Computational Aspects of Escher Tilings

by

Ellen Gethner

Ph.D. The Ohio State University, 1992

M.A. University of Washington, 1983

A.B. Smith College, 1981

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Department of COMPUTER SCIENCE

The University of British Columbia
Vancouver, Canada

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Abstract

At the heart of the ideas of the work of Dutch graphic artist M.C. Escher is the idea of automation. We consider one such problem that was inspired by some of his earlier and lesser known work [MWS96, Sc90, Sc97, Er76, Es86]. From a finite set of (possibly overlapping) connected regions within a unit square (Figure 1), is it possible to make a prototile with concatenated and colored copies of the original square tile (Figure 2), such that the pattern in the plane arising from tiling with the prototile

- uniformly colors connected components, and
- distinctly colors overlapping components (Figure 3)?

The answer is **yes**, that such a prototile exists for any (suitably defined) design confined to a unit square. We present a proof of existence and an efficient (and implementable) algorithm to construct prototiles. Moreover, in the existence proof, it will become apparent that a prototile for a given design may not be unique (up to concatenation). In such a situation, there are infinitely many "measurably different" prototiles. The secret of each design is encoded by either one or infinitely many (number theoretic) lattices; we will show how to extract all possible lattices by using techniques from graph theory and graph algorithms. Finally, from a certain point of view, the prototiles that we construct are canonical. We begin an analysis of the canonical prototiles by making a connection from lattices to binary quadratic forms to class number.

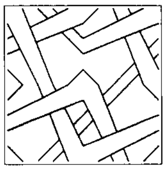

Figure 1: **A design for which there is a ...**

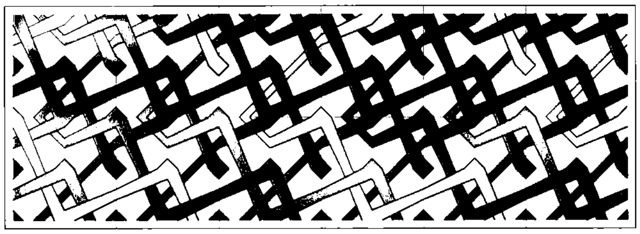

Figure 2: **...colored prototile...**

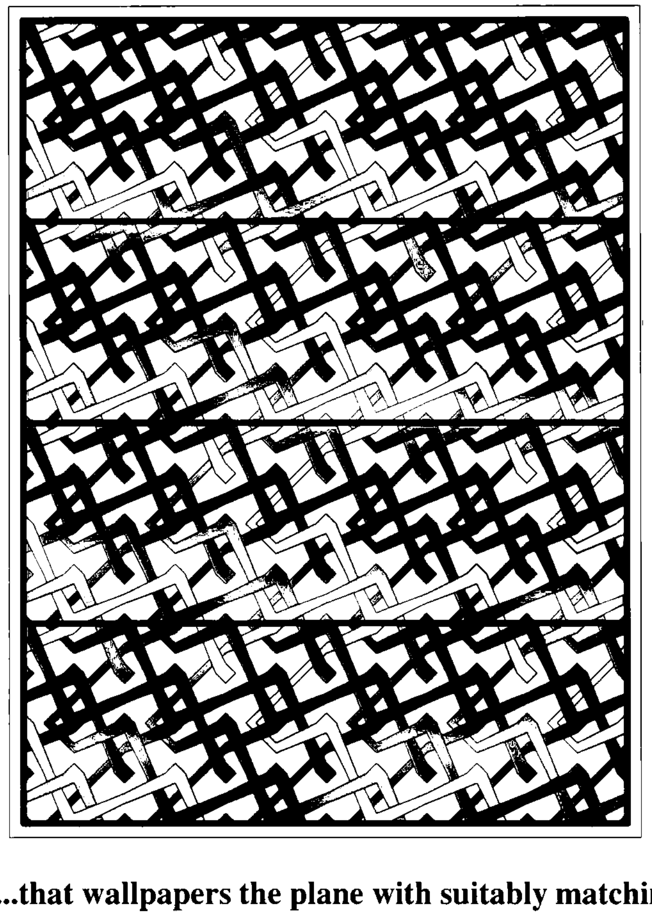

Figure 3: **...that wallpapers the plane with suitably matching colors.**

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Acknowledgements

M.C. Escher is a controversial figure in the Art World because of the innovative and mathematical methods that he employed. In the literature, one need not look far to find Escher ruminating about his peculiar position amongst the different cultures of science, mathematics and art; he never seems to have resolved to his own satisfaction where, in fact, he belonged. Vive la différence!

I am a Mathematician and now nearly have the good fortune to be called a Computer Scientist. Though many aspects of the two cultures are at odds, there are some wonderful exceptions; the group of Theoretical Computer Scientists is one such. It is an honor to count myself a member of that group. It seems fitting, therefore, that the problem addressed by this thesis was inspired by some of the earlier attempts of M.C. Escher to automate combinatorial patterns.

This work was not done in isolation: I am grateful to my fellow Theoreticians at the University of British Columbia for their collegiality and collaboration. In particular, I would like to thank Francois Anton, Alex Brodsky, Allen Clement, Stephane Durocher, and Mark McCann for many good conversations, both scholarly and otherwise. I thank my committee members Anne Condon, Will Evans, and Joel Friedman for their ideas and input, and for their support for this work. I am grateful to M.C. Escher, Rick Mabry, Doris Schattschneider, and Stan Wagon for their colorful combinatorial investigations that led to my current work. I thank the latter three for the use of their *Mathematica* package Escher.m, which was an invaluable investigative tool, and is also responsible for some of the graphics in this thesis. Special thanks to Doris Schattschneider for reading this manuscript and for her many insightful comments and suggestions. And I thank my family for their support for and acceptance of my nonstandard career choices.

Finally, I offer my deepest respect and appreciation to my supervisors, David Kirkpatrick and Nick Pippenger. I have greatly enjoyed and benefited from their breadth and depth of knowledge in concert with their humor and good cheer. The opportunity to work with them has been a rare and special privilege.

ELLEN GETHNER

The University of British Columbia
April 2002

To Maurits Cornelis Escher

Chapter 1

Preliminary Remarks

We begin by giving some examples of problems that have been studied in computer science and that are related either directly or indirectly to the problem that is the focus of this thesis. The areas that we will consider are **decidability of certain kinds of tiling problems**, and **computer graphics and aesthetics**.

1.1 Decidability

A problem, phrased as language membership, is said to be **decidable** [Pa94, Co86] if regardless of the input, there is procedure (algorithm) that outputs the correct answer of "yes" or "no" in a finite number of steps. Some examples are

- **Travelling Salesperson** Given a finite set of cities $C = \{c_1, \ldots, c_k\}$, a distance function $d : C \times C \rightarrow \mathbb{Z}$ (the integers), and a bound B, is there a tour that visits each city exactly once, ends where it began, and does so with an accumulated distance of at most B?

- **FSA** Do two particular finite state automata recognize the same language?

- **Tiling or Domino Problem** Given infinitely many copies of a finite set of unit square tiles with colored edges, can the plane be tiled by translation in such a way that the sides abut and colors match?

The bulk of this thesis considers a variation on the tiling problem whose origin lies in sketchbooks of M.C. Escher [Sc90, Er76]; it will be described in Chapter 2.

We mention some results that are related to the Domino problem. A conjecture of H. Wang [Wa74], that any set of tiles that tile the plane must admit a periodic tiling of the plane, was proved false by R. Berger [Be66] who produced a set of 20426 tiles that admit no periodic tiling; Berger later produced another such aperiodic set of tiles with cardinality 104. R. Robinson [Ro71] constructed an aperiodic set of tiles with only six elements. The Penrose tiles [Pe78, GS87] are a pair of quadrilateral tiles together with matching conditions that admit no periodic tiling of the plane. To date, it is not known if there exists a single tile (a connected region of $\mathbb{R}^2$) that only tiles the plane aperiodically. Problems of the tiling ilk have captured the attention of scholars in many areas, included among them mathematicians (geometry and number theory), computer scientists (decision problems, verification, graphics), artists (Alhambran designs), physicists and chemists (crystallography), and philosophers to name a few. See for example, [Ma76], [Ho79], [Es86], [Sc90], [Pe78],[Er76], [KS00a], [KS00b], and [MR01].

Outputting a "yes" or "no" answer upon any input to the original square tile problem is known to be equivalent to the halting problem, and so is undecidable [Pa94]. As such, not surprisingly, variations on the problem have surfaced. For example, Szegedy [Sz98] expands on the notion of tile T and allows as acceptable input a finite union of elements from $\mathbb{Z} \times \mathbb{Z}$ (lattice points in the plane), which is called a **finite cluster**. A tiling is then a covering of $\mathbb{Z} \times \mathbb{Z}$ with nonoverlapping translates of T. For an arbitrary such T no algorithm is known to determine whether or not T tiles $\mathbb{Z} \times \mathbb{Z}$, and Szegedy studies two special instances of the problem: $|T| = p$, where p is a prime number, and $|T|$=4. In both cases he gives an efficient algorithm that decides whether or not an arbitrary T with the appropriate cardinality tiles $\mathbb{Z} \times \mathbb{Z}$. He also generalizes the problem away from $\mathbb{Z} \times \mathbb{Z}$ to an abstract problem for arbitrary finitely generated abelian groups.

1.2 Graphics and Aesthetics

Many of the ideas behind the work of M.C. Escher lend themselves to computer automation. Two notable examples are

- C. Kaplan and D. Salesin's *Escherization* [KS00a],

- 1. D. Schattschneider's *Escher's Combinatorial Patterns,* [Sc97, Sc90] and
  2. R. Mabry, S. Wagon, and D. Schattschneider's *Automating Escher's Combinatorial Patterns* [MWS96].

The former paper presents an algorithm that, upon given any motif (in their case, a decorated subset of $\mathbb{R}^2$, referred to as a "closed Figure"), outputs a new motif that is "close" to the original and that tiles the plane (for examples, see [KS00b]). The technique used is simulated annealing, and their algorithm performs well on many convex or nearly convex motifs, and hence on many of Escher's original designs. The authors maintain that "Unlike most research projects in computer graphics, this one is motivated more by intellectual curiosity than by practical import."

The latter papers inspired the problem that is addressed in this thesis. Like the tiling problems mentioned in Section 1.1, the input will be information contained in a closed unit square, but with more complicated matching conditions than those imposed on the boundaries. The conditions arise from a problem of aesthetics and visualization, and in particular on some of the sketches found in Escher's notebooks [Sc90]. The problem is difficult to state in brief terms, but can be illustrated visually. We will do so in Chapter 2. At the outset, the problem may be viewed as a decidability problem (the question of the existence of a particular geometric object) that will eventually be answered in the affirmative. An affirmative answer to the question and an array of interesting examples provoke questions of classification. Moreover, underlying the work is that we seek an efficient (polynomial-time) algorithm that constructs what we will define as a Big Tile.

Chapter 2

Motivation and History

A problem inspired by M.C. Escher that has recently initiated a variety of papers [Da97, Ge02, MWS96, Sc97, Wa99] is described as follows [Es86, Sc90]. Produce a potato and a sharp knife. Cut the potato in half and square off the flat face of one of the halves. Carve an interesting design into the square face. Call this design a motif. Also consider the designs created by the cyclic group of rotations acting on the square-with-carved-design. Each such new design is called an **aspect** of the original motif. Ultimately, use the four aspects of the motif as ink stamps. Escher's idea, which he hoped to sell to a tiling company, was to make a square tile to be used for filling the plane with a periodic pattern. For example, in the grid shown in Figure 2.1 stamp each subsquare with any of the four aspects of the motif (one of Escher's own) in Figure 2.2.

There are four subsquares and thus $4^4 = 256$ tiles that can be produced. Take a particular tile T and create a pattern in the Euclidean plane by taking the image of T under $\mathbb{Z} \times \mathbb{Z}$. The result is a doubly periodic wallpaper pattern. See Figures 2.3 and 2.4. Let T_1 and T_2 be two tiles and W_1, W_2 be their respective doubly periodic wallpaper patterns. Tiles T_1 and T_2 are **inequivalent** if for every isometry σ we have $\sigma(W_1) \neq W_1$.

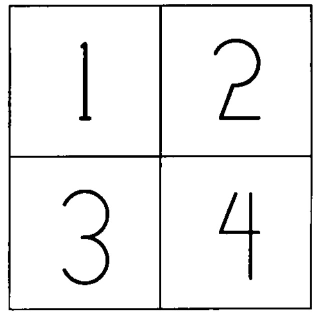

Figure 2.1: **Grid of subsquares to be used as a template for a 2×2 Escher tile**

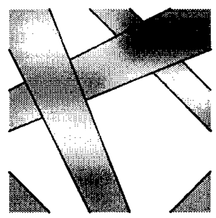

Figure 2.2: **A motif *M* designed by M. C. Escher**

Each tile yields a periodic wallpaper pattern, but some different-looking tiles yield wallpaper patterns that are equivalent up to isometry. Escher, who wanted to sell his idea to a tile company, wondered about the following question [Sc90, Sc97]: up to isometry, how many inequivalent 2×2 tiles in up to four aspects are there? The unexpected answer, which Escher calculated by brute force, is 23. Schattschneider in [Sc97] verified Escher's calculation by way of Burnside's Lemma. Gethner generalized Escher's result by giving an exact formula for the number of inequivalent $m \times m$ tiles in up to four aspects [Ge02] for any $m \in \mathbb{Z}^+$.

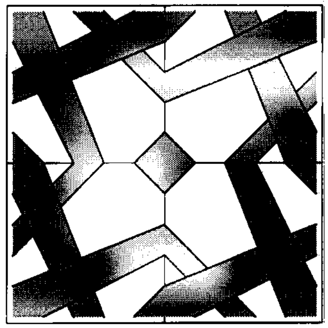

Figure 2.3: **2×2 Escher tile that uses all four aspects of the motif in Figure 2.2**

In particular

Theorem 2.0.1 (Gethner): *Let $N_4(m)$ denote the number of inequivalent $m \times m$ Escher tiles with motifs in up to four aspects. Then*

$$N_4(m) = \frac{1}{4m^2}[\sum_{k|m}(2k\phi(k) - \phi(k)^2)4^{m^2/k} + \sum_{k|m}(2^{r_k} - 2)4^{m^2/k}$$

$$+\sigma(m)(m^2 4^{m^2/4} + 3m^2 4^{m^2/2-1})], \tag{2.1}$$

where

$$\sigma(m) = \begin{cases} 1 & \textit{if } m \textit{ is even} \\ 0 & \textit{if } m \textit{ is odd,} \end{cases}$$

r_k is the number of (not necessarily distinct) prime divisors of k, and ϕ is the Euler ϕ function.

Escher also made use of the unused half of the potato: he carved a reflection of the original motif, thereby increasing the total number of aspects to eight. The number of inequivalent 2×2 tiles in up to eight aspects turned out to be 154, which was verified computationally by Davis [Da97].

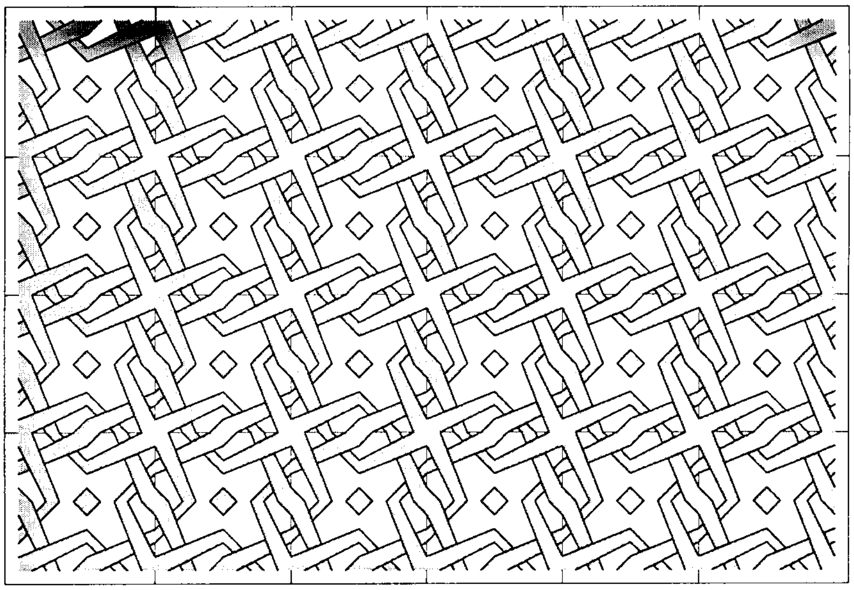

Figure 2.4: **Fragment of doubly periodic wallpaper pattern produced by the Escher tile in Figure 2.3**

Escher, Mabry, Schattschneider, and Wagon [MWS96, Sc90] were inspired to think about coloring and subsequently automating the coloring. The purpose of Section 2.1 is to make precise the set-up for the coloring question to which they alluded. The purpose of the remaining chapters is to lay the foundation for an affirmative answer to a question of existence, present an efficient and implementable algorithm that produces a Big Tile, to allude to aspects of classification, and to suggest possibilities for future research.

2.1 Statement of Problem

Escher's problem appears to start in the realm of geometry and in modern terms, in visualization. To give a precise statement of the problem, we need many definitions.

2.2 Weighty Definitions

Let $S_{i,j}$ be the subset of $\mathbb{R}^2$ given by $\{(x, y) : i - \frac{1}{2} \leq x \leq i + \frac{1}{2}, j - \frac{1}{2} \leq y \leq j + \frac{1}{2}\}$. That is, $S_{i,j}$ is a closed square region with area one and center (i, j) and $i, j \in \mathbb{Z}$. In the previous section, Escher used a motif that was composed of many individual motif pieces.

Definition 2.2.1 (Motif Piece): *A* **motif piece** (m, i, j) *is a connected subset of* $S_{i,j}$ *such that* $\partial S_{i,j} \bigcap m$ *is a* finite *union of closed intervals, where* $\partial S_{i,j}$ *is the boundary of* $S_{i,j}$.

In the long run, the problem we are after begins with some design composed of motif pieces inside a unit square. For the coloring problem that we will address, it does not matter what method the artist used to construct the design. The previous section simply describes one method (among infinitely many) that an artist could use.

Definition 2.2.2 (Escher tile): *An* **Escher tile** T *is given by* $T = \{(m_1, 0, 0), (m_2, 0, 0), \ldots, (m_k, 0, 0)\}$, *where* T *is a finite nonempty set of motif pieces satisfying*

1. $(m_i, 0, 0) \bigcap (m_j, 0, 0) \bigcap \partial S_{0,0} = \varnothing$ *whenever* $i \neq j$ *(no pair of motif piece intersect on the boundary),*
2. $(m_i, 0, 0) \not\subseteq \partial S_{0,0}$ *(no motif piece is contained in the boundary), and*
3. $(m_i, 0, 0) \not\subseteq \bigcup_{\substack{j_s \in A \\ j_s \neq i}} (m_{j_s}, 0, 0)$ *for* $A \subset \{1, \ldots, k\}$ *(no motif piece is contained in the union of other motif pieces).*

See Figure 2.5 for an illustration of one motif piece among many that define an Escher tile.

Remarks:

- An Escher tile is an artistic design contained in a closed unit square.
- Given an Escher tile $T = \{(m_1, 0, 0), (m_2, 0, 0), \ldots, (m_k, 0, 0)\}$ the motif pieces are distinct elements of a set, though as subsets of $\mathbb{R}^2$, it may be the case that $(m_i, 0, 0) \bigcap (m_j, 0, 0) \neq \varnothing$ for some $i \neq j$.
- In the ensuing discussions for which the location of an abstract motif piece (m_t, i, j) is not relevant, we denote (m_t, i, j) by a capital letter such as M or N or perhaps M_t or N_t.

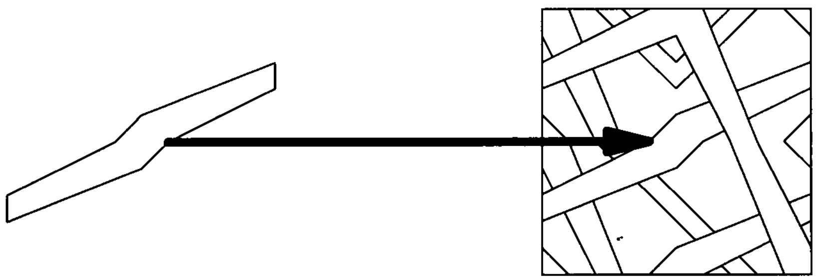

Figure 2.5: **One motif piece among many inside an Escher tile: Definitions 2.2.1 and 2.2.2**

Definition 2.2.3 (Wallpaper Pattern): *Given an Escher tile T, the* **Escher wallpaper pattern generated by** T *is the periodic plane pattern arising from taking the elementwise image of T under the map $\mathbb{Z} \times \mathbb{Z}$, and is denoted by $Wall(T)$.*

Notation: We will generate the map $\mathbb{Z} \times \mathbb{Z}$ by α and β, where for any point $(x, y) \in \mathbb{R}^2$, $\alpha((x, y)) := (x + 1, y)$ and $\beta((x, y)) := (x, y + 1)$. In general for integers r and s, $\alpha^r \beta^s((x, y)) = (x + r, y + s)$. For our purposes, given an arbitrary motif piece (m_t, i, j) we write $\alpha^r \beta^s((m_t, i, j)) = (m_t, i + r, j + s)$. See Figure 2.6 for a fragment of the wallpaper pattern generated by the Escher tile in Figure 2.5.

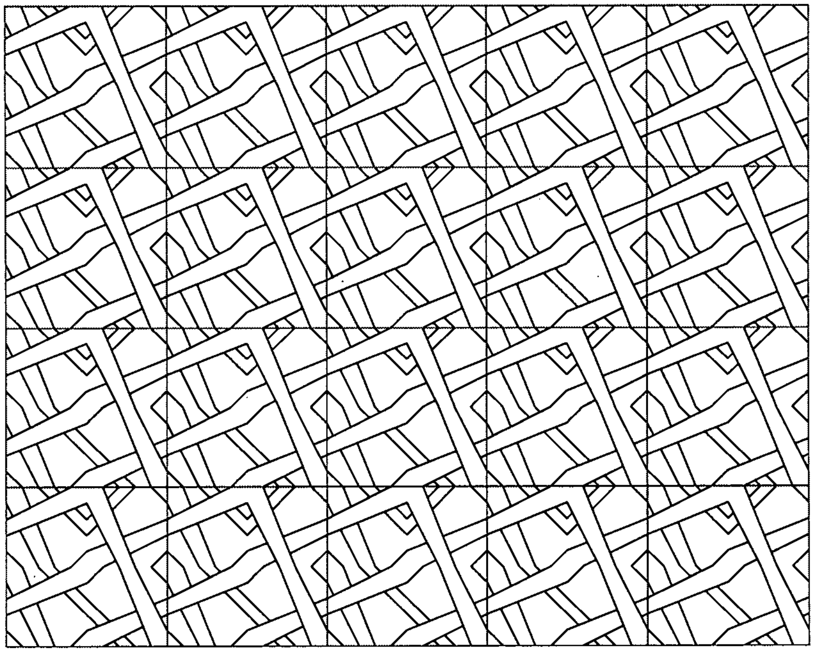

Figure 2.6: **Fragment of the Escher wallpaper pattern generated by the Escher tile in Figure 2.5**

With an Escher tile $T = \{(m_1, 0, 0), (m_2, 0, 0), \ldots, (m_k, 0, 0)\}$, the wallpaper pattern given by $Wall(T)$ is the set $\{\alpha^r\beta^s((m_1, 0, 0)) : r, s \in \mathbb{Z}\} \bigcup \{\alpha^r\beta^s((m_2, 0, 0)) : r, s \in \mathbb{Z}\} \bigcup \ldots \bigcup \{\alpha^r\beta^s((m_k, 0, 0)) : r, s \in \mathbb{Z}\}$. That is, $Wall(T)$ is simply the (necessarily infinite) set of all possible East-West and North-South integer translates of the elements of T.

Definition 2.2.4 (Location of a Motif Piece): *A motif piece* (m_t, r, s) *has* **location** (r, s)*, the center of the unit square in which* m_t *resides.*

In Definition 2.2.2, the definition of Escher tile, we choose the motif pieces to have location $(0, 0)$ for no other reason than convenience; any fixed location would serve the same purpose.

Definition 2.2.5 (Contiguous motif pieces): *A motif piece is* **contiguous with itself**. *Two motif pieces M and N in different locations are* **contiguous** *if $M \bigcap N \neq \varnothing$.*

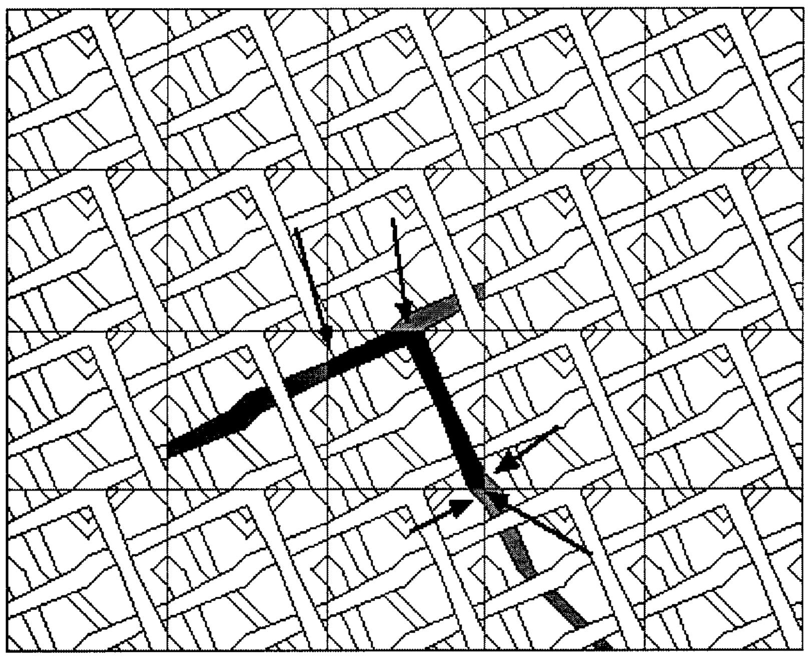

Figure 2.7: **The black motif piece is contiguous with five other motif pieces: Definition 2.2.5**

That is, a motif piece is contiguous with itself and two motif pieces in distinct locations are contiguous if and only if they intersect on the boundaries of translates of $S_{0,0}$; the intersection of distinct contiguous motif pieces is necessarily a finite union of vertical and horizontal line segments, some of which may be points [St81]. Figure 2.7 shows several contiguous motif pieces.

Definition 2.2.6 (Related motif pieces): *Two (not necessarily distinct) motif pieces M and N are said to be* **related** *if there exists a finite sequence of motif pieces $\{N_1, N_2, \ldots, N_t\}$ such that $M = N_1$, $N = N_t$ and N_i is contiguous with N_{i+1} for $i = 1, \ldots, t-1$.*

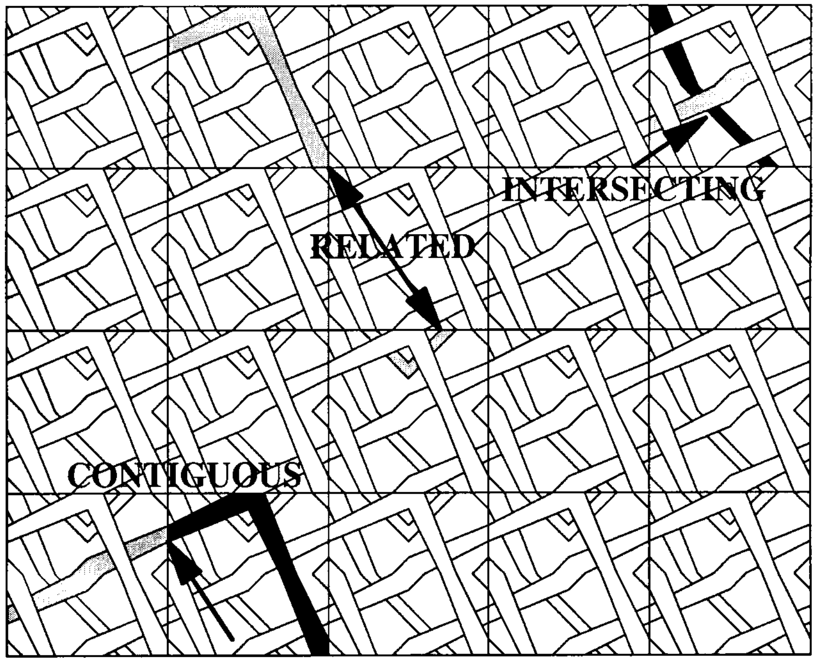

Figure 2.8: **Contiguous, related, and intersecting motif pieces: Definition 2.2.5 and Definition 2.2.6**

See Figure 2.8 for an example of contiguous, related, and intersecting motif pieces.

The subset of $\mathbb{R}^2$ given by $\bigcup_{i=1}^{t} N_i$ is necessarily connected, whereas two or more motif pieces in the same location that intersect are not necessarily related. Contiguous motif pieces are related, but related motif pieces are not necessarily contiguous.

Definition 2.2.7 (Wallpaper component): *Given a motif piece M, the* **wallpaper component generated by** M, *denoted* $W(M)$, *is the set* $\{N : N$ is a motif piece related to $M\}$.

Clearly "related" is an equivalence relation on the set of all motif pieces in $Wall(T)$ arising from a given Escher tile T. Consequently, an Escher wallpaper component is an equivalence class of motif pieces whose elementwise union is a connected subset of $\mathbb{R}^2$. Thus, the set of Escher wallpaper components partitions $Wall(T)$ into (possibly infinitely many) disjoint sets, each of whose elementwise union is a connected subset of $\mathbb{R}^2$.

It is important to continue to emphasize the distinction between set intersection and the intersection of motif pieces (subsets of $\mathbb{R}^2$), particularly in light of the next definition.

Definition 2.2.8 (Overlapping components): *Two distinct wallpaper components $W(M)$ and $W(N)$ are said to* **overlap** *if there exists $(m_s, i, j) \in W(M)$ and $(m_t, i, j) \in W(N)$ such that $(m_s, i, j) \bigcap (m_t, i, j) \neq \varnothing$.*

See Figure 2.9 for an example of a pair of distinct overlapping wallpaper components. The original Escher tile can be seen in any subsquare.

The following lemma is a direct consequence of Definition 2.2.8 and the $\mathbb{Z} \times \mathbb{Z}$ periodicity of $Wall(T)$.

Lemma 2.2.9 (Overlaps of $W((m_1, 0, 0))$): *Let $T = \{(m_1, 0, 0), \ldots, (m_k, 0, 0)\}$ be an Escher tile. Distinct wallpaper components $W((m_1, a, b))$ and $W((m_1, u, v))$ overlap if and only if $W((m_1, 0, 0))$ and $W((m_1, a-u, b-v))$ overlap.*

At last we have the means to give the definition of Big Tile, the idea of which leads to nontrivial questions of existence and classification. In a word, a Big Tile for an Escher tile T will be an $m \times n$ rectangular region consisting of mn concatenated copies of T with lower left subsquare centered at $(0, 0)$ and sides parallel to the standard axes; each motif piece in the Big Tile will be assigned a color from a set of cardinality Δ. Finally, the image of the Big Tile under $m\mathbb{Z} \times n\mathbb{Z}$ produces the original pattern $Wall(T)$ colored in such a way that wallpaper components are

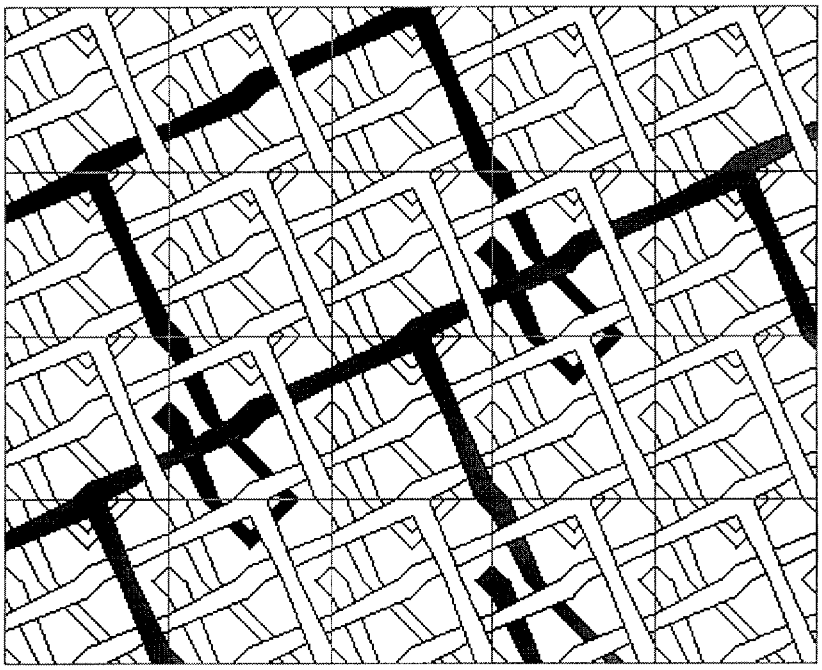

Figure 2.9: **Distinct and overlapping wallpaper components: Definition 2.2.8**

uniformly colored and distinct overlapping wallpaper components are colored with different colors. We give the formal definition next.

Definition 2.2.10 (Big Tile): *Given an Escher tile* $T = \{(m_1, 0, 0), \ldots, (m_k, 0, 0)\}$ *a* Δ*-colored* $m \times n$ **Big Tile for** T *denoted* $B_T(\Delta, m, n)$*, is laden with the following requirements. Let* $M, N \in Wall(T)$ *be arbitrary.*

- *There exists a function,* C_Δ*, that assigns some color from among* Δ *colors to each motif piece contained inside the* $m \times n$ *region that contains the Big Tile. That is* $C_\Delta : \{m_1, \ldots, m_k\} \times m\mathbb{Z} \times n\mathbb{Z} \to COLORS$ *is onto and* $|COLORS| = \Delta$*. We write* $(m_s, i, j)_\Delta$ *to signify the colored motif piece* m_s *in location* (i, j)*.*

- $B_T(\Delta, m, n) = \bigcup_{(i,j) \in \mathbb{Z}_m \times \mathbb{Z}_n} \bigcup_{s=1,\ldots,k} (m_s, i, j)_\Delta$*, and after "tiling" the plane with the image of* $B_T(\Delta, m, n)$ *under* $m\mathbb{Z} \times n\mathbb{Z}$*, we have*

1. *for every* (m_s, i_1, j_1) *and* $(m_t, i_2, j_2) \in W(M)$ *we have* $C_\Delta((m_s, i_1, j_1))$ $= C_\Delta((m_t, i_2, j_2))$ *(alternatively we write* $C_\Delta(W(M)) = C_\Delta(W(N))$*),*

2. *if* $W(M)$ *and* $W(N)$ *overlap and are distinct, then* $C_\Delta(W(M)) \neq C_\Delta(W(N))$*, and*

3. *there does not exist a* Δ*-colored Big Tile,* B'_T *such that* B_T *is composed of concatenated copies of* B'_T *(with sides abutting).*

The Main Questions:

1. Given an arbitrary Escher tile T, does a Big Tile B_T for T exist?

2. If B_T exists for a particular Escher tile, must it be unique?

3. If B_T exists and is not unique, what can be said about the set of Big Tiles for T? How do the number of colors and size of a Big Tile depend on the size of the input (number of motif pieces and boundary intersections)?

4. If the answer to Question 1 is "yes" then is there an efficient algorithm to construct B_T?

5. What can be said about the classification of the set of Big Tiles for an arbitrary Escher tile T?

A picture is worth 10,000 words. In the next section we give an example.

2.3 An Escher tile and Two Big Tiles

The Escher tile given in Figure 2.10 was designed by M.C. Escher and produced by a *Mathematica* package implemented by Mabry and Wagon in [MWS96, Wa99]. Recall that there are 154 inequivalent Escher tiles that arise from taking rotations and reflections of Escher's original motif (see Chapter 2) and arranging them in a 2×2 grid. By way of the same package it is shown by trial and error that a Big Tile exists for each of these 154 Escher tiles. Specifically, upon input of an Escher tile T together with a *correct guess* of the Big Tile dimensions, a graphical representation of a Big Tile is given as output. However, if an incorrect guess is made (say $m \times n$), then an $m \times n$ tile C_T will be returned: the image of C_T under $m\mathbb{Z} \times n\mathbb{Z}$ yields a wallpaper pattern whose wallpaper components are uniformly colored but for which there exists at least one pair of overlapping components that are colored the same.

See Figure 2.12 for an example of a 3-colored 1×3 Big Tile for the Escher tile in Figure 2.10.

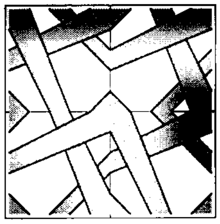

Figure 2.10: **Another Escher tile produced by the motif in Figure 2.2**

The Big Tile for the Escher tile of Figure 2.10 is not unique. A 4×4 Big Tile is shown in Figure 2.13 that requires four colors.

In fact, as will become evident in Section 5.1, there are infinitely many essentially different Big Tiles for the Escher tile in Figure 2.10.

2.4 Prior Work on Escher tiles

Aside from Escher's original sketches, only the paper by Mabry, Wagon, and Schattschneider [MWS96] has addressed the question of the decidability of the Big Tile problem, though they present the problem in terms of programming and visualization.

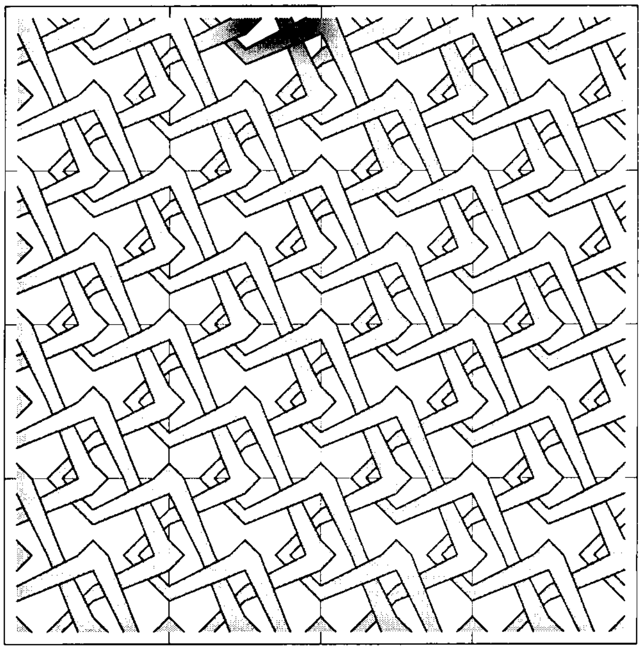

Figure 2.11: **Fragment of singly periodic wallpaper pattern produced by the Escher tile in Figure 2.10**

They use a brute-force approach to show that Big Tiles exist for each of 154 Escher tiles that arise from the motif and the reflection of the motif in Figure 2.2.

Their program works as follows:

1. Upon input of a given Escher tile T, make a guess as to the dimensions of a potential Big Tile for T. Suppose the guess is $m \times n$.

2. One copy of T is placed in each of the $m \times n$ subsquares, and every intersection of a motif piece with the boundary of its unit square in each of the $m \times n$ subsquares is labelled with a variable.

3. If two or more boundary intersections in a given subsquare belong to the same motif piece, then the same variable is assigned to each such boundary intersection.

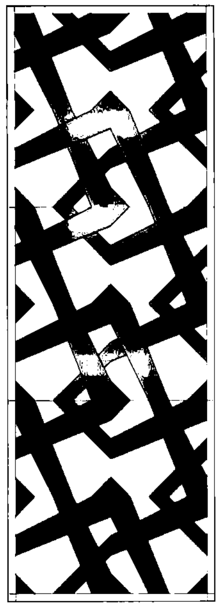

Figure 2.12: **A three-colored 1×3 Big Tile for the Escher tile in Figure 2.10**

4. Two adjacent copies of T may have nontrivial intersection along a boundary. Suppose copy A has a boundary intersection labelled x and adjacent copy B has a boundary intersection labelled y, and further suppose that the boundary intersections corresponding to x and y intersect. Assign $x = y$ and do so for all such contiguous motif pieces (see Figure 2.14). This subroutine ensures that motif pieces inside the $m \times n$ region that belong to a connected subset of $\mathbb{R}^2$ will be assigned the same color, and gives rise to a set of many equations and many unknowns (though the system is sparse) to be solved by techniques from linear algebra.

5. Construct a graph which contains a vertex for each wallpaper component, and two vertices are adjacent if and only if the corresponding connected regions inside the $m \times n$ region overlap.

6. Vertex-color this graph and if possible assign different colors to overlapping

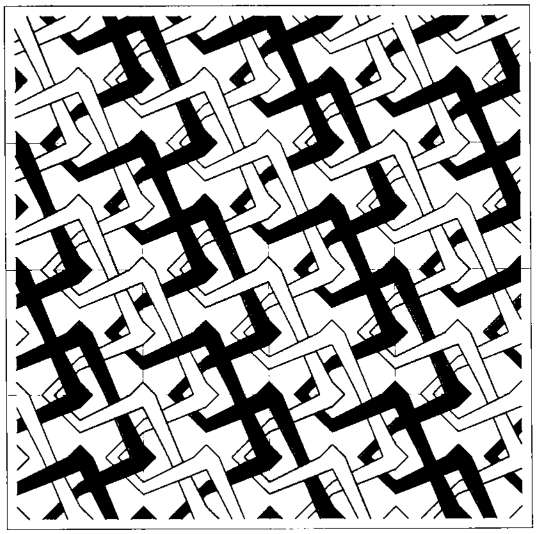

Figure 2.13: **A four-colored 4×4 Big Tile for the Escher tile in Figure 2.10**

wallpaper components.

7. If an incorrect Big Tile size is input, then the program returns as visual output a rectangular tile whose internal connected components are uniformly colored and whose opposing boundaries have correctly matching colors.

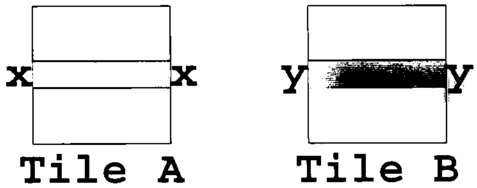

Figure 2.14: $x = y$

The authors construct a database of Big Tile sizes, one for each of the 154 Escher tiles constructed by way of Escher's original motif. They did so by brute force: they looked at large fragments of wallpaper and made guesses for Big Tile sizes. Most, though not all, of the Big Tile sizes yield minimally colored Big Tiles.

In the next chapter we discuss a method to prove the existence of a Big Tile for an arbitrary Escher tile.

Chapter 3

Escher tiles: Toolbox

3.1 Graph Theory

Two basic tools are required for the Big Tile existence theorem: the period graph of T and the overlap graph of T.

The period graph of T is a directed labelled multigraph whose vertices are in one-to-one correspondence with the motif pieces of T, and whose directed labelled edges identify contiguous motif pieces. That is, an Escher tile in the $(0, 0)$ position is surrounded by eight other Escher tiles whose centers are (i, j) with $i, j \in \{-1, 0, 1\}$ and can be thought of as being north, south, east, west, northwest, northeast, southwest, or southeast of the tile in the "home" position. In the period graph, two vertices m_s and m_t are adjacent if and only if $(m_s, 0, 0) \cap (m_t, i, j) = \varnothing$ for some $i, j \in \{-1, 0, 1\}$ (with at least one of i or j not equal to 0); the directed edge (m_s, m_t) is labelled with the vector $[i, j]$. See Figure 3.1 for an Escher tile together with it's eight surrounding Escher tiles. Figure 3.2 shows the period graph for the Escher tile in Figure 3.1.

The vertices of the overlap graph are equivalence classes of vertices of the period graph: two distinct period graph vertices will be equivalent exactly when the two corresponding motif pieces are related; the vertices of the overlap graph will be called E-vertices. Two equivalent vertices necessarily belong to the same Escher wallpaper component, and hence it is worthwhile early on to identify such occurrences. Two distinct E-vertices V_1 and V_2 will be adjacent in the overlap graph of T if and only if there is non-trivial intersection in $\mathbb{R}^2$ among motif pieces associated with V_1 and motif pieces associated with V_2. That is, adjacent E-vertices in the overlap graph belong to distinct overlapping Escher wallpaper components must receive different colors in any Big Tile for T.

Precisely,

Definition 3.1.1 (Period Graph of an Escher tile): *Let* $T = \{(m_1, 0, 0), (m_2, 0, 0), \ldots, (m_k, 0, 0)\}$ *be an Escher tile. The* **period graph of** T, *denoted* G_T, *is a labelled directed graph constructed by the following rules.*

1. **(Vertices)** $V(G_T)$, *the vertices of* G_T, *are in one-to-one correspondence with the motif pieces of* T. *Define* v_i *to be the vertex that corresponds to* $(m_i, 0, 0)$ *for* $i = 1, \ldots, k$.

2. **(Edges)** $E(G_T)$, *the edges of* G_T, *are directed and given by* $(v_s, v_t) \in E(G_T)$ *labelled with* $[i, j]$ *if and only if*

   - $(i, j) \neq (0, 0)$, *and*
   - $(m_s, 0, 0) \bigcap (m_t, i, j) \neq \varnothing$.

We write $\ell(v_s, v_t) = [i, j]$ *to denote the* **vector label** *of directed edge* (v_s, v_t). *On those occasions for which we must identify each coordinate of a vector label separately, we define* $\ell_1(v_s, v_t) = i$ *and* $\ell_2(v_s, v_t) = j$.

In essence, the period graph is a road map that tells one how to walk along a wallpaper component without fear of derailment.

The following proposition follows directly from Definition 3.1.1.

Proposition 3.1.2 (Vector Labels and Edges are Bidirectional Pairs): *Let T be an Escher tile with period graph G_T. Then $(v_s, v_t) \in E(G_T)$ if and only if $(v_t, v_s) \in E(G_T)$. Moreover, $\ell(v_s, v_t) = -\ell(v_t, v_s)$.*

Most of the time, we restrict our attention to an Escher tile whose period graph is connected, although in general a period graph need not be connected. Each connected component of a period graph corresponds to an Escher tile whose motif pieces are a subset of T; we make this idea precise in the next definition.

Definition 3.1.3 (Escher tile Induced by a Subgraph of the Period Graph): *Let T be an Escher tile with period graph G_T and suppose G_T has N connected components given by $G_1,\ldots,G_N$. Let $i \in \{1, \ldots, N\}$ and suppose $V(G_i) = \{v_{i_1}, v_{i_2},, \ldots, v_{i_s}\}$ for some $s \leq q$. Then the* **Escher tile induced by** *G_i is the set of motif pieces $\{(m_{i_1}, 0, 0)\ (m_{i_2}, 0, 0), \ldots, (m_{i_s}, 0, 0)\}$.*

Absent from the period graph is information about when and if distinct wallpaper components overlap. That is the purpose of the next two definitions.

Definition 3.1.4 (E-vertex): *Given the period graph G_T for an Escher tile T, v_i and v_j in $V(G_T)$ are said to be* **equivalent** *if and only if $(m_i, 0, 0)$ and $(m_j, 0, 0)$ are related. The equivalence class of $v_i \in V(G_T)$ under the relation related is said to be an* **E-vertex**, *and is denoted $[v_i]$.*

Note that an E-vertex represents a collection of motif pieces in a single Escher tile that belong to the same component in the wallpaper pattern. We have the machinery to define the overlap graph of T.

Definition 3.1.5 (Overlap Graph of an Escher tile): *Let T be an Escher tile with period graph G_T. The* **overlap graph of** T*, denoted O_T, is a simple, undirected, unlabelled graph constructed by the following rules.*

1. **(Vertices)** *The vertex set of O_T, denoted $V(O_T)$, is exactly the set of E-vertices of $V(G_T)$.*

2. **(Edges)** *Suppose $[v_i], [v_j] \in V(O_T)$ with $[v_i] \bigcap [v_j] = \varnothing$. In other words, $[v_i]$ and $[v_j]$ are distinct. Define $[v_i]$ to be adjacent to $[v_j]$ in O_T if and only if $\exists v_w \in [v_i]$ and $v_z \in [v_j]$ such that $(m_w, 0, 0) \bigcap (m_z, 0, 0) \neq \varnothing$.*

In particular, a pair of adjacent E-vertices corresponds to a pair of unrelated motif pieces contained in the unit square $S_{0,0}$, and therefore are elements of distinct **overlapping** wallpaper components. This information is crucial to have since eventually we must color such a pair of wallpaper components with distinct colors. Figure 3.3 shows the overlap graph for the Escher tile in Figure 3.1.

Often it will be helpful to use the undirected unlabelled graph that underlies the period graph.

Definition 3.1.6 (Agglomerated Period Graph): *Let G_T be the period graph for Escher tile T. The* **agglomerated period graph of** G_T *is the undirected graph obtained from G_T by dropping the order from the edges in $E(G_T)$, removing the vector labels and multiple edges. The agglomerated period graph is denoted by $\hat{G}_T$.*

Though the agglomerated period graph has no multiple edges, it may have loops. An analysis of all nontrivial simple cycles (including loops) in the agglomerated period graph will extract inherent periodicities of the wallpaper pattern.

Definition 3.1.7 (Agglomerated Spanning Tree): *Let T be an Escher tile with period graph G_T and agglomerated period graph $\hat{G}_T$. An* **agglomerated spanning tree of** G_T *is a spanning tree of $\hat{G}_T$, and is denoted by $\hat{S}_T$.*

Once we have an agglomerated spanning tree, useful information can be gained by reinstating the information about the edges of $\hat{S}_T$ that was suppressed when G_T was agglomerated.

Definition 3.1.8 (Fat Spanning Tree): *Let T be an Escher tile with agglomerated spanning tree $\hat{S}_T$. A* **fat spanning tree of** $\hat{S}_T$*, denoted S_T, is the agglomerated spanning tree with the multiplicities, directions, and vector labels inherited from the corresponding edges of the period graph G_T.*

See Figure 3.4 for the fat spanning tree for the period graph in Figure 3.2. It will be useful to keep track of the edges that were removed from both the period graph and agglomerated period graph when the agglomerated spanning tree and fat spanning tree were constructed.

Definition 3.1.9 (Agglomerated Removed Edges): *Let T be an Escher tile with agglomerated period graph $\hat{G}_T$ and agglomerated spanning tree $\hat{S}_T$. The* **removed edges of** $\hat{G}_T$ *is the set $E(\hat{G}_T) \setminus E(\hat{S}_T)$, and is denoted $\hat{R}_T$.*

That is, $\hat{R}_T$ is the set of edges that were removed from the agglomerated period graph when the agglomerated spanning tree was constructed.

Finally, we reinstate the directions, multiplicities and vector labels to elements of $\hat{R}_T$ in the next definition.

Definition 3.1.10 (Fat Removed Edges): *Let T be an Escher tile with agglomerated removed edges $\hat{R}_T$. The* **fat removed edges of** T*, denoted R_T, is $E(G_T) \setminus E(S_T)$. That is, R_T is the set $\hat{R}_T$ with the directions, multiplicities and vector labels reinstated.*

It is an easy consequence of Definitions 3.1.9 and 3.1.10 that $|R_T| = 2|\hat{R}_T|$.

Later on, for the purpose of coloring wallpaper components and in light of Lemma 2.2.9, it is important to identify

- $[r, s] \in \mathbb{Z}^2$ for which $W((m_1, 0, 0))$ does not overlap $W((m_1, r, s))$, and
- $[r, s] \in \mathbb{Z}^2$ for which $W((m_1, 0, 0)) = W((m_1, r, s))$.

The former vectors help to identify distinct wallpaper components that may legitimately be colored with the same color. On the other hand, the latter vectors are the periodicities inherent in the wallpaper pattern. We name these two kinds of vectors in the next definitions.

Definition 3.1.11 *(***Collision-Free Vector for** T**):** *Let T be an Escher tile whose period graph is connected. Any $[r, s] \in \mathbb{Z}^2$ for which $W((m_1, 0, 0))$ does not overlap $W((m_1, r, s))$ is a* **collision-free vector for** T*, and the set of all such vectors is denoted $A(T)$.*

At the other end of the spectrum, vectors that are "forbidden" (formally defined in Section 5.1) are essentially those vectors that are not collision-free. See Figure 3.10.

Definition 3.1.12 *(***Inherent Periodicities in** $Wall(T)$**):** *Let T be an Escher tile whose period graph is connected. Any $[r, s] \in \mathbb{Z}^2$ for which $W((m_1, 0, 0)) = W((m_1, r, s))$ is an* **inherent period of** $Wall(T)$.

The next chapter is devoted to extracting (inherent) periodicity properties and overlap information from the components of the graphs G_T and O_T. Sometimes G_T alone contains enough information to produce a Big Tile for T. On those occasions for which the graph O_T must be relied upon, the Big Tile problem gains more depth.

3.2 What the Graphs Have to Offer

Given an Escher tile T with period graph G_T, we associate a trail (any sequence of adjacent vertices) in G_T with a 2-dimensional vector value $[r, s] \in \mathbb{Z}^2$ by adding the vector labels of all edges in the trail. This idea will be made precise in Definition 3.2.1.

A trail in G_T encodes a walk on a fragment of a wallpaper component along (not necessarily distinct) contiguous motif pieces. Each step is either North, South, East, West, Northeast, Southeast, Southwest, or Northwest. Often we must keep track of the relative locations of motif pieces on such a walk. To do so, we define the vector value of a trail next.

Definition 3.2.1 (Vector value of a trail in G_T): *Let T be an Escher tile with period graph G_T and suppose $Tr = \{v_{a_1}, \ldots, v_{a_t}\}$ is a trail in G_T. The* **vector value** *of Tr is given by $\sum_{j=1}^{t-1} \ell(v_{a_j}, v_{a_{j+1}})$.*

That is, the vector value of a trail Tr is the sum over all vector labels of the directed edges in Tr. Furthermore, a trail $\{v_{a_1}, \ldots, v_{a_t}\}$ in G_T together with one motif piece (m_{a_1}, x, y) uniquely specifies a set of motif pieces, one for each vertex in Tr, that are related to (m_{a_1}, x, y). So, an Escher wallpaper walker who starts a walk on motif piece in location (x_1, y_1) and walks along trail Tr whose vector value is $[x_2, y_2]$ will finish the walk in location $(x_2 - x_1, y_2 - y_1)$.

Definition 3.2.2 (Set of Motif Pieces Induced by a Trail in G_T): *Let T be an Escher tile with period graph G_T and suppose $Tr = \{v_{a_1}, \ldots, v_{a_t}\}$ is a trail in G_T. For any $a, b \in \mathbb{Z}$, we define the* **set of motif pieces induced by** Tr **and** (m_{a_1}, a, b) *to be*

$$Tr((m_{a_1}, a, b)) := (m_{a_1}, a, b) \cup \bigcup_{i=2}^{t} \left(m_{a_i}, a - \sum_{j=1}^{i} \ell_1(v_j, v_{j+1}), b - \sum_{j=1}^{i} \ell_2(v_j, v_{j+1}) \right).$$

In other words, suppose we are standing on (m_{a_1}, a, b) and wish to walk along the sequence of contiguous (and hence related) motif pieces dictated by $Tr = \{v_{a_1}, \ldots, v_{a_t}\}$. We necessarily walk from (m_{a_1}, a, b) to a copy of m_{a_2}. The location of m_{a_2} is $(a - \ell_1(v_{a_1}, v_{a_2}), b - \ell_2(v_{a_1}, v_{a_2}))$. That is, we walk to a new location along a pair of contiguous motif pieces dictated by the vector label of (v_{a_1}, v_{a_2}). In general, suppose $(m_{a_{i-1}}, x, y)$ is the motif piece corresponding to $v_{a_{i-1}}$. Then $\big(m_{a_i}, x - \ell_1(v_{a_{i-1}}, v_{a_i}), y - \ell_2(v_{a_{i-1}}, v_{a_i})\big)$ is the motif piece corresponding to v_{a_i} in Tr. So, the coordinates of the location of m_{a_i} corresponding to $v_{a_1} \in Tr$ are obtained by keeping track of where we started (namely at (a, b)) and summing over all vector labels of consecutive edges in Tr up to and including the edge $(v_{a_{i-1}}, v_{a_i})$.

Definition 3.2.3 (Nontrivial and Trivial Circuits in $\hat{G}_T$): *Let Circ be a circuit in the period graph of an Escher tile. Circ is said to be* **nontrivial** *if the vector value of Circ in G_T is not $[0,0]$. A circuit whose vector value is $[0,0]$ is said to be* **trivial.**

It turns out that the non-trivial circuits in G_T (or lack thereof) will play an important role in the construction of a Big Tile. We will often exploit the association between circuits and vectors by referring to "linearly independent circuits" when the context is clear. Moreover, exploiting any spanning tree of the agglomerated period graph will give rise to a set of motif pieces, all related, that generate $Wall(T)$.

Definition 3.2.4 (Generating Motif Pieces): *Let $T = \{(m_1, 0, 0), \ldots, (m_k, 0, 0)\}$ be an Escher tile whose period graph G_T is connected. Define a* **set of generating motif pieces of** *T, denoted $gen(T, a, b)$, as follows.*

1. *Let S_T be the fat spanning tree of agglomerated spanning tree $\hat{S}_T$.*

2. *Let P_x be the unique path in $\hat{S}_T$ from v_1 to v_x for $x = 2, \ldots, k$ and $[i_x, j_x]$ be the vector value of P_x in S_T.*

Then **a set of generating motif pieces for** T *is* $gen(T, a, b) := \{(m_1, a, b), (m_2, a+i_2, b+j_2), \ldots, (m_k, a+i_k, b+j_k)\}$. *For* $s = 1, \ldots, k$, *we say that* $(m_s, a+i_s, b+j_s) \in gen(T, a, b)$ *is a* **generating motif piece** *for* T.

Most often we use $gen(T, 0, 0) = \{(m_1, 0, 0), (m_2, i_2, j_2), \ldots, (m_k, i_k, j_k)\}$. A set of generating motif pieces for an Escher tile is, in effect, a set of translated elements of T whose union is a rearrangement of the original motif pieces that forms a connected subset of $\mathbb{R}^2$. In fact an Escher tile whose period graph is connected is like a set of jumbled puzzle pieces that are unscrambled by gluing together the set of generating motif pieces with instructions from the fat spanning tree. Figure 3.5 is an example of an Escher tile together with a set of generating motif pieces. An example of an Escher tile, its period graph, a fat spanning tree and a set of generating motif pieces are given in Figures 4.1, 4.2, 4.4, and 4.5.

Definition 3.2.5 (Ghost Motif Piece and Ghost Vector): *Let* $T = \{(m_1, 0, 0), (m_2, 0, 0), \ldots, (m_k, 0, 0)\}$ *be an Escher tile whose period graph* G_T *is connected. Let* $\hat{S}_T$ *be an agglomerated spanning tree of* $\hat{G}_T$. *Suppose* $gen(T, 0, 0) = \{(m_1, 0, 0), \ldots, (m_k, i_k, j_k)\}$ *is a set of generating motif pieces that arise by way of* S_T. *A* **ghost motif piece of** $(m_s, i_s, j_s) \in gen(T, 0, 0)$ *is any motif piece* (m_s, a, b) *such that*

- $(a, b) \neq (i_s, j_s)$ *and*
- (m_s, a, b) *is contiguous with some element of* $gen(T, 0, 0)$.

The vector $[a - i_s, b - j_s]$ *is a* **ghost vector** *for* T.

A ghost motif piece (or simply ghost) is necessarily contiguous with a generating motif piece: the ghost vector that translates a ghost to it's generator will help identify inherent periods in the wallpaper pattern. For example in Figure 3.5, note that ghost vector $[2, 1]$ translates ghost $(m_3, -2, 1)$ to generator $(m_3, 0, 2)$.

The next lemma and corollary are immediate and useful consequences of Definition 3.2.5.

Lemma 3.2.6 (Finitely Many Ghost Vectors): *Let* $T = \{(m_1, 0, 0), (m_2, 0, 0), \ldots, (m_k, 0, 0)\}$ *be an Escher tile whose period graph is connected. Then there are only finitely many ghost vectors for* T*.*

Proof: Any set of generating motif pieces is contained within a square of side length k. By Definition 3.2.5, a ghost motif piece is contained within a square of side length $k + 2$ (expand the original by one unit in all directions). ∎

Corollary 3.2.7 (Ghost Vectors are Bounded in Length by $O(k)$): *Let* $T = \{(m_1, 0, 0), (m_2, 0, 0), \ldots, (m_k, 0, 0)\}$ *be an Escher tile whose period graph is connected. If* $[a, b]$ *is any ghost vector for* T *then* $|a|, |b| \leq k + 1$.

Finally, in the remainder of this section, we will show that the ghost vectors (inherent periodicities in their own right) are the building blocks for the inherent periodicities in the Escher wallpaper pattern $Wall(T)$.

Lemma 3.2.8 (Ghost Vectors as Building Blocks of Inherent Periodicities): *Let* $T = \{m_1, 0, 0), \ldots, (m_k, 0, 0)\}$ *be an Escher tile whose period graph is connected and suppose a set of generating motif pieces for* T *is* $gen(T, 0, 0) = \{(m_1, 0, 0), \ldots, (m_k, i_k, j_k)\}$. *Then* (m_s, a, b) *is related to* (m_s, c, d) *if and only if there exists* $x_1, \ldots, x_N \in \mathbb{Z}$ *such that*

- $$\begin{bmatrix} a \\ b \end{bmatrix} = x_1\mathbf{g_1} + \cdots + x_N\mathbf{g_N}, \tag{3.1}$$

  where $\{\mathbf{g_1}, \ldots, \mathbf{g_N}\}$ *is the set of ghost vectors for* T.

Moreover,

- *for any* $(m_s, a, b))$, $(m_s, c, d) \in Wall(T)$, *we have* $W((m_s\ , a, b)) = W((m_s,$ $c,\ d))$ *if and only if equation (3.1) holds for some* $x_1, \ldots, x_N$.

Proof: Motif piece (m_s, a, b) is related to (m_s, c, d) if and only if (m_s, a, b) and (m_s, c, d) belong to the same wallpaper component. Let $Walk = \{N_1 = (m_s, a, b),$ $N_2, \ldots, N_j = (m_s, c, d)\}$ be a sequence of contiguous motif pieces that describes a walk from (m_s, a, b) to (m_s, c, d). We know that

$$Wall(T) = \bigcup_{x,y \in \mathbb{Z}} gen(T, x, y),$$

where the union is disjoint, and thus *Walk* is represented by a (not necessarily distinct) sequence $gen(T, x_1, y_1)$, $gen(T, x_2, y_2)$, $\ldots$, $gen(T, x_Q, y_Q)$. In particular, *Walk* alternates between **subwalks within a set of generating motif pieces** and **an exit from that set of generating motif pieces** to the next $gen(T, x_\alpha, y_\alpha)$. The exit necessarily takes place from a generator in $gen(T, x_i, y_i)$ to its ghost in $gen(T, x_{i+1}, y_{i+1})$. Say the vector value of the walk from this generator to its ghost is $\mathbf{g_i}$.

We now determine the vector value of *Walk*. First, since $(m_s, a, b) \in gen(T,$ $x_1, y_1)$ we have $a = x_1 + i_s$ and $b = y_1 + j_s$. Similarly, $c = x_R + i_s$ and $d = y_R + j_s$. The subwalk in $gen(T, x_1, y_1)$ starts on (m_s, a, b) and must end on $(m_{t_1}, a - i_{t_1}, b -$ $j_{t_1})$, where $(m_{t_1}, x_2 + i_{t_1}, y_2 + j_{t_1}) \in gen(T, x_2, y_2)$ is a ghost of $(m_{t_1}, a - i_t, b - j_t)$. Let $\mathbf{g_{i_1}}$ be the associated ghost vector. Then the vector value of the walk from (m_s, a, b) to $(m_{t_1}, x_2 + i_{t_1}, y_2 + j_{t_1})$ is

$$\begin{bmatrix} x_1 + i_{t_1} - a \\ y_1 + j_{t_1} - b \end{bmatrix} + \mathbf{g_{i_1}} = \begin{bmatrix} x_1 + i_{t_1} - x_1 - i_s \\ y_1 + j_{t_1} - y_1 - j_s \end{bmatrix} + \mathbf{g_{i_1}} = \begin{bmatrix} i_{t_1} - i_s \\ j_{t_1} - j_s \end{bmatrix} + \mathbf{g_{i_1}}.$$

In general the walk from (m_s, a, b) to (m_s, c, d) has vector value given by

$$\begin{bmatrix} i_{t_1} - i_s \\ j_{t_1} - j_s \end{bmatrix} + \mathbf{g_{i_1}} + \begin{bmatrix} i_{t_2} - i_{t_1} \\ j_{t_2} - j_{t_1} \end{bmatrix} + \mathbf{g_{i_2}} + \cdots$$

$$+ \begin{bmatrix} i_{t_Q} - i_{t_{Q-1}} \\ j_{t_Q} - j_{t_{Q-1}} \end{bmatrix} + \mathbf{g_{i_{t_{Q-1}}}} + \begin{bmatrix} i_s - i_{t_Q} \\ j_s - j_{t_Q} \end{bmatrix}$$

$$= \mathbf{g_{i_1}} + \mathbf{g_{i_2}} + \cdots + \mathbf{g_{i_{Q-1}}},$$

as desired.

For the second claim, $W((m_s, a, b)) = W((m_s, c, d))$ if and only if (m_s, a, b) and (m_s, c, d) are related if (and by the first part of this proof) if and only if equation (3.1) holds. This completes the proof. ∎

Lemma 3.2.9 (Circuits in G_T are Linear Combinations of Ghost Vectors): *Let T be an Escher tile whose period graph G_T is connected. Suppose Circ $= \{v_{i_1}, \ldots, v_{i_s}\}$ is a circuit in $\hat{G}_T$ whose vector value is $[a, b]$ ($\neq [0, 0]$) and let $\{\mathbf{g_1}, \ldots, \mathbf{g_N}\}$ be the set of ghost periods for T. Then*

$$\begin{bmatrix} a \\ b \end{bmatrix} = x_1\mathbf{g_1} + \cdots + x_N\mathbf{g_N}.$$

Proof: Choose any $x, y \in \mathbb{Z}$. Motif pieces (m_s, x, y) and $(m_s, x + a, y + b)$ are contained in the set of motif pieces induced by *Circ* and (m_s, x, y). Therefore (m_s, x, y) and $(m_s, x + a, y + b)$ are related. By Lemma 3.2.8,

$$\begin{bmatrix} a \\ b \end{bmatrix} = x_1\mathbf{g_1} + \cdots + x_N\mathbf{g_N},$$

as desired. ∎

We summarize the results from this section.

- By Definition 2.2.6, (m_s, a, b) is related to (m_s, c, d) if and only if there is a walk along contiguous motif pieces from $(m_s,\ a,\ b)$ to $(m_s,\ c,\ d)$.

- By Definition 3.1.1, there is a walk along contiguous motif pieces from $(m_s,$ $a,\ b)$ to $(m_s,\ c,\ d)$ if and only if there is a circuit in $\hat{G}_T$ given by $\{v_s,\ v_{i_2},\ \ldots,$ $v_s\}$.

- By Lemma 3.2.8, (m_s, a, b) is related to (m_s, c, d) if and only if

$$\begin{bmatrix} c-a \\ d-b \end{bmatrix} = x_1\mathbf{g_1} + \cdots + x_N\mathbf{g_N},$$

  where $\mathbf{g_1}, \ldots, \mathbf{g_N}$ are the ghost vectors of T, and $x_1, \ldots, x_N \in \mathbb{Z}$.

We conclude that the vector value of any circuit in $\hat{G}_T$ is an integer linear combination of ghost vectors for T. Moreover, by the second claim in Lemma 3.2.8, all integer linear combinations of the ghost vectors for T exactly characterize the set of vectors $[A, B]$ such that $W((m_1, x, y)) = W((m_1, x + A, y + B))$.

Thus, the subspace of $\mathbb{Z}^2$ spanned by the ghost vectors $GV := \{g_1, \ldots, g_N\}$ describe the inherent periodicities in $Wall(T)$. In the next section we rely on this characterization of inherent periodicities to define, essentially, three kinds of Escher tiles (whose period graphs are connected):

- the subspace spanned by GV is trivial, or
- the subspace spanned by GV is one-dimensional, or
- the subspace spanned by GV is two-dimensional.

The machinery is in place: in the next section we outline how to use it.

3.3 Number Theory

The period graph of an Escher tile contains much information, some of which can be extracted by using number theory. The crux of the matter is that

- nontrivial circuits in the period graph can be viewed as vectors in $\mathbb{Z}^2$ (by way of their vector values),
- the vector value of any circuit in $\hat{G}_T$ is a linear combination of ghost vectors, GV, and
- the subspace spanned by GV has dimension 0,1 or 2.
- If the subspace of $\mathbb{Z}^2$ spanned by GV does not have full rank, then we supplement GV with either one or two collision-free vectors, as needed.
- Ultimately, we associate an Escher tile with a pair of linearly independent vectors from $\mathbb{Z}^2$. All integer linear combinations of such a vector pair forms a (number theoretic) lattice. This special lattice will be the main tool with which we construct a Big Tile for T.

The next three definitions serve to distinguish among the three possible ranks of the subspace spanned by the ghost vectors of an Escher tile.

Definition 3.3.1 (Component of G_T is trivially periodic): *Let $C(G_T)$ be a connected component of period graph G_T of an Escher tile T. If the Escher tile T' induced by $C(G_T)$ has no ghost vectors, then T' is said to be* **trivially periodic**.

An Escher tile that is trivially periodic will have a wallpaper component that is bounded by a disk in the plane. See Figure 3.6.

Definition 3.3.2 (Component of G_T is singly periodic): *Let $C(G_T)$ be a connected component of period graph G_T of an Escher tile T. Let T' be the Escher tile induced by $C(G_T)$. If the dimension of the subspace spanned by the ghost vectors for T' is one, then T' is said to be* **singly periodic**. *In particular, if $GV = \{x_1\mathbf{u}, x_2\mathbf{u}, \ldots, x_N\mathbf{u}\}$, then the* **natural period** *for T' is $gcd(x_1, \ldots, x_N)\mathbf{u}$.*

That is, the natural period, which is a priori an inherent period, is the smallest (in Euclidean length) vector $[p_1, p_2]$ for which there are related motif pieces in different locations, (m_s, a, b) and (m_s, c, d) such that

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} a - c \\ b - d \end{bmatrix}.$$

Moreover, an Escher tile that is singly periodic will have a connected component that is infinite, but that is contained in a band of finite width. See Figure 3.7.

Definition 3.3.3 (Component of G_T is doubly periodic): *Let $C(G_T)$ be a connected component of period graph G_T of an Escher tile T and suppose T' is the Escher tile induced by $C(G_T)$. If the dimension of the subspace spanned by the ghost vectors GV for T' is two, then T' is said to be* **doubly periodic**. *Any basis for GV is a pair of* **natural periods** *for T'.*

An Escher tile that is doubly periodic will have a component that is unbounded in all directions; it cannot be contained in any half-plane. See Figure 3.8 for an example of an Escher tile whose period graph has a doubly periodic component.

For the most part we will concentrate on Escher tiles whose period graphs have only one connected component. Figure 3.9 is a singly periodic Escher tile with natural period $[1, -3]$. Figure 3.10, a fragment of the wallpaper pattern generated by the Escher tile in Figure 3.9, shows three vectors: the natural period ($[1, -3]$), a collision-free vector ($[1, 0]$), and a forbidden vector ($[0, -1]$).

Next we use the natural period(s) of $C(G_T)$ to give a more concise description of wallpaper components than that given by Definition 2.2.7.

Lemma 3.3.4 (Wallpaper Component Description): *Let* $T = \{(m_1, 0, 0), \ldots, (m_k, 0, 0)\}$ *be an Escher tile and suppose* G_T *is is connected.*

- *Suppose* G_T *is singly periodic with natural period* $[p_1, p_2]$ *and let* $\hat{S}_T$ *be an agglomerated spanning tree for* $\hat{G}_T$ *that yields* $gen(T, 0, 0) = \{(m_1, 0, 0), (m_2, i_2, j_2), \ldots, (m_k, i_k, j_k)\}$. *Then* $W((m_1, r, s)) = \{\{(m_1, r + k_0p_1, s + k_0p_2), (m_2, r-i_2+k_0p_1, s-j_2+k_0p_2), \ldots, (m_k, r-i_k+k_0p_1, s-j_k+k_0p_2)\} : k_0 \in \mathbb{Z}\}$.

- *Suppose* G_T *is doubly periodic with natural periods* $[p_1, p_2]$ *and* $[q_1, q_2]$, *and let* $\hat{S}_T$ *be an agglomerated spanning tree for* $\hat{G}_T$ *that yields* $gen(T, 0, 0) = \{(m_1, 0, 0), (m_2, i_2, j_2), \ldots, (m_k, i_k, j_k)$. *Then* $W((m_1, r, s)) = \{\{(m_1, r + k_0p_1 + l_0q_1, s + k_0p_2 + l_0q_2), (m_2, r - i_2 + k_0p_1 + l_0q_1, s - j_2 + k_0p_2 + l_0q_2), \ldots, (m_k, r - i_k + k_0p_1 + l_0q_1, s - j_k + k_0p_2)\} : k_0, l_0 \in \mathbb{Z}\}$.

Proof: Case 1 (G_T is singly periodic with natural period $[p_1, p_2]$): By Definitions 2.2.7 and 3.1.1, in general, if (m_s, i_1, j_1) and $(m_t, i_2, j_2) \in W((m_1, r, s))$ are distinct, then there exists a nonempty path G_T from v_s to v_t. In our particular situation, $(m_1, i, j) \in W((m_1, r, s))$ with $(i, j) \neq (r, s)$ if and only if there exists a nonempty circuit, *Circ*, in G_T beginning (and ending) on v_1. Since G_T is singly periodic, the vector value of *Circ* is $k_0[p_1, p_2]$ for some $k_0 \in \mathbb{Z}$. Alternatively, $i = r + k_0p_1$ and $j = s + k_0p_2$, as needed.

Now suppose $(m_x, i, j) \in W((m_1, r, s))$ for some $x \in \{2, \ldots, k\}$. There is a path from (m_x, r, s) along contiguous motif pieces in $W((m_1, r, s))$ if and only if there is a corresponding path P from v_x to v_1. If P is the unique path from v_x to v_1 in $\hat{S}_T$, the agglomerated spanning tree of G_T then $i = -i_x + r$ and $j = -j_x + s$. If P is not the unique path from v_1 to v_x in $\hat{S}_T$ then suppose P' is. In other words the vector value of P' is $[i_x, j_x]$. The concatenation of P with the reversal of P' yields a circuit starting (and ending) at v_1 and passing through v_x. By necessity, the vector value of the circuit is $k_0[p_1, p_2]$ for some $k_0 \neq 0$. So, on one hand, the vector value of $Circ$ is $k_0[p_1, p_2]$ and on the other hand the vector value of $Circ$ is the difference of the vector values of P and P'. In all, the vector value of P is then $k_0[p_1, p_2] - [i_x, j_x] = [-i_x + k_0p_1, -j_x + k_0p_2]$. Alternatively, $i = r - i_x + k_0p_1$ and $j = s - j_x + k_0p_2$. In summary, the set of locations that contain translates of m_y in $W((m_1, r, s))$ is given by $\{(r-i_y+k_0p_1, s-j_y+k_0p_2 : k_0 \in \mathbb{Z}\}$ for $y = 1, \ldots, k$.

Case 2 (G_T is doubly periodic with natural periods $[p_1, p_2]$ and $[q_1, q_2]$): A simple modification of the proof of Case 1 gives us that the set of locations that contain translates of m_y in $W((m_1, r, s))$ is given by $\{(r - i_y + k_0p_1 + l_0q_2, s - j_y + k_0p_2 + l_0q_2 : k_0, l_0 \in \mathbb{Z}\}$ for $y = 1, \ldots, k$. ∎

Finally, we will use ghost motif pieces and vectors to produce the natural periods (if there are any) of an Escher tile.

Proposition 3.3.5 (Use Ghost Motif Pieces to Extract Natural Periods): *Let $T = \{(m_1, 0, 0), \ldots, (m_k, 0, 0)\}$ be an Escher tile whose period graph G_T is connected. Let GV be the set of ghost vectors for T. The natural periods of T can be extracted from GV in $O(k^2)$ time.*

Proof: By Lemma 3.2.6 and Corollary 3.2.7 there are $O(k^2)$ ghost vectors each of whose entries is bounded (in absolute value) by $O(k)$.

Suppose T is singly periodic and let the set of ghost vectors be $\mathrm{GV} = \{x_1\mathbf{u}, \ldots, x_t\mathbf{u}\}$. By Definition 3.3.2, the natural period of T is $\gcd(x_1, \ldots, x_t)\mathbf{u}$. Since

$t = O(k^2)$ and $x_i = O(k)$ for $i = 1, \ldots, k$, finding the greatest common divisor can be done in $O(k^2)$ time.

Suppose T is doubly periodic. By Definition 3.3.3, Lemma 3.2.6 and Lemma 3.2.9, a pair of natural periods for T is a basis for the vector space spanned by GV. Since $|GV| = O(k^2)$ and the entries of each element of GV are bounded in absolute value by $O(k)$, finding a basis can be done in $O(k)$ time [Co95]. ∎

The purpose of the next definition is to identify a particular translate of m_1 for an arbitrary Escher wallpaper component $W((m_1, r, s))$ whose component in the period graph is singly periodic. We have m_1 in location (r, s) belonging to $W((m_1, r, s))$. When a wallpaper component corresponds to graph component of G_T that is singly periodic (in a sense, the hardest case) we subtract integer multiples of the only periodicity $[p_1, p_2]$ from $[r, s]$ to find translates of m_1 in $W((m_1, r, s))$. The translate of m_1 closest to and above or on the $x-$axis will be a special copy of m_1 and its $y-$coordinate defined to be the **height** of $W((m_1, r, s))$. For example, when the height of $W((m_1, r, s))$ turns out to be 0, then $W((m_1, r, s))$ is a horizontal translate (or rightward shift) of $W((m_1, 0, 0))$.

Definition 3.3.6 (Height of a singly periodic wallpaper component): *Let T be a singly periodic Escher tile with natural period $[p_1, p_2]$. The* **height of** $W((m_1, r, s))$ *is* $s \pmod{p_2}$ *and is denoted* $height(W((m_1, r, s)))$.

Escher tiles for which the period graph has some connected components that are not doubly periodic provide flexibility in the outcome of a Big Tile. We will see that a doubly periodic Escher tile is encoded by a pre-determined number-theoretic lattice, and gives no choice as to the assignment of colors in a Big Tile, and is in a sense the easiest case to consider. The essence of the global solution, that of existence of a Big Tile for arbitrary Escher tiles, lies in the choice of one collision-free vector in the singly periodic case, and in the choice of a pair of linearly independent collision-free vectors in the trivially periodic case. One has the sense that there is a minimal (with respect to the associated lattice) such vector pair that will do the trick. Any vector pair that survives the minimality conditions and respects the inherent periodicities will be fair game for use in creating a Big Tile.

Not surprisingly, since we are after a rectangular Big Tile, and since we are going to use a lattice L to find one, it will be important to find the smallest rectangular sublattice of L.

Lemma 3.3.7 (Smallest Rectangular Sublattice of Lattice): *Suppose* $L = < [p_1, p_2], [q_1, q_2] >$ *is the lattice generated by* $[p_1, p_2], [q_1, q_2] \in \mathbb{Z}^2$ *and let* $\Delta = p_1 q_2 - p_2 q_1$. *The smallest rectangular sublattice of* L *is* $L' = < [\frac{\Delta}{|\gcd(p_2,q_2)|}, 0], [0, \frac{\Delta}{|\gcd(p_1,q_1)|}] >$.

Proof: If there are $x, y \in \mathbb{Z}$ such that

$$x \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + y \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix}$$

then

$$\begin{bmatrix} p_1 & q_1 \\ p_2 & q_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix}$$

in which case

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} q_2 & -q_1 \\ -p_2 & p_1 \end{bmatrix} \begin{bmatrix} r \\ 0 \end{bmatrix}$$

or alternatively,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{q_2 r}{\Delta} \\ \frac{-p_2 r}{\Delta} \end{bmatrix}.$$

Thus, $r = \frac{\Delta}{\gcd(p_2,q_2)}$ because $r \in \mathbb{Z}$ and $|r|$ is minimal. A similar argument shows that the smallest nonzero value of $|s|$ for which $[0, s] \in L$ is $s = \frac{\Delta}{\gcd(p_1,q_1)}$. This completes the proof. ∎

Our next task will be to gain some understanding about how to find one or two collision-free vectors for an Escher tile whose period graph is connected and either singly or trivially periodic. To find suitable collision-free vectors, we call upon O_T, the overlap graph of T; we explain the use of overlap graph by way of a detailed example.

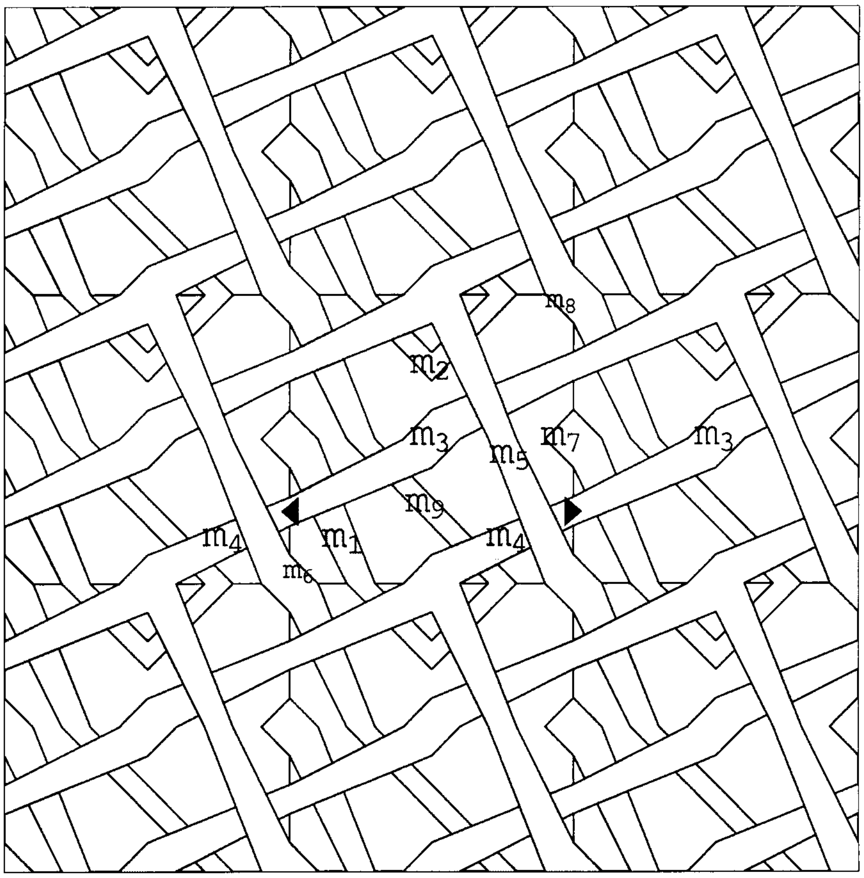

Figure 3.1: **An Escher tile in home position surrounded by eight other Escher tiles. Motif piece $(m_4, 0, 0)$ is contiguous with $(m_3, 1, 0)$ and $(m_3, 0, 0)$ is contiguous with $(m_4, -1, 0)$. Therefore, the period graph (see Figure 3.2) contains directed edge (m_4, m_3) with vector label $[1, 0]$ and directed edge (m_3, m_4) with vector label $[-1, 0]$ (as well as many other edges).**

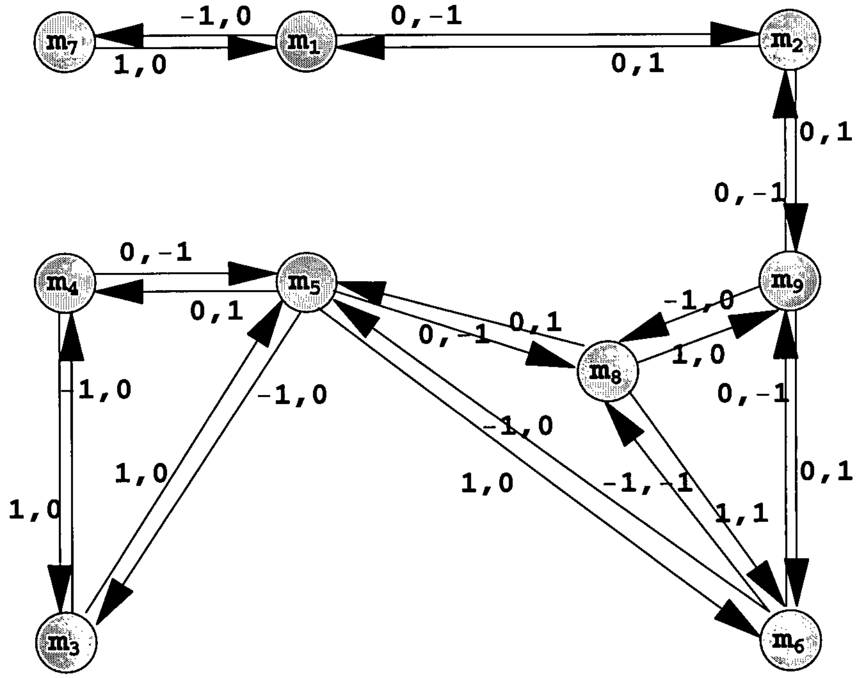

Figure 3.2: **The period graph for the Escher tile in Figure 3.1**

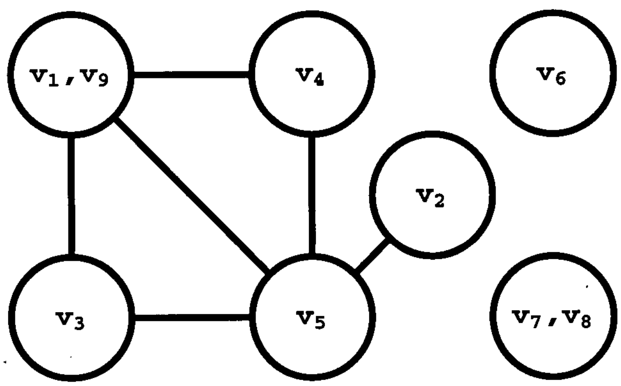

Figure 3.3: **The overlap graph for the Escher tile in Figure 3.1**

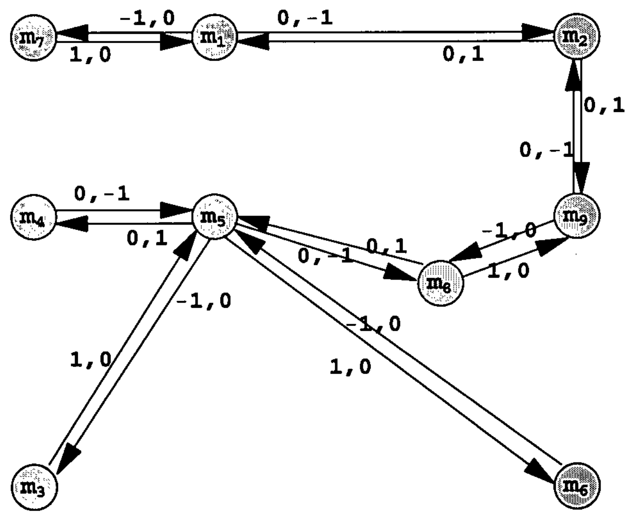

Figure 3.4: **A fat spanning tree of the period graph in Figure 3.2**

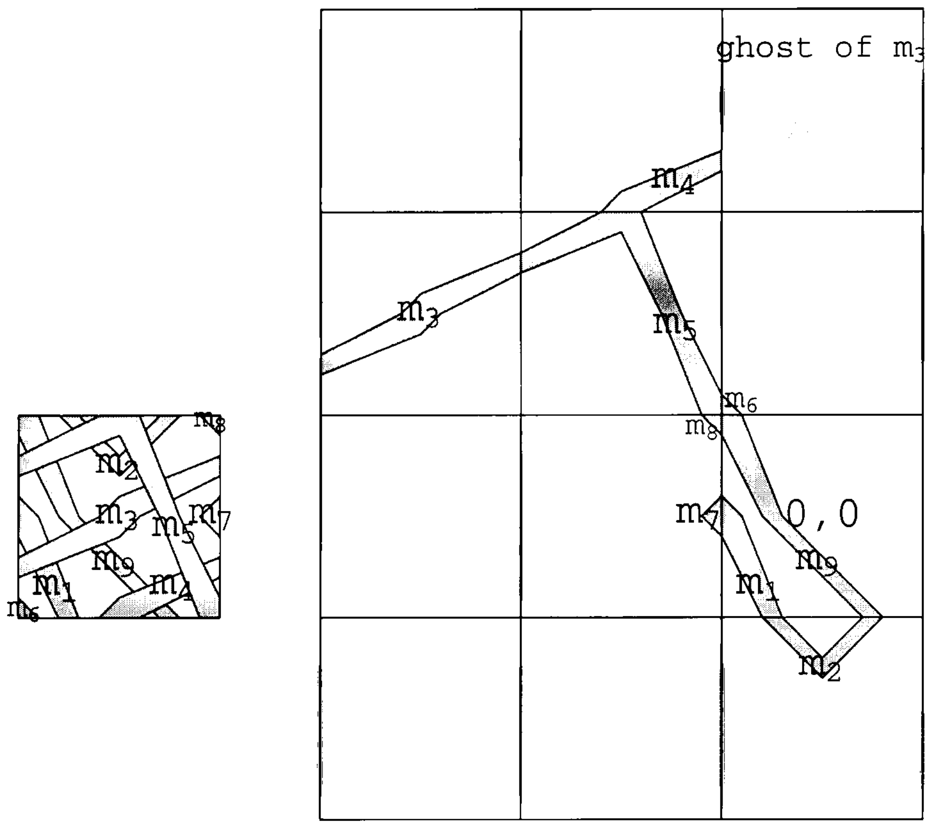

Figure 3.5: **Unravelling the Escher tile puzzle with a set of generating motif pieces. A ghost of** $(m_3, -2, 1)$ **is** $(m_3, 0, 2)$

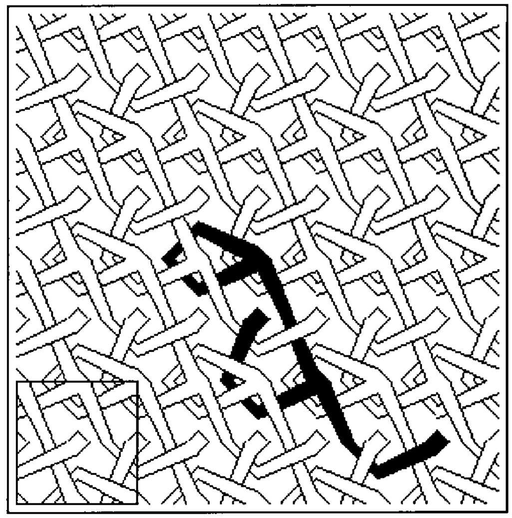

Figure 3.6: **Trivially periodic Escher tile**

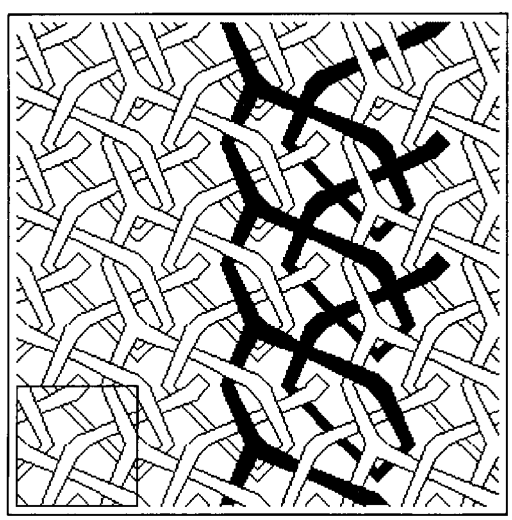

Figure 3.7: **Singly periodic Escher tile**

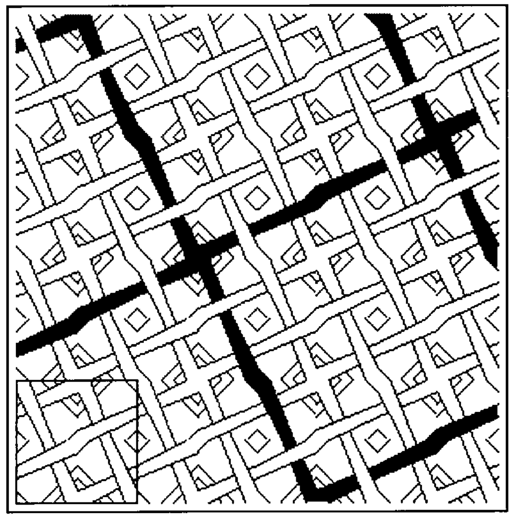

Figure 3.8: **Component that is doubly periodic**

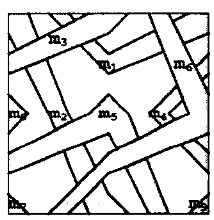

Figure 3.9: **Singly periodic Escher tile with natural period** $[1, -3]$

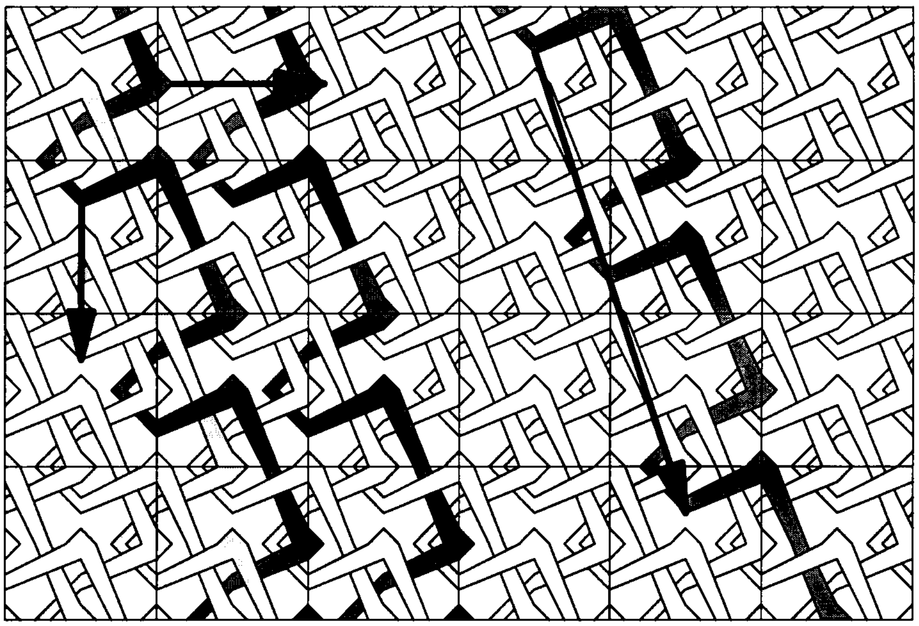

Figure 3.10: **Natural period** $[1, -3]$**; collision-free vector** $[1, 0]$**; forbidden vector** $[0, -1]$

Chapter 4

Detailed Example

Begin by inputting an Escher tile $T = \{(m_1, 0, 0), (m_2, 0, 0), \ldots, (m_9, 0, 0)\}$: see Figure 4.1. We keep track of the pairwise intersections of elements of T in the overlap graph O_T, and also of the intersections of elements of T with $\partial S_{0,0}$.

- Construct G_T, the period graph for T; see Figure 4.2.
- Let $\hat{S}_T$ be a spanning tree of the agglomerated period graph $\hat{G}_T$. An example is shown in Figure 4.3.
- We gain some insight by drawing the vertices and edges of the spanning tree in Figure 4.3 in the plane so that the vertices that represent motif pieces are in their correct physical locations under the assumption that m_1 is in location (0,0). See Figure 4.4.
- By way of the fat spanning tree S_T, a set of generating motif pieces (see Definition 3.2.4) is given by $gen(T, 0, 0) = \{(m_1, 0, 0), (m_2, 0, 1), (m_3, 2, 0),$ $(m_4, 1, 0)$, $(m_5, 1, 1)$, $(m_9, 1, 1)$, $(m_6, 2, 1)$, $(m_7, 3, 1), (m_8, 3, 2)\}$, which is shown in Figure 4.5.

- Let $\hat{R}_T$ be the set of edges that were removed from $\hat{G}_T$ when the spanning tree $\hat{S}_T$ was constructed. In this example, $\hat{R}_T = \{(v_4, v_9), (v_2, v_3)\}$. Let $R_T = E(G_T) \setminus E(S_T)$. That is, return the multiple edges and vector labels to elements of $\hat{R}$. Then $R_T = \{(v_4, v_9)$ *labelled* $[0, -1]$, (v_9, v_4) *labelled* $[0, 1]$, (v_2, v_3) *labelled* $[0, 1]$, (v_3, v_2) *labelled* $[0, -1]\}$. Thus, there are potentially four ghost motif pieces: $\{(m_4, 1, 0), (m_9, 1, 1), (m_3, 0, 2), (m_2, 2, -1)\}$. All of these motif pieces are contiguous with generating motif pieces, but not all give rise to ghost vectors as we shall see.

- We wish to determine if the motif piece corresponding to an endpoint of a removed edge is in a location that is different than its corresponding generating motif piece; when such a situation occurs, we have identified a ghost vector.

  - For example, $(m_4, 1, 0)$ lives directly below $(m_9, 1, 1)$, and is shown in Figure 4.6 in dark gray. Since $(m_4, 1, 1)$ lands in the same location as generating $(m_4, 1, 0)$, no information is gained.

  - Similarly, the ghost of $(m_9, 1, 1)$ lives directly above the generator $(m_4, 1, 0)$, and lands in the same location as the original $(m_9, 1, 1)$, so no information is gained. See Figure 4.7.

  - What about the ghosts of $(m_2, 0, 1)$ and $(m_3, 2, 0)$? The ghost of $(m_3, 2, 0)$ lives directly above $(m_2, 0, 1)$ and is shown in Figure 4.8.

  - Since the locations of $(m_3, 0, 2)$ and $(m_3, 2, 0)$ satisfy $[0 - 2, 2 - 0] = -[2, -2]$, we conclude that $[2, -2]$ is a ghost vector for the Escher tile in this example.

  - The last motif piece to check is the ghost of $(m_2, 0, 1)$. In general, the number of ghost motif pieces is at most $2|R_T|$. The ghost of $(m_2, 0, 1)$ lives directly below $(m_3, 2, 0)$ as shown by Figure 4.9.

- Since generator $(m_2, 0, 1)$ and ghost $(m_2, 2, -1)$ satisfy $[0-2, 1-(-1)] = [2, -2]$, we find (again) that $[2, -2]$ is a ghost vector of T.

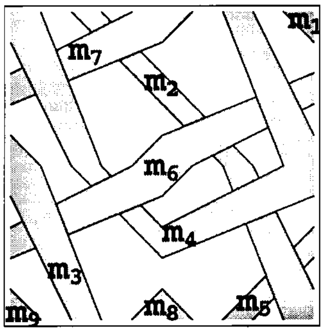

Figure 4.1: **An Escher tile: Big Tile to be determined**

We conclude that the Escher tile T is singly periodic with period $[2, -2]$ because there is only one ghost vector.

The previous example illustrates (one case of) a theorem guaranteeing that the natural period(s) of Escher tiles can be found in polynomial time.

We now have a method by which we can compute the E-vertices of the overlap graph.

Detect-related-motif-pieces-in-location-(0,0) algorithm:

If Escher tile T is doubly periodic, we need not worry about detecting related motif pieces because the Big Tile is unique and predetermined. In fact we do not need to use the overlap graph, as we shall see in Section 5.1.

In that case, first assume that T is singly periodic with natural period $[p_1, p_2]$. Recall that $gen(T, 0, 0) = \{(m_1, 0, 0), (m_2, i_2, j_2), \ldots, (m_k, i_k, j_k)\}$. Let (m_s, i_s, j_s), $(m_t, i_t, j_t) \in gen(T, 0, 0)$ with $s \neq t$. Since (m_s, i_s, j_s) is related to (m_t, i_t, j_t),

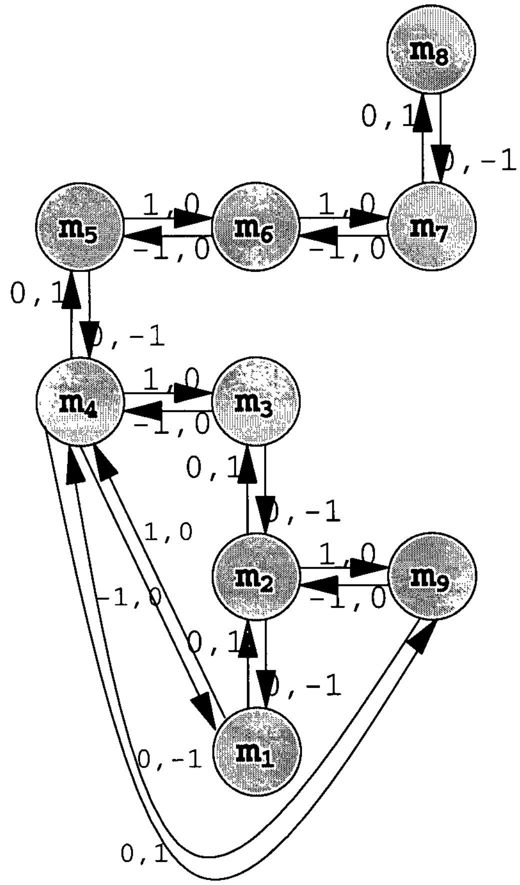

Figure 4.2: **Period graph, G_T, for T in Figure 4.1**

we have $(m_s, 0, 0)$ is related to $(m_t, i_t - i_s, j_t - j_s)$. Hence, $(m_s, 0, 0)$ is related to $(m_t, 0, 0)$ if and only if $(m_t, 0, 0)$ is related to $(m_t, i_t - i_s, j_t - j_s)$. By Lemma 3.3.4, $(m_t, 0, 0)$ is related to $(m_t, i_t - i_s, j_t - j_s)$ if and only if

$$\begin{bmatrix} i_s - i_t \\ j_s - j_t \end{bmatrix} = k_0 \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

for integer $k_0 \neq 0$.

Practically speaking, in the singly periodic case, we have reduced the problem of detecting which motif pieces in an Escher tile are related to that of comparing

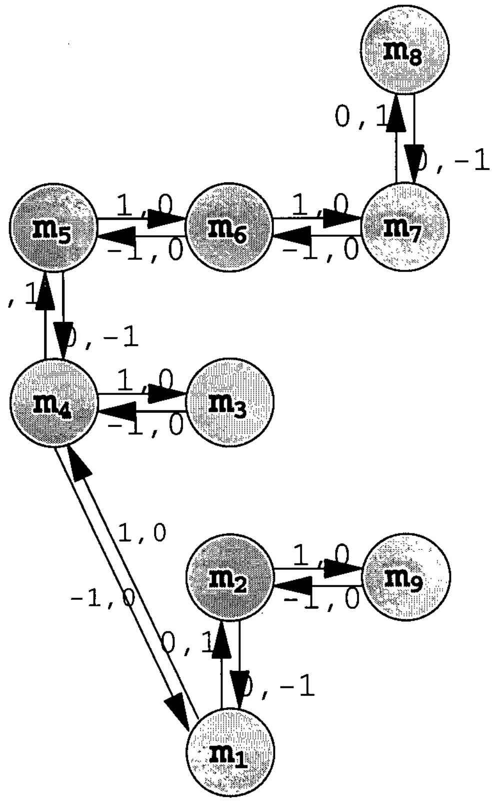

Figure 4.3: **A fat spanning tree, S_T, for G_T**

the locations of elements of $gen(T, 0, 0)$ with their corresponding ghosts.

In this example, $(m_5, 0, 0)$ and $(m_9, 0, 0)$ are related because $(m_5, 1, 1)$, $(m_9, 1, 1) \in gen(T, 0, 0)$ are related since $[1-1, 1-1] = [0, 0]$ is a multiple of $[2, -2]$ so that $v_5 \in [v_9]$. (In the trivially periodic case, the agglomerated spanning tree is unique and thus a set of generating motif pieces necessarily places related motif pieces in the same location.)

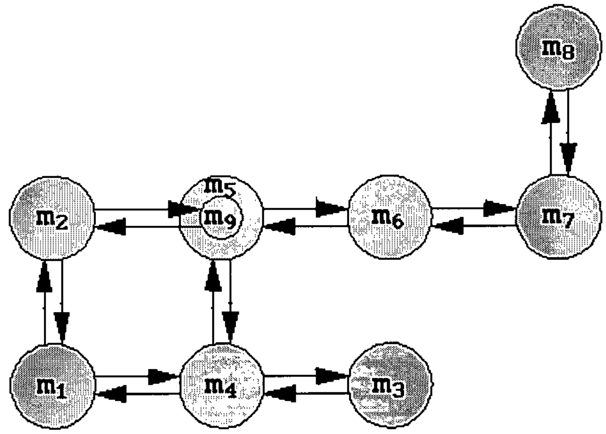

Figure 4.4: **The fat spanning tree, S_T with vertices embedded in correct locations relative to** $(m_1, 0, 0)$

4.1 Get a Collision-Free Vector

Next we seek a collision-free vector that is linearly independent with $[2, -2]$, the natural period of T. To find such a vector we will use the Overlap Graph of T. See Figure 4.10.

Remark: At most 14 distinct Escher wallpaper components overlap $W((m_1, 0, 0))$. (In general the "14" will be replaced by twice the number of edges in the overlap graph.)

Idea of Proof: By Definition 2.2.7, $W((m_1, 0, 0))$ is the set of all motif pieces related to $(m_1, 0, 0)$. By Lemma 3.3.4, the set of all translates of $(m_1, 0, 0)$ that belong to $W((m_1, 0, 0))$ have locations given by $\{(2k, -2k) : k \in \mathbb{Z}\}$. Similarly the set of translates of $(m_2, 0, 0)$ that belong to $W((m_2, 0, 1))$ have locations given by $\{(2k, 1 - 2k) : k \in \mathbb{Z}\}$. The set of translates of $(m_3, 0, 0)$ that belong to $W((m_1, 0, 0))$ have locations given by $\{(2 + 2k, -2k : k \in \mathbb{Z}\}$. The

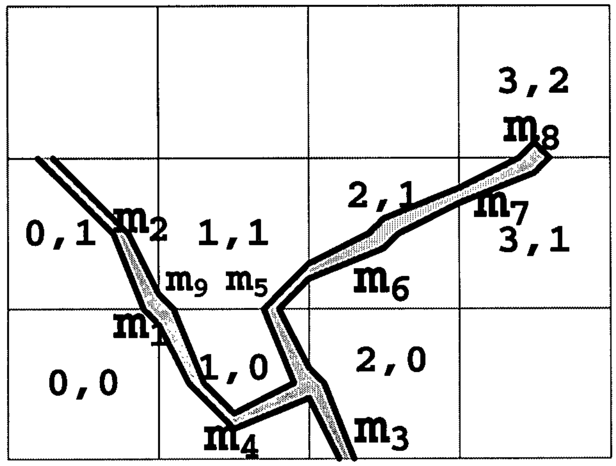

Figure 4.5: **Set of generating motif pieces** $gen(T, 0, 0)$

set of translates of $(m_4, 0, 0)$ that belong to $W((m_1, 0, 0))$ have locations given by $\{(1 + 2k, -2k) : k \in \mathbb{Z}\}$. The set of translates of $(m_5, 0, 0)$ and $(m_9, 0, 0)$ that belong to $W((m_1, 0, 0))$ have locations given by $\{(1 + 2k, 1 - 2k) : k \in \mathbb{Z}\}$. The set of translates of $(m_6, 0, 0)$ that belong to to $W((m_1, 0, 0))$ have locations given by $\{(2 + 2k, 1 - 2k) : k \in \mathbb{Z}\}$. The set of translates of $(m_7, 0, 0)$ that belong to to $W((m_1, 0, 0)$ have locations given by $\{(3 + 2k, 1 - 2k) : k \in \mathbb{Z}\}$. The set of translates of $(m_8, 0, 0)$ that belong to to $W((m_1, 0, 0))$ have locations given by $\{(3 + 2k, 2 - 2k) : k \in \mathbb{Z}\}$.

Thus, to find a wallpaper component that overlaps $W((m_1, 0, 0))$, it suffices to pick for example, $(m_4, 1 + 2k, -2k)$ and find all motif pieces that overlap $(m_4, 1 + 2k, -2k)$ (and in general, find all motif pieces that overlap elements of $gen(T, 0, 0)$.) With that goal in mind, note that $(m_i, 1 + 2k, -2k) \cap (m_4, 1 + 2k, -2k) \neq \varnothing$ if and only if $(m_4, 1, 0) \cap (m_i, 1, 0) \neq \varnothing$, which is true if and only if $(m_4, 0, 0) \cap (m_i, 0, 0) \neq \varnothing$. But this is precisely the information contained in the overlap graph O_T, namely which motif pieces in a given unit square have nontrivial intersection as subsets of $\mathbb{R}^2$. In all, for each E-vertex $[v]$ in the overlap graph,

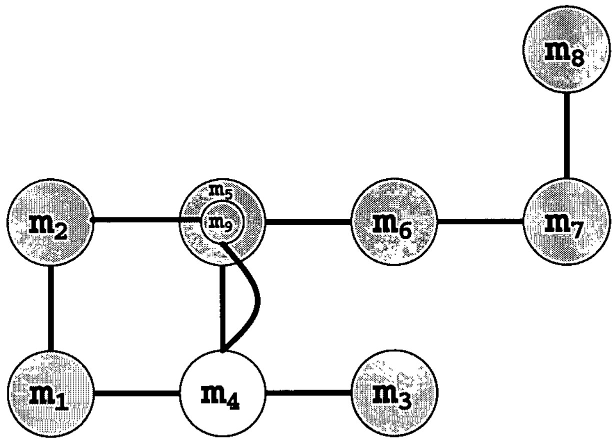

Figure 4.6: **A potential ghost of** m_4

there are $\deg([v])$ motif pieces that overlap the equivalence class of motif pieces contained in $[v]$. Hence, there are at most $\sum_{[v]\in O_T} \deg([v]) = 14$ distinct wallpaper components that overlap $W((m_1, 0, 0))$.

A nearly immediate consequence is that there exists a vector $[r, s]$ such that $W((m_1, 0, 0))$ and $W((m_1, r, s))$ do not overlap, which is what we are after. In fact, it won't be hard to generalize away from this example and characterize the set of all vectors $[r, s]$ for which $W((m_1, 0, 0))$ does not overlap $W((m_1, r, s))$ for any connected component of the period graph of an Escher tile T.

For now, though, we set our sights lower since we are after existence. Recall that the goal for the singly periodic example in this section is to find a second vector $[r, s]$ linearly independent with $[2, -2]$ and such that $W((m_1, r, s))$ does not overlap $W((m_1, 0, 0))$. It suffices to find the (at most) 14 wallpaper components that overlap $W((m_1, 0, 0))$, and then, for example, find a translate of $W((m_1, 0, 0))$ that is *not* in the set of wallpaper components that overlap $W((m_1, 0, 0))$. The following are the details of computations that enable us to find suitable collision-free vector(s).

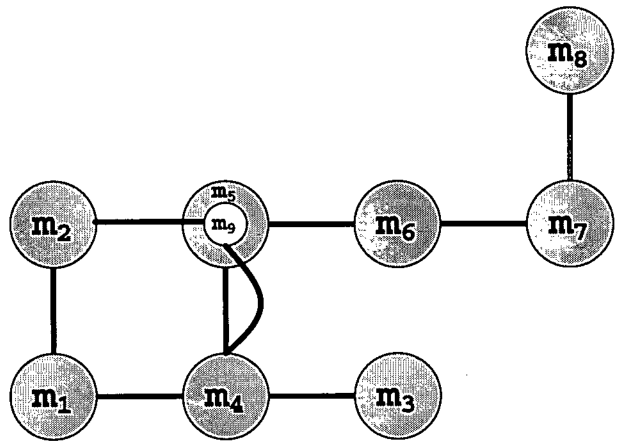

Figure 4.7: **A potential ghost of** m_9

The Actual Work:

Since $(m_1, 0, 0)$ and $(m_8, 0, 0)$ are not overlapped by any motif pieces, it suffices to study the pairwise intersections among $(m_2, 0, 1)$, $(m_3, 2, 0)$, $(m_4, 1, 0)$ $(m_{5,9}, 1, 1)$, $(m_6, 2, 1)$, and $(m_7, 3, 1)$. We use $gen(T, 0, 0)$ as a set of generators together with the natural period $[2, -2]$ to choose motif pieces (m_1, x, y) such that $y = 0$ or $y = 1$ (the height of the wallpaper component) as generators for all wallpaper components under consideration. This gives us an easy way to see when two wallpaper components are the same.

Motif piece $(m_2, 0, 1)$ **is overlapped by**

- $W((m_4, 0, 1)) = W((m_1, -1, 1))$,
- $W((m_6, 0, 1)) = W((m_1, -2, 0))$,
- $W((m_7, 0, 1)) = W((m_1, -3, 0))$, and
- $W((m_{5,9}, 0, 1)) = W((m_1, -2, 1))$.

Motif piece $(m_3, 2, 0)$ **is overlapped by**

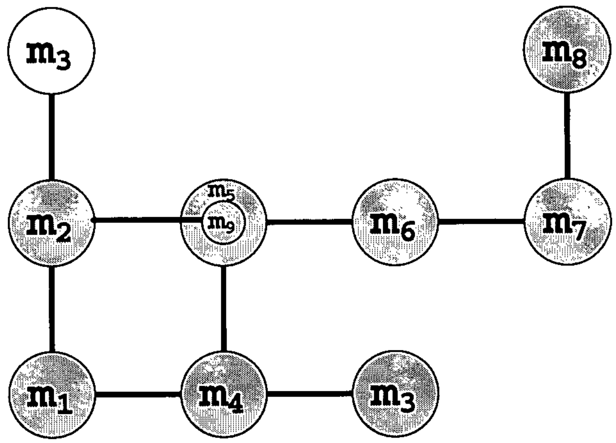

Figure 4.8: **The ghost of** m_3

- $W((m_6, 2, 0)) = W((m_1, 0, -1)) = W((m_1, -2, 1))$.

Motif piece $(m_4, 1, 0)$ **is overlapped by**

- $W((m_2, 1, 0)) = W((m_1, -1, 1))$,
- $W((m_6, 1, 0)) = W((m_1, -3, 1))$, and
- $W((m_7, 1, 0)) = W((m_1, -4, 1))$.

Motif piece $(m_{5,9}, 1, 1)$ **is overlapped by**

- $W((m_2, 1, 1)) = W((m_1, 1, 0))$.

Motif piece $(m_6, 2, 1)$ **is overlapped by**

- $W((m_2, 2, 1)) = W((m_1, 2, 0))$,
- $W((m_3, 2, 1)) = W((m_1, 0, 1))$, and
- $W((m_4, 2, 1)) = W((m_1, 1, 1))$.

Motif piece $(m_7, 3, 1)$ **is overlapped by**

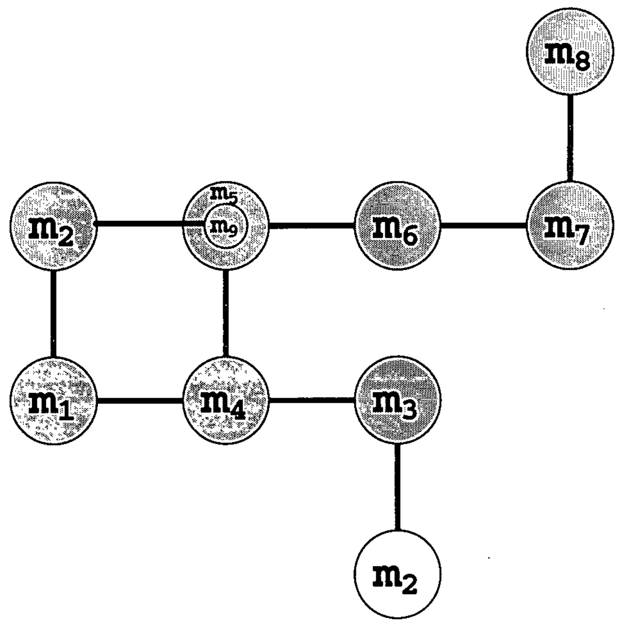

Figure 4.9: **The ghost of** m_2

- $W((m_2, 3, 1)) = W((m_1, 3, 0))$ and
- $W((m_4, 3, 1)) = W((m_1, 2, 1))$.

From a wallpaper component $W((m_1, a, b))$ that overlaps $W((m_1, 0, 0))$ (with $b = 0$ or $b = 1$ for this singly periodic example with natural period $[-2, 2]$) we know that any vector in the set $\{[a, b] + k[-2, 2] : k \in \mathbb{Z}\}$ is *not* available as a collision-free vector for Escher tile T. The previous list, once the repetitions have been eliminated, yields the complete list of vectors that are *not* available as collision-free vectors:

- $\{[-1, 1] + k[-2, 2] : k \in \mathbb{Z}\}$,
- $\{[-2, 0] + k[-2, 2] : k \in \mathbb{Z}\}$,
- $\{[-3, 0] + k[-2, 2] : k \in \mathbb{Z}\}$,

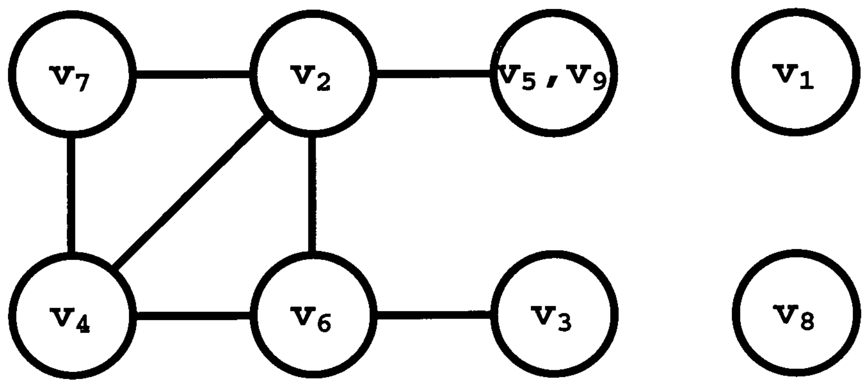

Figure 4.10: **The overlap graph**

- $\{[-2,1]+k[-2,2] : k \in \mathbb{Z}\}$,
- $\{[-3,1]+k[-2,2] : k \in \mathbb{Z}\}$,
- $\{[-4,1]+k[-2,2] : k \in \mathbb{Z}\}$,
- $\{[1,0]+k[-2,2] : k \in \mathbb{Z}\}$,
- $\{[2,0]+k[-2,2] : k \in \mathbb{Z}\}$,
- $\{[0,1]+k[-2,2] : k \in \mathbb{Z}\}$,
- $\{[1,1]+k[-2,2] : k \in \mathbb{Z}\}$,
- $\{[3,0]+k[-2,2] : k \in \mathbb{Z}\}$, and
- $\{[2,1]+k[-2,2] : k \in \mathbb{Z}\}$.

Since neither of the sets $\{[3,1]+k[-2,2] : k \in \mathbb{Z}\}$ nor $\{[-4,0]+k[-2,2] : k \in \mathbb{Z}\}$ are in the previous list, both $[3,1]$ and $[-4,0]$ are available as collision-free vectors. Moreover, the lattices $< [-2,2],[3,1] >$ and $< [-2,2],[-4,0] >$ are inequivalent, which will lead to the construction of two inequivalent 8-colored Big Tiles for the Escher tile of this example.

4.2 Big Tile Dimensions

We return to a more general setting: suppose the oracle has, upon input of Escher tile T, given you the pair of linearly independent vectors $[p_1, p_2]$, $[q_1, q_2] \in \mathbb{Z}^2$ that are collision-free and/or natural periods for T (depending on the input). Let $\Delta = |p_1 q_2 - p_2 q_1|$. The Big Tile, B_T, we will construct requires Δ colors. The dimensions of B_T will be $\frac{\Delta}{|\gcd(p_2,q_2)|}$ by $\frac{\Delta}{|\gcd(p_1,q_1)|}$; the oracle tells us that the dimensions were arrived at by finding the smallest rectangular sublattice of the lattice $< [p_1, p_2], [q_1, q_2] >$ (see Lemma 3.3.7).

Now, given any location (r, s) in the region $(0 \leq r < \frac{\Delta}{|\gcd(p_2,q_2)|}$ and $0 \leq s < \frac{\Delta}{|\gcd(p_1,q_1)|})$, assign a color to (m_1, r, s). The colors are chosen from a set of colors called a **palette**. The palette contains Δ colors, one for every pair (i, j), where $\overline{[i, j]}$ is a coset representative of $< [p_1, p_2], [q_1, q_2] >$, the lattice generated by $[p_1, p_2]$ and $[q_1, q_2]$. Visually, the palette is the half-open parallelogram spanned by $[p_1, p_2]$ and $[q_1, q_2]$. Loosely speaking, we color each copy of m_1 inside the palette with the color corresponding to the location of each such m_1. Extending the correct colors to the remaining motif pieces in the rest of the subsquares in the Big Tile region follows with some minor work that exploits some information contained in a spanning tree of the agglomerated period graph $\hat{G}_T$. We will illustrate the work by continuing the example.

4.3 Assigning The Colors

The Escher tile T in our example was singly periodic with period $[2, -2]$ and in section 4.1 we found that $[3, 1]$ could be used as a collision-free vector. A Big Tile parameterized by $[2, -2]$ and $[3, 1]$ requires $|2 \times 1 - (3 \times (-2))| = 8$ colors. The palette has dimensions 8×1, and linear algebra, elementary number theory, and a final use of the locations of the set of generating motif pieces found in section 4.1, Figure 4.5 are all that are needed to assign colors to all of the motif pieces whose locations are in the Big Tile region. In particular we can encode the coloring instructions in a pair of permutations: one horizontal and the other vertical. For example, the horizontal color permutation is given by (12345678) and can be extracted directly from the palette. The meaning of the permutation is: if motif piece (m_i, r, s) is known to be assigned color c, then $(m_i, r+j, s)$ must be assigned color $c + j \pmod 8$. The vertical coloring permutation is given by $(1, 1 + 1 \times 5 \pmod 8), 1 + 2 \times 5 \pmod 8), 1 + 3 \times 5 \pmod 8), 1 + 4 \times 5 \pmod 8), 1 + 5 \times 5 \pmod 8), 1 + 6 \times 5 \pmod 8), 1 + 7 \times 5 \pmod 8)) = (16385274)$. That is, if say motif piece (m_i, r, s) is known to have color c, then $(m_i, r, s+j)$ must be assigned color $c + 5j \pmod 8$. This information can be captured in a set of nine instructions to color all of the motif pieces in the Escher tile T with location (r, s): Let (m_i, t) mean "assign color t to motif piece m_i in location (r, s)." The coloring instructions for tile T in location (r, s) are given by $(T, r, s) = \{(m_1, 1 + r + 5s \pmod 8)), (m_2, 6 + r + 5s \pmod 8)), (m_3, 5 + r + 5s \pmod 8)), (m_4, 7 + r + 5s \pmod 8)), (m_5, 4 + r + 5s \pmod 8)), (m_6, 6 + r + 5s \pmod 8)), (m_7, 1 + r + 5s \pmod 8)), (m_8, 0 + r + 5s \pmod 8)), (m_9, 0 + r + 5s \pmod 8))\}$. The visual output of the eight-colored 8×8 Big Tile for the Escher tile given in Figure 4.1 is shown in Figure 4.11.

This method illustrates the outline of an efficient algorithm for producing a Big Tile for an arbitrary Escher tile T. Moreover, this example was chosen to illustrate another idea, that of the idea of essentially "different" Δ-colored Big Tiles for a given Escher tile.

Recall that $[3, 1]$ was a valid collision-free vector for the construction of a Big Tile for the Escher tile given in Figure 4.1. Another valid choice is $[-4, 0]$,

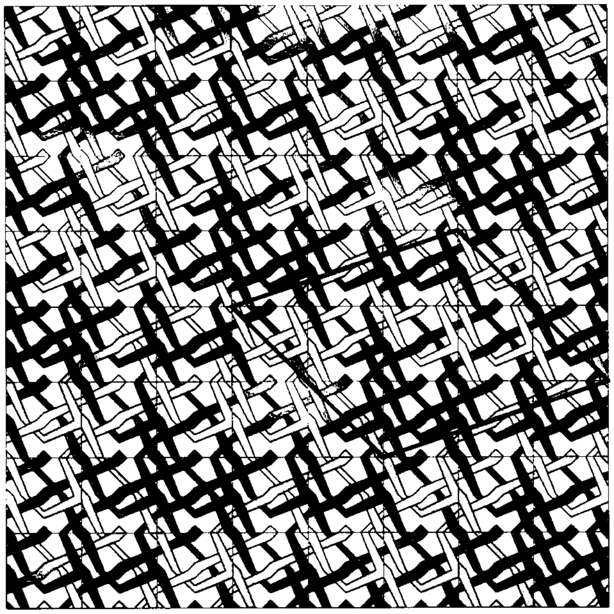

Figure 4.11: **An eight-colored 8× 8 Big Tile for the Escher tile in Figure 4.1. This corresponds to the lattice generated by** $\{[2,-2],[3,1]\}$.

and it is interesting to note that $|\det \begin{bmatrix} -2 & -4 \\ 2 & 0 \end{bmatrix}| = |\det \begin{bmatrix} -2 & 3 \\ 2 & 1 \end{bmatrix}| = 8$. Note also that the lattice generated by the vector pair $\{[2,-2],[-4,0]\}$ is not equivalent to the lattice generated by $\{[-2,2],[3,1]\}$ (this requires an argument using a basic tool from number theory [Ap76]). A different eight-colored Big Tile for T, corresponding to the lattice generated by $\{[2,-2],[-4,0]\}$, is shown in Figure 4.12.

In the next Chapter we will prove that a Big Tile exists for any Escher tile.

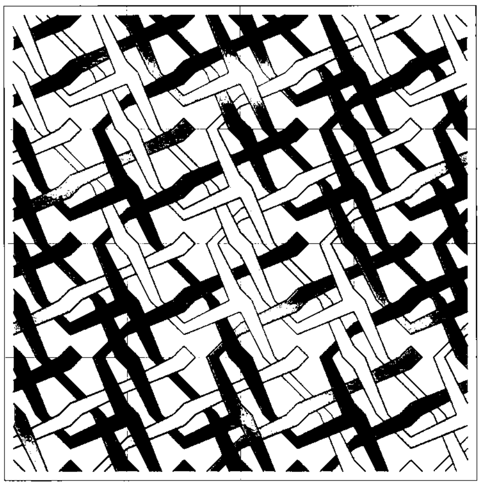

Figure 4.12: **A different eight-colored Big Tile for the Escher tile in Figure 4.1. This corresponds to the lattice generated by** $\{[2,-2],[-4,0]\}$.

Chapter 5

Mirabile Dictu: Existence Proof

The purpose of this chapter is to verify the allegations of previous chapters, namely that a Big Tile exists for an arbitrary Escher tile. First we study an Escher tile whose period graph is connected. That a Big Tile exists for an arbitrary Escher tile will follow with very little extra work. The Big Tile existence proof will be accomplished by constructing a Big Tile for T with tools from Chapter 3. In Chapter 6 we will massage the construction into an efficient algorithm. In Chapter 7 we will shed light on why the Big Tiles arising from our construction are canonical, and follow with a canonical representation of Big Tiles that will shed some light on the chromatic number of an Escher tile, and also under what circumstances there exist two measurably different Δ-colored Big Tiles for a fixed value of Δ. These ideas are intended to be catalysts for a future classification of Big Tiles.

5.1 A Few More Tools

It is of seminal importance to detect when two distinct wallpaper components overlap.

Lemma 5.1.1 (Find Collision-Free Vectors): *Let* $T = \{(m_1, 0, 0), \ldots, (m_k, 0, 0)\}$ *be an Escher tile whose period graph has one connected component and suppose* T *is either trivially or singly periodic.*

1. *If* T *is singly periodic with natural period* $[p_1, p_2]$*, then there exist infinitely many* $[q_1, q_2] \in \mathbb{Z}^2$*, linearly independent with* $[p_1, p_2]$*, such that*

$$\left(\bigcup_{M \in W((m_1,0,0))} M\right) \bigcap \left(\bigcup_{N \in W((m_1,q_1,q_2))} N\right) = \varnothing.$$

2. *If* T *is trivially periodic, then there exist infinitely many linearly independent vector pairs* $\{[p_1, p_2], [q_1, q_2] \in \mathbb{Z}^2\}$ *such that*

$$\left(\bigcup_{M \in W((m_1,0,0))} M\right) \bigcap \left(\bigcup_{N \in W((m_1,p_1,p_2))} N\right) = \varnothing,$$

   and

$$\left(\bigcup_{M \in W((m_1,0,0))} M\right) \bigcap \left(\bigcup_{N \in W((m_1,q_1,q_2))} N\right) = \varnothing.$$

Proof: **Case 1:** We draw upon the many and varied tools from Chapter 3.

We have:

- an Escher tile T with a connected period graph G_T that is singly periodic with natural period $[p_1, p_2]$,
- an agglomerated spanning tree $\hat{S}_T$ that gives rise to a set of generating motif pieces $gen(T, 0, 0) = \{(m_1, 0, 0), (m_2, i_2, j_2), \ldots, (m_k, i_k, j_k)\}$, and

- the overlap graph, O_T, which tells us if one equivalence class of motif pieces in any given location intersects another in the same location.

We need:

- a characterization of the set $F(T)$ of "forbidden" wallpaper components $\{W((m_1, a, b))\}$ that overlap $W((m_1, 0, 0))$. We will use O_T to find $F(T)$, which is represented by a set of forbidden vectors, $\tilde{F}(T)$.

We will show that $|F(T)|$ is finite, but more importantly, we will show that $\mathbb{Z}^2 \setminus \tilde{F}(T)$ is nonempty and in fact infinite.

The Procedure:

First we find $F(T)$. Suppose $W((m_1, a, b))$ overlaps $W((m_1, 0, 0))$. By Definition 2.2.8, two such wallpaper components overlap if and only if $\exists$ $(m_s,\ x,\ y) \in W((m_1, 0, 0))$ and $(m_t,\ x,\ y) \in W((m_1, a, b))$ such that $M \bigcap N \neq \varnothing$. For example if M is motif piece m_1 in some location, then the location must be $(k_0 p_1, k_0 p_2)$ for some $k_0 \in \mathbb{Z}$. In that case, $N = (m_s, k_0 p_1, k_0 p_2)$ for some $s \in \{1, \ldots, k\}$. By the construction of $W((m_1, 0, 0))$ (see Definition 2.2.7) $(m_1, k_0 p_1, k_0 p_2) \bigcap (m_s,\ k_0 p_1, k_0 p_2) \neq \varnothing$ if and only if $(m_1, 0, 0) \bigcap (m_s, 0, 0) \neq \varnothing$. In general, for $(m_t, a, b) \in W((m_1, 0, 0))$ we have $(a, b) = (-i_t + k_0\ p_1,\ -j_t + k_0 p_2)$, which means $(m_t, -i_t + k_0 p_1, -j_t + k_0 p_2) \bigcap (m_s,\ -i_t + k_0 p_1\ , -j_t + k_0 p_2) \neq \varnothing$ if and only if $(m_t, 0, 0) \bigcap (m_s, 0, 0) \neq \varnothing$. That is, wallpaper components that overlap $W((m_1, 0, 0))$ can be tracked to precisely those pairs of motif pieces in location $(0, 0)$ that intersect one another, which is exactly the information given by the overlap graph O_T, a finite graph. That is, to find a bound for the number of wallpaper components that overlap $W((m_1, 0, 0))$, it suffices to count the number of motif pieces that intersect each element $(m_s, i_s, j_s) \in gen(T, 0, 0)$; but (m_s, i_s, j_s) is intersected by exactly $deg([v_s])$ distinct motif pieces, where $[v_s]$ is an E-vertex of the overlap graph O_T. Thus, an upper bound for the number of wallpaper components that overlap $W((m_1, 0, 0))$ is

$$\sum_{[v_i] \in O_T} deg([v_i]),$$

which is twice the number of edges in O_T; this quantity does not always give the exact number of distinct wallpaper components that overlap $W((m_1, 0, 0))$ because two different pairs of intersecting motif pieces may belong to the same wallpaper component, as is the case in the detailed example in Chapter 4.

(Figure 5.1 shows a set of generating motif pieces $gen(T, 0, 0)$ for the Escher tile in Figures 3.9 and 3.5 and all of the motif pieces that overlap elements of $gen(T, 0, 0)$. See Figure 5.2 for the overlap graph of T.)

Now, suppose the finite set of forbidden wallpaper components that overlap $W((m_1, 0, 0))$ is given by $F(T) = \{W((m_1, a_1, b_1)), \ldots, W((m_1, a_t, b_t))\}$. Though each forbidden wallpaper component $W((m_1, a_s, b_s))$ is encoded by an infinite set of forbidden vectors, namely $\{[a_s + k_0 p_1, b_s + k_0 p_2] : k_0 \in \mathbb{Z}\}$, we can assume without loss of generality that $0 \leq b_s < p_2$ for $s = 1, \ldots, k$ by using $b_i = \text{height}(W((m_1, a_i, b_i))$ (Definition 3.3.6).

Consider $H = \{[M, 0] : M \in \mathbb{Z}\}$ (or else $\{[0, M] : M \in \mathbb{Z}\}$ if it so happens that $p_2 = 0$). Only finitely many elements of H can correspond to elements $F(T)$ leaving infinitely many vector candidates that can be used to identify Escher wallpaper components that do not overlap $W((m_1, 0, 0))$.

We conclude that there exist infinitely many vectors $[q_1, q_2] \in \mathbb{Z}^2$ such that $W((m_1, 0, 0))$ does not overlap $W((m_1, q_1, q_2))$. This completes the first (and hardest) case.

Case 2: Suppose T is trivially periodic. We modify the argument given in Case 1 to apply to this case. In particular, finding an upper bound for the number of wallpaper components that overlap $W((m_1, 0, 0))$ can be accomplished by examining all pairs of overlapping motif pieces in each location of the set of generating motif pieces, which is, again, twice the number of edges in the overlap graph so that the set of forbidden wallpaper components is finite. Say the set is $F(T) = \{W((m_1, a_1, b_1)), \ldots, W((m_1, a_t, b_t))$. Since T has no natural periods, the set of vectors $\{[a_1, b_1], \ldots, [a_t, b_t]\}$ uniquely identifies the forbidden vectors $\tilde{F}(T)$. In that case any vector in $[x, y] \in \mathbb{Z}^2 \setminus \tilde{F}(T)$ has the property that $W((m_1, x, y))$ does not overlap $W((m_1, 0, 0))$. This concludes Case 2 and the proof. ∎

We have, now, a constructive method to identify a pair of "good" linearly independent vectors $[p_1, p_2], [q_1, q_2] \in \mathbb{Z}^2$ with an Escher tile whose period graph is connected. The lattice $L = < [p_1, p_2], [q_1, q_2] >$ will be all we need to construct a Big Tile for T.

Lemma 5.1.1 gives fuel to the idea of constructing a Big Tile from a lattice. We make these ideas precise in the next several definitions.

Definition 5.1.2 (Forbidden Vectors and Determinants): *Let* $T = \{(m_1, 0, 0), \ldots, (m_k, 0, 0)\}$ *be an Escher tile whose period graph is connected, and is either singly or trivially periodic. By Lemma 5.1.1 the number of wallpaper components that overlap* $W((m_1, 0, 0))$ *is finite and can be characterized by a set of vectors* $\tilde{F}(T)$. *Let* $F(T) = \{W((m_1, a_1, b_1)), \ldots, W((m_t, a_t, b_t))\}$ *be the finite set of wallpaper components that overlap* $W((m_1, 0, 0))$.

- *If* T *is singly periodic with natural period* $[p_1, p_2]$, *then* $\tilde{F}(T) = \{[a_1 + k_0 p_1, b_1 + k_0 p_2], \ldots, [a_t + k_0 p_1, b_t + k_0 p_2] : k_0 \in \mathbb{Z}$ *and* $0 \leq b_i < p_2$ *for* $i = 1 \ldots, t\}$.

- *If* T *is trivially periodic then* $\tilde{F}(T) = \{[a_1, b_1], \ldots, [a_t, b_t]\}$.

We say that $\tilde{F}(T)$ **is the set of forbidden vectors for** T, *and*

- *if T is trivially periodic, for any $[r_1, r_2], [s_1, s_2] \in \tilde{F}(T)$, we say $|r_1 s_2 - r_2 s_1|$* **is a forbidden determinant for T.**

- *If T is singly periodic with natural period $[p_1, p_2]$ then for any $[q_1, q_2] \in \tilde{F}(T)$, we say $|p_1 q_2 - p_2 q_1|$* **is a forbidden determinant for T.**

Note that in the singly periodic case, the complement of $\tilde{F}(T)$ in $\mathbb{Z}^2$ is exactly the set of collision-free vectors for T.

In the singly periodic case there are infinitely many vectors that identify the finitely many forbidden wallpaper components that overlap $W((m_1, 0, 0))$ but as it turns out there are only finitely many forbidden determinants, as we demonstrate in the next lemma.

Lemma 5.1.3 (Finitely Many Forbidden Determinants): *Let T be an Escher tile whose period graph is connected and is either trivially periodic or else singly periodic with natural period $[p_1, p_2]$. Then there are only finitely many forbidden determinants for T.*

Proof: If T is trivially periodic then by the proof of Lemma 5.1.1 there are only finitely many forbidden vectors, and hence by Definition 5.1.2 there are only finitely many forbidden determinants. If T is singly periodic with natural period $[p_1, p_2]$, then by the proof of Lemma 5.1.1 the set of forbidden vectors for T is given by $\tilde{F}(T) = \{[a_1 + k_0 p_1, b_1 + k_0 p_2], \ldots, [a_t + k_0 p_1, b_t + k_0 p_2] : k_0 \in \mathbb{Z}$ and $0 \leq b_i < p_2$ for $i = 1, \ldots, t\}$. Since

$$|\det \begin{bmatrix} p_1 & a_i + k_0 p_1 \\ p_2 & b_i + k_0 p_2 \end{bmatrix} | = |\det \begin{bmatrix} p_1 & a_i \\ p_2 & b_i \end{bmatrix} |,$$

there are only finitely many forbidden determinants for T. ∎

We have enough vocabulary to narrow the field of allowable vectors even further; we wish to restrict to allowable vector pairs $\{[p_1, p_2], [q_1, q_2]\}$ such that $|\det \begin{bmatrix} p_1 & q_1 \\ p_2 & q_2 \end{bmatrix}|$ is strictly larger than any forbidden determinant.

Definition 5.1.4 (Color-Safe Determinant): *Let T be an Escher tile whose period graph is connected, and is either singly or trivially periodic. Let $F = \{\Delta_1, \ldots, \Delta_t\}$ be the set of forbidden determinants for T. Then a* **color-safe determinant for** T *is any integer Δ such that $\Delta > max_{\Delta_i \in F}\{\Delta_i\}$.*

Definition 5.1.5 (Color-Safe Lattice): *Let T be an Escher tile whose period graph is connected and suppose $A(T)$ is the set of collision-free vectors for T in the trivially and singly periodic cases.*

- *Suppose T is trivially periodic. Let $[p_1, p_2]$ and $[q_1, q_2] \in A(T)$ be linearly independent and suppose $|p_1q_2 - p_2q_1|$ is a color-safe determinant for T. Then the lattice $< [p_1, p_2], [q_1, q_2] >$ is said to be a* **color-safe lattice for** T.
- *Suppose T is singly periodic with natural period $[p_1, p_2]$ and let $[q_1, q_2] \in A(T)$. If $|p_1q_2 - p_2q_1|$ is a color-safe determinant for T then the lattice $< [p_1, p_2], [q_1, q_2] >$ is a* **color-safe lattice for** T.
- *Suppose T is doubly periodic with natural (linearly independent) periods $[p_1, p_2]$ and $[q_1, q_2]$. Then the lattice $< [p_1, p_2], [q_1, q_2] >$ is a* **color-safe lattice for** T.

It will be necessary to know that if $L =< [p_1, p_2], [q_1, q_2] >$ is a color-safe lattice then any sublattice of L is color-safe.

Lemma 5.1.6 (Sublattice of Color-Safe Lattice is Color-Safe): *Let T be an Escher tile with color-safe lattice L. Then for any sublattice $L' \subset L$ we have L' is color-safe for T. In particular, if $[r,s] \in L'$ then $W((m_1,0,0))$ does not overlap $W((m_1,r,s))$.*

Proof: Suppose $L' =< [a_1,a_2],[b_1,b_2] >$ is a sublattice of $L =< [p_1,p_2],[q_1,q_2] >$. Then $\exists x_1, y_1, x_2, y_2 \in \mathbb{Z}$ such that

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = x_1 \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + y_1 \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

and

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = x_2 \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + y_2 \begin{bmatrix} q_1 \\ q_2 \end{bmatrix},$$

which means

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = \begin{bmatrix} x_1p_1 + y_1q_1 & x_2p_1 + y_2q_1 \\ x_1p_2 + y_1q_2 & x_2p_2 + y_2q_2 \end{bmatrix}$$

and thus

$$|\det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} | = |\det \begin{bmatrix} x_1p_1 + y_1q_1 & x_2p_1 + y_2q_1 \\ x_1p_2 + y_1q_2 & x_2p_2 + y_2q_2 \end{bmatrix} |.$$

But

$$\begin{aligned} |\det \begin{bmatrix} x_1p_1 + y_1q_1 & x_2p_1 + y_2q_1 \\ x_1p_2 + y_1q_2 & x_2p_2 + y_2q_2 \end{bmatrix} | \\ = |(p_2q_1 - p_1q_2)(x_2y_1 - x_1y_2)| \\ \geq |p_2q_1 - p_1q_2| \end{aligned}$$

because $[a_1,a_2]$ and $[b_1,b_2]$ are linearly independent and x_1, y_1, x_2, $y_2 \in \mathbb{Z}$. In that case, since L is a color-safe lattice, $|p_2q_1 - p_1q_2|$ is a color-safe determinant and thus so is $|(p_2q_1 - p_1q_2)(x_2y_1 - x_1y_2)|$. We conclude that $L' =< [a_1,a_2],[b_1,b_2] >$ is color-safe. ∎

The following corollary is a direct consequence of Lemma 5.1.6.

Corollary 5.1.7 (Vector in a Color-Safe Lattice is Collision-Free): *Let T be an Escher tile T whose period graph is connected and for which L is a color-safe lattice. Then any* $\mathbf{v} \in L$ *is a collision-free vector for* T.

Finally, it will be useful to find an upper bound for the area of a color-safe lattice in terms of k, the number of motif pieces in an Escher tile. This area is the absolute value of the determinant of the matrix of generating vectors.

Proposition 5.1.8 (Color-Safe Lattice of Area at Most $O(k^2)$): *Let $T = \{(m_1, 0, 0), \ldots, (m_k, 0, 0)\}$ be an Escher tile whose period graph is connected.*

- *If T is doubly periodic with natural periods $[p_1, p_2]$ and $[q_1, q_2]$ then $< [p_1, p_2], [q_1, q_2] >$, the color-safe lattice, satisfies $|p_1 q_2 - p_2 q_1| \leq (k+1)^2$.*
- *If T is singly periodic with natural period $[p_1, p_2]$ then $< [p_1, p_2], [2k+2, 0] >$ is a color-safe lattice for T as long as $p_2 \neq 0$. If $p_2 = 0$, then $< [p_1, p_2], [0, 2k+2] >$ is a color-safe lattice for T.*
- *If T is trivially periodic, then $< [2k+3, 0], [0, 2k+3] >$ is a color-safe lattice for T.*

Proof: If T is doubly periodic, the claim follows from Corollary 3.2.7. Suppose T is singly periodic with natural period $[p_1, p_2]$ and without loss of generality assume $p_1 \neq 0$. By the proof of Lemma 5.1.1, we identify the finitely many wallpaper components that overlap $W((m_1, 0, 0))$ by examining the motif pieces that intersect each element of a set of generating motif pieces $gen(T, 0, 0) = \{(m_1, 0, 0), \ldots, (m_k, i_k, j_k)\}$. Suppose, for example, $(m_s, i_s, j_s) \in gen(T, 0, 0)$ and that $(m_s, i_s, j_s) \cap (m_t, i_s, j_s)$ for some $t \neq s$. Then (m_t, i_s, j_s) is related to $(m_1, i_s - i_t, j_s - j_t)$ and lies in the same forbidden wallpaper component $W((m_t, i_s, j_s)) = W((m_1, i_s - i_t, j_s - j_t))$. By the proof of Lemma 3.3.5, $|j_s - j_t| \leq |p_2|$. Hence, the height of $W((m_t, i_s, j_s))$ is

- $j_s - j_t$ if $j_s - j_t \geq 0$, or else
- $j_s - j_t + |p_2|$ if $j_s - j_t < 0$.

In the former case, the forbidden vector that we use to identify $W((m_t, i_s, j_s))$ is $[i_s - i_t, j_s - j_t]$, and in the latter case is $[i_s - i_t \pm p_1, j_s - j_t \pm p_2]$. In both cases, since by Corollary 3.2.7 we have $|p_1| \leq k + 1$, we have shown that any forbidden vector $[a, b]$ for T has the property that $|a| \leq 2k + 2$.

If T is trivially periodic, then by a similar argument, the forbidden vectors all have entries bounded (in absolute value) by k. Thus, we are assured that we may use one of $[2k + 3, 0]$ or $[0, 2k + 3]$ as a collision-free vector for T in the singly periodic case, and both of $[2k + 3, 0]$ or $[0, 2k + 3]$ as collision-free vectors in the trivially periodic case. In all cases we have shown that there exists a color-safe lattice whose fundamental region has area $O(k^2)$. ∎

We have worked hard to associate special lattices with an Escher tile T whose period graph is connected. The lattice $< [p_1, p_2], [q_1, q_2] >$ is an abelian group that has $|p_1q_2 - p_2q_1|$ cosets. In the next section we will color the cosets of a color-safe lattice, extend the coloring to the wallpaper components in $Wall(T)$, and extract a $|p_1q_2 - p_2q_1|$-colored Big Tile for T.

5.2 Coloring the Cosets

For the purpose of assigning colors to all of the motif pieces in $Wall(T)$ we define the **palette**, which requires as input a linearly independent vector pair from $\mathbb{Z}^2$. Subsequently we define the **colored tile**, which requires an Escher tile and a palette as input.

Definition 5.2.1 (Palette): *Suppose $[p_1, p_2]$ and $[q_1, q_2] \in \mathbb{Z}^2$ are linearly independent, let $\Delta = |p_1q_2 - p_2q_1|$ and $L =< [p_1, p_2], [q_1, q_2] >$. The* **palette correspond-**

ing to $[p_1, p_2]$, $[q_1, q_2]$ *is* $\{[i, j] + L : i, j \in \mathbb{Z}\}$*, and is denoted by* **Palette(**$[p_1, p_2]$, $[q_1, q_2]$**)**. *In other words, the palette is the set of* Δ *distinct cosets of the lattice* L. *Often we denote* $[i, j] + L$ *by* $\overline{[i, j]}$.

We were first introduced to the palette in Chapter 4, where visually speaking we think of the palette as the half-open parallelogram P spanned by $[p_1, p_2]$ and $[q_1, q_2]$ and the colors are labelled by the Δ lattice points contained in P. See Figure 5.3 for an example of a set of six distinct coset representatives of the palette corresponding to $[3, 0]$ and $[0, 2]$.

In Chapter 6, which contains a polynomial-time algorithm that constructs a Big Tile, we will benefit from knowing that for any arbitrary linearly independent $[p_1, p_2]$ and $[q_1, q_2] \in \mathbb{Z}^2$, we can always find a rectangular array of distinct coset representatives for $Palette([p_1,\ p_2],\ [q_1,\ q_2])$. This important fact will ultimately make the coloring instructions for a Big Tile constructed with $Palette([p_1\ p_2],\ [q_1,\ q_2])$ easy to specify.

Lemma 5.2.2 (Rectangular Array of Distinct Coset Reps): *Suppose* $[p_1, p_2]$ *and* $[q_1, q_2] \in \mathbb{Z}^2$ *are linearly independent, and let* $\Delta = |p_1 q_2 - p_2 q_1|$. *Then a set of distinct coset representatives for* $Palette([p_1,\ p_2],\ [q_1,\ q_2])$ *is given by* $\{(i, j) : 0 \leq i < \frac{\Delta}{|\gcd(p_2, q_2)|},\ \ 0 \leq j < |\gcd(p_2, q_2)|\}$.

Proof: Let $m = \frac{\Delta}{|\gcd(p_2, q_2)|}$ and $n' = |\gcd(p_2, q_2)|$. By Proposition 3.3.7, $|m|$ is minimal such that $[m, 0] \in L$ and that n' is at most as large as the smallest vertical shift in $Palette([p_1,\ p_2],\ [q_1,\ q_2])$. As such, all of the elements of $Palette([p_1,\ p_2],\ [q_1,\ q_2])$ are contained (with repetitions) within the infinite vertical strip $X = \{[i, j] : 0 \leq i < m,\ \ j \in \mathbb{Z}\}$. Now consider $Y = \{[i, j] : 0 \leq i < m,\ \ 0 \leq j < n'\}$ and note that $|Y| = \Delta$. By the minimality of m, any pair of elements in a row or column of Y must be distinct. Now, if all coset representatives of $Palette([p_1,$

p_2], [q_1, q_2]) in Y are distinct, we are done. Otherwise, without loss of generality, assume that $[0, 0]$ and $[i, j]$ both represent the same coset in $Palette([p_1, p_2], [q_1, q_2])$ for some $[i, j] \neq [0, 0]$ in Y and with i minimal. Consider the parallelogram spanned by $[m, 0]$ and $[i, j]$. Since n' is at most as large as the smallest vertical shift $< [p_1, p_2], [q_1, q_2] >$, we must have $i \neq 0$. For any $[a, b] \in \mathbb{Z}^2$ we have $\overline{[a + m, b]} = \overline{[a, b]}$ and hence the half-open parallelogram spanned by $[m, 0]$ and $[0, j]$ represents the same set of cosets as the half-open parallelogram spanned by $[m, 0]$ and $[i, j]$. In fact every point in Y is represented by some point in the parallelogram spanned by $[m, 0]$ and $[0, j]$. In that case, by the minimality of i, no other coset representatives of $Palette([p_1, p_2], [q_1, q_2])$ can be contained in X, which is a contradiction. We may conclude, then, that the elements of Y represent the Δ elements of $Palette([p_1, p_2], [q_1, q_2])$. ∎

The author wishes to thank Joel Friedman for the inspiration for this elegant proof.

Next we construct a **colored tile** for an Escher tile T whose period graph has only one connected component. The colored tile will consist of concatenated copies of T, sides abutting, whose motif pieces are colored by way of $Palette([p_1, p_2], [q_1, q_2])$, and whose dimensions depend upon p_1, p_2, q_1 and q_2.

Definition 5.2.3 (colored tile): *Let $T = \{(m_1, 0, 0), \ldots, (m_k, 0, 0)\}$ be an Escher tile and assume G_T, the period graph for T, is connected. Suppose $[p_1, p_2]$ and $[q_1, q_2] \in \mathbb{Z}^2$ are linearly independent, and let $L =< [p_1, p_2], [q_1, q_2] >$.*

1. *Set $\Delta = |p_1 q_2 - p_2 q_1|$, $m = \frac{\Delta}{|\gcd(p_2, q_2)|}$, and $n = \frac{\Delta}{|\gcd(p_1, q_1)|}$.*
2. *Let $\hat{S}_T$ be an agglomerated spanning tree of $\hat{G}_T$, and $gen(T, 0, 0) = \{(m_1, 0, 0), (m_2, i_2, j_2), \ldots, (m_k, i_k, j_k)\}$ a set of generating motif pieces obtained from the fat spanning tree S_T.*

3. *Define the coloring function* C_Δ. *Let* $C_\Delta : Wall(T) \to$ *Palette*$([p_1, p_2], [q_1, q_2])$ *by* $C_\Delta((m_s, i, j)) := \overline{[-i_s + i, -j_s + j]}$.

A **colored tile for** T *parameterized by* $[p_1, p_2]$ *and* $[q_1, q_2]$ *and denoted* $C_T([p_1, p_2], [q_1, q_2])$ *is the set* $\{(m_s, i, j, C_\Delta((m_1, -i_s + i, -j_s + j)) : i \in \mathbb{Z}_m, j \in \mathbb{Z}_n, s = 1, \ldots, k\}$, *where the fourth entry assigns a color from* $Palette([p_1, p_2], [q_1, q_2])$ *to* m_s *in location* (i, j) *by way of the coloring function* C_Δ.

Visually, $C_T([p_1, p_2], [q_1, q_2])$ is a $\frac{\Delta}{|\gcd(p_2, q_2)|} \times \frac{\Delta}{|\gcd(p_1, q_1)|}$ rectangle with lower left unit subsquare centered at $(0, 0)$, and sides parallel to the standard axes. Each unit subsquare centered at a lattice point contains a copy of T whose motif pieces are colored by the function C_Δ. The work behind the scenes of the construction of a colored tile is this: we place a copy of m_1 inside each unit subsquare centered at a lattice point contained in $Palette([p_1, p_2], [q_1, q_2])$ and color each such copy of (m_1, i, j) with the coset $[i, j] + L$. A copy of m_1 in location (i, j) is related to each of $\{(m_2, i_2 + i, j_2 + j), \ldots, (m_k, i_k + i, j_k + j)\}$, and C_Δ assigns color $[i, j] + L$ to this particular set of motif pieces related to (m_1, i, j). Note that the set $\{(m_1, i, j), (m_2, i_2 + i, j_2 + j), \ldots, (m_k, i_k + i, j_k + j)\}$ is simply $gen(T, i, j)$ or alternatively, is the set $gen(T, 0, 0)$ translated elementwise by the vector $[i, j]$.

See Figures 5.4 and 5.6 for an example of an Escher tile T and six translated (and colored) copies of $gen(T, 0, 0)$ that are derived from the palette corresponding to $[2, 1]$ and $[6, 0]$. The vector $[2, 1]$ is the natural periodicity of T and $[6, 0]$ is a collision-free vector for T. A Big Tile for T can be found in Figure 5.7.

We are ready to prove that a Big Tile exists for an Escher tile whose period graph has one connected component.

Theorem 5.2.4 (Big Tile for Escher tile with Connected Period Graph): *Let* $T = \{(m_1, 0, 0), (m_2, 0, 0), \ldots, (m_k, 0, 0)\}$ *be an arbitrary Escher tile whose period graph* G_T *is connected. Then there exists a Big Tile for* T.

Proof: The Escher tile T is trivially, singly, or doubly periodic. By Proposition 5.1.8, there exists a color-safe lattice for T in all three cases. Suppose one such is $< [p_1, p_2], [q_1, q_2] >$. Let $C_T = C_T([p_1, p_2], [q_1, q_2])$ be the colored tile for T colored by $Palette([p_1, p_2], [q_1, q_2])$. We claim that C_T is a Big Tile for T.

In order to verify the claim, by Lemma 2.2.9 it suffices to show

1. $C_\Delta(M) = C_\Delta(N)$ for every $M, N \in W((m_1, 0, 0))$, in which case we write $C_\Delta(W((m_1, 0, 0))$ to mean the one color from $Palette([p_1, p_2], [q_1, q_2])$ that is assigned to every motif piece in $W((m_1, 0, 0))$, and

2. if $C_\Delta(W((m_1, 0, 0)) = C_\Delta(W((m_1, i, j))$, then $W((m_1, 0, 0))$ and $W((m_1, i, j))$ do not overlap.

Part 1: The wallpaper components are uniformly colored by construction because by Lemma 3.3.7 the rectangular lattice $< [\frac{\Delta}{|\gcd(q_2, p_2)|}, 0], [0, \frac{\Delta}{|\gcd(q_1, p_1, \Delta)|}] >$ is a sublattice of $< [p_1, p_2], [q_1, q_2] >$. That is, the coloring of $Wall(T)$ assigned by way of the cosets of $< [p_1, p_2], [q_1, q_2] >$ is inherited by any sublattice of L.

Part 2: Suppose $C_\Delta(W((m_1, 0, 0))) = C_\Delta(W((m_1, i, j)))$. By the construction of C_T the colored tile generated by $[p_1, p_2]$ and $[q_1, q_2]$, we know that $[i, j] \in < [p_1, p_2], [q_1, q_2] >$. In that case $i = xp_1 + yq_1$ and $j = xp_2 + yq_2$ for some $x, y \in \mathbb{Z}$. Without loss of generality assume that at least one of $[p_1, p_2]$ or $[q_1, q_2]$ is a collision-free vector for T (if both are natural periodicities then $W((m_1, 0, 0)) = W((m_1, i, j))$, which has been disallowed). Suppose $[p_1, p_2]$ is not a natural period and $[q_1, q_2]$ is a natural period. Then $W((m_1, i, j)) = W((m_1, xp_1 + yq_1, xp_2 + yp_2)) = W((m_1, xp_1, xp_2))$. By Corollary 5.1.7, since $[xp_1, xp_2] \in L$, $W((m_1, 0, 0))$ does not overlap $W((m_1, xp_1, xp_2))$. Hence, wallpaper components that are colored the same do not overlap. ∎

At last we have the means to prove that a Big Tile exists for any Escher tile.

Theorem 5.2.5 (Big Tile for Arbitrary Escher tile): *Let T be an Escher tile. Then there exists a Big Tile for T.*

Proof: Let G_T be the period graph for T and suppose G_T has N connected components. Consider the N Escher tiles induced by the connected components G_T (see Definition 3.1.3): $T_1, T_2, \ldots, T_N$. By Theorem 5.2.4, there is a Big Tile for T_i for $i = 1, \ldots, N$. Let $B_{T_i} = B_{T_i}(\Delta_i, m_i, n_i)$ be a Δ_i-colored $m_i \times n_i$ Big Tile for T_i. For each use a new set of colors; being wasteful is not relevant for showing existence. Let $m = \text{lcm}(m_1, \ldots, m_N)$ and $n = \text{lcm}(n_1, \ldots, n_N)$. A Big Tile for T, then, consists of the concatenation of $\frac{m}{m_i} \times \frac{n}{n_i}$ copies of B_{i_T} inside an $m \times n$ rectangular region for $i = 1, \ldots, N$. The number of colors used will be $\sum_{i=1}^{N} \Delta_i$. ∎

Now that we know that a Big Tile exists for any Escher tile, we have two tasks ahead of us. First we describe an algorithm called **ColorFast**, that constructs a Big Tile for an Escher tile whose period graph is connected (which easily extends to any Escher tile). The algorithm will be linear in the total number of intersections of motif pieces along the boundary of $S_{0,0}$, the unit square in which T is located. Our last task, at least for purposes of this thesis, is to make a connection between irreducible Big Tiles and positive definite binary quadratic forms with integer coefficients. We view the Big Tile constructed in the proof of Theorem 5.2.4 as canonical, and it is the canonical Big Tiles for which we will make the connection to binary quadratic forms.

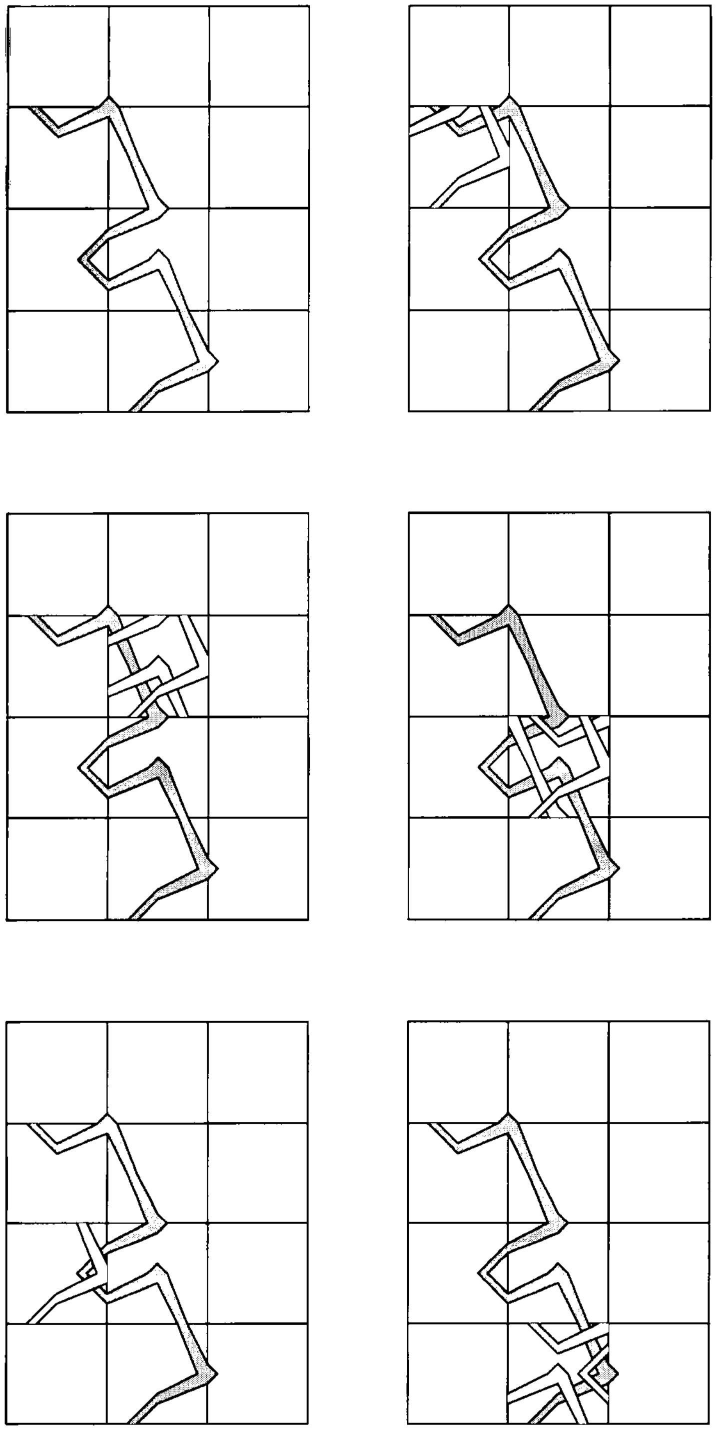

Figure 5.1: **Motif pieces that intersect elements of $gen(T, 0, 0)$ for the Escher tile in Figure 3.9**

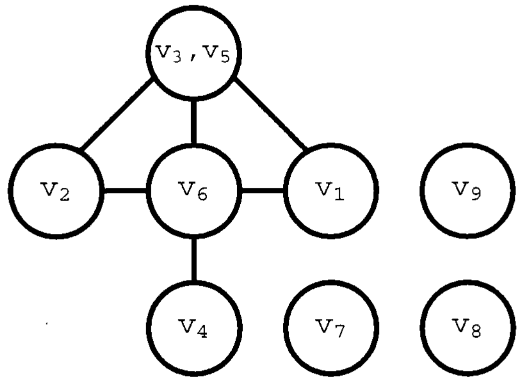

Figure 5.2: **Overlap graph for the Escher tile in Figure 3.9**

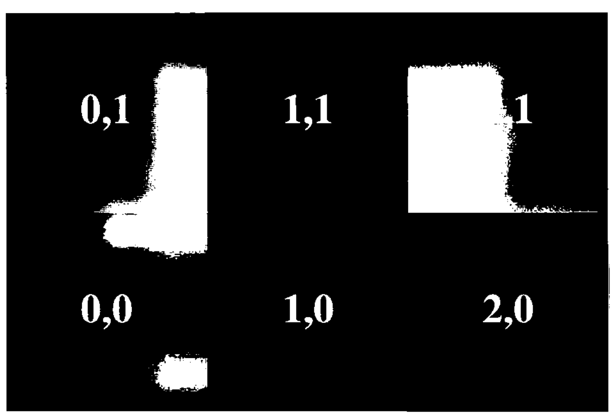

Figure 5.3: **Palette associated with** $[3, 0]$ **and** $[0, 2]$

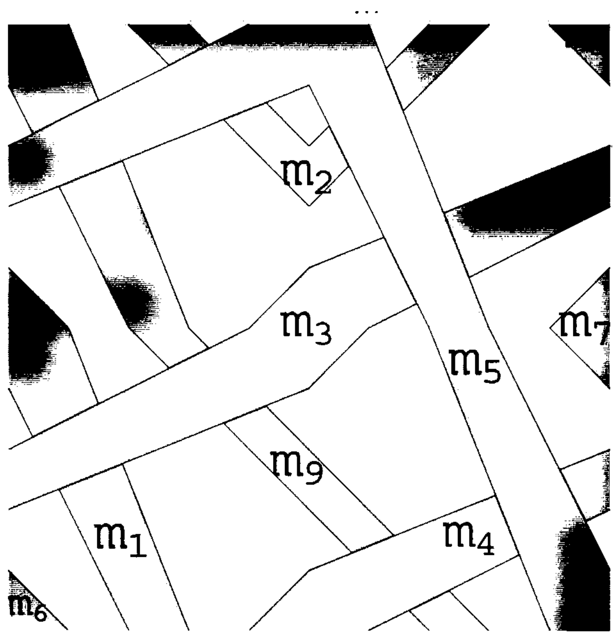

Figure 5.4: **A singly periodic Escher tile with natural period** $[2, 1]$

Figure 5.5: **A palette**

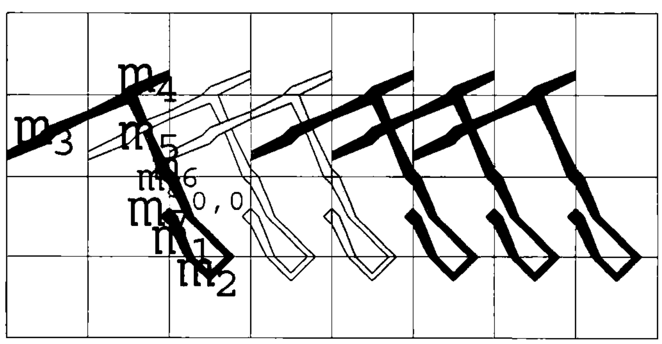

Figure 5.6: **Set of colored motif pieces that serve as generators for a colored tile: the palette and coloring function** C_Δ **are extracted from vectors** $[2, 1]$ **and** $[6, 0]$

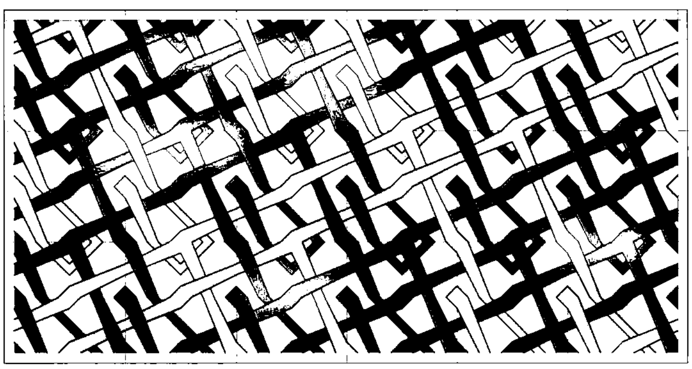

Figure 5.7: **A 6-colored Big Tile for the Escher tile in Figure 5.4**

Chapter 6

Efficient Big Tile Construction: The ColorFast Algorithm

The purpose of this Chapter is to describe a general polynomial-time algorithm, ColorFast, to construct a Big Tile for an Escher tile T whose period graph is connected. For the most part, ColorFast was used in the detailed example given in Section 4 and we will formalize that procedure with a few speedups along the way.

Abstractly, an Escher tile is specified by $T = \{(m_1, 0, 0), \ldots, (m_k, 0, 0)\}$, where $(m_i, 0, 0)$ is a connected subset of $S_{0,0}$ for $i = 1, \ldots, k$,, the pairwise intersection of the k motif pieces, and the set of intersections of each piece with $\partial S_{0,0}$ are given. The latter is assumed to be pre-sorted. For an implementation, one need only assume that each motif piece is a finite (connected) union of polygons. For the purpose of constructing a Big Tile for T, we only need to specify how many motif pieces there are, which pairs intersect, and the finite sorted set of boundary intersections for each. With that information, ColorFast will return the dimensions of a Big Tile, B_T, together with a finite set $COLORS$ of colors, and a function that takes a motif piece in B_T as input and returns an element of $COLORS$ to assign to that motif piece. In the end, it is up to the artist to design the Escher tile and construct the Big Tile, although the latter task could be delegated to, say, a tile company.

Most of the procedures that are called in the ColorFast algorithm will be familiar, but we give a brief description and the runtime for each. Let $n = max(k^2, I)$, where I is the total number of intersections of motif pieces of Escher tile T with $\partial S_{0,0}$.

- **An Escher tile** is input as $T = \{$*motifpieces,overlaps,boundaryintersections*$\}$, where *overlaps* has information containing the pairwise intersection of elements of $motifpieces$ and *boundaryintersections* contains for the bottom, top, right, and left sides, as well as NW, SE, NE, SW corners, a pre-sorted list of motif piece intersections along $\partial S_{0,0}$.

- **PeriodGraph** stores vertices (one for each motif piece in T) and directed edges. The directed edges can be obtained by comparing elements *boundaryintersections* in $O(n)$ time since *boundaryintersections* is presorted. Once this step has been accomplished we remove any multiple edges that contain the same vector label. Thus, the period graph will have $O(k^2)$ edges.

- **AgglomeratedPeriodGraph** can be obtained in $O(k^2)$ time by suppressing information (namely the vector labels and edge directions and multiplicities) contained in PeriodGraph.

- **AgglomeratedSpanningTree** can be obtained in $O(k^2)$ time by using a breadth first search tree [CLR89].

- **FatSpanningTree** takes as input a spanning tree of the agglomerated period graph $\hat{G}_T$ and reinstates the multiple edges, directions, and vector labels from the period graph G_T. This can be done in $O(k^2)$ time.

- **GeneratingMotifPieces** extracts, in $O(k)$ time, a location for each motif piece, from the fat spanning tree.

- **RemovedEdges** takes as input the agglomerated period graph $\hat{G}_T$ and an agglomerated spanning tree $\hat{S}_T$, and returns the edges that were removed from $\hat{G}_T$ when $\hat{S}_T$ was constructed. That is, RemovedEdges($\hat{G}_T, \hat{S}_T$) $= E(\hat{G}_T) \setminus E(\hat{S}_T)$. This can be done in $O(k^2)$ time.

- **FatRemovedEdges** is $E(G_T) \setminus E(S_T)$ and can be found in $O(k^2)$ time by reinstating information in RemovedEdges.

- **GhostMotifPieces** is a list of $O(k^2)$ edges and each endpoint produces a location relative to a related generating motif piece. This can be done in $O(k^2)$ time.

- **GhostVectors** finds all ghost vectors by comparing locations of each generating motif piece with the location of its potential ghost, by way of R_T and S_T, which takes time $O(k^2)$.

- **NaturalPeriods** returns the natural period(s) of T (if there are any). If T is singly periodic, the greatest common divisor of $O(k^2)$ scalars that are bounded by $O(k)$ must be computed (see Definition 3.3.2 and Lemma 3.2.7). If T is doubly periodic, find a basis of subspace of $\mathbb{Z}^2$ spanned by the ghost vectors of T. If T is trivially periodic, NaturalPeriods returns the empty set. In the worst case, this step takes time $O(k)^2$ [Co95].

- **Palette**. By Lemma 5.2.2, a set of distinct coset representatives for the lattice $< [p_1, p_2], [q_1, q_2] >$ is $reps := \{[i, j] : 0 \leq i < |\frac{\Delta}{\gcd(p_2, q_2)}|, 0 \leq j < |\gcd(p_2, q_2)|\}$, where $\Delta = |p_1 q_2 - p_2 q_1|$. The call to Palette($[p_1, p_2], [q_1, q_2]$) will return $reps$, which by Proposition 5.1.8 has size $O(k^2)$.

- **ColorSafeLattice** returns two vectors. If T is singly periodic with natural period $[p_1, p_2]$, ColorSafeLattice returns $\{[p_1, p_2], [2k+3, 0]\}$ if $p_2 \neq 0$ or returns

$\{[p_1, p_2], [0, 2k+3]\}$ if $p_2 = 0$. If T is trivially periodic, ColorSafeLattice returns $\{[2k+3, 0], [0, 2k+3]\}$. By Proposition 5.1.8, ColorSafeLattice returns a color-safe lattice. This step can be done in time $O(k^2)$.

- **BigTileInstructions** simply returns two positive integers m and n and a set of verbal instructions for the color to be assigned to each $(m_t, a, b) \in B_T$. Specifically, if $periods = \{[p_1, p_2], [q_1, q_2]\}$ then

  $$m = \frac{|p_1 q_2 - p_2 q_1|}{|\gcd(p_2, q_2)|}$$

  and

  $$n = \frac{|p_1 q_2 - p_2 q_1|}{|\gcd(p_1, q_1)|}.$$

  By Lemma 3.2.7, this step takes at most $O(\lg(k))$ steps since $|p_1|$, $|p_2|$, $|q_1|$, and $|q_2|$ are bounded by $O(k)$ [CLR89].

  The verbal instructions are as follows. Motif piece (m_t, a, b) is colored the same as $(m_1, a - i_t, b - j_t)$ (where $(m_t, i_t, j_t) \in gen(T, 0, 0)$). Therefore, the color assigned to $(m_1, a - i_t, b - j_t)$ is (i', j') where

  - $j' \equiv b - j_t \pmod{p_2}$ with, say, $b - j_t = qp_2 + j'$ and $0 \leq j' < p_2$, and
  - $i' = (a - i_t - qp_1) \pmod{m}$.

  In other words, given an arbitrary motif piece $(m_t, a, b) \in Wall(T)$, first we find a translate of $(m_1, 0, 0)$ that is related to (m_t, a, b) by way of (m_t, i_t, j_t) $\in gen(T, 0, 0)$. That is, $(m_1, a - i_t, b - j_t)$ is related to (m_t, a, b). Next we use the Euclidean Algorithm to find a quotient q and remainder j' such that $b - j_t = qp_2 + j'$, where by Definition 3.3.6, we have j' is the height of $W((m_t, a, b))$. Finally, it remains to specify the location of the copy of m_1 that belongs to $W((m_t, a, b))$ whose $y-$ coordinate is j'. This location is simply $((a - i_t - qp_1) \pmod m, b - j_t - qp_2)$. In summary, (m_t, a, b) is colored $((a - i_t - qp_1) \pmod m, b - j_t - qp_2)$.

We present the ColorFast algorithm next:

Algorithm ColorFast(T)

Initialization

- **input** $T = \{motifpieces, overlaps, boundaryintersections\}$

Procedure

1. $G_T \leftarrow$ PeriodGraph($motifpieces, boundaryintersections$)
2. $\hat{G}_T \leftarrow$ AgglomeratedPeriodGraph(G_T)
3. $\hat{S}_T \leftarrow$ SpanningTree($\hat{G}_T$)
4. $S_T \leftarrow$ FatSpanningTree($\hat{S}_T$)
5. $generators \leftarrow$ GeneratingMotifPieces(S_T)
6. $\hat{R}_T \leftarrow$ RemovedEdges(G_T, S_T)
7. $fatremovededges \leftarrow$ FatRemovedEdges($\hat{R}_T$)
8. $ghosts \leftarrow$ GhostMotifPieces($generators, fatremovededges$)
9. $ghostvectors \leftarrow$ GhostVectors($ghosts, generators$)
10. $periods \leftarrow$ NaturalPeriods($ghostvectors, generators$)
11. **if** $length(periods) = 2$

    (a) $colors \leftarrow$ Palette($periods$)

(b) **return** BigTileInstructions($colors$, $generators$)

12. **else**

    if $length(periods) = 1$

    (a) $colors \leftarrow$ Palette($periods \cup [2k+3, 0]$) or Palette($periods \cup [0, 2k+3]$)

    (b) **return** BigTileInstructions($colors$, $generators$)

13. **else**

    (a) $colors \leftarrow$ Palette($[2k+1, 0], [0, 2k+3]$)

    (b) **return** BigTileInstructions($colors$, $generators$)

14. **end**

In summary, the most costly part of ColorFast is the construction of the period graph which, in the worst case, takes $O(n)$ steps. All of the work following the construction takes at most $O(k^2)$ steps. Therefore, we have shown

Theorem 6.0.6 (ColorFast is a Polynomial-Time Algorithm): *Let* $T = \{(m_1, 0, 0), (m_2, 0, 0), \ldots, (m_k, 0, 0)\}$ *be an Escher tile whose period graph* G_T *is connected. Let* I *be the total number of intersections of motif pieces in* T *with* $\partial S_{0,0}$. *Let* $n = max(k^2, I)$, *where* k *is the number of motif pieces in* T. *The algorithm ColorFast takes time* $O(n)$.

Chapter 7

Toward a Classification

7.1 Irreducible and Reducible Big Tiles

To some extent, the Big Tiles constructed in Section 5.1 are canonical, by which we mean that given an arbitrary Δ-colored Big Tile for Escher tile T whose period graph is connected, we can always find a δ-colored Big Tile for T that exhibits the same structure as the Big Tiles constructed in Chapter 5.1 and with $\delta \leq \Delta$. This observation suggests the next definition, that of an irreducible Big Tile; we will adopt irreducible Big Tiles as our canonical representation.

Definition 7.1.1 (Irreducible and Reducible Big Tiles): *Let T be an Escher tile and B_T a Δ-colored $m \times n$ Big Tile for T and let $C_\Delta((m_s, i, j))$ denote the color assigned to motif piece (m_s, i, j). The Big Tile B_T is said to be* **irreducible** *if*

1. *for every $(m_s, i_1, j_1), (m_s, i_2, j_2) \in B_T$, whenever $C_\Delta((m_s, i_1, j_1)) = C_\Delta((m_s, i_2, j_2))$, $|i_1 - i_2| < m$ and $|j_1 - j_2| < n$ then $[i_1 - i_2, j_1 - j_2]$ is an integer linear combination of natural periods of T, and*

2. *$C_\Delta((m_s, i_1, j_1)) \neq C_\Delta((m_s, i_2, j_2)$ if either $i_1 = i_2$ or $j_1 = j_2$ for all $i_1, i_2 \in \mathbb{Z}_m$ and $j_1, j_2 \in \mathbb{Z}_n$ for all $s = 1, \ldots, k$.*

Otherwise, B_T is said to be **reducible**.

Thus, an irreducible Big Tile is one in which the same coloring of a motif piece does not occur in any row or any column. In particular, if T is trivially periodic, by Definition 7.1.1 every occurrence of motif piece m_s in an irreducible Big Tile for T is assigned a different color. Figures 2.12, 2.13, 4.11, and 4.12 are examples of irreducible Big Tiles. See Figure 7.2 for an example a reducible Big Tile for the Escher tile in Figure 7.1.

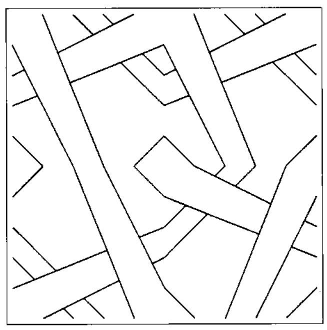

Figure 7.1: **An Escher tile**

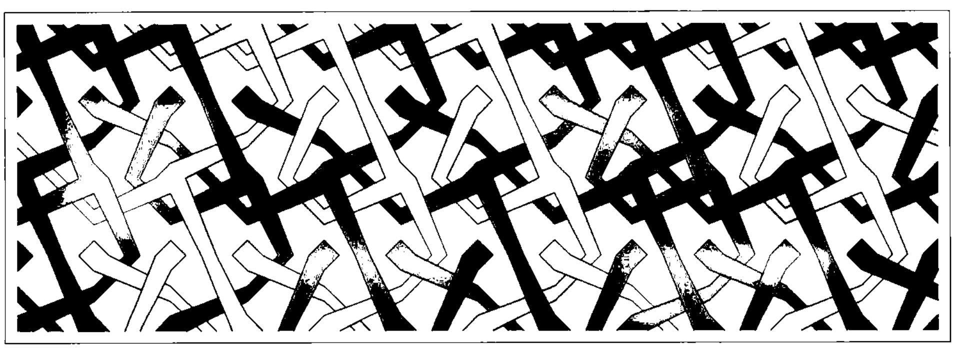

Figure 7.2: **A reducible Big Tile for the Escher tile in Figure 7.1**

Definition 7.1.2 (Lattice-Encoded Big Tile): *Let T be an Escher tile whose period graph is connected and $B_T = B_T(\Delta, m, n)$ a Δ-colored $m \times n$ Big Tile for T.*

The **lattice** $< [p_1, p_2], [q_1, q_2] > \in \mathbb{Z}^2$ **encodes** B_T *if* B_T *is equivalent to the colored tile generated by* $Palette([p_1, p_2], [q_1, q_2])$ *and* T.

Lemma 7.1.3 (Irreducible Big Tile is Encoded by a Lattice): *Let* $T = \{(m_1, 0, 0), \ldots, (m_k, 0, 0)\}$ *be an Escher tile whose period graph is connected and suppose and* $B_T := B_T(\Delta, m, n)$ *is an irreducible Big Tile for* T. *Then there exists a lattice that encodes* B_T.

Proof: Let a set of generating motif pieces for T be given by $gen(T, 0, 0) = \{(m_1, 0, 0), (m_2, i_2, j_2), \ldots, (m_k, i_k, j_k)\}$.

Case 1: Suppose T is trivially periodic. We claim that the lattice $L =< [m, 0], [0, n] >$ encodes B_T. By Definition 7.1.1, every translate of $(m_1, 0, 0)$ in B_T is colored with a different color so that a palette is given by the cosets of L. Since the period graph is connected, all motif pieces in B_T must be colored with some color from Palette($[m, 0], [0, n]$). Thus, $\Delta = mn$. Moreover, the locations of all motif pieces m_1 that are colored with $\overline{[0, 0]}$ must be $\{(xm, yn) : x, y \in \mathbb{Z}\}$. In general, because the period graph is connected, an arbitrary $(m_s, i, j) \in B_T$ is colored $\overline{[i - i_s, j - j_s]}$ (because (m_s, i, j) is related to $(m_1, i - i_s, j - j_s)$, which is colored $\overline{[i - i_s, j - j_s]}$. Trivially, the dimensions of B_T satisfy $m = \frac{mn}{|\gcd(0,n)|}$ and $n = \frac{mn}{|\gcd(m,0)|}$. Thus, B_T is the colored tile generated by Palette($[m, 0], [0, n]$) and T.

Case 2: Suppose T is singly periodic with natural period $[p_1, p_2]$ and for now assume $p_2 \neq 0$. We claim that B_T is the colored tile generated by $< [p_1, p_2], [m, 0] >$ and T.

First we show that $\Delta = |p_2 m|$: let $(m_1, i, j) \in Wall(T)$ be arbitrary and colored c. By Definitions 7.1.1 and 2.2.10, $\exists \bar{i}, \bar{j}$ such that $0 \leq \bar{i} < m$, $0 \leq \bar{j} < p_2$, and $(m_1, \bar{i}, \bar{j})$ is colored c. That is, without loss of generality we can assume j is the height of $W((m_1, i, j))$ and since B_T is a Big Tile of width m, we can use $j \pmod m$. Because $\bigcup_{M \in gen(T,0,0)} M$ is a connected subset of $\mathbb{R}^2$, the remaining motif pieces in $Wall(T)$ must be colored from the same set of colors. In that case, we see that $\Delta \leq mp_2$. Moreover, by Definition 7.1.1, if $0 \leq i_1, i_2 < m$ and $0 \leq j_1, j_2 < p_2$ and (m_1, i_1, j_1) and (m_1, i_2, j_2) are colored the same, then $[i_1 - i_2, j_1 - j_2] = [k_0p_1, k_0p_2]$ for some $k_0 \in \mathbb{Z}$. Since $0 \leq j_1, j_2 < p_2$ we have $k_0 = 0$, which means $i_1 = i_2$ and $j_1 = j_2$. In particular, $\Delta \geq p_2m$ and thus altogether $\Delta = |p_2m|$.

What can we say about the value of n? The previous argument shows that we can label the colors of B_T with the $\Delta = |mp_2|$ cosets of the lattice $< [m, 0], [p_1, p_2] >$. By Definition 7.1.1 for $0 < j < n$, we know that $(m_1, 0, j)$ cannot be colored $\overline{[0, 0]}$. Our argument so far has shown that the set of all locations of m_1 that are colored $\overline{[0, 0]}$ is given by $\{(xp_1 + ym, yp_2) : x, y \in \mathbb{Z}\}$. Thus, $|n|$ must be minimal (and nonzero) such that

$$\begin{bmatrix} 0 \\ n \end{bmatrix} = x \begin{bmatrix} m \\ 0 \end{bmatrix} + y \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

or equivalently,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{p_2m} \begin{bmatrix} p_2 & -p_1 \\ 0 & m \end{bmatrix} \begin{bmatrix} 0 \\ n \end{bmatrix}$$

$$\Rightarrow x = \frac{-p_1n}{|p_2m|} \quad \text{and} \quad y = \frac{mn}{|p_2m|} = \frac{n}{|p_2|}.$$

By the proof of Lemma 3.3.7, $n = \frac{|p_2m|}{|\gcd(p_1,m)|}$. Trivially, we have $m = \frac{|p_2m|}{|\gcd(p_2,0)|}$. We conclude that B_T, in this case, is the colored tile generated by Palette$([p_1, p_2], [m, 0])$, and hence by Definition 7.1.2 is encoded by the lattice $< [p_1, p_2], [m, 0] >$.

Case 3: Suppose T is doubly periodic with natural periods $[p_1, p_2]$ and $[q_1, q_2]$. We claim that B_T is encoded by the lattice $L =< [p_1, p_2], [q_1, q_2] >$. Since T is doubly periodic, by Definition 2.2.7, the wallpaper pattern $Wall(T)$ contains only finitely many wallpaper components, one for each of the $|p_1q_2 - p_2q_1|$ cosets of L. Let $(m_1, i, j) \in Wall(T)$ be arbitrary and suppose (m_1, i, j) is colored c. Then for any $[\tilde{i}, \tilde{j}] \in L$ we have $(m_1, \tilde{i}, \tilde{j})$ is colored c because $(m_1, \tilde{i}, \tilde{j})$ is related to (m_1, i, j). Moreover, if $W((m_1, 0, 0))$ and $W((m_1, a, b))$ are distinct (i.e., $[a, b] \notin L$), then $W((m_1, 0, 0))$ overlaps $W((m_1, a, b))$. Thus, all of the wallpaper components must be colored with $|p_1q_2 - p_2q_1|$ different colors and so we label each with a coset of L. So far, we have shown that $\Delta = |p_1q_2 - p_2q_1|$ and that the colors can be labelled with cosets. It remains to show that m and n satisfy the requirements of Definition 5.2.3. But by assumption, B_T is irreducible and hence m and n must be minimal such that

$$\begin{bmatrix} m \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + y_1 \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 \\ n \end{bmatrix} = x_2 \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + y_2 \begin{bmatrix} q_1 \\ q_2 \end{bmatrix},$$

for some $x_1, y_1, x_2, y_2 \in \mathbb{Z}$.The proof of Lemma 3.3.7 yields $m = \frac{\Delta}{|\gcd(p_2, q_2)|}$ and $n = \frac{\Delta}{|\gcd(p_1, q_1)|}$. Thus, B_T, in this case, is the colored tile generated by $Palette([p_1, p_2], [q_1, q_2])$ and T, and hence B_T is encoded by the lattice $< [p_1, p_2], [q_1, q_2] >$, as desired. ∎

In fact, the lattice that encodes an irreducible Big Tile is unique.

Corollary 7.1.4 (The Lattice that Encodes a Big Tile is Unique): *Let T be a doubly periodic Escher tile whose period graph is connected and suppose $L = < [p_1, p_2], [q_1, q_2] >$ encodes an irreducible Big Tile $B_T(\Delta, m, n)$. If $L' = < [r_1, r_2], [s_1, s_2] >$ is another lattice that encodes B_T. Then $L' = L$.*

Proof: By Definitions 7.1.2 and 5.2.3 and the proof of Lemma 7.1.3, we have $|r_1 s_2 - r_2 s_1| = \Delta$ and that $[m, 0]$, $[0, n] \in L$ and $[m, 0]$, $[0, n] \in L'$. Hence, $L = L'$, as desired. ∎

Finally, the following corollary is a direct consequence of Lemma 7.1.3.

Corollary 7.1.5 (Doubly Periodic Escher tile: Unique Big Tile): *Let T be an Escher tile whose period graph is connected. Then the Big Tile for T is unique and irreducible.*

7.2 Irreducible Big Tiles as Tools

7.2.1 In Search of the Chromatic Number of T

By Theorem 5.2.5, given any Escher tile T, there exists a Big Tile for T. Hence, it makes sense to define the chromatic number of T.

Definition 7.2.1 (Chromatic Number of an Escher tile): *Let T be an arbitrary Escher tile and $B = \{\Delta_1, \Delta_2, \ldots,\}$ be the set of all positive integers Δ for which there exists a Δ-colored Big Tile for T and without loss of generality assume Δ_i $< \Delta_{i+1}$ whenever $|B| > 1$ and Δ_i, $\Delta_{i+1} \in B$. By Theorem 5.2.5, B is nonempty. The* **chromatic number of** *T, denoted $\chi(T)$, is given by $\chi(T) = \Delta_1$. That is, the chromatic number of an Escher tile is the smallest (positive) integer Δ for which there exists a Δ-colored Big Tile for T.*

For example, three is the chromatic number of the Escher tile in Figure 2.10; an irreducible Big Tile for this coloring is in Figure 2.12. Irreducible Big Tiles were useful in the existence theorems in Chapter 5.1 (Theorems 5.2.4 and 5.2.5), and now will prove useful as a tool to investigate some special cases of Escher tiles for which we can actually compute $\chi(T)$. First we prove that when in search of

$\chi(T)$ we need look no further than the collection of irreducible Big Tiles for T. By Corollary 7.1.5, all Escher tiles for which the period graph is connected and doubly periodic have a unique Big Tile, which up to concatenation, is irreducible. In fact we can find the unique lattice that encodes the one irreducible Big Tile for T and thus, in this special case we can compute $\chi(T)$. As a next natural step in an analysis of the chromatic number, we analyze Escher tiles whose period graphs are connected and either trivially or singly periodic. The end result will be that a reducible $\Delta-$ colored Big Tile for T leads to an irreducible $\delta-$ colored Big Tile for T where $\delta \leq \Delta$. In other words, when searching for $\chi(T)$ for such an Escher tile, we need look no further than the collection of irreducible Big Tiles for T.

Lemma 7.2.2 (Reducible $B_T(\Delta, m, n) \rightarrow$ Irreducible $B_T(\delta \leq \Delta, m', n')$): *Let T be an Escher tile with connected period graph that is either trivially or singly periodic and suppose $B_T(\Delta, m, n)$ is a reducible Big Tile for T. Then there exists an irreducible Big Tile for T, $B_T(\delta, m', n')$ such that $\delta \leq \Delta$.*

Proof: **Case 1:** Let T be trivially periodic and suppose $0 \leq i_1, i_2 < m$, $0 \leq i_2, j_2 < n$, $(i_1, j_1) \neq (i_2, j_2)$, and (m_1, i_1, j_1) and (m_1, i_2, j_2) are colored with the same color (since $B_T(\Delta, m, n)$ is reducible, such a pair must exist). Suppose $i_1 - i_2 = p_1$ and $j_1 - j_2 = p_2$. Without loss of generality (by using an appropriate vector translation and by periodicity) we may assume $(m_1, 0, 0)$ and (m_1, p_1, p_2) with $0 \leq p_1 < m$ and $0 \leq p_2 < n$ are colored the same. Among all motif pieces colored the same as $(m_1, 0, 0)$ within $B_T(\Delta, m, n)$, choose p_1 to be minimal and positive (otherwise do the same for p_2). Then the set of lattice points within the parallelogram spanned by $[0, n]$ and $[p_1, p_2]$ contain copies of m_1 all colored with distinct colors by the minimality of p_1. Thus, $\Delta \geq p_1 n$. (If it so happens that $p_1 = 0$ simply re-run the argument with a minimal p_2 and use the parallelogram spanned by $[p_1, p_2]$ and $[m, 0]$ concluding that $\Delta \geq p_2 m$.) Now construct an irreducible Big

Tile for T using Palette($[p_1, p_2], [0, n]$). The resulting Big Tile requires $|p_1 n|$ colors and has area $\frac{|p_1 n|}{|\gcd(n,p_2)|} \frac{|p_1 n|}{|\gcd(p_1,0)|} = \frac{|p_1|n^2}{|\gcd(n,p_2)|}$.

Case 2: Suppose T is singly periodic with natural period $[p_1, p_2]$. If the copies of m_1 located at lattice points contained in the half-open parallelogram spanned by $[0, n]$ and $[p_1, p_2]$ (or use $[m, 0]$ if it so happens that $p_2 = 0$) are all colored with different colors then $\Delta \geq |mp_2|$ in which case simply construct the irreducible Big Tile $B_T(|mp_2|, \frac{|p_1 n|}{|\gcd(n,p_2)|}, \frac{|p_1 n|}{|\gcd(p_1,0)|})$, which has area $\frac{|p_1 n|}{|\gcd(n,p_2)|} \frac{|p_1 n|}{|\gcd(p_1,0)|} = \frac{|p_1|n^2}{|\gcd(n,p_2)|}$, just as in the previous case. Otherwise, without loss of generality (using a shift if necessary) there is some translate of m_1, say (m_1, q_1, q_2), contained in the half-open parallelogram spanned by $[p_1, p_2]$ and $[0, n]$ that is colored the same as $(m_1, 0, 0)$. Among all such (m_1, q_1, q_2) choose $|q_1|$ to be minimal. Then the translates of m_1 contained in parallelogram spanned by $[p_1, p_2]$ and $[q_1, q_2]$ are colored with different colors, which means that $\Delta \geq |p_1 q_2 - p_2 q_1|$. In that case simply construct an irreducible $|p_1 q_2 - p_2 q_1|$–colored Big Tile for T with $Palette([p_1,\ p_2],\ [q_1,\ q_2])$.
∎

The good news is that in any attempt to determine $\chi(T)$ it suffices to study only the irreducible Big Tiles for T. The bad news is that there is no guarantee that extracting an irreducible Big Tile from a reducible one will yield an irreducible Big Tile that is smaller in area than the reducible one.

Finally, by Lemma 5.1.8 and Lemma 7.2.2 we can give an upper bound for $\chi(T)$.

Theorem 7.2.3 (Upper Bound for $\chi(T)$): *Let $T = \{m_1, 0, 0), \ldots, (m_k, 0, 0)\}$ be an Escher tile whose period graph has N connected components. Then $\chi(T) \leq N(2k+3)^2$.*

Proof: Suppose the period graph of T has N connected components and let $T_1, \ldots,$ T_N be the Escher tiles induced by each component (Definition 3.1.3). By Proposition 5.1.8, for each T_i there is an irreducible Big Tile for T_i that requires at most $(2k+3)^2$ colors. Since there are N components, and by the construction given in Theorem 5.2.5, there is a Big Tile for T that requires at most $N(2k+3)^2$ colors. ∎

We can do somewhat better for an Escher tile whose period graph is connected and doubly periodic: we can compute the chromatic number exactly.

Theorem 7.2.4 (Chromatic Number of Doubly Periodic Escher tile): *Suppose T is an Escher tile whose period graph is connected, and let $[p_1, p_2]$ and $[q_1, q_2]$ be natural periods for T. Then $\chi(T) = |p_1q_2 - p_2q_1|$.*

Proof: Since T is doubly periodic, the Big Tile is unique and irreducible by Corollary 7.1.5. Thus, $\chi(T) = |p_1q_2 - p_2q_1|$. ∎

7.2.2 In Search of Inequivalent Δ-Colored Big Tiles

By Lemma 7.1.2 and Corollary 7.1.4, an irreducible Big Tile for Escher tile T corresponds to a unique lattice. Say this lattice can be generated by $\mathbf{u} = [u_1, u_2]$ and $\mathbf{v} = [v_1, v_2]$. Consider the **Gram Matrix** [Co95, Va91] of $\mathbf{u}$ and $\mathbf{v}$, namely $\begin{bmatrix} \mathbf{u}\cdot\mathbf{u} & \mathbf{u}\cdot\mathbf{v} \\ \mathbf{u}\cdot\mathbf{v} & \mathbf{v}\cdot\mathbf{v} \end{bmatrix}$. The **Binary Quadratic Form** (herein referred to as **BQF**) that we associate with B_T is $f(x,y) := [x,y] \begin{bmatrix} \mathbf{u}\cdot\mathbf{u} & \mathbf{u}\cdot\mathbf{v} \\ \mathbf{u}\cdot\mathbf{v} & \mathbf{v}\cdot\mathbf{v} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$. A straightforward calculation shows that

$$f(x,y) = (u_1^2 + u_2^2)x^2 + 2(u_1v_1 + u_2v_2)xy + (v_1^2 + v_2^2)y^2 \tag{7.1}$$

The **discriminant** of an arbitrary BQF $ax^2 + bxy + cy^2$ is given by $b^2 - 4ac$. If we let $\Delta = u_1v_2 - u_2v_1$, then the discriminant of f in equation 7.1 is $-4\Delta^2$.

Let B_T be an irreducible Big Tile for Escher tile T whose period graph is connected. We will make a connection between irreducible Δ-colored Big Tiles for T and BQFs with integer coefficients of discriminant $-4\Delta^2$. In particular, we wonder under what circumstances we can find two or more irreducible Δ-colored Big Tiles for T. That would entail finding two color-safe lattices $L = < [p_1, p_2],[q_1, q_2] >$ and $L' = < [p'_1, p'_2],[q'_1, q'_2] >$ for T such that $p_1q_2 - p_2q_1 = p'_1q'_2 - p'_2q'_1$ but for which $L \neq L'$. It is well understood in the realm of Analytic Number Theory (see for example [Ap76]) under what circumstances two lattices are equivalent, and we mention briefly how the mechanics work. To do so, we need a definition for the modular group, and an understanding of a particular group action of the modular group on the set of BQFs of a fixed discriminant.

Definition 7.2.5 (Modular Group): *The modular group* $\Gamma(1)$ *is*

$$\Gamma(1) := \left\{ \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} : \alpha, \beta, \gamma, \delta \in \mathbb{Z} \text{ and } \alpha\delta - \beta\gamma = 1 \right\}. \tag{7.2}$$

That is, the modular group is the set of all 2×2 matrices with integer entries and determinant 1. It is not hard to see that $\Gamma(1)$ is a group under matrix multiplication. Next we define an action of $\Gamma(1)$ on the set of BQFs with integer coefficients and of discriminant D.

In particular, let $f(x,y) = ax^2 + bxy + cy^2$, suppose $b^2 - 4ac = D$, and say that

$$M = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \Gamma(1).$$

Then the action of M on f is given by

$$f|M := f(\alpha x + \beta y, \gamma x + \delta y)$$

$$= a(\alpha x + \beta y)^2 + b(\alpha x + \beta y)(\beta x + \gamma y) + c(\gamma x + \delta y)^2$$

$$= (a\alpha^2 + b\alpha\gamma + c\gamma^2)x^2 + (2a\alpha\beta + b(\alpha\delta + \beta\gamma) + 2c\gamma\delta)xy + (a\beta^2 + b\beta\delta + c\delta^2)y^2.$$

We see that $f|M$ is a BQF with integer coefficients. Moreover, if we let $A = (a\alpha^2 + b\alpha\gamma + c\gamma^2)$, $B = (2a\alpha\beta + b(\alpha\delta + \beta\gamma) + 2c\gamma\delta)$, and $C = (a\beta^2 + b\beta\delta + c\delta^2)$, then the discriminant of $f|M$ is

$$B^2 - 4AC = (\alpha\delta - \beta\gamma)^2(b^2 - 4ac)$$

$$= D$$

since $\alpha\delta - \beta\gamma = 1$ and $b^2 - 4ac = D$. So, the described action of $\Gamma(1)$ on the set of BQFs of discriminant D is indeed a group action [Ga01]. What will be of use to us is that under the group action so described, the number of orbits is finite. Specifically [Bu89],

Theorem 7.2.6 (Finitely Many Orbits): *Let BQF_D be the set of all binary quadratic forms with integer coefficients of discriminant $D < 0$. Then the number of orbits of BQF_D under the action of $\Gamma(1)$ is finite.*

In particular,

Definition 7.2.7 (Class Number): *Let BQF_D be the set of all binary quadratic forms with integer coefficients of discriminant $D < 0$. The number of orbits of BQF_D under the action of $\Gamma(1)$ is called the* **class number of** *D, and denoted $h(D)$.*

In our situation, we make a connection between irreducible Δ-colored Big Tiles B_T with the set $BQF_{-4\Delta^2}$ as follows. We have

1. an Escher tile T whose period graph is connected,
2. an irreducible Δ-colored Big Tile B_T for T,

3. the unique lattice L generated by $[p_1, p_2]$ and $[q_1, q_2]$ that encodes B_T (satisfying $|p_1q_2 - p_2q_1| = \Delta$), and

4. a BQF of discriminant $-4\Delta^2$ to associate with B_T, namely $(p_1^2 + p_2^2)x^2 + 2(p_1q_1 + p_2q_2)xy + (q_1^2 + q_2^2)y^2$ by way of the Gram Matrix for $[p_1, p_2]$ and $[q_1, q_2]$.

Again, from [Ap76] we have

Theorem 7.2.8 (Equivalence of Lattices): *Two integer lattices $L = < [p_1, p_2], [q_1, q_2] >$ and $L' = < [p_1', p_2'], [q_1', q_2'] >$ are equivalent (meaning $L = L'$) if and only if there exists*

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \Gamma(1)$$

such that

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} p_1 & q_1 \\ p_2 & q_2 \end{bmatrix} = \begin{bmatrix} p_1' & q_1' \\ p_2' & q_2' \end{bmatrix}.$$

Inescapably, we are led to a method of differentiating between irreducible Δ-colored Big Tiles.

Definition 7.2.9 (Inequivalent Δ-Colored Big Tiles): *Let T be an Escher tile whose period graph is connected and suppose B_T and B'_T are both irreducible Δ-colored Big Tiles for T and encoded by lattices L and L' respectively. Then B_T* **is equivalent to** *B'_T if and only if L is equivalent to L'. Alternatively B_T* **is inequivalent to** *B'_T if and only if L is inequivalent to L'.*

The following theorem is a direct consequence of Theorem 7.2.8, the nature of the correspondence between a Δ-colored irreducible B_T and $BQF_{-4\Delta^2}$, and Definition 7.2.9.

Theorem 7.2.10 (Inequivalent Δ-Colored Big Tiles): *Let T be an Escher tile whose period graph is connected and suppose B_T and B_T' are both irreducible Δ-colored Big Tiles for T. If B_T is inequivalent to B_T', then $h(-4\Delta^2) > 1$.*

In fact, the two inequivalent and irreducible 8-colored Big Tiles in the example in Chapter 4 were constructed from the two inequivalent color-safe lattices

$$\begin{bmatrix} 2 & 3 \\ -2 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & -4 \\ -2 & 0 \end{bmatrix},$$

where $[2, -2]$ where the Escher tile T was singly periodic with natural period $[2, -2]$ and collision-free vectors $[-4, 0]$ and $[3, 1]$.

Faced with a particular D for which we want to compute $h(D)$, brute force methods are adequate [Bu89]. On the other hand, general results are much harder to come by. So, for our purposes, we simply use the class number as a tool that provides a necessary condition on Δ that tells us that there is a possibility of more than one irreducible Δ-colored Big Tile for T.

We end this chapter with one more theorem with thanks, again, to the class number.

Theorem 7.2.11 (Finitely Many Inequivalent Δ-Colored Irreducible Big Tiles): *Let T be an Escher tile whose period graph is connected, and let Δ be any positive integer. Then the number of inequivalent irreducible Δ-colored Big Tiles for T is finite and at most $h(-4\Delta^2)$.*

Chapter 8

Future Work

8.1 Generalizations

The following is a sample of ideas for generalizing the methods in this thesis.

1. Ask and answer the Escher Big Tile question for regular-hexagon tilings in the Euclidean plane.

2. Ask and answer the Escher Big Tile question for equilateral-triangle tilings in the Euclidean plane.

3. Ask and answer the Escher Big Tile question for regular tilings of the hyperbolic plane.

4. Replace the translation group $\mathbb{Z} \times \mathbb{Z}$ with another group (finitely generated and abelian? other wallpaper groups?) What can be asked and what can be answered?

5. The methods of this thesis should carry through to an analogue of the Big Tile existence question for 3-dimensional cubical tilings of $\mathbb{R}^3$.

8.2 Machinery Applied to Graph-Coloring and VLSI Design

The machinery that we built for constructing Big Tiles may shed light on an open problem that lives among the areas computational geometry, VLSI design, and graph coloring.

Problem: A **rectangle visibility graph** G (RVG) is a simple graph whose vertices are represented by nonoverlapping rectangles in the plane and whose sides are parallel to the $x-$ and $y-$axes; two vertices are adjacent if and only if they have either a horizontal or a vertical visibility [HSV99]. Rectangle visibility graphs are thickness-two graphs: they can be decomposed into (at most) two planar graphs G_1 and G_2, such that $V(G) = V(G_1)$, $V(G) = V(G_2)$, $E(G) = E(G_1) \bigcup E(G_2)$, and $E(G_1) \bigcap E(G_2) = \varnothing$. In general, a thickness-k graph is a simple graph that can be decomposed into k planar layers. Finding the thickness of an arbitrary graph is of interest to VLSI designers since thickness may be viewed as a measure of the minimum number of layers required to design a computer chip whose network is given by G and such that no two wires cross. The maximum value of the chromatic number of an arbitrary thickness-two graph is known to lie between 9 and 12 [Hu93, HR90, Ga80], and has an application to the testing of printed circuit boards [GJS76]. A complete graph on eight vertices, K_8, has a representation as an RVG. No RVG that requires more than eight colors is known [HSV99].

Conjecture: *Let G be an arbitrary RVG. Then* $\chi(G) \leq 8$.

It is not hard to show that the vertices of any RVG can be represented by rectangles whose corners have integer coordinates. As such, perhaps a modification of the period graph will be a tool that can be used to prove the conjecture. The edges will be weighted with vectors that have integer entries and contain information about type of visibility (horizontal or vertical, to the left of or to the right of etc), and the shortest distance between two vertices: the present best guess is that the vectors will live in $\mathbb{Z}^4$.

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Appendix A

Art Gallery

Over time we have learned to expect the unexpected. Figure A.1 shows examples of two Big Tiles (the Escher tile can be found in any subsquare) for which an increase in area does not correspond to an increase in the number of colors needed. The Big Tile on the left has area 18 and requires six colors, whereas the Big Tile on the right has area 16 and requires eight colors.

Figure A.2 shows an example of an Escher tile that at first glance seems to be composed of many unrelated motif pieces. Upon analyzing the overlap graph, we find that nearly all of the motif pieces are related, and that a Big Tile for T is 1×1 and requires only one color. A fragment of the Escher wallpaper is shown in Figure A.3.

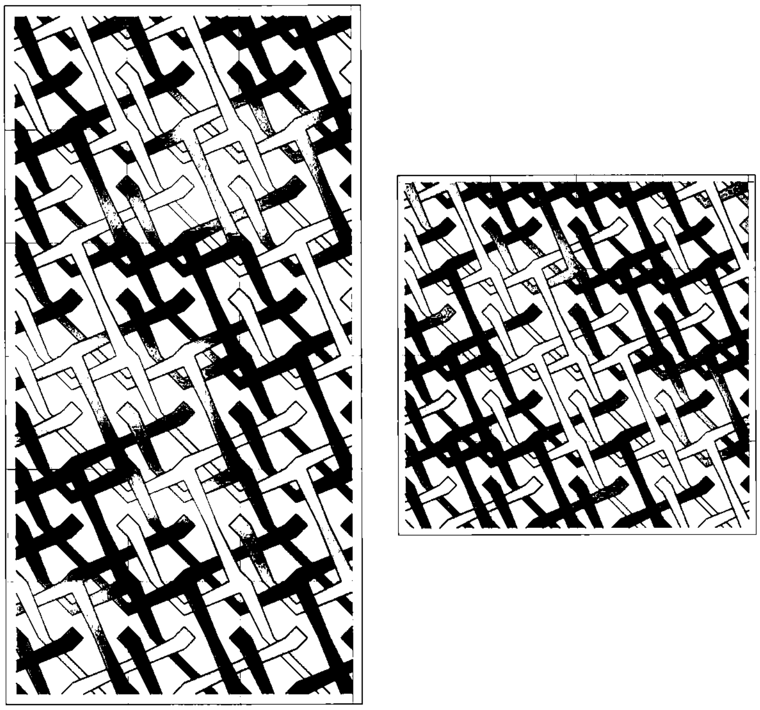

Figure A.1: **Minimal area need not correspond to minimal coloring**

Figure A.2: **A deceptive Escher tile**

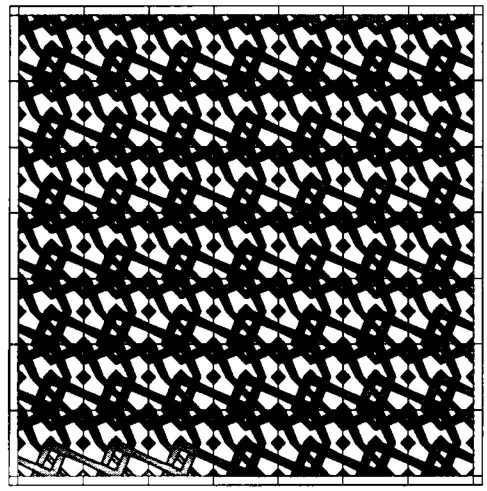

Figure A.3: **The deceptive tile unravelled**

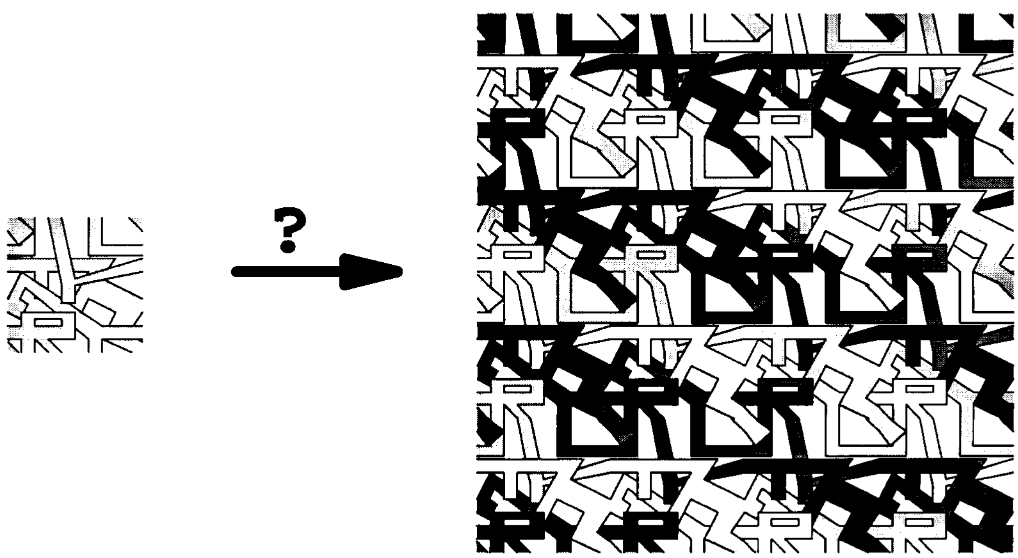

Figure A.4: **A hidden message**

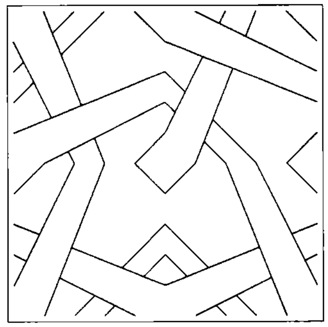

Figure A.5: **Escher tile whose period graph is disconnected**

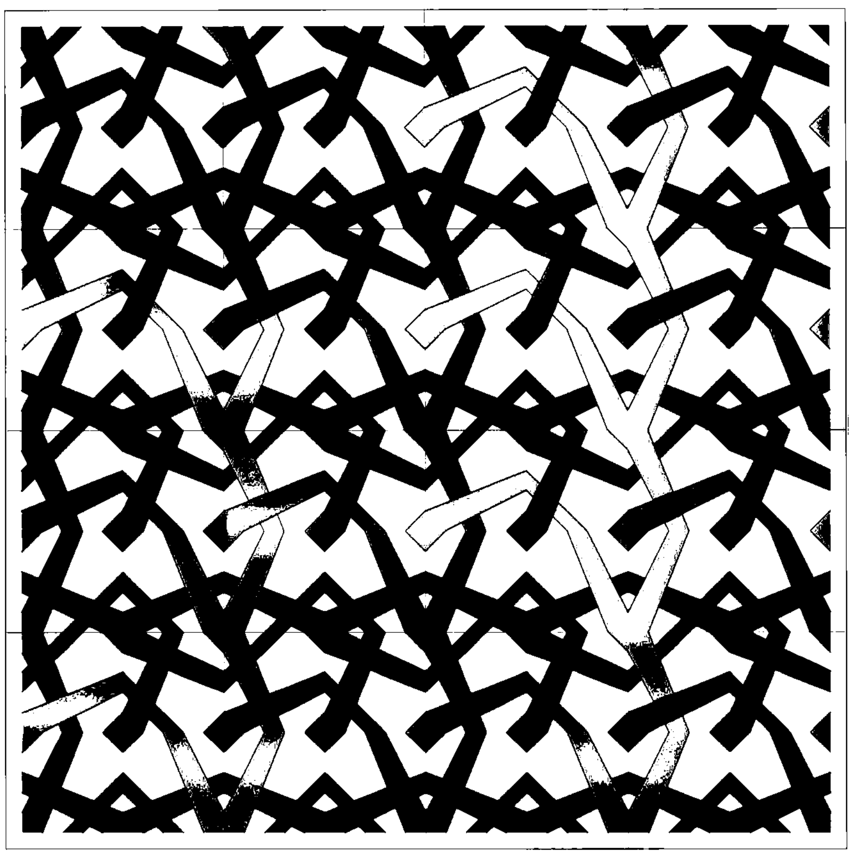

Figure A.6: **Fragment of wallpaper pattern for the Escher tile in Figure A.5**

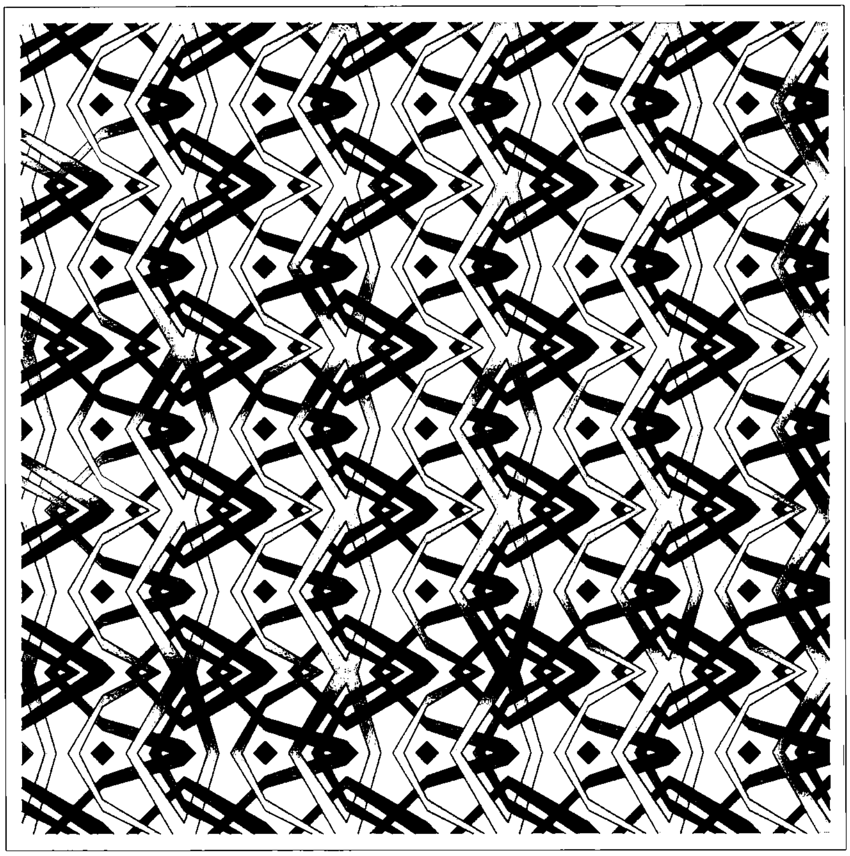

Figure A.7: **Another of Escher's designs**

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