenuated" by the same terms. Hence the value of the electric fields in media 1 and 3 may be given by:

$$\vec{E}_3 = \vec{E}_{30}(1 - e^{\beta_3 y}) + \vec{f(z)}\, e^{\beta_3 y}, \quad y \leq 0, \quad z = 0 \qquad (26)$$

$$\vec{E}_1 = \vec{E}_{10}(1 - e^{-\beta_1 y}) + \vec{f(z)}\, e^{-\beta_1 y}, \quad y \leq 0, \quad z = 0 \qquad (27)$$

where $\vec{f}(z)$ is the value of the electric field on y=0, $\vec{E}_{10}$ is the value of $\vec{E}_1$ at y= +∞, $\vec{E}_{30}$ is the value of E_3 at y= −∞ and $\beta = \sqrt{\mu\omega\sigma i}$ in media 1 and 3. It is thus related to the skin depth $\sqrt{\frac{2}{\mu\omega\sigma}}$.

The same idea may be applied to the case of H -polarization and a similar diffusion of H_x found to exist in the form

$$\vec{H}_3 = \vec{H}_{30}(1-e^{\beta_3 y}) + \vec{g}(z)e^{\beta_3 y} \quad , y \leq 0, \; z=0 \qquad (28)$$

$$\vec{H}_1 = \vec{H}_{10}(1-e^{-\beta_1 y}) + \vec{g}(z)e^{-\beta_1 y} \quad , y \geq 0, \; z=0 \qquad (29)$$

The rather arbitrary choice of $\vec{E}$ and $\vec{H}$ on z=0 may seem unjustified at this stage but it will be found that a function of this type is needed later in this thesis, even if it represents a crude approximation to the true case.

Summarizing, for the case of H -polarization three fields exist: H_x, E_y and E_z. The fields H_x and E_y appear to be continuous functions with continuous first derivatives each approaching constant and known values at $\pm\infty$ but the value of E_z appears to have a distribution with a maximum (cusp) value at y=0 and tending to zero at $\pm\infty$.

When considering the case of a wedge there is also a transmitted term to be added to the diffusion of an electric field which flows across the land-sea contact. As the angle α approaches zero it is found that there is less and less diffusion because of the difference in electric field intensities in the two media and more propagation of current as a simple transmitted and reflected ray problem. Hence in the limiting case of horizontal layers (L. Cagniard (1953), A.T. Price (1962)) there is only a problem of transmission and reflection with no diffusion due to a difference in electric field intensity.

BOUNDARY CONDITIONS

A solution of Maxwell's equations is given by the Fresnel equations at $y=\pm\infty$ for the air-land, air-sea contacts. If it is assumed that the initial wave is of the form: $\vec{E}_i = \vec{E}_o e^{i(\omega t - k_2 \hat{n}_o \cdot \vec{r})}$ $\quad H_i = \frac{k_2}{\omega\mu_2} \hat{n}_o \times \vec{E}_i$

then the transmitted and reflected waves will be of the form:

$$\vec{E}_t = \vec{E}_1 e^{i(\omega t - k_1 \hat{n}_1 \cdot \vec{r})}, \quad \vec{H}_t = \frac{k_1}{\omega\mu_1} \hat{n}_1 \times \vec{E}_t$$

and

$$\vec{E}_r = \vec{E}_2 e^{i(\omega t - k_2 \hat{n}_2 \cdot \vec{r})}, \quad \vec{H}_r = \frac{k_2}{\omega\mu_2} \hat{n}_2 \times \vec{E}_r$$

For E-polarization $\quad \vec{E}_1 = C_1 \vec{E}_o \quad$ and $\quad \vec{E}_2 = C_2 \vec{E}_o$

For H-polarization $\quad \vec{H}_1 = G_1 \vec{H}_o \quad$ and $\quad \vec{H}_2 = G_2 \vec{H}_o$

where C_1, C_2, G_1 and G_2 are known constants for the air-sea, air-land boundary (see Appendix 2). Hence the magnetic and electric field can be calculated at $y=\pm\infty$ for both H-polarization and E-polarization.

At any boundary,

$$\hat{n} \times (\vec{E}_i + \vec{E}_r) = \hat{n} \times \vec{E}_t \quad \text{and} \quad \hat{n} \times (\vec{H}_i + \vec{H}_r) = \hat{n} \times \vec{H}_t \qquad (30)$$

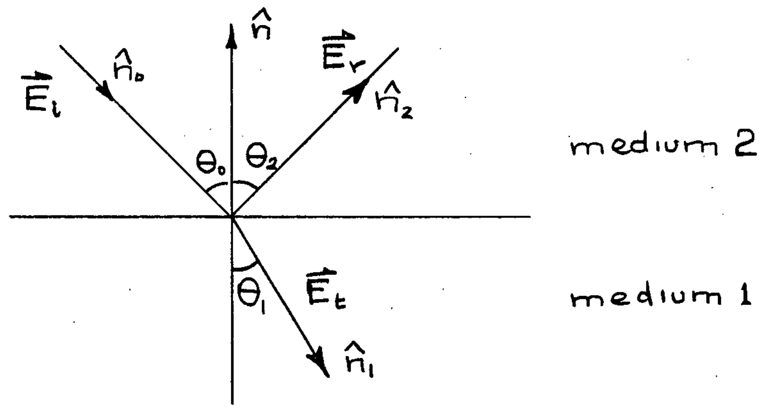


FIGURE V Incident, transmitted and reflected rays at a boundary.

For H -polarization these equations give:

$$\vec{H}_o e^{-im_2 y} + \vec{H}_r(y) = H_t(y)$$

where $\vec{H}_t(y)$ is the magnetic field inside the conductor. Let us call this function on z=0 $G_3(y)$ in land, $G_1(y)$ in sea and $G(y)$ in general. A similar case exists for E -polarization,

$$\vec{E}_o e^{-im_2 y} + \vec{E}_r(y) = \vec{E}_t(y)$$

Let the transmitted electric field on z=0 in land be called $C_3(y)$ and in sea $C_1(y)$ and $C(y)$, in general.

In the previous section an intelligent guess was made of the values of $G_1(y)$, $G_3(y)$, $C_1(y)$, and $C_3(y)$ but no use will be made of this until it is necessary.

At the land-sea contact for H-polarization the tangential component of the electric field is continuous, i.e. E_Z is continuous.

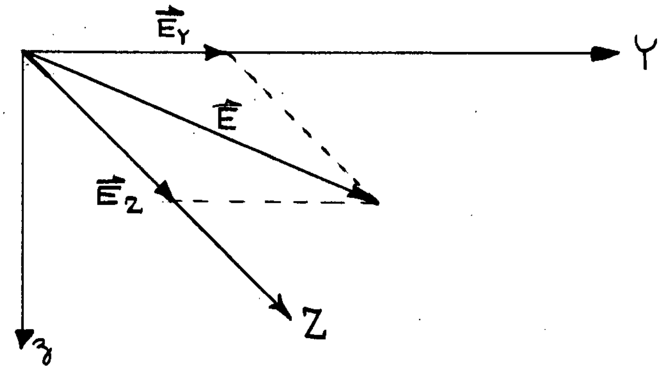


FIGURE VI General field vector in nonrectangular co-ordinates.

$$E_Z = (\operatorname{cosec}\alpha)\, E_z$$

and

$$\vec{E} = \frac{1}{\sigma} \nabla \times \vec{H}$$

so that $E_z = \frac{1}{\sigma}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) = -\frac{1}{\sigma}\frac{\partial H_x}{\partial y} = -\frac{1}{\sigma}\frac{\partial H_x}{\partial Y}$

Therefore $E_Z = -\frac{\operatorname{cosec}\alpha}{\sigma} \frac{\partial H_x}{\partial Y}$

and $\frac{1}{\sigma_3}\frac{\partial H_3}{\partial Y} = \frac{1}{\sigma_1}\frac{\partial H_1}{\partial Y}$

For E-polarization the tangential component of the magnetic field is continuous and a similar set of equations results.

Hence the boundary conditions required to solve Maxwell's equations are, for H-polarization:

H(i) The transmitted magnetic field, $G(y)$ = the magnetic field in the air on the boundary plane $Z = 0, -\infty < Y < +\infty$

H(ii) $H_1 = H_3 = g(Z)$ on $Y = 0$.

H(iii) $E_{tangential}$ is continuous, i.e.

$$\frac{1}{\sigma_3}\frac{\partial H_3}{\partial Y} = \frac{1}{\sigma_1}\frac{\partial H_1}{\partial Y} \qquad \text{on } Y = 0$$

H(iv) $H_1, H_3, \frac{\partial H_1}{\partial Z}, \frac{\partial H_3}{\partial Z}, \frac{\partial H_1}{\partial Y}$ and $\frac{\partial H_3}{\partial Y}$ all tend to zero as $Z \rightarrow +\infty$.

H(v) $H_1 = G_1 H_0, Y = +\infty$ and $H_3 = G_3 H_0, Y = -\infty$, which are all known, constant and finite.

For E-polarization the boundary conditions are:

E(i) The transmitted electric field $C(Y)$ = the electric field in the air on $Z = 0, -\infty < y < +\infty$.

E(ii) $E_3 = E_1 = f(Z)$ on $Y = 0$

E(iii) $\frac{\partial E_3}{\partial Y} = \frac{\partial E_1}{\partial Y}$ on $Y = 0$

E(iv) $E_1, E_3, \frac{\partial E_1}{\partial Z}, \frac{\partial E_3}{\partial Z}, \frac{\partial E_1}{\partial Y}$ and $\frac{\partial E_3}{\partial Y}$ all tend to zero as $Z \rightarrow \infty$ $(\alpha \neq 0)$.

E(v) $E_1 = C_1 E_0, Y \rightarrow +\infty$ and $E_3 = C_3 E_0, Y = -\infty$, which are all known, constant and finite.

It will be noted that in the case $\alpha = 90°$ $Y = y$ and $Z = z$ and the boundary conditions still hold true.

BOUNDARY CONDITIONS FOR $\alpha = 90°$

The boundary conditions for the general case have already been enumerated. These conditions will now be adapted to the case $\alpha = 90°$. In addition the boundary conditions across the land-sea and air contact will be given so that the values of $G(y)$ and $C(y)$ can be computed knowing the form of the electric and magnetic fields in the air and in the land-sea conductors and equating them on the boundary z=0.

For H-polarization:

H(i) The transmitted magnetic field equals the total magnetic field in the air which has the value $G(y)$ on the boundary plane $z = 0$, $-\infty < y < +\infty$.

H(ii) $H_1 = H_3 = g(z)$ on $y = 0$.

H(iii) The tangential component of the electric field is continuous, i.e.

$$\frac{1}{\sigma_3}\frac{\partial H_3}{\partial y} = \frac{1}{\sigma_1}\frac{\partial H_1}{\partial y} \quad \text{on} \quad y = 0.$$

H(iv) H_1, H_3, $\frac{\partial H_1}{\partial z}$, $\frac{\partial H_3}{\partial z}$, $\frac{\partial H_1}{\partial y}$ and $\frac{\partial H_3}{\partial y}$ all tend to zero as z tends to infinity.

H(v) $H_1 = G_1 H_0 e^{-\beta_1 z}$ for y equal to $+\infty$ and $H_3 = G_3 H_0 e^{-\beta_3 z}$ for y equal to $-\infty$, where the values of G_1 and G_3 are known, constant and finite.

H(vi) The reflected wave in air is propagated upwards, in the negative direction.

H(vii) The tangential components of the electric field are continuous across the land-sea and air contact, i.e.

$$E_{y_1} = E_{y_2}, \quad y > 0, \; z = 0$$
$$E_{y_3} = E_{y_2}, \quad y < 0, \; z = 0$$

For E-polarization:

E(i) The transmitted electric field equals the total electric field in the air which has the value $C(y)$ on the boundary plane $z = 0$, $-\infty < y < +\infty$.

E(ii) $E_3 = E_1 = f(z)$ on $y = 0$.

E(iii) The tangential components of the magnetic field are equal on the land-sea interface, i.e.

$$\frac{\partial E_3}{\partial y} = \frac{\partial E_1}{\partial y} \quad \text{on} \quad y = 0.$$

E(iv) E_1, E_3, $\frac{\partial E_1}{\partial z}$, $\frac{\partial E_3}{\partial z}$, $\frac{\partial E_1}{\partial y}$ and $\frac{\partial E_3}{\partial y}$ all tend to zero as z tends to infinity.

E(v) $E_1 = C_1 E_0 e^{-\beta_1 z}$ for y equal to $+\infty$ and

$E_3 = C_3 E_0 e^{-\beta_3 z}$ for y equal to $-\infty$, where the values of C_1 and C_3 are known, constant and finite.

E(vi) The reflected wave in the air is propagated upwards, in the negative direction.

E(vii) The tangential components of the magnetic field are continuous across the land-sea and air contact, i.e.

$$H_{y1} = H_{y2}, \quad y > 0, \quad z = 0$$

$$H_{y3} = H_{y2}, \quad y < 0, \quad z = 0$$

The continuity of the vertical components of the magnetic field is ensured by the condition of continuity of the electric field for E-polarization.

It is on the question of the correct boundary conditions that most of the earlier work has been in error. I. d'Erceville and G. Kunetz say that to solve the case of E -polarization one must assume that at the surface E_x is not dependent on y, which amounts to assuming that the vertical component of the magnetic field is zero on the surface. The fallacy of this assumption has already been pointed out in the "Introduction" to this thesis. D. Rankin, following the above authors assumes that, for the case of H -polarization, the vertical component of the electric field is zero on the surface. From this he concludes that

$$\frac{dH_x}{dy} = 0$$

If this were so then the function $G(y)$ would be equal to a constant, which is incorrect. Hence Rankin's assumption that the vertical component of the electric field is zero on the surface z=0 is incorrect. J.T. Weaver has also made the same incorrect assumption in stating that on z=0, $j_z = 0$ for H -polarization and $\frac{dH_y}{dy} = 0$ for E -polarization.

SOLUTION OF THE EQUATIONS FOR $\alpha = 90°$

H -polarization

When $\alpha = 90°$ the co-ordinate system is rectangular and the differential equation to be solved is thus simplified. Equation (11), viz.

$$\frac{\partial^2 H_x}{\partial y^2} + \frac{\partial^2 H_x}{\partial z^2} = i\eta^2 H_x \tag{50}$$

has to be satisfied with boundary conditions along y=0 and z=0. Because of the simplified differential equation it is possible to use a Fourier sine transform (see Appendix A5 for the general case). A Fourier cosine transform could also be used but it would be necessary to establish the boundary conditions in a slightly more complicated manner. The Fourier sine transform, with its inverse, can be expressed as follows

$$\left.\begin{aligned} \overline{\varphi(x)} &= \sqrt{\frac{2}{\pi}} \int_0^\infty \varphi(x) \sin\xi x \, dx \\ \text{and} \quad \varphi(x) &= \sqrt{\frac{2}{\pi}} \int_0^\infty \overline{\varphi(x)} \sin\xi x \, d\xi \end{aligned}\right\} \tag{51}$$

Application of the Fourier sine transform to the differential equation gives the following results:

$$\frac{d^2 \overline{H}_x}{dy^2} = \nu^2 \overline{H}_x - \xi \sqrt{\frac{2}{\pi}} H_x(y,0) \tag{52}$$

and

$$\nu^2 = \xi^2 + \beta^2 \qquad \text{where} \quad \beta^2 = i\eta^2 \tag{53}$$

If we now let $H_x(y,0) = G(y)$ it can be seen that there is only one unknown function whereas for the general case there are two (see Appendix A5).

The solution of this differential equation can be expressed as follows:

$$H_1 = \sqrt{\frac{2}{\pi}} \int_0^\infty \left[K_4 e^{-\nu_1 y} + \psi_1(y) \right] \sin \xi z \, d\xi \tag{54(a)}$$

$$H_3 = \sqrt{\frac{2}{\pi}} \int_0^\infty \left[K_1 e^{\nu_3 y} + \psi_3(y) \right] \sin \xi z \, d\xi \tag{54(b)}$$

where

$$K_4 = \overline{g(\xi)} - \psi_1(0) \quad , \qquad K_1 = \overline{g(\xi)} - \psi_3(0) \tag{55}$$

$$\psi_{1,3}(y) = \sqrt{\frac{2}{\pi}} \frac{\xi}{\nu_{1,3}^2} \sum_{n=0}^{\infty} \frac{1}{\nu_{1,3}^{2n}} \frac{d^{2n} G(y)}{dy^{2n}} \tag{56}$$

and

$$\overline{g(\xi)} = \frac{\nu_1 \sigma_3 \psi_1(0) + \nu_3 \sigma_1 \psi_3(0) + \sigma_3 \psi_1'(0) - \sigma_1 \psi_3'(0)}{\nu_1 \sigma_3 + \nu_3 \sigma_1} \tag{57}$$

E -polarization

Similar solutions exist for the case of E -polarization. No proof will be given but the results will be stated:

$$E_1 = \sqrt{\frac{2}{\pi}} \int_0^\infty \left[K_2 e^{-\nu_1 y} + \Phi_1(y) \right] \sin \xi z \, d\xi \ , \ y \geqslant 0 \quad (58(a))$$

$$E_3 = \sqrt{\frac{2}{\pi}} \int_0^\infty \left[K_3 e^{\nu_3 y} + \Phi_3(y) \right] \sin \xi z \, d\xi \ , \ y \leqslant 0 \quad (58(b))$$

where $$\Phi_{1,3}(y) = \sqrt{\frac{2}{\pi}} \frac{\xi}{\nu_{1,3}^2} \sum_{n=0}^{\infty} \frac{1}{\nu_{1,3}^{2n}} \frac{d^{2n} C(y)}{d y^{2n}} \quad (59)$$

with $$K_{2,3} = f(\xi) - \Phi_{1,3}(0) \quad (60)$$

and $$f(\xi) = \frac{\nu_1 \Phi_1(0) + \nu_3 \Phi_3(0) + \Phi_1'(0) - \Phi_3(0)}{\nu_1 + \nu_3} \quad (61)$$

Having obtained the basic formulae for the magnetic and electric fields, explicit expressions for $\psi_1(y)$, $\psi_3(y)$ $\Phi_1(y)$ and $\Phi_3(y)$ will be obtained.

The first approximation is that ψ and Φ are equal to one constant for $y \geqslant 0$ and to another for $y \leqslant 0$. This is too simple an assumption, and leads to false physical results (i.e. a step-function for the magnetic and electric fields for z=0). The next estimate would be to apply the function noted in the physical interpretation with the values

$$E_{30} = C_3 E_0$$

$$E_{10} = C_1 E_0$$

$$H_{30} = G_3 H_0$$

and $$H_{10} = G_1 H_0$$

The values of the ψ's and Ψ's can then be calculated completely as follows. Heaviside's operator formula for exponentials is

$$\frac{1}{F(D)}[e^{ax}] = \frac{e^{ax}}{F(a)} \quad , \quad F(a) \neq 0 \tag{61}$$

Applying this to $G(y)$ and $C(y)$ and noting that

$$\frac{1}{1 - \frac{\beta^2}{\nu^2}} = \frac{\nu^2}{\xi^2} \tag{62}$$

then $\psi_1(y), \psi_3(y), \Psi_1(y)$ and $\Psi_3(y)$ become

$$\psi_1(y) = \sqrt{\frac{2}{\pi}}\,\frac{\xi}{\nu_1^2}\left[H_{10} + (g(0) - H_{10})\,e^{-\beta_1 y}\,\frac{\nu_1^2}{\xi^2}\right] \tag{63(a)}$$

(A proof of this is given in Appendix A4).

Similarly

$$\psi_3(y) = \sqrt{\frac{2}{\pi}}\,\frac{\xi}{\nu_3^2}\left[H_{30} + (g(0) - H_{30})\,e^{\beta_3 y}\,\frac{\nu_3^2}{\xi^2}\right] \tag{63(b)}$$

$$\Psi_1(y) = \sqrt{\frac{2}{\pi}}\,\frac{\xi}{\nu_1^2}\left[E_{10} + (f(0) - E_{10})\,e^{-\beta_1 y}\,\frac{\nu_1^2}{\xi^2}\right] \tag{64(a)}$$

$$\Psi_3(y) = \sqrt{\frac{2}{\pi}}\,\frac{\xi}{\nu_3^2}\left[E_{30}+(f(0)-E_{30})e^{\beta_3 y}\frac{\nu_3^2}{\xi^2}\right] \qquad (64(b))$$

The question of the validity of Heaviside's method may be raised but the results can be verified by substituting equations (63(a),(b)) and (64(a),(b)) into equations (11) (12).

For the case y=0

$$\psi_{1,3}(0) = \sqrt{\frac{2}{\pi}}\,\frac{\xi}{\nu_{1,3}^2}\left[H_{10,30}+(g(0)-H_{10,30})\frac{\nu_{1,3}^2}{\xi^2}\right] \qquad (65(a))$$

$$\Psi_{1,3}(0) = \sqrt{\frac{2}{\pi}}\,\frac{\xi}{\nu_{1,3}^2}\left[E_{10,30}+(f(0)-E_{10,30})\frac{\nu_{1,3}^2}{\xi^2}\right] \qquad (65(b))$$

and $$\psi'_{1,3}(0) = \sqrt{\frac{2}{\pi}}\,\frac{(\mp\beta_{1,3})}{\xi}(g(0)-H_{10,30}) \qquad (66(a))$$

$$\Psi'_{1,3}(0) = \sqrt{\frac{2}{\pi}}\,\frac{(\mp\beta_{1,3})}{\xi}(f(0)-E_{10,30}) \qquad (66(b))$$

Substituting (63(a)) into equation (54(a)) and letting $y \to \infty$ the boundary condition $H(v)$ for $\alpha=90°$ is satisfied, i.e.

$$H_1 = \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\,\frac{\xi}{\nu_1^2}H_0 G_1 \sin\xi z\, d\xi = H_0 G_1 e^{-\beta_1 z}$$

also $$H_3 = H_0 G_3 e^{-\beta_3 z}, \quad E_1 = E_0 C_1 e^{-\beta_1 z} \text{ and } E_3 = E_0 C_3 e^{-\beta_3 z}.$$

This shows that the suggested solution is, at least, well behaved at the limits $y \to \pm\infty$.

The unknown function $g(0)$ may be found by equating the tangential components of the electric field at (0,0,0).

i.e. $$\frac{1}{\sigma_1}\frac{\partial H_1}{\partial y} = \frac{1}{\sigma_3}\frac{\partial H_3}{\partial y} \qquad \text{on } z=0$$

$$-\sigma_3\beta_1\left(g(0)-G_1H_0\right) = \sigma_1\beta_3\left(g(0)-G_3H_0\right)$$

$$g(0) = \frac{(G_1\sigma_3\beta_1 + G_3\sigma_1\beta_3)H_0}{\sigma_3\beta_1+\sigma_1\beta_3} \tag{67(a)}$$

Similarly,

$$f(0) = \frac{(C_1\beta_1 + C_3\beta_3)E_0}{\beta_1+\beta_3} \tag{67(b)}$$

Hence the functions $\psi_1(y)$, $\psi_3(y)$, $\Psi_1(y)$ and $\Psi_3(y)$ are all known and may be computed.

Values of the electric and magnetic field may now be found by direct substitution and when one lets $z \to 0$ (from the positive side) the integral should be equal to $G(y)$ for H-polarization and $C(y)$ for E-polarization if the analytical procedures employed are self consistent. Unfortunately, even in the case of this relatively simple estimate, the evaluation of the integral is extremely complex (due to the presence of branch points) and can only be evaluated by numerical means.

To ensure that $G(y)$ and $C(y)$ are the correct functions on the surface z=0, it is necessary to equate the magnetic and electric fields in the conducting media to the magnetic and electric fields in the air using the boundary conditions H(i), (vi) and (vii) for the magnetic field and boundary conditions E(i), (vi) and (vii) for the electric field.

Applying condition H(vii) for H-polarization it is found that:

$$\frac{1}{\sigma_1}\frac{\partial H_1}{\partial z} = -\frac{i}{\omega\epsilon}\frac{\partial H_2}{\partial z}\,, \qquad \text{on } z=0 \tag{68}$$

The value of H_1 is given by equation (54(a)), as z tends to zero from the positive side, and the value of H_2 is given in part by equation (21.), as z tends to zero from the negative side. Therefore:

$$\left.\begin{aligned} &\lim_{z\to 0+}\left\{\frac{1}{\sigma_1}\frac{\partial}{\partial z}\left(\sqrt{\frac{2}{\pi}}\int_0^\infty\left[K_4 e^{-\nu_1 y} + \Psi_1(y)\right]\sin\xi z\,d\xi\right)\right\} = \\ &-\lim_{z\to 0-}\left\{\frac{i}{\omega\epsilon}\frac{\partial}{\partial z}\left(\frac{1}{2\pi}\int_{-\infty}^{+\infty}dm\,e^{-i(my+nz)}\left[\int_0^\infty G_1(\eta)e^{im\eta}d\eta + \right.\right.\right. \\ &\left.\left.\left.\int_{-\infty}^0 G_3(\eta)e^{im\eta}d\eta\right] - H_0\left[e^{-i(m_2 y - n_2 z)} - e^{-i(m_2 y + n_2 z)}\right]\right)\right\} \end{aligned}\right\} \tag{69}$$

The order of differentiation and integration is interchangeable on the right hand side and so is the order of taking the limit and integration. Hence, assuming normal incidence:

$$\text{R.H.S.} = -\lim_{z \to 0-} \frac{i}{\omega\epsilon}\left[-in_2 H_0(e^{in_2 z}+e^{-in_2 z}) - \frac{i}{2\pi}\int_{-\infty}^{+\infty} n(m)\,dm\, e^{-i(my+nz)}\right.$$
$$\left.\times\left\{\int_0^\infty G_1(\eta)e^{im\eta}d\eta + \int_{-\infty}^0 G_3(\eta)e^{im\eta}d\eta\right\}\right]$$

$$= -\frac{2n_2 H_0}{\omega\epsilon} - \frac{1}{2\pi\omega\epsilon}\int_{-\infty}^{+\infty} n(m)e^{-imy}dm\left\{\int_0^\infty G_1(\eta)e^{im\eta}d\eta\right.$$
$$\left. + \int_{-\infty}^0 G_3(\eta)e^{im\eta}d\eta\right\}$$

where $n_2 = \sqrt{\mu\epsilon\omega^2}$, from equation (17), and the value of $G(\eta)$ is given by the estimate. Having obtained the value of the right hand side of the equation, to prove that the estimated value of $G(y)$ is correct, the value of the left hand side must be computed and proved to be equal to the right hand side.

$$\text{L.H.S.} = \lim_{z \to 0+}\left\{\frac{1}{\sigma_1}\frac{\partial}{\partial z}\left(\sqrt{\frac{2}{\pi}}\int_0^\infty [K_4 e^{-\nu_1 y} + \psi_1(y)]\sin\xi z\, d\xi\right)\right\}$$

Let $I_1(y,z) = \sqrt{\frac{2}{\pi}}\int_0^\infty [-\psi_1(0)e^{-\nu_1 y} + \psi_1(y)]\sin\xi z\, d\xi$

$$= \frac{2}{\pi}\int_0^\infty \left[\frac{\xi}{\nu_1^2}(H_{10}(1-e^{-\nu_1 y})\right]\sin\xi z\, d\xi \qquad (70)$$

which, as y tends to infinity, gives the boundary condition $H(v)$ upon integration but otherwise, is an integral with a branch point. If the integral I is given by

$$I = \sqrt{\frac{2}{\pi}}\int_0^\infty [K_4 e^{-\nu_1 y} + \psi_1(y)]\sin\xi z\, d\xi$$

then $\quad I = I_1 + I_2$

where $\quad I_2 = \sqrt{\frac{2}{\pi}} \int_0^\infty \overline{g(\xi)}\, e^{-\nu_1 y} \sin \xi z \, d\xi$

and
$$I_2 = \frac{2}{\pi} \int_0^\infty \left[\frac{(\nu_3 \sigma_3 H_{10} + \nu_1 \sigma_1 H_{30})/\nu_1 \nu_3 + \sigma_3 (g(0) - H_{10})(\nu_1 - \beta_1)/\xi^2 + \sigma_1 (g(0) - H_{30})(\nu_3 - \beta_3)/\xi^2}{\nu_1 \sigma_3 + \sigma_1 \nu_3} \right] \times \xi e^{-\nu_1 y} \sin \xi z \, d\xi \tag{71}$$

This integral can be subdivided into three more integrals, i.e. let $\quad I_2 = I_a + I_b + I_c$

where

$$I_a = \frac{2}{\pi} \int_0^\infty \frac{\xi (\nu_3 \sigma_3 H_{10} + \nu_1 \sigma_1 H_{30}) e^{-\nu_1 y} \sin \xi z \, d\xi}{(\nu_1 \sigma_3 + \nu_3 \sigma_1)\, \nu_1 \nu_3} \tag{71(a)}$$

$$I_b = \frac{2}{\pi} \int_0^\infty \frac{(g(0) - H_{10})(\nu_1 - \beta_1)\, \sigma_3\, e^{-\nu_1 y} \sin \xi z \, d\xi}{(\nu_1 \sigma_3 + \sigma_1 \nu_3)\, \xi} \tag{71(b)}$$

and
$$I_c = \frac{2}{\pi} \int_0^\infty \frac{(g(0) - H_{30})(\nu_3 - \beta_3)\, \sigma_1\, e^{-\nu_1 y} \sin \xi z \, d\xi}{(\nu_1 \sigma_3 + \sigma_1 \nu_3)\, \xi} \tag{71(c)}$$

These integrals possess complex branch-points making their mathematical evaluation extremely complicated. It is suggested that these equations can be better solved on a computer and the results plotted on a graph. If the results plotted from the right hand side of the equation fall on the curve derived from the left hand side then it may be assumed that the estimate satisfies all boundary conditions and differential equations.

The equations for E-polarization, which are similar to H-polarization, are

$$I_3 = \sqrt{\frac{2}{\pi}}\int_0^\infty \left[-\Psi_1(0)e^{-\nu_1 y} + \Psi_1(y)\right]\sin\xi z\,d\xi = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\xi}{\nu_1^2}\left[E_{10}(1-e^{-\nu_1 y})\right]\sin\xi z\,d\xi \qquad (72)$$

and $\quad I = I_3 + I_4$

where I_4 is given by

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \left[f(\xi_1)e^{-\nu_1 y}\right]\sin\xi z\,d\xi \quad , \qquad y \geq 0 \qquad (73)$$

and

$$I_4 = \frac{2}{\pi}\int_0^\infty \left[\frac{(\nu_3 E_{10} + \nu_1 E_{30})/\nu_1\nu_3 + (f(0) - E_{10})(\nu_1 - \beta_1)/\xi^2 + (f(0) - E_{30})(\nu_3 - \beta_3)/\xi^2}{(\nu_1 + \nu_3)}\right]$$

$$\times\, \xi\, e^{-\nu_1 y}\sin\xi z\,d\xi \qquad (74)$$

Again $\quad I_4 = I_d + I_e + I_f$

where

$$I_d = \frac{2}{\pi}\int_0^\infty \xi\frac{(\nu_3 E_{10} + \nu_1 E_{30})}{(\nu_1 + \nu_3)\,\nu_1\nu_3}\,e^{-\nu_1 y}\sin\xi z\,d\xi \qquad (74(a))$$

$$I_e = \frac{2}{\pi}\int_0^\infty \frac{(f(0) - E_{10})(\nu_1 - \beta_1)}{(\nu_1 + \nu_3)\,\xi}\,e^{-\nu_1 y}\sin\xi z\,d\xi \qquad (74(b))$$

and

$$I_f = \frac{2}{\pi}\int_0^\infty \frac{(f(0) - E_{30})(\nu_3 - \beta_3)}{(\nu_1 + \nu_3)\,\xi}\,e^{-\nu_1 y}\sin\xi z\,d\xi \qquad (74(c))$$

The numerical integration of each of these equations may be obtained by seperating the functions into real and imaginary parts and evaluating each integral on a computer.

The vertical magnetic field can be found, for the case of E-polarization, from equation (10) to be

$$H_z = -\frac{i}{\mu\omega}\frac{\partial E_x}{\partial y}$$

Therefore:

$$H_{z1} = \frac{i2}{\mu\omega\pi}\int_0^\infty \left[\frac{(\nu_3 E_{10}+\nu_1 E_{30})/\nu_1\nu_3 + (f(0)-E_{10})(\nu_1-\beta_1)/\xi^2 + (f(0)-E_{30})(\nu_3-\beta_3)/\xi^2}{(\nu_1+\nu_3)}\right]$$

$$\times\, \xi\,\nu_1\, e^{-\nu_1 y}\sin\xi z\, d\xi, \qquad y \geqslant 0 \tag{75}$$

A similar equation exists for $y \leqslant 0$.

If the estimate $C(y)$ is correct then

$$\left.\begin{aligned} H_{z1} &= \frac{i}{\mu\omega}\beta_1(f(0)-E_{10})\,e^{-\beta_1 y} \\ \text{and} \quad H_{z3} &= \frac{i}{\mu\omega}\beta_3(f(0)-E_{30})\,e^{-\beta_3 y}\end{aligned}\right\} \quad y \geqslant 0 \tag{76}$$

It will be noticed that the vertical magnetic field is independent of the frequency at y=0. β_1 is proportional to the square root of the frequency and so is $(f(0) - E_{10})$, and hence the only frequency dependent term is $\exp(-\beta_1 y)$. This result was obtained, assuming that

$$\frac{\partial E_1}{\partial z} = \frac{\partial E_3}{\partial z} = -i\,\omega B \tag{77}$$

on z=0, where $\mathbf{B}$ is constant, by J.T. Weaver (1963) and also by T. Watanabe in a derivation of equations for magnetotelluric modeling (private communication). J.T. Weaver has calculated the ratio of the vertical to horizontal magnetic field over the land-sea contact and his graphs are shown in Figure VII.

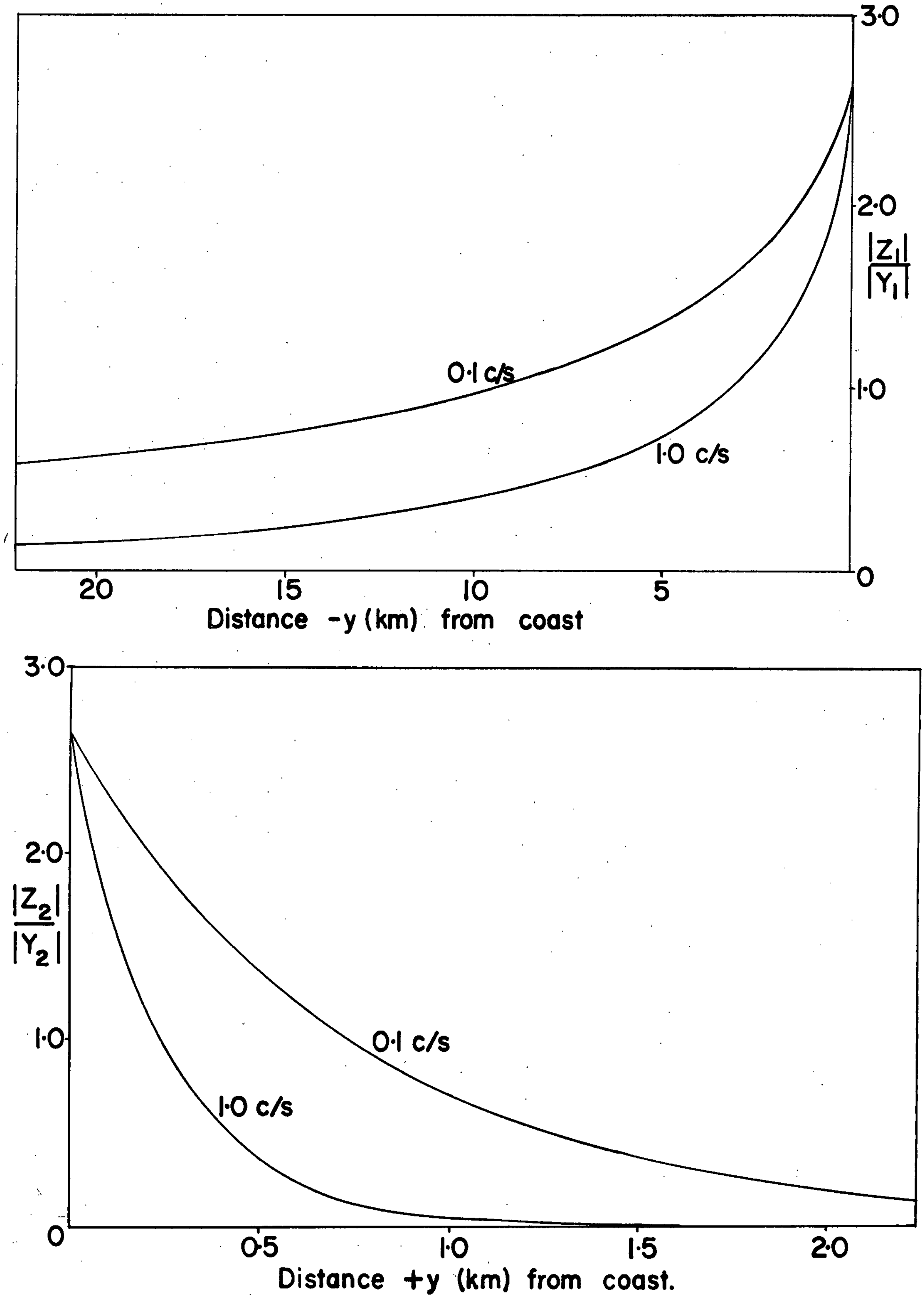


FIGURE VII Vertical magnetic anomaly, after J.T. Weaver.

PRACTICAL IMPLICATIONS

The vertical anomaly found in the Fraser River Delta has already been mentioned. This anomaly, it is thought, is due to relatively close-surface conductivity variations and the emphasis, if placed anywhere, must lie on periods in the range $0 < T \leqslant 50$ seconds. Longer periods occur less frequently and are less accurately plotted. Further the theory so far expounded of lateral skin-depth could not be interpreted for depths of over 5 km. in the sea at an absolute maximum and therefore longer periods would be of no interest here.

Geomagnetic bays, with a period of about 1 hour (3600 seconds) have been recorded by U. Schmucker on magnetograms in California. Stations were set up on a line perpendicular to the sea coast for a distance of just under 300 km. at San Francisco and La Jolla. Readings were also taken on the San Clemente Islands for the La Jolla traverse (see Figure IX). A study of the vertical component of the magnetic fields for well behaved bays, i.e. the electric field vector parallels the coast, showed that at the sea coast there was a pronounced disturbance but the further inland the records were taken the smaller the magnitude of the disturbance until at a distance of 100 km. or less the disturbance disappeared (see magnetogram records Figure VIII). If one assumes that the theory as previously described is

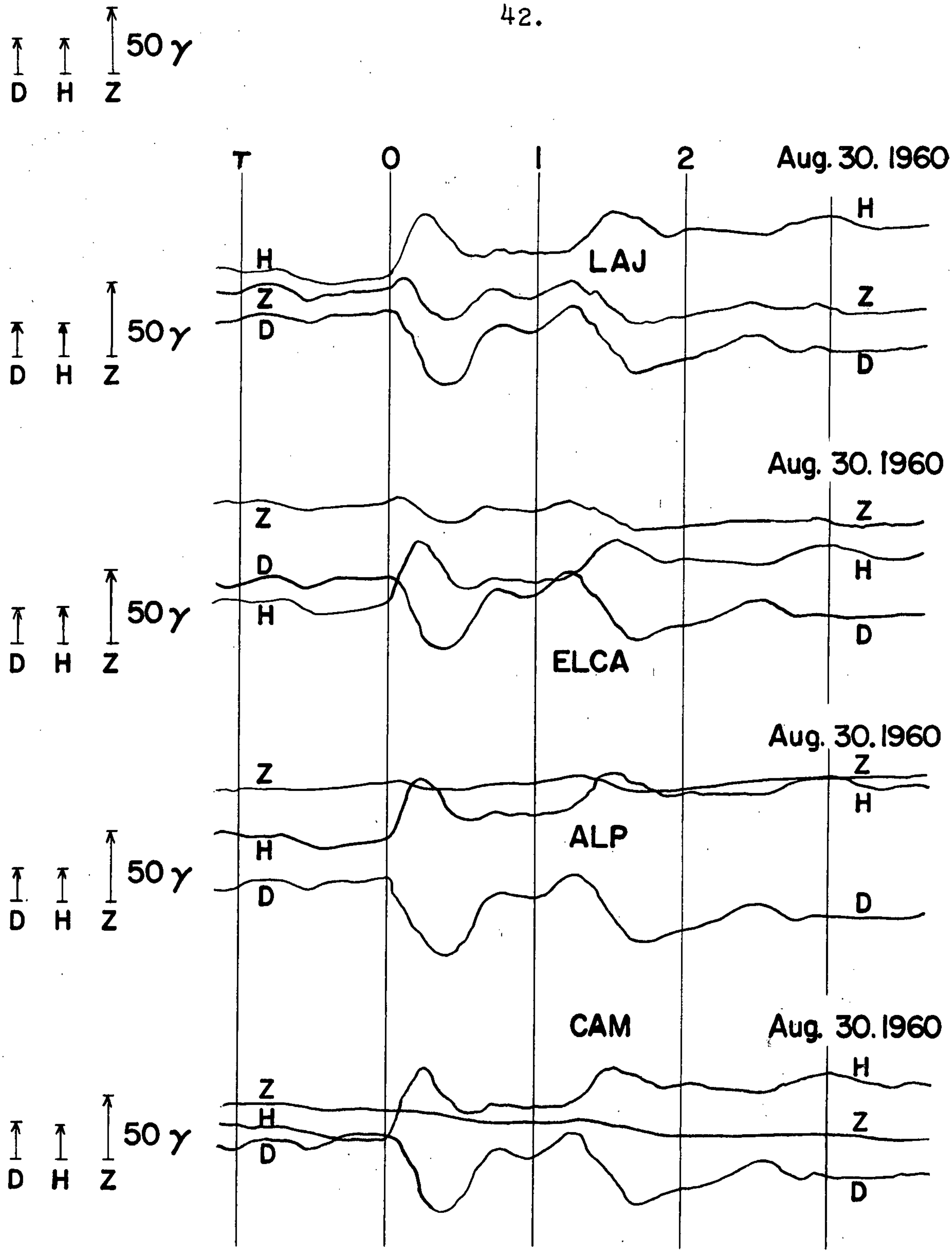


FIGURE VIII Magnetogram records, adapted from U. Schmucker.

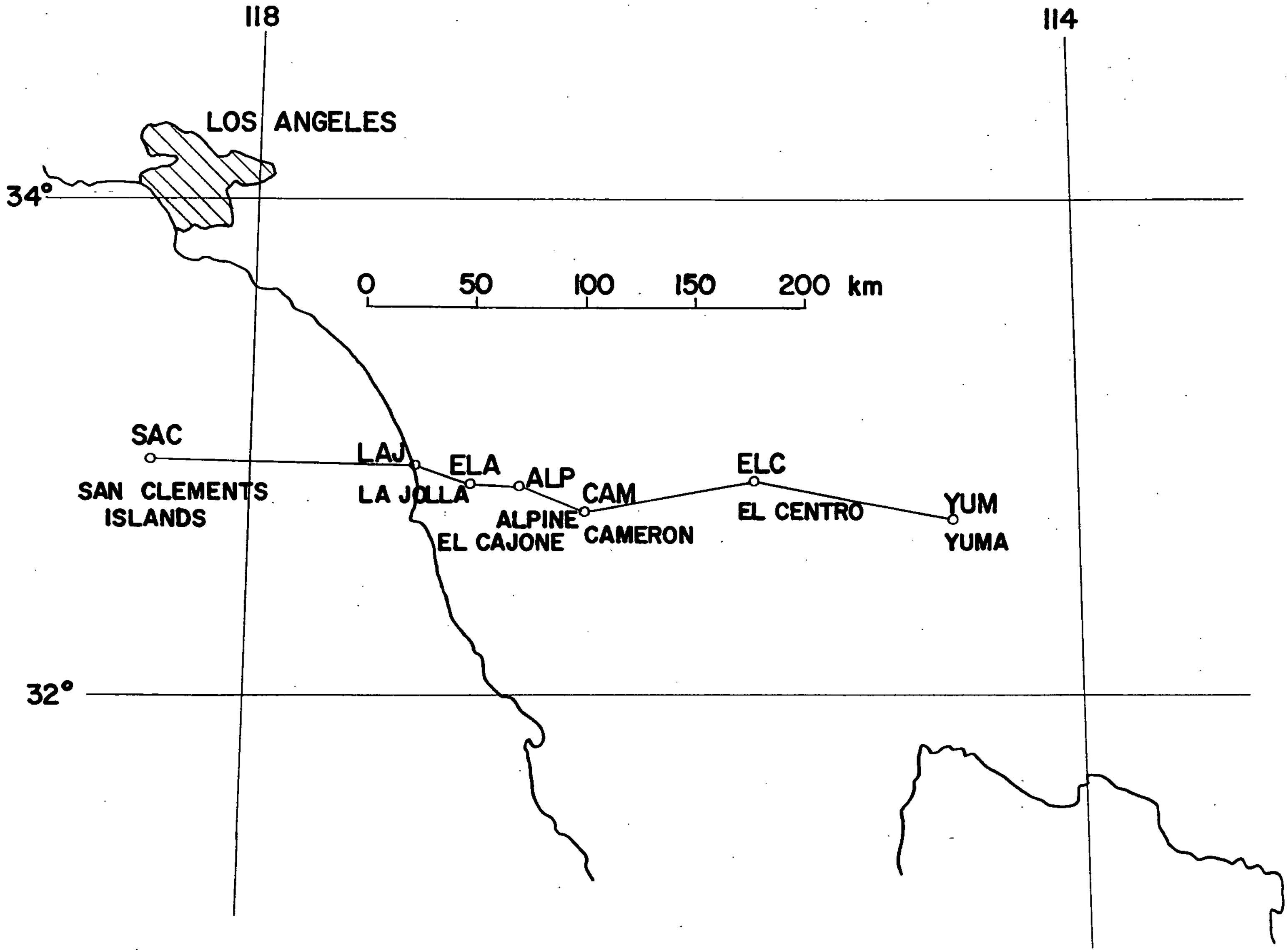


FIGURE IX   Map of Southern California.

correct one is led to the conclusion that there is a lateral diffusion current arising from lateral inhomogeneities in the conductivity. If this is the case then crude order of magnitude estimations may be carried out to find the approximate dimensions of this conductivity anomaly.

If $\mu = 1.257 \times 10^{-6}$ henry/m., $\omega = 3600$ cps, and $\mathfrak{z} = 100$ km. where $\mathfrak{z}$ is taken as the lateral dimension of the diffusion, the conductivity, σ , can be determined from the relation

$$\frac{k}{\sqrt{2}} = \frac{1}{\mathfrak{z}} = \sqrt{\tfrac{1}{2}\mu\omega\sigma}$$

σ is found to be of the order of 10^{-1} mho/m.

S.P. Srivastava's Ph.D. thesis (1962) contained a study of the subsurface conductivity of the planes of Alberta using the principles of magnetotellurics as expounded by L. Cagniard (1953). Srivastava concluded that for his survey the distribution of the conductivity with depth could be summarized as follows:

0 to 5 km., 10^{-1} mho/m.

5 to 90 km., 10^{-4} mho/m.

90 to 150 km., 10^{-1} mho/m.

T. Rikitake (1951) considered a two layer model, corresponding to Srivastava's second and third layers, but he placed his depth of contact closer to 400 km. than to 100 km.

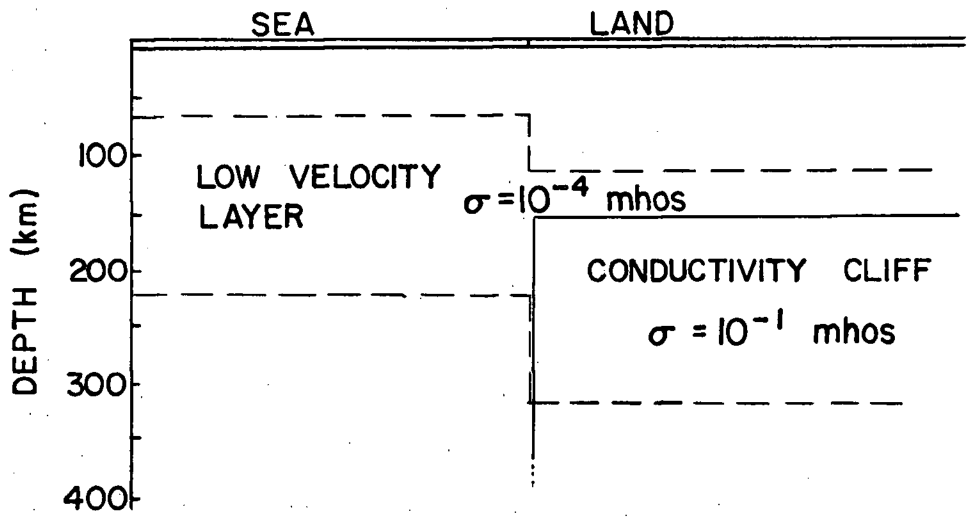


FIGURE X Schematic cross-section at the sea coast for an oversimplified conductivity cliff.

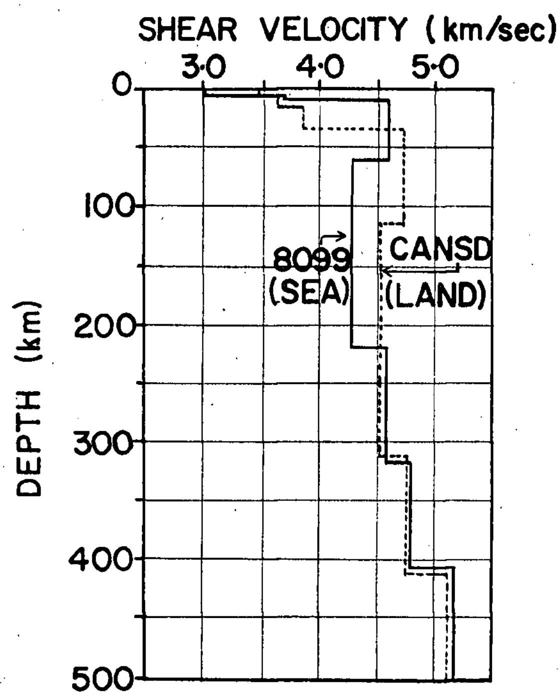


FIGURE XI Velocity depth curve (shear waves) under the Canadian Shield (Model CANSD) and under the ocean (Model 8099), after J. Brune and J. Dorman.

The initial value of 10^{-1} mho/m., ranging from a depth of 0 to 5 km., is not thick enough to appreciably influence the overall effective conductivity for the first 90 km. Therefore to a first approximation, the conductivity at a depth of 90 km. or more is much larger than the conductivity above, i.e.

$$\frac{\sigma_{50km}}{\sigma_{90km}} = \frac{1}{1000}$$

If this stratification of conductivity is extended to the Pacific Coast and values of the conductivities and depths there are compared with Srivastava's a striking similarity is found to exist. Two points must be emphasized:

(1) The height of the surface of the Earth above the conducting layer (10^{-1} mho/m.) would be of the order of 100 km. and hence the anomalous vertical field would flatten considerably, i.e. the value of the skin-depth would be of the order of 75 km. rather than 100 km. (The alternative to raising the depth would be to lower the conductivity contrast.)

(2) The vertical anomaly would arise, in the first approximation, from a "cliff" in the conductivity at depth; where the value of σ at 200 km. would be 10^{-1} mho/m. under the land and of the order of 10^{-4} mho/m. under the sea.

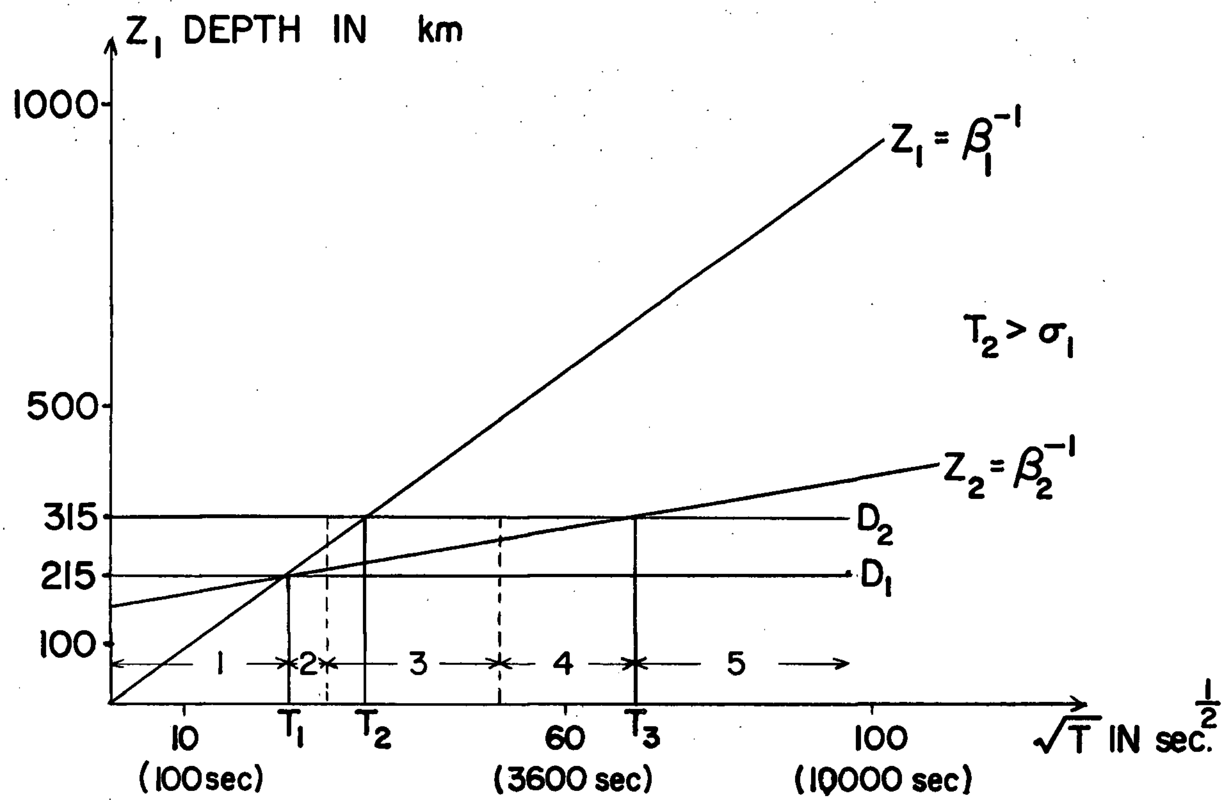


FIGURE XII(a) Idealized plot of skin depth vs. period for two conductivities.

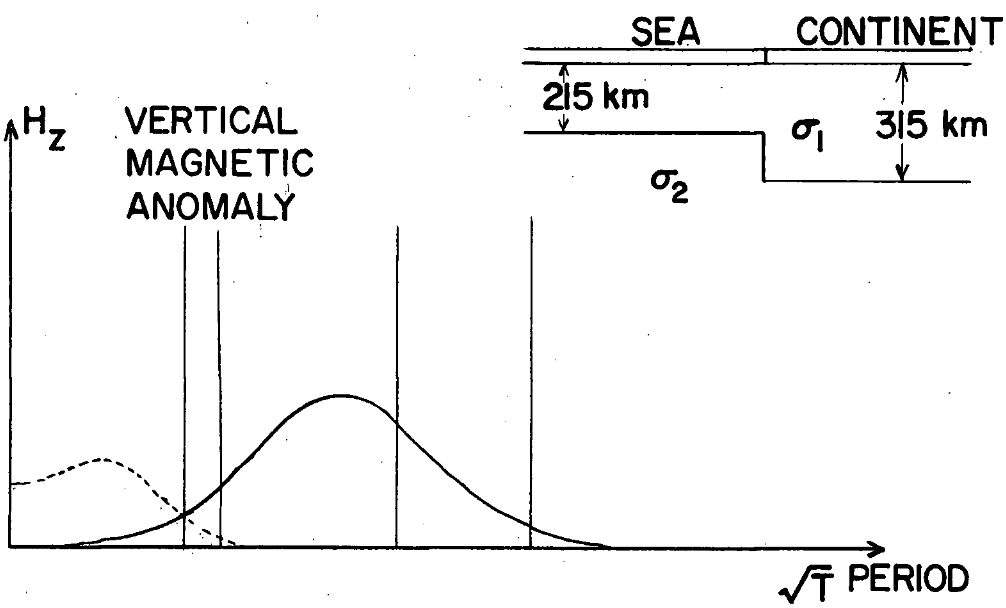


FIGURE XII(b) Probable variation of H_Z with period.

One is led to the conclusion that the variation of conductivity with depth might be as illustrated in Figure X. It would seem logical to try and check this result with other information about the Earth's mantle and crust derived from seismic sources. L. Sykes et al. (1962) have proposed a velocity depth curve for the mantle under the sea, and have suggested that there is a low velocity layer between the depths of 60 and 215 km. J. Brune and J. Dorman (1963) have suggested a velocity depth curve for the mantle and crust under the Canadian Shield. Their results show that there is a low velocity layer between the depths of 115 and 315 km. Hence the low velocity layer must go from deep to shallower depths as it passes from under the Continents to under the Ocean, instead of passing from shallow to deeper depths as might be expected from a simple study of skin depths and conductivity. Hence it must be concluded that either the shape of the low velocity layer and the "isoconds", or lines of equal conductivity, are independent of each other or that the layers and the isoconds are related and the picture of the conductivity distribution is oversimplified. It would seem logical to assume that the latter conclusion is correct but the possibility of the former cannot be completely ruled out. Assuming that the low velocity layer and the isoconds parallel each other the morphology of the vertical magnetic anomaly for various periods may be found from the diagrammatic graphs of depth vs. period and vertical field vs. period (see Figures XII(a) and (b)). It will be noticed that there are five regions on the graphs, which correspond to (1) no vertical field, (2) growth of the field, (3) field magnitude reaches a maximum and starts to decline, (4) decay of the field and (5) no field. These five regions would correspond to (1) micropulsations, (2) long period micropulsations, (3) and (4) geomagnetic bays and (5) storms. The depth $\mathfrak{z}$ plotted in the first graph, Figure XII(a), is the skin depth and hence is a function of conductivity. The two curves $\mathfrak{z}_1$ and $\mathfrak{z}_2$ are plots for two different planes of infinite depth and constant conductivity. By interpolation, it is possible to obtain an idea of the shape of the vertical magnetic anomaly as it varies with period for a given conductivity step function. This might be a first approximation to the junction of the velocity depth curve as it passes under the sea coast. Assuming that the depth to the bottom of the layer under the land is D_1 (315 km.) and under the sea is D_2 (215 km.), the second graph is obtained, Figure XII(b).

The interpretation of the lateral skin depth becomes somewhat clouded if this picture of the conductivity distribution is assumed to be correct. However, although one may not be able to make good estimates of the conductivity at 100 and 200 km. under the land and the sea, the basic concept should not be discarded. Information can be obtained from the attenuation of the vertical component, but the exact theory must be reworked for this special case. It is pertinent, at this point, to quote J. Dorman, M. Ewing and J. Oliver (1960), who suggested a structure for the upper mantle of the Pacific Basin from surface wave dispersion:

> ...Rayleigh waves indicate that the low velocity region of the upper mantle extends upwards to much shallower depth beneath the oceans than beneath the continents. Rayleigh wave dispersion data for the paths on the Pacific basin are interpreted to indicate that shear velocity below the M (Moho) decreases to about 4.3 km/sec at depths of about 60 km and that shear velocities are somewhat lower than in the sub-continental mantle down to about 400 km.

The greatest need is for readings of the magnetic field covering the whole spectrum of periods, from 1 second to 24 hours, for distances of 0 to 1000 km. from the coast on the surface of the sea.

Corroborative evidence of Figure XII(b) can be seen from the magnetogram records of U. Schmucker. Vertical magnetic anomalies arise from micropulsations because of the land-sea contact. Unfortunately these cannot be seen from Schmucker's records. There is an anomaly in the vertical component of geomagnetic bays, indicating the presence of some sort of step (it is more probably a sloping contact) in the conductivity, but no vertical component anomaly is found in storms, indicating that the conductivity is laterally homogeneous at greater depths.

CONCLUSIONS

The values of G_1 and G_3 (obtained from Appendix II) show that $G_1 \simeq 2$ and $G_3 \simeq 2$ and hence that to a first approximation the result $G_1 = G_3 = A$, a constant, as used by J.T. Weaver (1963), is valid. An improvement would be to write the function

$$\begin{aligned} G(y) &= G_1 \qquad & y > 0 \\ &= G_3 \qquad & y \leqslant 0 \end{aligned}$$

but this would be physically unrealistic. Hence the estimate of $G(y)$, which seems to be physically sound, has been suggested to give the solution additional accuracy even though it has not been checked mathematically.

It is found that

$$C_1 \propto \frac{1}{\sqrt{\sigma_1}} \qquad \text{and} \qquad C_3 \propto \frac{1}{\sqrt{\sigma_3}}$$

and hence $\quad C_1 : C_3 = \frac{1}{2} : 10^3$

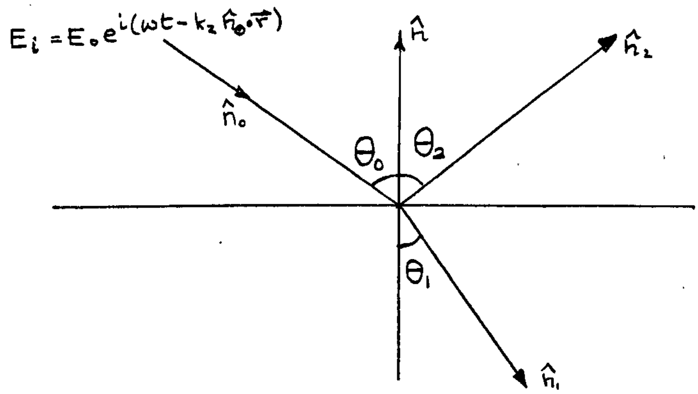
Thus the term C_3 is dominant. This fact is important in the computation of $C(y)$.

Graphs of the vertical magnetic field arising in the case of E-polarization have been calculated by J.T. Weaver for the approximations cited above and these have been included here with his very kind consent (see Figure VIII). The graphs have been calculated for $\omega = 0.01$ cps. and $\omega = 1$ cps. and are based on Weaver's equation assuming constant values across the discontinuity as opposed to a varying value of $C(y)$.

The theory of lateral skin depths has been tentatively applied to two cases, firstly to the vertical magnetic anomaly arising from the Fraser Delta experiment and secondly to the magnetogram records collected by U. Schmucker. Although there seems to be some doubt as to whether the theory may be freely applied in the latter case, one can conclude that there is some sort of lateral conductivity inhomogeneity. Its exact shape and form is not clear, but two models have been proposed, the first being a simple conductivity "cliff" and the second relying on information obtained by seismologists on the low velocity layer.

APPENDIX A1

Snell's Law applied to a Nonconductor-conductor Boundary



k_2 is the complex wave number in medium 2.

k_1 is the complex wave number in medium 1.

According to Snell's Law

$$k_1 \sin \theta_1 = k_2 \sin \theta_0$$

or $$(\alpha_1 + i\beta_1) \sin \theta_1 = (\alpha_2 + i\beta_2) \sin \theta_0 \qquad \text{A1(i)}$$

However, noting that, for air

$$\alpha_2 = \omega\sqrt{\mu_0 \epsilon_0} \;, \quad \beta_2 = 0 \qquad \text{A1(ii)}$$

and, for a conductor $\alpha_1 = \beta_1 = \sqrt{\dfrac{\omega \mu_1 \sigma_1}{2}}$ A1(iii)

we can express θ_1 as a function of α_1, α_2, β_1, and θ_0.

Hence

$$\sin \theta_1 = \sin \theta_0 \left[\omega\sqrt{\mu_0 \epsilon_0} \sqrt{\frac{2}{\omega \mu_1 \sigma_1}} \frac{1}{(1+i)} \right] \qquad \text{A1(iv)}$$

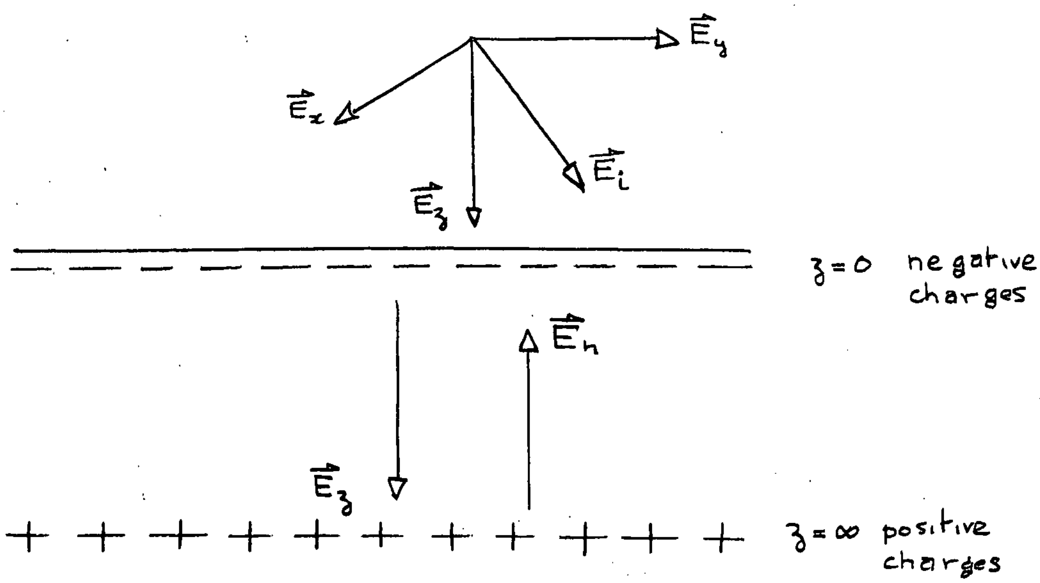
The condition, for which the displacement current is negligible is $|\sigma| \gg |i\omega\epsilon_0|$ and this is satisfied on the land-air and sea-air boundary. Using this condition in A1(iv) $\sin^2\theta_1$ is given by

$$\sin^2 \theta_1 = \sin^2 \theta_0 \left[\frac{\omega \epsilon_0 i}{-\sigma_1} \right] \simeq 0 \qquad \text{A1(v)}$$

assuming $\mu_0 = \mu_1$.

Since the expression for $\sin^2\theta_1$ is the product of two factors, its vanishing can equally well be the result of $\sin^2\theta_0 \simeq 0$ i.e. $\theta_0 \simeq 0$. Hence the condition that $\frac{\sigma}{\omega\epsilon_0} \gg 1$ has the same effect as assuming that the wave is propagated almost vertically and that only horizontal components of the electric and magnetic fields exist.

Physically it is possible to see this as follows: As soon as the wave passes into the conducting medium, the free charges, lying at random throughout the volume, migrate to the boundaries and set up an electric field, which is normal to the surfaces, E_n. This field opposes any vertical component of the field in the conductor, i.e. $E_n = E_z$.



APPENDIX A2

Fresnel Equations

If we assume an initial wave to be of the form

$$\vec{E}_i = \vec{E}_o e^{i(\omega t - k_2 \hat{n}_o \cdot \vec{r})} \quad , \quad \vec{H}_i = \frac{k_2}{\omega\mu_2} \hat{n}_o \times \vec{E}_i \qquad \text{A2(i)}$$

The transmitted and reflected waves will be of the form

$$\vec{E}_t = \vec{E}_1 e^{i(\omega t - k_1 \hat{n}_1 \cdot \vec{r})} \quad , \quad \vec{H}_t = \frac{k_1}{\omega\mu_1} \hat{n}_1 \times \vec{E}_t \qquad \text{A2(ii)}$$

and

$$\vec{E}_r = \vec{E}_2 e^{i(\omega t - k_2 \hat{n}_2 \cdot \vec{r})} \quad , \quad \vec{H}_r = \frac{k_2}{\omega\mu_2} \hat{n}_2 \times \vec{E}_r \qquad \text{A2(iii)}$$

The only unknown functions are C_1 and C_2 for E-polarization and G_1 and G_2 for H-polarization where C_1, C_2, G_1 and G_2 are defined by

$$\vec{E}_1 = C_1 \vec{E}_o \quad , \quad \vec{E}_2 = C_2 \vec{E}_o$$

and

$$\vec{H}_1 = G_1 \vec{H}_o \quad , \quad H_2 = G_2 \vec{H}_o$$

at the boundary

$$\hat{n} \times (\vec{E}_o + \vec{E}_2) = \hat{n} \times \vec{E}_1 \quad \text{and} \quad \hat{n} \times (\vec{H}_o + \vec{H}_2) = \hat{n} \times \vec{H}_1 \qquad \text{A2(iv)}$$

E-polarization

For E-polarization $\hat{n} \cdot \vec{E}_o = 0$

therefore $E_o + E_2 = E_1$

and $$\cos\Theta_0 E_0 - \cos\Theta_2 E_2 = \frac{\mu_2 k_1}{\mu_1 k_2}\cos\Theta_1 E_1$$

from equationA2(iv). Noting that

$$\Theta_0 = \Theta_2 \quad \text{and} \quad k_1\cos\Theta_1 = \sqrt{k_1^2 - k_2^2\sin^2\Theta_0}$$

we find

$$\vec{E}_1 = \frac{2\mu_1 k_2\cos\Theta_0}{\mu_1 k_2\cos\Theta_0 + \mu_2\sqrt{k_1^2 - k_2^2\sin^2\Theta_0}}\vec{E}_0$$

and

$$\vec{E}_2 = \frac{\mu_1 k_2\cos\Theta_0 - \mu_2\sqrt{k_1^2 - k_2^2\sin^2\Theta_0}}{\mu_1 k_2\cos\Theta_0 + \mu_2\sqrt{k_1^2 - k_2^2\sin^2\Theta_0}}\vec{E}_0$$

and hence the value of C_1 and C_2 may be obtained. It will be noticed that k_1 is complex but k_2 is real.

H-polarization

For H-polarization $\hat{n}\cdot\vec{H}_0 = 0$

and in this case $\hat{n}\cdot\vec{H}_0 = \hat{n}\cdot\vec{H}_1 = \hat{n}\cdot\vec{H}_2 = 0$

Noting that: $E_0 = -\frac{\omega\mu_2}{k_2}\hat{n}_0\times\vec{H}_0$, $\vec{E}_1 = \frac{-\omega\mu_1}{k_1}\hat{n}_1\times\vec{H}_1$

and $E_2 = -\frac{\omega\mu_2}{k_2}\hat{n}_2\times\vec{H}_0$

it is found that

$$\cos\Theta_0\vec{H}_0 - \cos\Theta_2\vec{H}_2 = \frac{\mu_1 k_2}{\mu_2 k_1}\cos\Theta_1 H_0 \quad \text{and} \quad H_0 + H_2 = H_1$$

from equation A2(iv), and hence solving for H_1 and H_2 we find,

$$H_1 = \frac{2\mu_2 k_1^2\cos\Theta_0}{\mu_2 k_1^2\cos\Theta_0 + \mu_1 k_2\sqrt{k_1^2 - k_2^2\sin^2\Theta_0}}\vec{H}_0$$

and $$H_2 = \frac{\mu_2 k_1^2\cos\Theta_0 - \mu_1 k_2\sqrt{k_1^2 - k_2^2\sin^2\Theta_0}}{\mu_2 k_1^2\cos\Theta_0 + \mu_1 k_2\sqrt{k_1^2 - k_2^2\sin^2\Theta_0}}\vec{H}_0$$

and hence the value of G_1 and G_2 can be obtained.

APPENDIX A3

Proof of div $\vec{E}=0$

(1) In the case of E-polarization the electric field has only one non-zero component, E_x, and so,

$$\operatorname{div} \vec{E} = \frac{\partial E_x}{\partial x}$$

which is zero because E_x is independent of x .

(2) In the case of H-polarization the electric field has two non-vanishing components E_y and E_z, which are connected with H_x as follows

$$\frac{\partial H_x}{\partial z} = (\sigma + i\omega\epsilon)\, E_y$$

$$-\frac{\partial H_x}{\partial y} = (\sigma + i\omega\epsilon)\, E_z$$

(These equations are derived from Maxwell's Equations)

But $\nabla \cdot \vec{E} = \dfrac{\partial E_y}{\partial y} + \dfrac{\partial E_z}{\partial z}$

and, $\dfrac{1}{(\sigma + i\omega\epsilon)} \left[\dfrac{\partial}{\partial y}\left(\dfrac{\partial H_x}{\partial z}\right) + \dfrac{\partial}{\partial z}\left(-\dfrac{\partial H_x}{\partial y}\right)\right] = 0$

therefore $\nabla \cdot \vec{E} = 0$

APPENDIX A4

Values of ψ and Ψ functions for $G(y)$

If $$\psi_1(y) = \sqrt{\frac{2}{\pi}}\,\xi \sum_{n=0}^{\infty} \frac{1}{\nu_1^{2n+2}} \frac{d^{2n} H_1}{dy^{2n}}$$

then replacing H_1 by the value of $G_1(y)$

$$\psi_1(y) = \sqrt{\frac{2}{\pi}}\,\frac{\xi}{\nu_1^2}\left[H_{10} + (g(0) - H_{10})e^{-\beta_1 y} \sum_{n=0}^{\infty} \frac{\beta_1^{2n}}{\nu_1^{2n}}\right]$$

If $\left|\frac{\beta_1^2}{\nu_1^2}\right| < 1$, the series is convergent and the summation can be simply expressed. If it is divergent (i.e. $\left|\frac{\beta_1}{\nu_1}\right| > 1$) then no solution can be found.

Now $\beta_1^2 = \omega\mu_1\sigma_1 i$ and $\nu_1^2 = \xi^2 + \omega\mu_1\sigma_1 i$

so that $\frac{\beta_1^2}{\nu_1^2} = \frac{1}{1 + \xi^2/\beta^2}$. Therefore for all $\xi \neq 0$ the series is convergent and

$$\sum_{n=0}^{\infty} \frac{\beta_1^{2n}}{\nu_1^{2n}} = \frac{\nu_1^2}{\nu_1^2 - \beta_1^2} = \frac{\nu_1^2}{\xi^2}$$

therefore

$$\psi_1(y) = \sqrt{\frac{2}{\pi}}\,\frac{\xi}{\nu_1^2}\left[H_{10} + (g(0) - H_{10})e^{-\beta_1 y}\frac{\nu_1^2}{\xi^2}\right]$$

similarly

$$\psi_3(y) = \sqrt{\frac{2}{\pi}}\,\frac{\xi}{\nu_3^2}\left[H_{30} + (g(0) - H_{30})e^{\beta_3 y}\frac{\nu_3^2}{\xi^2}\right]$$

$$\Psi_1(y) = \sqrt{\frac{2}{\pi}}\,\frac{\xi}{\nu_1^2}\left[E_{10} + (f(0) - E_{10})e^{-\beta_1 y}\frac{\nu_1^2}{\xi^2}\right]$$

and $$\Psi_3(y) = \sqrt{\frac{2}{\pi}}\,\frac{\xi}{\nu_3^2}\left[E_{30} + (f(0) - E_{30})e^{\beta_3 y}\frac{\nu_3^2}{\xi^2}\right]$$

APPENDIX A5

Solution of the Equations in General: H -polarization

A solution of the general case, when $0 < \alpha < 90^\circ$ will only be tentatively suggested and left in general terms, whereas the case for $\alpha = 90^\circ$ has been worked out in reasonable detail and conclusions can be drawn from this case about the type of field found in the general case.

The general equations differs from the case of $\alpha = 90^\circ$, because in addition to the second differentials with respect to the Y and the Z co-ordinates it has a cross-product term. This leads to difficulties in the solution of the partial differential equation. If the Laplace transform is used then two unknown functions result, which both need to be calculated.

If $$\varphi(x) = \int_{\Gamma - i\infty}^{\Gamma + i\infty} \frac{1}{2\pi i} e^{\xi x} d\xi \int_0^\infty \varphi(x) e^{-\xi x} dx \qquad \text{A5(i)}$$

is defined as the Laplace transform with its inverse, then applying it to the general equation (16), for H-polarization, it is found that

$$\frac{d^2 \overline{H}_x}{dY^2} - 2(\cos\alpha)\xi \frac{d\overline{H}_x}{dY} + (\xi^2 - i\eta^2 \sin^2\alpha)\overline{H}_x = \left[2\cos\alpha \frac{\partial H_x}{\partial Y} + \frac{\partial H_x}{\partial Z} + \xi H_x\right]_{Z=0}$$

or, writing $\nu^2 = \xi^2 - \beta^2 \sin^2\alpha$ and $H_x\big|_{Z=0} = G(Y)$,

$$\frac{d^2\overline{H}_x}{dY^2} - 2\xi\cos\alpha\frac{d\overline{H}_x}{dY} + \nu^2\overline{H}_x = 2\cos\alpha\frac{\partial G(Y)}{\partial Y} + \xi G(Y) + \left[\frac{\partial H_x}{\partial Z}\right]_{Z=0} \qquad \text{A5(ii)}$$

This equation is an ordinary differential equation in Y , and, placing

$$U(Y) = 2\cos\alpha\frac{\partial G(Y)}{\partial Y} + \xi G(Y) + \left[\frac{\partial H_x}{\partial Z}\right]_{Z=0} \qquad \text{A5(iii)}$$

we find $\frac{d^2\overline{H}_x}{dY^2} - 2\xi\cos\alpha\frac{d\overline{H}_x}{dY} + \nu^2\overline{H}_x = U(Y)$

A solution of this equation would be

$$\overline{H}_x = \underline{k}_2 e^{(\gamma_1+\delta_1)Y} + \underline{k}_1 e^{(\gamma_1-\delta_1)Y} + V(Y) \qquad \text{A5(iv)}$$

where $\gamma_1 = \xi\cos\alpha$ and $\delta_1 = \sin\alpha\sqrt{i\eta^2-\xi^2}$ A5(v)

Setting $\delta_1 = a+ib$ we get,

$$\frac{\delta_1}{\sin\alpha} = \frac{\eta^2}{\left[2\left(\xi^2+\sqrt{\xi^4+\eta^4}\right)\right]^{\frac{1}{2}}} + i\left[\frac{1}{2}\left(\xi^2+\sqrt{\xi^4+\eta^4}\right)\right]^{\frac{1}{2}}$$

Hence, when $Re\,\frac{\delta_1}{\gamma_1} > 1$, Assuming that $V(Y)$ is finite at $Y = +\infty$, the boundary condition H(v) can be applied to show that $\underline{k}_2 = 0$. The condition $Re\,\frac{\delta_1}{\gamma_1} > 1$ is $\xi^2 < \frac{\eta^2\sin^2\alpha}{2\cos\alpha}$, so that

ξ tends to a minimum value (zero) as α tends to zero, and ξ tends to a maximum value (infinity) as α tends to 90°. When ξ has a value outside this limit, i.e. $\xi^2 \geqslant \frac{\eta^2\sin^2\alpha}{2\cos\alpha}$ then

the problem becomes much more complicated. There will be two choices depending upon the function $\underline{k}_1(\xi)$. Either $\underline{k}_1 = \underline{k}_2 = 0$

or neither $\underline{K}_1$ nor $\underline{K}_2$ are zero, the condition as Y tends to infinity being satisfied by the subtraction of two infinities to produce a finite number. An intellegent guess might be that both $\underline{K}_1$ and $\underline{K}_2$ are zero but it can only be a guess. A theory will be developed assuming that $\underline{K}_2$ at least is zero inside the domain $\xi^2 < \dfrac{\eta^2 \sin^2\alpha}{2\cos\alpha}$ and a tentative approach to the problem will be given. $V(Y)$ will always be given by:

$$V(Y) = \frac{1}{\nu_1^2} \sum_{n=0}^{\infty} \left[-2\xi \cos\alpha \frac{d}{dY} + \frac{d^2}{dY^2} \right]^n U(Y) \frac{(-1)^n}{\nu_1^{2n}}$$

which is the particular integral of the differential equation, assuming that $U(Y)$ can be expressed as a Fourier series and that there are no singularities.

The values of $\underline{K}_1$ and $\underline{K}_3$, can be computed by applying boundary conditions H(ii)

$$\underline{K}_1 = \overline{g(\xi)} - V_1(0) \qquad \underline{K}_3 = \overline{g(\xi)} - V_3(0) \qquad \text{A5(vi)}$$

and applying H(iii)

$$\overline{g(\xi)} = \frac{\sigma_3 V_1(0)(\delta_1 - \gamma_1) + \sigma_1 V_3(0)(\delta_3 + \gamma_1) - \sigma_1 V_3'(0) + \sigma_3 V_1'(0)}{\sigma_1(\delta_3 + \gamma_1) + \sigma_3(\delta_1 - \gamma_1)} \qquad \text{A5(vii)}$$

It is apparent that the integral differential equation has been expressed as a function of $U(Y)$. $U(Y)$, in turn, is a function of the magnetic field on the surface $Z = 0$. But the value on $Z=0$ cannot be placed in the function $U(Y)$ until the partial derivative $\frac{\partial H_x}{\partial Z}$ has been computed. Hence, although $U(Y)$ is only a function of Y and has been calculated along $Z=0$, two seperate uncalculated functions appear in $U(Y)$. The function $G(Y)$ can be defined by the following conditions:

$$G(Y) = G_1 H_o \quad , \quad Y = +\infty$$

$$G(Y) = G_3 H_o \quad , \quad Y = -\infty$$

and $$\lim_{Y \to (-0)} \frac{1}{\sigma_3} \frac{\partial G(Y)}{\partial Y} = \lim_{Y \to (+0)} \frac{1}{\sigma_1} \frac{\partial G(Y)}{\partial Y}$$

and lastly $$\lim_{Y \to (-0)} G(Y) = \lim_{Y \to (+0)} G(Y) = g(0,0)$$

The function $\left[\frac{\partial H_x(Y,Z)}{\partial Z}\right]_{Z=0}$ is not nearly so easy to define; it appears that it can only be calculated after the values of H_1 and H_3 have been computed, which indicates that a second alternative solution of the partial differential equation is necessary before a complete solution of this case, using this method, can be found.

E-polarization

The relevant equation is

$$\frac{\partial^2 E_x}{\partial Y^2} - 2\cos\alpha \frac{\partial^2 E_x}{\partial Y \partial Z} + \frac{\partial^2 E_x}{\partial Z^2} = i\eta^2 \sin^2\alpha \, E_x$$

which can be "solved" in the same way as the partial differential equation for H-polarization using the Laplace transform and inversion.

The partial differentials can be converted to ordinary differentials to obtain the equation

$$\frac{d^2\bar{E}_x}{dY^2} - 2\xi\cos\alpha\frac{d\bar{E}_x}{dY} + \nu^2\bar{E}_x = T(Y) \qquad \text{A5(viii)}$$

where $\nu^2 = \xi^2 - \beta^2\sin^2\alpha$ and

$$T(Y) = \left[2\cos\alpha\frac{\partial C(Y)}{\partial Y} + \xi C(Y) + \frac{\partial E_x}{\partial Z}\right]_{Z=0} \qquad \text{A5(ix)}$$

The solution of this differential equation is the same as before, i.e.

$$\bar{E}_{x_1} = K_2 e^{(\gamma_1+\delta_1)Y} + K_1 e^{(\gamma_1-\delta_1)Y} + S_1(Y) \qquad \text{A5(x)}$$

where $\gamma_1 = \xi\cos\alpha$ and $\delta_1 = \sin\alpha\sqrt{i\eta^2 - \xi^2}$

and $$S_1(Y) = \frac{1}{\nu^2}\sum_{n=0}^{\infty}\left[-2\xi\cos\alpha\frac{d}{dY} + \frac{d^2}{dY^2}\right]^n T(Y)\frac{(-1)^n}{\nu^{2n}} \qquad \text{A5(xi)}$$

Once again applying condition E(v) makes $K_2 = 0$ if ξ is in the correct domain. The evaluation of K_1 and K_3 is obtained by applying the boundary conditions E(ii) and then E(iii).

$$K_1 = \bar{f}(\xi) - S_1(0) \quad \text{and} \quad K_3 = \bar{f}(\xi) - S_3(0)$$

Application of E(iii) gives the value of $\bar{f}(\xi)$ as

$$\bar{f}(\xi) = \frac{S_1(0)(\delta_1 - \gamma_1) + S_3(0)(\delta_3 + \gamma_1) + S_1'(0) - S_3'(0)}{\delta_1 + \delta_3}$$

when $\gamma_1 = \gamma_3$.

Once more the solution of the integral, differential equation is dependent on the value of $C(Y)$ and $\left.\frac{\partial E_x}{\partial Z}\right|_{Z=0}$. The value (or function) represented by $C(Y)$ may be expressed by a series of limiting conditions in the same way as $G(Y)$, i.e. $$C(Y) = C_1 E_0 \,, \qquad Y = +\infty$$

$$C(Y) = C_3 E_0 \,, \qquad Y = -\infty$$

$$\lim_{Y \to (-0)} \frac{\partial C(Y)}{\partial Y} = \lim_{Y \to (+0)} \frac{\partial C(Y)}{\partial Y}$$

and $$\lim_{Y \to (-0)} C(Y) = \lim_{Y \to (+0)} C(Y) = f(0,0)$$

However the problem of defining the value of $\left.\frac{\partial E_x}{\partial z}\right|_{z=0}$ still remains essentially unsolvable unless a subsiduary solution is added.

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