A VARIATIONAL FORMULATION AND FINITE ELEMENT TYPE SOLUTION FOR FREE OSCILLATIONS OF A ROTATING LATERALLY HETEROGENEOUS EARTH

by

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ABSTRACT

A variational type, finite element method is proposed for the study of normal modes of a rotating, laterally heterogeneous Earth. This method is very powerful and, in theory, overcomes all the limitations of the perturbation method. However, the present size of computer memory puts severe limitations on the use of this method.

The Lagrangian energy integral is derived for a general configuration of a perfectly elastic continuum in both spherical and oblate ellipsoidal coordinate systems. The assumed solutions for the displacements and the perturbation of the gravitational potential are formed by tensor products of the cubic Hermite basis functions of three coordinate variables. The undetermined coefficients of these assumed solutions are solved by the minimization of the Lagrangian energy integral by a Rayleigh-Ritz technique.

The results of the numerical solution of this approach show that (1) this algorithm is very efficient and promising in one-dimensionalized normal mode problems, (2) that the degenerate frequency of $nL$ splits into $(2L+1)$ components, even for the non-rotating, homogeneous spherical Earth model, due to the interpolation scheme of the azimuthal basis functions and (3) because the numerical spectral splitting is very large, the effects of self-gravitation and rotation cannot be examined clearly in this study. However, lateral heterogeneities which break the symmetry of the physical shape of the Earth greatly affect the normal mode spectra.
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CHAPTER 1

INTRODUCTION

In general there are two methods of dealing with normal mode differential equations of a dynamic system. In the first and more ordinary one, the differential equations are solved by assuming a mathematically reasonable solution for the given coordinate system and matching the boundary conditions to solve for the undetermined coefficients. Lamb (1882), and subsequently Love (1911), Takeuchi (1950), Alterman et al. (1958) and Backus and Gilbert (1967) solved the normal mode problems of Earth by this method.

In the second method, which was invented by Lagrange (1736 - 1813), and developed and applied by Gauss, Green, and others, the equations are expressed as the conditions that a certain quadratic function of the differential coefficients of the first order, integrated over the system, shall retain a stationary value when small variations are imposed on the variables. In a dynamical system, this function is the kinetic and potential energy (gravitational, elastic, rotational etc.) per unit volume. Its integral is the total energy of the system. When its stationary value is known, the equations of the system are obtained at once by application of the variations. The second procedure is therefore an easy and straightforward one, and has been applied to normal mode problems for the Earth by Rayleigh (1877), Stonley (1926), Jeffreys (1935), Backus and Gilbert (1967) and Wiggins (1973, 1976 (in press)). The scope of this thesis is principally concerned with the
numerical variational solution for normal modes of the most general and simple Earth model.

Since Lamb(1882) solved the normal mode problem of an elastic sphere and Benioff(1954) first suggested the free oscillation of the Earth from his observation of the Khamchatka earthquake of 4 Nov. 1952, there have been numerous theoretical solutions of different Earth models to interpret exactly the observations of normal modes of Earth.

1.1 Review Of Previous Works

Lamb(1882) was the first to examine theoretically the nature of the fundamental modes of vibration of an elastic sphere by solving the equation of motion with surface boundary conditions. Because of the originality and completeness of his paper, it is still referenced in recent text books. He showed how spheroidal and toroidal oscillations can be solved and estimated the gravest fundamental mode period of the Earth as 1hr. 18m.

Love(1911) considered a compressible and self-gravitating Earth and showed that gravity not only acts as a restoring force to shorten the eigenperiods but also causes a large initial state of stress throughout the Earth. Hoskin(1920) extended Love's approach to a radially heterogeneous Earth, taking the elastic parameters to be simple algebraic functions of radius.

Sezawa(1927) considered the propagation of Rayleigh waves on spherical surfaces, which correspond to the high order spheroidal oscillations.

Takeuchi(1950) solved the equation of motion to obtain
tidal deformations for the two Bullen's Earth models (1936, 1940). This was the first time ever an Earth model was used to calculate and to predict a geophysical phenomenon. His computation was performed with a desk calculator and the results were compatible with the observations.

Benioff (1954) first reported the actual observation of the free oscillation of the Earth after Khamchatka earthquake of 4 Nov. 1952.

Following this first observation, Alterman et al. (1958) made a most comprehensive study of free oscillations and earth tides for three Earth models: Bullen's model B and Bullard's models I and II. Their results, computed using the first generation of electronic computers, were in good agreement with Benioff's observations within the observational error. The normal mode periods of order up to 4 and the Love numbers \( h, k \) and \( \alpha \) are also calculated for various depths of earthquake sources for the Earth models. The possible effect of lateral heterogeneity of the mantle and the problem of core oscillations were also mentioned in their report.

Gilbert and MacDonald (1960) considered two Gutenberg and Jeffreys-Bullen Earth models and calculated the free periods of toroidal oscillations for various types of earthquake sources. They used an extended method of the Thomson-Haskell matrix approach. They also considered the correspondence between free oscillation and ray theory.

In 1961 Benioff, Press and Mith verified their previous report of observing the free oscillations of the Earth and also reported the splitting of spectral peaks from their analysis of
the strain and pendulum seismograph recordings made in California and Peru from the Great Chilean earthquake of 1960 May 22.

Backus and Gilbert (1961) and Pekeris et al. (1961) simultaneously considered the effect of Earth's rotation on the normal modes of the Earth, after Benioff et al. (1961) reported their observation of spectral doublets. Both these authors applied perturbation theory on the assumption that the rotation of the Earth is much smaller than the degenerate normal mode angular frequencies and they showed that the degenerate normal mode frequency in the absence of rotation was resolved by small rotational velocities into \((2\ell+1)\) modes where \(\ell\) is the angular order. Since the longest seismic oscillation of the Earth has a period of less than an hour this theory is applicable to all orders of free oscillations. Even though the observed spectral splitting of normal modes differed slightly depending on the seismograms used, both Backus and Gilbert (1961) and Pekeris et al. (1961) obtained exactly the same theoretical results. They also estimated the relative amplitudes of the \(m=(2\ell+1)\) modes for \(\ell=2\) and \(\ell=3\) to compare their results with the observations.

MacDonald and Ness (1961) reviewed the published normal mode observations and considered the effects of rotation by using a perturbation method. They also concluded that rotation removes the degeneracy and results in a splitting of spectral peaks of order \(\ell\) into \((2\ell+1)\) multiplets. Furthermore they predicted that rotation couples the toroidal and spheroidal oscillations.

Sato and Usami (1962) numerically solved the characteristic equation for the oscillation of a homogeneous elastic sphere as
a basic data for their later study of laterally heterogenous and spheroidal Earth models.

Jobert (1962) considered theoretical seismograms for torsional oscillations using Gutenberg's continental and Dorman's 8009 oceanic Earth models.

Usami and Sato (1962) considered torsional free oscillations of a homogeneous elastic spheroid. They concluded that ellipticity causes the $T^m_\ell$ mode to split into $(\ell+1)$ multiplets, while the Earth's rotation causes it to split into $(2\ell+1)$ multiplets.

Alsop (1963) calculated the periods of spheroidal oscillations numerically for four Earth models and also considered the phase and group velocities for the first two shear modes.

Usami, Sato, and Landisman (1965) and Landisman et al. (1970) considered the theoretical seismograms for both simplified and realistic Earth models and examined the relations between free oscillations and travelling waves. They also studied the equivalence of mantle and crustal Rayleigh waves to the spheroidal free oscillations and the equivalence of G-waves and Love waves to toroidal free oscillations.

Saito (1967) considered the radiation patterns of surface waves and free oscillations for vertically heterogeneous elastic media with arbitrary sources.

Backus and Gilbert (1967) described a geophysical inversion technique for exploring the collection of all Earth models which fit given gross Earth data. They applied the theory to the normal modes of the elastic-gravitational oscillation of the
Earth by utilizing Rayleigh's variational principle. This important work not only opened a new field of geophysical inversion, but also became a stepping stone for the later studies of normal modes by Dahlen (1968, 1972 etc.) and Luh (1973).

Wiggins (1968) considered Birch's Earth model and computed terrestrial variational tables of normal mode periods, rotational splitting parameters and energy functions. He also computed the overtone eigenfunctions for several orders to make the table more complete.

Dahlen (1968) considered the normal modes of a slowly rotating elliptical Earth. Using Rayleigh's variational principle (Backus and Gilbert (1967)) he computed approximate normal mode eigenfrequencies for slowly rotating and slightly aspherical and anisotropic Earth models. He found that for an arbitrary spheroidal or toroidal multiplet, the central member (m=0) of the multiplet is shifted slightly in frequency and that the other members of the multiplet are split apart asymmetrically by the effects of the Earth's rotation and ellipticity. He also reported that for the angular order numbers of $10 < \ell < 25$ the fundamental spheroidal eigenfrequency is very nearly equal to the fundamental toroidal eigenfrequency and it would be necessary or at least more convenient to use a quasi-degenerate perturbation theory to examine the effects of rotation and ellipticity on these multiplets. Later (1969) he examined more closely these modes ($10 < \ell < 25$) and $s_2 S_1$ and $s_3 S_3$. He found that some of the normal modes of a slowly rotating, slightly elliptical Earth model may consist of about half spheroidal type motion and half toroidal type motion. In
conclusion, he conveniently divided the normal modes into two categories. The first category included all those normal modes for which it is sufficient to neglect the effect of any lateral heterogeneity in the Earth with respect to ellipticity. Every normal mode in this category can be characterized by a single value of \( m \). The second category includes all those normal modes for which it is not possible to neglect the lateral heterogeneity. At least all the fundamental normal mode multiplets with angular order greater than \( \ell = 22 \) are included in this category. He said it is not possible to characterize normal modes in this second class by a single value of \( m \).

Dahlen did a very thorough job on the perturbation solution for the effects of the rotation and ellipticity of Earth.

Zharkov and Lyubimov (1970a) used the perturbation method to construct a theory of torsional oscillations of the Earth in a weakly spherically asymmetric Earth model. They expanded the small parameter increments \( \delta \mu \) and \( \delta \rho \) (shear modulus and density) in terms of spherical harmonics and determined the dependence of the variation of \( \Delta \omega \) on \( \delta \mu \) and \( \delta \rho \). They estimated \( \Delta \omega \) without using an electronic computer. Later (1970b) they applied the same technique to develop a theory of spheroidal vibrations of the Earth for a weakly spherically asymmetric Earth model and concluded, as Dahlen (1968) did, that the spreading of degenerate frequencies caused by the Earth's rotation is of order \( 1/\ell \) and decreases rapidly with increasing \( \ell \). For large \( \ell \) the spreading is determined mostly by the lateral heterogeneity of Earth.

Madariaga (1972) considered the effects of lateral
heterogeneities on the torsional free oscillations by means of the theory of perturbation of degenerate eigenfrequencies of a spherically averaged Earth. Like Zharkov and Lyubimov (1970a) he expanded the lateral heterogeneities in a series of spherical harmonics and deduced the selection rules by which the expansions of lateral heterogeneity selectively couples the eigenfunctions. Later, Madariaga and Aki (1972) applied the above solution to actual Earth models with continent-ocean and plate tectonic heterogeneity. They obtained multiplet widths less than 1% for $\lambda = 40$. This is much less than some observed multiplet widths of 1.7% of the degenerate frequency (Smith 1966). They also discussed some of the difficulties in the inversion of the split spectral peaks.

Recently Luh (1973) applied Rayleigh's variational principle to calculate the first order corrections to the eigenfrequencies of a spherically symmetric, non-rotating and elastic Earth model due to the lateral variations in the real Earth. His work is an extension of Madariaga's (1972) result to the remaining problem of quasi-degeneracy of the normal modes for a laterally heterogeneous Earth. Later Luh (1974) extended his research to a rotating laterally heterogeneous Earth model and obtained somewhat different computed values from those of Madariaga (1972). In a qualitative nature their results agreed. He concluded that, when $\lambda$ is small, second order rotational corrections may become comparable to first-order corrections due to the lateral perturbations and, as higher-order corrections are too formidable to solve, he recommended a variational method.
Wiggins(1973, 1976) applied an one dimensional finite element type Rayleigh-Ritz method with cubic Hermite polynomials to develop an algorithm to solve for normal mode periods, partial derivatives and rotational splitting parameters. However he did not consider the effects of ellipticity nor lateral heterogeneity.

Buland and Gilbert (1975, personal communication) have reported they are applying a variational method (1-D finite element method) to solve higher order normal mode problems.

Smylie (1975, personal communication) is also considering this variational method to solve core oscillation problem for rotating radially heterogeneous Earth models.

Dahlen (1976) studied the effects of lateral heterogeneity, rotation, and hydrostatic ellipticity of the Earth on the normal modes. Even though he expressed great uncertainties due to the treatment of the hydrostatic ellipticity, he concluded that there must exist significant lateral heterogeneities in the Earth's lower mantle and that the lateral heterogeneity in the upper mantle associated with the differences in structure beneath oceans and continents must persist to depths of at least a few hundred kilometers.
1.2 Review Of Perturbation Methods For The Effects Of Rotation, Ellipticity, And Lateral Heterogeneity.

The normal mode problem can be solved by the separation of the equations of motion, if the material properties and parameters vary only with radius, and retain the original spherical symmetry. Boundary conditions have to be satisfied at all internal boundaries as well as at the free surface boundary. However, if the material properties and parameters vary in such a way that the spherical symmetry of the system breaks down, the conventional method of solution cannot be applied to the system. The variations of material properties and parameters in the Earth are small and the classical perturbations have been applied to find the effects of these spherically unsymmetric material properties on the normal mode eigenfunctions of a spherically symmetric Earth model.

As briefly surveyed in the previous section, Backus and Gilbert (1961) and Pekeris et al. (1961) solved for the effect of rotation on the normal mode periods of Earth by a perturbation method assuming that the Earth's rotation is much smaller than the unperturbed normal mode frequency of Earth. They expanded the perturbed normal mode frequency by

\[ \frac{\omega^p_k}{\omega_k} = 1 + \delta_1 \left( \frac{\Omega}{\omega_k} \right) + \delta_2 \left( \frac{\Omega}{\omega_k} \right)^2 + \ldots \]

\[ \left( \frac{\Omega}{\omega_k} \right) \ll 1.0 \]

where \( \omega_k \) is the frequency of the unperturbed mode, \( \Omega \) is the angular velocity of rotation of Earth, and \( \omega^p_k \) is the non-degenerate perturbed normal mode frequency. They calculated the effects of rotation correct to second order in \( \Omega \) and derived the splitting parameters correct to zeroth order in \( \left( \frac{\Omega}{\omega_k} \right) \).
Dahlen (1968) applied the perturbation of parameters in Rayleigh's variational principle to investigate a rotating ellipsoidal Earth model. He obtained angular frequency multiplets of rotating ellipsoidal Earth model, correct to first order in ±(r), the ellipticity of Earth, and correct to second order in \( \frac{\Omega}{\omega_e} \);

\[
\frac{n\Omega_e}{\omega_e} = 1 + n\lambda_e + n\beta_e + n\gamma_e
\]

where \( n\lambda_e = n\lambda_e^r \left( \frac{\Omega}{\omega_e} \right)^2 + n\lambda_e^e \epsilon_a \)

\( n\beta_e = n\beta_e^r \left( \frac{\Omega}{\omega_e} \right)^2 \)

\( n\gamma_e = n\gamma_e^r \left( \frac{\Omega}{\omega_e} \right)^2 + n\gamma_e^e \epsilon_a \)

where the superscript \( r \) indicates a parameter depending only on the rotation \( \omega_e \) while the superscript \( e \) indicates a parameter depending on the ellipticity. For \( n\lambda_e^r, n\lambda_e^e, n\beta_e^r, n\gamma_e^r, \) and \( n\gamma_e^e \), see Dahlen (1968, Appendices A and B). Backus and Gilbert (1961) and Dahlen (1968) both concluded that for the lower order fundamental normal modes, at least \( \lambda < 25 \), the Earth's rotation and ellipticity are the dominant perturbing effects. They showed that the degeneracy of any multiplet \( nS_\lambda \) or \( nT_\lambda \) is in general completely removed for the rotating elliptic Earth. They also showed that the first order effect of ellipticity and the second order effect of rotation not only act to shift the entire multiplet but also cause the splitting of a multiplet to be asymmetrical.

Zharkov and Lyubimov (1970a, b), Madariaga (1972) and Madariaga and Aki (1972) applied the classical perturbation theory to study the effects of lateral inhomogeneity on the
normal mode solutions of Earth. They expanded the perturbation of the lateral heterogeneity in spherical harmonics:

\[
\delta \phi (r, \theta, \phi) = \sum_{s=1}^{\infty} \sum_{\ell=-s}^{s} \delta \phi_s^\ell (r) Y_s^\ell (\theta, \phi)
\]

and used the Wigner-Eckart theorem to simplify the complicated integrals and to compute the coupling coefficients between the normal modes and the expansion of the lateral heterogeneities. They showed that the expansion of the lateral heterogeneities in spherical harmonics permitted the separation of the radial and angular integrals of the scattering coefficients. From the angular integrals, they derived a selection rule which controls the coupling between the degenerate eigenfunctions and the lateral heterogeneities.

Following Dahlen (1968, 1969), Luh (1973) expanded the lateral heterogeneity as

\[
\delta \phi_j (r, \theta, \phi) = \sum_{LM} \delta \phi_{jLM} (r) Y_{LM} (\theta, \phi)
\]

where \( \delta \phi_j \) is \( \delta, \kappa, \mu \) and \( \delta \), for \( j=1, \ldots, 4 \). He then substituted this parameter perturbation into the first order perturbation equation to solve for the effects of lateral heterogeneity. He used the Wigner-Eckart theorem to simplify many complicated angular integrals involving the strain deviator tensor to more basic and manageable integrals. Later Luh (1974) included rotation and ellipticity to obtain the normal mode
solutions of a more general Earth model. For the case where the ellipticity $\xi(r)$ is caused by Earth's rotation and the Earth is in hydrostatic equilibrium, as Dahlen (1968) assumed, the perturbation is represented by

$$\delta f_j^s(r, \theta, \phi) = \left(\frac{4\pi}{5}\right)^{\frac{1}{2}} \left[ \frac{2}{3} r \xi(r) \frac{d f_i^j}{dr} - \frac{\sigma_2 r^2}{3} \delta_j^s \right] Y_{2,0}^s(\theta, \phi)$$

The ellipticity perturbation changes the locations of internal jump discontinuities in $(f, \kappa, \mu)$. So a spherical surface at $r=a$ becomes a surface of an ellipsoid with

$$r = a \left[ 1 - \left(\frac{4\pi}{5}\right)^{\frac{1}{2}} \frac{2}{3} \xi(a) Y_{2,0}(\theta, \phi) \right]$$

He then divided the Earth's surface into different subareas, to represent lateral inhomogeneities, beneath which $(f, \kappa, \mu)$ are functions of radius only. A surface function $f^s$ is introduced to describe each sub-area,

$$f^s(\theta, \phi) = \begin{cases} 1 & \text{on the } s^{th} \text{ subarea} \\ 0 & \text{otherwise} \end{cases} = \sum_{LM} f_{LM}^s Y_{LM}(\theta, \phi)$$

$$\delta f_j^s(r, \theta, \phi) = (\tilde{f}_j^s - \bar{f}_j^s)(r) f^s(\theta, \phi)$$

where $\tilde{f}_j^s$ and $\bar{f}_j^s$ denotes $f$, $\kappa$ and $\mu$ of the region and of the Earth model for $j=1,2,3$ respectively. Then the total perturbation used by Luh (1974) is, besides rotation, then
where

\[ \tilde{P}_{LM} (r) = \sum_{s} (\tilde{p}_{j}^s(r) - \bar{p}_{j}(r)) \gamma_{LM} (\theta, \phi) \]

Luh's Earth model MOD508 (obtained by inverting 508 gross Earth data) with the above perturbation representation can be said to resemble the real Earth more closely than any other Earth models studied before.

From the perturbation solution of the ordinary degenerate normal modes the following conclusions can be reached: 1) the effect of Earth's rotation is insignificant whenever \( \frac{n}{n+2} \) is small, 2) the effect of lateral heterogeneity on low \( n \) and low \( l \) is likewise insignificant and 3) the ellipticity corrections cannot be neglected at high \( l \) modes. For the closely coupled modes the quasi-degenerate solutions indicate that 1) the energy of each quasi-degenerate level is nearly evenly divided between two multiplets when their combined ordinary degenerate levels are crowded, 2) the strength of coupling is not related to the proximity of the unperturbed frequencies, but depends greatly on the size of the matrix elements that render coupling, 3) for closely coupled cases the differences between the ordinary degenerate perturbation splitting and those of quasi-degenerate lines are highly striking as predicted, 4) the Coriolis coupling of the form \( S_{x} - \lambda T_{2h} \) for small \( n \) and \( n' \) \( (n, n' < 5) \) is very strong and 5) most importantly, the usual mode designation loses it's
meaning as well as its identity when the coupling is strong.

1.3 Scope Of Discussion

As reviewed in the previous sections, the Earth models in normal mode studies have been improved greatly since Lamb (1882) first calculated the free oscillation periods of a homogeneous solid sphere. The application of the perturbation methods during the last decade also helped greatly in developing a more realistic understanding of the normal mode phenomenon. In summary, the most up to date and realistic Earth model we have considered in our theoretical study of free oscillations is,

1) slowly rotating \( \frac{\Omega}{\omega_0} \ll 1 \),
2) a slightly oblate ellipsoidal \( (e(a) \simeq 1/298.3) \),
3) radially heterogeneous with discontinuities in \( J \), \( K \), and \( \mu \) at arbitrary depths (including liquid and solid cores),
4) slightly laterally heterogeneous at the Earth's surface (continent and ocean lateral heterogeneity and plate tectonic lateral heterogeneity up to depths of 400 km),
5) and slightly laterally heterogeneous at the core-mantle boundary.

If the normal mode eigenfrequencies and eigenfunctions for the above Earth model are calculated by parameter perturbation technique using Rayleigh's variational principle (Luh (1974)), the following results can be summarized:

1) For the case of a spherical, non-rotating, elastic and isotropic Earth model, the \((2\ell+1)\) spheroidal modes
$S^m_{\ell}$ have a degenerate frequency of $\omega^S_{\ell}$ while the
$(2 + 1)$ toroidal modes $T^m_{\ell}$ have the degenerate
frequency of $\omega^T_{\ell}$.

2) The degeneracy of any multiplet $S^m_{\ell}$ or $T^m_{\ell}$ is
completely removed by the perturbing effects of
rotation, ellipticity or lateral heterogeneities.

3) For the range of angular order number from about
$\ell=10$ to about $\ell=25$, the fundamental spheroidal
eigenfrequency $\omega^S_{\ell}$ is very nearly equal to the
fundamental toroidal eigenfrequency $\omega^T_{\ell}$. In such
cases, the ordinary degenerate perturbation method is
not appropriate and it becomes necessary to use a
quasi-degenerate perturbation theory to examine the
effects of rotation, ellipticity and lateral
heterogeneity on the multiplets.

4) For the lower order (small $\ell$) fundamental modes, the
Earth's rotation and ellipticity are the dominant
perturbing effects. Every normal mode in this
category can be characterized by a simple $m$; to zeroth
order every such mode is either spheroidal $S^m_{\ell}$ or
toroidal $T^m_{\ell}$, or mixed $S^m_{\ell} - T^m_{\ell}$ or $S^m_{\ell} - S^m_{\ell}$, depending
on the quasi-degeneracy.

5) For the higher order normal modes, (angular order
greater than about $\ell=22$) the effects of lateral
heterogeneity is the most dominant perturbation. In
general, it is not possible to characterize the normal
modes by a single value of $m$ in this case.

However, the following difficulties and limitations are
also encountered:

1) When two multiplets nearly coincide in frequency, ordinary degenerate perturbation methods must be replaced by first order quasi-degenerate perturbation methods, introducing mathematical and computational complications.

2) When the effects of rotation becomes dominant, the first order perturbation calculations become insufficient and the higher order terms must be considered. This would introduce a great deal of horrendous algebraic manipulations and Luh (1974) recommended the use of a variational method for such cases.

3) If the condition \( \frac{\Omega}{\omega_2} << 1.0 \) cannot be satisfied as in the study of core oscillations, where \( \omega_2 \) is comparable to \( \Omega \) (Crossley (1975)), the above perturbation formalism fails.

4) The selection rule (Zharkov and Lyubimov (1970a,b), Madariaga (1972) and Luh (1973)) indicates that only the even angular orders of the lateral heterogeneity perturbation contribute to the splitting within a multiplet. The effects of the odd angular orders in the perturbation are not completely understood either theoretically or observationally.

5) As the overtone number \( n \) of a mode increases, the radial eigenfunction penetrates deeper into the Earth's interior and deep lateral heterogeneities become important. Evidence of the lateral
heterogeneities below the upper mantle are reported by Davies et al. (1972), Kanasewich et al. (1973) and Jordan et al. (1974). The lateral heterogeneities reported are rather arbitrary with respect to the Earth's surface lateral heterogeneity. Therefore, the lateral heterogeneity assumed at the core-mantle boundary, as considered by Luh (1974), is not most appropriate because he restricted the lateral heterogeneities at the core-mantle boundary to be a projection of the free surface lateral heterogeneities. Most recently Dahlen (1976) reported from his research that there exists a significant lateral heterogeneity in the Earth's lower mantle and that the Earth's surface lateral heterogeneities must extend at least a few hundred kilometers. He reached these two conclusions because his most up-to-date perturbation scheme, including the effects of Earth's surface lateral heterogeneity, rotation and the hydrostatic ellipticity, could not interpret the observed normal mode data very satisfactorily.

6) The usual mode designation and its identity lose their meaning for the cases of strongly coupled modes, and the interpretation of the normal modes by the modes of the spherically symmetric, non-rotating, isotropic Earth model is not proper.

These difficulties and limitations are, in general, due to complicated mathematical and computational reasons, and also to the assumptions made for the application of the perturbation
In the following, a new approach to solving the normal mode problem is proposed to overcome -at least in theory- all the above mentioned difficulties and to develop a truly general Earth model. This new variational type finite element method has its own difficulties (as will be explained in Chapter 3 and Chapter 6) but it is potentially more powerful and has fewer limitations than the above surveyed perturbation methods.

In chapter 2, after a brief review of the theoretical solution of normal modes, the one dimensional variational Lagrangian integral form of the equations of motion is derived. Then the derivation is extended to a three dimensional variational Lagrangian integral in a spherical coordinate system.

In Chapter 3, the Lagrangian integrals obtained in Chapter 2 are transformed into matrix forms for the numerical computations in Chapter 5.

Chapter 4 explains how the ellipsoidal Earth may be treated in the variational solution. Computational results and error analyses are included in Chapter 5, and the conclusions are presented in Chapter 6.
In this chapter, 1) the classical normal mode period equations, which will be used for test purposes, 2) the Lagrangian integrals for the one dimensional case and 3) the extension to a general three dimensional case are derived for the numerical calculations in the next chapter. The Earth model assumed is spherical, perfectly elastic, self-gravitating and radially heterogeneous. The convenient extensions for lateral heterogeneities and rotation are shown.

2.1 Generalized Equation Of Motion.

Let the Earth model consists of a self-gravitating, perfectly elastic continuum occupying an arbitrary bounded volume $V$ with surface $\partial V$. Assume further that it has a steady angular velocity of rotation $\Omega$ about its centre of mass, that $\rho_0(r)$ denotes the density, $T_0(r)$ denotes initial static stress field, and $\Phi_0(r)$ is the gravitational potential of the body occupying the volume $V$. If we assume a small time-dependent particle displacement field $u(r,t)$ for a small disturbance, this displacement will change both the volume $V(t)$ and the surface $\partial V(t)$ of the given continuum and will be accompanied by perturbations $\rho_1(r,t)$ in the density, $\Phi_1(r,t)$ in the gravitational potential, and $T_1(r,t)$ in the stress tensor at a fixed location.
Figure 1. shows the notations and the Earth model in a spherical coordinate system.

Figure 1  Spherical Coordinate System with variables used in this thesis.
The usual Lagrangian convention is to label a material particle by its location \( \bar{r} \) at time \( t=0 \). Then \( T_\sigma (r, t) + T_\varepsilon (r, t) \) is the value of the stress tensor at \( r \) in space at time \( t \). To first-order in \( u \), the incremental stress \( T(\bar{r}, t) \) is

\[
T = T_\varepsilon + \hat{u} \cdot \nabla T_\sigma
\]

and from constitutive relations

\[
T_{ij} = \sum_{kl} \sigma_{ij} \delta_{kl} u_k
\]

where \( \sigma_{ij} \) is the linear isentropic elastic coefficient (fourth-order tensor).

In general \( C_{ijkl} \) which is the linear elastic coefficient tensor with twenty one independent components, i.e.,

\[
C_{ijkl} = (\kappa - \frac{2}{3} \mu) \delta_{ij} \delta_{kl} + \mu (\sigma_{ik} \delta_{jl} + \sigma_{il} \delta_{jk})
\]

where \( \kappa(r) \) is isentropic bulk modulus and \( \mu(r) \) is the isentropic rigidity, then, \( T_{ij} \) is

\[
T_{ijkl} = C_{ijkl} \sigma_{kl} + \frac{1}{2} (T_{kl}^\circ \sigma_{kl}) \delta_{ij} - \frac{1}{2} T_{ij}^\circ \sigma_{kl} + \omega_{ik} T_{kj}^\circ - T_{ik}^\circ \omega_k
\]

and for the special case of \( T_\sigma (r) \) being a purely hydrostatic stress

\[
T_\sigma (r) = -p_0(r) I_0
\]
Then the most general and exact equations governing the motion about the equilibrium configuration are (after Dahlen (1972))

\[
(p_0 + p_i) \left[ \frac{D^2 u}{Dt^2} + 2 \nabla \times \frac{Du}{Dt} \right] = \nabla \cdot (T + T_0) - (p_0 + p_i) \nabla \left( \Phi_0 + \psi + \Phi_1 \right) + f
\]  

(7)

where \( f(r,t) \) is an external body force per unit volume, \( \frac{D}{Dt} \) is the Lagrangian time derivative, and

\[
\nabla^2 \Phi_1 = 4 \pi G f_1
\]  

(8)

In (7) the equilibrium condition for the initial steady state is

\[
p_0 \nabla \left( \Phi_0 + \psi \right) = \nabla \cdot T_0
\]  

(9)

\[
\nabla^2 \Phi_0 = 4 \pi G p_0
\]  

(10)

and the rotational potential due to the centripetal acceleration is

\[
\psi(r) = -\frac{1}{2} \left[ \nabla^2 r^2 - (\nabla \cdot \vec{r})^2 \right]
\]  

(11)

For the given equations of motion (7) and (8), if we neglect the second-order terms and subtract equation (9) from equation (7), we obtain

\[
T_{ij} = C_{ij} \kappa
\]  

(6)
Now from the linearized continuity equation

$$\dot{p} = - \nabla \cdot (p \ddot{u}) = (p \ddot{u}) \cdot \nabla$$  \hspace{1cm} (13)$$

we have

$$\rho \frac{\delta \ddot{u}}{\delta t} + 2 \rho \Omega \times \frac{\delta u}{\delta t} = \nabla \cdot T_E - \rho \nabla \dot{\Phi} - \rho (\dot{\Phi} + \psi) + f$$  \hspace{1cm} (14)$$

These relations will later be extended further to derive the Lagrangian integrals for the normal modes of the Earth.

2.2 Boundary Conditions

Biot (1965) showed how the variational principle and the boundary conditions may be deduced from general strain energy considerations. The approach shown below is similar to his derivation, but the general outline follows the approach by Dahlen (1972).

Consider an arbitrary small simply connected surface element $dA$ centered on an arbitrary material particle $r$ located on the undeformed surface $\partial V$ at time $t=0$. Let the unit outward normal to $dA$ be $\hat{n}(r)$ and denote the corresponding unit outward normal to the deformed surface $\partial V(t)$ by $\hat{n}(r,t)$ on $dA(t)$.

To first-order in $\ddot{u}$, the $\hat{n}(r,t)dA(t)$ on the deformed surface $\partial V(t)$ is related to $\hat{n}(r)dA$ on the corresponding undeformed surface $\partial V$ by the relation (Biot (1965), Dahlen (1972)).
The dilation term in (15) arises from the stretching of the surface element \( dA(t) \) and the other term arises from the deflection of the normal \( \hat{n}(r,t) \). Then, to first order, \( T_{ij} \) in equation (4) will become

\[
\tilde{T}_{ij} = C_{ijkl} \varepsilon_{kl} + \omega_{ik} \hat{w}_{kj} - \hat{T}_{ik} T_{kj} + \frac{1}{2} \alpha_{ik} \hat{T}_{kj}^2 + \frac{1}{2} \hat{T}_{ik} \alpha_{kl} \hat{T}_{lj}^2 - \hat{T}_{ik} \alpha_{kl} \hat{T}_{lj} \quad (16)
\]

and, in the special case of a hydrostatic initial stress \( T(r) = -p_o(r) I \),

\[
\tilde{T}_{ij} = T_{ij} - p_o ( \partial_{k} u_{j} + \partial_{j} u_{i} ) \quad (17)
\]

The equation of motion (14) will be, from (17) and (1),

\[
p_o \frac{\partial^2 u}{\partial t^2} + 2 p_o \nabla \times \frac{\partial u}{\partial t} = \nabla \cdot \tilde{T} - p_o \nabla \Phi - p_o u \cdot \nabla ( \nabla \cdot \Phi + \Phi ) \quad (18)
\]
1) $u(r,t)$ continuous

2) $\hat{n}(r) \cdot \bar{T}(r,t)$ continuous and $\hat{n}(r) \cdot \bar{T}(r,t) = 0$ on $\partial \Omega$

3) $\Phi_\alpha(r,t)$ continuous

4) $n(r) \cdot \nabla \Phi_\alpha(r,t) + 4\pi G \rho_o(r) \hat{n}(r) \cdot \bar{u}(r,t)$ continuous

These are the very general boundary conditions for the solid continuum case. If there is a liquid-solid boundary the boundary condition (19) of $u(r,t)$ is not necessarily continuous. Instead only $\hat{n} \cdot \bar{u}(r,t)$ is continuous and the shear stress vanishes across the boundary. Also, in the Alterman et al. (1959) and Wiggins (1968) type solution, the regularity of the integrated displacement solution becomes an important boundary condition at the centre of the Earth.

2.3 Classical Solution Of Normal Mode Problem Of Earth.

From the very general equations of motion (18), we will derive the solutions obtained by Alterman et al. (1959) to test the present variational solution.

In $V$, if a particle at a point $r$ in the deformed state was at $r-u$ in the equilibrium, the density at a point $r$ in the deformed state will be expressed by

$$\rho_0(\bar{r}) = \rho_0(\bar{r} - \bar{u}) - \rho_0(\bar{r}) \nabla \cdot \bar{u} = \rho_0 - \nabla \cdot (\rho_0 \bar{u})$$

(23)

to the first order of small quantities. Then, we have

$$\rho_0 = - \nabla \cdot (\rho_0 \bar{u})$$

(13)

and if we neglect the rotation of the Earth and the external
body force per unit volume \( f \), the equation of motion (18), to first order, becomes exactly identical to the ones given by Alterman et al. (1959) and Takeuchi and Saito (1972)

\[
P_0 \frac{\partial^2 u}{\partial t^2} = p_0 \nabla \Phi - g_0 \nabla \cdot (p_0 u) + \nabla \cdot \tilde{T}
\]

This can also be written as

\[
P_0 \frac{\partial^2 u}{\partial t^2} = p_0 \nabla \Phi - g_0 \nabla \cdot (p_0 u) + \left( \nabla \cdot \sigma_r, \nabla \cdot \sigma_\theta, \nabla \cdot \sigma_\phi \right) + \frac{1}{r} \left( \sigma_{\theta\theta} - \sigma_{\phi\phi}, \sigma_{r\theta} - \sigma_{r\phi} \cot \theta, \sigma_{r\phi} - \sigma_{\phi\phi} \tan \theta \right)
\]

For the torsional oscillation, the assumed solution of the form

\[
\begin{align*}
\phi_r &= 0 \\
\phi_\theta &= y_1(r) \frac{1}{2m_\theta} \frac{\partial Y_2(\theta, \phi)}{\partial \phi} e^{j\omega t} \\
\phi_\phi &= -y_1(r) \frac{\partial Y_2(\theta, \phi)}{\partial \theta} e^{j\omega t}
\end{align*}
\]

where \( Y_2(\theta, \phi) \) denotes a spherical harmonics of order \( l \). It is substituted into equation (25) and we have

\[
M\left( \frac{d^2 y_1}{dr^2} + \frac{2}{r} \frac{dy_1}{dr} \right) + \frac{d^2}{dr^2} \left( \frac{dy_1}{r} - \frac{y_1}{r} \right) + \left[ \omega^2 - \frac{L(l+1)}{r^2} \right] y_1 = 0
\]

which can be rewritten as two first-order differential equations, i.e.,

\[
\begin{align*}
\frac{dy_1}{dr} &= \frac{1}{r} y_1 + \frac{1}{M} y_2 \\
\end{align*}
\]
\[ \frac{d y'_r}{d r} = \left[ \frac{(l+1)(l-2) \mu}{r^2} - \omega^2 \frac{\epsilon}{r} \right] y'_r - \frac{3}{r} y'_\epsilon \]

where
\[ y'_\epsilon = \mu \left( \frac{d y'_1}{d r} - \frac{y'_1}{r} \right) \]

The equations in (28) are solved numerically by the University of British Columbia Computing Center subprogram DIFSY in Chapter 3 for comparison with the variational solutions. The boundary conditions to be satisfied in these cases are

\[ y'_\epsilon = \mu \left( \frac{d y'_1}{d r} - \frac{y'_1}{r} \right) = 0 \quad \text{at} \quad \begin{cases} r = a \\ r = b \end{cases} \quad (30) \]

where \( r = a \) is the free-surface and \( r = b \) is the core-mantle boundary.

For the non-rotating, perfectly spherical Earth model, the spheroidal oscillations are not coupled with the torsional oscillations and are identified with the following assumed solutions

\[ \begin{align*}
    u_r &= y'_1(r) \ Y_l(\theta, \phi) \ e^{i \omega t} \\
    u_\theta &= y'_3(r) \ \frac{\partial Y_l(\theta, \phi)}{\partial \theta} \ e^{i \omega t} \\
    u_\phi &= y'_3(r) \ \frac{1}{\sin \theta} \ \frac{\partial Y_l(\theta, \phi)}{\partial \phi} \ e^{i \omega t}
\end{align*} \quad (31) \]

The spheroidal oscillations involve dilation and cause perturbations in the gravitational potential \( \Phi(r,t) \), i.e.,
\[ \phi_1 = y_5(r) Y_l(\theta, \phi) e^{i\omega t} \]  

The non-vanishing dilation requires

\[
\begin{align*}
  y_2(r) &= (\lambda + \mu) + \frac{\lambda}{\mu} \left( 2y_1 - \lambda(l+1)y_3 \right) \\
  y_4(r) &= \mu \left( \frac{dy_5}{dr} - \frac{1}{r} y_3 + \frac{1}{r} y_1 \right)
\end{align*}
\]  

for the stress terms. The boundary condition for \( \phi_1 \) requires the continuity of

\[ \frac{dy_5}{dr} - 4\pi G \rho_0 y_1 \]  

at any surface of discontinuity and the sixth function is defined as

\[ y_6(r) = \frac{dy_5}{dr} - 4\pi G \rho_0 y_1 + \frac{l+1}{r} y_5 \]  

Then the following six first-order differential equations represent the spheroidally oscillating normal modes when they were solved simultaneously, i.e.,

\[ \frac{dy_1}{dr} = \frac{1}{\lambda + \mu} \left\{ y_2 - \frac{\lambda}{r} \left[ 2y_1 - \lambda(l+1)y_3 \right] \right\} \]
In normal mode research, Alterman et al. (1959) and, later, many others solved this system of first-order differential equations or slightly modified versions with the appropriate boundary conditions and the regularity of the integrated solutions at the centre of Earth. This has been the most popular method to solve the normal mode problem because of its relative simplicity. The above system of six first-order differential equations has to be
modified in the liquid outer core because of the vanishing of the shear modulus and the changes of the boundary conditions.

Now if we assume the Earth model is made of a set of uniform layers, as we assume in the actual numerical integration, then within each layer $\frac{du}{dr} = 0$ and equation (27) for toroidal oscillations reduce to

$$ r^2 \frac{d^2 y_1}{dr^2} + 2r \frac{dy_1}{dr} + \left[ r^2 \left( \frac{\omega^2 \rho}{\mu} \right) - \ell (\ell + 1) \right] y_1 = 0 $$

which is, in fact, a spherical Bessel differential equation with particular solutions,

$$ j_\ell (kr) = \sqrt{\frac{\pi}{2kr}} \ J_{\ell + \frac{1}{2}} (\ k \ r) \quad \quad (38) $$

where

$$ k = \ \frac{\omega^2 \rho}{\mu} $$

Then, for the given boundary condition that the shear stress vanishes at the free surface and at the core-mantle boundary, the characteristic equation is obtained as

$$ \frac{dj_\ell (kr)}{dr} - \frac{1}{r} j_\ell (kr) = 0 \quad \quad \text{at} \quad \int_{r=a} r=b $$

which leads to the recurrence relation

$$ (\ell - 1) \ J_{\ell + \frac{1}{2}} (ka) - (ka) \ J_{\ell + \frac{3}{2}} (ka) = 0 $$

This is a part of the characteristic equations solved by Sato et al. (1962) to obtain the basic data on the free oscillation of a homogeneous elastic sphere. The numerical solution of this equation is used in the next chapter for test purposes.
2.4 Lagrangian Integrals Of The One-dimensional Normal Mode Problem.

Now we will derive the Lagrangian integrals for the torsional and spheroidal oscillations by the inner product approach rather than the energy method. For torsional oscillation, we have from equation (27),

$$M\left(\frac{d^2 y_1}{dr^2} + \frac{2}{r} \frac{dy_1}{dr}\right) + \frac{dM}{dr}\left(\frac{dy_1}{dr} - \frac{y_1}{r}\right) + \left[\sqrt[3]{\frac{(l+1)m}{r^2}}\right] y_1 = 0 \quad (37)$$

If we let \( J_1 \) be the vector space consisting of all twice continuously differentiable vector fields \( \vec{y}(r) \) defined throughout the undeformed Earth model, for any two \( \vec{y}(r) \) and \( \vec{y}'(r) \) of \( J_1 \), an inner product on \( J_1 \) can be defined as

$$\langle \vec{y}, \vec{y}' \rangle = \int_0^a \left[ \vec{y}'(r) \cdot \vec{y}^*(r) \right] r^2 dr \quad (40)$$

where * denotes the complex conjugate. Now take the inner product of the equation (27) with another arbitrary vector \( \vec{y}'(r) \) of the inner product space

$$I_T = \int_0^a \left\{ M\left(\frac{d^2 y_1}{dr^2} + \frac{2}{r} \frac{dy_1}{dr}\right) + \frac{dM}{dr}\left(\frac{dy_1}{dr} - \frac{y_1}{r}\right) + \left[\frac{(l+1)m}{r^2}\right] y_1 \right\} y_1' r^2 dr$$

which, after a few algebraic steps, can be written
This is the Lagrangian integral for the torsional oscillations solved in Chapter 3 as a test of the method before the three dimensional formulation is begun. From the boundary condition that shear stresses at the free surface and at the core-mantle boundary vanish, the last term in the equation (41) is identically zero. The last term of equation (41) is, in fact, the natural boundary condition in the variational solution. The numerical solution of equation (41) is carried out by using the cubic Hermite polynomial basis as will be shown fully in Chapter 3.

By a similar derivation, the Lagrangian integral for the spheroidal oscillation is
In the above expression, the terms with $\omega^2$ are associated with the kinetic energy of the system and terms without $\omega^2$ are associated with potential (elastic and gravitational) energy.

2.5 Lagrangian Integral For The Generalized Three-dimensional Problem

In general there are two ways of deriving Lagrangian energy integrals for a given dynamic system. One way is to start with the definition of kinetic and potential energy terms of the particles in the motion. The other way is by forming an inner
product of the equation of motion and an arbitrary solution vector defined in the same vector or functional space as the equation of motion. We will follow the inner product approach since it is more simple and convenient mathematically.

Define the Fourier transforms of the various first order terms, \( u(\tilde{r}, t) \), \( \Phi_0(\tilde{r}, t) \), \( \rho_0(\tilde{r}, t) \), \( \vec{T}(\tilde{r}, t) \) and \( f(\tilde{r}, t) \),

\[
u(\tilde{r}, \omega) = \int_{-\infty}^{\infty} u(r, t) e^{i\omega t} dt \quad (43)
\]

etc.

And, for a fixed frequency \( \omega \), let \( \mathcal{D}_3 \) be the vector space consisting of all twice continuously differentiable vector fields \( u(\tilde{r}, \omega) \) defined throughout the undeformed Earth model. Then for any two \( u(\tilde{r}, \omega) \) and \( u'(\tilde{r}, \omega) \) of \( \mathcal{D}_3 \), define an inner product on \( \mathcal{D}_3 \) as

\[
\langle u, u' \rangle = \iiint dV \left[ \rho_0(r) \cdot \frac{\partial}{\partial t} \mathbf{u}'(r, \omega) \cdot \mathbf{u}^*(r, \omega) \right] \quad (44)
\]

The Fourier transformed linearized equation of motion is

\[
\rho_0 \mathbf{u}' - 2 \rho_0 \omega \mathbf{u} \times \mathbf{u} = \rho_0 \nabla \Phi_0 \mathbf{1} + \rho_0 \mathbf{u} \cdot \nabla \left[ \nabla (\Phi_0 + \Psi) \right] - \nabla \vec{T} + \mathbf{f} \quad (45)
\]

Now by taking the inner product of equation (45) with \( u(r, \omega) \) of the inner product space \( \mathcal{D}_3 \)

\[
\langle u', \rho_0 \frac{1}{\rho_0} \left\{ \rho_0 \nabla \Phi_0 + \rho_0 \mathbf{u} \cdot \nabla \left[ \nabla (\Phi_0 + \Psi) \right] - \nabla \vec{T} \right\} \rangle =
\]

\[
= \omega^2 \langle u', u \rangle - 2 \omega \langle u', \mathbf{j} \times \mathbf{u} \rangle + \langle u', \rho_0^{-1} \mathbf{f} \rangle \quad (46)
\]

First in the left hand side of equation (46), consider the evaluation of the volume integral defining the bilinear
functional, \( \mathcal{K}(u', u) \) as,

\[
\mathcal{K}(u', u) = \iint \int dV \left[ - u' \partial_i \bar{T}_{ij}^* + p_0 u' \partial_i \bar{\Phi}_i^* + p_0 u' u_j^* \partial_i \bar{\Phi}_j^* \left( \Phi_0 + \psi \right) \right]
\]  

(47)

Then an application of Gauss's theorem (Backus and Gilbert 1967)

\[
\iiint dV \ u' \frac{\partial \nu}{\partial \chi_i} = \int \int \int dA \ \nu \ \nu \ \partial_i \ - \ \iint \int dV \ \frac{\partial u}{\partial \chi_i} \ \nu
\]  

(48)

The potential energy bilinear functional becomes

\[
\mathcal{K}(u', u) = \iint \int dV \left[ \partial_j u' \bar{T}_{ij}^* + p_0 u' \partial_i \bar{\Phi}_i^* + p_0 u' u_j^* \partial_i \bar{\Phi}_j^* \left( \Phi_0 + \psi \right) \right] - \int \int \int dA \ \nu j \left[ u' \bar{T}_{ij}^* \right] \]  

(49)

Now from Green's first identity

\[
\iint \int \nabla \Phi \cdot \nabla \psi \ dV = \int \int \int \Phi \frac{\partial \psi}{\partial \chi_i} dA - \int \int \int \nabla \psi \ n \ dV
\]  

(50)

the gravitational perturbation term can be written as

\[
\iint \int \nabla \Phi_i^* \cdot \nabla \Phi_i^* dV = \int \int \int \Phi_i^* \frac{\partial \Phi_i^*}{\partial \chi_i} dA + \int \int \int dV \left[ \Phi_i^* \ n \ \partial_i \Phi_i^* \right]
\]  

(51)

Now

\[
\frac{\partial \Phi_i^*}{\partial \chi_i} = \hat{n} \cdot \nabla \Phi_i^* = \hat{n} \partial_i \Phi_i^*
\]  

(52)

and the Fourier transformed continuity equation and Poisson's equation are
By substituting (52), (53) and (54) into (51) we find

\[ \iiint \nabla \Phi_i \cdot \nabla \Phi_i^* \, dV = \iiint \Phi_i \, \eta_j \cdot \phi_j \, \Phi_i^* \, dV + \]

\[ + \iiint \Phi_i \, [4\pi G \cdot (\nabla \cdot (\rho u^*))] \, dV \] (55)

Another application of Gauss's theorem

\[ \iiint \Phi_i \, \nabla \cdot u^* \, dV = \iiint \Phi_i \, \rho \, \bar{u}^* \cdot \hat{n} \, dA + \]

\[ - \iiint \nabla \Phi_i \cdot \rho \, \bar{u}^* \, dV \] (48)

gives

\[ \iiint \nabla \Phi_i \cdot \nabla \Phi_i^* \, dV + 4\pi G \iiint \nabla \Phi_i \cdot \rho \, \bar{u}^* \, dV = \]

\[ = \iiint \Phi_i \, \eta_j \cdot \phi_j \, \Phi_i^* \, dA + 4\pi G \iiint \Phi_i \, \eta_j \, u_j^* \, \rho \, dA \]

and
By adding (49) and (56),

\[ \chi^* (u', u) = \iiint dV \left[ \alpha_i u_i \hat{\Phi}^* \hat{\Phi} + \int \int dA \frac{1}{4\pi G} \partial_i \hat{\Phi}^* \partial_i \hat{\Phi} \right] + \]

\[ + \iiint dV \left[ p_i u_i \hat{\Phi}^* + p_0 u_i \alpha_i \hat{\Phi} \right] + \]

\[ + \iiint dV \left[ \alpha_i u_i \hat{\Phi}^* \right] + \]

\[ + \iiint dV \left[ p_i u_i \hat{\Phi}^* \right] \]

\[ + \iiint dV \left[ p_0 u_i \alpha_i \hat{\Phi} + \hat{\Phi} + \phi \right] \]

\[ - \iiint dA \left\{ u_i \partial_j \hat{\Phi}^* - \hat{\Phi} \left( \frac{1}{4\pi G} \partial_i \hat{\Phi}^* + p_0 u_i \right) \right\} \]  

(57)

In this equation the first integral is the elastic potential energy, the second and third integrals are the perturbed gravitational potential energy and the fourth integral is the initial gravitational energy plus the rotational potential energy. The last surface integral is the boundary integral term for the surfaces of discontinuities. Therefore, the Lagrangian
The first variation of the above Lagrangian integral with respect to the complex displacement \( \bar{u}(r) \) is zero, to the first-order in \( \Delta \bar{u}(r) \). This is the precisely the form of Rayleigh's variational principle. It can be seen that all energy integrals in the Lagrangian are bilinear functionals and are Hermitian symmetric. Consequently, the
Lagrangian integral derived above is an elastic gravitational normal mode problem and, by application of the principle of least action, the sum of the bilinear functionals is identically zero when evaluated at any \( u(r,t) \) which is an eigenfunction of the Earth model \( V \) with eigenfrequency \( \omega \). This means that the above derived Lagrangian integral can be interpreted as a principle of conservation of energy for the elastic-gravitational mechanical system \( V \) for normal modes.

The above Lagrangian integral is solved by restricting \( u(r,t) \) to be real, in which case the Coriolis term integral

\[
2\omega \oint_{V} dV \rho_o \left[ \dot{u} \cdot (\vec{r} \times \ddot{u}') \right]
\]

vanishes. The numerical aspects of the solution are considered in detail in Chapter 3.

2.6 Boundary Conditions In Variational Principles Of Normal Mode Problems.

In general, there are two essentially different types of boundary conditions, natural and geometric, in the variational formulations of boundary value problems (Courant et al. (1970)).

For a given functional \( I_r \) (or \( I_{sy} \) or \( I_{zp} \)),

\[
I_r = \int_{r_i}^{r_e} F(r, u, u', u'') dr
\]

(59)

(where \( u \) and \( u \) are the first and second derivatives of \( u \) with respect to \( r \)) the variation of \( I_r \), \( \delta I_r \) is,
\[
\delta I_T = \int_{r_1}^{r_2} \left[ \frac{\partial F}{\partial u} - \frac{d}{dr} \left( \frac{\partial F}{\partial u'} \right) + \frac{d^2}{dr^2} \left( \frac{\partial F}{\partial u''} \right) \right] \delta u \, dr + \\
+ \left[ \frac{\partial F}{\partial u'} - \frac{d}{dr} \left( \frac{\partial F}{\partial u''} \right) \right] \delta u \bigg|_{r_1}^{r_2} + \\
+ \left[ \left( \frac{\partial F}{\partial u''} \right) \delta u \right] \bigg|_{r_1}^{r_2} = 0 \tag{60}
\]

and, since the variation \( \delta u \) is arbitrary, to satisfy the condition

\[
\delta I_T = 0 \tag{61}
\]

we have the conditions,

\[
\left[ \frac{\partial F}{\partial u} - \frac{d}{dr} \left( \frac{\partial F}{\partial u'} \right) + \frac{d^2}{dr^2} \left( \frac{\partial F}{\partial u''} \right) \right] = 0 \tag{62}
\]

\[
\left[ \frac{\partial F}{\partial u'} - \frac{d}{dr} \left( \frac{\partial F}{\partial u''} \right) \right] \delta u \bigg|_{r_1}^{r_2} = 0 \tag{63}
\]

\[
\left[ \left( \frac{\partial F}{\partial u''} \right) \delta u' \right] \bigg|_{r_1}^{r_2} = 0 \tag{64}
\]

Condition equation (62) is the Euler equation which is satisfied by the energy functionals of the system. Expressions (63) and (64) are called the natural boundary conditions. The natural boundary conditions satisfied at the free surface or at the solid-liquid boundary are, in general, called free boundary conditions.
For the one-dimensionalized torsional oscillation problem, we have, from the Lagrangian integral (41),
\[
\left[ M \left( \frac{d y_i}{d r} - \frac{y_i}{r} \right) \right]_{a_i}^{a} = 0 \quad (65)
\]
which is automatically satisfied from equation (30) at the free surface and at the core-mantle boundary, because of the vanishing of the shear stresses at these boundaries (from section 2.2 and section 2.3).

Since, in the numerical solution in Chapter 3, the minimization of the Lagrangian automatically imposes the natural boundary conditions we do not have to directly impose the boundary conditions at the free surface and at the core-mantle boundary.

For the generalized three-dimensional case, we have, from equation (58),
\[
\int dA \, \gamma_j \left\{ u_i \, \tilde{\tau}_{ij}^* - \left[ \phi_j' \left( \frac{\alpha G}{\pi} \delta_{ij} + p_0 u_i^* \right) \right]_{-}^{+} \right\} = 0 \quad (66)
\]
This natural boundary condition is satisfied at the free surface and at the core-mantle boundary as in the torsional oscillation case. From equations (20) and (22)
\[
\gamma_j \, u_i \, \tilde{\tau}_{ij}^* = 0 \quad \text{from \hspace{1em} B.C.} \quad (60)
\]
and the natural boundary condition (66) is identically satisfied and zero.

If these natural boundary conditions were not satisfied we must impose the condition that the geometric boundaries satisfy

\[ \delta I_T = 0 \quad \text{or} \quad \delta I_{3\rho} = 0 \quad \text{etc.} \quad (61) \]

In the process of solving the Lagrangian integrals, with some appropriate basis functions in each coordinate direction, we encounter internal boundary conditions at each nodal point and nodal surface. These internal boundary conditions are satisfied by simply matching the displacements and the stress terms at all nodal points or nodal surfaces. This is very straightforward in the finite element type solution process.
CHAPTER 3

NUMERICAL FORMULATION OF NORMAL MODES OF SIMPLE EARTH MODELS

In the variational type finite-element method, we assume a trial solution consisting of appropriately chosen piecewise, continuously differentiable polynomial basis functions and minimize the Lagrangian energy functional with the proper boundary conditions. From a mathematical point of view, this is an extension of the Rayleigh-Ritz-Galerkin technique. The basic idea of this method is an old one, first proposed by Ritz (1900-1938) in the 1930's, but the use of the piecewise basis functions in the solution is more recent (Birkhoff et al. (1966)).

3.1 Variational TypeFinite-element Algorithm With Cubic Hermite Polynomial Basis Functions.

As shown in Chapter 2, we have a Lagrangian energy integral $I_T$ (or $I_{xp}$ or $I_{sp}$) from equations (41), (42) and (58)

$$I_T = \int_{r_0}^{r_a} F(r, \ddot{u}, \ddot{u}') \, dr = - \omega^2 T(r, \ddot{u}, \ddot{u}') + V(r, \ddot{u}, \ddot{u}')$$

(59)

where $T(r, \ddot{u}, \ddot{u})$ is the kinetic energy functional and $V(r, \ddot{u}, \ddot{u})$ is the potential energy functional in the Fourier transformed domain. Now, if the integrand $F(r, \ddot{u}, \ddot{u})$ satisfies Euler's equation (62) and the natural boundary conditions (63) and (64) are satisfied, then there is an approximate solution of the form
\[ \tilde{u}^A = \sum_{k=1}^{N} a_k \varphi_k(r) \]

where \( \{ \varphi_k(r) \}_{k=1}^{N} \) denotes \( N \) linear basis functions in the piecewise continuous, twice continuously differentiable function space, \( PC^2 \), and \( \tilde{u}(r) \in PC^2 \).

The trial solution \( \tilde{u}(r) \) minimizes the Lagrangian energy functional \( I_T \) (or \( I_{sp} \) or \( I_{sp} \)), in the limit,

\[ \lim_{N \to \infty} I_T(\tilde{u}^A) = I_T(\tilde{u}) \tag{63} \]

Then we have \( I_T \) (or \( I_{sp} \) or \( I_{sp} \)) is minimum if, from equations (59), (62) and (63),

\[ \frac{\partial I_T}{\partial a_k} = -\omega^2 \frac{\partial T(r, \tilde{u}^A, \tilde{u}'^A)}{\partial a_k} + \frac{\partial V(\tilde{r}, \tilde{u}^A, \tilde{u}'^A)}{\partial a_k} = 0 \tag{64} \]

which gives a generalized eigenvalue problem of type

\[ \begin{bmatrix} [V] & -\omega^2 [T] \end{bmatrix} [a] = 0 \]

where elements of the matrices \([V]\) and \([T]\) will be given later for the different cases. This generalized eigenvalue problem is solved by the QZ-algorithm (Moler and Stewart (1973)), which is briefly described in Appendix A.

For the basis functions which will be used in the trial solution (62), we have several choices. The most frequently used basis functions in the finite-element method are linear functions, quadratic functions, splines (cubic, B-spline and L-spline) and cubic Hermite polynomials. There are higher order basis functions but the high cost of the numerical implementation is often compromised by the accuracy of the
desired solution. When Wiggins (1973) solved the normal mode problem for the Earth by the Rayleigh-Ritz scheme he used Hermite polynomials as basis functions. Linear functions are most often used in engineering problems but due to the large error bounds and the nature of the coordinate system used in the present problem, they are not suitable. The a priori error bounds are given in Table 1 for the three most often used basis functions.

Table 1. Theoretical a priori error bounds for the linear, cubic spline and cubic Hermite interpolations in one-dimensional problem.

<table>
<thead>
<tr>
<th>Interpolation functions</th>
<th>a priori error bounds</th>
</tr>
</thead>
<tbody>
<tr>
<td>linear interpolation</td>
<td>( \max_{r \in (r_k, r_{k+1})}</td>
</tr>
<tr>
<td>cubic spline interpolation</td>
<td>( \max_{r \in (r_k, r_{k+1})}</td>
</tr>
<tr>
<td>cubic Hermite interpolation</td>
<td>( \max_{r \in (r_k, r_{k+1})}</td>
</tr>
</tbody>
</table>

Cubic Hermite polynomial basis functions are used in the present numerical scheme, because they have the smallest a priori error bounds among the three shown in Table 1, and because they are piecewise continuous and twice continuously differentiable. They have computational simplicity and satisfy the necessary continuity conditions for stresses at designated boundaries.

Cubic Hermite interpolation in one-dimension is defined as
\[ \theta_{H}(\Delta r) \bar{u} = \sum_{k=1}^{N+1} \left[ a_{k} h_{k}(r) + a_{k}^{'} h_{k}^{'}(r) \right] \]

given

\[ \Delta r: 0 = r_{1} < r_{2} < \ldots < r_{N} < r_{N+1} \]

where \( H(\Delta r) \) is a \( 2(n+1) \)-dimensional vector space, and the cubic Hermite basis, \( h_{k}(r) \), is unique in \( H(\Delta r) \) such that \( h_{k}(r_{l}) = \delta_{kl}, \)
\( 1 \leq k, l \leq N+1 \), and \( Dh_{k}(r_{l}) = 0, 1 \leq k, l \leq N+1 \), and \( h_{k}(r_{l}) = 0, 1 \leq k, l \leq N+1 \), and \( Dh_{k}^{'}(r_{l}) = \delta_{kl}, 1 \leq k, l \leq N+1 \). The shape of the cubic Hermite basis function is shown in the Figure 2 for an arbitrary interval \( (r_{k}, r_{k+1}) \). The definition of the cubic Hermite basis function is (from Schultz(1973))

\[ \tilde{h}_{k}(r) \equiv \begin{cases} \frac{2 r^{5}}{r_{i}^{5}} + \frac{3 r^{3}}{r_{i}^{3}} + 1 & 0 \leq r \leq r_{i} \\ 0 & r_{i} \leq r \leq 1.0 \end{cases} \]

\[ \tilde{h}_{k}(r) \equiv \begin{cases} -2 \frac{(r-r_{k-1})^{3}}{(r_{k}-r_{k-1})^{3}} + 3 \frac{(r-r_{k})^{2}}{(r_{k}-r_{k-1})^{2}} & r_{k-1} \leq r \leq r_{k} \\ 2 \frac{(r-r_{k})^{3}}{(r_{k+1}-r_{k})^{3}} - 3 \frac{(r-r_{k+1})^{2}}{(r_{k+1}-r_{k})^{2}} & r_{k} \leq r \leq r_{k+1} \\ 0 & r \in [0,1]-[r_{k+1},r_{k+1}] \end{cases} \]

for \( 1 \leq k \leq N \), and
\[ h_{NH}(r) = \begin{cases} -2 \frac{(r-r_N)^3}{(1-r_N)^5} + 3 \frac{(r-r_N)^2}{(r_{NH} - r_N)^2} & r_N \leq r \leq 1.0 \\ 0 & 0 \leq r \leq r_N \end{cases} \]

and

\[ h'_1(r) = \begin{cases} \frac{r(r_1 - r)^2}{r_1} & 0 \leq r \leq r_2 \\ 0 & r_2 \leq r \leq 1.0 \end{cases} \]

\[ h'_k(r) = \begin{cases} \frac{(r-r_{k-1})(r-r_k)}{(r_k - r_{k-1})^2} & r_{k-1} \leq r \leq r_k \\ \frac{(r-r_k)(r_{k+1} - r)^2}{(r_{k+1} - r_k)^2} & r_k \leq r \leq r_{k+1} \\ 0 & r \in [0,1] - [r_{k-1}, r_{k+1}] \end{cases} \]

for \( 1 \leq k \leq N \), and

\[ h'_{NH}(r) = \begin{cases} \frac{(r-r_N)^2(r-1)}{(1-r_N)^2} & r_N \leq r \leq 1.0 \\ 0 & 0 \leq r \leq r_N \end{cases} \]
Figure 2  Cubic Hermite Basis function in a normalized interval of $(r_k, r_{k+1})$. 

C.H. POLYNOMIAL BASE

CH(X1)  CH(X2)

CH1(X1)  CH1(X2)

AMPLITUDE

NORMALIZED R.S OR P INTERVAL
3.2 One-dimensional Variational Solution Of Torsional Oscillation

As a first step to the three dimensional generalized variational solution of normal modes of the Earth, the one dimensional torsional mode problem is solved with cubic Hermite polynomial basis functions.

From Chapter 2, we have the Lagrangian energy functional (41) for the torsional oscillation, i.e.,

\[ I_T = -\omega^2 T + V \]

where

\[ T = \int_{r_b}^{r_a} \rho y_i^2 r^2 dr \]

\[ V = \int_{r_b}^{r_a} \frac{\rho(\alpha-1)(\ell+1)}{r^2} y_i^2 r^2 dr + \]

\[ + \int_{r_b}^{r_a} \left[ (\frac{dy_i}{dr})^2 - \varepsilon (\frac{dy_i}{dr})(\frac{u_i}{r}) + \left( \frac{y_i}{r} \right)^2 \right] r^2 dr \]

If we assume a trial solution of the form (66),

\[ \tilde{y}_i^A(r) = \sum_{k=1}^{\mu+1} \left[ C_k \tilde{h}_k(r) + C_k' \tilde{h}'_k(r) \right] \]

for the N layers in the solid mantle region between the core-mantle boundary \( t \) and the free-surface boundary \( a \), then from the relation (64), we have
\[ \left( \frac{\partial V}{\partial C_k} \right)_k = 2 \sum_{\ell=1}^{2(N+1)} \int_{r_b}^{r_a} \left\{ \frac{m}{r} \left( \frac{h_k}{r} - \frac{h'_k}{r} \right) + \frac{m}{r^2} \frac{h_k}{r^2} \right\} r^2 dr \]

and

\[ \left( \frac{\partial T}{\partial C_k} \right)_k = 2 \sum_{\ell=1}^{2(N+1)} \int_{r_b}^{r_a} \int_{r_b}^{r_a} r^2 h_k h_j dr \]  

The integration limits are from \( r_b \) to \( r_a \), but, in actual practice, it is performed from \( r_{k-1} \) to \( r_{k+1} \) for the \( k \)'th matrix element. This gives the same result since for the \( k \)'th matrix element the basis functions are zero outside the interval \( (r_{k-1}, r_{k+1}) \). The basis functions also overlap each other in this integration interval as shown in the Figure 3. From the nature of the piecewise continuity and the overlapping of the basis functions in the actual integration interval, the matrix eigenvalue problem,

\[ \begin{bmatrix} V \end{bmatrix} - \omega^2 \begin{bmatrix} T \end{bmatrix} \begin{bmatrix} C \end{bmatrix} = 0 \]  

involves symmetric block tridiagonal matrices \( [V] \) and \( [T] \). The elements of the matrices \( [V] \) and \( [T] \) are

\[ [T_{ij}] = \int_{r_b}^{r_a} \int_{r_b}^{r_a} r^2 h_i h_j dr \]  

and

\[ [V_{ij}] = \int_{r_b}^{r_a} \int_{r_b}^{r_a} r^2 \left\{ \frac{m}{r} \left( \frac{h'_i h'_j}{r} - \frac{h_i h'_j}{r} + \frac{h_i h'_j}{r^2} \right) + \frac{m(l-1)(l+2)}{r^2} \right\} r^2 dr \]  

The matching of boundary conditions, as graphically shown in
Figure 3, results in the overlapping of the block matrices in $[V]$ and $[T]$. The schematic example is shown in Figure 4. An elegant and detailed treatment of boundary conditions in one dimensional variational problem is given by Wiggins (1976).

![Figure 3 The Cubic Hermite Basis functions overlapping in an arbitrary integration interval $(r_k, r_{k+1})$.](image)

The integrations in the one dimensional problem are done analytically and thus exactly, which is not possible for the more complicated problems.
\[ [V] \text{ or } [T] = \]

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Figure 4 The boundary condition matching in the energy matrices \([V]\) and \([T]\).
As a test problem, the normal modes of torsional oscillation for a simple spherical Earth model with homogeneous, isotropic one-layered mantle is solved by two methods; the Wiggins (1968) solution technique and the present variational solution with two different nodal points. For the better understanding of errors, the variational solutions are first integrated with N=2 and then with N=10. The radius of the Earth is normalised such that the core-mantle boundary is at 0.5454 and the free surface is at 1.0. The density and the shear modulus are also normalised to 1.0. Then the equations in (28) (Wiggins (1968)) are integrated using subroutine DIFSY of the University of British Columbia Computing Center, with the error control set to EPS = 1.0x10^-7.

For the variational solutions, the mantle is divided into two and ten layers giving total radial nodal points 3 and 11 respectively including the boundary nodes. The matrices [Vij] and [Tij] are calculated, obtaining block-tri-diagonal matrices [V]'s and [T]'s, then the generalized eigenvalue problem is solved by the QZ-algorithm with EPS=1.0x10^-7.
Table 2. Comparison of the torsional free oscillation frequencies found from the one dimensional variational solutions and Alterman et al. (1959) type solution.

<table>
<thead>
<tr>
<th>Toroidal Mode</th>
<th>Variational Type Solution (2-layer)</th>
<th>Variational Type Solution (10-layer)</th>
<th>Alterman et al. Type Solution (DIFSY)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( l ) ( n )</td>
<td>( 5.8079 )</td>
<td>( 5.8078 )</td>
<td>( 5.8075 )</td>
</tr>
<tr>
<td>2</td>
<td>( 66.5671 )</td>
<td>( 66.5563 )</td>
<td>( 66.5525 )</td>
</tr>
<tr>
<td>4</td>
<td>( 209.9236 )</td>
<td>( 209.6510 )</td>
<td>( 209.6486 )</td>
</tr>
<tr>
<td>6</td>
<td>( 472.3389 )</td>
<td>( 448.4006 )</td>
<td>( 448.3534 )</td>
</tr>
<tr>
<td>8</td>
<td>( 1480.8981 )</td>
<td>( 782.8127 )</td>
<td>( 782.6623 )</td>
</tr>
<tr>
<td>10</td>
<td>( 2143.7765 )</td>
<td>( 1212.9269 )</td>
<td>( 1212.5294 )</td>
</tr>
<tr>
<td>3</td>
<td>( 14.3167 )</td>
<td>( 14.3162 )</td>
<td>( 14.3156 )</td>
</tr>
<tr>
<td>1</td>
<td>( 79.2740 )</td>
<td>( 79.2621 )</td>
<td>( 79.2161 )</td>
</tr>
<tr>
<td>2</td>
<td>( 221.5732 )</td>
<td>( 221.3311 )</td>
<td>( 221.3256 )</td>
</tr>
<tr>
<td>3</td>
<td>( 483.5133 )</td>
<td>( 459.7209 )</td>
<td>( 459.6725 )</td>
</tr>
<tr>
<td>4</td>
<td>( 1494.0799 )</td>
<td>( 793.9963 )</td>
<td>( 793.8447 )</td>
</tr>
<tr>
<td>4</td>
<td>( 25.3522 )</td>
<td>( 25.3512 )</td>
<td>( 25.3504 )</td>
</tr>
<tr>
<td>6</td>
<td>( 142.9385 )</td>
<td>( 142.9118 )</td>
<td>( 142.8970 )</td>
</tr>
<tr>
<td>8</td>
<td>( 281.0638 )</td>
<td>( 280.6191 )</td>
<td>( 280.7970 )</td>
</tr>
<tr>
<td>10</td>
<td>( 540.1837 )</td>
<td>( 516.7423 )</td>
<td>( 516.6863 )</td>
</tr>
<tr>
<td>12</td>
<td>( 1559.4941 )</td>
<td>( 850.1181 )</td>
<td>( 849.9599 )</td>
</tr>
<tr>
<td>14</td>
<td>( 72.3378 )</td>
<td>( 72.3336 )</td>
<td>( 72.3325 )</td>
</tr>
<tr>
<td>16</td>
<td>( 172.1587 )</td>
<td>( 172.1249 )</td>
<td>( 172.1076 )</td>
</tr>
<tr>
<td>18</td>
<td>( 309.5371 )</td>
<td>( 309.2009 )</td>
<td>( 309.1710 )</td>
</tr>
<tr>
<td>20</td>
<td>( 567.1080 )</td>
<td>( 543.6046 )</td>
<td>( 543.5441 )</td>
</tr>
<tr>
<td>22</td>
<td>( 138.9997 )</td>
<td>( 138.9891 )</td>
<td>( 138.9875 )</td>
</tr>
<tr>
<td>24</td>
<td>( 500.0233 )</td>
<td>( 499.8205 )</td>
<td>( 499.8184 )</td>
</tr>
<tr>
<td>26</td>
<td>( 1072.3134 )</td>
<td>( 1070.8317 )</td>
<td>( 1070.8239 )</td>
</tr>
<tr>
<td>28</td>
<td>( 1854.6703 )</td>
<td>( 1849.5201 )</td>
<td>( 1849.5105 )</td>
</tr>
</tbody>
</table>
The results of the above three solution procedures are tabulated in Table 2. The detailed error analysis will be treated with the three dimensional solution results in the Chapter 5. However, we can easily see that, from the results in Table 2, the error increases greatly as \( n \) increases for the both cases of \( N=2 \) and \( N=10 \). The absolute error also increases as \( l \) increases. Both results were expected, but the error increase with \( n \) in the cases of \( N=2 \) (number of layers) merits special attention at this stage, because, in the three dimensional solution, the nodal points in each coordinate axes are limited by computer core storage.

3.3 Three Dimensional Solution For Non-rotating, Non-gravitating, Spherical Simple Earth Model

From the previous chapter, we have the following Lagrangian integral for the non-rotating, non-gravitating, isotropic, spherical Earth model: (equations (46) and (49))

\[
\mathcal{I}_3^0 = \omega^2 \int \int \int p_0 (r, \theta, \phi) \rho \cdot \bar{u} \cdot \bar{u}^* dV + \\
- \int \int \int \left\{ \sum \eta_j \bar{u}_j \bar{e}_j \bar{e}_j \bar{u}^*_k - \int \sum \eta_j \bar{u}_j \bar{T}_j^* \right\} dA
\]

(\eta_2)

For this simple Earth model, the Lagrangian energy integral includes only the kinetic energy term, the elastic potential energy term, and the surface boundary integral term. As in the one dimensional case, any solution \( u \) which minimizes this Lagrangian integral will be an eigen-solution of the system. To apply a trial solution consisting of the cubic Hermite
polynomial basis in a spherical coordinate, we can rewrite the equation (72) in matrix form, i.e.,

$$I_3 = \int_{V} p_0(r, \theta, \phi)^{p.c.} \left[ \mathbf{V} \right]^T \left[ \mathbf{u} \right] r^2 \sin \theta \, dr \, d\theta \, d\phi$$

$$- \int_{V} \left[ \mathbf{u} \right]^T \left[ \mathbf{D} \right]^T \left[ \mathbf{E} \right] \left[ \mathbf{D} \right] \left[ \mathbf{u} \right] r^2 \sin \theta \, dr \, d\theta \, d\phi$$

(73)

where the superscript \textit{p.c.} means 'piecewise continuous', and

$$\left[ \mathbf{u} \right] = \left[ u_r, u_\theta, u_\phi \right]^T$$

$$\left[ \mathbf{D} \right] = \begin{bmatrix}
\frac{3}{r^2} & 0 & 0 \\
\frac{1}{r} & \frac{1}{r^2} & 0 \\
\frac{1}{r} & \frac{1}{r} & \frac{1}{r^2} \\
0 & \frac{1}{r \sin \phi} & \frac{1}{r^2 \sin \phi} \\
\frac{1}{r^2 \sin \phi} & 0 & (\frac{1}{r^2} - \frac{1}{r}) \\
\frac{1}{r^2} & (\frac{1}{r^2} - \frac{1}{r}) & 0
\end{bmatrix}$$

(superscript $^T$ means the transpose of a matrix) and
The last term in equation (72) vanishes on the application of surface boundary condition. In a three dimensional, spherical coordinate system, we introduce a trial solution consisting of tensor products of three coordinate basis functions similar to the ones introduced in the previous section. We let \( H_{3\text{ph}} \) be

\[
H_{3\text{ph}} = H(\Delta r) \circ H(\Delta \theta) \circ H(\Delta \phi)
\]

where

\[
\begin{align*}
H(\Delta r) &\rightarrow \left[ h(r), h'(r) \right] \\
H(\Delta \theta) &\rightarrow \left[ r h(\theta), r h'(\theta) \right] \\
H(\Delta \phi) &\rightarrow \left[ r \sin \theta h(\phi), r \sin \theta h'(\phi) \right]
\end{align*}
\]

The \( H(\Delta r) \), \( H(\Delta \theta) \), and \( H(\Delta \phi) \) are the cubic Hermite polynomial basis functions in \( r \), \( \theta \) and \( \phi \) coordinate directions and each
forms a $2(N+1)$, $2(M+1)$ and $2(L+1)$-dimensional vector space. Then we will have an interpolation scheme with $H(\phi)$ in $8(N+1)(M+1)(L+1)$-dimensional piecewise tricubic polynomial vector space, where $(N+1)$, $(M+1)$ and $(L+1)$ are the number of nodes in each coordinate direction.

Then we will have a trial solution of form

$$[u] = [H][c] = \begin{bmatrix} h_1, h_2, \ldots, h_8 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{bmatrix}$$

(78)

where $c$'s are the undetermined coefficients and

$$h_1 \equiv h(r) h(\theta) h(\phi)$$
$$h_2 \equiv h'(r) h(\theta) h(\phi)$$
$$h_3 \equiv h(r) h'(\theta) h(\phi)$$
$$\vdots$$
$$h_8 \equiv h'(r) h'(\theta) h'(\phi)$$

On the substitution of (74), (75), (76) and (78) into (73), we have the Lagrangian integral
\[ I_{3p}^* = \mathbf{w}^T \begin{bmatrix} c \end{bmatrix}^T \int_0^1 \int_0^{2\pi} \begin{bmatrix} I \end{bmatrix}^T [H] [H] [c] \begin{bmatrix} \gamma^2 \sin \theta \end{bmatrix} d\theta d\phi + \]

\[ \begin{bmatrix} c \end{bmatrix}^T \int_0^1 \int_0^{2\pi} [H]^T [D]^T [E] [D] [H] [c] \begin{bmatrix} \gamma^2 \sin \theta \end{bmatrix} d\theta d\phi \]

(80)

For the minimum of \( I_{3p}^* \),

\[ \frac{\partial I_{3p}^*}{\partial \begin{bmatrix} c \end{bmatrix}^T} = 0 \]

gives, again, a generalized eigenvalue problem of type

\[ \left\{ \begin{bmatrix} V \end{bmatrix} - \omega^2 \begin{bmatrix} T \end{bmatrix} \right\} \begin{bmatrix} c \end{bmatrix} = 0 \]

which is solved by the QZ-algorithm as mentioned above.

The boundary conditions are satisfied at each nodal surface by matching the displacement basis functions as well as their partial derivatives with respect to each coordinate variable. The basic principle of matching the displacements and their partial derivatives is exactly the same as in the one-dimensional problem, only more complicated and tedious.

For the normalized simple Earth model with \( a = 1.0 \), \( \lambda = 1.0 \) and \( \mu = 1.0 \), the results are tabulated in Table 3 with the results of the classical solutions from equation (39). In the variational solution the Earth model has 2 nodes, 3 nodes and 4 nodes respectively along \( r \), \( \theta \) and \( \phi \) axis in the spherical coordinate. The 168x168 matrices \( \begin{bmatrix} V \end{bmatrix} \) and \( \begin{bmatrix} T \end{bmatrix} \) are symmetric and banded with bandwidth 144.

In the three dimensional variational solution, the eigenvalues are calculated all at once without any
discrimination. As shown in the previous section, the
eigenvalues for the overtones with higher \( n \) will depart
considerably from the true values but the eigenvalues for small
\( n \) and \( l \) modes are expected to be reasonably close to the upper
limit of true values. One difficulty in the three dimensional
solution is to identify each of the modes as they are
characterized in the conventional solution. If the rotation or
other perturbing effects are included in the solution, the
identification will become even more difficult than in the
present simple case due to coupling effects.

3.4 Effects Of Rotation In The Normal Modes Of
Self-gravitating Spherical Earth Model

If we add self-gravitation and rotation, we will have the
general Lagrangian energy integral, from equation (58),
\[ I_{3D} = - \omega^2 T_{3D} + V_{3D} \]  
where
\[ T_{3D} = \omega^2 \iiint dV \rho_0 \ddot{u} \cdot \ddot{u}^* \]
\[ V_{3D} = I_A + I_B + I_C + I_D \]
\[ I_A = \iiint dV \left[ -\nabla \cdot \ddot{u} \cdot \ddot{u}^* \right] \]
\[ I_B = \iiint dV \left[ \rho_0 \ddot{u} \cdot \ddot{u}^* \sigma \delta_j \left( \delta_0 + \Psi \right) \right] \]
In the above equation, the last term is identically zero upon the application of the boundary conditions. If we include the self-gravitation we have an extra unknown besides the displacement fields. The general unknown vector $\bar{u}$ will, again, be interpolated by cubic Hermite basis functions. Then the $\bar{u}$ is

$$[u] = [u_r, u_\theta, u_\phi, \Phi_1] = \begin{bmatrix} h_1, h_2, \ldots, h_8 & \ldots & \ldots \end{bmatrix} \begin{bmatrix} c_1 \vdots \vdots \vdots \end{bmatrix}$$

which reduces to equation (74) during the integration of kinetic energy terms. The matrix $[D]$ will, now, be
\[
[D] = \begin{bmatrix}
\frac{2}{\delta r} & 0 & 0 & 0 \\
\frac{1}{r} & \frac{1}{r} \delta \Theta & 0 & 0 \\
\frac{1}{r} & \frac{1}{r} \cot \Theta & \frac{1}{r^2} \delta \phi & 0 \\
0 & \frac{1}{r^2} \delta \phi & \left( \frac{1}{r} \delta \Theta - \frac{r}{r^2} \cot \Theta \right) & 0 \\
\frac{1}{r \delta \Theta} & 0 & \left( \frac{r}{r^2} - \frac{1}{r^2} \right) & 0 \\
\frac{1}{r \delta \phi} & \left( \frac{2}{\delta r} - \frac{1}{r} \right) & 0 & 0
\end{bmatrix} \quad (8.2)
\]

For the isotropic materials \( [E] \) is same as before. The matrix form of the energy integrals for the kinetic and elastic strain terms is the same as before (equation (80)). The gravitational and the rotational potential terms will be

\[
\iiint dV \left[ \rho \dot{u} \cdot \ddot{u} \delta \xi \delta \eta \delta \zeta \left( \Phi_b + \psi \right) \right] =
\]

\[
= \iiint dV \rho \left[ u \right]^T \begin{bmatrix}
\frac{2}{\delta r} & \frac{1}{r} \delta \Theta & \frac{1}{r^2} \delta \phi & 0 \\
\frac{1}{r \delta \Theta} & \frac{1}{r^2} \delta \phi & \frac{1}{r^2} \delta \phi & 0 \\
\frac{1}{r^2} \delta \phi & \frac{1}{r^2} \delta \phi & \frac{1}{r^2} \delta \phi & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \left( \Phi_b + \psi \right) \left[ u \right] \quad (8.3)
\]

where \( \Phi_b \) and \( \psi \) are functions only of \( r \), in this case, i.e.,
The gravitational perturbation integral terms can also be written in matrix form as follows:

\[
\begin{array}{c}
\Phi_0(r) = -\iiint_V \frac{G p_0(r)}{r} \, dV = \frac{4\pi G}{r} \int_0^r p_0(r) r \, dr \\
\psi_0(r) = \frac{1}{2} \left[ \Omega^2 r^2 - (\mathbf{\vec{r}} \cdot \mathbf{\vec{r}})^2 \right]
\end{array}
\]

The gravitational perturbation integral terms can also be written in matrix form as follows:

\[
\begin{array}{c}
\iiint_V dV \left[ p_0 \sum_{i} \partial_j \Phi_i^* + p_0 \sum_{i} \partial_j \Phi_i \right] = \\
\left[ \begin{array}{c} \frac{d}{dr} \\ 0 \end{array} \right] \\
\begin{array}{c}
\begin{bmatrix}
0 & 0 & 0 & \frac{2}{r} \\
0 & 0 & 0 & \frac{2}{r} \\
0 & 0 & 0 & \frac{1}{r^2} \\
0 & 0 & 0 & 0
\end{bmatrix}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\frac{d}{\delta_1} & \frac{1}{r^2} & \frac{1}{r} & \frac{1}{r^2} \\
0 & 0 & 0 & 0
\end{bmatrix}
\end{array}
\]
During the evaluation of \([V]\), most of the volume integrals were evaluated analytically or by series expansion. However, some surficial part of the volume integral could not be carried out by this simple method and a numerical scheme was used. The integration schemes using the series expansion and the numerical methods are explained in Appendix B. In general the evaluation of the volume integrals requiring numerical treatment is very costly as expected and makes the whole algorithm very expensive.

3.5 Effects Of Simple Lateral Heterogenieties

In the perturbation technique, lateral heterogeneity was applied as a series expansion of surface spherical harmonics. In the variational type finite element method, the lateral heterogeneities, which may be represented by \(f(r, \theta, \phi)\), \(\lambda(r, \theta, \phi)\) or \(\mu(r, \theta, \phi)\), are applied during the evaluation of volume integral for each elements.
Therefore, the constants \( f_0(r, \theta, \phi) \), \( \lambda(r, \theta, \phi) \) or \( \mu(r, \theta, \phi) \) may have discontinuities across any nodal surfaces without disturbing the internal boundary conditions for the displacement, stress or gravitational perturbation. A general element block is shown in Figure 5. However in present research the elementary block is more simple with 4 nodes: \( r = (0, 1) \), \( \theta = (0, \pi) \) and \( \phi = (0, \pi/2) \). In the present calculation the constants \( f_0(r, \theta, \phi) \), \( \lambda(r, \theta, \phi) \) and \( \mu(r, \theta, \phi) \) are assumed to be stepwise constant in each element and have discontinuities at the nodal surfaces if they were different in each block. All other computations are identical to those in the previous section. A spline interpolation scheme using cubic Hermite polynomial can be computer-programmed for the interpolation of
\( \rho(\tau, \theta, \phi) \), \( \lambda(\tau, \theta, \phi) \) and \( \mu(\tau, \theta, \phi) \), but this would make the complicated program even more horrendous to handle and was not implemented.
CHAPTER 4

ELLIPSCIDAL EARTH MODEL

After the epoch-making Chilean earthquake of May 22 1960, the splitting of spectral peaks were first interpreted as the effect of the rotation of the Earth by Pekeris et al. (1961), Backus and Gilbert (1961) and MacDonald and Ness (1961). Another cause, proposed by Usami and Stc (1962), to explain the splitting was the ellipticity of Earth. They solved the equation of motion in a spheroidal coordinate system to explain that the torsional oscillation $T_x$ mode splits into $(\ell + 1)$ modes by the ellipticity of the Earth. Later Caputo (1963) applied the perturbation technique to examine the effects of the ellipticity on the normal modes. The more conveniently treated application of perturbation technique is developed by Backus and Gilbert (1967) and Dahlen (1968), and the effects of the ellipticity of the Earth are extensively studied by Dahlen (1969) using this scheme.

4.1 Recent Development In Research On The Effects Of The Ellipticity Of The Earth On The Normal Modes.

Most recently Dahlen (1975) applied the normal mode perturbation theory in correcting great circular surface wave phase velocity measurements for the effects of the rotation and the hydrostatic ellipticity of the Earth. He estimated the apparent great circular path length as
\[ L_{\text{app}}(T, \theta) = 2\pi a \left[ 1 - \chi_1(T) \omega \theta - \chi_2(T)(1 - 3 \omega^2 \theta) \right] \] (87)

where \( a \) is the mean radius of the Earth, \( \theta \) is the co-latitude and \( \chi_1(T) \) and \( \chi_2(T) \) are respectively the normal mode multiplet rotational and elliptical splitting parameters of the Earth. He calculated the splitting parameters \( \chi_1(T) \) and \( \chi_2(T) \) for the two Earth models 1066A and 1066E (Dziewonski and Gilbert (1972)) for the period range 150-300s. At a period of 300s, \( \chi_2(T) \) for Rayleigh waves on the model 1066B with two sharp upper mantle discontinuities (420Km and 670Km) deviates from the asymptotic limit of \( e(a)/6.0 \) by more than a factor of 4. (see Figure 1 of Dziewonski and Sailor (1976)) This stunning result was received with a doubtful mind. In a search for a possible explanation of Dahlen's (1975) results, Dziewonski and Sailor (1976) found that the major contribution to the ellipticity splitting parameter comes from the treatment of the internal boundary conditions at the sharp discontinuities in the upper mantle and they concluded that the result is erroneous.

Woodhouse (1976) proved analytically that this result is indeed improperly calculated and found the cause to be the wrong application of the Rayleigh's principle. Backus and Gilbert (1976) and Dahlen (1968, 1969 and 1975) used, in the parameter perturbation technique, a Rayleigh's principle applied to the Lagrangian for the upper mantle discontinuities:

\[ \delta \omega^2 \int_V \mathbf{p} u^2 dV = \int_{A_0} \left[ \mathcal{L} \right]^+ \delta \mathbf{h} ds \] (88)

where \( \delta \mathbf{h} \) is the displacement vector of the appropriate
boundary, $A_p$ is the surface of discontinuities and $[^L_l]$ denotes the discontinuities in the Lagrangian $L$, across $A_p$ with positive contribution from the side of $A_p$ corresponding to $\delta^p_k$ positive. Woodhouse (1976) showed rigorously that Rayleigh's principle must be applied to the Hamiltonian instead of to the Lagrangian for the internal discontinuities, i.e.,

$$\omega \int_{A_p} \sum \left[ L - \frac{\partial L}{\partial u^*_i} \bar{u}_{ij} n_i n_j \right] \delta^p_k dS \tag{89}$$

At the external boundary, the integrand of equation (89) reduces to $[^L_l]$, the same as equation (88), because $\frac{\partial L}{\partial u^*_i} n_i$ vanishes and equation (88) produces correct result. The splitting parameters are calculated with this new formula by Dzwewonski and Sailor (1976) and they are shown in their Figure 1; their results compare closely to the results by the conventional method.

The splitting parameters computed by Dahlen (1976) for the combined effects of the rotation, ellipticity and lateral heterogeneity are weakly affected by the above mentioned error but they have to be corrected accordingly as well as his previous results (Dahlen 1968, 1969). In reference to this new development, Dahlen (1976, p92) specifically points out that there is still great uncertainty in the calculation of the effects of the ellipticity on the normal modes.

4.2 Ellipsoidal Coordinate System

To derive a Lagrangian energy functional for the oblate ellipsoidal Earth model, we consider an oblate ellipsoidal coordinate system, assuming the centre of the Earth to be the
origin and the polar axis of the Earth to be the axis of rotation.

Figure 6 Ellipsoidal Coordinate System.

\[
0 \leq \varsigma \leq \gamma_0 \\
0 \leq \eta \leq \pi \\
0 \leq \phi \leq 2\pi
\]
We then have the following relation between the Cartesian \((x,y,z)\), spherical \((r, \theta, \phi)\) and the oblate spheroidal \((\xi, \eta, \phi)\) coordinates

\[
\begin{align*}
\xi &= r \cos \theta \cos \phi = c \cos \xi \sin \eta \cos \phi \\
\eta &= r \sin \theta \sin \phi = c \sin \xi \sin \eta \sin \phi \\
\phi &= r \cos \theta = c \sinh \xi \cos \eta
\end{align*}
\]

(90)

where \(c\) is the distance between the origin and the focus. The contours of constant \(\xi\) form a group of spheroids. The spheroid with \(\xi = \xi_0\) is the surface of the Earth and \(\xi = 0\) is a circular disc. The polar axis is represented by \(\eta = 0\), and the contours of constant \(\eta\) are a group of hyperboloids of one sheet whose foci are given by \(A\) and \(A'\). The condition \(\eta = \frac{\pi}{2}\) gives the equatorial plane excluding the circular disc \(AA'\). Thus the equatorial and polar radii are

\[
\begin{align*}
a_1 &= c \cosh \xi \\
a_2 &= c \sinh \xi_0
\end{align*}
\]

(91)

where \(\xi_0\) is the value of \(\xi\) on the surface of the Earth. For the numerical values,

\[
a_1 = 6378.380 \text{ Km}
\]

(92)

\[
\alpha = \frac{(a_1 - a_2)}{a_1} = \frac{1}{297.0} = \varepsilon(a)
\]

and

\[
\begin{align*}
a_2 &= 6356.912 \text{ Km} \\
c &= 522.981 \text{ Km}
\end{align*}
\]
The variable \( \phi \) represents the longitudinal coordinate, as in the previous spherical coordinate, varying from 0 to \( 2\pi \) radians. If we let
\[
C \to 0 \quad , \quad \xi \to \infty \quad \text{and} \quad \eta \to 0
\]
under the condition that
\[
\cosh \xi = r
\]
we obtain a formula which holds in the case of a spherical Earth.

4.3 Lagrangian Energy Integral For The Oblate Spheroidal Coordinate System

In a generalized curvilinear orthogonal coordinate system the strain relations are given as follows (Takeuchi and Saito (1972)),
\[
E_{\xi \xi} = \frac{1}{h_{\xi}} \frac{\partial u_{\xi}}{\partial \xi} + \frac{u_{\eta}}{h_{\xi} h_{\eta}} \frac{\partial h_{\eta}}{\partial \eta} + \frac{u_{\phi}}{h_{\xi} h_{\phi}} \frac{\partial h_{\phi}}{\partial \phi}
\]
\[
E_{\eta \eta} = \frac{1}{h_{\eta}^2} \frac{\partial u_{\eta}}{\partial \eta} \left( \frac{u_{\phi}}{h_{\phi}} \right) + \frac{u_{\eta}}{h_{\phi}} \frac{\partial h_{\phi}}{\partial \phi} \left( \frac{u_{\eta}}{h_{\eta}} \right)
\]
where the coordinate transformation scale factors \( h_{\xi}, h_{\eta} \) and \( h_{\phi} \) are of the form
and equation (90) gives
\[
\begin{align*}
\eta_x^2 &= (\frac{\partial \xi}{\partial x})^2 + (\frac{\partial \eta}{\partial x})^2 + (\frac{\partial \phi}{\partial x})^2 \\
\eta_y^2 &= (\frac{\partial \xi}{\partial y})^2 + (\frac{\partial \eta}{\partial y})^2 + (\frac{\partial \phi}{\partial y})^2 \\
\eta_\phi^2 &= (\frac{\partial \xi}{\partial \phi})^2 + (\frac{\partial \eta}{\partial \phi})^2 + (\frac{\partial \phi}{\partial \phi})^2
\end{align*}
\]  

Now the components of strain, in oblate spheroidal coordinates, are
\[
[e_e] = [D_e][ue]
\]

where the subscript 'e' means ellipsoidal coordinate and

\[
[D_e] = \begin{bmatrix}
\frac{1}{\eta_x} & \frac{1}{\eta_y} & \frac{1}{\eta_\phi} \\
\frac{1}{\eta_x} & \frac{1}{\eta_y} & \frac{1}{\eta_\phi} \\
\frac{1}{\eta_x} & \frac{1}{\eta_y} & \frac{1}{\eta_\phi}
\end{bmatrix}
\]
In the generalized form of Hooke's law, it is assumed that each of the six components of stress is a linear function of all the components of strain. Also for an isotropic medium, it is independent of the choice of coordinates and all transformations from one set of coordinates to another one are invariant. Thus we have from equation (76)

\[ [E_e] = [E] = \begin{bmatrix}
\lambda + \mu & \lambda & \lambda & 0 & 0 & 0 \\
\lambda & \lambda + \mu & \lambda & 0 & 0 & 0 \\
\lambda & \lambda & \lambda + \mu & 0 & 0 & 0 \\
0 & 0 & 0 & m & 0 & 0 \\
0 & 0 & 0 & 0 & m & 0 \\
0 & 0 & 0 & 0 & 0 & m
\end{bmatrix} \]  

(98)

The basis functions we used in the Chapter 3 also have to be transformed into the new oblate spheroidal coordinate system, i.e., from equation (77),

\[ H_{eq} \equiv H(\Delta \xi) \circ H(\Delta \eta) \circ H(\Delta \phi) \]  

(71), (99)

where

\[ H(\Delta \xi) \ldots [h_\xi h(\xi), h_\xi h'(\xi)] \]

\[ H(\Delta \eta) \ldots [h_\eta h(\eta), h_\eta h'(\eta)] \]
\[
H(\Delta \phi) \cdots [h_\phi h(\phi), h_\phi h'(\phi)]
\]

and \(h_\xi, h_\eta, \text{ and } h_\phi\) are coordinate transformation scale factors
and \(h(\xi), h'(\xi), \cdots\) are defined in Chapter 3 by equations (67)
and (68). The solution vector \([\tilde{u}_e]\) will then be, as defined in
equation (78),
\[
[\tilde{u}_e] = [He][Ce]
\]

\[
= \begin{bmatrix} h_\xi, h_\eta, \cdots, h_\phi, h_\xi, h_\eta, \cdots, h_\phi, \\
\vdots & \ddots & \vdots
\end{bmatrix}
\begin{bmatrix} \tilde{u}_e, \tilde{u}_e, \cdots, \tilde{u}_e
\end{bmatrix}
\]

where \(h, h, \cdots\) are
\[
h_\xi = h(\xi) h(\eta) h(\phi)
\]
\[
h_\eta = h'(\xi) h(\eta) h(\phi)
\]
\[
\vdots
\]

Then, again from equation (58), the general Lagrangian energy
functional in an oblate spheroidal figure with volume \(V\) and the
boundary \(\partial V\) is
\[
I_e = V_e - \omega^2 T_e
\]

where
\[ I_{e} = \omega^{2} \int_{0}^{2\pi} \int_{0}^{\pi} \int_{0}^{r_{0}} \tilde{u}_{e} \cdot \tilde{u}_{e} \, h_{\phi} h_{\phi} h_{\phi} \, d\xi \, d\eta \, d\phi \]

(for this integral evaluation only, \( F_{e} \) in \( \tilde{u}_{e} \) is set to zero)

\[ V_{e} = I_{E} + I_{F} + I_{G} + I_{H} \]

where

\[ I_{E} = \iiint_{V_{e}} [C_{e}]^T [H_{e}]^T [D_{e}]^T [E_{e}] [D_{e}] [H_{e}] [C_{e}] h_{\xi} h_{\eta} h_{\phi} \, d\xi \, d\eta \, d\phi \]

\[ I_{F} = \iiint_{V_{e}} f [U_{e}]^T [D_{e}] (F_{e} + Y_{e}) [U_{e}] h_{\xi} h_{\eta} h_{\phi} \, d\xi \, d\eta \, d\phi \]

\[ I_{G} = \iiint_{V_{e}} f \left[ [U_{e}]^T [D_{e}] [U_{e}] + [U_{e}]^T [D_{e}] [U_{e}] \right] \, dV \]

and

\[ I_{H} = \iiint_{V_{e}} \frac{1}{4\pi Q} \left[ [U_{e}]^T [D_{e}] [D_{e}] [U_{e}] \right] \, dV \]

In a generalized orthogonal coordinate system, the gradient, divergence and Laplacian are defined as

\[ \nabla_{e} f = \frac{1}{h_{\xi}} \frac{\partial f}{\partial \xi} \hat{\xi} + \frac{1}{h_{\eta}} \frac{\partial f}{\partial \eta} \hat{\eta} + \frac{1}{h_{\phi}} \frac{\partial f}{\partial \phi} \hat{\phi} \]

\[ \nabla_{e} \cdot \vec{V} = \frac{1}{h_{\xi} h_{\eta} h_{\phi}} \left[ \frac{2}{\partial \xi} \left( h_{\phi} h_{\phi} v_{\xi} \right) + \frac{2}{\partial \eta} \left( h_{\phi} h_{\phi} v_{\eta} \right) + \frac{2}{\partial \phi} \left( h_{\phi} h_{\phi} v_{\phi} \right) \right] \]

\[ ^{(102)} \]
\[ \nabla_e^2 \Phi = -\frac{1}{\mathcal{H}_1 H_1 \Phi} \left[ \frac{\partial^2}{\partial \xi^2} \left( \frac{H_1 H_2 \Phi}{\partial \xi^2} \right) + \frac{\partial}{\partial \eta} \left( \frac{H_1 H_2 \Phi}{\partial \eta} \right) + \frac{1}{\mathcal{H}_1 H_1 \Phi} \left( \frac{\partial}{\partial \Phi} \right) \right] \]

and the \([De_1], [De_2]\) and \([De_3]\) in the equation (101) are, from equation (102),

\[ [De_1] = \begin{bmatrix}
\frac{1}{\mathcal{H}_1 H_1 \Phi} \frac{\partial^2}{\partial \xi^2} & \frac{1}{\mathcal{H}_1 H_2 \Phi} \frac{\partial^2}{\partial \xi \partial \eta} & \frac{1}{\mathcal{H}_1 H_2 \Phi} \frac{\partial^2}{\partial \xi \partial \Phi} & 0 \\
\frac{1}{\mathcal{H}_1 H_2 \Phi} \frac{\partial}{\partial \eta} & \frac{1}{\mathcal{H}_1 H_2 \Phi} \frac{\partial^2}{\partial \eta^2} & 0 & 0 \\
\frac{1}{\mathcal{H}_1 H_2 \Phi} \frac{\partial}{\partial \Phi} & \frac{1}{\mathcal{H}_1 H_2 \Phi} \frac{\partial^2}{\partial \Phi^2} & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \]

\[ [De_2] = \begin{bmatrix}
0 & 0 & 0 & \frac{1}{\mathcal{H}_1 H_2 \Phi} \frac{\partial}{\partial \Phi} \\
0 & 0 & 0 & \frac{1}{\mathcal{H}_1 H_2 \Phi} \frac{\partial}{\partial \Phi} \\
0 & 0 & 0 & \frac{1}{\mathcal{H}_1 H_2 \Phi} \frac{\partial}{\partial \Phi} \\
0 & 0 & 0 & 0
\end{bmatrix} \]

and

\[ [De_3] = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\frac{1}{\mathcal{H}_1 H_2 \Phi} \frac{\partial}{\partial \xi} & \frac{1}{\mathcal{H}_1 H_2 \Phi} \frac{\partial}{\partial \eta} & \frac{1}{\mathcal{H}_1 H_2 \Phi} \frac{\partial}{\partial \Phi} & 0
\end{bmatrix} \]

Now the gravitational potential, \(\Phi_e\), at equilibrium state, satisfies

\[ \nabla_e^2 \Phi_e = \psi \pi \varphi \]

and in our calculation
\[ \Phi_e = G \iiint_{\Omega_e} \frac{\rho}{R} \, \rho \, \xi \, \eta \, \phi \, d\xi \, d\eta \, d\phi \]  

(105)

where \( R \) is the distance to the observation point from the centre of mass, and, at the surface of an oblate ellipsoidal Earth, \( R \) becomes

\[ R_0 = C \cosh 2\xi_0 \]

The potential due to the centripetal acceleration of the oblate spheroidal Earth, rotating with angular velocity \( \Omega \) about its centre of mass is

\[ \psi_e = -R^2 C \sinh \eta \left( \cosh \eta \cos^2 \phi + \sinh ^2 \eta \sin^2 \phi \right) \]  

(106)

Now if we substitute all the relations into equation (101) and perform the volume integrations and minimize the Lagrangian \( I_e \) with respect to \( [C_e]^T \), we again have a generalized matrix eigenvalue problem of the same form as equation (65)

\[ \begin{bmatrix} \mathcal{V}_e \end{bmatrix} - w^2 \begin{bmatrix} \mathcal{T}_e \end{bmatrix} \begin{bmatrix} C_e \end{bmatrix} = 0 \]  

(107)

Again this equation can be solved for the eigenvalues and eigenvectors by using the QZ-algorithm. (Appendix A)

In equation (101), the matrices resulting from the integration of each individual elements can be assembled, as described in Chapter 3, by matching the displacements and their derivatives along the nodal points and nodal surfaces. For the boundaries which require the matching of stress components the matrices can be conveniently transformed as described in detail by Wiggins (1976). In the oblate spheroidal Earth model, we have
a circular surface with radius C at the centre of Earth instead of a point. The regularity and boundary conditions at this central disc are satisfied by just matching displacements and their derivatives (or stresses) along this surface for every element in contact with it.

Most of the boundary treatment will be same as in the spherical Earth model, but, if we have a spherical boundary at some radius inside the ellipsoidal Earth, then we will have a very complicated boundary matching because, in the oblate spheroidal Earth model, the $\xi = \xi_\lambda$ gives a spheroidal surface rather than a spherical surface. In such cases, we can represent the spherical boundary in the spheroidal coordinate variables resulting in an extra matrix transformation and compute the boundary terms for matching. Or we can represent the gap between the spherical surface and the oblate spheroidal surface with same $a_\lambda = a$ as an extra element and proceed the solution.
CHAPTER 5

NUMERICAL RESULTS

In this chapter, the numerical formulations of the variational normal mode solutions, described in the previous chapters, are computed for the simple three-dimensional Earth models. The Earth models and parameters are normalized into dimensionless variables. Since the Earth models considered in this research are perfectly elastic spheres, the average values of the density \( \bar{\rho} \), elastic shear modulus \( \bar{\mu} \), elastic bulk modulus \( \bar{\lambda} \), s-wave velocity \( \bar{v}_s \) and p-wave velocity \( \bar{v}_p \) are defined initially as

\[
\bar{\rho} = \frac{\int_b^a \rho^2 r^2 dr}{\int_b^a r^2 dr},
\]

\[
\bar{\mu} = \bar{\rho} \cdot \bar{v}_s^2,
\]

\[
\bar{\lambda} = \bar{\rho} \left( \bar{v}_p^2 - \bar{v}_s^2 \right),
\]

\[
\bar{v}_s^2 = \frac{\int_b^a v_s^2 \rho r^2 dr}{\int_b^a \rho r^2 dr},
\]

and

\[
\bar{v}_p^2 = \frac{\int_b^a v_p^2 \rho r^2 dr}{\int_b^a \rho r^2 dr}.
\]

The average normalizing values for these parameters are, for Bullen's Earth model \( \beta \),
\[ \bar{\rho} = 5.5 \times 10^{-6} \text{ gm/m}^3 \]
\[ \bar{\mu} = 1.46 \times 10^8 \text{ dynes/m}^2 \]
\[ \bar{\lambda} = 3.47 \times 10^8 \text{ dynes/m}^2 \]

and the average Earth's radius is taken as
\[ \bar{r} = \bar{a} = 6371 \times 10^6 \text{ m} \]

The normalizing factors for the gravitational constants and time are

\[ \bar{t} = \bar{r} / \bar{v}_p \]
\[ \bar{G} = 1 / \bar{\rho} \bar{r}^2 \]
\[ \bar{\omega} = \text{rad} / \bar{t} \]

After the normalized dimensionless parameters are substituted into the energy integrals, they are evaluated for each element sectionalized by the nodal points in the three coordinates. This matrix has a dimension of 96x96 for the perfectly elastic, non-rotating, non-gravitating sphere and 128x128 for the gravitating and/or rotating Earth model.
Figure 7 Whole Earth energy matrix assembly.
The letters 'c' represent the elements of the matrix involving the integration of the basis functions which include the origin (center of the Earth) and the letters 'p' indicate the inclusion of the basis functions which have nodes on the polar axes. The matrix blocks with numbers are the only general point energy matrix elements. If the Earth consisted of eight equal elements with origin, as shown in Figure 5, and each block had names such as A, B, C, ... H the total energy matrix would, after boundary matching, look like the one in Figure 7. The resulting matrix will be 168x168 with half bandwidth of 144 for the non-rotating, non-gravitating Earth model and 224x224 with half bandwidth 192 for the gravitating and/or rotating Earth models. Once the potential and kinetic energy matrices are assembled the eigenvalues are found by the QZ-algorithm described in Appendix A.

5.1 Normal Modes Of A Non-gravitating Homogeneous Elastic Sphere With Simple Heterogeneities

The dimensionless normal mode frequencies were calculated for the homogeneous elastic spherical Earth model with elements as shown in Figure 7. The lower order eigenvalues are given in Table 3. For the Earth models without gravitation and rotation, we have 168 eigenvalues.
Figure 8 Surface displacement fields of the normal modes of $^oT^*$ and $^oS^*$. The displacement fields plotted in the upper half of page are the ones computed from the variational solution and the ones in the lower half of the page are the theoretical displacement fields.
### Y AXIS LATITUDE (RADIAN)

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### Correspondence between printer symbols and contour levels

- 6 - Y.00000 TO 6.00000
- 0 - 6.00000 TO 6.00000
- 1 - 6.00000 TO 6.00000
- 2 - Y.00000 TO 6.00000

![Diagram](image-url)

Up for T°
### V Axis Latitude (Radians)

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### Correspondence Between Printer Symbols and Contour Levels

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- 2 = 0.4 x 10^-4 to 0.4 x 10^-3
- 1 = 0.4 x 10^-3 to 0.4 x 10^-2
- 0 = 0.4 x 10^-2 to 0.4 x 10^-1

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### Correspondence Between Printer Symbols and Contour Levels

- 4 = 0.4 x 10^-6 to 0.4 x 10^-5
- 3 = 0.4 x 10^-5 to 0.4 x 10^-4
- 2 = 0.4 x 10^-4 to 0.4 x 10^-3
- 1 = 0.4 x 10^-3 to 0.4 x 10^-2
- 0 = 0.4 x 10^-2 to 0.4 x 10^-1
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Correspondence between various symbols and contour levels:

- G: 0.000000 to 0.000001
- H: 0.000001 to 0.000002
- I: 0.000002 to 0.000003
- J: 0.000003 to 0.000004
- K: 0.000004 to 0.000005

**U° for OS°**
Figure 9  Radial surface displacement fields of $S_2^+$.  
The plot in the upper half of the page is from the variational solution and the one in the lower half is the theoretical displacement field.
U for other symbols.
Figure 10 Surface displacement field $u_\phi$ of mode $S_3'$. The plot in the upper half of the page is from the variational solution and the one in the lower half is the theoretical displacement in $\phi$ direction.
One way to identify a particular mode from these spectra is to plot the eigenfunctions for a specific band of eigenvalues and compare the plots with the eigenfunction plots of the classical spherical harmonic solutions. Since the size of the energy matrix is very large, this plot itself is very costly and, in Figures 8, 9 and 10, the surface eigenfunctions are plotted for $u_r$, $u_\theta$ and $u_\phi$ components of $T^\circ_2$ and $S^\circ_2$, the $u_r$ for $S^\circ_4$ and the $u_\phi$ component of $S_3^\circ$ to show the relative surface displacements. In these figures, the upper diagrams are the displacements from the variational solution and the lower ones are the displacements from the theoretical spherical harmonic solutions. The models have 2, 3 and 4 nodes along radial, colatitude, and azimuthal coordinates, which is very crude. The eigenfunction plots of the theoretical solutions and the solutions from these crude models naturally have minor differences but the general outlines of the modes are easily identified. One fact we have to notice is that as $\lambda$ increase the eigenfunction plot from the variational solution shows a gradual phase shift in both the azimuthal direction and also the colatitude direction (Figure 10). One of the most significant features of the variational numerical solution is that the degenerate eigenfrequency $\omega_2$ splits into $(2m+1)$ spectral components as shown in Figure 12. This fact can easily be explained by the interpolation scheme of the $h(\phi)$ for the azimuthal component of the eigenvibration.
Figure 11 φ-coordinate interpolation scheme for $\chi_4^n$ with different $m$ values.
As shown in Figure 11, the interpolation function \( h(\phi) \) and \\
\( h'(\phi) \) only depend on the number of nodes in the \( \phi \) coordinate \\
and this set of cubic Hermite basis functions, in the interval \\
\( 0 \leq \phi \leq 2 \), interpolate all \((2m+1)\) \( \phi \) variations of each \\
eigenfunction and, depending on the accuracy of interpolation \\
for each of the \((2m+1)\) modes, the degenerate normal mode \\
frequency splits into \((2m+1)\) levels. The degree of splitting is \\
not uniform and, to explain this, a very detailed error analysis \\
is necessary. However, the \( \omega_k^N \) frequency is not affected \\
either by this numerical spectral splitting or by the effects of \\
other perturbations studied.

To examine the effects of the lateral heterogeneity, four 

simple models are considered:

- **0**: The earth as a sphere without any perturbations.
- **A1**: The \( f, \mu \) and \( \lambda \) for the block A (Figure 7) are 
  1.3% greater than the rest of the sphere.
- **A3**: The \( f, \mu \) and \( \lambda \) for the block A are 2% 
  greater.
- **A1A7**: The block A and G have \( f, \mu \) and \( \lambda \) values 1% 
  higher than the rest of the model.
- **HEM**: The upper hemisphere has \( f, \mu \) and \( \lambda \) values 
  1% higher than those of the lower hemisphere.

In these types of crude lateral heterogeneities, we can handle 
only large scale perturbations physically and the \( f, \mu \) and 
\( \lambda \) were varied all together by 1% and up to 2%. 
Table 3 Normalised Dimensionless Normal Mode Frequencies For The Models 0, A1, A3, A1A7 And HEMI.

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<tr>
<th>Theo.Sol.for Unpert.Model</th>
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<td>3.99907</td>
<td>3.99913</td>
<td>4.00406</td>
<td>3.99807</td>
<td>4.00689</td>
</tr>
</tbody>
</table>

In the real Earth, the most probable lateral heterogeneity of the above size scale would be much less than 1%, and the lateral heterogeneities considered in all the previous perturbation techniques are less than 1%.
The normalised dimensionless normal mode frequencies for these models and the theoretical solution are given in Table 3. The theoretical solutions are obtained by solving the characteristic equations for the torsional (equation (39)) and spheroidal oscillation frequencies were obtained by the routine by Wiggins (1976). As predicted in Chapter 3, from the results of the one dimensional solution, the errors increase greatly as \( l \) or \( m \) increases and consequently only the lower order torsional and spheroidal oscillations are listed in Table 3.

The spectral splitting of the above listed modes are graphically shown in Figures 12 \( (\sigma_S^m) \), 13 \( (\sigma_S^3) \) and 14 \( (\sigma_T^m) \). From these Figures, the following facts can be observed (the lines with a dot on the left hand side are the zeroth degree multiplet lines): (1) The numerical spectral splitting of the model '0' is much larger than expected, reaching up to 4% of the degenerate frequencies \( \omega^S_2 \) and \( \omega^T_1 \) (the vertical lines represent 1%). (2) The effects of the lateral heterogeneities, of the models considered, seem much smaller than the numerical splitting. (3) In \( \sigma_S^m \) and \( \sigma_S^3 \) modes, the lateral heterogeneities of types A3 and HEMI have the most effects on the normal modes. The zeroth degree mode frequency, as well as the \( m \)'th degree spectral lines, shift up to 0.3% in both models. (4) In torsional oscillation, the most spectral shift occurs for the model A3 and greatly reduced effects are exhibited for the model HEMI. For \( \sigma_T^m \) mode, some of the azimuthal splitting frequencies with different \( m \) are missing or unidentifiable.
Figure 12 Spectral splitting of $S_2^m$ mode for the Earth models Al, A3, AlA7 and HEMI.

The lines with a dot on the left hand side are the zeroth degree multiplet lines and the others are for $m = -2, -1, 1$ and 2. The vertical line represents a relative magnitude 1% of degenerate frequency.
Figure 13. Spectral splitting of $S_3$ mode for the Earth models A1, A3, A1A7 and HEMI.

The lines with a dot on the left side are the zeroth degree multiplet lines and the other lines are for $m = -3, -2, -1, 1, 2$ and 3.

The vertical line represents a relative magnitude 1% of the degenerate frequency.
Figure 14 Spectral splitting of $T_2^m$ mode for the models A1, A3, A1A7 and HEMI.

The lines with a dot on the left side are the zeroth degree multiplet lines and the other lines represent $m = -2, -1, 1$ and 2.

The vertical line represents a relative magnitude 1% of the degenerate frequency.
5.2 Normal Modes Of Gravitating, Rotating And Laterally Heterogeneous Sphere

As a next step, self-gravitation and rotation are added into our model. The angular velocity of rotation and gravitational constant are normalised as mentioned at the beginning of this chapter. The values are

\[ \Omega = 0.03857 \]
\[ G = 0.1290 \times 10 \]

The models considered are:

- \( G_0 \) .... only the self-gravitation is added to the previous model '0'.
- \( GW \) .... the normalised \( \Omega \) is added to the model \( G_0 \).
- \( GW_{A1} \) .... the lateral heterogeneity of type A1 is combined with the model \( GW \).

The normalised dimensionless frequencies are tabulated in Table 4 for these models and the spectral splitting diagrams are shown in Figures 15 and 16 for the modes \( T_2 \) and \( S_2 \). As shown in these figures, the effects of the self-gravitation and rotation are very small. The numerical spectral splitting is dominant and even the combined rotation and the lateral heterogeneity effects on the splitting are negligible.

From previous results of similar research using perturbation techniques, the effects of lateral heterogeneities on the normal modes are expected to be small and the effects of the rotational potential would be large for the lower order modes.
Table 4 Normalised Dimensionless Normal Mode Frequencies
For The Models 0, GO, GW And GWA1.

<table>
<thead>
<tr>
<th>Theo. Sol. for Unpert. Model</th>
<th>0</th>
<th>GO</th>
<th>GW</th>
<th>GWA1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m = -2 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(-1)</td>
<td>2.47532</td>
<td>2.47893</td>
<td>2.46371</td>
<td>2.46735</td>
</tr>
<tr>
<td>( 0.501 )</td>
<td>2.50129</td>
<td>2.50491</td>
<td>2.50107</td>
<td>2.50111</td>
</tr>
<tr>
<td>( T_2 )</td>
<td>2.52273</td>
<td>2.52335</td>
<td>2.52383</td>
<td>2.52388</td>
</tr>
<tr>
<td>( S_2 )</td>
<td>2.54039</td>
<td>2.54201</td>
<td>2.54281</td>
<td>2.54299</td>
</tr>
<tr>
<td>( T_1 )</td>
<td>2.56278</td>
<td>2.56440</td>
<td>2.56049</td>
<td>2.55711</td>
</tr>
<tr>
<td>( S_1 )</td>
<td>2.75133</td>
<td>2.75523</td>
<td>2.75028</td>
<td>2.74865</td>
</tr>
<tr>
<td>( S_1 )</td>
<td>2.76808</td>
<td>2.77296</td>
<td>2.77613</td>
<td>2.77434</td>
</tr>
<tr>
<td>( S_0 )</td>
<td>2.78408</td>
<td>2.78451</td>
<td>2.78446</td>
<td>2.78455</td>
</tr>
<tr>
<td>( S_2 )</td>
<td>2.80064</td>
<td>2.80327</td>
<td>2.79571</td>
<td>2.79613</td>
</tr>
<tr>
<td>( T_1 )</td>
<td>2.81771</td>
<td>2.81783</td>
<td>2.82159</td>
<td>2.82178</td>
</tr>
<tr>
<td>( S_1 )</td>
<td>3.87275</td>
<td>3.87438</td>
<td>3.87328</td>
<td>3.87425</td>
</tr>
<tr>
<td>( T_3 )</td>
<td>3.88723</td>
<td>3.88868</td>
<td>3.89025</td>
<td>3.88906</td>
</tr>
<tr>
<td>( T_1 )</td>
<td>3.89506</td>
<td>3.89668</td>
<td>3.89668</td>
<td>3.89670</td>
</tr>
<tr>
<td>( S_3 )</td>
<td>3.91256</td>
<td>3.89810</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( S_2 )</td>
<td>3.97555</td>
<td>3.977174</td>
<td>3.97465</td>
<td>3.97368</td>
</tr>
<tr>
<td>( T_3 )</td>
<td>3.97804</td>
<td>3.979713</td>
<td>3.97911</td>
<td>3.97828</td>
</tr>
<tr>
<td>( T_1 )</td>
<td>3.98082</td>
<td>3.98245</td>
<td>3.98277</td>
<td>3.98122</td>
</tr>
<tr>
<td>( S_3 )</td>
<td>3.98792</td>
<td>3.98780</td>
<td>3.98901</td>
<td>3.98815</td>
</tr>
<tr>
<td>( S_2 )</td>
<td>3.99409</td>
<td>3.99512</td>
<td>3.99493</td>
<td>3.99516</td>
</tr>
<tr>
<td>( S_3 )</td>
<td>3.99689</td>
<td>3.99852</td>
<td>3.99890</td>
<td>3.99949</td>
</tr>
<tr>
<td>( S_3 )</td>
<td>3.99907</td>
<td>4.00089</td>
<td>4.00059</td>
<td>4.00571</td>
</tr>
</tbody>
</table>
Figure 15 Spectral splitting of $^{1}T_2$ mode for the models GO, GW and GWA1.

The lines with a dot on the left side are the zeroth degree multiplet lines and the other lines represent $m = -2, -1, 1$ and 2. The vertical line represents a relative magnitude 1% of the degenerate frequency.
Figure 16 Spectral splitting of $S_2^{-}$ mode for the models GO, GW and GWA1.

The lines with a dot on the left side are the zeroth degree multiplet lines and the other lines represent $m = -2$, $-1$, $1$ and $2$. The vertical line represents a relative magnitude $1\%$ of the degenerate frequency.
However the results of this research indicate that the effects of self-gravitation and rotation are very small relative to the numerical spectral splitting and no reasonable conclusions can be obtained. The extreme lateral heterogeneities of types A3 and HEMI, however, seem to affect the normal mode frequencies greatly even with large numerical spectral splitting. Thus the real Earth must be well symmetrically balanced in the $f, M$ and $\lambda$ distribution and the degree of lateral heterogeneities ($\leq 1.0\%$) considered in the perturbation techniques are well justified from a physically realistic viewpoint.

5.3 Ellipsoidal Earth Model

The ellipsoidal Earth models can also be studied by exactly the same approach as above and the uncertainties created by the perturbation solutions can be cleared up. However, the test runs of the integration routines calculating the energy matrices which involved the hyperbolic functions, were extremely expensive and after a brief search for more economical algorithms, the ellipsoidal Earth model study was left for future investigation. An integration method, which might provide a solution for this particular case, was proposed by Wiggins (1975, personal communication). In this method a number of exact numerical integrations are carried out as a function of radius on a regularly spaced grid in $\theta$ and $\phi$. The intermediate regions are then interpolated to complete the desired integration for the given block. This idea seems most appropriate but it requires a great deal of trial computing runs and numerical tests and it was not applied.
5.4 Remarks On Computational Error

During the problem formulation stage, a group of available basis functions were studied and, from the a priori error criteria (and the convenient numerical manipulability), cubic Hermite polynomial basis functions were chosen for this research.

The numerical computations are all performed in double precision and consequently the results are very accurate. The eigenvalue evaluation routines are supposed to have a minimum accuracy of $10^{-2}$ for the normal matrices of size 50x50. The matrices tried in this research are much larger than 50x50, but the results are expected to be reasonably accurate even though no quantitative test was made. According to Birkoff et al. (1966) the accuracy of the eigenvalue is $O(h^5)$, $h = \max_x (x_{cm} - x_L)$, and the accuracy of the eigenfunction is $O(h^3)$ for the one-dimensional eigenvalue problem. The results of the one-dimensional torsional oscillation problem solved in this research (Table 2) indicate this variational method with cubic Hermite polynomial basis is very accurate and powerful, even in the extreme case of 2 layer models. Errors increase rapidly as $\lambda$ and $n$ increase. In the three-dimensional variational solution, the error bounds are expected to be very large and this is shown in the results of this chapter. There is no theoretical estimation of error bounds for the three dimensional finite element method and we can only compare the numerical results with known solutions.
Even though no attempt is made to estimate the errors quantitatively, the results in Table 3 and 4 indicate the maximum errors in the eigenfrequencies of the modes with orders 2 and 3, reach up to 14 % of the theoretical degenerate solutions. This will increase rapidly as the order $\lambda$ and $\eta$ increase.
CHAPTER 6

CONCLUSIONS

The task of developing computational techniques for calculating the free oscillations of a realistic Earth model has been carried out recently by a number of authors. Their approach has been the application of perturbation techniques to the classical solution of the normal mode problem for a spherically symmetric, non-rotating, isotropic and radially heterogeneous Earth model. However, there remain several problems which cannot be solved by the conventional perturbation techniques. In this thesis, a variational type finite element solution of the free oscillations of Earth is proposed. The work clearly demonstrates that this technique is very powerful and efficient in one-dimensional problems and is also feasible for three-dimensional normal mode problems. In principle, it overcomes all the difficulties associated with the use of perturbation methods.

Firstly, the variational approach allows the effects of the rotation of the Earth to be included without restriction. In the perturbation method, one has to assume that $\frac{2}{\Omega} \ll 1.0$, a condition which cannot be satisfied in the study of core oscillations with very long periods.

Secondly, the use of the variational technique theoretically removes the restriction that lateral heterogeneities in the lower mantle must be projections of those introduced near the surface. The perturbation method requires that inhomogeneities be confined to near surface depths (in the order of a few hundred kilometers). While the approach presented
here does not include such restrictions in its theoretical development, present computational facilities do not allow its application to realistic models of Earth's heterogeneities.

Thirdly, uncertainties in the calculation of the effects of the Earth's ellipticity on the normal modes can be removed by the use of the variational solution formulation. Recently published work by a number of authors who used perturbation techniques has presented some conflicting results for the effects of ellipticity. Theoretically this conflict can be resolved but, again, the computational facilities limit the application of the variational type finite element method with cubic Hermite polynomial basis functions.

In the thesis, variational solutions for several simple Earth models are presented. Based on these results, which are given in Chapter 5, the following conclusions can be made: (1) In the variational type finite element solution of normal modes, the degenerate normal mode frequency \( \nu \omega_k \) splits into \((2m+1)\) components even when there are no perturbations of rotation, or lateral heterogeneities. This is a result of the characteristics of the numerical solution scheme, (2) In general the numerical spectral splitting of the degenerate frequency is large enough for the crude Earth models considered in this research that it obscure the effects of rotation and small lateral heterogeneities, (3) Self-gravitation seems to have a very small effect on the normal mode spectra of the models considered, (4) Several lateral heterogeneities (\( > 1.0\% \) but \(< 2.0\% \)), which break the physical symmetry of the Earth model, shift the spectra in general up to \(0.4\%\) including the zeroth degree
frequency.

Thus this research provides a theoretical basis for eventually obtaining more precise knowledge about the effects of rotation, ellipticity and lateral heterogeneities of Earth on the normal modes. However, computation of realistic results is not possible now. Either we must await the development of a new generation of computers, or a new computer algorithm must be formulated such that it would greatly reduce the number of unknowns yet retain the capabilities of handling realistic Earth models.

However, within the confines of current computing capabilities, some immediate extensions of the three-dimensional version of the present research are possible:

1) Splitting parameters can be calculated as a function of $\Omega$ as $\Omega$ becomes large.

2) The spectral splitting can be calculated for Earth models that contain only odd (or even) angular orders of lateral heterogeneities. This will verify the results from the perturbation technique using selection rules (Dahlen 1968, 1969, and Luh 1973, 1974).

3) A fast integration program may be implemented for the normal mode solutions of ellipsoidal Earth models.

4) A new application of the present technique to the rotational dynamic of Earth may be developed.

As well, the one-dimensional version of the formulation may be applied to core oscillations and Earth tide problems with better efficiency and accuracy than any other conventional numerical techniques.
In conclusion, the variational type finite element algorithm using cubic Hermite basis functions is a powerful and efficient method to study the normal modes of Earth. However, the requirement for a large computer memory to solve the three-dimensional problem for realistic Earth models restricts its application to one-dimensional problems with spherical harmonics or spheroidal harmonics.
BIBLIOGRAPHY AND REFERENCES


Zharkov, B.N. And Lyubimov, V.M., 1970b. Theory of Spheroidal Vibrations for a Spherically Asymmetric Model of the
APPENDIX A

Qz-algorithm for the solution of $Ac = \lambda Bc$.

Recently there has been great progress in the development of new algorithms to solve generalized matrix eigenvalue problems of the form $Ac = \lambda Bc$ (Wilkinson (1965), Wilkinson and Reinsch (1971), Moler and Stewart (1973), Van Loan (1975) and Ward (1975)). However, the $A$ and $B$ matrices are symmetric and banded in our case and the algorithm developed by Moler and Stewart (1973) is chosen over other available algorithms to solve the eigenvalues and eigenvectors, because of its generality, stability and accuracy with reasonable efficiency. If the matrices $A$ and $B$ are rectangular the VZ-algorithm developed by Van Loan (1975) can be used.

1). In the first step $E$ is reduced to upper triangular form, one column at a time, by Householder row transformations. These transformations are simultaneously applied to $A$. $A$ is then reduced to upper Hessenberg form by row rotations while maintaining the triangular form of $B$. This is possible if the elements of $A$ are annihilated in a precise order, columnwise from the bottom, as for example when $n=5$

\[
\begin{bmatrix}
  x & x & x & x & x \\
  x & x & x & x & x \\
  (3) & x & x & x & x \\
  (2) & (5) & x & x & x \\
  (1) & (4) & (6) & x & x
\end{bmatrix}
\]
Each rotation annihilating an element of A introduces one non-zero element on the subdiagonal of B. This element is then annihilated by a column rotation which when applied to A does not disturb any of the newly introduced zeroes. By accumulating these rotations in a dummy space T, full information is saved for later use in back transforming the eigenvectors of the reduced system into those of the original system.

2). Once A is upper Hessenberg and B is upper triangular form, an iterative technique of the explicit QZ-algorithm is used to reduce A to upper triangular form while maintaining the triangularity of B. Let \( C = AB^{-1} \), then apply QR-algorithm with shift \( \sigma \). Then \( Q \) is determined as an orthogonal transformation such that the matrix

\[
E = Q(C - \sigma I)
\]

is upper triangular. The next iterate \( C \) is defined as

\[
C' = RQ^T + \sigma I = QCQ^T
\]

and is known to be upper Hessenberg form. If we set

\[
A' = QAZ
B' = QBZ
\]

where \( Z \) is any unitary matrix, then

\[
A' B' = QAZZ^T B^{-1} Q = QAE^{-1} Q^T = C'
\]

If matrix \( Z \) is chosen so that \( B \) is upper triangular form, \( A' = C' B' \) is upper Hessenberg form. In the process of iteration, we can avoid calculating \( C = AB^{-1} \) explicitly. Then we have the iteration scheme

\[
A_{k+1} = Q_A Y Z Y
E_{k+1} = Q_B Y Z Y
\]
During the iteration, \( B \) stays as an upper triangular matrix and a sequence of upper Hessenberg matrices is formed for \( A \) which converges to a quasi-triangular matrix, that is, an upper Hessenberg matrix whose eigenvalues are the eigenvalues of 1x1 or 2x2 principal submatrices. The origin shifts at each iteration are the eigenvalues of the lowest 2x2 principal minor. Whenever a lowest 1x1 or 2x2 principal submatrix finally splits from the rest of the matrix, the algorithm proceeds with the remaining submatrix. This process is continued until the matrix has split completely into submatrices of order 1 or 2.

3). The next step is to check 1x1 and 2x2 principal submatrices. The eigenvalues of 1x1 problems are the ratios of the corresponding elements of \( A \) and \( B \). The eigenvalues of 2x2 problems might be calculated as the roots of a quadratic equation, and may be complex even for real \( A \) and \( B \). However, in this QZ-algorithm the 2x2 principal submatrices are further reduced to 1x1 if possible and eigenvalues are derived.

The computer program (FORTRAN IV), kindly provided by Dr. R. Mathon, Dept. Of Computer Science, University of Toronto, consists of four subroutines for each of the sections described above and one for the back-substitution to obtain the generalized eigenvectors. This FORTRAN program has been tested before by others and found to have good stability and accuracy. It has been tested again with EVPOWR (Power method) of the University of British Columbia Computing Center for a simple one
dimensional torsional oscillation problem. The test results show the program is very good in efficiency and accuracy for the moderate size matrices of A and B, but becomes very expensive for large matrices as expected. The more detailed aspects of the technique can be found in the paper by Moler and Stewart (1973).
APPENDIX E

Integration Methods Used In The Numerical Computations

The integrations along radial coordinate are performed analytically on the different combinations of cubic Hermite polynomial basis functions. The material constants \( f, \mu \) and \( \lambda \) could be interpolated using the same cubic Hermite polynomial basis functions if they are functions of \( r \), but in the present cases they are assumed constants between nodes (or nodal surfaces) with discontinuities at the nodes (or at the nodal surfaces). This analytical scheme is the most desirable feature of the present basis functions because of its accuracy and economy.

However, for the integration of the azimuthal and colatitude variables it is complicated. Again for simplicity, the material constants \( f, \mu \) and \( \lambda \) are assumed to be constants between each integration interval and thus pose no problems. The trigonometric functions introduced by the use of a spherical coordinate system make some of the integration difficult. Integrations of the form

\[
K \int_0^{\phi} \sin \theta \, d\theta
\]

are computed by an analytic approach. The remaining integrations cannot be computed analytically so that

\[
K \int_0^{\phi} \cos \theta \, d\theta
\]

and
are expanded in series involving Bernoulli numbers (Gradshteyn and Ryzhik (1965)) i.e.,

\[ K \int \frac{\theta^p}{\sin \theta} \, d\theta = K \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k} B_{2k}}{(2k)! (p+2k)(2k)!} \theta^{p+2k} \quad (p \geq 1, |\theta| < \pi) \]

\[ K \int \frac{\theta^p}{\sin \theta} \, d\theta = K \left\{ \frac{\theta^p}{p} + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2 (2^k - 1)}{(p+2k)(2k)!} B_{2k} \theta^{p+2k} \right\} \]

During the numerical evaluations of these integrals, each term with \( k > i \) in the expansion is added to the sums up to \( k = i \) until the sums up to \( k = i \) asymptotically approach the true values and the \((k=i+1)\)'th term is negligible compared to the sums. The largest term \((k=i+1)\) neglected has an absolute magnitude of \( 10^{-6} \) or less. This scheme is much more economical than available numerical integration methods. The singularities involved with \( \cot \theta \) or \( \frac{1}{\sin \theta} \) are approached as close as \( \theta = 1.0 \times 10^{-5} \) and the integrals are well approximated.

In most cases, the above method stably and accurately approximates the integrals, but for the cases when convergence is not good a numerical integration method, SQUANK of the University of British Columbia Computing Center, is used. The double precision version of SQUANK is used as a back-up subroutine for greater accuracy. The algorithm of SQUANK is based on Simpson's integration rule applied over intervals with varying width. For more detailed description of SQUANK, the reader is referred to the write-up for SQUANK at the University of British Columbia Computing Centre.