THE SPATIAL FILTERING FROM CORE-SCALE CONDUCTIVITY TO THE CONDUCTIVITY MEASURED BY A SLUG TEST

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A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF APPLIED SCIENCE

in

THE FACULTY OF GRADUATE STUDIES
GEOLICAL SCIENCES

We accept this thesis as conforming to the required standard

THE UNIVERSITY OF BRITISH COLUMBIA
August 1995
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Abstract

In the thesis, I investigate the measurement process of the slug test, an important and efficient engineering technique employed to probe aquifer properties. The objective is to determine the scale at which a slug test 'measures' hydraulic conductivity. To determine the scale, I conceptualize slug test measurement process in a system and filtering framework. I quantify, through numeric experiments, the spatial filtering effect of the slug test upon the core-scale hydraulic conductivity field. I examine the role of specific storage on this filtering effect.

I develop two approaches to identify the system filter that epitomizes the filtering system — measurement process: spectral analysis with Wiener filtering and numerical perturbation. In heterogeneous media, system linearity is analyzed and the inverse estimation of the system output is evaluated. With the spectral approach, I optimally estimate the system filter in the presence of measurement noise. In homogeneous media where the spectral approach cannot be used, I apply the numerical perturbation approach. The spectral analysis and numerical perturbation are two conceptually different but complementary approaches.

I analyze the nonparametric form of the system filter in heterogeneous media of various spatial variability and specific storage values. I determine the empirical parametric expression of equivalent filter width versus specific storage in homogeneous media. It is found that:

1. In both heterogeneous and homogeneous media, the equivalent filter width increases linearly with respect to \( \log \frac{1}{\sqrt{S_s}} \);

2. In homogeneous media, the filter amplitude decreases away from the wellbore by an approximate \( \frac{1}{r} \) law if \( S_s \leq 10^{-4}[L^{-1}] \);
3. A preliminary analysis of the role of heterogeneity characteristics upon the slug-test filtering was undertaken. The slug-test filtering was not strongly influenced by heterogeneity characteristics for log-conductivity variances less than 0.7. At variances less than 0.3, anisotropy in the conductivity field caused anisotropy in the slug-test filtering. These effects were not quantified in this thesis.

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Preface

The thesis subject is the spatial filtering from core-scale conductivity to the conductivity measured by a slug test. In Chapter 1, we introduce a spatial filtering approach to interpret a slug test measurement. Some relevant concepts are discussed from a filtering perspective. In Chapter 2, we present numerical methods to simulate slug tests in heterogeneous and homogeneous media. The conceptualization of the slug test in a system and filtering framework is given in Chapter 3, where system filtering properties and the inverse estimation of system outputs are discussed in detail. In Chapter 4, a spectral analysis through Wiener filtering is developed to identify spatial filter in heterogeneous media. As a result of computation load constraints, the specific storage value examined in the heterogeneous conductivity filtering is limited to $S_s = 10^{-n}[L^{-1}]$, $n = 1, \ldots, 5$. The effects of other $S_s$ with values of $n = 6, \ldots, 10$ on the conductivity filtering are investigated in homogeneous media where the spectral analysis method becomes invalid.

A numerical perturbation method is developed, which is the topic of Chapter 5. In Chapter 5, empirically determined expressions of influence radius and equivalent filter width versus specific storage are presented. Consequently, Chapters 4 and 5 contain conceptually different but complementary approaches. In Chapter 6, the conductivity filtering in heterogeneous media with different specific storages and various spatial variabilities are explored. In Chapter 7, the major results are summarized in conclusions and future work is recommended, also the possible engineering application and theoretical significance of the spatial filtering approach are discussed.
Acknowledgment

First of all I wish to thank my advisor, Roger Beckie. Without his help this thesis would not have been written. His help always extends to more than research, and I appreciate his various discussions on culture and history around the world.

I would like to thank my committee member Les Smith. His inspiration on the practical applications of my thesis results is of great value. My committee member Tad Ulrych deserves my heart-thanks. His thorough review and comments on the system and filtering and on the spectral method leads my thesis to a more robust version. I am indebted to Greg Dipple for his kindness being my committee chair. I would like to acknowledge my fellow students in the Hydrogeology group.

I also wish to thank my former advisor, Xibin Zhu, and my colleagues at the Chinese Academy of Geological Sciences. It was their help that provided me knowledge on some advanced topics in hydrogeology.

Support from my family is of great help for the completion of the thesis. I am especially grateful to my parents whom I can always count on. My gratitude to my grandmother who passed away during my study abroad will be cherished deep in heart and last forever. My sisters, Lichun, Liyuan, Lixin and Lifeng receive special thanks for helping Mom and Dad all the time.

If there were one thought that I could best express by words in this thesis, I would devote it to colleague and my wife, Yanhong; that thought would embrace my whole-hearted appreciation for all that she has suffered through, and my sincere thankfulness for all that she has done.
This thesis is dedicated
to the memory of my
grandmother

XIVING HOU
(17 May 1915 — 4 December 1993)
Chapter 1

INTRODUCTION

In this chapter, we introduce the slug test measurement from a spatial filtering perspective. We first discuss some relevant concepts: scale, averaging, and parameter upscaling, under a filtering framework. The literature review of the work on parameter upscaling is relegated to Appendix A. The more relevant work on the measurement — pumping test is reviewed in Section 1.1.3.1, with focus on the influence of the aquifer heterogeneity on the measured conductivity. In Section 1.2 we review the literature on the slug test and we introduce a spatial filtering approach to interpret the slug test measurement. A brief thesis overview is given in Section 1.3.

1.1 Scale and averaging

In this section we discuss the scale and averaging in the real world and the engineering world from a filtering perspective, and we present a filtering interpretation to parameter upscaling.

1.1.1 An opening example

Fig.1.1.1 shows a common scale problem. The shadowed zones schematically represent the sampling zones of well test, core test and tracer test, which all are the basic techniques utilized to probe aquifer properties. The grids represent an engineering model which is to be employed to interpret real-world processes and predict field problems. Now we see that different engineering sampling techniques measure aquifer properties on different
scales. Then the scale problem is: How can one utilize the measurement data in the engineering model? To solve the problem, one needs to understand the scale relationship between different measurements and the scale relationship between measurement and real world processes. In the thesis, we develop a spatial filtering approach that quantitatively characterizes the scale relationship from one scale to another. In the following, we first discuss the concepts of system and filtering, then we return to the scale issue.

1.1.2 Scale and averaging: A filtering perspective

A system can be viewed as any process that results in the transformation of signals. Thus a system has an input signal and an output signal which is related to the input signal through the system transformation [Oppenheim et al., 1983, p35]. If the transformation changes the relative amplitudes of the frequency components in a signal or perhaps eliminate some frequency components entirely, the transformation is referred to as filtering [Oppenheim et al., 1983, p397]. If the system is linear and space invariant,
the filtering process can be represented in a convolution form,

\[ Y(x) = \int_{-\infty}^{\infty} H(x - x') X(x') \, dx', \]  

(1.1)

where \( X \) and \( Y \) represent the input signal and the output signal respectively, and \( H(x) \) is called filter function, its Fourier transform is called transfer function. The so-called system identification is to find the filter or transfer function, e.g., the impulse response of a linear time/space invariant system.\(^2\) The filter function epitomizes the system, hence characterizes the relationship between input and output. Mathematically, the convolution represents a summation or superposition. Physically, the convolution signifies an averaging process, e.g., the measured conductivity \( [Y(x)] \) by well test is somehow averaged by \( [H(x)] \) from the core-scale conductivities \( [X(x)] \). In the thesis averaging is congruent in meaning to filtering.

The real world exhibits variabilities on infinite scales, and it is frequently perceived to be nonergodic/nonlinear. The process in the real world can never be exactly reproduced in its counterpart, the engineering world, where scientific knowledge holds valid only on finite scales and only limited scale ranges are of practical concern. The engineering world is often assumed to be linear, stationary, ergodic and Gaussian [Newland, 1993, p80]. These assumptions are not true in cases where sufficient measurements are not available and the measurements are obtained on small scales. However, if more measurements are performed and averaged to larger scales, we are heading for the engineering assumption. It is averaging process that bridges engineering knowledge with reality. Averaging is always linked with the issue of scale, and averaging itself is a transformation of scale — scaling, usually upscaling. The scale and averaging are shown in Fig. 1.1 and discussed in the following.

\(^1\)Note that the formulas in the thesis are mixture of 2D and 3D, where a single argument appears, it should be considered a vector coordinate. The slug test results in the rest chapters are 2 dimensional.

\(^2\)"/" denotes "and/or".
There are two possible broad groups of averaging operating in the real world and engineering world: *natural averaging* and *engineering averaging*. Natural averaging is postulated to be the real-world averaging that involves scale transformation. For example, a regional-scale fluvial fan is a result of the transformation of small-scale sediments (e.g., igneous fragments, metamorphic clasts etc.), the averaging processes may involve water process, temperature and pressure and so on. Natural averaging may occur on geological time scales, thus it is hard to understand through experiments,\(^3\) but we can interpret it in a filtering framework. First, the aquatic sedimentary environment is represented by a system operating on different scales; second, the unsorted rock, i.e., the source, as input

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\(^3\)Note that the natural averaging has meaning on spatial scale greater than that of representative elementary volume (REV) pertaining to the soil or porous media composed by mixtures of powders, where a fractal phenomenon seems have been discovered [Ghilardi *et al.* 1993, *Katz and Thompson* 1985]. These authors explain self-similar heterogeneity by a fractal behavior due to growth of crystals within the pore space and relate fractal dimension to morphogenetic kinetics. Here, the natural averaging operates on the sediments which are already-formed, apparently on larger scales than those of crystals.
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signals; last, the observed aquifer structures constitute system output. Consequently, the system filtering/averaging signifies the scale relationship between the source rocks and the aquifers. Natural averaging is beyond the scope of this thesis, we shall not discuss it further.

The engineering averaging is of the most concern and it may be excited by various measurement techniques. A measurement process can be defined as, in a mathematical notation, a mapping from the real-world process to an engineering framework, the notation has an identical basic form as (1.1),

\[
Y(x)^P = \int_{-\infty}^{\infty} H(x - x')^P X(x') \, dx'
\]  

(1.2)

where \( Y^P \) is a measure of \( X \) and \( H^P \) denotes the filter function associated with measurement process \( P \) [Baveye and Sposito, 1984; Cushman, 1984; Cushman, 1986]. We use a terminology, the equivalent filter width of the filter function, to represent the scale of measurement \( P \), \( \lambda_P \), defined as [Bracewell, 1986]

\[
\lambda_P = \frac{\int_{-\infty}^{+\infty} H(x)^P \, dx}{H(0)^P},
\]

(1.3)

that is, the area of the function divided by its central coordinate. Note that eq.(1.3) is a one dimensional expression.\(^4\)

If time is included, eq.(1.2) extends to

\[
Y(x,t)^P = \int_{-\infty}^{\infty} H(x - x', t - t')^P X(x', t') \, dx' \, dt'.
\]

(1.4)

For example, a slug test 'measures' different aquifer volume at different times, the conductivity mapped from the core-scale(wellbore-scale) conductivities, by model of

\(^4\)Note that the equivalent filter width differs from filter width. If we consider filtering as a random signal process, filter width(\( \lambda \)) has the same meaning as the correlation range(\( a \)) of the signal's variogram, and as the integral scale of its covariance function(\( \tau \)), i.e., \( a = \tau = \lambda \). In this thesis, we save the terms 'filter width' to describe filters, while 'correlation range' for variogram and 'integral scale' for covariance of random signals.
Cooper et al. [1967], is a volume and time average. As noted by Cushman [1986], one can not ‘measure’ constitutive variables, e.g., conductivity, storativity, and one can only measure fundamental variables, e.g., hydraulic head, flux. Constitutive variables can be computed from the constitutive equations in a closure model, e.g., the model of Cooper et al. [1967]. This averaging of (1.2) and (1.4) is namely spatial or spatial-time averaging, more specifically, we call it measurement averaging or filtering.

In practice, all the variabilities can never be entirely observed, since this would require enormous measurements up to the sampling volume covering, for instance in mining engineering, the field of variability from 1μm to 100km [Journel and Huijbregts, 1978, p149]. Hence, measurements are only taken at some selected localities pertaining to some optimal sampling network with a risk-cost-benefit objective [Freeze et al., 1990; Bogardi et al., 1985]. The spatial structure of the measured variable inferred from measured data is then utilized in a variety of geostatistical, statistical or interpolation methods [Deutsch and Journel, 1992; Cressie, 1991; Journel and Huijbregts, 1978; Marsily, 1986], to estimate the variable state in unsampled region. This type of estimation bears an essence of averaging, thus termed stochastical/statistical estimation averaging, more generally, we call it estimation averaging or filtering.

The estimation averaging may be performed on the grid-scale of the engineering model,

\[
Y(x)^S = \int_{-\infty}^{\infty} H(x - x')^P H(x - x')^S X(x') \, dx',
\]

(1.5)

where \(Y^S\) is the estimated \(X\) in the whole domain and \(H^S\) is the filter epitomizing the

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5 Here the closure model refers to the one that does not require an explicit description of the dynamics on scale smaller than \(\lambda_p\) [Beckie, Aldama and Wood, 1994]. In the following discussion, the ‘measure’ implies what is meant by Cushman [1986].

6 The variable state in the unsampled region is somehow an average of the variable state measured in the sampled region. In the case of unconditional simulation, it still bears an essence of averaging because the inferred model structure is established in an averaging mean sense.
estimation procedure. If on a scale smaller than a grid, a block averaging is then applied,

\[ Y(x)^B = \int_{-\infty}^{\infty} H(x - x')^P H(x - x')^S H(x - x')^B X(x') \, dx', \quad (1.6) \]

where \( Y^B \) denotes the averaged block estimates of \( X \), and \( H^B \) is block averaging filter. This process belongs to the so-called upscaling, which shall be discussed in next section.

Now we see that the estimation averaging always invokes measurement averaging, thus, its accuracy depends upon the accuracy of the measurement averaging. This fact seems frequently be ignored in the literature. For example, on the conditional simulation work of Delhomme [1979], two assumptions have been made: 1) the simulated values of conductivity have the same correlation structure as the ‘true’ conductivity field, and 2) the simulated conductivity at the measurement (e.g., pumping test) location are consistent with the ‘observed’ values. It is clear that the so-called ‘true’ or ‘observed’ conductivity is inverted from the drawdown recorded at pumping and observation well [Theis, 1935; Cooper and Jacob, 1946], representing the spatial/spatial-time averaged aquifer property through the associated measurement filters. Thus, the inverted conductivity is neither ‘true’ nor ‘observed’, instead, an averaged value. Consequently the accuracy of the estimation averaging is limited by the accuracy of measurement averaging.

Now it becomes clear that the engineering world only possesses finite scales, not only because the measurement process can only sample scales of limited ranges, but the dynamics become invalid beyond certain scale — the resolving scale. For example, Darcy’s law is not valid on molecular scales and remains unproved in large-scale heterogeneities. The effect of unresolved small-scale dynamics on resolved large-scale dynamics can be lumped into phenomenological parameters on the resolving scale. For example,

- molecular process \( \longrightarrow \) fluid continuum \( \longrightarrow \) porous media continuum
- energy conservation \( \mu = \frac{1}{\varepsilon} \frac{d}{dn} \) (Newtonian fluid) \( K = \frac{\partial^2}{\mu} k \)
where $\mu$ is viscosity $[\text{ML}^{-1}\text{T}^{-1}]$, $\zeta$ for tangential stress $[\text{ML}^{-1}\text{T}^{-2}]$ and $\frac{du}{dn}$ represents velocity gradient. The phenomenological parameter $\mu$ lumps the molecular moments $\zeta$ and $\frac{du}{dn}$, while the conductivity $K$ integrates the effects of the fluid $\frac{\partial u}{\partial x}$ and the pore $k$. Indeed, it has been proved that Darcy's law can be directly derived by averaging the Navier-Stokes equations Keller [1980].

Beven [1989] recognized the effects of the interaction between the subgrid-scale (unresolved) and the grid-scale (resolved) on the resolved dynamics, in the evaluation of the physically based models, and he concluded that the present physically based models essentially still belong to traditional lumped conceptual models. The subgrid-scale ~ grid-scale interactions in groundwater flow models have been recently, for the first time, evaluated by Beckie, Aldama and Wood [1994]. These authors adopted some averaging concepts from turbulence continuum and statistic physics literature and employed a spatial filtering approach (references therein). They found that if the resolved and subgrid scales of the variability of the porous medium are separated by a spectral gap, Darcy's law manifests an accurate resolved scale model structure. The so-called spectral gap $\Delta f$ equal to $\delta - \gamma$ should meet the condition,

$$\Delta f > 2\gamma,$$

where in frequency space, $[-\gamma, \gamma]$ represents the resolved scale range (or wavelength equal to $f^{-1}$), and $[-\mu, -\delta]$ and $[\delta, \mu]$ denote the subgrid scale ranges on two sides of zero frequency.

The preceding paragraphs complete the discussion of Fig. 1.1. To further our understanding, we graphically show in Fig. 1.2 the scale and engineering averaging relationships between real world, measurement, model and decision analysis. The two circles represent measurement filtering, and estimation filtering and model upscaling. Note that these graphs demonstrate scale versus correlation range ($\lambda \sim a$), and they differ from the REV
Figure 1.2: The $\lambda$(scale) $\sim \alpha$(correlation range) of continuously filtered real world(A), measurements(C), model(E), decision analysis(F), and filtering processes(B, D).
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illustrations of Bear [1972] (Fig.1.3.2), Cushman [1984] (Fig.1), and Baveye and Sposito [1984] (Fig.1). What the authors show is averaged parameter versus the scale of averaging region or volume. Here, Fig. 1.1A manifests the conjecture that correlation range continuously increases with real world scales, λₘ, but decreasing $\frac{\theta(\lambda_m)}{\theta(\lambda)}$. This can be best explained by a hypothetical filtering [Vanmarcke, 1983]. Consider an infinite continuous signal, we filter this signal by a bell-shaped filter. We then can see an increasing correlation in the filtered signal from a filtering with larger filter width and this increase slows down with the further increase of filter width. Then graph A is a plot of the filter width versus the correlation length of the filtered signal.

Fig. 1.1C shows the measurement filter width λₚ versus the correlation range of the sampled signal. Here we have assigned a representative scale to each measurement, accordingly, we observe the same scale relationship as that in graph A except here both λₚ and $a_p$ are in discrete form. Note that the ‘soft-data-technique’ may be able to achieve ‘soft’ continuous relationships with ‘experience filters’. Also note that $a_p$ doesn’t start from the origin point because the measurements can not sample the scale smaller than λₚ, thus the correlation is uncertain below some cut-off value. This differs from the correlation range of a conventional variogram of point data support.

The model resolving scale can be represented by its grid block scale, λₖ. Shown in Fig. 1.1E, the λₖ and $a_B$ of a model are relatively more uniform than those of the measurements, the decision variables and the real world. The scale of the decision variable, shown in Fig. 1.1F, λ₆, varies for different decision variables and for different engineering applications. The realistic λ₆ and a₆, being concerned by the decision maker, should be consistent with the λₚ and $a_B$ of the measurement and the λₖ and $a_B$ of the model. In other words, the measurement should be designed to have the λₚ and $a_B$ consistent with λ₆ and a₆, so does the model.
Meanwhile, model dynamics should be scaled up. In water resources context, dynamics upscaling has been pioneered by several authors, Hassanizadeh and Gary [1979a, b]; Baveye and Sposito [1984]; Cushman [1984, 1986]; and Beckie, Aldama and Wood [1994]. The basic idea employs a filtering approach, by applying appropriate filters to filter the small scale mass, momentum, energy and entropy balance equations, upscaling to a large macroscopic scale amenable for field measurement and model representation. These filters can be any or all of the measurement filter \( H^P \), estimation filter \( H^S \), and block scaleup filter \( H^B \). Dynamics upscaling is beyond the scope of the thesis, thus it shall not be discussed further. Parameter upscaling is discussed in the following.

### 1.1.3 Parameter upscaling: A filtering interpretation

Filtering from a smaller scale to a larger scale, through the filters \( H^P, H^S, H^B \), is exactly the essence of upscaling. Parameter upscaling is an engineering compromise. The measurement filter \( H^P \) is utilized as an instrumental technique scaling-up and scanning-out unmeasurable real-world small-scale variability. On the other hand, the estimation filter \( H^S \), upscaling to the whole domain, is a practical concession to cost and technical constraints. Furthermore, if the block-scale of the engineering model is greater than the data support of measurement and estimation filtering, the block scaleup filter \( H^B \) is employed to provide data of the model-input format. Consequently, upscaling conceptually unifies effective parameter, equivalent parameter and measurement process, since all are scale-dependent concepts and related to some basic small-scale-unit, e.g., core permeability.

In Appendix A, centering on upscaling we review the work on effective and equivalent parameter, in particular on conductivity, as a limiting case and a general case of upscaling, respectively.
1.1.3.1 Parameter measurement: a special case of upscaling

The measured parameter can be considered as a special case of upscaling, in the sense that the measured parameter is a result of the upscaling through technical measurement. In this section we specifically review the work on pumping tests in relating measured conductivity to the small-scale conductivities. We denote $Y_m$ the logarithm of the measured conductivity, and $Y$ the logarithm of the small-scale conductivity. In next section we discuss the work on slug test — the theme of the thesis.

In the work on pumping test in confined aquifer, Butler [1991] found that $Y_m$ is dependent on the spatial location of observation well employed in the analysis methods of Theis [1935] and Cooper and Jacob [1946]. In the analysis of angular dependence, he found that the variability of $Y_m$ increases with $\sigma^2_Y$ (variance) and the distance from pumping well, so is the radial dependence, but the radial dependence is much greater than the angular dependence. He stated that the dependence should be weakened with decreasing correlation of $Y$.

Using a semi-analytical method, Butler and Liu [1991] worked on pumping test sited on the center of an infinite linear strip of one material embedded in a matrix of differing hydraulic properties. The drawdown contours are shown for several cases: $K^{strip}/K^{matrix} = 0.1, 100, 10000$ (see their Fig. 5, 7, 8). Sensitivity analysis is employed to investigate the influence time of the strip on drawdown, depending on strip width and the $K^{strip}/K^{matrix}$ ratio. Various settings of pumping and observation wells relative to the strip location are discussed. It is found that the strip can only affect the changes in drawdown for a finite time, meaning that after certain time further changes will be independent of $K^{strip}$, although the total drawdown will always be dependent on $K^{strip}$.

---

7 The spatial dependence includes angular and radial dependence which refer the direction and the distance of observation well to pumping well, respectively.

8 $K$ denotes conductivity.
This is similar to the sensitivities of zones lying radially outward from the pumping well and observation wells of Butler and McElwee [1990].

Butler [1990] found that the log-log [Theis, 1935] and semilog [Cooper and Jacob, 1946] methods yield dissimilar estimates due to their emphasis on properties in different portions of the aquifer. The difference is proportional to $\sigma_T^2$, and inversely proportional to the outward distance from pumping well if observation well record is used. Butler [1990] discussed two interpretations: (i) Using pumping well drawdown: the near-well conductivity strongly impacts the $K$ from the log-log method ($K^{ll}$), whereas the $K$ from semilog method ($K^{sl}$), using large time portion, can not “feel” the effect of the near-well conductivity, and it represents the property of the aquifer at the front of depression zone. (ii) Using observation well drawdown: the effect of the near-well conductivity on $K^{ll}$ lessens with increasing outward distance from pumping well.

Furthermore, Butler [1990] stated that the close agreement between the slug test and large-time $K^{sl}$ at several observation well should be considered as strong support for representing the aquifer as a uniform at the slug-test or larger scale. One crucial point Butler [1990] did not explicitly elucidate is: the dissimilar estimates of $K^{ll}$ or $K^{sl}$, at the pumping well and the observation well, result from the differing flow patterns around the pumping and observation well, i.e., the filtering power and volume around the pumping and observation well are different [Oliver, 1993].

Butler and McElwee [1990] utilized the sensitivity analyses of observation well drawdown to $T$ and $S$, denoted $U_T$ and $U_S$. By the analysis of sensitivity in the zonated aquifers simulated numerically, they found that, only during a finite interval of time, $U_T$ is sensitive to the near-pumping well material, being consistent with the analytical results [Barker and Herbert, 1982; Butler, 1988]. This is also true for $U_S$. They concluded that drawdown at an observation well placed at a distance from pumping well is relatively insensitive to material lying between the pumping well and the observation well, as
only the early-time responses are dependent on the properties of the interwell material. Consequently, observation wells are superior to more distant wells for the purposes of characterizing property variability in a radial-flow field. Their findings can be examined by the filters of the pumping and observation wells [Oliver, 1993].

In a uniform aquifer, McElwee and Yukler [1978] analytically derived the sensitivities:

\[
U_T = -\frac{s}{T} + \frac{Q}{4\pi T^2} \exp \left(-\frac{r^2 S}{4Tt}\right) \quad (1.8)
\]

\[
U_S = -\frac{Q}{4\pi TS} \exp \left(\frac{r^2 S}{4Tt}\right) \quad (1.9)
\]

\[
s = \frac{Q}{4\pi T} \int_{r^2 S/4Tt}^{\infty} \frac{e^{-u}}{u} \, du \quad (1.10)
\]

One interesting point can be drawn from the formulas is that systems with lower \( \tau \), will have higher sensitivities and thus will, in general, be more amenable to inversion analyses, given a sufficient time of pumpage has elapsed [McElwee, 1987].

Oliver [1990] characterized the relationship between drawdown and permeability as a Frechet kernel function of the radial flow in pumping tests,

\[
S_i = \int_1^{\infty} G_i(x) \left(1 - \frac{1}{K_D(x)}\right) \, dx; \quad K_D(x) = \frac{K(x)}{\bar{K}} \quad (1.11)
\]

where \( G_i(x) \) is the kernel and \( (1 - \frac{1}{K_D(x)}) \) is the presumed permeability distribution which is radially symmetric but nonuniform, and \( \bar{K} \) is the average permeability. He solved for the kernel by a perturbation method. This kernel function was used by Oliver [1992] to derive the permeability distribution of eq.(1.11). Oliver [1993] generalized the \( (1 - \frac{1}{K_D(x)}) \) permeability pattern to nonradially symmetric porous media and analytically solved the associated Frechet kernel by a perturbation method.

Desbarats [1994] defines the effective conductivity measured from 3D flow pumping test analysis as a weighted “power average”,

\[
K_{eV}^w = \frac{1}{W} \int_V K(x)^w \, dV, \quad W = \int_V \frac{dV}{[r(x)]^2} \quad (1.12)
\]
where \( V \) is drainage volume. The \( \frac{1}{r(x)} \) weight was first seen in the work of Cardwell and Parsons [1945] in the determination of the upper and lower bounds of \( K_v \). Desbarats [1992b] also employs the \( \frac{1}{r(x)} \) weight in the definition of \( K_v \) in 2D radial flow. The power exponent \( w \) in eq. (1.12) is obtained by fitting \( K_v \) to numerical simulations.

### 1.2 The measurement filter: slug test

In Section 1.2.1 we first review the slug test literature, then we discuss the slug test from a spatial filtering perspective in Section 1.2.2.

#### 1.2.1 Review of slug test literature

Hvoslev [1951] may have been the first to point out that a slug test can be used to estimate in situ hydraulic conductivity. Hvoslev [1951] develops models for a variety of well, piezometer, and standpipe geometries with full or partial penetration in an infinite, semi-infinite below, or confined homogeneous, anisotropic aquifer. The assumption of no aquifer storage in the models of Hvoslev [1951] was evaluated by Chirlin [1989]. Demir and Narasimhan [1994] developed an improved interpretation strategy of the slug tests of Hvoslev [1951].

Cooper et al. [1967] presents the mathematical model of the slug test of a finite wellbore radius in infinite confined homogeneous media. By the analogy to the heat conduction problem of Carslaw and Jaeger [1959], Cooper et al. [1967] give the analytical solution of hydraulic head at wellbore and in aquifer. Papadopulos et. al [1973] extended the type curves of Cooper et al. [1967] to those with \( S_a = 10^{-n}, n = 6, \ldots, 10 \) [L\(^{-1}\)].

\(^9\)For simplicity, the dimension unit of specific storage is hereafter omitted. In other words, we consider an aquifer of unit thickness.
Sageev [1986] discussed the influence of wellbore skin and dimensionless wellbore storage\textsuperscript{10} on the early-time, intermediate time and late time type curves. Karasaki et. al [1988] presented analytical solutions of slug tests in bounded porous media, fractured rocks, with linear and radial flow. Their results show that slug tests suffer problems of nonuniqueness to a greater extent than other well tests.

Guyonnet et. al [1993] empirically determined the distance traveled by a given head perturbation $H/H_0$ through regression analysis for the slug test in unbounded homogeneous media. For $H/H_0 = 1\%$,

\begin{align}
\frac{r(t)}{r_w} &= 3.54 \left( \frac{Tt}{Sr_w^2} \right)^{0.462} \\
\frac{r^{\text{max}}}{r_w} &= 8.37 \left( \frac{\pi r_c^2}{2\pi r_w^2 S} \right)^{0.495}
\end{align}

where $r(t)$ the traveled by the $H/H_0 = 1\%$ perturbation at time $t$, and $r^{\text{max}}$ is the maximum traveled distance.

In the analysis of conductivity measured by a slug test in confined heterogeneous conductivity field with square blocks on wellbore scale, Harvey [1992] employ the power-average definition $K_{eV}$ of eq.(A.II.152) and determine the exponent $w$ by fitting $K_{eV}$ to the results of Monto Carlo simulations. The integral of eq.(A.II.152) is evaluated by summing for all conductivity block values within a cylinder of radius $r^{\text{opt}}$ with power $w^{\text{opt}}$ where $r^{\text{opt}}$ and $w^{\text{opt}}$ are the values that provide best fit of $K_{eV}$ to numerical results. Harvey [1992] states that (i) $r^{\text{opt}}$ linearly varies with $\frac{r_w}{\sqrt{bS_s}}$; (ii) $w^{\text{opt}}$ decreases with $\frac{r_w}{\lambda_h \sqrt{bS_s}}$, where $b$ is aquifer thickness, $r_w$ wellbore radius, $S_s$ specific storage, and $\lambda_h$ horizontal correlation.

Beckie and Wang [1994] developed a spatial filtering approach to relate the conductivity measured by a slug test to the core-scale conductivities(on wellbore scale). This

\footnotetext[10]{Wellbore storage is $\frac{\pi r_c^2}{2\pi r_w^2 S}$, where $S$ is storativity and $r_c$ and $r_w$ are well casing radius and well screen radius.}
shall be discussed in the next section.

1.2.2 The slug test filter

Unlike any methods reviewed above in determining the effective or equivalent parameter or the parameter measured from aquifer tests, the spatial filtering method explicitly links the core-scale conductivity to the measured conductivity through a spatial filter, which quantifies the contributions of core-scale conductivity at different locations to the measured conductivity,

\[ Y_m(x) = \int G(x - x') Y_c(x') \, dx' \]  

(1.15)

where \( Y_c(x), Y_m(x) \) are the logarithm conductivities of the core-scale field and measured field, and \( G(x) \) is the spatial filter. The filtering representation and filter identification are the subjects of the following chapters.

In order to obtain a stable slug test filter,\(^\text{11}\) the spatial filtering approach requires a great number of measurements, and the more the measurements, the better the stability of the filter. In the thesis, the number of slug tests is at the magnitude of \( 10^3 \sim 10^5 \). Note that the number of measurements is far more than the number of Monto Carlo runs conducted by various workers [Harvey, 1992; Desbarats, 1994; Butler, 1990], who intend to relate the statistics of \( Y_m \) to that of \( Y_c \). Thus, the numerical spatial filtering approach may provide results of superior stability. On the other hand, the noise in the measurement process can be evaluated and removed through an optimal Wiener filtering technique, whereas it would pose difficulties for the Monto Carlo method. Furthermore, the spatial filtering method is versatile to account for conductivity field of high variance, multimodal distribution (e.g., bimodal sand-shale) and nonstationarity and non-linear transient flow, whereas these conditions are hard to be accommodated by analytical

\(^{11}\)Here 'stable' means that the adding of new measurements will not improve or degrade the identified slug test filter.
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methods (e.g., first-order perturbation, spectral method) without simplifications. The engineering application and theoretical significance of the spatial filtering approach shall be summarized in Chapter 7.

1.3 Thesis overview

The thesis subject is the spatial filtering from core-scale conductivity to the conductivity measured by a slug test and the effect of specific storage on the conductivity filtering through numeric experiments. In Chapter 2, we present numerical methods to simulate slug test in heterogeneous and homogeneous media. The conceptualization of slug test in a system and filtering framework is given in Chapter 3, and in which system filtering properties and the inverse estimation of system outputs are discussed in detail. In Chapter 4, a spectral analysis through Wiener filtering is developed to identify spatial filter in heterogeneous media. As a result of computation load constraint, the specific storage value examined in the heterogeneous conductivity filtering is limited to $S_s = 10^{-n}$, $n = 1, \ldots, 5$. The effects of other $S_s$ with values of $n = 6, \ldots, 10$ on the conductivity filtering are investigated in homogeneous media where the spectral analysis method becomes invalid, a numerical perturbation method is developed, which is the topic of Chapter 5. In Chapter 5, empirically deterministic expressions of influence radius and equivalent filter width versus specific storage are presented. Consequently, concerning the aquifer considered in Chapter 4 and 5, the two chapters contain conceptually different but complementary approaches. In Chapter 6, the conductivity filtering in heterogeneous media with different specific storages and various spatial variabilities are explored. In Chapter 7, the major results are summarized in conclusions and future work is recommended, also the possible engineering application and theoretical significance of the spatial filtering approach are discussed.
Chapter 2

SIMULATION OF SLUG TEST

In this chapter, we introduce the boundary value problem associated with the slug test and its numerical solution. For the research in subsequent chapters, we are required to simulate the slug tests in the following domains:

a) Two dimensional, heterogeneous conductivity field;

b) Two dimensional, homogeneous conductivity field with a small region of inhomogeneity, or a so-called perturbed homogeneous field.

c) One dimensional, homogeneous conductivity field.

We utilize three different finite difference models to simulate slug tests. Considering the constraints in the simulation, i.e., 1) the statistical requirement of random conductivity, 2) the hydraulic gradient distribution excited by the slug test, and 3) the efficiency, we discretize the domains with three appropriate type of grids, shown graphically in Fig. 2.3. Details of these discretizations will be discussed in the corresponding sections.

This chapter proceeds as follows: First, we present the slug test set-up and its analytical solution. Second, we introduce the three FDM models, the approximation and discretizations techniques. For illustration convenience, we first discuss the 2D heterogeneous case, then the 2D perturbed homogeneous case, and lastly the 1D homogeneous simulation.
Figure 2.3: Domain discretization of slug tests. Graph A is a rectangular discretization in heterogeneous conductivity field; B is a radial discretization in homogeneous field with a small region of inhomogeneity (only half shown); C is a radially symmetric discretization in homogeneous field.
2.1 The boundary value problem of a slug test

Fig. 2.4 shows the schematic diagram of a slug test of a finite wellbore diameter in an infinite confined aquifer. The flow, initial and boundary conditions will be described by a mathematical model after the next paragraph.

To facilitate the numerical simulation of above problem in heterogeneous media, we simplify the problem by assuming that the vertical flow is negligible, by considering that the wellbore is screened throughout the whole thickness of the aquifer, and that the aquifer stretches to infinity. It is then not unreasonable to assume the horizontal hydraulic gradient to be much greater than the vertical hydraulic gradient, i.e., \( \frac{\partial h}{\partial r} \gg \frac{\partial h}{\partial z} \). Consequently, the assumption of negligible vertical flow has the effect to reduce the flow dimension from 3D to 2D. Other practical issues are excluded from this thesis, e.g., the effects of initial excess head and wellbore damage, the double straight line and inertial effects, and the effects of entrapped air and oscillatory responses etc.
The simplified model is represented mathematically,

\[ \nabla \cdot (K \cdot \nabla h) = S_s \frac{\partial h}{\partial t}, \quad r \geq r_w, \quad (2.16) \]

\[ \int_0^{2\pi} K \frac{\partial h}{\partial r} r d\theta dr = \pi r_c^2 \frac{dH}{dt}, \quad r = r_w, \quad (2.17) \]

\[ h(r_w, t) = H(t), \quad (2.18) \]

\[ H = H_0, \quad t = t_0, \quad r = r_w, \quad (2.19) \]

\[ h(r, 0) = 0, \quad r \geq r_w, \quad (2.20) \]

where,

- \( K \): conductivity.
- \( S_s \): specific storage.
- \( H_0 \): initial excess head.
- \( H(t) \): water level at wellbore at time \( t \).
- \( h(r, t) \): head in testing aquifer at distance \( r \), time \( t \).
- \( r_c \): well casing inner radius.
- \( r_w \): well screen inner radius.

Note that, by the principle of superposition, the initial conditions can be generalized to any case.

Cooper et al. [1967] presented an analytical solution of a slug test in homogeneous medium. This solution was taken from heat conduction literature [Carslaw and Jaeger, 1959]. Hydraulic head at wellbore is solved as

\[ H(t) = \frac{8}{\pi^2} \frac{H_0 \alpha}{\Delta u} \int_0^\infty e^{-\frac{\alpha^2 u}{u \cdot \Delta u}} \]

\[ \Delta u = [u J_0(u) - 2 \alpha J_1(u)]^2 + [u Y_0(u) - 2 \alpha Y_1(u)]^2, \]

\[ \alpha = \frac{r_w^2}{r_c^2} S_s, \quad \beta = \frac{K \cdot t}{r_c^2}, \]
Chapter 2. SIMULATION OF SLUG TEST

where \( S_s \) and \( K \) represent specific storage and conductivity, respectively; and \( \alpha \) and \( \beta \) are of the same dimension \([L^{-1}]\). The so-called dimensionless wellbore storage [Carslaw and Jaeger, 1959; Sageev, 1986; Karaski et al., 1988; Guyonnet et al., 1993] is 

\[
\frac{\pi r_c^2}{2 r_c b} S = \frac{\alpha}{2 S_s b},
\]

where \( S \) is storativity and \( b \) is aquifer thickness. In the thesis we assume \( r_c = r_w \), and use a different terminology. We call the \( \alpha = S_s \) at wellbore the wellbore specific storage, or for short, wellbore storage. Because the wellbore is a hollow cylinder only containing water, thus wellbore storage equals to 1 \([L^{-1}]\).\(^1\)

For slug test in the simplified 2D heterogeneous aquifer, the solutions can be obtained numerically. In the next section, we will introduce a FDM model in such an aquifer and discuss in detail the numerical approximation errors and techniques.

2.2 2D FDM in rectangular coordinates

For the work that follows, we must simulate slug tests in heterogeneous conductivity fields. The conductivity fields are generated by geostatistical simulation methods which utilize regular grids. In numerical models, the fields are most often discretized in blocks regularly spaced in cartesian coordinates. For rectangular grids with constant mesh spacing, finite difference and finite element methods have no advantage over each other. In the thesis, we select the finite difference method because of its appealing simplicity in discretization and the amenability for an efficient symmetric matrix. The derivation of the FDM equation is depicted in Appendix B.I. Here only the aspects affecting the accuracy of the FDM method are discussed.

\(^1\)For simplicity, the dimension unit of specific storage is hereafter omitted. In other words, we consider an aquifer of unit thickness.
2.2.1 Numerical approximation and discretization errors

The discrete output of numeric models are utilized to approximate the continuous response of the physical systems. It is well known that we can use the sampling theorem to reproduce a continuous function through discrete sampling [Oppenheim, 1983], hence it is possible to use the sampling theorem as a guide for the time and spatial discretization of numeric models. However, this is very hard to achieve when the physical response is governed by a set of intractable partial differential equations predicated on the boundary and initial conditions. Thus, here we follow a 'try-evaluate-try' methodology which rests on the proper physical interpretation of model outputs. We minimize or exclude the time and spatial approximation errors until there is a best fit between the model outputs and the expected outputs possibly obtained with the accurate analytical solutions of some idealized systems, which in this case is the slug test in homogeneous media.

Even though the slug test is one of the simplest hydrogeologic tests, its numeric simulation is far from a trivial task. The complexity arises from the extremely dynamical response and the Dirac initial conditions of hydraulic head across the wellbore and the interactive role of conductivity and specific storage in the recovery. These aspects are discussed in the next 4 subsections and a summary of numerical errors is given at the end.

2.2.1.A Wellbore approximation

In the 2D FDM model, the wellbore is represented with one grid block to facilitate computer programming (see Fig. 2.3 A). We numerically derived in Appendix B.I that the wellbore grid is equivalent to a real well by assigning wellbore storage \( S_w^{well} = 1 \) and wellbore conductivity \( K^{well} \) several magnitudes higher than that in the aquifer.
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Figure 2.5: A: Discretizing wellbore into a number of fine blocks. B: A round well is approximated by a square block.

This treatment of the wellbore is particularly important for the simulation of slug tests. In this case, a general discretization formula can be applied at the wellbore with the only need of assigning $S_{well}^*$ and $K_{well}$ to the wellbore grid. More importantly, it is feasible to assign any number of blocks as wellbore so as to readily achieve a finer wellbore discretization that is shown in Fig. 2.5 A, or, investigate the spatial discretization errors, which will be discussed in the next section.

Approximating a round wellbore with radius $r_a$ by a square grid block with dimension $\Delta x$ related to $r_a$ by

$$\pi r_a^2 = \Delta x^2, \quad r_a = 0.564 \Delta x,$$

(2.22)

will introduce numerical error even though it implements the mass balance equation (2.17) exactly. Shown in Fig. 2.5 B, it overestimates the hydraulic gradient $\frac{\partial h}{\partial x} w_2P_2$ while underestimating $\frac{\partial h}{\partial x} w_1P_1$. This shape effect becomes less significant for a finer discretization
proximal to wellbore.

2.2.1.B Spatial discretization

Figure 2.6: Head distribution at initial time and head response at wellbore.

Figure 2.6 A shows head distribution at initial time. The Dirac-function-like feature of the head distribution across the inner boundary requires the space discretization proximal to wellbore to be infinitely small at the very early time. As time progresses, this requirement eases.

The resultant numeric error can be reduced by refining the spatial discretization, simply by assigning a number of blocks at wellbore and employing a fine discretization in the vicinity of wellbore. Fig. 2.7 shows the simulated response curves of two discretization schemes denoted as a and b. Curve a represents the discretization of evenly spaced grids of a wellbore diameter (hereafter denoted as $\Delta x^w$), while curve b stands for the grids of $1/13$ of a $\Delta x^w$. We see that curve b matches the analytical solution very closely. This confirms that the misfit of curve a is a result of coarse spatial discretization. The type of spatial discretization error shown by curve a can also be observed in the work of Demir
Figure 2.7: Wellbore response comparison: numerical solution against analytical solution. a: space step equal a wellbore diameter; b: space step equal $\frac{1}{13}$ of a wellbore diameter. All other parameters of $a$ and $b$ are identical.

and Narasimhan [1994] who utilize an integrated finite difference method to simulate slug test and treat the wellbore as a capacitor (see their Fig. 5).

The $S_s$ of Fig. 2.7 is $10^{-1}$. With other $S_s$ values, Fig. 2.8 displays the close match between the analytical solutions and the numerical results of refined meshes. The evenly spaced grids are of $1/5, 1/4, 1/3, 1/2$ of a $\Delta x^w$, for $S_s$ equals to $10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}$, respectively. For $S_s < 10^{-5}$, the numerical errors of the grids of a $\Delta x^w$ become negligible. This means the spatial discretization error decreases with decreasing $S_s$. It is because a small $S_s$ invokes a large influence zone, hence small hydraulic gradient, and mitigates
Figure 2.8: Wellbore response comparison: numerical solution against analytical solution. The solid curves are analytical solutions; The dashed curves are simulated results. Specific storage ranges from $10^{-4}$ (upper right) to $10^{-1}$ (lower left). Their space steps are $\frac{1}{3}, \frac{1}{4}, \frac{1}{5}$, of a wellbore diameter respectively.

However, such grid refinements increase the number of degrees of freedom significantly. For example, with $S_s = 10^{-1}$, and considering the simulation of a domain of $64 \times 64 \Delta x^w$, the number of freedom after refining the grid to $1/13$ of a $\Delta x^w$ will be $64 \times 64 \times 13 \times 13 = 692224$ blocks. For smaller $S_s$, although using a large grid block of a $\Delta x^w$ may be feasible, the drastic increase of flow domain requires overwhelming grid size (the number of grids). Computation limitations suggest it is better to use the spatial discretization of one grid block for the wellbore. The resultant numerical errors can be mostly offset by employing
an effective radius which is introduced next.

We define the effective radius as that which yields the best match between a numerical simulation in a homogeneous aquifer and the analytical results (Cooper et al., 1967; Beckie and Wang, 1994). Beckie and Wang [1994] found \( r_e = 0.59\Delta x^w \) for \( S_s \leq 0.15 \), while Harvey [1992] set \( r_e = 0.427\Delta x^w \) and Peaceman [1978] used \( r_e = 0.19\Delta x^w \). The differences between these workers arise from the methods used to approximate the wellbore.

![Figure 2.9: Effective screen radii in terms of wellbore diameter versus logarithmic specific storages.](image)

Figure 2.9 illustrates \( r_e \) versus \( S_s \). We see that all effective radii are greater than the apparent screen radius \( r_a = 0.564\Delta x^w \). Using \( r_e > r_w \) reflects the attempt to increase flux. On the other hand, it implies that in general the numerical errors inhibit flux to the wellbore. This is true because the approximation of \( \frac{\partial h}{\partial r} \) by \( \frac{\Delta h}{\Delta r} \) underestimates the spatial gradient, particularly near the wellbore. We also see that, for \( S_s \) smaller than

\[ \Delta x^w = \Delta z, \] if the wellbore contains only one grid block.
10^{-6}, the effective radii approaches the apparent screen radius. This is consistent with our observation that the spatial discretization errors are negligible for $S_s < 10^{-5}$.

The numerical spatial discretization error varies with time. The time discretization will be discussed in the next section. Here we only show that if we evaluate $r_e$ at each time step, we end up with a time-dependent effective radius $r_e(t)$.

Figure 2.10: Effective screen radii in terms of wellbore diameter ($\Delta x^w$) are a function of time. The smaller $S_s$, the flatter the $r_e(t)$, the faster $r_e(t)$ reach $0.564 \Delta x^w$. (The last time step is, $t_{55} = \frac{\beta r_e^2}{K} = \frac{10x(0.564 \Delta x^w)^2}{K}$).

Figure 2.10 demonstrates the time dependence of the estimated $r_e(t)$, for $10^{-5} \leq S_s \leq 10^{-1}$. We see that $r_e(t)$ curves approach the apparent well screen radius $0.564 \Delta x^w$ at later time steps, when the hydraulic gradient becomes flat enough to assuage discretization error. We also see that the smaller specific storages, the faster $r_e(t)$ falls off, and the closer $r_e(t)$ stays to the apparent well screen radius $0.564 \Delta x^w$. For $S_s < 10^{-5}$, the $r_e(t)$ curves overlap the $0.564 \Delta x^w$ line. The reason for this is again the effect of $S_s$ on the
hydraulic gradient distribution.

2.2.1.C Time discretization

We have shown, in Figure 2.6 B, the wellbore response with respect to time. Starting from time $t_0$, the response is continuous, this fact eases the time discretization. The static hydraulic heads are almost recovered as $\beta = \frac{KQ}{r_0^2}$ approaches 100 [Papadopulos, 1973] and the recovery becomes independent of $S_s$.

Considering that the recovery becomes insignificant at later time for $S_s > 10^{-5}$, we simulate our slug tests in heterogeneous media until $\beta = 10$. The total real time $t_{max}$ is calculated by

$$t_{max} = \frac{\beta r_w^2}{K_G}$$  \hspace{1cm} (2.23)

where $K_G$ is the geometric mean of the heterogeneous core-scale conductivity. We discretize $t_{max}$ to steps geometrically increasing by a constant factor $t_f$. Thus the first time step $t_1$ should be

$$t_1 = \frac{t_{max} (1 - t_f)}{1 - (t_f)^{nt}}$$  \hspace{1cm} (2.24)

where $nt$ is number of time steps. If $\beta$ and $t_f$ are specified, the $t_1$ depends only on $nt$. The greater the specific storage, the faster response, thus the more time steps are required. For a given $S_s$, we simply adjust $nt$ until the time discretization error dissipates. In the thesis we choose $nt = 55$ for $t_f = 1.15$ or $nt = 100$ for $t_f = 1.1$, and we find that time discretization error becomes negligible for $10^{-10} \leq S_s \leq 10^{-1}$.

2.2.1.D The hypothetical remote boundary

Mathematically, the model solution requires a remote boundary condition at an infinite distance from the wellbore. The effect of a finite domain be minimized if we place the
boundary sufficiently far away from the wellbore. However, this results in an excessive computational burden because the grids are evenly spaced and of finite length.

To get around this, an approximation technique is employed to divide the whole domain into a heterogeneous inner zone with blocks equally spaced and a homogeneous outer zone with blocks of length increasing by a geometric factor each step away from the inner zone [Harvey, 1992]. The homogeneous conductivity is set to the geometric average of the inner zone. The diagram was shown in Fig. 2.3 A.

The concept of influence zone is frequently seen in the well testing literature, yet no universal definition can be found. It is due to the 'dynamic' essence of what is meant by 'influence'. For slug tests, for example, it is called "range" of the slug test [Barker and Black, 1983]; "radius of investigation" [Sageev, 1986]; "the maximum distance traveled by a perturbation $H/H_0 = 1\%, 5\%, 10\%$" [Guyonnet et al., 1993]. A more rigorous definition will be given in Chapter 4, here we temporarily adopt the $H/H_0 = 1\%$ definition of Guyonnet et al. [1993].

The influence zone at large enough time (e.g., $\beta = 100$) is determined by $S_s$ and is independent of $K$. Guyonnet [1993] experimentally determined that area of influence zone is proportional to specific storage, $r/r_w \propto \left[\pi r^2_c/(2\pi r^2_w b S_s)\right]^{0.495}$. Our initial results show that, for $10^{-10} \leq S_s \leq 10^{-1}$, 20 blocks is enough to exclude the boundary effect if the geometric factor is set to 2. The number of blocks at inner zone is obtained by enlarging the inner zone until wellbore response is not affected by the presence of the outer homogeneous zone. It is found that 64 by 64 blocks is sufficient for $10^{-3} \leq S_s \leq 10^{-1}$. More blocks are needed for smaller $S_s$ and computation becomes expensive.

In the preceding discussions, we only used the wellbore response curve to investigate the numerical errors. This is because wellbore response is most dynamical so that its simulation possesses the maximum error. We are assured that the simulation error within flow domain should be trivial if the simulated wellbore response curve is accurate enough.
to have matched the analytical solution.

In summary, the sources of numerical errors include: (a) The temporal and spatial
discretization by approximating \( \frac{\partial h}{\partial r} \frac{\partial h}{\partial t} \) with \( \frac{\Delta h}{\Delta r} \), \( \frac{\Delta h}{\Delta t} \). Using a \( K \)-dependent discretiza-
tion, the temporal approximation can be reduced to a negligible level. But the spatial
discretization error cannot be easily eliminated due to computation constraint. (b) The
boundary approximation. The wellbore shape effect is unavoidable but not significant,
while the effect of the remote boundary can be minimized. Errors of type (a) consistently
cause an underestimation of the hydraulic gradient, which can be offset by applying an
effective radius greater than the apparent radius. Errors of type (b), for approximation
of a round wellbore, cause under- or overestimation of hydraulic gradient, while for the
remote boundary, it may positively contribute the wellbore flux if not placed far away.

2.3 2D FDM in radial coordinates

Simulation of a slug test in homogeneous confined aquifer with a region of inhomogene-
ity (Fig. 2.3 B) can be conducted by a 2D FDM model in radial coordinates. The radially
distributed mesh conforms with the flow pattern invoked by a slug test and thus is much
more resistant to space discretization errors, and is more efficient in reducing the number
of degrees of freedoms.

The mathematical model is exactly the same as that of the slug test in heterogeneous
aquifer introduced in the previous section. We only need transform the heterogeneous
model to radial coordinates and perform the same finite differencing. We rewrite the flow
equation (2.16) as

\[
\frac{1}{r} \frac{\partial}{\partial r} \left[ r K(r, \theta) \frac{\partial h}{\partial r} \right] + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left[ K(r, \theta) \frac{\partial h}{\partial \theta} \right] = S_s \frac{\partial h}{\partial t}. \tag{2.25}
\]

The detailed FDM equations are derived in Appendix 2.3.

In the FDM model, besides the boundaries of the 2D heterogeneous model, one more
boundary of $\theta$ is needed. We set the $\theta$ boundary to be opposite to the perturbed inhomogeneity and treat it as a nonflow boundary. Note that there is only one inhomogeneity and it is placed along a centerline of the grid. The nonflow boundary is physically valid because the symmetric flow pattern invoked by the inhomogeneity warrants no crossing flux. Note that we can not simply simulate only some slices of the radial domain and incorporate the $\theta$ boundary effects by means of superposition like that of pumping wells. Because the boundary effects on the inhomogeneity affecting the flux to the slug well differs from the boundary effects on the pumping or of any other source terms. Thus, we need simulate the whole flow domain, which was shown in Fig. 2.3 B. Note that only half of the domain is illustrated.

The spatial discretization, unlike the constant grid spacing in the 2D heterogeneous model, can be easily refined until sufficient accuracy is achieved. The $\frac{\partial h}{\partial r}$ discretization is obtained empirically as a function of $S_a$. The $\frac{\partial h}{\partial \theta}$ discretization of 50 equal $\Delta \theta$ is found to sufficient. The time discretization formula is identical to that of the 2D heterogeneous model, but for better accuracy 100 time steps and a factor of 1.1 are used. Apparently, the wellbore shape effect does not exist in the radial model.

Fig. 2.11 demonstrates the excellent agreement between the numerically simulated wellbore response and the analytical results. The minor graphical misfit is due to the inadequate data points in the original paper [Cooper et al., 1967].

2.4 1D FDM in radial coordinates

We use a 1D FDM homogeneous model to reproduce the analytical solution of Cooper et al. [1967] and Papadopulos et al. [1973]. The 1D FDM model is very important to efficiently reproduce the wellbore response curve needed in the inverse estimation of the measured conductivity discussed in Chapter 3. Even though the analytical solution
can be evaluated by the numerical inversion of Laplace transform with the algorithm of Stehfest [1970], which is found to be of great use in hydrologic applications [Moench and Ogata, 1984; Butler and Liu, 1991; Guyonnet et al., 1993; Harvey, 1992], the efficiency of this algorithm is still not adequate for the problem at hand. For example, considering 256×256 slug tests are recorded with each comprising 55 time points, and assume the inverse fitting procedure (discussed in Chapter 3) converges by 10 searches, then the number of calls to the Stehfest's [1970] algorithm is \(3.6 \times 10^7\), this is overwhelming! The example is just a moderate case of those discussed in Chapter 3.
However, the homogeneous model can be easily solved by the 1D FDM. The straightforward boundary conditions can be readily implemented. The spatial discretization is identical to the $\frac{\partial h}{\partial r}$ discretization of the 2D perturbed homogeneous model, and the time discretization is also identical. We rewrite the flow equation (2.16) in a one dimensional form,

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial h}{\partial r} \right) = \frac{S_s}{K} \frac{\partial h}{\partial t}. \quad (2.26)$$

The FDM formulas are derived in Appendix B.III.

Fig. 2.12 demonstrates an excellent agreement between the numerically simulated response curve and the analytical results. The minor graphical misfit is due to the inadequate data points in the original papers [Cooper et al., 1967; Papadopulos, 1973]. The $\beta (nt = 55, t_f = 1.15)$ value of these response curves equals to 10. This is appropriate in the inversion of the response curve of the 2D heterogeneous model, while a value of $50 (nt = 100, t_f = 1.1)$ should be used in the inversion of the 2D response curve from perturbed homogeneous model.
Figure 2.12: Simulated wellbore response (dashed lines) versus analytical solution (solid lines). From lower left to upper right, $S_s$ decreases from $10^{-1}$ to $10^{-10}$. For all simulated curves, $\beta = \frac{K_0 l^2}{r_w} = 10$. 
Chapter 3

SYSTEM AND FILTERING

Centering on system and filtering, we will introduce three topics in this chapter:

a) The concepts of system, filtering, and the definition of a slug test conductivity filtering system;
b) The properties of the slug test system filtering;
c) The method to obtain the system outputs.

3.1 The concepts of system and filtering

A system can be viewed as any process that results in the transformation of signals. Thus a system has an input signal and an output signal which is related to the input signal through the system transformation [Oppenheim et al., 1983, p35]. If the transformation changes the relative amplitudes of the frequency components in a signal or perhaps eliminates some frequency components entirely, the transformation is referred to as filtering [Oppenheim et al., 1983, p397]. If the filtering system is space invariant and linear, the filtering process may be represented in a convolution form,

\[
Y(x) = \int_{-\infty}^{\infty} H(x - x') X(x') \, dx'
\]

(3.27)

where \( H(x) \) is called system function or filter function, and its Fourier transform is called transfer function. The convolution is in essence a summation or superposition. A filtering system is shown in Fig. 3.1.
Chapter 3. SYSTEM AND FILTERING

The system function epitomizes the system behavior, hence characterizes the relationship between the input and the output. In a linear time/space invariant system, the system function is the system response to a Dirac input, i.e., the impulse response of the system. But in a nonlinear system, the system function can not be easily obtained, and in which case we can probe the system properties by an input of various signals and the observation of the output signals.

3.2 The slug test conductivity filtering system

In this section we first physically define the slug test filtering system, then we discuss its appropriate mathematical representation.

3.2.1 Physical definition

In Chapter 2 we have introduced the slug tests in heterogeneous core-scale(of a wellbore diameter $\Delta x^w$) conductivity fields, $K_{\text{core}}$, and the numerical simulation of the wellbore response curve. If we fit the wellbore response curve by that of a homogeneous conductivity, $K^{\text{slug}}$, of the model of Cooper et al. [1967], we obtain a uniform conductivity $K^{\text{slug}}$ which is thought to be somehow a representation of the heterogeneous conductivity $K_{\text{core}}$. Then one may ask: How good is the representation? What volume of $K_{\text{core}}$ is
being sampled? and what is the quality of sampling (the extent of representativity) on each part of the sampled $K^{\text{core}}$? More generally, what is the scale relationship between $K^{\text{core}}$ and $K^{\text{slug}}$?

We utilize a spatial filtering approach to explore the scale relationship between $K^{\text{core}}$ and $K^{\text{slug}}$, by conceptualizing the scale relationship in a system and filtering framework. We regard the slug test as a conductivity filtering system which receives $K^{\text{core}}$ input and produces $K^{\text{slug}}$ output. The slug test filtering system is graphically shown in Fig. 3.13.

![Diagram of slug test filtering system](image)

**Figure 3.13:** Slug test filtering system receives core-scale conductivity field and produces measured-conductivity field on a larger scale.

The slug test filter is what we are after. If we know the input $K^{\text{core}}$ and the output $K^{\text{slug}}$, then we can readily identify slug test filter with the well-established theory of system identification, which shall be discussed in the next chapter.

The input $K^{\text{core}}$ is generated by the geostatistical simulation methods of GSLIB [37] or a spectral method elucidated in Chapter 4. The output $K^{\text{slug}}$ can be obtained by two steps: First, performing slug tests at different locations in the $K^{\text{core}}$ field and recording the wellbore response curves. This is done by the numerical simulation model described in Chapter 2. Second, inverting the wellbore response curves to the conductivity values
Chapter 3. SYSTEM AND FILTERING

\( K^{\text{slug}} \). The two steps are shown in Fig. 3.14. The inversion procedure will be discussed in Section 3.5.

![Slug test scenario diagram](image)

Figure 3.14: Slug test scenario.

The above discussion has only been concerned about the filtering of the conductivity field. It is known that specific storage is also heterogeneous in reality, thus a similar filtering may exist: \( S_{s}^{\text{core}} \rightarrow S_{s}^{\text{slug}} \). However, it is reported that specific storage is less variable than conductivity [Freeze, 1975; Dagan, 1989]. On the core-scale, it is not unreasonable to assume \( S_{s} \) to be uniform. This assumption will greatly facilitate the analysis of \( K^{\text{core}} \rightarrow K^{\text{slug}} \) filtering. With both \( S_{s}^{\text{core}} \) and \( K^{\text{core}} \) heterogeneous, the inverse method must accommodate the following difficulties:

1. The inverse estimation of \( K^{\text{slug}} \) and \( S_{s}^{\text{slug}} \) can be nonunique due to the interactivity of \( K^{\text{slug}} \) and \( S_{s}^{\text{slug}} \) in the inversion. The heterogeneity effects of \( K^{\text{slug}} \) can be absorbed in \( S_{s}^{\text{slug}} \) and may appear pseudo-homogeneous; vice versa for \( S_{s}^{\text{slug}} \). It also means that the bias of one of them can propagate to the other, thus \( K^{\text{slug}} \) and \( S_{s}^{\text{slug}} \) may not be representative of \( K^{\text{core}} \) and \( S_{s}^{\text{core}} \).
2. With heterogeneous $S_{\text{core}}$, the influence zone may be of irregular shape and differing area, in which the filtering power distribution\(^1\) may be very erratic.\(^2\) This contradicts with the preferred technical requirement of the spatial filtering approach.\(^3\)

For simplicity, better accuracy, and the facilitation of the technical requirement of conductivity filtering, we proceed with a uniform core-scale specific storage, and we examine the role of specific storage in the measurement process by varying $S_{\text{core}}$ uniformly instead of heterogeneously.

The conductivity filtering is transient process, resulting from the transient head recovery at the wellbore and the transient propagation of the hydraulic head perturbation, $\Delta h^p(r,t)$, which is

$$\Delta h^p(r,t) = \begin{cases} 
H_0, & r = r_w, \\
0, & r > r_w, \\
H(t), & r = r_w, \\
h(r,t), & r > r_w.
\end{cases}$$

where the notations are the same as those in Chapter 2. At initial time $t = t_0 = 0$, $\Delta h^p(r,t) = 0$ for $r > r_w$, no filtering occurs; as it starts off, the head perturbation propagates outwards from the wellbore with deceasing magnitude, $H(t) \searrow h(r,t) \searrow 0$, thus, the filtering area enlarges with time while the filtering power proximal to the...

---

\(^1\)Filtering power is equivalent to filter amplitude.

\(^2\)In Chapter 2 we mentioned that specific storage determines influence zone. For homogeneous $S$, the influence zone is symmetric, enlarging with decreasing $S$. For heterogeneous $S$, one can infer the influence zone to be asymmetric and of differing area for differing $S$, in which case, being convolved by the heterogeneous effects of $K_{\text{core}}$, the filtering power distribution may be drastically different for slug tests performed from one location to the other.

\(^3\)The conductivity filter estimated with the spectral analysis method of Chapter 4 is an equivalent filter that can be interpreted as the one averaged from the individual filters for each slug tests sampling different core-scale conductivities; this requires the conductivity filterings of each slug tests should not be drastically different, otherwise the spectrally estimated filter may not be representative or meaningful. The different, irregular filter area and power distribution due to heterogeneous $S_{\text{core}}$ may result in fundamentally different filterings for each slug test. Thus it may reduce the representativeness of the filter identified by the spectral method.
wellbore decreases with time. This phenomenon shall be elaborated in the perturbation method of Chapter 5. It is pictorially shown in Fig. 3.15, where the filter functions are illustrated by three hypothetical Gaussian probability density functions

$$f(x, y) = \frac{6}{\pi \lambda_x \lambda_y} e^{-\left[\frac{1}{\lambda_x^2} + \frac{1}{\lambda_y^2}\right]}$$

(3.28)

where, \(\lambda_x = \lambda_y = 8, 16, 32\), which are the characteristic filter widths.

Figure 3.15: Spatial conductivity filtering is a transient process. The top three show the core-scale field under filtering at different time \(t_1\), \(t_2\) and \(t_3\), where \(t_3 > t_2 > t_1\). The bottom are the corresponding filter functions.

The transient filter can be obtained by inverting the response curve in different time intervals. The inverted \(K^\text{slug}\) in each interval lumps the spatial filtering of \(K^\text{core}\) in that

\[\text{Here we use such a function is just for illustration convenience because of its similarity to the slug test filtering behavior, see Fig. 5.37, and explanations on page 115.}\]
interval. However, this is rarely the case in engineering practice in which one observes a response in a very short time period, and one may frequently find that some parts of data record contain noise, e.g., the very early-time response. Thus, it is a usual practice to invert the response curve as a whole. This is because that the practical concern is on the gross sampling volume of a slug test in the whole duration of measurement.

Considering its practical applicability, in the thesis research we adopt the method of the inversion of the whole response curve. The inversion method is equivalent to integrating the slug test filtering over the whole duration of the measurement. The identified filter function is a spatial function that represents the total sampling volume and bears different filtering power in different parts of the volume. Note that the transient nature colligates with the spatial-varying property, meaning that the spatial filtering proximal to the wellbore represents the early-time filtering, while the distant spatial filtering displays the later-time filtering. Consequently, the spatial filter illustrates a temporally integrated filtering. We will revisit the subject of temporal filtering by the perturbation approach in Chapter 5.

3.2.2 Mathematical representation

The conductivity filtering can be mathematically represented as,

$$ Y_m(x) = \int G(x - x') Y_c(x') \, dx' + N(x) \quad (3.29) $$

where $Y_c(x)$, $Y_m(x)$ are the logarithm conductivities of the core-scale field $K_{core}$ and measured field $K_{slug}$, $G(x)$ is the filter function and $N(x)$ is the noise. The noise term accounts for numeric model misfit and random truncation error resulting from the slug test simulation method(Section 2.2, 2.3, 2.4) and the inversion routine of the wellbore response curve.
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The reasons for choosing logarithmic conductivity in eq. (3.29) include 1) logarithm K has a normal distribution [41]; 2) logarithm K is less variable than K; 3) the logarithm conductivity occurs naturally in the flow equation,

\[ \frac{\partial^2 h}{\partial x_i^2} + \frac{\partial \ln K}{\partial x_i} \frac{\partial h}{\partial x_i} = \frac{1}{K} S_\ast \frac{\partial h}{\partial t}, \quad i = 1, 2. \]  

(3.30)

It can be seen that the resistivity \( \frac{1}{K} \) shows up in the r.h.s of above equation. For a uniform flow in a vertically homogeneous aquifer, the effective K is a harmonic average where \( \frac{1}{K} \) is the basic element. The conductivity of the form, \( \frac{1}{K}, \frac{1}{K_D} \), are utilized in the literature [Gelhar, 1994; Oliver, 1990, 1992]. For the slug test filtering, it is not essential which form of K is used in eq. (3.29) as long as the form is physically meaningful and mathematically convenient. So is the expression of the filter function. One may write \( G(x) \) as

\[ G(x) = F(x) \frac{1}{x^2} \]  

(3.31)

to explore the \( \frac{1}{x^2} \) behavior reported in the pumping test literature [Desbarats, 1994; Cardwell and Parsons, 1945]. We use eq. (3.29) in the thesis.

3.3 System inputs: core-scale fields

We can probe the system filtering property by the input \( K^\text{core} \) fields of some characteristic spatial variability(e.g., the variance, correlation, and variogram model). Because the system filter characterizes the scale relationship between the input and output, the system filter should demonstrate some features pertaining to the characteristics of input and the output. We postpone the discussion of input design to Chapter 6 until we have learned more aspects about the filtering process.

\(^5\)\( K_D \) is the conductivity normalized by the geometric mean.
3.4 Properties of System filtering

In this section, we explore the system and filtering properties.

3.4.1 Stability and nonuniqueness

Initial results show that the filter function in eq. (3.29) satisfies

\[ \int \int G(x,y) \, dx \, dy \approx 1, \]  

(3.32)

thus,

\[ \max \{ \log K_m(x,y) \} \leq \max \{ \log K_c(x,y) \} \leq +\infty \]  

(3.33)

This means that the slug test system produces a bounded output from a bounded input, i.e., it is a stable system.\(^6\)

The system inversion, that is inverting the system output \( K^{\text{slug}} \) to its input \( K^{\text{core}} \), is nonunique. Because there are uncountable realizations of \( K^{\text{core}} \) fields that can be filtered by a function \( G(x,y) \) to give an identical \( K^{\text{slug}} \).

3.4.2 Space invariant property

The space invariant property is similar to the time invariant property in the context of geophysical or electric engineering and it is identical to the one in image analysis, meaning that the filter function is invariant with respect to spatial translation. For the conductivity filtering, the space invariant property requires the filter volume\(^7\) and filter amplitude \( G(x) \) versus \( x \) remain the same regardless of the slug test locations in the \( K^{\text{core}} \) field. Apparently, this is not strictly true. For example, \( K^{\text{core}} \) filtering is limited

\(^6\)Note: \( \log \) denotes base 10 logarithm in this thesis.

\(^7\)Filter volume is the integral of filter amplitude in the influence zone.
proximal to the wellbore if the wellbore is surrounded by an annular low conductivity clay ring; while the filtering may propagate to a large volume in an isotropic sand aquifer of small variance.\(^8\)

Thus, the conductivity filter can only be approximately space invariant under certain hydrogeologic conditions, e.g., isotropic media of small variance. In an absolute sense, the filtering pattern of \(K^{\text{core}} \rightarrow K^{\text{slug}}\) should vary for the slug tests performed at different locations in a heterogeneous \(K^{\text{core}}\) media. But, the mean or the average of the individual filters can be considered to be space invariant if the filtering function process is stationary,\(^9\)

\[
G(x - x'; \omega) = \langle G(x - x') \rangle + g(x'; \omega) \tag{3.34}
\]

where \(G(x - x'; \omega)\) is a realization of filter function representing the conductivity filter of slug test performed at location \(x'\). Bracket denotes mean, \(\langle G(x - x') \rangle\) is the constant mean filter and \(g(x'; \omega)\) is the variation of \(G(x - x'; \omega)\) at \(x'\). The mean of \(G(x - x'; \omega)\) will approach the constant \(\langle G(x - x') \rangle\) if we increase the number of slug test measurements, in other words, the mean of \(g(x'; \omega)\) turns to zero.

If the filtering process is not stationary, \(\langle G(x - x') \rangle\) may not approach a constant as more measurements are averaged. For example, in a sand-shale field \([\text{Desbarats}, 1987; \text{Bachu and Guthiell}, 1990]\), the filtering will exhibit two distinctive patterns for slug tests performed on the sand and on the shale. The mean filter \(\langle G(x - x') \rangle\) may not be meaningful. The system identification method described in Section 4.1 of Chapter 4 is developed under the space invariant assumption. The identified filter would always be an averaged filter\(^{10}\) from the ones of invoked by each individual slug tests. The

---

\(^8\)In these two cases, the influence zone may be the same because it is independent of conductivity if \(\beta = Kt/r_w^2\) is sufficiently large. What differs is the filter amplitude within the influence zone.

\(^9\)The filter function can be considered to be a random process. At each location \(x'\), there is a filter function realization(labeled \(\omega\)), \(G(x - x'; \omega)\).

\(^{10}\)It is called an effective filter if the filtering process is stationary and the number of measurements is sufficient; or less strictly, called an equivalent filter if the filtering process is nonstationary or the number
representativeness of the filter depends on the extent of nonstationarity.

### 3.4.3 Linearity

The linearity of the filtering process can be evaluated by two kinds of tests. The first is to test the scaling property or homogeneity, that is

$$ a \, Y_{c1}(x,y) \rightarrow a \, Y_{m1}(x,y), $$

(3.35)

where \( a \) is a scaling factor. The second is to test the additivity that is

$$ Y_{c1}(x,y) + Y_{c2}(x,y) \rightarrow Y_{m1}(x,y) + Y_{m2}(x,y), $$

(3.36)

where \( Y_{c1}(x,y), Y_{c2}(x,y) \) are two different core-scale fields, and \( Y_{m1}(x,y), Y_{m2}(x,y) \) are the corresponding measured fields. If a process exhibits both homogeneity and additivity, then it is a linear process [Oppenheim et al., 1983, p43].

The linearity analysis of the signals \( K, Y = \lg K, R = \frac{1}{K} \) is shown in the Table 3.1. The subscripts \( c \) and \( m \) notate the core-scale conductivity and the measured conductivity, respectively. We use \( K^\circ, Y^\circ, R^\circ \) and \( K^\oplus, Y^\oplus, R^\oplus \) denote the scaling property and additivity of variables \( K, Y, R \). In the slug test filtering, \( ^\circ \) is more straight forward than \( ^\oplus \), particularly, the \( K^\circ \) property. So we first discuss \( K^\circ \), then use it to analysis the other properties. Note that scaling a random variable \( x \) by a factor \( a \) changes the mean from \( \mu_x \) to \( a\mu_x \), variance from \( \sigma_x^2 \) to \( a^2\sigma_x^2 \); and adding two random variables \( x \) and \( y \) ends up with a mean \( \mu_{x+y} = \mu_x + \mu_y \), and a variance \( \sigma_{x+y}^2 = \sigma_x^2 + \sigma_y^2 + 2\sigma_x\sigma_y\rho_{xy} \), where \( \rho_{xy} \) is the correlation coefficient.

The \( K^\circ \) can be justified from the finite difference eq. (2.16), (2.17), (B.I.155), (B.III.171), i.e., scale \( K_c \) by a factor \( a \) is equivalent to scale \( K_m \) by the same factor. It corresponds to vary the mean \( \mu_Y \) to \( \mu_Y + a \). The \( Y^\oplus \) is only satisfied under the ideal condition of \( Y_c \) of measurements is insufficient.
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Table 3.1: Linearity analysis of $K$, $\lg K$ and $\frac{1}{K}$.

<table>
<thead>
<tr>
<th></th>
<th>scaling property</th>
<th>additivity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K$</td>
<td>$a (K_c)_m = (a K_c)_m$</td>
<td>$(K_{c1})<em>m + (K</em>{c2})<em>m \overset{?}{=} (K</em>{c1} + K_{c2})_m$</td>
</tr>
<tr>
<td>$Y = \lg K$</td>
<td>$\lg {((K_c)_m)^a} \overset{?}{=} \lg {((K_c)^a)_m}$</td>
<td>$\lg [(K_{c1})<em>m] + \lg [(K</em>{c2})<em>m] \overset{?}{=} \lg [(K</em>{c1}, K_{c2})_m]$</td>
</tr>
<tr>
<td>$R = \frac{1}{K}$</td>
<td>$\frac{a}{(K_c)_m} = \frac{1}{(K_c)^a}_m$</td>
<td>$\frac{1}{(K_{c1})<em>m (K</em>{c2})<em>m} \overset{?}{=} \frac{1}{(K</em>{c1} + K_{c2})_m}$</td>
</tr>
</tbody>
</table>

being homogeneous, by using the property of $K^\otimes$. The extent of $Y^\otimes$ for heterogeneous $Y_c$ can be explored by slug tests in $Y_c$ fields with variances varying from very small (close to homogeneous) to large values. $R^\otimes$ is satisfied thanks to the $K^\otimes$ property.

The $K^\otimes$, $Y^\otimes$, $R^\otimes$ properties can be first appreciated in the homogeneous case in which all are satisfied. For heterogeneous fields, the extent of additivity can be explored by varying $\sigma^2_K$ or $\sigma^2_Y$ from very small to large values. Note that $Y^\otimes$ is satisfied if one of the fields is homogeneous, implying that if the variance of one field is small enough, regardless of the variances of the other field, $Y^\otimes$ can be approximated.

Now, we explore the extent of $Y^\otimes$ and $Y^\otimes$ in heterogeneous core-scale fields. A spectral method, which shall be discussed Chapter 4, page 95, is used to generate the core-scale fields, which have the spectrum

$$S(f_x,f_y) = \frac{2 \sigma^2 \pi \lambda_x \lambda_y}{(1 + 4\pi^2 f_x^2 \lambda_x + f_y^2 \lambda_y)^{3/2}}.$$  \hspace{1cm} (3.37)

where $\sigma^2$ is variance and $\lambda_x = \lambda_y$ is correlation length.

For the tests of $Y^\otimes$, 7 fields are generated with $\sigma^2 = 0.01, 0.1, 0.2, 1, 2, 4, 6$, same $\lambda_x = \lambda_y = 4\Delta x^w$ and field size $= 256 \times 256$ blocks$(\Delta x^w \times \Delta x^w)$. The last six fields are obtained by scaling the first field by an appropriate factor to change its variance. Slug tests are performed in every other blocks in a central area of $128 \times 128$ blocks in each field. $S_s$ is chosen to be 0.01. Test results are shown in Fig. 3.16. It is evident that the extent of $Y^\otimes$ decreases with variance.
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For the tests of $Y^\circ$, 2 sets of fields are generated. Each set consists of 3 fields that have $\sigma^2 = 0.01, 0.2, 1$ and the same $\lambda_x = \lambda_y = 4\Delta x^w$. The amplitude spectra between each set are identical, but the phase spectra between each set are chosen to be different. Then, respectively add up those fields of same variances between the two sets. Other parameters are the same as those of $Y^\circ$ test fields. Test results are shown in Fig. 3.17. Clearly, we see the extent of $Y^\circ$ decreases with increasing variance.

In other circumstances with different the specific storages and different spatial variability of $Y_c$ fields, one would expect different linearity observations.

3.5 System outputs: inverse estimation of $K$ and $S$

The inverse estimation of parameters in heterogeneous media by fitting the type curves of some homogeneous models, e.g., models of Theis [1935]; Cooper and Jacob [1946]; Cooper et al. [1967], is the usual practice in various engineering disciplines. The aim is to characterize the heterogeneous properties by a few simple parameters. Even though being replete in practice, the methodology has not been adequately evaluated for the effectiveness of the inverted parameters with respect to spatial variability.

Here, we only discuss the inversion of the measured response curve of the slug test in heterogeneous media. The inversion is to fit the measured response curve with that of the homogeneous model of Cooper et al. [1967]. The parameters pertaining to the fitting curve are the equivalent parameters (or so-called measured parameters) that 'somehow' epitomize heterogeneous media. The 'somehow' shall be explored in great depth in the following Chapters by the spatial filtering approach. Here we only interpret the 'somehow' from the inversion-process perspective. This is not only of great practical significance, but also is very important for the validity of the spatial filtering approach which works best for the outputs containing minimal bias.

In the following, we will investigate the role of the inverse estimation variables (i.e.,
Figure 3.16: Scaling property $Y^\otimes$: $(Y_c \rightarrow Y_m) \implies (a Y_c \rightarrow a Y_m)$, where $\sigma_{Y_c}^2 = 0.01$. 
Figure 3.17: Additivity $Y^\oplus$: $(Y_{c1} \rightarrow Y_{m1}, Y_{c2} \rightarrow Y_{m2}) \implies (Y_{c1} + Y_{c2} \rightarrow Y_{m1} + Y_{m2})$. 

\[\sigma_{Y_{c1}}^2 = \sigma_{Y_{c2}}^2 = 0.01\]

\[\sigma_{Y_{c1}}^2 = \sigma_{Y_{c2}}^2 = 0.2\]

\[\sigma_{Y_{c1}}^2 = \sigma_{Y_{c2}}^2 = 1.0\]
conductivity and specific storage) in the process of response curve inversion, and interpret the inverse estimates with respect to the spatial variability. We introduce these issues in three step-by-step subsections: 1) The inversion objective function and its relationship to the estimation variables. 2) The estimates of two different inversion methods and their comparison. 3) Interpret the estimates of the two inversion methods with respect to the spatial variability and the role of the estimation variables in the inversion. In all of the subsections, we discuss two inversion methods which may be both appropriate: a) Estimate conductivity only, with specific storage equal to true value; b) Concurrently estimate conductivity and specific storage.

3.5.1 Objective function versus estimation variables

We have discussed that the system output is obtained by fitting the measured response curve to that of the homogeneous model of Cooper et al. [1967]. The associated curve-fitting objective function, \( \varepsilon(\log K, \log S) \),\(^{11}\) can be defined as

\[
\varepsilon(\log K, \log S) = \sum_{i=1}^{nt} \left[ H_m(t_i) - H_a(t_i; K, S) \right]^2,
\]

where \( H_m(t_i) \) is the measured wellbore response curve from the simulation model discussed in Chapter 2; \( H_a(t_i; K, S) \) is the analytical solution of model of Cooper et al. [1967], which is reproduced by the 1D FDM model introduced in Chapter 2, and \( t_i \) is the time with \( nt \) time steps. We find such \( \log K \) and \( \log S \) (equal to \( \log K_{slug} \) and \( \log S_{slug} \) respectively) that minimizes the objective function. This is done with a steepest decent algorithm in the MINOS [71] optimization package.

From the flow equation in heterogeneous media, eq. (2.16), we know that the wellbore response curve is a function of \( K_{core} \) and \( S_{core} \). Thus the response curves of various shapes

\(^{11}\)Here \( \log K \) and \( \log S \) are utilized as estimation variables because they are less variable than \( K \) and \( S \). But we still use \( K \) and \( S \) as the input parameters to the 1D model of the fitting curve in eq. 3.38.
can be encountered in the field measurements. Then it is important to recognize that when we fit the measured curve, represented by the objective function of eq. (3.38), the estimation variables play different roles in the control of the fitting curve. Hereafter we denote the measured response curve RC and the fitting curve FC.

In the inversion procedure, the changes of the estimation variables \( \lg K \) and \( \lg S_s \), i.e., \( \lg K + \delta_{lg} K \) and \( \lg S_s + \delta_{lg} S_s \), are equivalent to rescaling of \( K \) and \( S_s \) in the 1D model by \( 10^{\delta_{lg} K} \) and \( 10^{\delta_{lg} S_s} \) respectively. From the r.h.s. of finite difference eq.(B.III.171) of the 1D model,

\[
\frac{S_s}{K} \frac{h^n_t - h^{n-1}_t}{\Delta t}
\]

we see that scaling \( K \) will shift FC horizontally, because the scaling factor \( 10^{\delta_{lg} K} \) can be scaled to the time, \( \Delta t \), i.e., \( \lg t + \delta_{lg} K \), thus, horizontally shifting the FC of \( \lg t \) versus \( H/H_0 \) by \( \delta_{lg} K \). We also see that scaling \( S_s \) by \( 10^{\delta_{lg} S_s} \) will change the FC shape, because \( 10^{\delta_{lg} S_s} \) can be scaled to \( \frac{h^m_t - h^{m-1}_t}{\Delta t} \), the time derivative that determines the curve shape. Thus, in conclusion, \( \lg K \) determines the FC time coordinates while variable \( \lg S_s \) controls the FC shape.

In order to understand the relationship between the objective function and the estimation variables \( \lg K \) and \( \lg S_s \), a hypothetical test is conducted. First, inputting a pair of \( K \) and \( S_s \), denoted as \( K^{true}, S_s^{true} \), to the 1D FDM model, yields \( H_a(t_i; K^{true}, S_s^{true}) \); second, perturbing \( K^{true}, S_s^{true} \) by an amount of \( \delta_{lg} K, \delta_{lg} S_s \), gives \( H_a(t_i; K^{true} + 10^{\delta_{lg} K}, S_s^{true} + 10^{\delta_{lg} S_s}) \); lastly calculating the objective function as

\[
\epsilon(\lg K, \lg S) = \epsilon(\lg K^{true} + \delta_{lg} K, \lg S_s^{true} + \delta_{lg} S_s) = \sum_{i=1}^{n} \left[ H_a(t_i; K^{true} + 10^{\delta_{lg} K}, S_s^{true} + 10^{\delta_{lg} S_s}) - H_a(t_i; K^{true}, S_s^{true}) \right]^2,
\]

\( \delta_{lg} K \in [\!-0.15,0.15\!] \), \( \delta_{lg} S_s \in [\!-0.15,0.15\!] \).
Figure 3.18: Objective function versus logarithm conductivity and specific storage: $\varepsilon(\log K, \log S_s)$ versus $(\log K, \log S_s)$. Top graph, $\varepsilon(\log K^{true}, \log S_s^{true}) = \varepsilon(2, -1) = 0$; bottom graph, $\varepsilon(\log K^{true}, \log S_s^{true}) = \varepsilon(2, -10) = 0$. Contour level: 0.4, for $\varepsilon(\log K, \log S_s) \geq 0.4$; 0.0025, for $\varepsilon(\log K, \log S_s) \leq 0.2$. 
Figure 3.18 shows the objective function versus logarithm conductivity and specific storage, i.e., $e(\log K, \log S_s)$ versus $(\log K, \log S_s)$. Two extreme cases of the $e(\log K, \log S_s)$ versus $(\log K, \log S_s)$ relationship are demonstrated in Fig. 3.18, where $\log S_s^{\text{true}} = -1, -10$.

In graph A, $\log S_s^{\text{true}} = -1$, we see that the gradient $\frac{\partial e}{\partial \log K} > \frac{\partial e}{\partial \log S_s}$, thus the $\log K$ optimum is more easily found and has a higher precision (i.e. closer to its true value) than $\log S_s$. In graph B, $\log S_s^{\text{true}} = -10$, $\frac{\partial e}{\partial \log K} > \frac{\partial e}{\partial \log S_s} \approx 0$, which means a wide range of $\log S_s$ produces the same objective function. Any optimization method terminates the optimal searching if the objective function approaches the specified cut-off criteria. Thus, the $\log S_s$ optima could happen to be randomly picked up from the objective function contour that has the same value as the cut-off criteria, and the $\log S_s$ optima contain different amount of biases depending on the searching pattern of the specific optimization method used. We can also see that $\frac{\partial e}{\partial \log K}$ of graph B is greater than that of graph A, hence, with given cut-off criteria, the $\log K$ estimation of graph B contains less bias. For intermediate specific storages, $-9 \leq \log S_s \leq -2$, the objective function lines will rotate approximately from NW 30° (graph A) to N90° (graph B).

In summary, with a given cut-off criteria, we observed: 1) $\log K$ optimum is more easily found and has a higher precision than $\log S_s$ optimum. 2) For smaller specific storages, the $\log K$ contains less bias whereas $\log S_s$ estimates is more strongly biased. Note that these observations are based on the fitting to the response curve of a homogeneous aquifer, and they are not necessarily true for the fittings to those response curves of heterogeneous aquifers. For example, for the fit to a response curve of some particular shape, $\frac{\partial e}{\partial \log S_s}$ may be greater than $\frac{\partial e}{\partial \log K}$ so that $\log S_s$ is more easily found and has a better precision.

The fit to the response curves of heterogeneous aquifers can be easily carried out by a simple numeric routine, but the interpretation of $K^{\text{slug}}$ and $S_s^{\text{slug}}$ in terms of the spatial heterogeneity is not straightforward, and it is very complex to quantify the role of the estimation variables in the inversion process. In the following, in Section 3.5.2 we first
present the estimates of two different inversion methods and their comparisons, then in Section 3.5.3 we interpret the estimates of the two inversion methods with respect to the spatial variability and the role of the estimation variables in the inversion.

3.5.2 Comparison of the estimates of different inversion methods

Here, we introduce two inversion methods denoted as a and b: a) Estimate $\log K$ only with $\log S_a = \log S_a^{true}$ (hereafter we denote the estimate $\log K$ as $\log K^a$). b) Concurrently estimate $\log K$ and $\log S_a$ (hereafter we denote the estimates as $\log K^b$ and $\log S_a^b$). We know that the RC measured in a heterogeneous conductivity field usually does not exactly conform to the FC of an equivalent homogeneous media. Thus, both the estimates of the two methods are more or less biased. For method a, the $\varepsilon(\log K)$ of the shape-misfit between $H_{m}(t_i)$ and $H_a(t_i; K)$ can not be reduced, whereas for method b the shape-misfit can be minimized by adjusting the shape-control factor $\log S_a^b$. However, the $\log S_a^b$ deviating from $\log S_a^{true}$ is a biased estimate, and the bias may pollute the $\log K^b$ with larger bias than $\log K^a$ due to the nonuniqueness (or interactivity) of $\log K^b$ and $\log S_a^b$ in the inversion.

We perform slug tests and measure the RCs in an isotropic heterogeneous $K^{core}$ field with an exponential variogram of $\sigma_{lgK}^2 = 0.25$ and correlation length of $\lambda = 4 \Delta x^w$ (wellbore diameter),

$$\gamma(h) = \sigma_{lgK}^2 \cdot [1 - e^{(-\frac{h}{\lambda})}].$$

(3.40)

The field is generated by a sequential Gaussian simulation method of GSLIB [37], and it comprises 256 x 256 square blocks. A number of $S_a$ values (true $S_a$) are assigned to the field, $S_a = 10^{-1}, 0.05, 10^{-2}, 0.005, 10^{-3}, 10^{-4}, 10^{-5}$. Slug tests are performed in every blocks in an central area of 64 x 64 blocks.

Figure 3.19 shows the comparison of the histograms of the $\log K^a$ and the ones of $\log K^b$ and $\log S_a^b$, with true $S_a$ varying from $10^{-5}$ to $10^{-1}$. We see that: 1) Both $\log K^a$ and $\log K^b$ shrink towards the mean of the core-scale field, $\mu_{lgK^{core}} \approx 0$. 2) The shrinking is skewed,
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with the high $K$ end shrinking more strongly than the low $K$ end, and $\lg K^b$ being more skewed than $\lg K^a$. The skewness becomes more pronounced with decreasing true $S_s$. 3) The distribution of $\lg S^b_s$ is not normal to the true $S_s$, spreading out from the true $S_s$ and asymmetrically skewing towards the low value. Larger spreading occurs when the true $S_s$ decreases.

Fig. 3.20 displays the cross plots $\lg S^b_s$ versus $\lg K^b$, with true $S_s$ varying from $10^{-5}$ to $10^{-1}$. We see that: 1) Most $\lg S^b_s$ estimates are biased,$^{12}$ and the magnitude of bias increases with decreasing true $S_s$. 2) For decreasing true $S_s$, the high conductivities shrink more strongly towards the mean $\mu_{\lg K^\text{core}} \approx 0$, but, the low conductivities are somehow persistent.

Figure 3.21 illustrates the comparisons of the $\lg K^a$ with the $\lg K^b$ by the cross plots $\lg K^a$ versus $\lg K^b$. We can roughly see that: 1) For true $S_s = 10^{-1}$, $\lg K^a$ is mostly smaller than $\lg K^b$, but for $S_s < 10^{-1}$ the high value of $\lg K^a$ is greater than that of $\lg K^b$. 2) For decreasing true $S_s$, the high conductivities shrink more strongly towards the mean $\mu_{\lg K^\text{core}} \approx 0$, but, the low conductivities are somehow persistent; and the high conductivities in $\lg K^b$ is more strongly reduced to the $\mu_{\lg K^\text{core}} \approx 0$ than is the $\lg K^a$.

Fig 3.22 shows the cross-plots of $[(\lg K^a) - (\lg K^b)]$ versus $[(\lg S^b_s) - (\lg S^\text{true}_s)]$. We see that: 1) When $\lg S^b_s$ equals to $\lg S^\text{true}_s$, $\lg K^b$ collapses to $\lg K^a$. 2) Whenever $\lg S^b_s < \lg S^\text{true}_s$, $\lg K^b > \lg K^a$. 3) The derivative of $[(\lg S^b_s - \lg S^\text{true}_s)]$ with respect to $[(\lg K^a) - (\lg K^b)]$ increases with decreasing $\lg S^\text{true}_s$, which means a given amount of bias of $\lg S^b_s$ will cause less bias in the $\lg K^b$ estimation with decreasing $\lg S^\text{true}_s$.

In summary, these four types of graphs showed two major points: 1) For decreasing $S_s$, $\lg K^a$ and $\lg K^b$ skewly shrink towards $\mu_{\lg K^\text{core}} \approx 0$, while $\lg S^b_s$ asymmetrically spreading out from the true $S_s$. The low-value conductivity is more persistent than the high-value conductivity. 2) Whenever $\lg S^b_s < \lg S^\text{true}_s$, $\lg K^b > \lg K^a$.

$^{12}$The bias refers to $|\lg S^b_s - \lg S^\text{true}_s|$. 
Figure 3.19: Comparison of histograms of the lg $K$ estimated with true lg $S_s$ and the ones of the lg $K$ concurrently estimated with lg $S_s$. 
Figure 3.20: Cross-plots of $\log K$ concurrently estimated with $\log S_s$ versus estimated $\log S_s$. The dashed lines represent the true $\log S_s$. 
Figure 3.21: Comparison of $\lg K$ estimated with true $\lg S_s$ with $\lg K$ concurrently estimated with $\lg S_s$. The inclined lines are of unit slope.
3.5.3 Interpretation of the estimates of different inversion methods

In order to explain the causes of the skewed histograms and the differences between the two types of inverse estimates, first in Section 3.5.3.A we discuss some extreme values of the inverse estimates of the measured data. Then in Section 3.5.3.B we investigate the effects of spatial conductivity variability and the effects of specific storage on the inverse estimates. Lastly, in Section 3.5.3.C we explore the influence of low and high conductivity at different locations from the wellbore. A summary is given at the end of
Section 3.5.3.C.

3.5.3.A Extreme values of conductivity estimated with true $S_s$

We discuss the maximum and minimum of the conductivity estimates inverted with method $a$. We denote the core-scale conductivities corresponding to these extrema as $K^{\text{high}}$ and $K^{\text{low}}$ respectively. The RCs(response curves) of these $K^{\text{high}}$ and $K^{\text{low}}$ are also inverted with method $b$. The skewness, $K^\text{M}$, and the differences between the estimates of the two inverse methods, $\Delta K$ and $\Delta S_s$, are defined in table 3.2. The skewness is a parameter used to describe the extent of skewing of the histogram of the measured fields with respect to their means. Table 3.2 shows the estimates from the RCs simulated

Table 3.2: The maximum and minimum estimates, skewness $K^\text{M}$, $\Delta K$ and $\Delta S_s$.

<table>
<thead>
<tr>
<th>$S_s^{\text{true}}$</th>
<th>$K$</th>
<th>$K^\text{M}^*$</th>
<th>$K$</th>
<th>$S_s$</th>
<th>$K^\text{M}$</th>
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*: all are logarithmic data;

*: $K^\text{M} = K^{\text{high}} + K^{\text{low}} - 2 \mu_{K^{\text{core}}}$, for $K^{\text{high}} > 0 > \mu_{K^{\text{core}}} = -0.0589$;

$K^\text{M} = K^{\text{high}} + K^{\text{low}}$, for $K^{\text{high}} < 0 < \mu_{K^{\text{core}}}$; $\mu_{K^{\text{core}}}$ = mean of core-scale field;

†: $\Delta K = (K$ of varying $K$ only) - $(K$ of varying $K$ and $S_s)$;

‡: $\Delta S_s = (S_s$ of varying $K$ and $S_s$) - $(S_s^{\text{true}})$. 
with total time $t_{\text{max}} = \frac{\beta r_n^2}{K_n}$, $\beta = 10$, where $K_n$ equals the smaller of the geometrical mean of the core-scale conductivities at the inner zone of the simulation model, and the geometrical mean of the conductivities at the four blocks around wellbore. Such a $t_{\text{max}}$ is able to catch the whole recovery in $K^{\text{high}}$ and $K^{\text{low}}$ as well.

A negative $K^m$ means the histogram skews to values less than the mean. We see that all the estimates are negatively skewed, the extent of skewness is proportional to decreasing true $S_*$. $\Delta K$ and $\Delta S_*$ represent the differences of the inversion methods $a$ and $b$. They are seen to be positively correlated. Graphic illustration of $K^m$ versus $S_*$ and $\Delta K$ versus $\Delta S_*$ shall be given in the next subsection where a series of comparisons are displayed.

We graphically show, in the left column of Fig. 3.23, the RCs and the FCs(fitting curves) of $K^{\text{high}}$ and $K^{\text{low}}$ by varying $K$ only; in the right column, the RCs against the FCs by varying both $K$ and $S_*$. We see: 1) For $K^{\text{low}}$, FC of method $a$ always crosses the RC at approximately $H/H_0 = 0.5$, and FC is lower than RC at $H/H_0 < 0.5$ while high than RC at $H/H_0 > 0.5$; The FC of method $b$ fits the RC quite well. 2) For $K^{\text{high}}$, the relative positioning pattern of the FC and RC in method $a$ is opposite to that of $K^{\text{low}}$ except the very early-time FC of $S_* = 0.1$. FC of method $b$ does not closely fit RC for $S_* \geq 0.001$. 
solid line: simulated response curve
dashed line: fit with varying $K$ only

$S_s = 10^{-1}$

$K^\text{high}$

$K^\text{low}$

$H/H_0$

$\log_{10}(t)$

$S_s = 10^{-2}$

$K^\text{high}$

$K^\text{low}$

$H/H_0$

$\log_{10}(t)$

continued on next page
solid line: simulated response curve
dashed line: fit with varying $K$ only

solid line: simulated response curve
dashed line: fit with varying $K$ and $S_s$

$K^\text{high}$

$K^\text{low}$

$S_s = 10^{-3}$

$S_s = 10^{-4}$

continued on next page • • •
Figure 3.23: The solid lines are the simulated response curves. The dashed lines are the fitting response curves. Left column represents the fit by varying $K$ only. Right column is the fit by varying both $K$ and $S_s$. 
3.5.3.B Effects of spatial conductivity variability with different specific storages on the inverse estimates

In this section, we investigate the $K^{pt}$ versus $S_s$ and $\Delta K$ versus $\Delta S_s$ relationships observed in the inversion of $K^{high}$ and $K^{low}$ through a series of tests in conductivity fields of hypothetical variability. Because the $K^{core}$ (core-scale conductivity) proximal to the wellbore has the strongest influence on $K^{slug}$ (measured conductivity from slug test), the near-wellbore $K^{core}$ corresponding to the maximum $K^{slug}$ may be the highest $K^{core}$ conductivities; and due to the existence of correlation in $K^{core}$, one may expect the $K^{core}$ somehow varies, away from the wellbore, from the near-wellbore high conductivities to some low conductivities, and vice versa for the $K^{core}$ associated with the minimum $K^{slug}$. Consequently, we design a hypothetical radially symmetric spatial variability varying, according to some function, from $\log K^{core}(r = r_w) = \pm 2$ to $\log K^{core}(r = r_{max}(S_s)) = 0$, where $r_{max}(S_s)$ is the influence radius dependent on $S_s$. This variability should cover the practical ranges involved in the previous sections.

The following is the designed continuous functions,

$$a' : Y_c(r) = \pm 2 \left[ 1 - \left( \frac{r - r_w}{r_{max} - r_w} \right)^\alpha \right]$$  \hspace{1cm} (3.41)

$$b' : Y_c(r) = \pm 2 \left( 1 - \frac{r - r_w}{r_{max} - r_w} \right)$$  \hspace{1cm} (3.42)

$$c' : Y_c(r) = \pm 2 \left[ 1 - \left( \frac{r - r_w}{r_{max} - r_w} \right)^\alpha \right]$$  \hspace{1cm} (3.43)

The $a'$ and $c'$ are symmetric about the middle point of $b'$. The nice property of these functions is the exponent $\alpha$ which is very flexible to adjust the variability. A series of discretized functions can be obtained if we use $i$ of $r(i)$ as the varying variable, the above becomes,

$$a : Y_c(\Delta r(i)) = \pm 2 \left[ 1 - \left( \frac{i - 1}{n - 1} \right)^\alpha \right]$$  \hspace{1cm} (3.44)
Chapter 3. SYSTEM AND FILTERING

\[ b : Y_c(\Delta r(i)) = \pm 2 \left(1 - \frac{i-1}{n-1}\right) \quad (3.45) \]
\[ c : Y_c(\Delta r(i)) = \pm 2 \left[1 - \left(\frac{i-1}{n-1}\right)^a\right] \quad (3.46) \]
\[ i = 1, \ldots, n, \]

where \( \Delta r(i) \) represents the ring \( r(i) - r(i-1) \), and \( n \) is number of rings. If we set \( n = 100 \), the function

\[ d : Y_c(\Delta r(i)) = \pm \log\left(\frac{1}{n}\right), \quad i = 1, \ldots, 100. \quad (3.47) \]

exhibits another kind of variability. Functions \( a, b, c, d \) are used in the tests, which are shown in top two graphs in Fig. 3.24. Test \( a \) represents a \( \log K \) slowly varying from the wellbore, test \( c \) a \( \log K \) fast varying from the wellbore, \( b \) a linear log variation. \( d \) represents a exponent linear variation, i.e., \( K \) varies linearly.

In equations (3.44) to (3.47), we denote the positive and negative \( Y_c(\Delta r(i)) \) as \( K^> \) and \( K^< \), respectively. We perform slug tests in \( K^> \) and \( K^< \) with different \( S_s \) values, see the upper-left graph of Fig. 3.24. For each \( S_s \), the summation of the measured conductivity in \( K^> \) and that in \( K^< \) is what we called skewness \( K^M \). A non-zero \( K^M \) is a measure of non-linearity(non-additivity), in which case the measured conductivity of \( K^> \) and \( K^< \) are not symmetric about the mean.

We conduct slug tests and the analyses of \( K^M \) for all of the hypothetical variability cases \( a, b, c, d \) and for \( S_s = 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5} \). For results comparability, we choose the \( r_{\text{max}} \) of \( S_s = 10^{-5} \) and an empirical grid spacing of \( \Delta r(i) \) for all the \( S_s \) cases. The conductivity in the ring \( \Delta r(i) \) is given by the above formulas.

The bottom two graphs of Fig. 3.24 show the skewness \( K^M \) versus \( S_s \) of these tests, compared with the \( K^M \) of the measured data observed in Section 3.5.3.A, for both inversion method \( a \) and \( b \). The skewness represents the extent of additivity \( Y^\oplus \) with respect to different conductivity variations and specific storage values. For a perfect \( Y^\oplus \), we
should observe $K^m = 0$. In method $a$, we see: 1) All $K^m < 0$, meaning that $Y^n$ is not satisfied. 2) The linear-variation test $b$ is the closest to the measured data. The $\lg K$ of test $a$ varies too slowly to mimic the measured case, while $c$ and $d$ too fast. 3) The $K^m$ of the slowly-varying test $a$ is the smallest for $S_s \geq 10^{-4}$. This is consistent with the rule that the extent of $Y^n$ is proportional to decreasing variance. 4) The $K^m$ of the linear exponent test $d$ does not change with $S_s$, it may be due to the fact that the fast-varying portion of $d$ is influenced by the slug tests of all the $S_s$ cases, and it may follow a long slowly-varying portion in which the extent of $Y^n$ is similar, i.e., $K^m$ is similar for all $S_s$ cases.

Fig. 3.25 shows $\Delta K$ versus $\Delta S_s$. All the graphs demonstrate the positive correlation. Test $b$ is the closest to the measured data case, which is consistent with that in our observation in $K^m$ versus $S_s$.

Fig 3.26 shows the FC and RC of test $d$. Three cases are considered: 1) $K^> : K^1 > K^2 > \cdots > K^n = K^o$; 2) $K^o : K^1 = K^2 = \cdots = K^n = K^o$; 3) $K^< : K^1 < K^2 < \cdots < K^n = K^o$.

We see: 1) For $K^<$, FC of method $a$ always crosses the response curve (RC) at approximately $H/H_0 = 0.5$, and FC is lower than RC at $H/H_0 < 0.5$ while high than RC at $H/H_0 > 0.5$; FC of method $b$ fits RC quite well. 2) In method $a$, the relative positioning of FC and RC of $K^>$ is opposite to that of $K^<$. 3) The skewness $K^m$ is displayed by the difference (horizontal distance between the FC of $K^>$ and $K^o$) — (horizontal distance between the FC of $K^<$ and $K^o$). The 1) and 2) are consistent with our observation in the FC and RC of the measured data case, which means the designed test is reasonable. The FC and RC of tests $a$, $b$, $c$ and $d$ are listed in Table 3.3.

The results of tests $a$, $b$, $c$ and $d$ are listed in Table 3.3.,
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\[ \log_{10}(K'(i)) = \]

- Curve a: \[ \alpha = 5; i = 2, ..., n; n = 100 \]
- Curve b: \[ \alpha = 5; i = 2, ..., n; n = 100 \]
- Curve c: \[ \alpha = 5; i = 2, ..., n; n = 100 \]
- Curve d: \[ \alpha = 5; i = 2, ..., n; n = 100 \]

Figure 3.24: Conductivity variation functions and skewness \( K'_{MB} \) of tests a, b, c, d.
Figure 3.25: $\Delta K \sim \Delta S_s$ of conductivity variation functions $a$, $b$, $c$, $d$. 
Chapter 3. SYSTEM AND FILTERING

Case 1 ($K^o$): $K^1 > K^2 > \ldots > K^n = K^o$
\[ \log_{10}(K^i) = \log_{10}(K^o) - \log_{10}(i/n) \]
\[ \log_{10}(K^o) = 0, \ n = 100 \]

Case 2 ($K^o$): $K^1 = K^2 = \ldots = K^n = K^o$
\[ \log_{10}(K^i) = \log_{10}(K^o) \]
\[ \log_{10}(K^o) = 0, \ n = 100 \]

Case 3 ($K^o$): $K^1 < K^2 < \ldots < K^n = K^o$
\[ \log_{10}(K^i) = \log_{10}(i/n) + \log_{10}(K^o) \]
\[ \log_{10}(K^o) = 0, \ n = 100 \]

continued on next page
solid line: simulated response curve
dashed line: fit with varying $K$ only

solid line: simulated response curve
dashed line: fit with varying $K$ and $S_s$

$S_s = 10^{-2}$

$S_s = 10^{-3}$

continued on next page
Figure 3.26: Fitting and response curves of $K^>$, $K^<$ and $K^o$ of test $d$. 
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<td>0.1126</td>
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<td>0.1797</td>
<td>-3.4237</td>
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<td>0.1609</td>
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<tr>
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<td>-0.1704</td>
<td>-9.7245</td>
<td></td>
<td>-0.2766</td>
</tr>
</tbody>
</table>

* all are logarithmic data;  
* $K^{M} = K^{>} + K^{<}$;  
† $\Delta K = (K$ of varying $K$ only) - $(K$ of varying $K$ and $S_{s}$);  
‡ $\Delta S_{s} = (K$ of varying $K$ and $S_{s}$) - $(S_{i}^{true})$.  

Table 3.3: $K^{M}$, $\Delta K \sim \Delta S_{s}$ of tests a, b, c, d.
3.5.3 C Influence of low and high conductivity at different locations from the wellbore on the inverse estimates

In this section, we discuss the influence of low and high conductivity at different locations from the wellbore on the inverse estimates. In other words, we shall explore the $K^M$ versus $S_*$ and $ΔK$ versus $ΔS_*$ relationships in space. We employ a numeric perturbation idea, shown in Fig. 3.27, perturbing the homogeneous conductivity $Yc(x)$ by an amount of $δ_{Yc(x)}$,

$$δ_{Yc(x)} = \begin{cases} 
0, & x \notin Ω_p, \\
\log(\frac{1}{a}), & x \in Ω_p \\
0 < a < 1 
\end{cases}$$  \hspace{1cm} (3.48)

where $Ω_p$ is the area of one grid block. We use $K^{[−]}$ and $K^{[+]}$ to notate negative perturbation $Yc(x) − δ_{Yc(x)}$ and positive perturbation $Yc(x) + δ_{Yc(x)}$, respectively. The perturbation is carried out, one block per time, on all blocks on one centerline, using the FDM introduced in Chapter 2. We fix the true $S_* = 0.1$ and observe the spatial relationships $K^M$ versus $r_D$, $ΔK$ versus $r_D$, and $ΔS_*$ versus $r_D$, where $r_D = r/r_w$ is the dimensionless perturbation location. For $S_*$ equal to other $S_*$ values, similar observations should be expected.

Several typical RCs of $K^{[−]}$ and $K^{[+]}$ near the wellbore or distant from wellbore are shown in Fig. 3.27. The positioning of FC to RC, of $K^{[−]}$ and $K^{[+]}$, for both inversion method $a$ and $b$, should exhibit the same pattern as we observed in the $K^{low}$ and $K^{high}$ of measured data case or the $K^<$ and $K^>$ of the test cases. Compared with the type curves of the homogeneous RC, we expect, for inversion method $b$, $S_*$ shall be underestimated in the FC of near-wellbore low $K$, while overestimated in the FC of distant low $K$, and we should observe the opposite in the high $K$ case. The location that $S_*$ is correctly estimated is where such a perturbation has identical influence on the early- and later-time RC, i.e,
it doesn't alter the shape of the RC of the homogeneous case, only horizontally shifts it.

The perturbation is independent of $Y_c$ according to the $K^\varnothing$ property ($Y_c(x) \pm \delta Y_c(x)$) is equivalent to $10^{\pm \delta Y_c(x) K_c}$, thus we can assume $Y_c = 0$ without losing generality. We choose $a = 0.99, 0.95, 0.9, 0.8, 0.7, 0.6, 0.5$ and $S_s = 0.1$.

The results of the tests $K^{[1]}$ and $K^{[4]}$ are shown in Fig. 3.28 and Fig. 3.29. These plots are the inverse estimates versus the perturbation location. We see: 1) Shown in the first 4 graphs, $K^m$ decreases with $a$, and $K^m \approx 0$, for $a \geq 0.9$. It means that $K^\varnothing$ is approached if the changes of conductivity by the perturbation are less than 10% of the homogeneous conductivity. This somehow quantifies the previous linearity analysis that $K^\varnothing$ is approximated if the variance is sufficiently small (close to homogeneous). 2) The comparison of the results of inversion method $a$ and $b$ are shown in the last two graphs. The graph $\Delta K(K^{[\pm]})$ versus $r_D$ shows $[K(K^{[\pm]})^a - K(K^{[\pm]})^b]$ versus $r_D$. In comparison of the graph $\Delta S_s(K^{[\pm]})$ versus $r_D$ which is $[S_s(K^{[\pm]})^b - \text{true} S_s]$ versus $r_D$, it is clear that $\Delta K(K^{[\pm]})$ and $\Delta S_s(K^{[\pm]})$ are positively correlated. 3) At the only location, $r_D = 1.8$, $\Delta S_s(K^{[\pm]}) = 0$, i.e., $S_s(K^{[\pm]})^b$ is correctly estimated, which is where the perturbation has identical influence on the early- and later-time RC, thus doesn't alter the shape of the RC of the homogeneous case. At this location, $\Delta K(K^{[\pm]}) = 0$, i.e., $K(K^{[\pm]})^a = K(K^{[\pm]})^b$. For other true $S_s$ values, one may expect that the $r_D$ where $S_s(K^{[\pm]})^b$ is correctly estimated shall increases with decreasing true $S_s$.

To summarize the hypothetical tests in the preceding two sections, in Section 3.5.3.B we observed:

1. $K^m$ versus $S_s$ of method $a$ is negatively skewed, and the extent of skewness of the $K^m$ versus $S_s$ with the linear log conductivity variation is proportional to decreasing true $S_s$. $K^m$ versus $S_s$ of method $b$ exhibits erratic changes which result from the noises of $\lg K^b$ due to the interactivity $\lg S_s^b$ versus $\lg K^b$ in the inverse estimation.
2. $\Delta K$ versus $\Delta S_s$ conforms a positive correlation.

In Section 3.5.3.C we observed:

1. $K^m$ versus $r_D$ decreases with perturbation magnitude, and $K^m$ versus $r_D$ approaches zero if the perturbation about is less than 10% of the unperturbed conductivity.

2. $\Delta K(K^{[\pm]})$ versus $r_D$ positively correlates with $\Delta S_s(K^{[\pm]})$ versus $r_D$.

The explanation for the first observations is that the slug test filtering system exhibits nonlinearity. The extent of nonlinearity is proportional to the variance and the decreasing $S_s$ of the heterogeneous aquifer. The reason for the second observations is the interactivity of $\lg S_s^b$ versus $\lg K^b$ in the inverse estimation in which $\lg S_s^b$ controls the shape of FC in the fitting to the RC which can be of various shapes due to the heterogeneity. In rest of this thesis, we choose the inversion method $a$ to estimate $Y_m$. 
Figure 3.27: Perturbation set-up, and the response curves of low/high $K$ perturbations near wellbore and distant from wellbore.
Measured conductivity and specific storage in homogeneous field with a small area of perturbation versus dimensionless perturbation location \( r_D = r/r_w \).

**Notations:**
1) solid lines: negative perturbation \( K[-] = \log(K) - \log(1/a) \)
2) dashed lines: positive perturbation \( K[+] = \log(K) + \log(1/a) \)

**Graph labels:** from the zero line to its two sides, \( a = 0.99, 0.95, 0.9, 0.8, 0.7, 0.5 \).

Figure 3.28: \( K^\pm \sim r_D \), \( K^\star \sim r_D \), where \( r_D = r/r_w \), of inversion scheme \( a \) and \( b \). True \( S_s = 0.1 \).
Figure 3.29: $\Delta K^\pm \sim r_D$, $\Delta S_s \sim r_D$, where $r_D = r/r_w$, of inversion scheme $a$ and $b$. True $S_s = 0.1$. 
Chapter 4

SYSTEM IDENTIFICATION: SPECTRAL ANALYSIS

Spectral analysis is known as the estimation of the power spectrum of a stationary random process, and it is also applicable to deterministic as well as nonstationary process with appropriate modifications [Mohanty, 1986; Priestly, 1981]. Using spectral estimates, in Section 4.1 we identify the system filter by a Wiener filtering method. The Wiener filtering method requires the estimation of power spectrum, so we discuss the spectrum estimation methods in Section 4.2. In the last section, we test the Wiener filtering approach by the estimation of a number of hypothetical filters.

4.1 Nonparametric filter function identification: Wiener filtering

We have conceptualized in Section 3.2.2 that the spatial filter can be represented as

\[ Y_m(x) = \int G(x-x') Y_c(x') \, dx' + N(x) \]  

where where \( Y_c(x) \), \( Y_m(x) \) are the logarithm conductivities of the core-scale field and measured field, \( G(x) \) is the filter function and \( N(x) \) is the noise. In this section, we use a Wiener filtering approach to deconvolve the system filter \( G(x) \).

Taking Fourier transform of (4.49) and using hat to indicate Fourier transformation, we get

\[ \hat{Y}_m = \hat{G} \hat{Y}_c + \hat{N}. \]  

Let’s define,

\[ \hat{Y}_m = \hat{G} \hat{Y}_c = \hat{Y}_m - \hat{N}, \]  

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\[ \hat{G} = \frac{\hat{Y}_m \hat{Q}}{\hat{Y}_c} \]  

(4.52)

where \( \hat{Y}_m, \hat{G} \) are optimal estimates of \( \hat{Y}_m, \hat{G} \), and \( \hat{Q} \) is an optimal filter that minimizes \( \| \hat{G} - \hat{G} \|^2 \) which is

\[
\| \hat{G} - \hat{G} \|^2 = \left| \frac{\hat{Y}_m \hat{Q}}{\hat{Y}_c} - \frac{\hat{Y}_m}{\hat{Y}_c} \right|^2,
\]

\[
= \frac{1}{\left| \hat{Y}_c \right|^2} \left( \left| \hat{Y}_m \hat{Q} \right|^2 \left| \hat{Q} \right|^2 + \left| \hat{N} \right|^2 \left| \hat{Q} \right|^2 \right).
\]

(4.53)

Differentiating equation (4.53) with respect to \( \hat{Q} \), and setting the result equal to zero gives

\[ \hat{Q} = \frac{\left| \hat{Y}_m \right|^2}{\left| \hat{Y}_m \right|^2 + \left| \hat{N} \right|^2}. \]

(4.54)

This is the formula for the optimal filter \( \hat{Q} \). Thus the optimally estimated transfer function is

\[ \hat{G} = \frac{\hat{Y}_m \hat{Q}}{\hat{Y}_c} \]

\[ = \frac{\hat{Y}_m \left| \hat{Y}_m \right|^2}{\hat{Y}_c \left| \hat{Y}_m \right|^2 + \hat{Y}_c \left| \hat{N} \right|^2}. \]

(4.55)

Multiplying by \( \hat{Y}_c^* \), dividing by \( \left| \hat{Y}_m \right|^2 \), and substituting \( \left| \hat{Y}_m \right|^2 \) with \( \left| \hat{Y}_c \right|^2 \left| \hat{G} \right|^2 \), yields

\[ \hat{G} = \frac{\hat{Y}_m \hat{Y}_c}{\left| \hat{Y}_c \right|^2 + \left| \hat{N} \right|^2 \left| \hat{G} \right|^2}. \]
We rewrite above formulation as

\[ \hat{G}(f) = \frac{S_{cm}(f)}{S_{cc}(f)} \]  

(4.56)

where \( S_{cm}(f) \) is the cross spectral density of \( Y_c \) and \( Y_m \), \( S_{cc}(f) \), \( S_{NN}(f) \), \( S_{GG}(f) \) are spectral densities of \( Y_c \), \( N \), \( G \), respectively, and \( f \) is spatial frequency. Eq. 4.56 can also be deduced from the similar work of Sondhi [1972] and Helstrom [1967] in image analyses.

The Wiener filtering is performed by an iterative procedure, shown in Figure 4.30.

4.2 Spectrum estimation

The spectral estimate of minimum bias and lowest variance is the goal of various spectrum estimation methods. Centering on bias and variance, in this section we first discuss the sources of bias, and the dilemma existing in the spectrum estimation; secondly, we discuss the ‘traditional’ periodogram and data tapering method; lastly, a multitaper method, which is an improved version of the periodogram and data tapering method, is presented and verified through some hypothetical filtering tests.

4.2.1 Bias and dilemma of spectrum estimation

We define the bias, \( B_S(f) \), of a spectrum estimate \( \tilde{S}_N(f) \), as [Mohanty, 1986],

\[ B_S(f) = E\{\tilde{S}_N(f) - S_N(f)\}, \]  

(4.57)

where \( S_N(f) \) is the true spectrum. Understanding the sources of biases in a spectrum estimation procedure by the spectral method will be of great help to minimize the bias and improve the resolution or accuracy of the spectral estimation.

The typical steps in a spectrum estimation procedure and their effects on the spectral estimates are illustrated in Fig. 4.31. We see that bias can be introduced in every stage
Chapter 4. SYSTEM IDENTIFICATION: SPECTRAL ANALYSIS

Y spectrum
\[ S_{cc}(f) = \tilde{Y}_c(f) \tilde{Y}_c^*(f) \]

Cross-spectrum
\[ S_{cm}(f) = \tilde{Y}_c(f) \tilde{Y}_m(f) \]

Initial signal/noise ratio
\[ \frac{S_{NN}(f)}{S_{GG}(f)} = \text{initial value} \]

Transfer function
\[ \tilde{G}(f) = \frac{S_{cm}(f)}{S_{cc}(f) + \frac{S_{NN}(f)}{S_{GG}(f)}} \]

Filter spectrum
\[ S_{GG}(f) = \tilde{G}(f) \tilde{G}^*(f) \]

Estimated \( \hat{Y}_m(f) \)
\[ \hat{Y}_m(f) = \tilde{G}(f) \hat{Y}_m(f) \]

\[ \sum_{j=0}^{\infty} \tilde{G}(f)^{j+1} \]

Noise
\[ \hat{N}(f) = \hat{Y}_m(f) - \hat{Y}_m(f) \]

Noise spectrum
\[ S_{NN}(f) = \hat{N}(f) \hat{N}^*(f) \]

\[ E = \frac{\sum_{j=0}^{\infty} \tilde{G}(f)^{j+1} - \sum_{j=0}^{\infty} \tilde{G}(f)^{j}}{\sum_{j=0}^{\infty} \tilde{G}(f)^{j+1}} \]

\[ E \leq 10^{-6} \]

Figure 4.30: Filter identification by an iterative Wiener filtering procedure.
in the spectrum estimation procedure, which typically are the data tapering in the lag-window, discrete Fourier transformation (DFT), spectrum averaging in spectral-window, and spectrum smoothing.

Data tapering discards part of data record and unevenly weights the data in the lag window. Aliasing occurs when the Nyquist frequency, $\frac{1}{2\Delta}$, is smaller than the cut-off frequency, $f_c$, in the original signal, i.e.,

$$\frac{1}{2\Delta} < f_c.$$

The implicit DFT periodicity folds the spectral energy at $f > \frac{1}{2\Delta}$ back to the energy at $[0, \frac{1}{2\Delta}]$. Spectrum averaging by the spectral window function, the DFT of the lag window, causes uneven averaging in the main lobe of the spectral window, while in the small side lobes it introduces leakage from spectral peaks. Spectrum smoothing is performed
through a moving-window in which adjacent estimates are arithmetically averaged.

The quality of an estimated spectrum can be judged through the evaluation of estimation bias and spectrum variance. The smaller bias and the lower variance, the better the PSD estimate. However, these two goals usually cannot be achieved simultaneously by a single data taper method. This can be seen from the fundamental result [Newland, 1993, p108],

\[
\frac{\sigma}{m} \approx \frac{1}{\sqrt{B_e N}}, \quad (4.58)
\]

\[
B_e = \frac{1}{\int_{-\infty}^{\infty} \tilde{A}^2(f) \, df}, \quad \text{and} \quad \int_{-\infty}^{\infty} \tilde{A}(f) \, df = 1, \quad (4.59)
\]

where \( \sigma, m, B_e, N \) are standard deviation, mean, effective spectral-window width and the length of data record. \( \tilde{A}(f) \) is the spectral window function. Eq.(4.58) holds valid if spectrum changes slowly over frequency intervals of order \( 1/N \), i.e. provided that the record length is long enough to resolve adjacent spectral peaks. Eq.(4.58) also applies to the measurement made with an analogue spectral analyzer and applies approximately to the measurement of the autocorrelation function.

From Eq.(4.58), we can now appreciate the common dilemma that is, for a high resolution spectrum (i.e. small bias), \( B_e \) must be small, while for good statistical reliability (i.e. low variance) \( B_e \) must be large compared with \( \frac{1}{N} \), the Rayleigh frequency. For an example, the Blackman-Tukey lag window [Mohanty, 1986],

\[
\lambda (k) = \begin{cases} 
\frac{1}{2} \left( 1 + \cos \left( \frac{\pi k}{M} \right) \right), & |k| \leq M, \\
0, & \text{otherwise}.
\end{cases} \quad (4.60)
\]

It gives a bias of [Priestley, 1981],

\[
B_S(w) = \frac{\pi^2}{4M^2} \frac{d^2}{dw^2} S(w), \quad (4.61)
\]
and a variance as

$$\lim_{N \to \infty} [N \text{Var}(\tilde{S}(w))] = \begin{cases} M \frac{3}{4} S(w)^2, & w \neq 0, \pm \pi, \\ M \frac{6}{\pi} S(w)^2, & \text{otherwise}, \end{cases} \quad (4.62)$$

where \( w \) is angular frequency and \( S(w) \) is the power spectrum density. We can see that when \( M \) is large (\( B_e \) is small), the bias is small and the variance is large. If \( M \) is small (\( B_e \) is large), the bias is large and the variance is small. In general, a single-taper methods are plagued by the trade-off between bias and variance, however, multitaper method provides "tricks-of-the-trade" and yield robust spectrum estimates, which we shall discuss after the next section.

### 4.2.2 Periodogram and data tapering

The periodogram can be defined as [Mohanty, 1986],

$$I_N(f) = \frac{1}{N} \left| \tilde{X}_N(f) \right|^2 \quad (4.63)$$

where \( \tilde{X}_N(f) \) is the DFT of data series \( X_N(n) \). The periodogram is an asymptotically unbiased estimate of the true power spectrum density, \( S_N(f) \), in the sense that [Mohanty, 1986],

$$\lim_{N \to \infty} E\{I_N(f)\} = S_N(f), \quad (4.64)$$

but, it is an inconsistent PSD estimate because

$$\lim_{N \to \infty} \text{Var}\{I_N(f)\} \neq 0, \quad (4.65)$$

which means the spectrum variance does not drop off as we increase the number of measurements.

Although the periodogram does not have any data tapering bias, i.e., it uses the exact whole data record, the periodogram can be shown to be equal to the true PSD
$S(f)$ convolved with an implicit spectral window that is the DFT of a boxcar function
$\Pi_N(n)$ of a unit height and a length of N,

$$\Pi_N(n) = 1, \quad n = 0, 1, 2, ..., N - 1$$

$$\hat{H}(f) = \frac{\sin(\pi f N)}{\pi f} = \frac{N \sin(x)}{x}, \quad x = \pi f N$$

(4.66)

$$\tilde{S}(f) = \int_{-\infty}^{\infty} |\hat{H}(f - f')|^2 S(f') df'$$

(4.67)

where $\hat{H}(f)$ is the sinc function and $\tilde{S}(f)$ is the estimated PSD. Eq.(4.67) shows that
$\hat{H}(f)$ unevenly weights $S(f)$ and causes leakage to the roll-off frequency.

Therefore we can use a tapered data window to smooth the data at each end of the
record before it is analyzed. This has the effect of sharpening the spectral window, hence
reducing leakage. The tapers, Hanning, cosine, Parzen, etc., belong to this kind. The
PSD estimate of the tapered data relates to periodogram,

$$\tilde{S}(f) = \int_{-\infty}^{\infty} |\hat{A}(f - f')|^2 I_N(f') df'$$

(4.68)

$$\int_{-\infty}^{\infty} \hat{A}(f - f') df' = 1,$$

(4.69)

where $\hat{A}(f)$ is the DFT of the data taper $a(n)$. However, as long as only a single taper
is used, there will be a trade-off between the resistance to spectral leakage and the
variance of spectral estimate. This problem can be addressed by the multitaper technique
[Thomson, 1982; Park et al., 1987].

4.2.3 Multitaper spectral analysis

The multitaper method was first presented by Thomson [1982] and was applied to seis-
mosgrams by Park et al. [1987]. The multitaper method has the lowest variance and
minimum bias compared with several single-taper methods [Park et al., 1987].
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The multitaper method multiplies the data by not one, but several leakage-resistant tapers. The statistical information discarded by the first taper is partially recovered by the second taper, the information discarded by the first two tapers is partially retrieved by the third taper, and so on [Thomson, 1982]. The multitaper spectrum, \( \tilde{S}(f) \), is formed as a weighted sum of several eigenspectra, \( |\tilde{Y}_k(f)|^2 \), which are separately estimated from each tapered data [Park et al., 1987],

\[
\tilde{Y}_k(f) = \sum_{n=0}^{N-1} a_n^k X_N(n) e^{-i2\pi fn}
\]

\[
\tilde{S}(f) = K^{-1} \sum_{k=0}^{K-1} \frac{|\tilde{Y}_k(f)|^2}{\lambda_k}
\]

where \( X_N(n) \) is the original data record and

\[
\lambda_k(N, W) = \frac{\int_{-W}^{W} |\tilde{A}^k(f)|^2 df}{\int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} |\tilde{A}^k(f)|^2 df}
\]

and \( k \) index the order of eigenspectra, \( a_n^k \) are the eigentapers and \( W \) is half of the spectral window width. Therefore the multitaper spectrum is already a smooth estimate; it has less variance than respective eigenspectra which have been designed to reduce bias (leakage), and it is also a consistent estimator.

For the estimation of eigenspectra \( \tilde{Y}_k(f) \), equation (4.73) can be interpreted as the fraction of energy that derives from the frequency interval \( |f - f'| \leq W \) while \( 1 - \lambda_k \) is spectral energy that leaks in from outside the band. Because each taper samples the record in a different manner, the net leakage can be reduced by the offsetting effects of the different eigenspectra leakages.

Only a few low order tapers can be used, as the higher-order tapers allow an unacceptable level of spectral leakage. Also the summation of too many eigenspectra will
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decrease the resolution. The preliminary results show that we should use less than 5 tapers whose \( \lambda_k \) are greater than 0.95.

The arithmetic average results of eq.(4.71) can be improved by an adaptive multitaper spectral estimate which uses frequency-dependent weights, \( d_k(f) \) [Thomson, 1982],

\[
d_k(f) = \frac{\sqrt{\lambda_k} \ S(f)}{\lambda_k \ S(f) + E\{B_k(f)\}},
\]

and produces the optimal weighted spectrum, \( \tilde{S}(f) \),

\[
\tilde{S}(f) = \frac{\sum_{k=0}^{K-1} |d_k(f) \ \tilde{Y}_k(f)|^2}{\sum_{k=0}^{K-1} |d_k(f)|^2},
\]

where \( S(f) \) is the true value of the spectrum and \( B_k(f) \) is the spectral energy at frequency \( f \) that leaks in from outside the frequency band \([f - W, f + W]\). \( E\{B_k(f)\} \) can be approximated by \( \sigma^2 \epsilon (1 - \lambda_k) \), where, \( \sigma^2 \epsilon = \sum_{n=0}^{N-1} X_N(n)^2 \) is total energy of the original data. When the spectrum has a steep slope, the higher-order eigenspectra are down-weighted and the adaptive spectral estimates tend towards the least biased eigenspectral estimates [Park et al., 1987].

4.2.3.A Evaluation: variance and bias

The prolate spheroidal taper sequences are calculated with inputs of \( N \) and \( P \), where \( P = NW \) is the time-bandwidth product of each taper. \( 2W = 2P/N \) is the frequency-bandwidth in which spectral energy concentrates. So, the spectral window of each taper is \( 2P \times \frac{1}{N} \) wide, i.e., it has \( 2P \) Rayleigh frequencies.

In the preceding paragraphs we mentioned that variance and leakage are functions of spectral window width and the summation of different eigenspectra. Here we investigate
the role of \( P \) and the number of tapers, \( K \), in the estimation procedure in terms of variance, leakage and total spectral energy. They are respectively defined as

\[
Var(\tilde{S}(f)) = \begin{cases} 
\frac{\sigma_e^2}{K^2} \sum_{k=0}^{K-1} \frac{1}{\lambda_k^2}, & \text{multitaper [Park et al., 1987]} \\
\frac{\sigma_e^2}{(\sum_{k=0}^{K-1} \lambda_k)^2}, & \text{adaptive multitaper [Park et al., 1987]}
\end{cases}
\] (4.76)

\[
E_L = 1 - \frac{K}{\sum_{k=0}^{K-1} \frac{1}{\lambda_k}} \quad [\text{Park et al., 1987}], \quad (4.77)
\]

\[
E_N = \frac{\sum_{f=0}^{N-1} \tilde{S}(f)}{\sigma_e^2}, \quad (4.78)
\]

\[
\text{where } \sigma_e^2 = \sum_{n=0}^{N-1} X_N^2(n). \quad (4.79)
\]

We conduct the investigation in a random field that is generated by the spectral method introduced in Section 4.2.3.B. We choose the variance of the random field, \( \sigma_{ik}^2 K = 0.25 \), the correlation length \( \tau_x = \tau_y = 4 \Delta x^w \) (wellbore diameter), and field size = \( 256 \times 256 \) blocks \( \Delta x^w \times \Delta x^w \). We estimate the spectrum with both the multitaper and adaptive multitaper methods.

Figure 4.32 shows the variance, leakage and the total spectral energy with respect to \( P \) and \( K \). From top to bottom, they are the variance \( Var(\tilde{S}(f)) / \sigma_e^2 \), the logarithm of leakage \( E_L \), and dimensionless total energy \( E_N \) of the multitaper and adaptive multitaper methods. Note that by the definition in eq. (4.77), the two methods produce the same amount of leakage. The good similarity between A1 and B1 shows that the effects of \( P \) and \( K \) on variances of the multitaper and adaptive multitaper are almost the same. A2B2 is the logarithm leakage and it clearly shows that leakage decreases rapidly as \( P \) becomes larger. This is because more energy concentrates in \([f - \frac{2P}{N}, f + \frac{2P}{N}]\). It also shows that leakage increases as \( K \) tends larger. It is because the higher order tapers are less resistant to leakages by their side lobes. Especially if \( P \) is greater than 4, this
Figure 4.32: Variance, leakage and total spectral energy with respect to spectral window width, \( P = NW \), and the number of prolate tapers, \( K \), of the multitaper (left column) and adaptive multitaper methods (right column). A2B2 is the leakage for both methods.
phenomenon is much more pronounced.

In A3, B3, first, we can see that Parseval’s theorem is not satisfied. This is because the multitaper formulas were not required to satisfy Parseval’s theorem. Secondly, we can see that the adaptive multitaper method overestimates the total energy $\sigma^2$, because of its attempt to compensate the energy lost to leakage by boosting the coefficients of the higher-order eigenspectra in the weighted sum, eq.(4.71). A3 tells us that the multitaper method will overestimate $E_N$ as $P$ and $K$ increase due to the high frequency energy gathered by the side lobes of the higher order tapers. The multitaper method underestimates $E_N$ for small $P$ and small $K$ because in this case the $\frac{1}{\lambda_k}$ weighting is not adequate. Thirdly, comparing A3, B3 with A2B2, we can see that the more the leakage, the more overestimation of the total energy.

4.2.3.B Verification: hypothetical case

We test the robustness of the multitaper spectral method in two steps. First, we use a spectral method to generate a random field with a known hypothetical spectral density function. It is given by

\[
S(f_x, f_y) = \frac{2 \sigma^2 \pi \lambda_x \lambda_y}{(1 + 4\pi^2 f_x^2 \lambda_x + f_y^2 \lambda_y)^\frac{3}{2}}
\] (4.80)

where $\sigma^2$ is the variance of the random field and $f_x, f_y$ are frequencies in $x$ and $y$ direction, and $\lambda_x, \lambda_y$ are the correlation lengths in $x$ and $y$ direction. The random field is generated by following steps:\footnote{This is coded by Dr. R. D. Beckie, Dept. of Geol. Sci., University of British Columbia.}

- spectrum $\rightarrow$ amplitude $\rightarrow$ complex number $\rightarrow$ To physical space

\[
S(f_x, f_y) \quad |r| = \sqrt{S(f_x, f_y)} \quad r = |r| \cos \theta + i |r| \sin \theta \quad \text{IDFT}
\]

random phase $\theta \in [0, 2\pi]$
The parameters are chosen to be, $\sigma^2 = 1.05$, $\lambda_x = \lambda_y = 18\Delta x^w$, field size $512 \times 512$ blocks. To avoid the symmetry of the spectrally generated field, only one quarter of the field is used. We estimate the spectrum of random field by the multitaper method, and we compare the estimated spectrum with the hypothetical spectrum. Furthermore, we filter the field by a hypothetical Gaussian filter function,

$$G(x, y) = \frac{6}{\pi \lambda_x \lambda_y} e^{-6 \left( \frac{x^2}{\lambda_x^2} + \frac{y^2}{\lambda_y^2} \right)}, \quad (4.81)$$

in Fourier space it is

$$\tilde{G}(f_x, f_y) = e^{-\frac{1}{2} \pi^2 \left( \lambda_x^2 + \lambda_y^2 \right)}, \quad (4.82)$$

where $\lambda_x, \lambda_y$ are the filter widths. Then we also estimate the spectra of the filtered fields.

We denote the primitive random field and the filtered field as $K_p, K_f$. We filter $K_p$ by the Gaussian filter with different filter widths, $\lambda_x = \lambda_y = 8, 16, 32\Delta x^w$ respectively and denote them as $K_f^8, K_f^{16}, K_f^{32}$. We estimate the PSDs by the multitaper method with different $P$ and $K$ parameters and we denote the multitaper estimate with $P = n$ and $K = m$ as $P_nK_m$. We compare them with the periodogram and the true spectrum. The results are illustrated in Figure 4.33.

Note that in Fig. 4.33, we have averaged the 2D spectra over the frequencies $f_x, f_y$ at distance $f_r$, to reduce the dimension to 1D, i.e., the 1D spectra are the averaged slices of the 2D spectra,

$$\tilde{S}(f_r) \bigg|_{f_r = \sqrt{f_x^2 + f_y^2}} = \frac{1}{M} \sum_{f_x = 0}^{\frac{1}{\Delta f}} \sum_{f_y = 0}^{\frac{1}{\Delta f}} \tilde{S}(f_x, f_y) \quad (4.83)$$

where $M$ is number of data points averaged. Furthermore, we scale the frequency by the the grid block scale $\Delta x$, i.e., $f \Delta x$, resulting in dimensionless frequency. For simplicity, the 'dimensionless' is omitted. All frequency space plots in the rest of the thesis are graphed in dimensionless frequency.
target spectra of primitive field and filtered fields and Rayleigh frequencies

primitive field

filtered by $\lambda = 8$

*continued on next page*
Figure 4.33: Comparisons of multitaper spectra, periodogram and true spectra. The primitive field is filtered by a Gaussian function $G(x,y) = \frac{6}{\pi \lambda_x \lambda_y} \exp[-6(\frac{x^2}{\lambda_x^2} + \frac{y^2}{\lambda_y^2})]$, where filter width $\lambda_x = \lambda_y = 8, 16, 32 \Delta x$. 

**Filtered by $\lambda = 16$**

**Filtered by $\lambda = 32$**
Of the estimates for the primitive field, the periodogram and the multitaper spectra $P_2K_1$, and $P_5K_3$ match very well with the true spectrum $S(f)$. However, $P_{30}K_6$ deviates from $S(f)$ at frequencies smaller than 0.1. For the filtered field $K_f^8$, the $P_2K_1$ and $P_5K_3$ still fit the true spectrum, but the $P_{30}K_6$ deviates further from $S(f)$ and the periodogram deviates $S(f)$ at $f > 0.1$. The $P_{30}K_6$ and periodogram of $K_f^{16}, K_f^{32}$ continue to deviate from $S(f)$. The $P_2K_1$ of $K_f^{16}$ starts jumping above $S(f)$ at $f > 0.09$ while in $K_f^{32}$ the deviation begins at $f > 0.03$.

The primitive and the filtered fields are 256 by 256 pixels each. So the Rayleigh frequency is $\frac{1}{256}$. The top graph in Fig. 4.33 shows all the true spectra and the Rayleigh frequencies of 2D data series with $N$ by $N$ data points, where $N = 2^3, 2^4, 2^5, 2^6, 2^7, 2^8$ respectively. Rayleigh frequency $\frac{1}{256}$ produces the best resolution and the resolution of $K_p^{64}$ turns out the poorest.

### 4.3 Testing the Wiener filtering method

We test the system identification technique in Section 4.1 by using it to invert the hypothetical Gaussian filter function that was used in Section 4.2.3.B in filtering the primitive random field $K_p$. We use the $K_p$ and the filtered fields $K_f$ as the system input and outputs. The PSDs estimates are as those of Section 4.2.3.B. We use the iterative procedure of Fig. 4.30 to identify the filter function.

Figure 4.34 shows the estimated filter functions in spatial and frequency domains against the hypothetical Gaussian functions. Note that the filter functions are averaged over the data points $(x,y)$ at distance $r$,

\[
G(r) \bigg|_{r=\sqrt{x^2+y^2}} = \frac{1}{M_1} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} G(x, y) \quad (4.84)
\]
and the transfer functions are averaged over the frequencies $f_x, f_y$ at frequency $f_r$,

$$\hat{G}(f_r) \bigg|_{f_r = \sqrt{f_x^2 + f_y^2}} = \frac{1}{M_2} \sum_{f_x=0}^{\frac{1}{2}A} \sum_{f_y=0}^{\frac{1}{2}A} \hat{G}(f_x, f_y)$$

(4.85)

where $M_1, M_2$ are number of data points averaged. We can see that for $\lambda_x = \lambda_y = 8, 16 \Delta x^w$, the Wiener filtering method almost back out the hypothetical function exactly. The misfits for $\lambda_x = \lambda_y = 32 \Delta x^w$ are because the Rayleigh frequency $\frac{1}{256}$ is too coarse for the resolution of the spectrum of $K_j^{22}$, see Fig. 4.33. For better estimation, we need increase field size to decrease the Rayleigh frequency.
Figure 4.34: Left column: comparisons of the estimated filter function with the hypothetical filter function \( G(x, y) = \frac{6}{\pi \lambda_x \lambda_y} e^{-\frac{6}{\pi^2} \left( \frac{x^2}{\lambda_x^2} + \frac{y^2}{\lambda_y^2} \right)} \); right column: comparisons of the estimated transfer function with the hypothetical transfer function \( \tilde{G}(f_x, f_y) = e^{-\frac{3}{8} \pi^2 \left( \frac{f_x^2}{\lambda_x^2} + \frac{f_y^2}{\lambda_y^2} \right)} \); where filter width \( \lambda_x = \lambda_y = 8, 16, 32\Delta x \) respectively.
Chapter 5

SYSTEM IDENTIFICATION: PERTURBATION METHOD

In this chapter, we utilize a numerical perturbation method to identify the system filter. A general introduction to perturbation methods is given in Section 5.1. The mathematical formulas of the numerical perturbation method are derived in Section 5.2. The perturbation method requires the inverse conductivity estimation, already discussed in Chapter 3, from the response curve measured in a perturbed conductivity field, and the accuracy of the method is dependent on the inversion procedures. Thus, in Section 5.2.1 one more inversion scheme is analyzed and compared with the scheme a and b of Chapter 3. The area and the amount of perturbation are very crucial for the accuracy of the perturbation method. The area and amount of perturbation which possess the minimal bias and the highest accuracy are numerically determined in Section 5.3. The system filters with different specific storages and the appropriate perturbation schemes are discussed in Section 5.4.

5.1 Introduction

Perturbation methods are widely used to identify the statistical relationship between the fluctuations of target variable, $\phi(x)$, with the variations of its input parameter, $p(x)$. Both $\phi(x)$ and $p(x)$ are often seen as random processes that can be expressed as an expectation or mean, plus a perturbation, i.e., $\phi(x) = E[\phi(x)] + \epsilon_\phi$, $p(x) = E[p(x)] + \epsilon_p$, where $\epsilon_\phi$ and $\epsilon_p$ denote the perturbation and they are required to be relatively small compared to their mean values for the convergence of the perturbation.
In stochastic subsurface hydrology, perturbation approaches are extensively employed to derive the statistic dependence of hydraulic head variance on the variance of logarithm conductivity and the relationship between effective conductivity and the small scale heterogeneity \cite{Gelhar, 1994}. The assumption of a constant mean hydraulic gradient, i.e., uniform mean flow in space, greatly facilitates the perturbation analyses \cite{Gelhar, 1994}. However, this condition may not be met in reality, for instance, in the nonuniform radial flow to a pumping well.

In the nonuniform flow study, Oliver \cite{1990} characterized the relationship between drawdown and permeability as a Frechet kernel function of the radial flow in pumping tests,

\[
S_i = \int_1^\infty G_i(x) \left(1 - \frac{1}{K_D(x)}\right) dx; \quad K_D(x) = \frac{K(x)}{K}
\]  

(5.86)

where \(G_i(x)\) is the kernel and \(1 - \frac{1}{K_D(x)}\) is the presumed permeability distribution which is radially symmetric but nonuniform, and \(K\) is the average permeability. He solved the kernel by the perturbation

\[
S_D = P_{D_0} + \epsilon P_{D_1} + \epsilon^2 P_{D_2} + O(\epsilon^3)
\]  

(5.87)

where the perturbed drawdown \(S_D\) is approximated by the sum of pressure terms \(P_{D_i}\) of order smaller than 3, and \(\epsilon\) is a small perturbation. This kernel function was used by Oliver \cite{1992} to derive the permeability distribution in eq.(5.86). Oliver \cite{1993} generalized the \(1 - \frac{1}{K_D(x)}\) permeability pattern to nonradially symmetric porous media and analytically solved the associated Frechet kernel by a perturbation method similar to eq.(5.87).

Another method similar to the perturbation idea is the sensitivity analysis, which can be regarded as a special case of the perturbation approach in which \(\phi(x)\) and \(p(x)\) usually are deterministic, i.e., \(\epsilon_\phi = \epsilon_p = 0\). Suppose the functional dependence of \(\phi(x)\)
on $p(x)$ is in the form

$$
\phi(x) = g(p(x)),
$$

(5.88)

then for variation of $p(x)$ about some value, say its mean $\mu_p$, denoted as $\delta_p = p(x) - \mu_p$, $\phi(x)$ can be approximated by a series in terms of $\mu_p$ and $\delta_p$, for instance, the Taylor expansion is

$$
\phi(x) = g(\mu_p) + \frac{\partial g(p(x))}{\partial p(x)} \delta_p + \frac{\partial^2 g(p(x))}{2\partial p(x)^2} \delta_p^2 + O(\delta_p^3).
$$

(5.89)

The $\frac{\partial g(p(x))}{\partial p(x)}$ term in above equation is the sensitivity of $p(x)$ to $\phi(x)$, the associated analysis is referred as sensitivity analysis.

Yeh [1986] reviewed several commonly used sensitivity methods. Wang [1992] used a sensitivity method to identify the principle contamination sources to the shallow aquifer underlying a chemical-industry city where various polluting factors had been detected. The groundwater quality management model of Dong [1994] is also an application of sensitivity analysis, i.e., the objective function is calculated from the response matrix that results from the variation of the flux and concentration under planning.

### 5.2 Numerical perturbation formula

The pros-and-cons of analytical perturbation versus numerical ones are broadly discussed by Gelhar [1994]. The chief disadvantage of analytical methods is their dependence on oversimplified conditions which may be far from reality. Numerical methods can account for practical situations in much greater detail. In the meantime, the generic meaning of numerical results could be as good as the analytical ones if appropriate numerical methods are used.

Here the task is to use a perturbation method to deconvolve the system filter from
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eq.(3.29), which is

\[ Y_m(x) = \int_{-\infty}^{\infty} G(x - x') Y_c(x') \, dx' + N(x) \]  

(5.90)

where \( Y_m(x) \) is inverted from the response curve \( H(t) \) measured in core-scale field \( Y_c(x) \). \( N(x) \) accounts for model misfits mainly due to discretization errors. We shall use the high accuracy 2D FDM model for the simulation, \(^1\) thus the noise term becomes negligible and the above equation simplifies to

\[ Y_m(x) = \int_{-\infty}^{\infty} G(x - x') Y_c(x') \, dx'. \]  

(5.91)

One may attempt to apply analytical perturbation method and directly derive \( H(t) \) for a given \( Y_c(x) \), \(^2\) and then use the \( H(t) \) solution of Cooper et al. [1967] to obtain \( Y_m \), at last the resulting functional dependence between \( Y_m \) and \( Y_c(x) \) gives the filter \( G(x) \), i.e.,

\[ Y_c(x) \rightarrow H(t) \rightarrow Y_m(x) \rightarrow G(x) \]

But the inverse step \( H(t) \rightarrow Y_m(x) \) is very complex to derive analytically, \(^3\) as can be seen from the \( H(t) \) equation,

\[ H(t) = \frac{8}{\pi^2} \frac{H_0 \alpha}{\beta} \int_0^{\infty} e^{-\alpha^2} \frac{du}{u \cdot \Delta u} \]  

(5.92)

\[ \alpha = \frac{r_w^2}{r_c^2} S_s, \quad \beta = \frac{10^{-t}}{r_c^2}, \]

\[ \Delta u = [u \cdot J_0(u) - 2 \alpha J_1(u)]^2 + [u \cdot Y_0(u) - 2 \alpha Y_1(u)]^2. \]

The complexity of above equation dictates the impossibility of analytically obtaining the functional relationship \( Y_m \sim H(t) \) without some simplifications, that may in turn

\(^1\)We have checked its accuracy in Chapter 2.

\(^2\)Perform perturbations on \( Y_c(x) \), like that of Oliver [1993].

\(^3\)We have discussed the numerical inversion of \( H(t) \rightarrow Y_m(x) \) in Chapter 3.
cause intolerable errors so as to plague the analytical attempts. Contrarily, the steps $Y_c(x) \rightarrow H(t)$ and $H(t) \rightarrow Y_m(x)$ are convenient to be performed numerically.

With the numerical perturbation, $Y_m(x)$ can be readily obtained from the inversion estimation of RC (response curve), and $Y_c(c)$ is known, and what we are after is the spatial filter $G(x)$. Let us first discuss the simple case of homogeneous $Y_c(x)$, in which $Y_m(x) = Y_c(x)$, then we get

$$\int_{-\infty}^{\infty} G(x-x')dx' = 1 \quad (5.93)$$

it means the filter volume equals to 1 for slug test at location $x$. But we still don’t know the filter amplitude distribution around $x$, and for that purpose we must lift the integral $\int_{-\infty}^{\infty}$, which is the deconvolution task of the numerical perturbation, discussed in the following.

We perturb $Y_c(x)$ by an amount of $\delta Y_c$ in an area of $\Omega_p$, Fig.5.35, the change in $Y_m(x)$ can be written as

$$\delta Y_c = \begin{cases} 0, & x \notin \Omega_p, \\ \text{constant}, & x \in \Omega_p. \end{cases}$$

Figure 5.35: The radial mesh for the simulation of perturbation experiments.

4Here the numeric perturbation scheme is slightly different from the traditional analytical method. Analytical perturbation is usually performed in the whole parameter domain by a given statistic description of the perturbation, e.g., $\mu_{ep}$, $\sigma_{ep}^2$. Here we numerically perturb, only once a time, in a small area $\Omega_p$ in the parameter space. This can be achieved easily by a discrete numerical method, while in contrast, such a perturbation is onerous to be solved analytically.
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\[ \delta Y_m(x) = \langle D_{Y_m}, \delta Y_c \rangle + O\left[\delta Y_c(x)\right] \quad (5.94) \]

where \( \delta Y_m \) is the change in measured conductivity from the perturbation \( \delta Y_c \), and \( \langle D_{Y_m}, \delta Y_c \rangle \) is the Frechet derivative

\[ \langle D_{Y_m}, \delta Y_c \rangle = \int_{-\infty}^{\infty} G(x - x') \delta Y_c(x') \, dx', \quad (5.95) \]

where \( \delta Y_c \) is

\[ \delta Y_c(x) = \begin{cases} 
0, & x \notin \Omega_p, \\
\text{constant,} & x \in \Omega_p. 
\end{cases} \quad (5.96) \]

\( O\left[\delta Y_c(x)\right] \) represents the high order terms. Note that eq.(5.95) is actually the notation for

\[ \langle D_{Y_m}, \delta Y_c \rangle = \int_{-\infty}^{\infty} G(x - x') \left[ Y_c(x) + \delta Y_c(x') \right] \, dx' - \int_{-\infty}^{\infty} G(x - x') Y_c(x) \, dx'. \quad (5.97) \]

substituting eq.(5.91) to eq.(5.97), we have

\[ Y_m(x) + \delta Y_m(x) = \int_{-\infty}^{\infty} G(x - x') \left[ Y_c(x) + \delta Y_c(x') \right] \, dx' + O\left[\delta Y_c(x)\right]. \quad (5.98) \]

This is the initial perturbation equation.

Clearly, the filter function is dependent on the flow pattern of slug test \(^5\) and the perturbation will alter the local flow pattern. Thus the filter function is also being perturbed. The perturbed filter function, \( G^p(x) \), can be written as

\[ G^p(x) = G(x) + \delta G(x) \quad (5.99) \]

\(^5\)Because \( Y_m \) is inverted from RC and is dependent on the flow pattern.
where $\delta G(x)$ is the filter variation invoked by the $\delta Y_c(x)$ perturbation. Thus the high order term in eq.(5.98) equals

$$O \left[ \delta^2 Y_c(x') \right] = \int_{-\infty}^{\infty} \delta G(x-x') \left[ Y_c(x) + \delta Y_c(x') \right] \, dx'$$  \hspace{1cm} (5.100)

If we assume $\delta G(x) = 0$, i.e., $O \left[ \delta^2 Y_c(x') \right] = 0$ so that eq.(5.94) reduces to

$$\delta Y_m(x) = \int_{-\infty}^{\infty} G(x-x') \delta Y_c(x') \, dx'$$  \hspace{1cm} (5.101)

$\delta Y_c(x)$ is constant over $\Omega_p$, denoted as $\delta Y_c|_{\Omega_p}$ and zero everywhere else, eq.(5.96), thus we have

$$\delta Y_m(x) = \int_{\Omega_p} G(x-x') \, \delta Y_c \, dx'$$

$$= \delta Y_c|_{\Omega_p} \int_{\Omega_p} G(x-x') \, dx'$$  \hspace{1cm} (5.102)

which is

$$\int_{\Omega_p} G(x-x') \, dx' = \frac{\delta Y_m(x)}{\delta Y_c|_{\Omega_p}}$$  \hspace{1cm} (5.103)

If we further assume that $G(x)$ in $\Omega_p$ to be approximately uniform, denoted as $G|_{\Omega_p}$, then

$$G|_{\Omega_p} \approx \frac{1}{\Omega_p} \frac{\delta Y_m(x)}{\delta Y_c|_{\Omega_p}}$$  \hspace{1cm} (5.104)

This is the final perturbation formula, meaning that:

*the filter amplitude in $\Omega_p = (the \ change \ of \ the \ conductivity \ measured \ at \ wellbore \ located \ at \ x \ in \ a \ homogeneous \ aquifer \ where \ the \ conductivity \ in \ \Omega_p \ is \ perturbed \ by \ \delta Y_c|_{\Omega_p} \ ) \div [(the \ perturbation \ magnitude \ \delta Y_c|_{\Omega_p} \ ) \times (the \ perturbation \ area \ \Omega_p)].*

In the above derivation, there are two operations on the filter integral,

1) Move the interval from $[-\infty, +\infty]$ to $\Omega_p$, eq.(5.91) − eq.(5.103)

$$\int_{-\infty}^{\infty} G(x-x') \, dx' \longrightarrow \int_{\Omega_p} G(x-x') \, dx'$$

2) Assume $G$ is constant in $\Omega_p$, thus taking it out from under the integral.

$$\int_{\Omega_p} G(x-x') \, dx' \longrightarrow G|_{\Omega_p} \int_{\Omega_p} dx = G|_{\Omega_p} \Omega_p$$
The assumptions made for each operation are: 1) Filter function does not change under perturbation, i.e., $\delta_G(x) = 0$. 2) $G(x)$ is approximately uniform in $\Omega_p$. These assumptions can be asymptotically approached by proper choice of $\delta y_c$ and $\Omega_p$, which is crucial for the perturbation equation (5.104) to be valid, and which is very complex to be determined analytically. We use numerical techniques.

In Section 5.2.1 we compare different inverse estimation procedures of perturbed conductivity, and in Section 5.3 we employ numerical techniques to determine the area and the magnitude of the numerical perturbation.

5.2.1 The inverse estimation procedures of perturbed conductivity

We define the perturbed logarithm conductivity, $Y_p$, as

$$Y_p = Y_c + \delta y_c$$

$$= \lg(K_c) + \lg(a)$$

where $\delta y_c$ is the logarithmic perturbation. Thus the perturbed conductivity, $K_p$, is

$$K_p = a K_c.$$ 

(5.106)

We define the dimensionless perturbation amount as

$$d_k = \frac{|K_c - K_p|}{K_c} = |1 - a|,$$

(5.107)

which represents the extent of perturbation.

If we perturb $d_k$, once a time, at the blocks along the same line shown in Fig.5.35, then there may be three inversion methods for the RC measured for each perturbation: a) Estimate only $\lg K$ with $\lg S_s$ equal to true value; b) Concurrently estimate $\lg K$ and $\lg S_s$; c) Estimate $\lg K$ at each time step with $\lg S_s$ equal to true value. We have discussed method $a$ and $b$ for the RC measured in heterogeneous conductivity fields.
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Now we compare the estimates by method c with those of a and b. Here forth, we use superscripts a, b, and c to denote the estimates from these methods.

The results are shown in Fig. 5.36 where the unperturbed \( \lg K \) and \( \lg S_3 \) equal to 2 and \(-1\) respectively, and the perturbation \( d_k \) is 0.5.

With method a, Fig. 5.36A shows \( \lg K^a \) from each perturbation versus the corresponding dimensionless perturbation locations \( r_D = r/r_w \). Because \( d_k \) equal to 0.5 is a negative perturbation, the measured \( \lg K \) is less than the unperturbed \( \lg K \). Away from wellbore, the perturbation area \( \Omega_p \) increases, the perturbation strength, approximated by \( \Omega_p \times d_k \) also increases, while the influence of the inhomogeneity on RC decreases, the net effect results in the curve shape shown in Fig. 5.36A.

With method b, graph B and C of Fig. 5.36 demonstrate \( \lg K^b \sim r_D \) and \( \lg S^b_3 \sim r_D \). We can see that \( \lg K^b \) is biased because of the deviation of \( \lg S^b_3 \) from the true value, and wherever \( \lg K^b \) is underestimated, \( \lg S^b_3 \) is overestimated, and vice versa. Thus we do not choose this method, as we discussed in Chapter 3. We also discussed in Chapter 3 that method a may possess bias because the perturbed wellbore response curve can not be entirely matched by that of a homogeneous model, but this error is far smaller than that from method b, and also it can be minimized by an optimal choice of a minimum nontrivial \( \delta_y \) and a proper \( \Omega_p \).

With method c, Fig. 5.36D displays \( \lg K^c \sim r_D \) where \( \lg K^c \) at \( r_D \) is the arithmetical average of the \( \lg K^c \) at \( r_D \) inverted at all time steps, i.e., the geometric average of \( K^c \).

---

\(^6\)Here we need to discuss these inversion methods because they result in different \( \lg K \) estimates, hence, the different filter functions associated with these estimates. We did not use method c in Chapter 3, see the arguments therein, but here for the inversion of the RC of the theoretical perturbed conductivity fields, the estimates from method c can be readily obtained, and they can be used to calculate the filter function at each time step or for all time at certain spatial location.

\(^7\)We proved that in Chapter 3 that the perturbation is independent of the unperturbed \( \lg K \), thus we can choose any value without losing generality. The other \( \lg S_3 \) cases will be discussed in Section 5.4.

\(^8\)Another reason to reject method b is that in the perturbation case the \( \delta_y \) and the \( \delta_y^c \) usually are small compared to the unperturbed \( Y_e \), and the small changes of \( Y_m \) due to \( \delta_y \) may be severely damped by the interactivity of \( Y_m^* \sim \lg S^* \) in the inversion, which will be shown in Section 5.3.

\(^9\)The choice of such an average seems more arbitrary than with some theoretical proof. The possible
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The comparison of above \( \lg K \) estimates are shown in Fig. 5.36E. We see that \( \lg K^a \) \( \lg K^b \) intersect at the point P2 of graph B, where \( r_D = 1.8 \). This is the only point at the influence zone where the \( b \) estimation is unbiased.

The \( \lg K^a \) intersects the \( \lg K^c \) at two locations, where \( r_D = 1.3, 5.1 \). If we use \( \lg K^a \) as the standard and evaluate the deviation from it as under- or overestimation, then \( \lg K^c \) is overestimated at \([0, 1.3) \) and \((5.1, 50] \), underestimated at \([1.3, 5.1] \). Note that method \( b \) and \( c \) overestimate the influence radius \( R_e, 10 \) where \( R^b_e = 30, R^c_e = 50 \), whereas \( R^a_e = 19.5 \).

The \( \lg K^b \) also intersects the \( \lg K^c \) at two points. Interestingly, the most overestimated point of method \( c \) overlaps the most underestimated point of method \( b \), where \( r_D \) equals to 1.1. This radius might be interpreted as a "numeric-estimation wellbore skin" radius, in the sense that the estimated conductivity at such radius has abnormal high or low values. Note that if a positive perturbation is used, the \( \lg K^b \) will be most overestimated at \( r_D = 1.1 \), as we illustrated in Chapter 3.

The above \( \lg K \) estimates give rise to different filter functions that are shown in Fig. 5.36F. Their intersections are the same as that in graph E.

If we do not average the estimates from method \( c \), we can appreciate the transient filtering property from two aspects: Firstly, illustrated in Fig. 5.37A, at each point in the influence zone, the filter function changes with time. At locations closer to wellbore, the filtering has a higher power and happens earlier. The filter function at all locations decay off to zero until the end of total time of which \( \beta = Kt/r_w^2 \) = 50. Secondly, shown in Fig. 5.37B, the filter function in dimensionless space \( r_D \in [1, R_e] \) varies with time. At earlier time most of the filtering power concentrates around wellbore and theoretical consideration is that the temporal average is co-phased with spatial average, which is reported in various literature that a geometric average is appropriate.

The concept of influence radius was discussed in Chapter 2, here the influence radius is scaled to be dimensionless by \( r_w \).
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\[ \log_{10}(K') \sim r_D \]

\[ \log_{10}(K') \sim r_D \]

\[ \log_{10}(S_s') \sim r_D \]

continued on next page
Figure 5.36: The lg $K$ inverted by different methods and the associated filter functions.
Figure 5.37: Spatial filter is a function of time. A shows the filter function at locations \( P_1, P_2, P_3 \) at time \([0, \beta], \beta = 50\). B shows the filter function at time \( t_1, t_2, t_3 \) in dimensionless space \( r_D, r_D \in (1, R_e] \).
decreases rapidly, while at later time the averaging power spreads to a larger volume and its amplitude drops off much more slowly.

For the same reason discussed in Chapter 3, we shall invert the $\lg K$ with the whole response curve by method $a$ in following discussions.

5.3 The area and the amount of perturbation

Analytical methods allow one explicitly determine limits of perturbation through evaluating the approximations used in the perturbation and the final analytical expressions. For the numerical perturbation, however, we don’t have explicit results, thus we have to decide the area and the amount of perturbation by a “try-evaluate-try” approach that is guided by the proper interpretation of the intermediate numerical results with respect to the physical behavior of the slug test system.

From eq.(5.104) we know that, physically, $\Omega_P$ can not be too large otherwise $G(x)$ is not close to be uniform in $\Omega_P$ and such a perturbation may strongly alter the flow pattern so that it invalidates the assumption of eq.(5.101); Also, $\Omega_P$ can not be too small otherwise the perturbation becomes trivial; mathematically, if $\Omega_P \rightarrow 0$, then $G(x) \rightarrow +\infty$, it causes overestimation; if $\Omega_P \rightarrow +\infty$, then $G(x) \rightarrow 0$, it introduces underestimation.

We select $\Omega_P$ by two steps. Firstly, we adjust the block sizes of the 2D radial FDM model of a slug test in a homogeneous aquifer until the numerical RC matches the RC of analytical solution. This means the space discretization is suitable to resolve the hydraulic gradient in the influence zone. The same criteria is used to choose $\Omega_P$, i.e., a suitable $\Omega_P$ is the one in which hydraulic head is approximately uniform. So secondly, we simply select blocks along one line, opposite to which is a nonflow boundary. This nonflow boundary is for the angular variable $\theta$, and it is physically valid because the
perturbation invokes symmetric flow pattern that warrants no crossing flux.

The perturbation $d_k$ can not be too large, otherwise it alters flow pattern drastically, e.g., one limiting case, if $Ss = 1$, and $d_k = 100$, it turns the perturbed block into an open well. Preliminary tests suggest it is better to choose $d_k < 1$, i.e., $0 < a < 2$. On the other hand, $d_k$ can not be too small in which case the changes of the perturbed wellbore response curve might be close in magnitude to the tolerance of the numerical equation solver, hence the calculated filter function might be instable. Furthermore, if $d_k$ too small, the influence of perturbation at blocks farther away from wellbore would not affect wellbore response, thus resulting in an underestimation of the volume and averaging power of the filter function.

We find the optimal perturbation amount through a number of tests of different $d_k$ values. Two criteria should be met for an appropriate perturbation: 1) The integral of the filter approximately equals to 1, \( \int_{-\infty}^{\infty} G(x-x')dx' \approx 1 \)

2) The skewness $K^m \to 0$, where $K^m = (Y_m$ of negative perturbation) + ($Y_m$ of positive perturbation). Under this condition, the same filter function $G|_{n_p} \approx \frac{1}{n_p} \frac{\delta y_m(x)}{\delta y_c|_{n_p}}$ should be obtained no matter what kind of perturbation is performed.

Fig. 5.38 shows the results, estimated with method $a$ and $b$, of $d_k = 0.5, 0.1, 0.05, 0.01$, where $0 < a < 1$, unperturbed $\lg K = 2$, and unperturbed $\lg S_s = -1$. \(^{12}\) Fig. 5.38A and Fig. 5.38B show $\lg K^a \sim r_D$ and the associated filter functions $G^a \sim r_D$. We see that the filter functions converge to the one of $d_k = 0.01$, and we shall show in the next section that the filter integral approximately equals to 1. Thus we choose the optimal $d_k$.

\(^{11}\)Because the perturbation is usually only a small fraction of the unperturbed homogeneous conductivity, the the filtering should be very close the homogeneous case. The filtering of a unit filter volume is called mean-preserving filtering.

\(^{12}\)Here we use negative perturbations. In Chapter 3 we illustrated that the skewness $K^m \to 0$ if $a > 0.9$, i.e., $d_k < 0.1,$
Figure 5.38: Perturbation with different $d_k$. The $\log K$ is estimated with $\log S_a$ equal to true value.
Figure 5.39: Perturbation with different $d_k$. The $\log_{10} K$ is concurrently estimated with $\log_{10} S_s$. 
as the one to which the filter functions, estimated with a range of $d_k$ values, converges to 1. \textsuperscript{13} Here the optimal perturbation amount $d_k$ is 0.01.

In contrast, with method $b$ the $\lg K^b \sim r_D$ and the associated filter functions $G^b \sim r_D$ appear inconvergent, shown in Fig. 5.39. This inconvergency is caused by the interactive property of $\lg K^b$ and $\lg S_s^b$ estimation that we analyzed in Section 5.2.1. This further justifies our judgment that method $b$ is not appropriate as the inversion procedure for the perturbation.

Clearly, for different specific storages, the perturbation affects wellbore response in a different manner. Smaller $S_s$ has stronger influence at farther locations while larger $S_s$ is more sensitive around wellbore. Thus the selection of $d_k$ should be $S_s$-dependent. So it is no wonder that a suitable $d_k$ for a small $S_s$ may be too large for the perturbation with a large $S_s$. The $S_s$ in Fig. 5.38 is 0.1. For $S_s < 0.1$ we may need a $d_k$ larger than 0.01. This is the topic of the next section.

5.4 Filter functions of different specific storages

We use several statistical measures to characterize the filter functions of different specific storages. The volume of filter function, $V_G(r_D)$ is defined as

$$V_G(r_D) = \int_{1+\xi}^{R_e} 2 \pi G(r_D) r_D dr_D,$$

where $r_D$ is dimensionless distance, and $\xi$ is a infinitely small positive number, i.e., $\xi \rightarrow 0^+$. Because the 2D radial FDM we used is mesh-centered, the first node is on the inner boundary where $r_D$ equals to 1, so that $G(1)$ is zero and $G(1+\xi)$ represents filtering power at the node closest to wellbore in the aquifer. $R_e$ is the dimensionless influence

\textsuperscript{13}In the $\lg K^a \sim r_D$ plots, we can observe that the curves of different $d_k$ values converge to a curve of some $d_k$ value, and filter integral associated with the $d_k$ should approximately equal to 1.
radius defined as
\[ \frac{\partial [V_G(r_D)]}{\partial (r_D)} \rightarrow 0, \quad (5.109) \]
which means the filter volume stabilizes at and beyond \( R_e \). This may be a more rigorous definition than that by Guyonnet et al. [1993]\(^{14} \) because: 1) the \( \frac{\partial [V_G(r_D)]}{\partial (r_D)} \) explicitly incorporates the concept of gradient which manifests the rate of increase of the filter volume; 2) the \( G(r_D) \) in the expression of \([V_G(r_D)]\) explicitly tells the contribution of the conductivity at \( r_D \) to the conductivity inverted from the wellbore response curve.

The dimensionless equivalent filter width is defined as
\[ \lambda_e = \frac{V_G(R_e)}{G(1 + \xi)} \quad (5.110) \]

In the next Sections 5.4.1 and 5.4.2, we shall compare the filter functions that are estimated with two perturbation schemes: 1) Fixed perturbation magnitude, and 2) \( S_s \)-dependent perturbation.

**5.4.1 Fixed perturbation magnitude**

We fix the perturbation magnitude, \( d_k = 0.5 \), for \( S_s = 10^{-n} \), \( n = 1, 2, \ldots, 10 \). Consequently, from the discussions in the previous section we expect filter functions to be slightly overestimated for large \( S_s \) and slightly underestimated for small storage.

To distinguish the different filter shapes, the filter functions are plotted in logarithm coordinates, \( \lg [G(r_D)] \sim \lg r_D \), in Fig. 5.40A. It displays that, as \( S_s \) decreases, the filter support area increases, and the amplitude of the filter away from the wellbore decreases less rapidly.

Fig. 5.40B is a plot of \( V_G(r_D) \sim \lg r_D \). It shows 1) \( V_G(r_D) \) ceases increasing as \( r_D \) approaches \( R_e \), 2) With larger \( S_s \), \( V_G(r_D) \) grows faster while the influence area becomes smaller, and 3) \( V_G(R_e) \) approaches a unity volume for \( S_s \) decreasing from \( 10^{-1} \)

\(^{14}\)Their definition is "the maximum distance traveled by a perturbation \( \frac{H}{H_0} = 1\%, 5\%, 10\%".\]
to $10^{-8}$. The volume overestimation ($V_G(R_e) > 1$) is what we have predicted, e.g., in Fig. 5.38A1 (where $S_a = 10^{-1}$), the filter function of $d_k = 0.5$ is overestimated. The $V_G(R_e)$ of $S_a = 10^{-9}, 10^{-10}$ is less than 1. This indicates the perturbation of $d_k = 0.5$ is too small so that the filter functions are underestimated.

Fig. 5.41A demonstrates the linear relationship between $\lambda_e$ and $\log(\sqrt[3]{S_e})$. Note that since the overestimation or underestimation of $V_G(R_e)$ and $G(1 + \xi)$ occurs concurrently,\(^{15}\) thus $\frac{V_G(R_e)}{G(1 + \xi)} = \lambda_e$ is a more reliable representation of the filtering property than the $G(r_D)$ and $V_G(r_D)$ of Fig. 5.40 are.

Fig. 5.41B is a plot of $\log(R_e)$ versus $\log(\sqrt[3]{S_e})$. The $R_e$ is taken from Fig 5.40B. It shows a good linear relationship.

### 5.4.2 $S_a$-dependent perturbation

Following the same reasoning of Fig. 5.38A1, we perform the perturbation tests of different $d_k$ for $S_a = 10^{-n}, n = 1, 2, \ldots, 10$. Some of the test results, $G(r_D) \sim r_D$ and $V_G(r_D) \sim \log(\sqrt[3]{S_e})$, are graphically shown in Fig. 5.42 - 5.45 whose $S_a$ equals to $10^{-1}, 10^{-3}, 10^{-5}, 10^{-7}$ respectively. The curve of $d_k = 0.5$ can be used as a reference for comparison across these graphs.

In the graph A of Figures 5.42 - 5.45, the optimal $d_k$ values are found to be 0.01, 0.05, 0.1, 0.3, respectively. The smaller the $S_a$, the larger the optimal $d_k$. This tells us that aquifers of high specific storage are more sensitive to the conductivity variation. In Fig. 5.42, 5.43, we can see that, for the perturbations smaller than the optimal $d_k$, the filter functions become instable because the hydraulic head changes from perturbation are likely of the same magnitude as random truncation errors of the matrix solver.

In the graph B of Fig. 5.42 - 5.45, we can see that the total averaging power of the filter function calculated with the optimal $d_k$ approaches unity. We also can see that for

\(^{15}\) Because $V_G(R_e)$ is an spatial integral of $G(1 + \xi)$. 
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the filter functions calculated with $d_k$ greater than the optimal value, the greater is the $d_k$, the more overestimation is the $V_G(R_e)$; Note that they have identical influence radius; This conforms to the reality that the influence radius is independent of conductivity and it is only a function of specific storage.

Note that, in graph B of Fig. 5.42 - 5.45, we have approximated the integral of $V_G(r_D)$ defined in eq.(5.108) by two steps. We first interpolate the estimated filter functions into $10^6$ data points evenly log-spaced; Secondly we sum up these discrete points. The associated interpolation error could contribute both positively and negatively to the averaging power $V_G(r_D)$, depending on the shape of the filtering functions, but in general they should be insignificant.

Fig. 5.46A shows the $V_G(r_D) \sim \lg\left(\frac{1}{\sqrt{S_s}}\right)$ calculated with the optimal $d_k$ for $S_s = 10^{-n}$, $n = 1, 2, \ldots, 10$. Fig. 5.46B shows the comparison of the $\lambda_e$ of Fig. 5.46A with the $\lambda_e$ of Fig. 5.40B estimated with $d_k = 0.5$. This graph shows 1) $\lambda_e \sim \lg\left(\frac{1}{\sqrt{S_s}}\right)$ obeys a linear relationship, 2) the filter width of the filter functions calculated by $d_k = 0.5$ are more underestimated for decreasing $S_s$. This can be explained with the $V_G(r_D) \sim \lg\left(\frac{1}{\sqrt{S_s}}\right)$ pattern in Fig. 5.40 and by the $\lambda_e$ definition in eq.(5.110).
Figure 5.40: The spatial filter for different specific storages, $S_s = 10^{-n}$, $n = 1, 2, \ldots, 10$. For all perturbations, $d_k = 0.5$. A shows the filter function versus dimensionless distance in log space. B is the volume of filter function integrated over dimensionless distance $r_D$, $V_G(r_D) = \int_{1+\xi}^{R_e} 2\pi G(r_D) r_D \, dr_D$. 
Figure 5.41: A is the equivalent filter width defined as $\lambda_e = \frac{V_0(R_e)}{C(1+\varepsilon)}$. B is the influence radius $R_e$ defined as $\frac{\partial V_0(r_D)}{\partial (r_D)} \to 0$.
Figure 5.42: $S_* = 10^{-1}$. A shows that $G(r_D) \sim r_D$ converges to the $d_k = 0.01$ curve. If $d_k$ is too small ($d_k < 0.01$), solutions become unstable. B is the corresponding $V_G(r_D) \sim \lg(\frac{1}{\sqrt{S_*}})$ curves.
Figure 5.43: $S_s = 10^{-3}$. A shows that $G(r_D) \sim r_D$ converges to the $d_k = 0.05$ curve. If $d_k$ is too small ($d_k < 0.05$), solutions become unstable. B is the corresponding $V_G(r_D) \sim \lg \left( \frac{1}{\sqrt{S_s}} \right)$ curves.
Figure 5.44: $S_x = 10^{-5}$. A shows that $G(r_D) \sim r_D$ converges to $d_k = 0.1$ curve. B is the corresponding $V_G(r_D) \sim \log\left(\frac{1}{\sqrt{S_x}}\right)$ curves.
Figure 5.45: $S_z = 10^{-7}$. A shows that $G(r_D) \sim r_D$ converges to $d_k = 0.3$ curve. B is the corresponding $V_G(r_D) \sim \log\left(\frac{1}{\sqrt{S_z}}\right)$ curves.
continued on next page
Figure 5.46: A shows the $V_G(r_D) \sim \lg r_D$ curves calculated with the optimal $d_k$ for $S_s = 10^{-n}$, $n = 1, 2, \ldots, 10$. B is a comparison of the $\lambda_e \sim \lg(\frac{1}{\sqrt{S_s}})$ of A with those estimated with $d_k = 0.5$ for all $S_s$ values. C is the corresponding $\lg [G(r_D)] \sim \lg r_D$. 
5.4.3 Parametric representations

In this section, we present the parametric representations of the filter functions, equivalent filter widths and influence radius, determined from the optimal $d_k$ for different specific storages in Section 5.4.2.

The filter functions $\log[G(r_D)] \sim \log r_D$, shown in Fig. 5.46C exhibit straight-line patterns in the intermediate portions which constitute the major parts of whole curves. Here we only fit the straight-line portions by a parametric representation,

$$G(r_D) = G(1 + \xi) \frac{1}{(r_D)\xi}, \quad \xi \to 0,$$

(5.111)

where the $G(1 + \xi)$ is the filter amplitude at the wellbore edge, and $p$ is an exponent.

<table>
<thead>
<tr>
<th>$S_s$</th>
<th>$10^{-1}$</th>
<th>$10^{-2}$</th>
<th>$10^{-3}$</th>
<th>$10^{-4}$</th>
<th>$10^{-5}$</th>
<th>$10^{-6}$</th>
<th>$10^{-7}$</th>
<th>$10^{-8}$</th>
<th>$10^{-9}$</th>
<th>$10^{-10}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G(1 + \xi) \times 10^{-2}$</td>
<td>18.338</td>
<td>7.908</td>
<td>4.882</td>
<td>3.472</td>
<td>2.840</td>
<td>2.412</td>
<td>2.165</td>
<td>1.950</td>
<td>1.885</td>
<td>1.753</td>
</tr>
<tr>
<td>$p$</td>
<td>3.0</td>
<td>2.3</td>
<td>2.1</td>
<td>2.05</td>
<td>2.0</td>
<td>2.0</td>
<td>2.0</td>
<td>2.0</td>
<td>2.0</td>
<td>2.0</td>
</tr>
</tbody>
</table>

Table 5.4 shows 1) $G(1 + \xi)$ decreases with decreasing $S_s$, and 2) $G(r_D) \sim r_D$ approximately approaches a $\frac{1}{\sqrt{S_s}}$ decreasing law for the $S_s$ smaller than $10^{-4}$. Now we fit both $G(1 + \xi) \sim \frac{1}{\sqrt{S_s}}$ and $p \sim \frac{1}{\sqrt{S_s}}$, shown in Fig. 5.47. The fitted curves are

$$G(1 + \xi) = 0.2S_s^{0.25} + 0.2S_s^{0.75} + 0.01753$$

(5.112)

$$p = 3.0\sqrt{S_s} + 2.0$$

(5.113)

Thus eq. 5.111 becomes

$$G(r_D) = \left(0.2S_s^{0.25} + 0.2S_s^{0.75} + 0.01753\right)\frac{1}{(r_D)^{3.0\sqrt{S_s}+2.0}}.$$

(5.114)

The above equation describes the direct relationship between $G(r_D)$ and $S_s$. 
Figure 5.47: A: Fitting $G(1 + \xi) \sim \frac{1}{\sqrt{S_s}}$ by $G(1 + \xi) = 0.25^{0.25} + 0.25^{0.75} + 0.01753$; B: Fitting $p \sim \frac{1}{\sqrt{S_s}}$ by $p = 3.0\sqrt{S_s} + 2.0$. 
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The approximate straight-line portion of $\lambda_e \sim \lg\left(\frac{1}{\sqrt{S_e}}\right)$, shown in Fig. 5.46B, can be parameterized as

$$\lambda_e = \lambda^\circ - \left[\lg\left(\frac{1}{\sqrt{S_e}}\right)\right]^\circ + 14.58 \ \lg\left(\frac{1}{\sqrt{S_e}}\right)$$

(5.115)

$$\lambda^\circ = 5.47, \ \left[\lg\left(\frac{1}{\sqrt{S_e}}\right)\right]^\circ = 0.5$$

(5.116)

The approximate straight-line portion of $R_e \sim \lg\left(\frac{1}{\sqrt{S_e}}\right)$, shown in Fig. 5.41, can be parameterized as

$$\lg R_e = (\lg R_e)^\circ - \left[\lg\left(\frac{1}{\sqrt{S_e}}\right)\right]^\circ + \lg\left(\frac{1}{\sqrt{S_e}}\right)$$

(5.117)

$$R_e = \frac{1}{\sqrt{S_e}} \ \left[10(\lg R_e)^\circ - \left[\lg\left(\frac{1}{\sqrt{S_e}}\right)\right]^\circ\right]$$

(5.118)

$$(\lg R_e)^\circ = 1.29, \ \left[\lg\left(\frac{1}{\sqrt{S_e}}\right)\right]^\circ = 0.5$$

(5.119)
In this Chapter we investigate the filter functions in heterogeneous core-scale fields of various spatial variabilities and specific storages, utilizing the Wiener filtering methodology. To explore the effects of $S_s$ on the spatial filtering, in Section 6.1 we perform slug tests in isotropic $K^{core}$ fields of $64 \times 64$, $128 \times 128$ and $256 \times 256 (\Delta x^w \times \Delta x^w)$ with $S_s = 10^{-n}, n = 1, 2, \ldots, 5$. In Section 6.2, we present the effects of the spatial variability of conductivity on the spatial filtering. We consider the spatial variability of variance, correlation length, variogram model and distribution pattern. The investigations in this chapter are not designed to accommodate every cases met in practice, but intended to disclose the general roles of $S_s$ and $K^{core}$ variabilities on the spatial filtering.

6.1 Filter function for different specific storages

In Chapter 2 we discussed that specific storage determines the influence zone, or, the filtering area of the slug test. In Chapter 5 we manifested the relationships $R_e \sim S_s$ and $G(r_o) \sim S_s$ by a perturbation method. In chapter 4 we presented a Wiener filtering methodology to determine the filter functions in heterogeneous fields. Now we utilize the Wiener filtering method and conduct a systematic investigation of the role of $S_s$ on the filtering in heterogeneous fields.
6.1.1 Slug tests in isotropic core-scale fields: different field sizes, different specific storages

We perform slug tests in an isotropic exponential $K^{core}$ field with $S_s = 10^{-n}$, $n = 1, 2, \ldots, 5$. The field is generated by a sequential Gaussian simulation method of GSLIB [37] with a variogram

$$\gamma(h) = c \cdot [1 - e^{(-h/a)}],$$

(6.120)

where $c$ is variance ($c = \sigma^2_Y$), and $a$ is the correlation length in $X$ and $Y$ direction ($a = \lambda_x = \lambda_y$). The parameters are chosen to be, $\sigma^2_Y = 0.25$, $\lambda_x = \lambda_y = 4\Delta x^w$, field size $= 512 \times 512$ blocks ($\Delta x^w \times \Delta x^w$). The cases of other parameter values or spatial variabilities are to be discussed in the next section. With different specific storages, slug tests are performed in this field at areas of $64 \times 64$, $128 \times 128$ and $256 \times 256$ blocks, respectively, listed in table 6.5. Sampling distance equal to 1 means slug test performed at every block in the selected area, equal to 2 means at every other blocks.

Table 6.5: Slug tests with different specific storages.

<table>
<thead>
<tr>
<th>specific storage</th>
<th>slug test area</th>
<th>sampling distance</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^{-1}, 0.05, 10^{-2}, 0.005, 10^{-3}, 10^{-4}, 10^{-5}$</td>
<td>$64 \times 64$</td>
<td>1</td>
</tr>
<tr>
<td>$10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}$</td>
<td>$128 \times 128$</td>
<td>2</td>
</tr>
<tr>
<td>$10^{-1}, 10^{-2}, 10^{-3}$</td>
<td>$256 \times 256$</td>
<td>2</td>
</tr>
</tbody>
</table>

Fig. 6.48 shows the grey-scales and histograms of the core-scale and the measured field of size $64 \times 64$. The grey-scales clearly demonstrate the smoothing effect of the filtering process while this effect is embodied in the histograms as the shrink of data ranges. The smoothing is stronger for smaller specific storages due to a larger area of aquifer being averaged.

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1For illustration convenience, the units of correlation length and field blocks may be omitted in the rest of this Chapter.
Figures 6.49, 6.50, 6.51 are the plots of the variograms of the core-scale and the measured fields of size $64 \times 64$, $128 \times 128$ and $256 \times 256$. We see that the averaging reduces the variance and increases the correlation range. The changes in mean and variance versus specific storage are plotted in Fig. 6.52. We see that the spatial filtering with different specific storages is approximately mean-preserving, i.e., the means of the measured fields ($\mu_{Y_m}$) are close to the mean of the core-scale field ($\mu_Y$). The deviation of $\mu_{Y_m}$ from $\mu_Y$ may result from the differences in the spatial filtering of each slug test excited in differing aquifer area of dissimilar local heterogeneity. The variance decreases with decreasing $S_a$, because smaller $S_a$ invokes a larger averaging area, i.e., a stronger 'smoothing effect'. Note that the means and variances of the measured fields with $S_a = 10^{-4}, 10^{-5}$ appear very similar. This is because the influence radii of slug tests with $S_a \leq 10^{-4}$ have reached the whole inner heterogeneous zone($65 \times 65$ blocks) simulated by the slug test FDM model, thus their averaging effects become similar.
Figure 6.48: The grey-scales and histograms of core-scale and measured fields with different specific storages. The core-scale field has an isotropic exponential variogram. Field size $64 \times 64$ blocks.
Figure 6.49: The variograms of core-scale and measured fields of size $64 \times 64$. 
Figure 6.50: The variograms of core-scale and measured fields. Field size $128 \times 128$. 

X-direction variogram of core-scale and measured fields

Y-direction variogram of core-scale and measured fields
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Figure 6.51: The variograms of core-scale and measured fields. Field size $256 \times 256$. 
Figure 6.52: The mean and variance of core-scale and measured fields with different specific storages. Field size $64 \times 64$, $128 \times 128$, $256 \times 256$. 
6.1.2 Comparisons of $K^{core}$ and $K^{slug}$ spectra

In this section we analyze the spectra of $K^{core}$ and $K^{slug}$ fields of different field size and different specific storages. The strength of spectral analyses is the visualization of the information contained in the data record in the frequency domain. The power spectrum density (PSD) illustrates the distribution of variance over spatial frequency [Gelhar, 1994, p31]

$$R(0) = \sigma^2 = \int_{-\infty}^{+\infty} S(f) \, df$$  \hspace{1cm} (6.121)

where $R(0)$ is the autocovariance of zero lag. It also means that spectrum volume equals to the total energy of the data record — the summation of the square of the data at every point, divided by data record length. For a two dimensional periodogram, the PSD at zero frequency equals to the square of the volume of the 2D data series in physical space.

In the following spectrum estimation by the multitaper method, the data mean is removed. The above properties may not be exactly met due to the way of data tapering and the removing of the mean. The estimated spectra may only demonstrate these properties approximately.

The spectra of the core-scale fields and the measured fields with different specific storages are shown in Fig. 6.53. The graphs in Fig. 6.53 demonstrate: 1) The sharpening effects of the spectra for decreasing specific storages. In physical space it means the increasing of correlation range, because of the stronger averaging effect of smaller $S_s$. The first frequency shown is the Rayleigh frequency.\footnote{The zero frequency is removed from these graphs due to the logarithmic plotting requirement. This will be the case for the rest log scale PSD plots in this thesis.} 2) The Spectra of $S_s = 10^{-4}, 10^{-5}$ are very similar, this is due to the same reason for the similarity of the mean or variance between these measured fields, which we observed Fig. 6.52.

The comparisons of the spectra of the core-scale and the measured fields of size 64 ×
64, 128 × 128, 256 × 256 blocks are shown in Fig. 6.54. The target spectrum is eq. 4.80 that is

\[
S(f_x, f_y) = \frac{2 \sigma^2 \pi \lambda_x \lambda_y}{(1 + 4\pi^2 f_x^2 \lambda_x + f_y^2 \lambda_y)^{3/2}}.
\]

Eq. 6.122 is the Fourier transform of the covariance function associated with the variogram eq. 6.120,

\[
\text{Cov}(h) = c e^{(-h^2/a^2)},
\]

where \( c = \sigma^2 \) and \( a = \lambda_x = \lambda_y \). Thus the parameters of the target spectrum should be: \( \sigma^2 = 0.25, \lambda_x = \lambda_y = 4\Delta x^w \).

We see that the spectra of the core-scale fields deviate from the target spectrum at \( f > 0.2 \). This is caused by the inherent routines of GSLIB [37] for random field generation, not by the multitaper spectrum estimation method. We shall illustrate our explanation shortly.

In Fig. 6.53 and Fig. 6.54 we also see that the spectra of the measured fields of size 64 × 64 have tails at \( f > 0.38 \) where energy drops off much less rapidly. Three possible causes for the tails are: 1) they are somehow transferred from the tail of the spectra of the core-scale fields; 2) there are some numerical errors in the measured fields, resulting from the slug test simulation model (Chapter 2) and the \( Y_c \) inversion method (Chapter 3); and 3) they come with the multitaper method.

These aspects are investigated through a series of hypothetical filtering tests. Firstly, filter the core-scale field of size 256 × 256 with the Gaussian filter of eq. 4.81 of filter widths \( \lambda_x = \lambda_y = 4, 8, 16, 32\Delta x^w \), respectively. Secondly, select the filtered fields of size 64 × 64, 128 × 128 and 256 × 256, and estimate their spectra by the multitaper method.

\[^{3}\text{The fields of 64 \times 64, 128 \times 128 are chosen from the center area of the field of 256 \times 256.}\]
Fig. 6.55 shows the spectra of core-scale fields and the hypothetically filtered fields by different filter widths. It displays that the spectra of the fields filtered with $\lambda = 4$ have no tail; while the rest have tails. Consequently, 1) the foregoing observed tails are not necessarily affected by the tail of the spectra of the core-scale fields; 2) these tails do not have to be caused by numerical errors in the measured fields (i.e., these hypothetical filtering can be assumed containing negligible errors, but tails still observed.); 3) they are not likely coming with the spectrum estimation method (i.e., otherwise, one might observe tails consistently.).

In Fig. 6.55, we also see that the spectral energy at the Rayleigh frequency, of different field sizes, has a very similar pattern as that in Fig. 6.53. This indicates that the nature of conductivity filtering of the slug test may be similar to a Gaussian filtering. From Fig. 6.55, we see that with increasing field size, the spectra of the filtered fields match better to the target spectra. This is evidently demonstrated in Fig. 6.56. From Fig. 6.56, we see that the pattern of spectral energy distribution at low frequency $f < 0.07$ is very similar to that of Fig. 6.53, even though spectral estimates have low reliability at low frequencies.

Furthermore, a random field is generated with the spectral method introduced in Section 4.2.3.B on page 95. The parameters are chosen to be the same as the above core-scale field. Similarly, we perform the hypothetical filtering and estimate the spectra. Results are shown in Fig. 6.57, 6.58.

The formats of Fig. 6.57, 6.58 are identical to those of Fig. 6.55, 6.56. Here we see that the PSD of the spectrally generated core-scale field matches the target spectrum very well. Other spectra characteristics are very similar to those of Fig. 6.55, 6.56. This indicates that the tails shown on the spectrum of the core-scale field generated by the GSLIB method should not be a problem and the core-scale field has no non-workable statistics.
Now, the reason accounted for the spectral tails has to be related to the scale of filtering. Fields filtered with large filter width, equivalent to slug testing with a small $S_s$, will have corresponding large correlation lengths in physical space and sharp spectra in frequency space; consequently, it requires a fine resolution, i.e., a small Rayleigh frequency. Fig. 6.59 shows the Rayleigh frequencies and the target spectra of the above core-scale and filtered fields. On the other hand, these spectral tails have very small magnitude, in which case the log-log spectrum graphs at high frequency may be somehow misleading, for instance, the target spectra goes straightly down to $\frac{1}{\infty}$, whereas the spectrum volume (total energy) is finite, i.e., the energy above certain frequency should be no more than zero. Thus if the spectra is plotted in non-log space, one should observe the spectral energy approaching zero beyond certain frequency.
Figure 6.53: Spectra of the core-scale fields and the measured fields with different specific storages. The core-scale field has an isotropic exponential variogram with $\sigma^2_{ig} K = 0.25$, $\lambda_x = \lambda_y = 4\Delta x^\omega$. 
Figure 6.54: Comparisons of the spectra of core-scale fields and measured fields of size $64 \times 64$, $128 \times 128$, $256 \times 256$ blocks.
Figure 6.55: Spectra of the core-scale fields and the hypothetically filtered fields with different filter widths. The core-scale fields are generated with a sequential Gaussian method of GSLIB, with an exponential variogram, $\sigma^{2}_{K} = 0.25$, $\lambda_x = \lambda_y = 4\Delta x^w$. 
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spectra of core-scale and filtered fields

core-scale: generated with GSLIB
filtered: \( \lambda = 4, 8, 16, 32 \)

Figure 6.56: Comparisons of the spectra of the core-scale field generated with GSLIB and the hypothetically filtered fields of size 64 x 64, 128 x 128, 256 x 256 blocks.
spectra of core-scale and filtered fields
core-scale: generated with a spectral method
filtered: $\lambda = 8, 16, 32$

Figure 6.57: Spectra of the core-scale fields and the hypothetically filtered fields with different filter widths. The core-scale field is generated with a spectral method with an exponential variogram, $\sigma^2_{\log K} = 0.25$, $\lambda_x = \lambda_y = 4\Delta x^w$. 
spectra of core-scale and filtered fields
core-scale: generated with a spectral method
filtered: \( \lambda = 4, 8, 16, 32 \)

Figure 6.58: Comparisons of the spectra of the spectrally generated core-scale and the hypothetically filtered fields of size 64 \( \times \) 64, 128 \( \times \) 128, 256 \( \times \) 256 blocks.
Figure 6.59: The Rayleigh frequency and the target spectra of core-scale and filtered fields.
6.1.3 Comparisons of filter spectra, filter functions

Fig. 6.60 shows the filter spectra with different specific storages. This time the frequency coordinate is in non-log form. We can observe the sharpening effects of the spectra for decreasing specific storages, in physical space meaning that the filtering occurs in a large influence zone. Note that the spectral energy at zero frequency equals to the square of the filter volume because the filter spectrum is estimated as a periodogram in the iterative Wiener filtering approach. We see that this energy decreases with $S_s$. This is because smaller $S_s$ introduces lower filtering power and a larger influence zone, however, the effective filtering is limited to the $65 \times 65$ blocks in the inner heterogeneous zone of the simulation model.

Fig. 6.61 shows the comparisons of the filter spectra of the fields of size $64 \times 64$, $128 \times 128$, $256 \times 256$. Their differences at low frequency were caused by the role of the Rayleigh frequency in the spectrum estimation, which is different for the measured fields that have different correlation lengths. We see that the filter spectra of the measured fields have a good match for $S_s = 0.1$. This means the Rayleigh frequency $\frac{1}{64}$ is good enough for the spectrum resolution of the measured fields with $S_s = 0.1$. For $S_s = 0.001$, we see that even the Rayleigh frequency $\frac{1}{256}$ still appears to be too coarse, which is evidenced by the fact that the energy at zero frequency is less than 1.

The filter function comparisons are illustrated in Fig. 6.62. The first three graphs show the comparisons of the filter functions associated with the slug tests of size $64 \times 64$, $128 \times 128$ and $256 \times 256$ with the filter functions identified by the perturbation method in Chapter 4. The filter functions by the spectral method are radially averaged. For a fair comparison to the spectral results that only have discrete values at dimensionless radius (scaled by wellbore radius) $^4$, $r_D = 2, 4, \cdots 64$, the filter functions from

$^4$From now on, for the convenience for plotting and description, the wellbore radius is approximated by half of a block size, it actually, in the slug test simulation model, equals to $0.63\Delta x$, $0.599\Delta x$, $0.582\Delta x$. 

perturbation method are averaged as

\[ G(r_D) = \frac{\int_{r_D-1}^{r_D+1} G(s) \, ds}{2} \]  \hspace{1cm} (6.124)

Note that the filter functions from these two methods are in fact not accurately comparable. The perturbation results are obtained from the small perturbations in a homogeneous aquifer, thus the flow pattern in the perturbed aquifer is very close to that of a homogeneous case, whereas the filter identified by the spectral method can be interpreted as the one averaged from all individual filter functions of every slug test performed in the heterogeneous core-scale field. Also note that the radius \( r_D \) in the filter function of the spectral method are greater than its true value (see the footnote of page 153), the filter functions would match more closely to the perturbation results if the accurate \( r_D \) is used.

The first three graphs show that the filter functions of the fields of size 128 x 128 and 256 x 256 match very well and they are closer to the filter functions of the perturbation method than are the filter functions of the field of size 64 x 64. The last three graphs display the filter function comparisons for different storages. Note that their differences can not be well shown on the scale of a same plot. For instance, the influence radii \( R_e \) or the equivalent filter widths \( \lambda_e \) appear to be very similar, whereas from the perturbation results we see that they are very different, e.g., \( R_e = 19.5, 60, 195, \lambda_e = 5, 13, 20 \), for \( S_s = 10^{-1}, 10^{-2}, 10^{-3} \).

Fig. 6.63 shows the filter functions in space where the central point corresponds to the location of the wellbore. The filtering power is magnified by \( 10^4 \). The 20-contour demarks an area where the filter is well resolved. We see that filter function of a smaller \( S_s \) has less spurious off-center peaks, because filter is better resolved for a smoother field. Also, the the filter in the field of size 256 x 256 has less off center peaks than that of the field for \( S_s = 0.1, 0.01, 0.001 \), respectively. See Fig. 2.9
of size 128 × 128. Because the filter identified by the spectral method can be interpreted as the one averaged from all individual filter functions of every slug test performed in the core-scale field, thus, the more slug tests are performed, the more averaging the spectral method does to the filter function. The corresponding filter functions by the perturbation method are shown in Fig. 6.64.

It can be seen from Fig. 6.64 and Fig. 6.63 that the 20 contour line of $S_s = 10^{-2}$ is larger than that of $S_s = 10^{-1}$ while the 20 contour of $S_s = 10^{-2}$ and $S_s = 10^{-3}$ stay approximately the same. For $S_s$ decreasing from $10^{-1}$ to $10^{-3}$, the 220 contour line shrinks. Filter functions from the perturbation method clearly demonstrate the differences of the averaging power less than 1 for different specific storages while such differences become very spurious in the spectral method.

This existence of such an averaged filter identified by the spectral method can be directly evaluated through the scatter cross-plots between the measured field, $Y_m$, and the estimated field from Winer filtering, $\tilde{Y}_m$, which is the result of the core-scale field filtered by the optimally estimated filter function(see Fig. 6.65, 6.66, 6.67). If these scatter points gather around the line of $Y_m = \tilde{Y}_m$, a universal filter for all the individual slug test exists. Note that the measured $Y_m$ contains some noises from the simulation model and the inversion procedure. These noises do not form up to a trend, for instance, in the inversion procedure different $Y_m$ estimates can fit the requirement of the objective function. Thus the cross-plots of $Y_m$ versus $\tilde{Y}_m$ need not to be exactly on the $Y_m = \tilde{Y}_m$ line. If the cross-plots obviously skewed away from the $Y_m = \tilde{Y}_m$ line, a universal filter does not exist (the filter is poorly resolved) or the filtering is nonlinear, or, there are some trends in the scale-scale and measured $Y_c$ data. This may be the case where the core-scale field has a large variance or exhibits a multimodal distribution, e.g., the bimodal sand-shale lense field.

Fig. 6.68 displays the comparison of the equivalent filter width versus specific storage
(\(\lambda_e \sim \frac{1}{\lg(\sqrt{S_s})}\)) from the spectral analysis and the perturbation method. In the perturbation method, the equivalent filter width is defined in eq. (5.110) which is

\[
\lambda_e = \frac{V_G(R_e)}{G(1 + \xi)}, \quad \xi \to 0^+.
\]  

(6.125)

In the spectral analysis method, the filter volume \(V_G(R_e)\) is directly obtained by the value of transfer function at zero frequency, i.e., \(V_G(R_e) = \hat{G}(0)\). \(V_G(R_e)\) is found to be close to 1, meaning that the filtering process is approximately mean-preserving. \(G(1 + \xi)\) equals \(G(2)\), because the closest node to the center of the wellbore is located at dimensionless distance \(r_D = 2r_w/r_w = 2\). This explains why the equivalent filter widths of the spectral method are greater than that of the perturbation method.\(^5\)

We also see that the \(\lambda_e\) of Spectral(128 x 128)\(^6\) is greater than that of Spectral(64 x 64), because, in calculating \(\hat{G}(0)\), Spectral(128 x 128) has a finer Rayleigh frequency, which promotes the filter volume, i.e., \(\hat{G}(0)\). Furthermore, we see that the \(\lambda_e\) of Spectral(128 x 128) and Spectral(256 x 256) become similar, this reflects the fact that refining the Rayleigh frequency becomes insignificant in promoting \(\hat{G}(0)\). On the other hand, this tells us that the \(\lambda_e\) of Spectral(256 x 256) can be considered as the one of the highest accuracy. We fit the \(\lambda_e\) of Spectral(256 x 256) versus \(S_s\), i.e., \(\lambda_e \sim \lg(\frac{1}{\sqrt{S_s}})\), by a parametric representation,

\[
\lambda_e = \lambda^o - [\lg(\frac{1}{\sqrt{S_s}})]^o + 14.58 \lg(\frac{1}{\sqrt{S_s}})
\]  

(6.126)

\(\lambda^o = 12.5, \quad [\lg(\frac{1}{\sqrt{S_s}})]^o = 0.5.\)

Note that this relationship is obtained through the slug tests in the fields with variance \(\sigma^2_{\lambda_e} = 0.25\), and correlation length \(\lambda_y = 4\Delta x^w\).

---

\(^5\)Because in the perturbation method in Chapter 5 we have observed that \(V_G(R_e) \approx 1\) and \(G(1 + \xi) \approx G(1)\), for \(\xi \to 0^+\).

\(^6\) 'Spectral' denotes that \(\lambda_e\) is identified by the spectral analysis method, '128 x 128' represents the slug testing field size.
In the following, we present a systematic exploration of the spatial variability of $Y_c$ on the spatial filtering.
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conductivity filter spectra
of slug tests
for different specific storages

filter spectrum
of log$_{10}$K field of size 256 x 256
$S_S = 10^{-3} - 10^{-1}$

filter spectrum
of log$_{10}$K field of size 64 x 64
$S_S = 10^{-5} - 10^{-1}$

Figure 6.60: Filter spectra of the slug tests in the conductivity fields of different specific storages.
Figure 6.61: Comparisons of the filter spectra in fields of size 64 x 64, 128 x 128, 256 x 256 blocks.
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Filter function \( S_8 = 10^{-1} \)

Filter function \( S_8 = 10^{-2} \)

Filter function \( S_8 = 10^{-3} \)

continued on next page
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Figure 6.62: Averaged filter functions for different specific storages.
Figure 6.63: Filter functions for different specific storages from spectral method. Enlarged by $10^4$. Contour level = 200.
Figure 6.64: Filter functions for different specific storages from perturbation method. Enlarged by $10^4$. Contour level of inner zone = 200; contour level of outer zone = 0.2.
cross-plots of measured fields versus estimated fields from Wiener filtering

field size: 256 x 256
$S_s = 10^{-3} \sim 10^{-1}$

A: $S_s = 10^{-1}$

B: $S_s = 10^{-2}$

C: $S_s = 10^{-3}$

Figure 6.65: Scatter plots of measured fields versus estimated fields from Wiener filtering. Field size 256 x 256. The inclined lines are of unit slope.
cross-plots of measured fields versus estimated fields from Wiener filtering
field size: 128 x 128
S_s = 10^{-5} \sim 10^{-1}

Figure 6.66: Scatter plots of measured fields versus estimated fields from Wiener filtering. Field size 128 × 128. The inclined lines are of unit slope.
cross-plots of measured fields versus estimated fields from Wiener filtering

field size: 64 x 64
$S_0 = 10^{-5} \sim 10^{-1}$

Figure 6.67: Scatter plots of measured fields versus estimated fields from Wiener filtering. Field size $64 \times 64$. The inclined lines are of unit slope.
Figure 6.68: Comparison of the equivalent filter width versus specific storage ($\lambda_e \sim \lg\left(\frac{1}{\sqrt{S_s}}\right)$) from the spectral analysis and the perturbation method.
6.2 Filter functions in conductivity fields of different spatial variabilities

The statistical parameters describing the spatial variability of a conductivity field may include the mean, variance, correlation length, variogram model, and stochastic realization. By the scaling property $K^\oplus$ and the spatial filtering representation of eq. (3.29), the mean should not affect the filtering, thus it needs not be considered. The other effects of the spatial variability on the conductivity filtering are discussed in the following sections.

6.2.1 Variance, variogram model and stochastic realization

We investigate the role of the variance, variogram and stochastic realization in the filtering process by the slug tests in isotropic conductivity fields ($Y_c$) of multiple realizations.

6.2.1.A Multiple realizations of isotropic fields

The multiple realizations of isotropic fields are generated by a sequential Gaussian simulation method of GSLIB [37] with both the exponential variogram (6.120) and a Gaussian variogram [37],

$$\gamma(h) = c \cdot \left[ 1 - e^{-\frac{h^2}{2\sigma^2}} \right],$$

(6.127)

The parameters are chosen to be $a = \lambda_x = \lambda_y = 8\Delta x^w$, $c = \sigma_{Y_c}^2 = 0.3$, 0.7, field size = $256 \times 256$ blocks. With $S_s = 0.001$, sampling distance = 2, we perform slug tests in a center area of $128 \times 128$ blocks in these fields. (Table 6.6).

Fig. 6.69 shows the estimated filter functions in the above fields. The last graph is the plot of the respective averaged filter functions from the first three graphs. Their close-match indicates that the variance, variogram model and multiple realization only have very small effects on the filtering process. This shows their influence on the changes of the filtering pattern of individual slug tests are averaged out by the spectral method.
Table 6.6: Slug tests in isotropic core-scale fields from multiple realizations.

<table>
<thead>
<tr>
<th>variogram model</th>
<th>variance($\sigma_{Y_c}^2$)</th>
<th>number of realization</th>
</tr>
</thead>
<tbody>
<tr>
<td>exponential</td>
<td>0.3</td>
<td>3</td>
</tr>
<tr>
<td>exponential</td>
<td>0.7</td>
<td>3</td>
</tr>
<tr>
<td>Gaussian</td>
<td>0.3</td>
<td>3</td>
</tr>
</tbody>
</table>

Thus the universal filter from spectral analysis is approximately independent of variance, variogram model and statistical realization. Note that this observation is obtained from isotropic fields with modest variance $\sigma_{Y_c}^2 \leq 0.7$. The role of anisotropy will be discussed in the next section.
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Continued on next page
Figure 6.69: Filter functions estimated in multiple core-scale conductivity realizations, with different variances and variogram models. The last graph is a comparison of the averaged filter functions shown in the first three graphs.
6.2.2 Core-scale conductivity of different correlation length

We explore the role of the correlation length in the filtering with two slug test schemes. One is slug tests in isotropic fields with different correlation lengths, the second is slug tests in anisotropic fields with different correlation length ratios.

6.2.2.A Isotropic, different correlation length

The isotropic fields are generated by a sequential Gaussian simulation method of GSLIB [37] with the exponential variogram of equation (6.120). The parameters are chosen to be $a = \lambda_x = \lambda_y = 3, 6, 12\Delta x^w, c = \sigma_Y = 0.3$, field size = $256 \times 256$ blocks. With $S_x = 0.001$, sampling distance = 2, we perform slug tests in a center area of $128 \times 128$ blocks.

Fig. 6.70 shows the comparisons of the estimated filter functions. Their close-match demonstrates that the power distribution of the $Y_c$ filtering is approximately independent of the correlation length of isotropic fields.

Fig. 6.71 shows the scatter plots of measured fields versus estimated fields from Wiener filtering. Here we see that the filter is better resolved with increasing correlation length. This reflects the fact that the similarity of the spatial filtering among individual slug tests increases with correlation length.
Comparisons of filter functions in isotropic IgK fields of different correlation lengths \( \lambda \).

Figure 6.70: Comparisons of filter functions in isotropic fields of different correlation lengths, \( \lambda_x = \lambda_y = 3, 6, 12 \Delta x^w \).
scatter plots of measured fields versus estimated fields from Wiener filtering
core-scale \( \lg K \) fields:
isotropic, different correlation \( \lambda \)
\( S_s = 0.01 \)

Figure 6.71: Scatter plots of the measured fields versus the estimated fields from Wiener filtering. The core-scale fields have isotropic variograms of correlation lengths \( \lambda = 3, 6, 12 \Delta x^w \).
6.2.2.B Anisotropic, different correlation length ratios

The anisotropic fields are generated by a sequential Gaussian simulation method of GSLIB [37] with the exponential variogram (6.120). The parameters are chosen to be: $\sigma_Y^2 = 0.3$, $\lambda_Y = 3\Delta x^w$, field size = 256 x 256 blocks. The correlation length ratios are listed in Table 6.7. With $S$ = 0.01 and sampling distance = 1, slug tests are performed in a center area of 64 x 64 blocks in these fields.

Table 6.7: Slug tests in anisotropic conductivity fields.

<table>
<thead>
<tr>
<th>$\lambda_x/\lambda_y$ ratios</th>
<th>$\lambda_y = 3\Delta x^w$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_x/\lambda_y$</td>
<td>1 3 5 10 20 30 40</td>
</tr>
</tbody>
</table>

Fig. 6.72 shows the grey-scales and histograms of the core-scale and the measured fields. We see that the averaging effect in the $X$ direction is stronger than that in the $Y$ direction, and it is more pronounced for increasing $\lambda_x/\lambda_y$ ratio.

Fig. 6.73 shows the reduced variance, $\sigma_Y^2 - \sigma_{Y_m}^2$, versus the correlation length ratio $\lambda_x/\lambda_y$. It demonstrates that the stronger the anisotropy, the less the variance reduction. Because the anisotropy somehow bars the flux to the wellbore in the $Y$ direction, and forces the averaging to occur in a relatively long and narrow zone, so that the net averaging volume may be relatively smaller and variance is less reduced.

Fig. 6.74 shows filter functions on the the center-lines in the $X$ and $Y$ directions. We see that $X$-direction averaging expands to a larger area and $Y$-direction averaging stays approximately the same. This phenomenon is shown in space in Fig. 6.75.

The scatter plots in Fig. 6.76 indicates the existence of a universal filter for the individual filtering of the slug tests performed in the corresponding fields. Note that all the $Y_c$ fields are generated with a fixed $\lambda_y = 3\Delta x^w$, if we increase $\lambda_y$ but keep the
same $\lambda_x/\lambda_y$ ratios, such averaged filters identified with the spectral method may not be meaningful, i.e., the averaging patterns of slug tests performed at different locations may be very dissimilar, hence, degrades the spectral results.
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isotropic
core-scale and
measured fields
\( \lambda_x = \lambda_y = 6 \, r_w \)

anisotropic
core-scale and
measured fields
\( \lambda_x / \lambda_y = 3, \lambda_y = 6 \, r_w \)

continued on next page • • •
Figure 6.72: The grey-scales and histograms of the core-scale and the measured field of different correlation length ratios. The fields shown are 64 × 64 blocks each.
Figure 6.73: Graph A: reduced variance \( \sigma_Y^2 - \sigma_{Y_m}^2 \) versus correlation length ratio \( \lambda_x / \lambda_y \).
Graph B: the mean of measured fields \( \mu_Y \) versus the mean of core-scale fields \( \mu_Y \).
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Figure 6.74: The filter functions on the center lines in the X and Y directions in anisotropic conductivity fields.
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Figure 6.75: Filter functions of different anisotropy ratios. Enlarged by $10^4$. 
Figure 6.76: Scatter plots of the measured fields versus the estimated fields from Wiener filtering. Anisotropy ratio of the core-scale fields: $\lambda_x / \lambda_y = 1, 3, 5, 30 \Delta x^w$. The inclined lines represent unit slope.
6.2.2.C Lense field

We can generate a bimodal lense field by several steps: Firstly, generating two isotropic fields, denoted as $Y_{CA}^a$, $Y_{CB}^b$, with a sequential Gaussian simulation method of GSLIB [37]; secondly, generating the lense shapes with a Boolean simulation method [37]; lastly reducing the mean of $Y_{CA}^a$ and projecting it through the Boolean lense shapes to $Y_{CB}^b$. Two lense fields are generated by this approach, see Table 6.8. Slug tests are performed in the two lense fields with different specific storages, see Table 6.9.

Table 6.8: Statistic parameters for two lense fields.

<table>
<thead>
<tr>
<th></th>
<th>variograms of</th>
<th>correlation of</th>
<th>reduced mean of</th>
<th>variance of</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$Y_{CA}^a$, $Y_{CB}^b$</td>
<td>$Y_{CA}^a$, $Y_{CB}^b$</td>
<td>$Y_{CA}^a$, $Y_{CB}^b$</td>
<td>$Y_{CA}^a$, $Y_{CB}^b$</td>
</tr>
<tr>
<td>field one</td>
<td>exponential</td>
<td>8</td>
<td>-2</td>
<td>0.3</td>
</tr>
<tr>
<td>field two</td>
<td>Gaussian</td>
<td>4</td>
<td>-4</td>
<td>0.8</td>
</tr>
</tbody>
</table>

Table 6.9: Slug tests in two lense fields.

<table>
<thead>
<tr>
<th>$S_s$</th>
<th>slug testing area</th>
<th>sampling distance</th>
</tr>
</thead>
<tbody>
<tr>
<td>field one</td>
<td>$10^{-1}$</td>
<td>$128 \times 128$</td>
</tr>
<tr>
<td></td>
<td>$10^{-2}$</td>
<td>$128 \times 128$</td>
</tr>
<tr>
<td>field two</td>
<td>$10^{-1}$</td>
<td>$64 \times 64$</td>
</tr>
<tr>
<td></td>
<td>$10^{-2}$</td>
<td>$64 \times 64$</td>
</tr>
</tbody>
</table>

Fig. 6.77 shows that the grey-scales and histograms of the core-scale and the measured fields. We see that the filtering occurs to both the background aquifer(sands) and foreground lenses(clay lenses), but there are only limited averaging between the sands and the lenses.

Fig. 6.78 displays the estimated filter functions. We discussed in the previous sections that the filter function from the spectral method can be interpreted as an averaged filter.
Clearly the filtering for slug tests in the lenses and those in the sands are fundamentally different. Thus, the averaged filter function by the spectral method is not representative to the individual filtering patterns of each slug test and it is poorly resolved.

Fig. 6.79 demonstrates the scatter plots of the measured fields versus the estimated fields from Wiener filtering. It can be seen that the optimally estimated filter functions honor the filtering pattern in the sand aquifer. The pattern of the scatter plots reflects the principle filtering fashion, which is affected by the extent of the lense heterogeneity. The 128 × 128 field is relatively less heterogeneous than the 64 × 64 field. The 64 × 64 field basically comprises half lenses and half sands, resulting in two distinctive centers in the scatter plot.
Figure 6.77: The grey-scales and histograms of the core-scale and the measured field with lense heterogeneties. Size of Lense Field one = 128 × 128; size of Lense Field two = 64 × 64.
Figure 6.78: Poorly resolved filter functions in two lense fields. Enlarged by $10^4$. 
Figure 6.79: Scatter plots of the measured fields versus the estimated fields from Wiener filtering. The inclined lines are of unit slope.
Chapter 7

CONCLUSIONS

7.1 The spatial filtering from core-scale conductivity to the conductivity measured by a slug test

1. **System and filtering.** We conceptualize the scale relationship between the core-scale conductivity and the conductivity measured by a slug test in a system and filtering framework. The slug test is a stable, space-variant, and nonlinear filtering system. The extent of space-variant property and nonlinearity increases with increasing $\sigma_{Y_c}^2$ (variance of core-scale field), and increases with decreasing $S_s$ (specific storage).

2. **System outputs – inverse parameter estimation.** $(\lg K^b - \lg K^a) \sim (\lg S_s^b - \lg S_s^{true})$ are positively correlated due to the role of $\lg S_s^b$ in controlling the shape of the fitting curve to match the response curve measured in a slug test. Superscripts $^a$ and $^b$ denote the inverse estimation methods which estimate $K$ alone and concurrently estimate $K$ and $S_s$, respectively.

3. **Role of $S_s$ in filtering.** In *heterogeneous* media\(^1\), the spatial filtering, from core-scale conductivity to the conductivity measured by a slug test, is dependent on the specific storage. With decreasing $S_s$, the filtering area increases and the filter amplitude proximal to wellbore decreases, and the filtering power distribution, from wellbore to influence radius, decays off less rapidly. Such filtering processes are approximately mean-preserving, and it reduces variance and increases the correlation

\(^1\)Here "media" refers to core-scale conductivity fields.
length of the measured conductivity fields. The changes in the variance and in the
correlation length is proportional to decreasing $S_s$.

For the slug tests in the fields with variance $\sigma_{Yc}^2 = 0.25$, and correlation length
$\lambda_x = \lambda_y = 4\Delta x^w$ (wellbore block diameter), it is found that equivalent filter width
versus $S_s$, $\lambda_e \sim \log(\frac{1}{\sqrt{S_s}})$,

$$\lambda_e = \lambda^o - \left[\log\left(\frac{1}{\sqrt{S_s}}\right)\right]^o + 14.58 \log\left(\frac{1}{\sqrt{S_s}}\right)$$

$\lambda^o = 12.5$, $\left[\log\left(\frac{1}{\sqrt{S_s}}\right)\right]^o = 0.5$.

4. Effects of spatial variability on filtering. In isotropic heterogeneous media, in tests
with $\sigma_{Yc}^2 \leq 0.7$, Gaussian and exponential variogram, correlation length $3\Delta x^w \leq
\lambda \leq 12\Delta x^w$, and a number of realizations for each set of parameters, it is found
that spatial filtering is approximately independent of variance, variogram model
and stochastical realizations; it is also independent of the correlation length, but,
for fields with larger correlation, the filter function is better resolved in the spectral
analysis method. In anisotropic heterogeneous media, the spatial filtering shows an
anisotropic pattern proportional to the anisotropy ratio.

5. Role of $S_s$ in filtering. In homogeneous media, the spatial filtering is dependent on
$S_s$. The filtering function has the form,

$$G(r_D) = G(1 + \xi) \frac{1}{(r_D)^p}, \quad \xi \rightarrow 0^+, \quad r_D = r/r_w$$

where the $G(1 + \xi)$ is the filter amplitude at the wellbore edge, and $p$ is an exponent:

<table>
<thead>
<tr>
<th>$S_s$</th>
<th>$10^{-1}$</th>
<th>$10^{-2}$</th>
<th>$10^{-3}$</th>
<th>$10^{-4}$</th>
<th>$10^{-5}$</th>
<th>$10^{-6}$</th>
<th>$10^{-7}$</th>
<th>$10^{-8}$</th>
<th>$10^{-9}$</th>
<th>$10^{-10}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G(1 + \xi) \times 10^{-2}$</td>
<td>18.338</td>
<td>7.908</td>
<td>4.882</td>
<td>3.472</td>
<td>2.840</td>
<td>2.412</td>
<td>2.165</td>
<td>1.950</td>
<td>1.885</td>
<td>1.753</td>
</tr>
<tr>
<td>$p$</td>
<td>3.0</td>
<td>2.3</td>
<td>2.1</td>
<td>2.05</td>
<td>2.0</td>
<td>2.0</td>
<td>2.0</td>
<td>2.0</td>
<td>2.0</td>
<td>2.0</td>
</tr>
</tbody>
</table>
where: 1) \( G(1 + \xi) \) decreases with decreasing \( S_s \), and 2) \( G(r_D) \sim r_D \) approaches approximately a \( 1/r_D^2 \) decreasing law for \( S_s \) smaller than \( 10^{-4} \). The direct relationship between \( G(r_D) \) and \( S_s \) is

\[
G(r_D) = \left( 0.2S_s^{0.25} + 0.2S_s^{0.75} + 0.01753 \right) \frac{1}{(r_D)^{3.6/\sqrt{S_s}+2.5}}. \tag{7.130}
\]

Equivalent filter width versus \( S_s \), \( \lambda_e \sim \lg(\frac{1}{\sqrt{S_s}}) \),

\[
\lambda_e = \lambda^0 - [\lg(\frac{1}{\sqrt{S_s}})]^o + 14.58 \lg(\frac{1}{\sqrt{S_s}}) \tag{7.131}
\]

\( \lambda^0 = 5.47, \quad [\lg(\frac{1}{\sqrt{S_s}})]^o = 0.5 \)

Influence radius versus \( S_s \), \( R_e \sim \lg(\frac{1}{\sqrt{S_s}}) \),

\[
\lg R_e = (\lg R_e)^o - [\lg(\frac{1}{\sqrt{S_s}})]^o + \lg(\frac{1}{\sqrt{S_s}}) \tag{7.132}
\]

\[
R_e = \frac{1}{\sqrt{S_s}} 10^{(\lg R_e)^o - [\lg(\frac{1}{\sqrt{S_s}})]^o} \tag{7.133}
\]

\( (\lg R_e)^o = 1.29, \quad [\lg(\frac{1}{\sqrt{S_s}})]^o = 0.5 \)

7.2 Possible engineering application

The spatial filtering methodology is equally applicable to many other engineering measurement techniques which invoke the spatial averaging of the measured properties. The measurement filters quantify the spatial significance of the sub-scale properties on the measured-scale properties. Here, we illustrate this idea in two types of problems in the following.

7.2.1 The Q-problem

The Q-problem refers to the one whose primary concern is the flux rate under certain conditions. The scale under consideration in the Q-problem is the scale of the aquifer
that contributes the fluxes. The filtering property tells us that the measured conductivity (equivalent or effective conductivity) is strongly affected by near-wellbore heterogeneity, which thus is very crucial in the control of flux rate. Then direct implications may be in the siting of pumping wells or any other engineering facilities extracting water from aquifers.

On the other hand, it is no wonder that the equivalent or effective conductivity estimated with traditional geostatistical methods, e.g., Kriging, conditional simulation, etc., which usually do not incorporate the filters invoked by engineering activities, might be far different from the one estimated with the filtering function, and thus result in errors in flux rate prediction. For instance, the statistically interpolated grid-scale conductivity may incur large error in the prediction of pump yield.

### 7.2.2 The $L$-problem

The $L$-problem refers to the one whose primary concern is the mass transport distance under certain conditions. In the $L$-problem the small-scale (or core-scale) conductivity is of practical interests, because the transport distance is often seen to be strongly impacted by the small-scale conductivity variations. The small-scale conductivity can not be measured or at least are too expensive to be measured, but if the filter function (from the small-scale to the measured scale) is known, one can deconvolve the measured data back to the small-scale using spectral analysis.\footnote{The result of the deconvolution is a spectrum or a variogram which can be used to generate small-scale fields.}

On the other hand, when the measurements are insufficient, one may attempt to employ some geostatistical models to simulate the the parameter values in the unsampled region, conditioning on the measured data. Since we now know from measurement filtering that the measured conductivity is a spatial average of the small-scale conductivity,
thus the simulation does not have to: 1) exactly utilize the variogram inferred from measured data record, or 2) exactly honor the measured data at the conditioning locations. In this case, deconvolution (to small-scale or to remove noise), or some soft-data techniques appear necessary. The soft-data techniques may incorporate all the available information of different characteristics and of various sources (like hard data and experience), thus may be more appropriate to relieve some engineering intractability.

7.3 Theoretical significance

As we discussed in the introduction in Chapter 1, measurement filter is not only able to transfer parameter from small-scale to larger scale, but it can be employed in upscaling the fundamental flow and transport equations. The equations are often obtained from small-scale laboratory experiments and will become invalid beyond certain scale ranges. The measurement filter can be used to enlarge or transfer the application scale ranges of the equations, to accommodate the scale of field problems.

7.4 Future work

1. In the spatial filtering of heterogeneous core-scale conductivity, for each slug test we have only simulated an inner zone of $65 \times 65$ blocks ($\Delta x^w \times \Delta x^w$), and $S_s = 10^{-n}$, $n = 1, 2, \ldots, 5$, due to the computational limitation. The simulation area and $S_s$ value can be expanded if superior computer resources becomes available, or, more efficient numerical techniques (e.g., multigrid method) are developed. In unconfined aquifer where vertical flow becomes significant, 3D model should be used.

2. The filter function has not yet been parameterized in terms of the variance, correlation length and anisotropy of core-scale field. A generic form of filter functions possibly can be obtained if more tests could be performed with more computational
resource. So at this time, the nonparametric representation may not be readily applied by field workers. But the generic formulas in homogeneous fields obtained with the perturbation method should serve as a good clue for the heterogeneous case. On the other hand, the similarity between the slug test filtering and Gaussian filtering,

\[
G(x, y) = \frac{6}{\pi \lambda_x \lambda_y} e^{-\frac{6}{\lambda_x} \left[ \frac{x^2}{\lambda_x^2} + \frac{y^2}{\lambda_y^2} \right]}
\]  

(7.134)

where \( \lambda_x \) and \( \lambda_y \) are the characteristic filter widths which are somehow related to the variance, correlation length, anisotropy, and specific storage of the core-scale field, may ease the application dearly.

3. Only slug test filter is considered in the thesis. Other aquifer tests or engineering measurement techniques which invoke the spatial averaging of the intended-to-be-measured properties are worth being investigated and made readily applicable to field workers and theoreticians.
Bibliography


Appendix A

EFFECTIVE AND EQUIVALENT PARAMETERS

We have illustrated in eq.(1.5) that the real world has already been mapped to an engineering framework after the measurement filtering $H^P$ and estimation filtering $H^S$. However, $H^S$ filtering is a probabilistic or stochastic process, and it generates infinite number of realizations in which each realization has equal significance. These realizations may require exorbitant computation load when they are utilized in the simulation of engineering problems (e.g., Monte Carlo simulation), even though they may well suit the scale-requirement of the decision variable. The consequent compromise is the use of effective parameters, which are defined under ergodicity as the representative values of the mean behavior through an ensemble of realizations [Sanchez-Vila et al., 1995].

Equivalent parameters are defined as spatial averages obtained on a single realization [Sanchez-Vila et al., 1995], not necessarily requiring ergodicity or stationarity. The equivalent parameters are associated with certain initial and boundary states. Consequently, equivalent parameter is deterministic. If the equivalent parameter scale equals to the size of a grid block, we call it a block equivalent parameter, or shortly, a block parameter. Apparently, block parameters are dependent upon the block geometry and block boundary states, but this dependence can be lifted under certain conditions, e.g., increasing the block size (reducing the boundary effects) and imposing stationarity on the upper-block scale. Indeed, under stationarity and ergodicity, when the block size is enlarged to infinity, or more strictly, the block scale is large enough relative to global heterogeneity scale, the block parameter approaches the effective parameter. Consequently,
block parameter a generalization of the effective parameter.

A.I Effective parameter: a limiting case of upscaling

The engineering importance of the real world heterogeneity has motivated the study of effective parameters, of which conductivity receives the most concern. Among the pioneering work on several disciplines, Warren and Price [1961], using a Monte Carlo method, studied the flow in heterogeneous media in petroleum engineering; Matheron [1967] relates the average permeability to the harmonic, geometric and arithmetic mean of the local permeabilities, in the geostatistics context. The Monte Carlo simulation of Freeze [1975], followed by Smith and Freeze [1979] led to the explosion of heterogeneity study from various aspects in water resources research.

Following the pioneering works, enormous efforts have been devoted to the study of effective conductivity in heterogeneous media (hereafter $K_{ef}$ denotes effective conductivity). With both numerical and analytical vehicles, one is able to relate the statistics of $K_{ef}$ to those of small scale conductivities assumed to be known. Analytical tools include the versatile first-order perturbation method [Schwydler, 1962; Matheron, 1967; Bakr et al., 1978; Gutjahr et al., 1978; Gutjahr and Gelhar, 1981], self-consistent method [Dagan, 1979; 1989], and the method of volume-averaging and spatial moments [Kitanidis, 1990]. In order to account for more complicated flow and porous media and initial and boundary effects, another group of authors used semi-analytical method [Desbarats, 1992a; 1992b; 1994], and Monte Carlo method [Freeze, 1975; Warren and Price, 1961; Desbarats, 1987; Gomez-Hernandez and Gorelick, 1989]. The review shall not be exhaustive but comprise the major contributions. Other more detailed summary can be found in Gelhar [1994], Gomez-Hernandez and Gorelick [1989], Sanchez-Vila et al. [1995], and Desbarats [1987, 1992a].
Appendix A. EFFECTIVE AND EQUIVALENT PARAMETERS

Under the condition of uniform steady mean flow in infinite isotropic media of small variance, the effective conductivity $K_{ef}$ equals to

$$K_{ef} = \bar{K_G} \exp \left[ \sigma_{\ln K}^2 \left( \frac{1}{2} - \frac{1}{n} \right) \right]$$  \hspace{1cm} (A.I.135)

where $K_G$ is the geometric mean and $n = 1, 2, 3$ are flow dimensions [Gutjahr et al., 1978; Bakr et al., 1978; Gutjahr and Gelhar, 1981; Dagan, 1979; 1989]. The 2D solution was extended to any $\sigma_{\ln K}^2$ by Gelhar [1986]. The 3D solution is the first-order approximation of the $K_{ef}$ first postulated by Matheron [1967].

In transient flow Freeze [1975] states $K_{ef}$ is time-dependent. In the slowly varying flow excited by a instantaneous point ‘slug injection’ in conductivity field of small $\sigma^2_f (Y = \ln K)$, Dagan [1982] found that $K_{ef}$ varies from arithmetic mean at initial time to steady state value during relaxation time (which must elapse after some change in the system so that $K_{ef}$ can be defined). The flow condition is similar to that of a slug test of Cooper et al. [1967], except that $K$ and $S_s$ at the injection location are of that of aquifer as usual, whereas at the wellbore of a slug test $K = \infty$, $S_s = 1$.

Kitanidis [1990] studied the effective conductivity in the flow condition similar to Dagan [1982] but in periodic conductivity fields. Kitanidis [1990] derived $K_{ef}$ by the generalization of block conductivity $K_{eV}$.\(^2\) He represented $K_{eV}$ in a matrix form (eq.56),

$$\begin{align*}
(D_{ij})_{eV} &= -\frac{1}{2} \frac{1}{l_1 l_2 l_3} \int_V \left[ d_i \cdot \Delta g_j + d_j \cdot \Delta g_i \right] dx + \overline{D_{ij}}, \hspace{1cm} (A.I.136)
\end{align*}$$

where $D$ is diffusivity ($K/S$), overbar denotes volumetric average, $l$ is the side lengths of rectangular parallelepipeds, and $g$ is a function of local spatial coordinates. $(K_{ij})_{eV}$ can be obtained by solving the matrix if $S_s$ is constant. In an isotropic media, above equation

\(^1\)Dimensions of $K$ and $S$ are [LT\(^{-1}\)] and [L\(^{-1}\)] respectively, they will be omitted in the following discussion.

\(^2\) $K_{eV}$ will be discussed in next section.
Appendix A. EFFECTIVE AND EQUIVALENT PARAMETERS

reduces to

\[(K_{ij})_e^V = - \frac{1}{2 l_1 l_2 l_3} \int_V K \left[ \frac{\partial g_i}{\partial x_i} + \frac{\partial g_j}{\partial x_j} \right] dx + \overline{K} \delta_{ij} \quad (A.I.137)\]

where, \(\overline{K} = \frac{1}{V} \int_V K(x) dx\) \hspace{1cm} (A.I.138)

\(\delta_{ij}\) is the Kronecker delta, \(\delta_{ij} = 1\) if \(i = j\) and \(\delta_{ij} = 0\) otherwise. For perfectly stratified media, above further entails

\[(\cdot^3_3) e^V = (K_{22})_e^V = \overline{K} = K_A, \quad (A.I.139)\]

\[(K_{33})_e^V = \left[ \frac{1}{l_3} \int_0^{l_3} \frac{du}{K(u)} \right]^{-1}, \quad (A.I.140)\]

where \(K_A\) denotes arithmetic average. For the media with small variations, the volume-averaging effective conductivity can be expressed in a general equation.

\[(K_{ij})_e^V = - \exp \left( \overline{Y} \right) \sum_k \frac{k_i k_j}{k^2} |\Psi(k)|^2 + \exp \left( \overline{Y} \right) \left[ 1 + \sigma_Y^2/2 \right] \delta_{ij} \quad (A.I.141)\]

where \(|\Psi(k)|^2\) is the power spectrum of the periodic function, and \(k^2 = k_1^2 + k_2^2 + k_3^2\). If \(V \to \infty\), \(K_e^V \to K_{ef}\), by replacing the summation of a integral, above equation reduces to

\[(K_{ij})_{ef} = - \exp \left( \overline{Y} \right) \int \frac{k_i k_j}{k^2} S(k) dk + \exp \left( \overline{Y} \right) \left[ 1 + \sigma_Y^2/2 \right] \delta_{ij} \quad (A.I.142)\]

where \(S(k)\) is the PSD of \(Y\). This is identical to the result of Gutjahr et al. [1978] which followed a ensemble—averaging approach, whereas here follows a purely deterministic volume—averaging approach. This essentially means the result for a periodic medium by taking a limit (the period tending to infinity) is applicable to a random medium. \((K_{ij})_{ef}\) can be reduced to those of Gutjahr et al. [1978], Bakr et al. [1978], and Gutjahr and Gelhar [1981], also confirms the 3D effective conductivity hypothesis of Matheron [1967], but involving no assumption other than isotropic conductivity variability.
Kitanidis [1990] showed the assumptions of steady flow [Gutjahr et al., 1978; Bakr et al., 1978; Gutjahr and Gelhar, 1981] and unsteady flow [Dagan, 1982]) are not essential in the definition or determination of effective parameters. Instead, the essential assumption is that of 'gradually varying' flow. That is the scale of fluctuations of the head must be large in comparison to the typical length scale of fluctuations of local conductivity so that all values of conductivity are sampled (eq. 47).

Kitanidis [1990] also found that shortly after the slug injection the net spreading rate of slug mound depends on the value of conductivity in the neighborhood of the injection and varies with time, and this is when the 'gradually varying' condition is not satisfied, hence, this is when effective conductivity is not meaningful. This spreading continues up to the relaxation time and converges to a constant, and this time can be approximated by \( \bar{t} S_a/K, \) where \( \bar{t} \) is the scale of conductivity fluctuations, \( K \) is a typical value of conductivity. These observations are in general agreement of Dagan [1982].

In his study of 2D radial flow in an annular region drained by a pumping well with fixed head drawdown at \( (r = r_w) \) and zero drawdown at some 'large distance' \( r = r_m, \) Matheron [1967] found that in the unconditional case, \( K_{ef} \to K_H \) for \( r_m/\lambda \to \infty; \) and \( K_{ef} \to K_A \) for \( r_w/\lambda \to 0; \) while in the conditional case — with \( K = K_w \) at the well, \( K_{ef} \to K_H \) for \( r_m/\lambda \to \infty; \) and \( K_{ef} \to K_w \) for \( r_w/\lambda \to 0, \) where \( \lambda \) is conductivity correlation length, \( K_A \) and \( K_H \) are the arithmetic and homonic mean, respectively.

Gomez-Hernandez and Gorelick [1989] performed Monte Carlo simulations under a quite complicated conditions: unconfined aquifer in bounded domain with a variety of boundary conditions, and distributed recharges and recharge from river, and having pumping wells, they found \( K_H < K_{ef} < K_G. \) They used the \( K_{ef} \) defined in \( p \) norm to fit the \( K_{ef} \) calculated from MC simulation,

\[ K_p = \langle K^n \rangle^{1/p} \]  

(A.I.143)
Appendix A. EFFECTIVE AND EQUIVALENT PARAMETERS

The numerically determined $p$ values are confirmed in the analytical derivation of Ababou and Wood [1990], directly relating to the statics of $Y(= \ln K)$,

$$K_p = \begin{cases} 
K_G \exp \left( p \frac{\sigma_Y^2}{2} \right), & p = 1, \\
K_G \left( K_G / K_H \right)^p, & p = 0, \\
K_G \left( K_A / K_G \right)^p, & p = -1 \end{cases}$$

(A.I.144)

Ababou and Wood [1990] also expanded the 2D $K_{ef}$ results of Matheron [1967] by combining the perturbation solutions from Matheron [1967, section 25] and Schwydler [1963]. They found the effective conductivity for a Gaussian-shaped correlation function as

$$K_{ef} \approx K_G \exp \left\{ \frac{\sigma_Y^2}{2} \left[ \frac{\left( \ln \left( r_G / r_w \right) \right)^2}{\left( \ln \left( r_m / r_w \right) \right)^2} - 1 \right] \right\}$$

(A.I.145)

$$r_G = r_w \exp \left\{ \sqrt{2 \left[ 1 + \left( \frac{\ln \left( r_w / \lambda \right)}{\ln \left( r_m / \lambda \right) \right)^2 \right]} \right\}$$

(A.I.146)

With the assumption $r_w \ll \lambda \ll r_m$, it yields

$$\begin{cases} 
K_{ef} \leq K_G, & r_m \geq r_G \\
K_{ef} \geq K_G, & r_m \leq r_G \\
K_{ef} = K_G, & r_m = r_G \end{cases}$$

(A.I.147)

where $r_G$ is a critical depression cone radius called "geometric" radius. Besides above findings, they stated: (i) For fixed $\lambda$, $K_{ef} \to K_H$, for $r_m \to \infty$; $K_{ef}$ decreases and becomes closer to $K_H$ as $r_w$ increases but remains smaller than $\lambda$; (ii) $K_{ef} \to K_A$, for $r_w \to 0$; $K_{ef}$ increases as $r_m$ decreases but remains larger than $\lambda$; and (iii) For fixed $r_w$ and $r_m$, the $K_{ef}$ increases as $\lambda$ increases within $[r_w, r_m]$. This somehow explains the $K_H < K_{ef} < K_G$ of Gomez-Hernandez and Gorelick [1989]. Note that Dagan [1979] also found this $K_H < K_{ef} < K_G$ relationship in unbounded isotropic fields of small variance, but $K_{ef}$ will exceed $K_G$ for larger variances.

Using first order perturbation analysis, Naff [1991] studied the 3D steady mean-radial-flow to a pumping well sited in infinite medium. Naff [1991] found: (i) At distance
Appendix A. EFFECTIVE AND EQUIVALENT PARAMETERS

\[ p = r / \lambda > 2 \text{ or } 3, \quad K_{ef} \text{ remains constant, depending on the anisotropy } \mu = \lambda / \lambda_3; \]

(ii) At the wellbore, \( K_{ef} = K_A \), regardless of \( \mu \); (iii) At large distance, \( K_{ef} \) varies from \( K_H(2\text{D flow}) \) to \( K_A(3\text{D flow in a well-stratified medium}) \). In general the results agree closely with the asymptotes of Matheron [1967]. Interestingly, Naff [1991] stated the variance of specific discharge \( q_{ef} \) is approximately proportional to \( r^{-2} \).

A.II Equivalent parameter: a general case of upscaling

As characterized by Sanchez-Vila et. al [1995], there are four major types of block upscaling methods reported in literature: (i) Darcian definition; (ii) Method of Rubin and Gomez-Hernandez [1990]; (iii) Power averaging method of Desbarats [1992a]; and (iv) Method of Indelman and Dagan [1993a, b] and Indelman [1993]. Sanchez-Vila et. al [1995] state that all the methods seem to provide striking agreements in statistic terms. Here we only discuss method (i) and (iii), and one more method – weighted averaging method [Dykaar and Kitanidis, 1992a, b]

The upscaling by the Darcian approach directly applies to the equation of fluid flow in porous media at small scale

\[ \nabla \cdot [K(x) \nabla \phi(x, t)] = S \frac{\partial \phi(x, t)}{\partial t}, \quad \text{(A.II.148)} \]

upscaling to a large-scale equation, where the small-scale effects are lumped into the effective parameter \( K_{eV} \). The proportionality of a upscaled Darcy’s law equals

\[ K_{eV} = - \frac{\nabla (\phi(x))}{(q(x))}, \quad \text{(A.II.149)} \]

and the upscaled flow equation is

\[ \nabla \cdot [K_{eV} \nabla \phi(x, t)] = S \frac{\partial \phi(x, t)}{\partial t}. \quad \text{(A.II.150)} \]

\[ ^3\text{This may be somehow explained by the } r^{-2} \text{ in } K_{eV} = \frac{1}{V} \int_V K(x)^w dV, \quad W = \int_V \frac{dV}{r(x)^w} \text{ of Desbarats [1994], where } V \text{ is the drainage volume of pumping test and } w \text{ is an empirical exponent. By applying Darcy's law, } q_{ef}(r) = K_{ef}(r) J_{ef}(r) \text{ where } J \text{ is gradient, the } r^{-2} \text{ reaches } q_{ef}. \]
The upscaling should meet the scale requirement [Dagan, 1986; Gelhar, 1986],

\[ l \ll L \ll D \]  \hspace{1cm} (A.II.151)

where \( l \) is the small-scale, \( L \) is the scale of the averaging volume, and \( D \) is the global heterogeneity scale.

Desbarats [1992a] empirically defines the effective conductivity as the “power average” of random function \( K(x) \) over a volume \( V \)

\[ K_{eV} = \left( \frac{1}{V} \int_V K(x)^w dV \right)^\frac{1}{w} \]  \hspace{1cm} (A.II.152)

and analyzes the exponent \( w \) as that best fits numeric simulations.

Dykaar and Kitanidis [1992a, b] defines \( K_{eV} \) in as a weighted spatial average in a volume \( V \),

\[ K_{eV}(x_c) = \frac{1}{V} \int_V K(x_c + \xi) W(\xi) d\xi. \]  \hspace{1cm} (A.II.153)

where \( x_c \) is the centroid of \( V \) and \( W(\xi) \) a weight function matrix obtained by solving flow equations. This is similar to the definition of the measurement filter of eq.(1.2) where the filter is equivalent to the weight function. The idea of weighted average was described by a number of authors [Matheron, 1965; Marie, 1967; Anderson and Jackson, 1967] (cf. the discussion in Baveye and Sposito [1984]).

Dykaar and Kitanidis [1992a, b] found that in a 2D flow in conductivity fields of a Gaussian model with \( \sigma^2 = 2 \), \( 4 \) and \( 5 \leq L/l \leq 80 \), a \( V \) of \( 80\lambda \) (integral scale) is needed for \( K_{eV} \) to reach a relative constant value, while in 3D a \( V \) of \( 30\lambda \) is required for \( K_{eV} \) to reach its asymptotic value. The heterogeneities of scale smaller than \( 2\lambda \) (for 2D) and \( 1.3\lambda \) (3D) do not significantly contribute to \( K_{eV} \).
Appendix B

FINITE DIFFERENCE EQUATIONS OF SLUG TEST

B.I 2D FDM in rectangular coordinates

Forward finite difference approximation of equation (2.16), Figure B.I.80,

\[
\frac{1}{d_{xx}} \left[ K_{i,j+\frac{1}{2}} \left( \frac{h_{i,j+1}^m - h_{i,j}^m}{d_{xp}} \right) - K_{i,j-\frac{1}{2}} \left( \frac{h_{i,j+1}^m - h_{i,j}^m}{d_{xm}} \right) \right] \\
+ \frac{1}{d_{yy}} \left[ K_{i+\frac{1}{2},j} \left( \frac{h_{i+1,j}^m - h_{i,j}^m}{d_{yp}} \right) - K_{i-\frac{1}{2},j} \left( \frac{h_{i,j}^m - h_{i-1,j}^m}{d_{ym}} \right) \right]
\]

Figure B.I.80: Block centered finite difference in Cartesian coordinates.

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Appendix B. Finite Difference Equations of Slug Test

\[ \Delta h_{i,j} = S_s \frac{h_{i,j}^m - h_{i,j}^{m-1}}{\Delta t} \]  
\[(B.I.154)\]

Multiplying by \( d_{xx}d_{yy} \), to get the mass balance equation at block \((i,j),\)

\[ d_{yy}[K_{i,j+\frac{1}{2}}(\frac{h_{i,j+1}^m - h_{i,j}^m}{d_{xp}}) - K_{i,j-\frac{1}{2}}(\frac{h_{i,j+1}^m - h_{i,j}^m}{d_{xm}})] + d_{xx}[K_{i+\frac{1}{2},j}(\frac{h_{i+1,j}^m - h_{i,j}^m}{d_{yp}}) - K_{i-\frac{1}{2},j}(\frac{h_{i,j}^m - h_{i-1,j}^m}{d_{ym}})] = d_{xx}d_{yy}S_s\frac{h_{i,j}^m - h_{i,j}^{m-1}}{\Delta t} \]
\[(B.I.155)\]

Rearrange the above, to get,

\[ \left[\left(\frac{K_{i,j+\frac{1}{2}}}{d_{xp}} + \frac{K_{i,j-\frac{1}{2}}}{d_{xm}}\right)d_{yy} + \left(\frac{K_{i+\frac{1}{2},j}}{d_{xp}} + \frac{K_{i-\frac{1}{2},j}}{d_{ym}}\right)d_{xx} + \frac{d_{xx}d_{yy}S_s}{\Delta t}\right]h_{i,j}^m \]
\[- \left(\frac{d_{yy}}{d_{xp}}K_{i,j+\frac{1}{2}}h_{i,j+1}^m - \frac{d_{yy}}{d_{xm}}K_{i,j-\frac{1}{2}}h_{i,j-1}^m\right) \]
\[- \left(\frac{d_{xx}}{d_{yp}}K_{i+\frac{1}{2},j}h_{i+1,j}^m - \frac{d_{xx}}{d_{ym}}K_{i-\frac{1}{2},j}h_{i-1,j}^m\right) = \frac{d_{xx}d_{yy}S_s}{\Delta t}\frac{h_{i,j}^m - h_{i,j}^{m-1}}{} \]
\[(B.I.156)\]

where,

\[ d_{xm} = \frac{dx(j) + dx(j - 1)}{2} \]
\[ d_{xp} = \frac{dx(j) + dx(j + 1)}{2} \]
\[ d_{xx} = \frac{d_{xm} + d_{xp}}{2} \]
\[ d_{ym} = \frac{dy(i) + dy(i - 1)}{2} \]
\[ d_{yp} = \frac{dy(i) + dy(i + 1)}{2} \]
\[ d_{yy} = \frac{d_{ym} + d_{yp}}{2} \]
Appendix B. **FINITE DIFFERENCE EQUATIONS OF SLUG TEST**

\[ K_{i+\frac{1}{2},j} = \frac{(d_{yy} + d_{yp})K_{i+1,j}K_{i,j}}{d_{yp}K_{i,j} + d_{yy}K_{i+1,j}} \]

\[ K_{i-\frac{1}{2},j} = \frac{(d_{yy} + d_{ym})K_{i-1,j}K_{i,j}}{d_{ym}K_{i,j} + d_{yy}K_{i-1,j}} \]

\[ K_{i,j+\frac{1}{2}} = \frac{(d_{xx} + d_{xp})K_{i,j+1}K_{i,j}}{d_{xp}K_{i,j} + d_{xx}K_{i,j+1}} \]

\[ K_{i,j-\frac{1}{2}} = \frac{(d_{xx} + d_{xm})K_{i,j-1}K_{i,j}}{d_{xm}K_{i,j} + d_{xx}K_{i,j-1}} \]  

(B.I.157)

Finite differencing wellbore equation (2.17), yields the discretized mass balance equation at wellbore, \((i, j) = (i_{wellbore}, j_{wellbore})\),

\[
d_{yy} \left[ K_{i,j+\frac{1}{2}} \left( \frac{h_{i,j+1}^m - h_{i,j}^m}{d_{xp}} \right) - K_{i,j-\frac{1}{2}} \left( \frac{h_{i,j+1}^m - h_{i,j-1}^m}{d_{ym}} \right) \right]
+ d_{xx} \left[ K_{i,j+\frac{1}{2}} \left( \frac{h_{i,j+1}^m - h_{i,j}^m}{d_{yp}} \right) - K_{i,j-\frac{1}{2}} \left( \frac{h_{i,j}^m - h_{i,j-1}^m}{d_{xm}} \right) \right]
\]

\[
= d_{xx}d_{yy} \frac{h_{i,j}^m - h_{i,j}^{m-1}}{\Delta t} \]  

(B.I.158)

Equation (B.I.158) is the same as equation (B.I.155) except \(S_s = 1\) in the right hand side term. This physically means that wellbore is filled with water.

Equation (B.I.158) is the same as equation (B.I.155) except \(S_s = 1\) in the right hand side term. This physically means that wellbore is filled with water.

Rearrange the above, get,

\[
\left[ \frac{K^w_{i,j+\frac{1}{2}}}{d_{xp}} + \frac{K^w_{i,j-\frac{1}{2}}}{d_{xm}} \right] d_{yy}^w + \left( \frac{K^w_{i+\frac{1}{2},j}}{d_{yp}} + \frac{K^w_{i-\frac{1}{2},j}}{d_{ym}} \right) d_{xx}^w + \frac{d_{xx}^w d_{yy}^w}{\Delta t} h_{i,j}^m
\]

\[
- \left( \frac{d_{yy}^w}{d_{xp}} K^w_{i,j+\frac{1}{2}} \right) h_{i,j+1}^m - \left( \frac{d_{yy}^w}{d_{xm}} K^w_{i,j-\frac{1}{2}} \right) h_{i,j-1}^m
\]
Figure B.I.81: Wellbore specific storage equals to 1.

\[ \Delta V = \Delta x \Delta y \Delta H S_s = \Delta x \Delta y \Delta H \]

\[ S_s = 1 \]

Equation (B.I.159) is the same as equation (B.I.156) except \( S_s = 1 \) and

\[
\begin{align*}
\frac{d^{w}}{d_{xx}} & = \frac{dx(j - 1)}{2} \\
\frac{d^{w}}{d_{xp}} & = \frac{dx(j + 1)}{2} \\
\frac{d^{w}}{d_{xx}} & = dx(j) \\
\frac{d^{w}}{d_{ym}} & = \frac{dy(i - 1)}{2} \\
\frac{d^{w}}{d_{yp}} & = \frac{dy(i + 1)}{2} \\
\frac{d^{w}}{d_{yy}} & = dy(i)
\end{align*}
\]

(B.I.160)

\[ K_{i+\frac{1}{2},j} = K_{i+1,j} \]
where, $w$ indexes coefficients associated with wellbore $(i,j)$.

To keep symmetry, need modify equations at nodes to the east, south, west and north of wellbore $(i,j)$:

East node $(i_e, j_e) = (i, j + 1)$

$$
K_{i_e, j_e}^w = K_{i-1, j}
$$

$$
K_{i, j_e + 1}^w = K_{i, j+1}
$$

$$
K_{i, j_e - 1}^w = K_{i, j-1}
$$

(B.I.161)

South node $(i_s, j_s) = (i - 1, j)$

$$
d_{yp} = \frac{dy(i_s)}{2}
$$

$$
d_{yy} = d_{ym} + \frac{d_{yp}}{2}
$$

$$
d_{i_s + \frac{1}{2}, j_s} = K(i_s, j_s)
$$

(B.I.162)

West node $(i_w, j_w) = (i, j - 1)$

$$
d_{xp} = \frac{dx(j_w)}{2}
$$

$$
d_{xx} = d_{xp} + \frac{d_{xm}}{2}
$$

$$
d_{i_w, j_w + \frac{1}{2}} = K(i_w, j_w)
$$

(B.I.163)
Appendix B. FINITE DIFFERENCE EQUATIONS OF SLUG TEST

North node \((i_n, j_n) = (i + 1, j)\)

\[
\begin{align*}
  d_{ym} &= \frac{dy(i_n)}{2} \\
  d_{yy} &= d_{ym} + \frac{dy}{2} \\
  d_{i_n-\frac{1}{2},j_n} &= K(i_n, j_n)
\end{align*}
\]

(B.I.165)

Coefficient definitions from (B.I.160) to (B.I.165) reflect attempts to approximate the inner boundary which encloses a non-aquifer area filled with water. Comparing the corresponding coefficient terms in (B.I.156) and (B.I.159), it immediately shows that if setting \(S_s\) in (B.I.156) equal to 1 and conductivity at block \((i, j)\) several magnitude greater than that of surrounding blocks, (B.I.156) collapses to (B.I.159). This is a much easier way to implement inner boundary. Another big advantage of this technique is that it makes it feasible to assign any number of blocks as wellbore so as to get a finer wellbore discretization, thus reducing numerical error, see Section 2.2.1.

**B.II 2D FDM in radial coordinates**

Mesh-centered finite difference equation, Figure B.II.82,

\[
\frac{1}{r_i \Delta i} \left( K_{i+\frac{1}{2},j} \frac{r_{i+\frac{1}{2}}}{r_{i+1} - r_i} \frac{h_{i+1,j}^m - h_{i,j}^m}{\Delta i} - K_{i-\frac{1}{2},j} \frac{r_{i-\frac{1}{2}}}{r_i - r_{i-1}} \frac{h_{i,j}^m - h_{i-1,j}^m}{\Delta i} \right)
\]

\[
+ \frac{1}{r_i^2 \Delta \theta} \left( K_{i,j+\frac{1}{2}} \frac{h_{i,j+1}^m - h_{i,j}^m}{\Delta \theta} - K_{i,j-\frac{1}{2}} \frac{h_{i,j}^m - h_{i,j-1}^m}{\Delta \theta} \right)
\]

\[
= S_s \frac{h_{i,j}^m - h_{i,j}^{m-1}}{\Delta t}
\]

(B.II.166)

Multiplying by \(\frac{1}{r_i \Delta i}\) to obtain symmetry,
Appendix B. FUTURE DIFFERENCE EQUATIONS OF SLUG TEST

Figure B.II.82: Mesh centered finite difference in radial coordinates.

\[
\begin{align*}
\left( \frac{K_{i+\frac{1}{2},j} r_{i+\frac{1}{2}}}{l_{i+\frac{1}{2}}} + \frac{K_{i-\frac{1}{2},j} r_{i-\frac{1}{2}}}{l_{i-\frac{1}{2}}} \right) + \frac{\Delta l_i}{r_l \Delta \theta^2} \left( K_{i,j+\frac{1}{2}} + K_{i,j-\frac{1}{2}} \right) + \frac{S_s r_l \Delta l_i}{\Delta t} \right] \frac{h_{i,j}^m}{h_{i,j}^m} \\
- \frac{K_{i+\frac{1}{2},j} r_{i+\frac{1}{2}}}{l_{i+\frac{1}{2}}} h_{i+1,j}^m - \frac{K_{i,j+\frac{1}{2}} \Delta l_i}{r_l \Delta \theta^2} h_{i,j+1}^m - \frac{K_{i-\frac{1}{2},j} r_{i-\frac{1}{2}}}{l_{i-\frac{1}{2}}} h_{i-1,j}^m - \frac{K_{i,j-\frac{1}{2}} \Delta l_i}{r_l \Delta \theta^2} h_{i,j-1}^m \\
= \frac{S_s r_l \Delta l_i}{\Delta t} h_{i,j}^{m-1} \quad (B.II.167)
\end{align*}
\]

where,

\[
\begin{align*}
K_{i,j+\frac{1}{2}} &= \frac{2}{K_{i,j}} \frac{K_{i,j+1} K_{i,j}}{K_{i,j+1}} \\
K_{i,j-\frac{1}{2}} &= \frac{2}{K_{i,j}} \frac{K_{i,j-1} K_{i,j}}{K_{i,j-1}} \\
K_{i+\frac{1}{2},j} &= \frac{(\Delta l_{i+1} + \Delta l_i) K_{i+1,j} K_{i,j}}{\Delta l_{i+1} K_{i,j} + \Delta l_i K_{i+1,j}} \\
K_{i-\frac{1}{2},j} &= \frac{(\Delta l_i + \Delta l_{i-1}) K_{i,j} K_{i-1,j}}{\Delta l_i K_{i-1,j} + \Delta l_{i-1} K_{i,j}}
\end{align*}
\]
Finite differencing wellbore equation, at \((i, j) = (1, j)\), Figure B.II.83,
Appendix B. FINITE DIFFERENCE EQUATIONS OF SLUG TEST

\[ r_c \Delta \theta \frac{h_{i+1,j}^m - h_{i,j}^m}{l_{i+\frac{1}{2}}} + r_a \frac{K_{i,j+\frac{1}{2}}^m}{r_c} \frac{h_{i,j+1}^m - h_{i,j}^m}{r_c \Delta \theta} - r_a \frac{K_{i,j-\frac{1}{2}}^m}{r_c} \frac{h_{i,j}^m - h_{i,j-1}^m}{r_c \Delta \theta} = \frac{\Delta \theta}{2\pi} \pi r_c^2 \frac{h_{i,j}^m - h_{i,j}^{m-1}}{\Delta t} \]  

(B.II.169)

Multiplying by \( r_b \), rearrange, coupling to domain flow equation,

\[ \left[ \frac{r_b r_c}{2\Delta t} \right] - \frac{r_b}{l_{i+\frac{1}{2}}} K_{i+\frac{1}{2},j}^m + \frac{r_a r_b}{r_c^2} \Delta \theta^2 (K_{i,j+\frac{1}{2}} + K_{i,j-\frac{1}{2}}) \right] h_{i,j}^m \]

\[ - \frac{r_b}{l_{i+\frac{1}{2}}} K_{i,j+\frac{1}{2}}^m h_{i+1,j}^m - \frac{r_a r_b}{r_c^2} \Delta \theta^2 h_{i,j+1}^m - \frac{r_a r_b}{r_c^2} \Delta \theta^2 h_{i,j-1}^m \]

\[ = \frac{r_b r_c}{2\Delta t} h_{i,j}^{m-1} \]  

(B.II.170)

where,

\[ r_a = \frac{r(2) + r(1)}{2} \]

\[ r_b = \frac{r(2) - r(1)}{2} \]

\( K_{i,j+\frac{1}{2}}, K_{i,j-\frac{1}{2}}, K_{i+\frac{1}{2},j} \) have same definition as (B.II) except \( \Delta l_i = r_b \) which means that wellbore is a non-aquifer area.

B.III 1D FDM in radial coordinates

The finite difference equation,

\[ \frac{1}{rl_i \Delta l_i} \left( r_{i+\frac{1}{2}} \frac{h_{i+1}^m - h_{i}^m}{l_{i+\frac{1}{2}}} - r_{i-\frac{1}{2}} \frac{h_{i}^m - h_{i-1}^m}{l_{i-\frac{1}{2}}} \right) = \frac{S_s}{K} \frac{h_{i}^m - h_{i}^{m-1}}{\Delta t} \]  

(B.III.171)
Appendix B. FINITE DIFFERENCE EQUATIONS OF SLUG TEST

Rearrange the above,

\[
\frac{1}{r_l \Delta l_i} \left[ \left( \frac{r_{i+\frac{1}{2}}}{l_{i+\frac{1}{2}}} + \frac{r_{i-\frac{1}{2}}}{l_{i-\frac{1}{2}}} + \frac{r_i \Delta l_i \Delta t}{\Delta t} \right) h_i^m - \frac{r_{i+\frac{1}{2}}}{l_{i+\frac{1}{2}}} h_{i+1}^m - \frac{r_{i-\frac{1}{2}}}{l_{i-\frac{1}{2}}} h_{i-1}^m \right] = \frac{S_s}{K \Delta t} h_i^{m-1} \quad (B.III.172)
\]

where,

\[
\begin{align*}
l_{i+\frac{1}{2}} &= r(i + 1) - r(i) \\
l_{i-\frac{1}{2}} &= r(i) - r(i - 1) \\
r_{i-\frac{1}{2}} &= r(i - 1) + r(i) \quad \frac{2}{2} \\
r_{i+\frac{1}{2}} &= r(i) + r(i + 1) \quad \frac{2}{2} \\
\Delta l_i &= r(i + 1) - r(i - 1) \quad \frac{2}{2} \\
r_l &= r_{i-\frac{1}{2}} + \Delta l_i \quad (B.III.173)
\end{align*}
\]

Finite differencing wellbore equation,

\[
2\pi r_c K \frac{h_{i+1}^m - h_i^m}{l_{i+\frac{1}{2}}} = \pi r_c^2 \frac{h_i^m - h_i^{m-1}}{\Delta t} \quad (B.III.174)
\]

Rearrange, get

\[
\left(2K + \frac{r_c l_{i+\frac{1}{2}}}{\Delta t} \right) h_i^m - 2K h_{i+1}^m = \frac{r_c l_{i+\frac{1}{2}}}{\Delta t} h_i^{m-1} \quad (B.III.175)
\]

where \(l_{i+\frac{1}{2}}\) has same definition as \(B.III.173\).