Shortest Paths on Uncertain Terrains

by

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Abstract

In this dissertation, we introduce the concept of uncertain terrains first suggested by Jörg Sack. We then examine the problem of finding the shortest path that stays on these terrains given certain assumptions about the terrains. We show that this problem is NP-hard under two fairly natural assumptions (meaning that we do not expect any polynomial time algorithm that finds these paths to be discovered).

Contents

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Chapter 1

Introduction

In this dissertation, we will examine shortest paths on uncertain terrains. Uncertain terrains are a new model for dealing with errors in measuring the elevation data of a terrain. It is conceivable that we could use uncertain terrains anywhere that we use regular terrains in computational geometry. However, we will restrict ourselves to studying various versions of the two point shortest path problem that arise on uncertain terrains.

1.1 An Example

Suppose Alice has made a map of the mountainous terrain near her house. She can store this map as a polyhedron where no two points share the same x and y coordinates. Her altimeter is strange in that it gives an error estimate for each point that it measures, so Alice records the elevation of each point that she measures as a range of values.

Now Bob calls Alice and asks her to meet him. Alice would like to get to Bob's house as quickly as possible, so she takes out her new map in order to find the shortest route from her house to Bob's.

Since the terrain could be any of the terrains allowed by the ranges of Alice's map, this is not yet a well-specified problem. Alice needs to make some assumption about how the terrain lies. She might be optimistic and assume that the terrain will be laid out to make every path as short as possible, or she can be pessimistic and find the path that is the shortest she can guarantee assuming that the terrain is laid out in the worst way possible.

1.2 Related Work

The subject of shortest paths on terrains was first explored by Sharir and Schorr [12] in 1986. It has been a fairly actively researched topic since then, with notable papers by Mitchell *et al*, Kapoor, and Chen and Han [11, 9, 4]. The basic idea of the algorithms presented in these papers is to "unfold" the terrain in such a way that the terrain is flat and a straight line connects the start and end points. The fastest of these algorithms (presented in the paper by Kapoor) finishes in $O(n \log^2 n)$, where n is the complexity of the terrain. One fine survey on this subject is by Joseph Mitchell [10].

In this thesis, we will primarily use techniques developed by Canny and Reif [3], who showed that finding the Euclidean shortest path in three dimensions with polyhedral obstacles is NP-hard, to prove that Alice's problems are NP-hard.

1.3 Definitions of Terms and Symbols

A *polyhedral terrain* is a polyhedron that satisfies the property that no two points have the same x and y coordinates. It is assumed that the faces of the terrain are flat interpolations of the vertices.

An *uncertain terrain* is a polyhedral terrain where the x and y coordinates are known but elevations of vertices are given as ranges.

A vertex with more than a single point in its range of possible elevations is called an *uncertain vertex.*

For a vertex v of the uncertain terrain, v^+ is the (3 dimensional) vertex at the maximum of v's elevation range (also known as an up vertex), and v^- is the vertex at the minimum of v's elevation range (a down vertex).

A *consistent terrain* (with respect to some uncertain terrain) is a polyhedral terrain whose vertices lie within the ranges specified by the uncertain terrain.

We will use the term *upper terrain* to mean the consistent terrain of up vertices and *lower terrain* to mean the consistent terrain of down vertices.

A *path* on a terrain is a sequence of x, y coordinates (the z value is obtained from the terrain) connected by line segments that do not cross terrain edges. For a path to be well-defined on an uncertain terrain, the elevations of the vertices of the terrain must be set.

$(x, y, [z_{down}, z_{up}])$ denotes the uncertain vertex with x coordinate x, y coordinate y, and a z coordinate that can be anywhere in the closed interval between z_{down} and z_{up}.

1.4 Layout of the Thesis

This thesis is presented in 5 chapters. This one that you have just finished reading is the introduction. Chapter 2 presents a general framework for some hardness proofs that are to follow. Chapter 3 deals with uncertain terrains using the assumption that the terrain makes every path on it as long as possible. In it, we show that the problem of finding the shortest path is NP-hard. In Chapter 4, we show a hardness result for finding the shortest path on an uncertain terrain assuming that the length of the path is the shortest of all the possible terrains. Finally, we give some conclusions and open problems in Chapter 5.

Chapter 2

Hardness Proof Overview

In chapters 3 and 4, we create a polynomial-time reduction the 4SAT problem to the problem of finding the length of the shortest paths on the uncertain terrains under the guaranteed and optimistic assumptions respectively. Since 4SAT is NP-complete [7], this implies that the decision problem associated with finding the shortest paths is NP-hard. We will only show that the problems are NP-hard, since the algebraic complexity of a potential shortest path might be high enough that verifying a solution could take an exponential amount of time [2]. Therefore, the problem is not obviously in NP, implying that it is not obviously NP-complete.

We use a reduction that is similar to Asano *et al* [1], who were in turn inspired by Canny and Reif [3]. In this chapter, we give an overview of how the proofs work, leaving the details to the later chapters. This is because the proofs are so similar that there would be a great amount of duplication otherwise.

We assume that we are given a 4SAT formula Φ and wish to answer the question, "is Φ satisfiable?"

We use a series of modular "gadgets" in our construction. There are three "building-block" gadgets – the *splitter*, the *collector*, and the *shuffler*. These aid in the construction of the "filter" gadgets: the *literal filter*, the *clause filter* and the *formula filter*.

All the gadgets have an input edge and an output edge. The output edge of one gadget is the input edge of the next gadget in the series. For every edge (or implicit edge defined by the range of an uncertain vertex), there are a finite number of intervals that can be reached by an optimistic or guaranteed shortest path of at most a certain length (which we prescribe for each input and output edge) from the start vertex. We call these intervals *portals.*

We will see that a path can reach a portal in less than its prescribed distance only if it travels through each preceding gadget in a "locally optimal" manner and does not get lengthened in any filters. The output portal corresponding to a given input portal is the set of points that can be reached on the output edge from the input portal by a (guaranteed / optimistic) shortest path of length at most some specified threshold.

The building block gadgets share the property that every portal on the input edge has at least one corresponding portal on the output edge. The idea is that all paths across a gadget between corresponding portals are almost exactly the same length and a path between portals that do not correspond to each other are much longer.

2.1 Splitter

For every portal on the input edge of the splitter, there are two on the output edge. The distance separating portals on the splitter's input edge is the same as the distance separating portals on the output edge, so the output edge must be twice as long as the input edge. If we join n splitters together such that the output edge of one is connected to the input edge of the next, there are 2^n times as many portals on the output edge of the final splitter as there were on the input edge of the first.

2.2 Collector

The collector is the reverse of the splitter. Every portal on the input edge corresponds to one portal on the output edge, but each portal on the output edge corresponds to two portals on the input edge. Therefore, the number of portals is halved. Also, if the guaranteed or optimistic shortest path to one of the two portals on the input edge is shorter than the other path, the length of the shortest path to the corresponding output portal is the sum of the shorter length and the length added by the collector.

2.3 Shuffler

For each portal on the input edge of the shuffler, there is only one corresponding portal on the output edge. However, the portals on the output edge that correspond to portals on the left half of the input edge are interleaved with portals that correspond to portals on the right half of the input edge.

If we assign a binary string to each portal on the input edge in numerical order, all the portals with a 0 in their first bit will be contiguous and all the portals with a 1 in their first bit will also be contiguous. On the output edge of the shuffle, all the portals with a 0 in their *second* bit will be contiguous, as will all the portals with a 1 in their second bit. This phenomenon generalizes, so if we assign the binary strings to the portals on the input edge of the first of a series of j shufflers, on the output edge of the jth shuffler, the jth bit is the bit that has all 0s and 1s contiguous. Note that n shufflers in a row will shuffle the paths back into their original order.

When the shuffler interleaves the portals on the output edge, it also halves the distance separating portals on its output edge. Each subsequent shuffler halves the distance again and, unlike a splitter, the paths may not go through a shuffler in reverse. Therefore, if we are to put the paths through many shufflers, we must make sure that they start out in well-separated portals.

2.4 Literal Filter

Each literal filter corresponds to a literal in the formula Φ. If a path corresponds to a truth assignment that does not satisfy that literal, the literal filter will make it so long that it can never be a shortest path. On its input edge, it has 2^n portals. Each of these is assigned an

n bit binary string in numerical order. This binary string corresponds to a truth assignment for the n variables in Φ.

If the literal filter corresponds to $\overline{x_j}$, for example, it must lengthen all paths that start at a portal with a 1 in its jth bit. To do this, we make the paths go through j shufflers in a row, so that all the paths with a 1 in their jth bit are contiguous. Then, we put a barrier blocking all the paths with a 1 in their jth bit, but leave the paths with a 0 in their jth bit unblocked. To go around the barrier causes any blocked path to lengthen by an unacceptable amount. Then we make the paths go through $n - j$ more shufflers so that they are in their original order.

2.5 Clause Filter

The clause filter represents a clause of our formula. Each clause in our formula has four literals, so each clause filter has four literal filters. Again, a correspondence is set up between the 2^n portals on the input edge of the clause filter and the 2^n possible truth assignments of the variables. The paths are directed through two splitters, so each path is split into four copies. Each copy is then directed through one of the literal filters. Finally, two collectors join the copies back together.

The collectors choose the shortest of the four copies to propagate, so if a path is not blocked by at least one of the literal filters, it is still a potential shortest path on the output edge of the clause filter. This gives us the disjunction of the literals that we desire in a clause filter.

2.6 Formula Filter

The formula filter also sets up the correspondence between the 2^n portals and truth assignments. Then it directs the paths in series through one clause filter per clause in the formula. This creates a conjunction of disjuncts because a path must correspond to a truth assignment that satisfies *all* the clauses in order to get through the formula filter in a length that allows it to continue on a length that could potentially be a guaranteed / optimistic shortest path.

2.7 Gaps

Everywhere that the terrain is not explicitly defined is called a *gap*. We do not want any paths to cross gaps and yet they must be legal parts of the terrain. To enforce this restriction, we make all faces in the gaps have at least one point whose elevation is nearly infinite. This means that we add one point per gap. Any path that crosses one of the faces in a gap will be nearly infinitely long, disqualifying it as a candidate guaranteed / optimistic shortest path.

2.8 Portal Widths

When describing the geometry of the gadgets, it will often be advantageous to have some faces be vertical or otherwise violate the conditions of a legal terrain. When we make the terrain legal by giving these faces slope that is infinitesimally different from vertical, the distance of the locally optimal route between two pairs of corresponding portals will be slightly different. The distance that defines a portal on the output edge must be the same for all the portals, so the paths between corresponding portals with slightly less distance between them have a degree of freedom in where they can hit the output edge. This adds a small amount of width to some portals. We must ensure that the widths of the portals stay small enough and that portals are well-separated so that it is not possible for a path to change from one portal to another and still be a candidate guaranteed / optimistic shortest path.

2.9 Solving 4SAT

The method of finding a solution to the question of whether Φ is satisfiable should now be becoming clear. We start with a point s at $(0, 0, 0)$ (or any other point that has a reasonable number of bits in its representation). Then we set up n splitters in a row, using walls to ensure that any shortest path must go through them. This creates the 2^n portals that the formula filter requires. We put the formula filter so that its input edge is the output edge of the last splitter in the series. Attached to the output edge of the formula filter, we erect n collectors in series. We put t in the single portal on the output edge of the final collector.

If there is an algorithm that can determine if there is a path from s to t that does not get lengthened by the barrier in any literal filter, we can use that algorithm to solve 4SAT.

Chapter 3

Guaranteed Shortest Paths

3.1 Introduction

Alice wants to get to Bob's house as quickly as possible. However, she also knows that Bob is very unforgiving about people coming to his house late. Therefore, Alice wants to be able to give Bob a guarantee about when she will be arriving. To do this, she assumes that the terrain is in its worst possible configuration for any path that she takes across it. If she finds the path that is the shortest using this assumption, she knows that this is the best path that she can take while making sure that she fulfills her promise to Bob.

3.1.1 Problem Statement

Given an uncertain terrain and two points s and t in the xy-plane, we want to find a path from s to t that has the shortest length among all paths from s to t where the path length is measured on the consistent terrain that maximizes its length. Cast as a decision problem, this becomes, "Is there a path from s to t whose maximum length over all consistent terrains is at most K?"

3.2 Terrains to Consider

Lemma 3.1. *The maximum length of a path is realized on some terrain each of whose vertices is up or down.*

Proof. Suppose we have a fixed path on the terrain that makes it the longest with vertex i at a non-extreme elevation in its range. Let path segments $p_1, \ldots, p_k$ be the path segments that vertex i's height affects. Let the length of each segment be $l_{p_1}, \ldots, l_{p_k}$ and $\sum_{j=1}^{k} l_{p_j}$ be their total length. If we can increase this total length, we have a contradiction. Consider each l_{p_j} as a function of z_i. Then we have (using barycentric coordinates (a_1, a_2, a_3) and (b_1, b_2, b_3) to represent the endpoints of the segment p_j, where $\sum_{i=1}^{3} a_i = 1$ and $a_i \geq 0$ for all i and similarly for all b_i)

$$l_{p_j}(z_i) = \sqrt{((a_1x_i + a_2x_2 + a_3x_3) - (b_1x_i + b_2x_2 + b_3x_3))^2 + ((a_1y_i + a_2y_2 + a_3y_3) - (b_1y_i + b_2y_2 + b_3y_3))^2 + ((a_1z_i + a_2z_2 + a_3z_3) - (b_1z_i + b_2z_2 + b_3z_3))^2}$$

With all constants incorporated into K_1, K_2, and K_3, this is

$$l_{p_j}(z_i) = \sqrt{K_1 + ((a_1z_i + K_2) - (b_1z_i + K_3))^2}$$

and

$$l'_{p_j} = \frac{((a_1z_i + K_2) - (b_1z_i + K_3))(a_1 - b_1)}{\sqrt{K_1 + ((a_1z_i + K_2) - (b_1z_i + K_3))^2}} = \frac{(a_1 - b_1)^2 z_i + (K_2 - K_3)(a_1 - b_1)}{\sqrt{K_1 + ((a_1z_i + K_2) - (b_1z_i + K_3))^2}}$$

For each l_{p_j} as a function of z_i, we have the derivative monotonically increasing. This implies that the derivative has at most one root in the range of z_i. This root, if it exists, implies a minimum in the original function (because the second derivative is positive). Thus $\sum_{j=1}^{k} l'_{p_j}$ is monotonically increasing. Thus it has at most one root as well, which must also be a minimum. Therefore, the maximum must be at one extreme end of the range of z_i. This contradicts the supposition that the path was on the terrain that made it the longest. □

3.3 Hardness Proof

The filter gadgets will be constructed as described in Chapter 2. However, we still need to describe the building-block gadgets. In the guaranteed shortest path case, there are a few extra building-block gadgets that we must also describe.

3.3.1 90° Turn

The 90° turn gadget is used to change the orientation of a group of paths by 90° in the xy plane, while adding the same amount to the length of each path in the group. It is represented schematically in Figure 3.1.

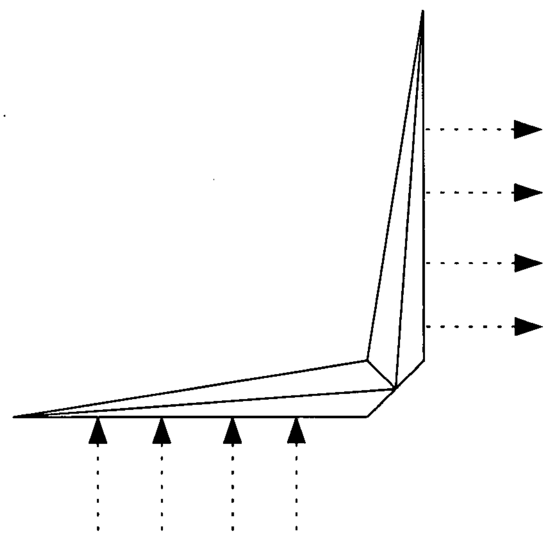

Figure 3.1: Top-down view of a 90° turn gadget showing several possible paths (dotted lines) entering and exiting.

The 90° gadget can be made from a $2w \times w$ rectangle of paper in which one short side is the input edge and the other is the output edge. Fold the paper down the middle in order to make two $w \times w$ squares (vertical walls) that are joined at a 90° angle. Then fold each square along a diagonal so that the top corner goes to the middle of the bottom edge.

Given that this gadget is constructed from a rectangle, the shortest path from the input edge to the output edge is simply the "folded" version of the straight line across the rectangle. That is, if we drew a straight line lengthwise across the rectangle and then folded the paper as described above, this line would represent the shortest path through the gadget. Since none of the vertices are uncertain vertices, these shortest paths are also the guaranteed shortest paths.

3.3.2 Mover

A mover gadget increases (or decreases) the x component of a path moving in a direction parallel to the y axis by an amount proportional to its width. It is constructed from two 90° turn gadgets that turn in opposite directions. It is shown in Figure 3.2. See Appendix A for origami instructions that describe how paper could be folded to make one of these gadgets.

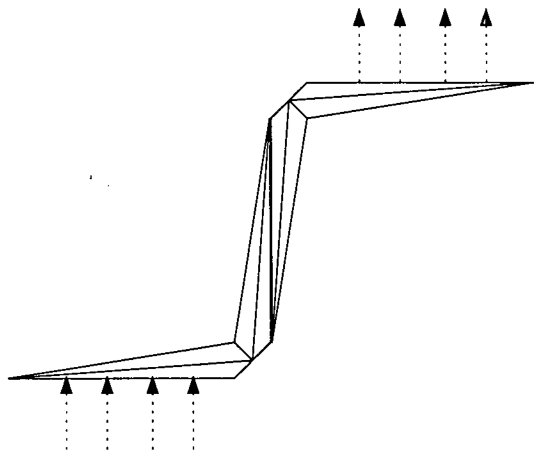

Figure 3.2: Mover gadget

3.3.3 Wide Splitter

The wide splitter gadget is shown in Figure 3.3. It has one input edge and two output edges. The input edge leads to an edge that links two uncertain vertices. The left uncertain vertex (vertex a) has a range of $[0, w]$ (where w is the length of the input edge) and the right uncertain vertex (vertex b) has a range of $[-w, 0]$. All the other vertices have height

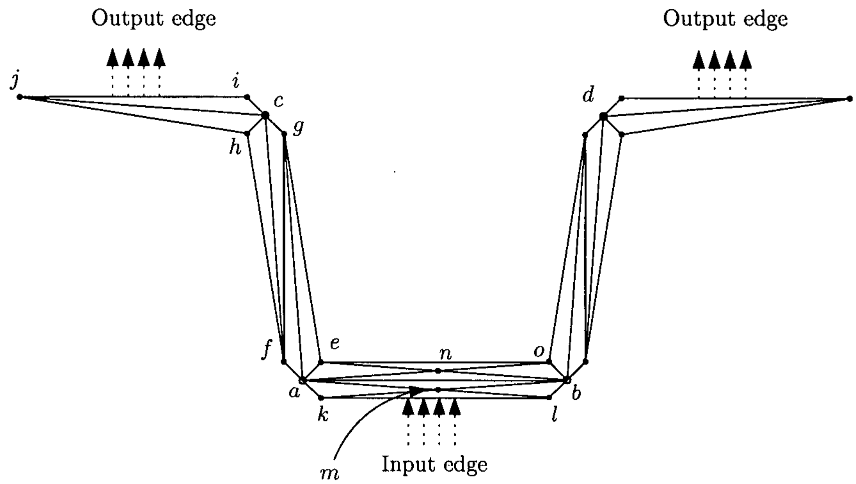

Figure 3.3: Wide Splitter

0 except c and d, which have height w. The remainders of mover gadgets are attached to each side of this contraption as shown.

We must show that any path P that travels through the correct part of the wide splitter as if it were a path through the appropriate version of the mover gadget is the guaranteed shortest path through the splitter. That is, P is not only the shortest way through the splitter, but the terrain is the worst possible for P.

Suppose the path leaves on the left output edge. When both uncertain vertices are up, P is the shortest path across the gadget. Thus, if P is in its longest version on this terrain, it is the guaranteed shortest path between these two points (since any other path is longer at least on the terrain where both uncertain vertices are up). It remains to show that the path is longest on this terrain. We will do this for the middle region of the input edge. As a consequence of this analysis, we show exactly what we mean by "middle region."

There are four cases to consider. When both uncertain vertices are up, the distance function with respect to the entry point t on the input edge, is $d(t) = 4w$. See Figure 3.4. When vertex a is down and vertex b is up, the distance function is $d(t) = 2t + 2w$, which is less than $4w$ for all $0 \leq t < w$. See Figure 3.5. When both uncertain vertices are down, the distance function is $d(t) = 2t + \sqrt{2t^2} + 2w$. See Figure 3.6. This is less than $4w$ for all

$0 \leq t < \frac{2w}{2+\sqrt{2}} \approx 0.5857w$. Finally, when vertex a is up and vertex b is down, the distance function is

$$d(t) = |w - 2t| + \sqrt{t^2 + ((w-t) - |w-2t|)^2} + w + 2w$$

This is less than $4w$ for all $0 \leq t < \frac{2w}{3}$. See Figure 3.7.

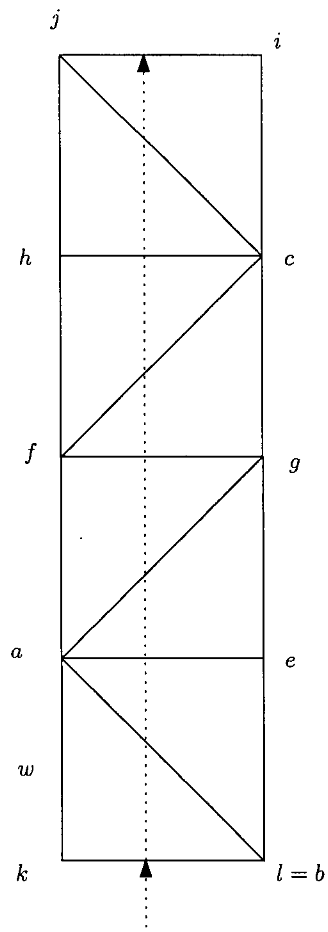

Figure 3.4: Wide splitter unfolded when a and b are up

If the path leaves on the right output edge, the limits are mirrored around $t = \frac{w}{2}$ When both vertices are down, the distance function is $d(t) = 4w$. When vertex a is down and vertex b is up, the distance function is $d(t) = 2w - 2t + 2w$, which is less than $4w$ for all $0 < t \leq w$. When both uncertain vertices are up, the distance function is

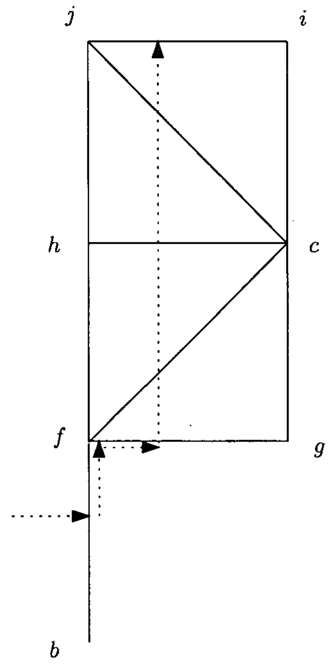

Figure 3.5: Wide splitter unfolded when a is down and b is up

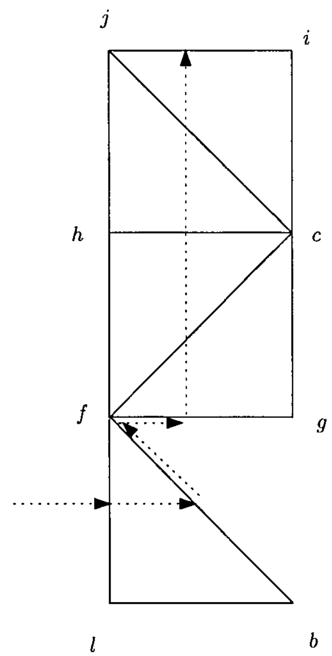

Figure 3.6: Wide splitter unfolded when a and b are down

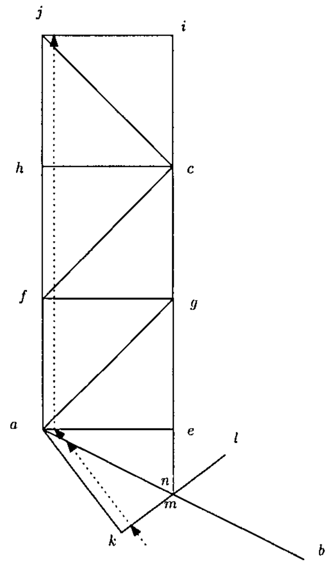

Figure 3.7: Wide splitter unfolded when a is up and b is down. We have omitted triangles lbm and obn, which would cause intersections.

$d(t) = (w - t) + \sqrt{2(w-t)^2} + (w - t) + 2w$, which is less than $4w$ for all $w - \frac{2w}{2+\sqrt{2}} \approx 0.41421w < t \leq w$. Finally, when vertex a is up and vertex b is down, the distance function is

$$d(t) = |w - 2(w-t)| + \sqrt{(w-t)^2 + ((w-(w-t)) - |w-2(w-t)|)^2} + w + 2w$$

This is less than $4w$ for all $\frac{w}{3} < t \leq w$.

Thus, the middle region of the wide splitter is $w - \frac{2w}{2+\sqrt{2}} < t < \frac{2w}{2+\sqrt{2}}$.

3.3.4 Narrow Splitter

A narrow splitter simply takes the paths created by the wide splitter and uses mover gadgets to move the paths so that every consecutive pair of paths have the same distance between them. See Figure 3.8.

The width of the movers relative to the width, w, of the wide splitter is $w + \frac{2w}{2+\sqrt{2}} - \frac{w+\delta}{2}$. They are placed such that the tips of the output edges (as shown in Figure 3.8) are δ apart.

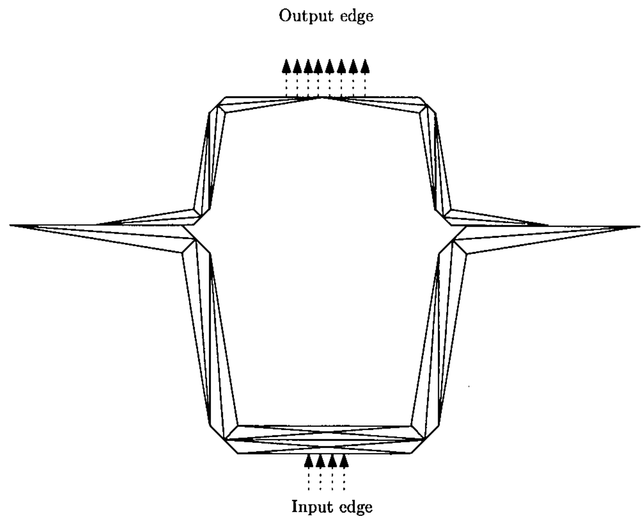

Figure 3.8: Narrow Splitter

3.3.5 Shuffler

A shuffler is like a reversed narrow splitter except that the mover gadgets have a slightly different width and the shuffler is slightly off-center with respect to the middle path entering the input edge. This causes the paths from one half of the group of paths to interleave with the other half.

For the shuffler, we describe the width and placement of the movers relative to the output edge. The width of this edge is w. Let x_0 and x_1 be the endpoints of the range of points that the paths are allowed to go through on the output edge and δ be the amount of space between each portal on the input edge. Then the movers must each have width $w + \frac{x_0 - x_1}{2} - \frac{\delta}{4}$. They must be placed δ apart so that each directs exactly half of the paths.

We attach a wide collector to the output edges of the movers. In order for the portals to interleave, we put the center of the wide collector $\frac{\delta}{2}$ units to the right of the center of both of the movers.

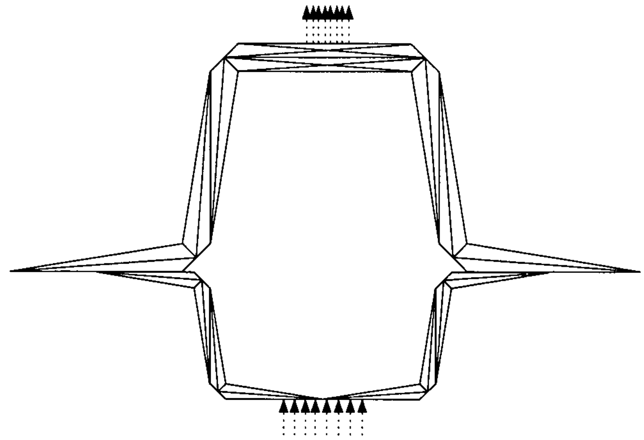

Figure 3.9: Shuffler

3.4 Making Walls Non-vertical

As mentioned in Section 2.8, we have used vertical walls in some of the gadgets in order to simplify the explanation of them. This is not legal, so we change the walls so that they are very nearly vertical. This causes the portals to have nonzero width. We must ensure that the portals do not become too large, otherwise it would be possible for a path to cross a gadget in a "good" distance and end in a portal that does not correspond to the one it started in.

3.4.1 90° Turn

Suppose the 90° turn gadget is situated as shown in Figure 3.10. Then when we unfold it, we get the quadrilateral pictured in Figure 3.11. Notice that one side is slightly longer than the other, implying that the output edge is at a slight angle with respect to the input edge. Therefore, shortest paths traveling from input to output edge vary in length depending on where they start on the input edge.

We will not calculate the difference in the distance that paths travel because it turns out to be unimportant. No 90° turn gadget is ever used on its own in our construction – they are always used in pairs that turn in opposite directions. Doing this makes both edges have the same length, which makes the output edge parallel to the input edge when the gadget is unfolded. Therefore, while these gadgets used alone would cause the portals to widen, when they are used in our construction, they do not.

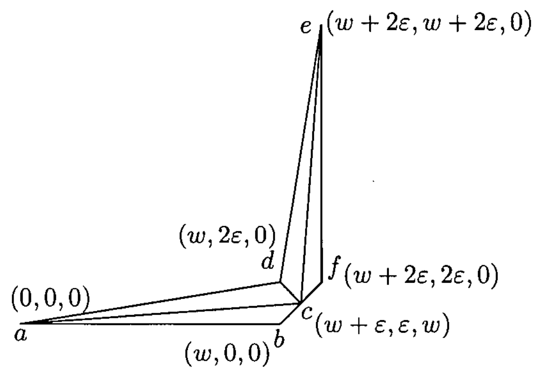

Figure 3.10: 90° turn gadget with relative coordinates

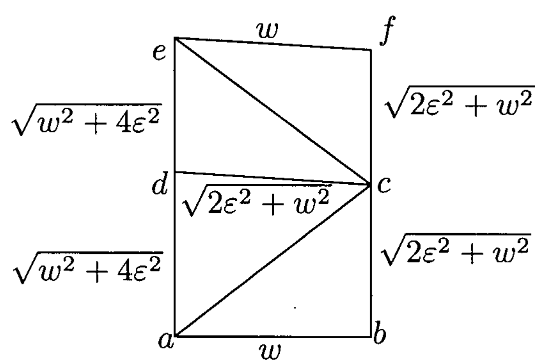

Figure 3.11: 90° turn gadget unfolded

Wide Splitter

The wide splitter, on the other hand, does cause a slight widening of the portals since the length to the output edge for paths entering on one side of the wide splitter is slightly more than the length to the output edge for paths entering on the other side. The reason for this is the small flat section just after the input edge. When the paths turn to the left, the flat section is largest on the right side of the splitter. When they turn to the right, the flat section is largest on the left side. Since the paths can only enter on the middle region of the wide splitter, the difference between the length of the shortest locally optimal path that travels through the splitter and the longest locally optimal path that travels through the splitter is less than $(\sqrt{2}\gamma + \sqrt{w^2 + 2\gamma^2 + 2\gamma w} - \sqrt{w^2 + 4\gamma^2})/3$ (see Figure 3.12). We can upper-bound this value by $(\sqrt{2}\gamma + 2\gamma)/3$.

If we let the length of the path across the splitter be the length of the longest possible locally optimal path, then the length allowed across the splitter is $l = 2\sqrt{w^2 + 2\gamma^2} + 2\sqrt{w^2 + 4\gamma^2} + (2(2 + \sqrt{2})\gamma)/3$. The base of the right triangle with l as its hypotenuse and $l - (\sqrt{2} + 2)\gamma/3$ as its height is

$$\sqrt{\frac{(2+\sqrt{2})\gamma}{3}(4\sqrt{w^2 + 2\gamma^2} + 4\sqrt{w^2 + 4\gamma^2} + (2 + \sqrt{2})\gamma)}$$

Therefore, if $\gamma \leq \frac{\varepsilon^2}{4(2+\sqrt{2})\sqrt{w^2+1}}$, each portal can increase in width by at most 2ε.

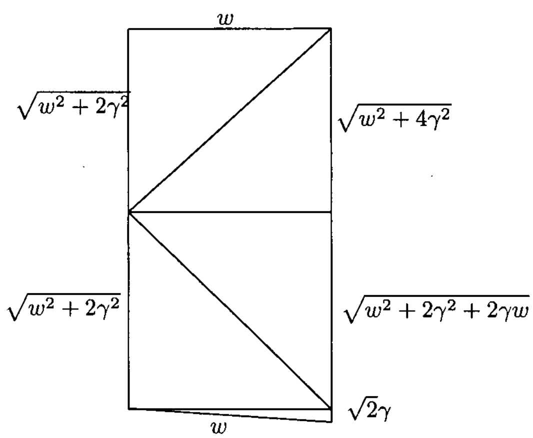

Figure 3.12: Wide splitter gadget unfolded. We attach an unfolded 90° turn gadget to the top, with the long side of that gadget attached to the short side of the splitter.

Since the rest of the gadgets are created from the wide splitter (as well as movers), there is nothing else that can cause the portals to widen. We can make portals start very far apart in the first splitter (a polynomial number of bits gives us a lot of room to work with), so portals will not overlap at any point in our construction. Thus we have sketched a proof to

Theorem 3.1. *The guaranteed shortest path problem is NP-hard. That is, we can find a path that has length* $\leq K$ *if and only if we can find a setting of variables that satisfies the formula* Φ *in 4-CNF form.*

Chapter 4

Optimistic Shortest Paths

4.1 Introduction

The assumption that the terrain will be the worst possible for any path is not a cheery one, and Alice is a cheery girl. She decides that she is going to find the shortest path to Bob's house under the assumption that the terrain is going to be the best for any path that she chooses.

Notice that, unlike with guaranteed shortest paths, we must consider the infinitude of possible terrains when we are deciding which terrain is the best for each path. For example, if it is possible for the uncertain vertices to be situated so that the terrain appears to be perfectly flat from the path's point of view, then we will assume that the vertices are at these elevations.

4.1.1 Problem Statement

Given an uncertain terrain and two points s and t in the xy-plane, we want to find a path from s to t that has the shortest length among all paths from s to t where the path length is measured on the consistent terrain that minimizes its length. Cast as a decision problem, this becomes, "Is there a path from s to t on some consistent terrain of length at most K?" We call this the optimistic shortest path since we are measuring a path on the consistent terrain that minimizes its length.

4.2 The Construction

4.2.1 Rotator

A basic operation that our gadgets must perform is to rotate the orientation of the group of portals by 90° in the xz plane. This is important as a building block of other gadgets and is performed by a *rotator*, which is like a square of paper folded along its diagonal. See Figure 4.1 for an example of a rotator of width 1 located at $(0, 0, 0)$ with a thickness parameter γ that is essentially 0. Note that vertex d is an uncertain vertex. Any path that enters this gadget at $(x, 0, 0)$ is shortest if it goes vertically up from its input edge to the diagonal edge and then horizontally from there to the vertex d, which is at height x (approximately – see Lemma 4.1 for a more precise discussion).

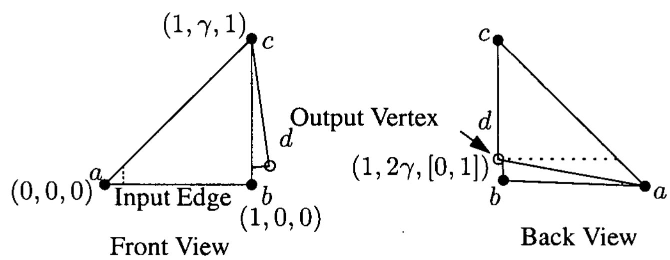

Figure 4.1: Front and back view of a rotator showing shortest paths across it as dotted lines

Since the shortest paths leave the rotator in a vertical line, we have changed the orientation of the line of portals from horizontal to vertical.

4.2.2 Path Splitter

To create the splitter, shown in Figure 4.2, we construct the uncertain terrain so that any shortest path from s to t must enter at or near the uncertain vertex d, cross either edge ef or fg, and cross output edge eg. The height of the input vertex d closely determines the point at which the path crosses the output edge.

Note that splitters are like two rotators that have been joined together.

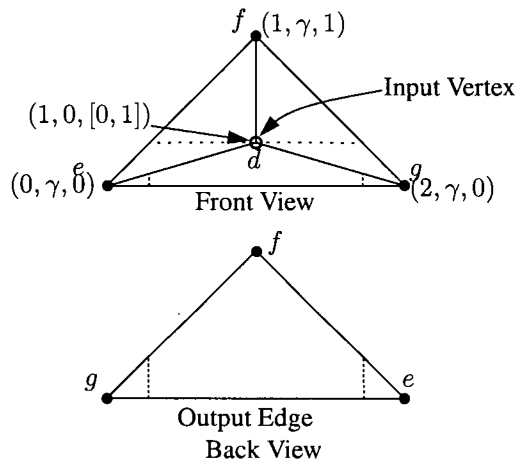

Figure 4.2: The splitter, showing the doubling of the number of path classes

4.2.3 Creating Portals

Given these gadgets, we are ready to describe the first part of the construction. In front of the initial point s, we construct a splitter that is high enough that any shortest path from s to t must go around the wall to the left or right. This creates two portals. Following this wall is a sequence of n connected rotators and splitters, joined at their common vertex d and increasing in size by a factor of 2 each time. When we join the rotators and the splitters at the vertex d, we leave a gap between vertex a on the rotator and vertex e on the splitter that cannot be crossed. (See section 2.7).

The output edge of the last splitter may be reached from s via 2^n shortest routes. Each of these shortest routes intersects the final output edge at a different location along the output edge. We number the resulting portals along this edge from 0 to $2^n - 1$ and associate them with the 2^n possible truth assignments to the variables of the formula Φ. The rest of the construction lengthens the paths in those portals that correspond to unsatisfying truth assignments.

4.2.4 Path Shuffler

As with the splitter, the shuffler can be seen as being composed of two rotators put together. Assume that the paths are δ apart. We will split the group of paths into two parts, so that half go to one side and half go to the other. When the group has been split, one of the halves will be raised by ($w/2 + \delta/4$, where $2w$ is the width input edge) and the other half will be lowered the same amount. The paths will then be rotated so that they interleave vertically.

We put a gap between the two groups of paths. This is shown in Figure 4.3 as the gray triangle.

The shortest paths leave the rotators through point a or point b in Figure 4.3. They then travel along edge ac or bc to point c. Point c is the input vertex to a reversed rotator (not shown) that rotates the group of paths back to horizontal.

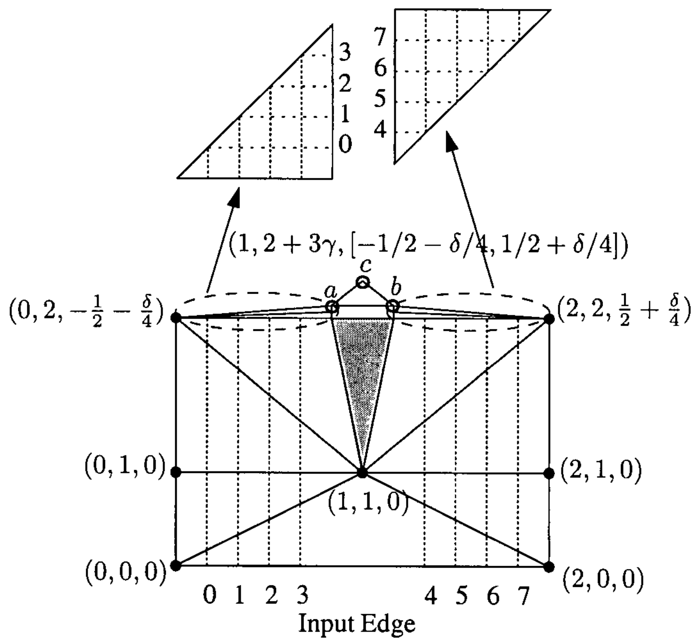

Figure 4.3: The shuffler viewed from above. The circled regions contain rotators, shown projected into the xz plane above the shuffler at the correct relative heights. The vertex marked c is the output vertex.

Using the gadgets described in this section, we construct the splitters following the procedure outlined in Chapter 2. That is, we construct the literal filters using n shufflers and a barrier, then we construct clause filters using four literal filters, and finally we make formula filters by putting the clause filters in series. As before, we must make sure that portals do not overlap.

4.3 The Hardness of Optimistic Shortest Paths

We will now show that each gadget increases the length of the shortest path traveling through it by a specified amount. We will also show that a shortest path must traverse each gadget in a locally optimal fashion. That is, if it enters a gadget via input portal i it must exit via a corresponding output portal.

Lemma 4.1. *For a rotator of width w, there exists a path from a point in $([x_0, x_1], 0, 0)$ on the input edge to the output vertex of length less than or equal to $w + 2\gamma$ only if the path exits through the output vertex in the range $(w, 2\gamma, [x_0 - \varepsilon, x_1 + \varepsilon])$ where $\gamma = \frac{\varepsilon^2}{8w}$.*

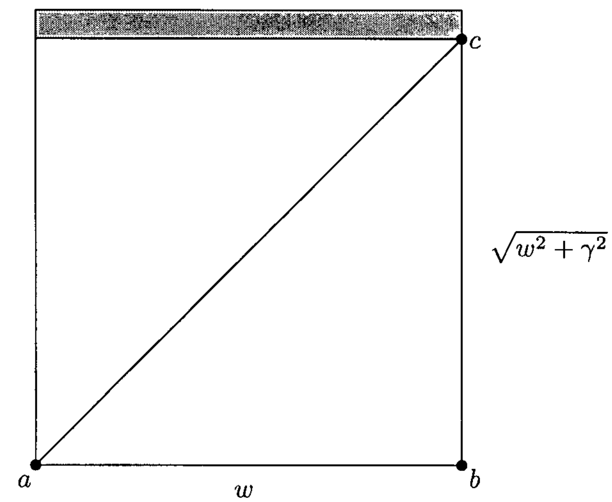

Figure 4.4: The unfolding has three fixed vertices and the fourth inside the shaded strip.

Proof. In order to find the locally optimal path across a rotator, first we consider the unfolding of the rotator when the output vertex is at its lowest possible elevation and $\gamma = 0$. The shortest path from the point $(x_0, 0, 0)$ is clearly the straight line across to $(x_0, w, 0)$. If we refold the rotator, this point goes to $(w, 2\gamma, x_0)$. Since points can only exit via the output vertex, the output vertex must move to that point.

Now consider the case when $\gamma \neq 0$. Whereas the unfolding of the rotator was a square with edge length w, it is now a quadrilateral. In the unfolding (see Figure 4.4), the triangle abc forms a right-angled triangle with leg lengths w and $\sqrt{w^2 + \gamma^2}$. The fourth vertex d falls within a strip of width γ above the $w \times \sqrt{w^2 + \gamma^2}$ rectangle. We can construct an approximation of the quadrilateral that is a $w \times w$ square with a 2γ strip added to the top to represent the uncertain location of d; the output vertex d must be inside this strip.

If we consider a path of length $w + 2\gamma$ across this quadrilateral from an arbitrary input point, the width of the region that the output point could potentially be in is $4\sqrt{\gamma(w + \gamma)}$ (the length of the chord at distance w from the input point in a circle centered at the input point of radius $w + 2\gamma$). Therefore, if $\gamma = \frac{\varepsilon^2}{8w}$, this region has a width that is less than 2ε. Thus, if the input point starts in range $[x_0, x_1]$, it will exit in the range $[x_0 - \varepsilon, x_1 + \varepsilon]$. □

Lemma 4.2. *For a shuffler of width $2w$, there exists a path from a point in $([x_0, x_1], 0, 0)$ on the input edge to the output vertex of length less than or equal to $2w + \sqrt{w^2 + (w/2 + \delta/4)^2} + 2\gamma$ only if the output vertex is either in $(w, 2w + 3\gamma, [x_0 - w/2 - \delta/4 - \varepsilon, x_1 - w/2 - \delta/4 + \varepsilon])$ or in $(w, 2w + 3\gamma, [x_0 + w/2 + \delta/4 - \varepsilon, x_1 + w/2 + \delta/4 + \varepsilon])$, depending whether $x_1 < 1$ or not.*

Proof. The locally optimal motion for a path entering a shuffler in a straight line in the vertical projection to the rotator across from it. Once it is at the rotator, Lemma 4.1 applies. □

From Lemma 4.1 and Lemma 4.2, we can deduce the maximum distance possible for any locally optimal path that corresponds to a truth assignment that satisfies the formula Φ. This distance will always be less than the minimum distance of any path that is either not locally optimal or that corresponds to a truth assignment that does not satisfy the formula Φ.

We ensure that no path crosses from a portal representing one truth assignment to a portal that represents another by making the initial wall very wide. This creates a large initial separation between portals. Since the width of a portal only grows by 2ε for each gadget, the portals are always separated by enough so that any path that goes from an input portal to a non-corresponding output portal is too long to satisfy our optimistic shortest path length bound. Thus we have

Theorem 4.1. *The optimistic shortest path problem is NP-hard. That is, we can find a path that has length $\leq K$ if and only if we can find a setting of variables that satisfies the formula Φ in 4-CNF form.*

Chapter 5

Conclusions

We have outlined two proofs showing that finding a shortest path on uncertain terrains given the optimistic and pessimistic assumptions are a hard problems. The method used to prove that these problems are hard has been shown to be useful and fairly generalizable.

5.1 Open Problems / Future Work

One natural problem arising from the results presented here is to find an ε-approximation algorithm that solves the guaranteed and optimistic shortest path problems with a path that is no longer that $(1+\varepsilon)$ times the length of the shortest path. These algorithms exist (with a running time inversely proportional to ε) for the 3D Euclidean shortest path [5] and for the d_1-optimal motion of a rod [1] problems – problems which have hardness proofs very similar to those that were shown.

The "discrete" case of the guaranteed shortest path and optimistic shortest path problems is the same problem with the restriction that the paths must travel on edges of the terrain. In the guaranteed shortest path case, the fact that vertices only need to be considered at extreme elevations implies that this can be turned into a very manageable graph. It might be that this will allow a polynomial algorithm to be found.

Another interesting area for future work would be to look at possibility of using linear programming to solve these problems in the L_1 (Manhattan) metric. This type of solution does not work in the L_2 (Euclidean) metric because of the square roots and the square operations in the distance measurements.

Exploring other geometric problems like the Voronoi diagram or the convex hull problem on uncertain terrains might also be interesting, but seem like they would almost certainly be NP-hard. This is because many of these problems depend on being able to find the shortest path between points in a reasonable amount of time and we have just shown that this is not likely to be possible.

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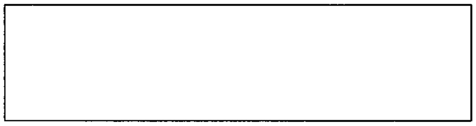
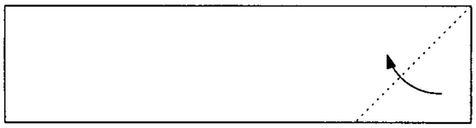
Appendix A

Origami Instructions

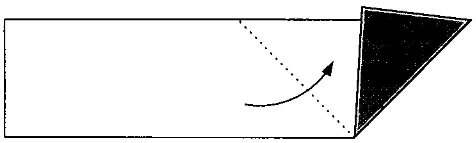
The following instructions are for making the mover gadget described in Chapter 3. All of the other gadgets in Chapter 3 are compositions of this gadget, so it can be used to make any of them.



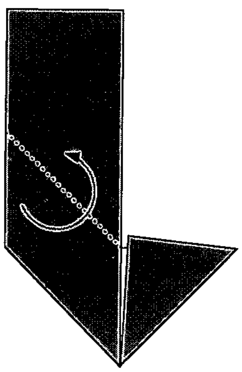
1. Start with a 1 × 4 strip of paper.



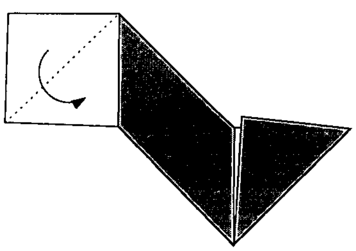
2. Fold right corner up to the top edge.



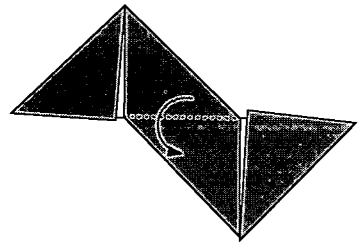
3. Fold the rest of the strip up.



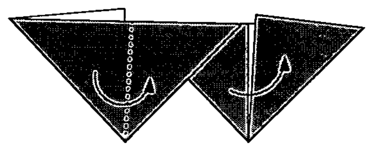
4. Fold the top part of the strip over to make the pictured triangle.



5. Fold the leftmost part of the strip into the final triangle.



6. Fold the top half over onto the bottom half.



7. To finish, fold the right quarter and the left quarter until they are at right angles with the middle half.