S-R-T Division Algorithms As Dynamical Systems

by

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Abstract

S-R-T division, as it was discovered in the late 1950s [4, 19, 23], represented an important improvement in the speed of division algorithms for computers at the time. A variant of S-R-T division is still commonly implemented in computers today. Although some bounds on the performance of the original S-R-T division method were obtained, a great many questions remained unanswered. In this thesis, S-R-T division is described as a dynamical system. This enables us to bring modern dynamical systems theory, a relatively new development in mathematics, to bear on an older problem. In doing so, we are able to show that S-R-T division is ergodic, and is even Bernoulli, for all real divisors and dividends.

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Chapter 1

Introduction and Background

1.1 Introduction to S-R-T Division

"S-R-T division" roughly refers to a class of non-restoring, binary division algorithms that have been designed for floating-point computers [3, 5, 6, 7, 14, 22]. The term "non-restoring" refers to the fact that partial remainders are allowed to range freely through the interval $(-1, 1)$, rather than being restored to the positive realm before proceeding to the next step. This feature reduces uses of the adder by about fifty percent. An equally important feature of this algorithm is the "S-R-T" optimization from whence the algorithm gets its name. In the late 1950's, Sweeney [4], Robertson [19], and Tocher [23] independently made the observation that whenever a partial remainder is in the range $(-\frac{1}{2}, \frac{1}{2})$, there will be one or more leading zeros that can be shifted through in a very short amount of time (usually one cycle). The more leading zeros in a given step, the more the algorithm can avoid costly uses of the adder. A further development of this original algorithm, which is still called S-R-T division, is the algorithm most often implemented in modern floating-point units. In modern S-R-T division, a fixed number of quotient digits are produced every cycle as opposed to a variable number [5, pp. 37–62].

An example of modern S-R-T division in use is Intel's first release of the Pentium™ CPU with its infamous "Pentium Bug," which was really just a small error in its S-R-T division implementation. This thesis will restrict its attention to the original version of S-R-T division.

To present the simplest type of S-R-T division, we begin with a few definitions for an algorithm similar to that presented by Shively [22, pp. 3–4]:

(a) n represents the number of iterations performed in the algorithm.

(b) p_0 is the dividend (or initial partial remainder) normalized so that $p_0 \in [\frac{1}{2}, 1)$.

(c) $p_i \in (-1, 1)$, $i \in \mathbb{N}$, is the partial remainder after the ith step.

(d) D is the divisor normalized to $[\frac{1}{2}, 1)$.

(e) $q_i \in \{-1, 0, 1\}$ $(i \in \{0, \ldots, n-1\})$ is the quotient digit generated by the ith step.

(f) $Q_n = \sum_{i=0}^{n-1} \frac{q_i}{2^i}$ is the "rounded off" quotient generated after n steps of the algorithm.

Given the above definitions, after n steps of the division algorithm, we would like it to be true that

$$p_0 = DQ_n + \varepsilon(n)$$

where $\varepsilon(n)$ is a term that goes to zero as n goes to infinity.

A recurrence relation for the S-R-T division algorithm can be stated as

$$p_{i+1} = \begin{cases} 2p_i & : \quad |p_i| < \frac{1}{2} \\ 2(p_i - D) & : \quad |p_i| \geq \frac{1}{2} \text{ and } p_i \geq 0 \\ 2(p_i + D) & : \quad |p_i| \geq \frac{1}{2} \text{ and } p_i < 0\,, \end{cases}$$

and

$$q_i = \begin{cases} 0 & : \quad |p_i| < \frac{1}{2} \\ 1 & : \quad |p_i| \geq \frac{1}{2} \text{ and } p_i \geq 0 \\ -1 & : \quad |p_i| \geq \frac{1}{2} \text{ and } p_i < 0 \,. \end{cases}$$

By observing that

$$p_{i+1} = \begin{cases} 2(p_i - (0)D) & : \quad |p_i| < \frac{1}{2} \\ 2(p_i - (1)D) & : \quad |p_i| \geq \frac{1}{2} \text{ and } p_i \geq 0 \\ 2(p_i - (-1)D) & : \quad |p_i| \geq \frac{1}{2} \text{ and } p_i < 0 \,, \end{cases}$$

we can rewrite the definition of p_{i+1} as

$$p_{i+1} = 2(p_i - q_i D) \,.$$

After n steps have been completed, we have

$$p_n = 2^n p_0 - 2^n q_0 D - 2^{n-1} q_1 D - \cdots - 2^1 q_{n-1} D \,,$$

and then after dividing by 2^n and solving for p_0 we find that

$$\begin{aligned} p_0 &= \frac{p_n}{2_n} + \frac{q_0 D}{2^0} + \frac{q_1 D}{2^1} + \cdots + \frac{q_{n-1} D}{2^{n-1}} \\ &= D \sum_{i=0}^{n-1} \frac{q_i}{2^i} + \frac{p_n}{2_n} = D Q_n + \frac{p_n}{2_n} \,. \end{aligned}$$

Now let $\varepsilon(n) = p^n / 2^n$ and let $Q^* = \lim_{n \to \infty} Q_n$. Since $|p_n| < 1$, in the limit as n goes to infinity

$$p_0 = D Q^* \,.$$

The quotient bits being generated are not in a standard binary representation, but it is a simple matter to convert the answer back to standard binary without using any expensive operations. Figure 1.1 shows a simple pseudo state-machine (really a push-down automaton) that converts positive floating-point numbers in the $\{-1, 0, 1\}$ representation into binary.

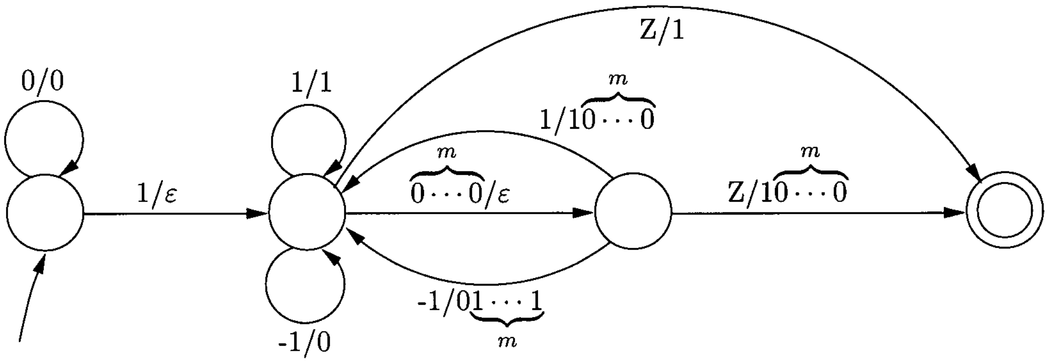

Figure 1.1: A pseudo state-machine for converting sequences of $\{-1, 0, 1\}$ into sequences of $\{0, 1\}$ (binary). We assume that the input sequence corresponds to a positive number. The letter 'Z' is used to indicate that the end of the sequence has been reached, and the symbol ε represents the null string. We represent a run of m zeros as $0\cdots0$ and a run of m ones as $1\cdots1$. Sequences of symbols should be read from left to right. For example, the expression $1/1\overbrace{0\cdots0}^{m}$ means: if a 1 is encountered in the input sequence, write a 1 followed by m zeros.

The above conversion automaton implies that conversion happens after the calculation is completed. In reality, the conversion from the generated quotient bits to standard binary is done in hardware on-the-fly, using registers to convert runs of zeros into runs of zeros or ones in parallel, or by performing a single subtraction.

Figure 1.2 shows an example of using the S-R-T division algorithm to divide 0.67 by 0.75. The steps that produce non-zero quotient bits have been shown. In this example, after six uses of the adder, the quotient $(0.89\overline{3})$ has been determined to four digits of precision.

$p_0 = 0.67 \quad = \quad 0.67$		
$p_1 = 2(0.67 - D) \quad = -0.16$	$q_0 = 1$	$Q_0 = 1$
$p_4 = 2(2^2(-0.16) + D) = \quad 0.22$	$q_3 = -1$	$Q_3 = 0.875$
$p_7 = 2(2^2(0.22) - D) \quad = \quad 0.26$	$q_6 = 1$	$Q_6 = 0.890625$
$p_9 = 2(2^1(0.26) - D) \quad = -0.46$	$q_8 = 1$	$Q_8 = 0.89453125$
$p_{11} = 2(2^1(-0.46) + D) = -0.34$	$q_{10} = -1$	$Q_{10} = 0.8935546875$
$p_{13} = 2(2^1(-0.34) + D) = \quad 0.14$	$q_{12} = -1$	$Q_{12} \doteq 0.8933105469$

Figure 1.2: An example of S-R-T division when the dividend $p_0 = 0.67$, and the divisor $D = 0.75$. The quotient Q^* is $0.89\overline{3}$.

Now, with this simple system of division in hand, we might want to ask certain questions about its performance. For example, we could ask "How many bits of precision are generated per iteration of the algorithm on average?" To answer this question, we must look at the magnitude of $|Q^* - Q_n| = |p_n/2^n|$. The number of bits of precision on the nth step is then $n - \log_2 p_n$. In the worst case, p_n is close to 1, and therefore we get at least one bit of precision per iteration of the algorithm, regardless of the values of D or p_0. Of course, a designer of actual floating-point hardware probably wants to know the expected performance based on the expected values of p_n. To answer the many variants of this type of question, it is clear that we must know something about the distribution of partial remainders over time. The remainder of this thesis is devoted to extending what is known about the answer to this type of question as it relates to S-R-T division and its variants.

1.2 S-R-T Division as a Dynamical System

The example in figure 1.2 makes it clear that keeping track of the signs of successive partial remainders is irrelevant in determining how many times the adder will be used for a particular calculation. For this reason, we only need to consider the magnitudes of successive partial remainders. We now give a reformulation of S-R-T division that will allow us to look at division as a dynamical system.

Definition 1 (S-R-T Division Transformation). For $D \in [\frac{1}{2}, 1)$, we define the function $T_D : [0,1) \to [0,1)$ as

$$T_D(x) = \begin{cases} 2x & : \quad 0 \leq x < \frac{1}{2} \\ 2(D - x) & : \quad \frac{1}{2} \leq x < D \\ 2(x - D) & : \quad D \leq x < 1 \;. \end{cases}$$

This transformation of the unit interval represents the successive partial remainders that arise as S-R-T division is carried out by a divisor D on a dividend x. D is normalized to $[\frac{1}{2}, 1)$. The dividend x is normalized to $[\frac{1}{2}, 1)$ initially, while each of the successive partial remainders $T_D^n(x)$ $(n \in \mathbb{N})$ subsequently ranges through $[0, 1)$.

By using the characteristic function for a set Δ defined as

$$1_\Delta(x) = \begin{cases} 1 & : \quad x \in \Delta \\ 0 & : \quad x \notin \Delta \;, \end{cases}$$

we can rewrite T_D as

$$T_D(x) = 2x \cdot 1_{[0,\frac{1}{2})}(x) \; + \; 2(D-x) \cdot 1_{[\frac{1}{2},D)}(x) \; + \; 2(x-D) \cdot 1_{[D,1)}(x) \;. \tag{1.1}$$

If we plot equation 1.1 on the unit interval, we obtain a very useful visualization of our transformation. Figure 1.3 shows the plot of $T_{0.75}(x)$ combined with a plot of the successive partial remainders that arise while dividing 0.67 by 0.75. This is the same system that was presented earlier in figure 1.2. Notice that a vertical line in the interval $[\frac{1}{2}, D)$ corresponds to a subsequent flip in the sign of the next partial remainder.

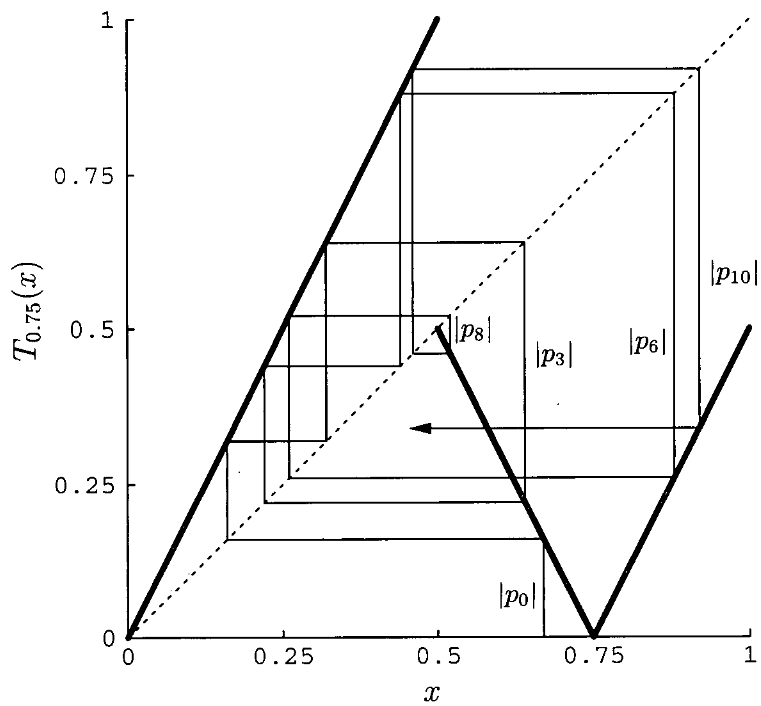

Figure 1.3: An example of following partial remainder magnitudes graphically for $D = 0.75$ and $p_0 = 0.67$. The heavy solid lines represent the transformation $T_{0.75}$, while the abscissa of the thin vertical lines represent successive partial remainder magnitudes.

Figure 1.3 shows an example of following the trajectory of a single partial remainder for a particular divisor. After ten applications of the $T_{0.75}$, there is not any obvious regular pattern, although we expect to see one eventually since the quotient is rational in this case. Of course, most numbers are not rational and we can deduce that for most numbers, the transformation will never exhibit a repeating pattern. In figures 1.4 and 1.5, we see that a very small change in the value of the initial partial remainder quickly produces large differences in the observed behaviour of the subsequent partial remainders. Our system appears to be chaotic (it certainly has sensitive dependence on initial conditions and is topologically transitive), and, if this the case, we will gain little understanding by studying the trajectories of individual partial remainders. The logical next step is to study the behaviour of distributions of points over the whole interval.

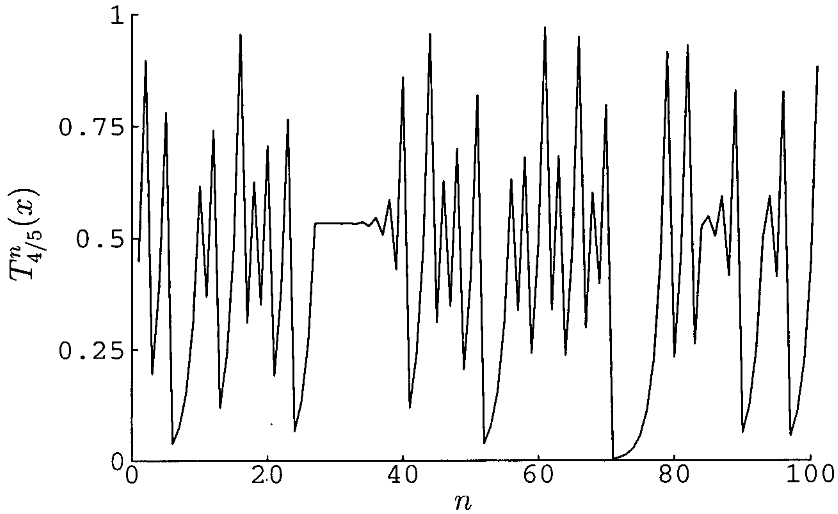

Figure 1.4: The result of applying $T_{4/5}$ to $x = \frac{\pi}{7}$ one hundred times.

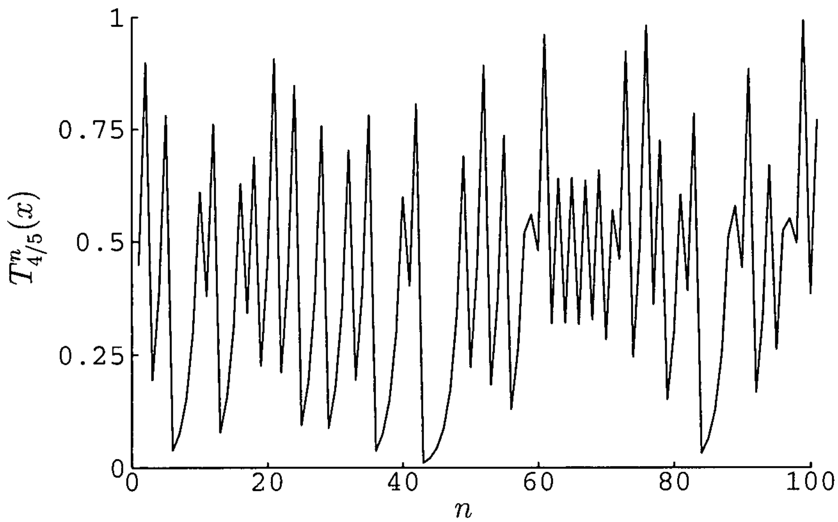

Figure 1.5: The result of applying $T_{4/5}$ to $x = \frac{\pi}{7} + 0.00001$ one hundred times.

The area of understanding the behaviour of ensembles of points under repeated transformation is the realm of dynamical systems theory. For the remainder of this thesis, we assume a certain amount of familiarity with the fundamentals of dynamical systems theory (or ergodic theory), which requires some basic understanding of measure theory. We will include a few helpful background material definitions as they are needed, but mostly we will provide references. A very good introduction to the study of chaotic systems is Lasota and Mackey's book *Chaos, Fractals, and Noise* [11]. For a more detailed introduction to ergodic theory (along with the necessary measure theory needed to understand this thesis), Peter Walters's book *An Introduction to Ergodic Theory* [24] and Karl Petersen's book *Ergodic Theory* [18] are highly recommended.

Definition 2 (Probability Space). If $\mathcal{B}$ is a σ-algebra on subsets of a set X and if m is a measure on $\mathcal{B}$ where $m(X) = 1$, then the triple $(X, \mathcal{B}, m)$ is called a *probability space.* (See [24, pp. 3–9] and [11, pp. 19–31] for a good overview of basic measure theory and Lebesgue integration.)

Definition 3 (Stationary Distribution). Let $(X, \mathcal{B}, m)$ be a probability space, let P be the Perron-Frobenius operator associated with a non-singular transformation $T : X \to X$, and let L^1 denote the L^1 space of $(X, \mathcal{B}, m)^\dagger$. If $f \in L^1$ is such that $Pf \overset{\circ}{=} f$,[‡] then f is called a *stationary distribution* of T.

Definition 4 (Perron-Frobenius operator). For a probability space $(X, \mathcal{B}, m)$, the *Perron-Frobenius* operator associated with a non-singular transformation $T : X \to X$ is defined by

$$\int_B Pf(x)\,\mathrm{d}m = \int_{T^{-1}(B)} f(x)\,\mathrm{d}m, \quad \text{for } B \in \mathcal{B}\ .$$

For a piecewise $C^{2\S}$ transformation T with n pieces, we can give an explicit formula for the Perron-Frobenius operator. Let $A = \{A_1, A_2, \ldots, A_n\}$ be the partition of X which separates T into n pieces. For $i \in \{1, \ldots, n\}$, let $t_i(x)$ represent the

[†]For a probability space $(X, \mathcal{B}, m)$, the L^1 space of $(X, \mathcal{B}, m)$ is the set of $f : X \to \mathbb{R}$ satisfying $\int_X |f(x)|\,\mathrm{d}m < \infty$.

[‡]The $\circ$ symbol will be used to indicate that a given relation holds except possibly on a set of measure zero.

[§]C^2 denotes the set of all functions with two continuous derivatives.

natural extension of the ith C^2 function $T(x)|_{A_i}$. The Perron-Frobenius operator for T is then

$$Pf(x) = \sum_{i=1}^{n} \left| \frac{\mathrm{d}}{\mathrm{d}x} t_i^{-1}(x) \right| f(t_i^{-1}(x)) \cdot 1_{t_i(A_i)}(x) \; .$$

In particular, for T_D (as in equation 1.1),

$$\begin{aligned} Pf(x) &= \tfrac{1}{2} f(\tfrac{1}{2}x) \cdot 1_{[0,1)}(x) \\ &\quad + \tfrac{1}{2} f(D - \tfrac{1}{2}x) \cdot 1_{(0,2D-1]}(x) \; + \; \tfrac{1}{2} f(D + \tfrac{1}{2}x) \cdot 1_{[0,2-2D)}(x) \; . \end{aligned} \tag{1.2}$$

With equation 1.2 we can show precisely what happens to an initial distribution of points (described by an integrable function) after they are repeatedly transformed under T_D. Figures 1.6 and 1.7 show what happens to two different initial distribution of points after five applications of the Perron-Frobenius operator associated with $T_{3/5}(x)$. By the fifth application, the distributions look remarkably similar. One might guess that they are both approaching the same final distribution. This situation is in marked contrast to chaotic behaviour observed in figures 1.4 and 1.5.

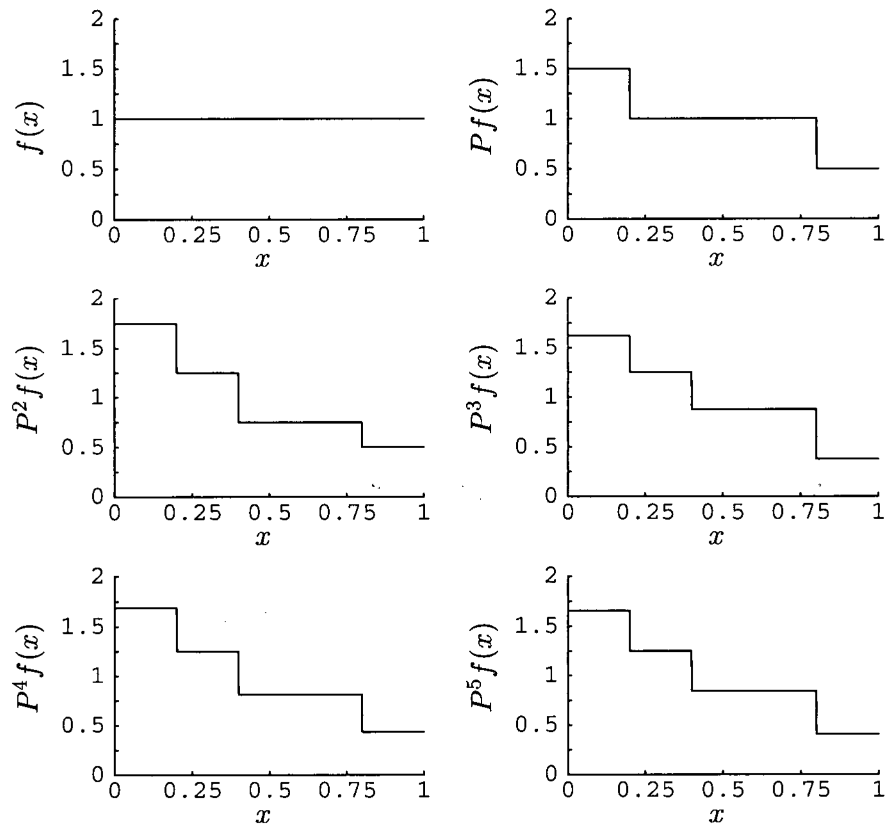

Figure 1.6: The result of applying the Perron-Frobenius operator P associated with $T_{3/5}$ to $f(x) = 1$ six times.

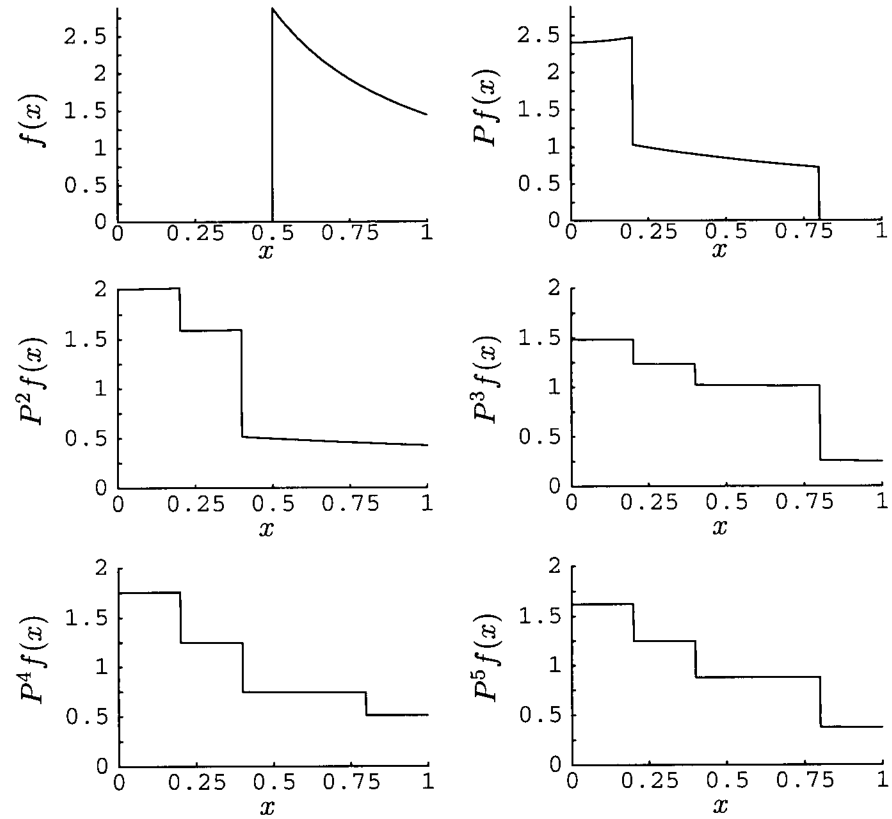

Figure 1.7: The result of applying the Perron-Frobenius operator P associated with $T_{3/5}$ to $f(x) = \dfrac{1}{\log 2}\displaystyle\int_{1/2}^{1} \dfrac{\mathrm{d}x}{x}$ six times.

1.3 Shift Average for $D \in [\frac{3}{4}, 1)$

An exact equation for the stationary distribution when $D \in [\frac{3}{4}, 1)$ was first given by Freiman [6] and is restated by Shively [22] as

$$f(x) = \frac{1}{D}1_{[0,2D-1)}(x) + \frac{1}{2D}1_{[2D-1,1)}(x)\,. \tag{1.3}$$

To verify that this is a stationary distribution function, we begin by applying the Perron-Frobenius operator as given in equation 1.2 to equation 1.3 and verifying that $Pf(x) \triangleq f(x)$. So then, applying P to f we get

$$\begin{aligned} Pf(x) &= \frac{1}{2}\left(\frac{1}{D}1_{[0,2D-1)}(\tfrac{1}{2}x) + \frac{1}{2D}1_{[2D-1,1)}(\tfrac{1}{2}x)\right)1_{[0,1)}(x) \\ &\quad + \frac{1}{2}\left(\frac{1}{D}1_{[0,2D-1)}(D-\tfrac{1}{2}x) + \frac{1}{2D}1_{[2D-1,1)}(D-\tfrac{1}{2}x)\right)1_{(0,2D-1]}(x) \\ &\quad + \frac{1}{2}\left(\frac{1}{D}1_{[0,2D-1)}(D+\tfrac{1}{2}x) + \frac{1}{2D}1_{[2D-1,1)}(D+\tfrac{1}{2}x)\right)1_{[0,2-2D)}(x)\,. \end{aligned}$$

Assuming that $D \in [\frac{1}{2}, 1)$, and observing that $x \in [0,1)$,

$$\begin{aligned} Pf(x) &= \frac{1}{2}\left(\frac{1}{D}1_{[0,4D-2)}(x) + \frac{1}{2D}1_{[4D-2,1)}(x)\right)1_{[0,1)}(x) \\ &\quad + \frac{1}{2}\left(\frac{1}{D}1_{(2-2D,1)}(x) + \frac{1}{2D}1_{(0,2-2D]}(x)\right)1_{(0,2D-1]}(x) \\ &\quad + \frac{1}{2}\left(\frac{1}{2D}1_{[0,2D-1)}(x)\right)1_{[0,2-2D)}(x)\,. \end{aligned}$$

Finally, assuming that $D \in [\frac{3}{4}, 1)$, we have

$$\begin{aligned} Pf(x) &= \frac{1}{2D}1_{[0,1)}(x) + \frac{1}{2D}1_{(2-2D,2D-1]}(x) + \frac{1}{4D}1_{(0,2-2D]}(x) + \frac{1}{4D}1_{[0,2-2D)}(x) \\ &= \frac{3}{4D}1_{[0,2-2D)}(x) + \frac{1}{4D}1_{(0,2-2D]}(x) \\ &\quad + \frac{1}{2D}1_{[2-2D,2D-1)}(x) + \frac{1}{2D}1_{(2-2D,2D-1]}(x) \\ &\quad + \frac{1}{2D}1_{[2D-1,1)}(x) \\ &\triangleq \frac{1}{D}1_{[0,2D-1)}(x) + \frac{1}{2D}1_{[2D-1,1)}(x) = f(x)\,. \end{aligned}$$

One of the primary uses of having a formula for the distribution of partial remainders is for calculating the *shift average* for a given divisor. The shift average is the average number uses of the shift register (single shift or multiplication by two) between uses of the adder. Under the assumption that a register shift is a much faster operation than using the adder, the shift average gives a useful characterization of the expected performance of our algorithm for a given divisor. With equation 1.3, we know the fraction of bits that require the use of the adder. To calculate the average number of zero bits generated between non-zero bits (bits requiring use of the adder), we take the reciprocal of the fraction of bits that require the adder. We calculate the shift average for a divisor $D \in [\frac{3}{4}, 1)$ to be

$$s(D) = \frac{1}{1 - \frac{1}{2D}} = \frac{2D}{2D - 1}\,. \tag{1.4}$$

Since have not proven that the stationary distributions from S-R-T division are unique, we have no way of knowing whether or not a shift average calculation in equation 1.4 is correct. To prove that all stationary distributions are unique, we need to show that T_D is ergodic for all $D \in [\frac{1}{2}, 1)$. Freiman [6] shows that T_D is ergodic for rational D, but we extend this result for real D. In the next section we show that all T_D are Bernoulli and it is known that having the Bernoulli property implies ergodicity.

Before concluding this chapter with a definition for ergodicity, we will briefly comment on the derivation of stationary distributions for $D \in [\frac{1}{2}, \frac{3}{4})$. For $D \in [\frac{3}{5}, \frac{3}{4})$, the stationary distribution functions have been derived, and their associated shift average functions have been shown to be constantly three [6, 22]. The layout of stationary distribution functions in the region $D \in [\frac{1}{2}, \frac{3}{5})$ has several surprising properties and is far from being fully understood. We discuss the calculation of shift averages as an interesting area for future investigation in Chapter 4.

Definition 5 (Ergodic [11]). Let $(X, \mathcal{B}, m)$ be a probability space and let a non-singular transformation $T : X \to X$ be given. Then T is *ergodic* if for every set $B \in \mathcal{B}$ such that $T^{-1}(B) = B$, either $m(B) = 0$ or $m(X \setminus B) = 0$.

Chapter 2

Bernoulli Property

In this chapter, we will prove that the class of transformations of the interval that characterizes the S-R-T division for all real divisors D has the property that each transformation T_D is Bernoulli. Although the basic concept of a Bernoulli shift (the things to which transformations having a Bernoulli property are isomorphic to) is not difficult, a complete definition requires enough auxiliary concepts from measure theory (concepts not used anywhere else in this thesis) that we chose to refer the interested reader to [17, 18, 21, 24] and other selections listed in the Bibliography. Neither an understanding of Bernoulli shifts, nor a formal definition of what it means to be Bernoulli is required to follow the proofs in this chapter. Having said this, we should mention informally the connection between Bernoulli shifts and transformations having the Bernoulli property.

The transformation T_D is an non-invertible endomorphism of the unit interval. This means that from a given partial remainder we can predict all future partial remainders, but we cannot uniquely predict past partial remainders. There is a natural way (called the natural extension) to make our transformation invertible (an automorphism) on a larger space. Specifically, each non-invertible transformation T_D having the Bernoulli property has an extension to an automorphic transformation, isomorphic to a two-sided Bernoulli shift [18, pp. 13,276]. From the way that entropy for a transformation is defined, the entropy for an automorphic Bernoulli transformation associated with a non-invertible Bernoulli transformation is the same as the entropy for the non-invertible Bernoulli transformation. By proving that all transformations T_D are Bernoulli, and by proving that entropy of each T_D is the same, we will be able to conclude that the natural extensions of S-R-T division algorithms are isomorphic to each other for all divisors.

2.1 Proof of Bernoulliness

Definition 6 (of Bowen [1], Expanding). We will say that a transformation T on an interval is *expanding* if it has the property that $\sup_{n>0} \mu(T^n U) = 1$ for all open intervals U with $\mu(U) > 0$, where μ is any normalized measure that is absolutely continuous with respect to Lebesgue measure.

Definition 7 (Straddle). Let U be an interval of reals (either open, closed, or half open) and let $p \in \mathbb{R}^+$. If $p \in U^\circ$,[†] then we say that U *straddles* p.

Theorem 1. *The S-R-T division transformation is expanding for all real divisors.*

Proof. Let $(X, \mathcal{B}, m)$ be a probability space where $X = [0, 1)$, $\mathcal{B}$ is the Borel σ-algebra on X and m is the Lebesgue measure on $\mathcal{B}$[‡]. Let $T_D : X \to X$ be the S-R-T division transformation for a given normalized divisor D as defined in equation 1.1.

[†]The symbol $\circ$ as the exponent of an interval denotes an open version of the interval.
[‡]For an interval $[a, b]$, the Lebesgue measure is defined as $m([a, b]) = b - a$.

Let us define an infinite sequence of intervals $\mathcal{U} = \{U_i\}_{i\in\mathbb{N}}$ as

$$U_1 = U \quad \text{and}$$

$$U_{i+1} = \begin{cases} T_D(U_i) & : \quad U_i^\circ \subseteq [0, \frac{1}{2}) \qquad \text{or } U_i^\circ \subseteq [\frac{1}{2}, 1) \\ T_D(U_i \cap [0, \frac{1}{2})) & : \quad U_i^\circ \not\subseteq [0, \frac{1}{2}) \text{ and } U_i^\circ \not\subseteq [\frac{1}{2}, 1) \text{ and} \\ & \qquad\qquad m(U_i \cap [0, \frac{1}{2})) \geq m(U_i \cap [\frac{1}{2}, 1)) \\ T_D(U_i \cap [\frac{1}{2}, 1)) & : \quad U_i^\circ \not\subseteq [0, \frac{1}{2}) \text{ and } U_i^\circ \not\subseteq [\frac{1}{2}, 1) \text{ and} \\ & \qquad\qquad m(U_i \cap [0, \frac{1}{2})) < m(U_i \cap [\frac{1}{2}, 1)) \,. \end{cases}$$

Property 1. *For all U_i such that $\frac{1}{2} \notin U_i^\circ$ and $D \notin U_i^\circ$, $m(U_{i+1}) = 2m(U_i)$.*

Proof. If a U_i° is a subset of either $[0, \frac{1}{2})$, $[\frac{1}{2}, D)$, or $[D, 1)$, then we are in the first case of the $\mathcal{U}$ definition and we apply T_D directly. Since each of the three cases of the T_D expand an interval by a factor of two, it is clear that $m(T_D(U_i)) = m(U_{i+1}) = 2m(U_i)$.

Property 2. *For all U_i where $D \notin U_i$, $m(U_{i+1}) \geq m(U_i)$.*

Proof. Assume that $D \notin U_i$. If $\frac{1}{2} \notin U_i$, then according to Property 1, U_{i+1} doubles. Otherwise, $\frac{1}{2} \in U_i$ and therefore, to find U_{i+1}, we must consider the second and third cases of the $\mathcal{U}$ sequence. In the worst case, $m(U_i \cap [0, \frac{1}{2})) = m(U_i \cap [\frac{1}{2}, D))$, and regardless of which half we choose, $m(U_i \cap [0, \frac{1}{2})) = m(U_i \cap [\frac{1}{2}, D)) = \frac{1}{2}m(U_i)$. By applying T_D to this truncated interval, we double what we halved so that $m(U_i) = m(U_{i+1})$.

By way of contradiction, let us assume that there exists a sequence of $\mathcal{U}$ that never expands to fill X. Such a sequence can never include the point D and the following Property will hold:

Property 3. *There exists N such that for all $i \geq N$*

(a) $m(U_i \cap [0, \frac{1}{2})), m(U_i \cap [\frac{1}{2}, 1)) > 0$ *(in other words, all subsequent intervals must straddle $\frac{1}{2}$), and*

(b) $m(U_i \cap [0, \frac{1}{2})) < m(U_i \cap [\frac{1}{2}, 1))$ *(in other words, all subsequent U_i must be such that the right half of U_i is not discarded by the definition of $\mathcal{U}$).*

Proof of Property 3(a) Property 1 says that the only way not to double is to straddle $\frac{1}{2}$. Therefore, at a minimum, it must be the case that $\frac{1}{2}$ is eventually included every time or else the interval will double a sufficient number of times to include D which would be a contradiction.

Proof of Property 3(b) If $m(U_i \cap [0, \frac{1}{2})) \geq m(U_i \cap [\frac{1}{2}, 1))$, then we have $U_i = (\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon')$ where $\varepsilon > \varepsilon'$. Now $U_{i+1} = T_D(U_i) = T_D(\frac{1}{2} - \varepsilon, \frac{1}{2}) = (1 - 2\varepsilon, 1)$. But, since D is not in U_{i+1}, $\frac{1}{2}$ cannot be in U_{i+1} and Property 3(a) fails, resulting in a contradiction.

By Property 3, we will eventually be in a situation where $U_i = (\frac{1}{2} - \varepsilon', \frac{1}{2} + \varepsilon)$, $\varepsilon' < \varepsilon$, and Property 3 will hold for every subsequent interval. So then

$$U_{i+1} = T_D(\tfrac{1}{2} - \varepsilon', \tfrac{1}{2} + \varepsilon) = T_D[\tfrac{1}{2}, \tfrac{1}{2} + \varepsilon) = (2D - 1 - 2\varepsilon, 2D - 1]$$

by Property 3(b). But again by Property 3,

$$U_{i+2} = T_D(2D - 1 - 2\varepsilon, 2D - 1] = T_D[\tfrac{1}{2}, 2D - 1] = [2 - 2D, 2D - 1]\,.$$

It is now clear that $\frac{1}{2}$ is at the midpoint of U_{i+2} and that we must now pick the left half of the interval which contradicts Property 3(b). Therefore, D will eventually be included in an interval and the sequence will expand to fill all of X. $\square$

We can now prove that the S-R-T division process is weak-mixing, and therefore Bernoulli, by two theorems of Bowen [1].

Theorem 2 (of Bowen [1]). *Let T be a piecewise C^2 map of $[0, 1]$, μ be a smooth T-invariant probability measure, and $\lambda = \inf_{0 \leq x \leq 1} |f'(x)| > 1$. If the dynamical system (T, μ) is weak-mixing, then the natural extension of (T, μ) is Bernoulli.*

We mention here that the *natural extensions* of (T, μ) is the associated automorphic transformation that we alluded to at the beginning of this chapter. See Petersen [18, p. 13] for an exact definition.

Theorem 3 (of Bowen [1]). *With T and μ as in Theorem 2, (T, μ) will be weak-mixing if T is expanding.*

Theorem 4 (of Lasota and Yorke [10]). *Let $(X, \mathcal{B}, m)$ be a probability space and let $T : X \to X$ be a piecewise C^2 function such that $\inf |T'| > 1$. If P is the Perron-Frobenius operator associated with T, then for any $f \in L^1$, the sequence $(\frac{1}{n}\sum_{k=0}^{n-1} P^k f)_{n=1}^{\infty}$ is convergent in norm to a function $f^* \in L_1$. The limit function f^* has the property that $Pf^* = f^*$ and consequently, the measure $\mathrm{d}\mu^* = f^* \,\mathrm{d}m$ is invariant under T.*

Having established that T_D is expanding, we now use the above three theorems to prove the central result of this thesis.

Theorem 5. *T_D is Bernoulli.*

Proof. From the definition of T_D, we see that T_D is C^2 and that $\inf_{0\leq x\leq 1} |T_D{}'(x)| = 2 > 1$ since $|T_D{}'(x)| = 2$ for all x for which the derivative is defined. Since $\inf_{0\leq x\leq 1} |T_D{}'(x)| > 1$, by Theorem 4 there exists at least one μ such that μ is a smooth T_D-invariant probability measure. By Theorem 1, we see that Theorem 3 holds. Hence, (T_D, μ) is weak-mixing and, by Theorem 2 (T_D, μ) is Bernoulli. $\square$

2.2 Entropy of T_D

Knowing that all T_D are Bernoulli is a very useful property because we can use entropy as a complete invariant to show isomorphism amongst the two-sided Bernoulli shifts associated with T_D that have the same entropy. This comes from the contribution of Ornstein to the Kolmogorov-Ornstein Theorem.

Theorem 6 (of Kolmogorov [8, 9] and Ornstein [16]). *Two Bernoulli shifts are isomorphic if and only if they have the same entropy.*

The purpose of this section is to calculate the entropy of T_D. We begin with a multi-part definition of entropy along with some supporting definitions that follow the development presented by Walters [24, pp. 75-87].

Definition 8 (Partition). A *partition* of $(X, \mathcal{B}, m)$ is a disjoint collection of elements of $\mathcal{B}$ whose union is X.

Definition 9 (Join). Let $\mathcal{P}$ and $\mathcal{Q}$ be finite partitions of $(X, \mathcal{B}, m)$. Then $\mathcal{P} \vee \mathcal{Q} = \{P \cap Q : P \in \mathcal{P}, \text{ and } Q \in \mathcal{Q}\}$ is called the *join* of $\mathcal{P}$ and $\mathcal{Q}$. Note that $\mathcal{P} \vee \mathcal{Q}$ is also a finite partition of $(X, \mathcal{B}, m)$.

Definition 10 (Entropy of a partition). Let $(X, \mathcal{B}, m)$ be a probability space and let $\mathcal{P} = \{P_1, \ldots, P_k\}$ be a finite partition of $(X, \mathcal{B}, m)$. The *entropy of the partition* is defined as

$$H(\mathcal{P}) = -\sum_{i=1}^{k} m(P_i) \log m(P_i) .$$

Definition 11 (Entropy of a transformation with respect to a partition). Suppose $T : X \to X$ is a measure-preserving transformation of the probability space $(X, \mathcal{B}, m)$. If $\mathcal{P}$ is a finite partition of $(X, \mathcal{B}, m)$, then

$$h(T, \mathcal{P}) = \lim_{n \to \infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{P} \right)$$

is called the *entropy of T with respect to partition $\mathcal{P}$*.

Definition 12 (Entropy of a transformation). Let $T : X \to X$ be a measure-preserving transformation of the probability space $(X, \mathcal{B}, m)$ and suppose $h(T) = \sup h(T, \mathcal{P})$, where the supremum is taken over all finite partitions $\mathcal{P}$ of $(X, \mathcal{B}, m)$. Then $h(T)$ is called the *entropy of T*.

The following definitions and theorems involving C-maps and PC-maps are taken from a paper of Ledrappier [12] and have been streamlined for our argument.

Definition 13 (of Ledrappier [12], C-map). A real function f defined on an interval $[a,b]$ is said to be a *C-map* if f is continuously differentiable and its derivative f' has the following properties:

(a) f' satisfies a Hölder condition[§] of order $\varepsilon > 0$.

(b) There are only a finite number of points $x \in [a,b]$ where $f'(x) = 0$. We denote them by $a \leq a_1 < a_2 \ldots < a_n \leq b$ with $f'(a_i) = 0$ for $0 < i \leq n$.

(c) There exist positive numbers k_i^- (k_i^+) such that $\left|\log \frac{|f'(x)|}{|x-a|^{k_i^{-(+)}}}\right|$ is bounded in a left (right) neighborhood of a_i.

Definition 14 (of Ledrappier [12], PC-map). A map $f : [0,1) \to [0,1)$ is called a *PC-map* if there exists a finite partition $0 < b_1 < b_2 \ldots < b_m < 1$ such that f is a C-map from $[b_j, b_{j+1}]$ into $[0,1)$, for any j.

Theorem 7 (of Ledrappier [12]). *Let f be a PC-map. If μ is an a.c.i.m. (absolutely continuous invariant measure), then Rohlin's formula [20] is true:*

$$h(f) = \int \log |f'| \, \mathrm{d}\mu.$$

Theorem 8. *The entropy $h(T_D)$, of T_D for $D \in [\frac{1}{2}, 1)$ is equal to $\int \log |T_D'| \, \mathrm{d}\mu =$ $\log 2$.*

Proof. We begin by showing that T_D is a PC-map. By the definition of a PC-map, T_D is a PC-map if each of the three functions $T_D|_{[0,\frac{1}{2})}$, $T_D|_{[\frac{1}{2},D)}$, and $T_D|_{[D,1)}$ is a C-map.

Trivially, each T_D restricted to any of the three domains $[0,\frac{1}{2})$, $[\frac{1}{2}, D)$, or $[D,1)$ satisfies a Hölder condition of order $\varepsilon = 1$ because each piece of T_D is just a

[§]A function $f(x)$ defined on an interval $[a,b]$ satisfies a Hölder condition of order $\varepsilon \in \mathbb{R}^+$ if there exists $c \in \mathbb{R}^+$ such that for any two points $p_1, p_2 \in [a,b]$, $|f(p_1) - f(p_2)| \leq c\,|p_1 - p_2|^{\varepsilon}$.

line of slope two. Thus condition (a) of Definition 13 is satisfied. Condition (b) is satisfied because there are no points for which the derivative is equal to zero within a given line segment. Therefore, condition (c) is trivially satisfied. Thus each of the three segments of T_D are C-maps and by Definition 14, T_D is a PC-map.

Now, since each T_D is Bernoulli, there exists a unique a.c.i.m., call it μ, for each T_D. By Theorem 7, we can use Rohlin's formula to calculate the entropy:

$$h(T_D) = \int \log |T_D{}'| \, \mathrm{d}\mu = \log 2 \int \mathrm{d}\mu = \log 2.$$

□

With the proof of Theorem 8 we have established isomorphism amongst the automorphic transformations (or natural extension) associated with simple S-R-T division transformations by an application of the Kolmogorov-Ornstein Theorem. The key to obtaining this result was being able to show that T_D has Bowen's expanding property. In Chapter 3, we extend the results of this chapter to a more general type of S-R-T division.

Chapter 3

Extensions to Multi-Divisor S-R-T Division

3.1 Multi-Divisor S-R-T Division

A common optimization to the S-R-T division algorithm is the inclusion of additional divisors to increase the shift average. In this section, we prove that all such division algorithms with reasonable assumptions on the separation of the divisor multiples have the expanding property. It will be useful to define precisely a class of multi-divisor S-R-T division transformations.

Definition 15. Let $\boldsymbol{\alpha} \in \mathbb{R}^n$ be such that

(a) $0 < \alpha_1 < \alpha_2 < \ldots < \alpha_n$, and

(b) For all $x, D \in [\frac{1}{2}, 1)$, there exists $i \in \{1, \ldots, n\}$ such that $|\alpha_i D - x| < \frac{1}{2}$.

We define $\mathfrak{A}_n$ to be the set of all $\boldsymbol{\alpha} \in \mathbb{R}^n$, satisfying conditions (a) and (b). Also, $\mathfrak{A} = \bigcup_{n \in \mathbb{N}} \mathfrak{A}_n$.

Definition 16 (Peaks and Valleys). Given an $\boldsymbol{\alpha} \in \mathfrak{A}_{n \geq 2}$, the point of intersection between two lines $f(x) = 2(x - \alpha_i D)$ and $g(x) = 2(\alpha_{i+1} D - x)$ will be called a *peak* and is denoted by $\boldsymbol{\psi_i} = (\frac{1}{2}D(\alpha_{i+1}+\alpha_i), D(\alpha_{i+1}-\alpha_i))$. For convenience, we will refer to the abscissa as $\psi_i^x = \frac{1}{2}D(\alpha_{i+1}+\alpha_i)$, and to the ordinate as $\psi_i^y = D(\alpha_{i+1}-\alpha_i)$. The point of intersection of the two lines $f(x) = 2(\alpha_i D - x)$ and $g(x) = 2(x - \alpha_i D)$ is $(\alpha_i D, 0)$ and will be called a *valley*.

Definition 17. For a $D \in [\frac{1}{2}, 1)$ and $\boldsymbol{\alpha} \in \mathfrak{A}$, define the transformation $T_{D,\boldsymbol{\alpha}}(x) : [0,1) \to [0,1)$. For $\boldsymbol{\alpha} \in \mathfrak{A}_1$, we get the familiar transformation

$$T_{D,\boldsymbol{\alpha}}(x) = \begin{cases} 2x & : \quad 0 \leq x < \frac{1}{2} \\ |2(D-x)| & : \quad \frac{1}{2} \leq x < 1\,. \end{cases}$$

For $\boldsymbol{\alpha} \in \mathfrak{A}_2$,

$$T_{D,\boldsymbol{\alpha}}(x) = \begin{cases} 2x & : \quad 0 \leq x < \frac{1}{2} \\ |2(\alpha_1 D - x)| & : \quad \frac{1}{2} \leq x < \psi_1^x \\ |2(\alpha_2 D - x)| & : \quad \frac{1}{2} \leq x \text{ and } \psi_1^x \leq x < 1\,. \end{cases}$$

For $\boldsymbol{\alpha} \in \mathfrak{A}_{n\geq 3}$,

$$T_{D,\boldsymbol{\alpha}}(x) = \begin{cases} 2x & : \quad 0 \leq x < \frac{1}{2} \\ |2(\alpha_1 D - x)| & : \quad \frac{1}{2} \leq x < \psi_1^x \\ |2(\alpha_i D - x)| & : \quad \frac{1}{2} \leq x \text{ and } \psi_i^x \leq x < \psi_{i+1}^x \\ |2(\alpha_n D - x)| & : \quad \frac{1}{2} \leq x \text{ and } \psi_{n-1}^x \leq x < 1\,. \end{cases}$$

Definition 18. Define $\mathfrak{M}_n = \{T_{D,\boldsymbol{\alpha}} : D \in (\frac{1}{2}, 1], \boldsymbol{\alpha} \in \mathfrak{A}_n\}$ and define $\mathfrak{M} = \bigcup_{n\in\mathbb{N}} \mathfrak{M}_n$. We call $\mathfrak{M}_n$ the set of all *n-divisor S-R-T division transformations* and we call $\mathfrak{M}$ the set of *multi-divisor S-R-T division transformations.*

Condition (b) in Definition 15 guarantees that the division algorithm generates a new quotient bit every step. Although the condition makes intuitive sense, it is not immediately obvious if an $\boldsymbol{\alpha}$ satisfies the condition just by inspection. Lemma 10 below provides an easier way to check.

Lemma 9. *If $\boldsymbol{\alpha} = (\alpha_1)$, then condition (b) of Definition 15 is satisfied if and only if $\alpha_1 = 1$.*

Proof. If $\alpha_1 = 1$, then $\max_{D,x\in[1/2,1)} |\alpha_1 D - x| < \frac{1}{2}$. Now consider the cases when $\alpha_1 \neq 1$ and $\varepsilon \in \mathbb{R}^+$. If $\alpha_1 = 1+\varepsilon$, then when $D = \frac{1}{1+\varepsilon}$ and $x = \frac{1}{2}$, $|\alpha_1 D - x| = 1 - \frac{1}{2} = \frac{1}{2} \not< \frac{1}{2}$. On the other hand, if $\alpha_1 = 1-\varepsilon$, then when $D = \frac{1}{2}$ and $x = 1 - \frac{\varepsilon}{2}$, $|\alpha_1 D - x| = 1 - \frac{\varepsilon}{2} - (1-\varepsilon)\frac{1}{2} = \frac{1}{2} \not< \frac{1}{2}$. ☐

Lemma 10. *An $\boldsymbol{\alpha} \in \mathfrak{A}_n$ that satisfies condition (a) of Definition 15 also satisfies condition (b) if and only if for some $i, j \in \{1, \ldots, n\}$ (possibly $i = j$), either*

(i) $\alpha_i \in (0, \frac{1}{2}]$ and $\alpha_j \in [1, 1+\alpha_i]$, or

(ii) $\alpha_i \in [\frac{1}{2}, 1]$ and $\alpha_j \in [1, 3\alpha_i]$.

Proof (Sketch). Lemma 9 has shown that a single component α of $\boldsymbol{\alpha}$ with $\alpha = 1$ is sufficient to ensure that the range of $f(x) = 2\,|\alpha D - x|$ is equal to $[0,1)$ as x and D range over $[\frac{1}{2}, 1)$. It is easy to see based on the proof of Lemma 9 that if there does not exist $i \in \{1, \ldots, n\}$ such that $\alpha_i = 1$, then there must exist $i, j \in \{1, \ldots, n\}$ $(i < j)$ where $\alpha_i < 1$ and $\alpha_j > 1$.

Let us assume that i is the largest value where $\alpha_i < 1$, and let us assume that j is the smallest value where $\alpha_j > 1$ (therefore $j = i+1$). We make this assumption because no other scalars of D will have an influence on whether or not condition (b) is satisfied. Consider the case where $\alpha_i \in (0, \frac{1}{2}]$. In this case where $\alpha_i \in (0, \frac{1}{2}]$, when D is close enough to 1, some of the line $f(x) = 2(x - \alpha_i D)$ appears in the region (denoted R) where $\frac{1}{2} \leq x < 1$, $0 \leq T_{\boldsymbol{\alpha}}(x) < 1$. When a portion of the line $f(x)$ appears in region R, we must put restrictions on α_j in terms of α_i so that the peak $\boldsymbol{\psi}_1$ is always in R. ψ_i^y is greatest when $D = 1$. We find the maximum allowable value of α_j by setting $D = 1$ and solving $\psi_i^y = 1$ for α_j:

$$\psi_i^y = 1 \quad \Rightarrow \quad D(\alpha_j - \alpha_i) = 1 \quad \Rightarrow \quad \alpha_j = \alpha_i + 1\,.$$

Therefore, if $\alpha_i \in (0, \frac{1}{2}]$, then $\alpha_j \in [1, 1+\alpha_i]$.

In the case where $\alpha_i \in [\frac{1}{2}, 1]$, for large enough values of D, the line $f(x) = 2(x - D\alpha_i)$ crosses the line $x = 1$ in the range $[0, 1)$. Because of this, we must loosen the restriction that $\alpha_j \in [1, 1 + \alpha_j]$. It is straightforward to calculate that $f(x)$ begins to cross the line $x = 1$ in the range $[0, 1)$ when $D = \frac{1}{2\alpha_i}$. We can ensure that as D becomes smaller, the peak $\boldsymbol{\psi_i}$ will always be in region R by solving $\psi_i^y = 1$ for α_j when $D = \frac{1}{2\alpha_i}$:

$$\psi_i^y = 1 \quad \Rightarrow \quad D(\alpha_j - \alpha_i) = 1 \quad \Rightarrow \quad \frac{1}{2\alpha_i}(\alpha_j - \alpha_i) = 1 \quad \Rightarrow \quad \alpha_j = 3\alpha_i\,.$$

Therefore, if $\alpha_i \in [\frac{1}{2}, 1]$, then $\alpha_j \in [1, 3\alpha_i]$. □

Definition 19 (Separation). For $\boldsymbol{\alpha} \in \mathfrak{A}_n$, we define the *separation* in $\boldsymbol{\alpha}$ as

$$\mathit{separation}(\boldsymbol{\alpha}) = \max_{i \in \{1, \ldots, n-1\}} \frac{\alpha_{i+1}}{\alpha_i}\,.$$

Limiting the separation is a convenient way to restrict the subset of $\mathfrak{A}$ being considered. If $\mathit{separation}(\boldsymbol{\alpha}) = r$, we say that "the divisor multiples in $\boldsymbol{\alpha}$ are separated by at most a factor of r."

Figure 3.1 shows an example of multi-divisor S-R-T division. This example is performing the same calculation as in figure 1.2, but it has computed the dividend with twice as many digits of precision with the same effective number of uses of the adders. We say "effective" because in multi-divisor S-R-T division, there are several adders working in parallel. In a real implementation of multi-divisor S-R-T division, the values for $\boldsymbol{\alpha}$ must be carefully chosen so that not too much overhead is required to select a good partial remainder. There is also a tradeoff between the amount of overhead in choosing a good partial remainder and the precision to with which a good partial remainder is selected.

$p_0 = 0.67 = 0.67$		
$p_1 = 2(0.67 - \alpha_2 D) = -0.16$	$q_0 = \alpha_2$	$Q_0 = 1$
$p_4 = 2(2^2(-0.16) + \alpha_1 D) = -0.155$	$q_3 = -\alpha_1$	$Q_3 = 0.90625$
$p_7 = 2(2^2(-0.155) + \alpha_1 D) = -0.115$	$q_6 = -\alpha_1$	$Q_6 = 0.89453125$
$p_{11} = 2(2^3(-0.115) + \alpha_3 D) = 0.035$	$q_{10} = -\alpha_3$	$Q_{10} \doteq 0.8933105469$
$p_{16} = 2(2^4(0.035) - \alpha_1 D) = -0.005$	$q_{15} = \alpha_1$	$Q_{15} \doteq 0.8933334351$
$p_{24} = 2(2^7(0.005) + \alpha_1 D) = -0.155$	$q_{23} = -\alpha_1$	$Q_{23} \doteq 0.8933333456$

Figure 3.1: An example of S-R-T division where three multiples of the divisor are used. In this example the dividend $p_0 = 0.67$, and the divisor $D = 0.75$ with divisor multiples $\boldsymbol{\alpha} = (0.75, 1, 1.25)$. The quotient Q^* is $0.89\overline{3}$.

3.2 Proof of Bernoulliness

In this section, we will show that all multi-divisor S-R-T division transformations are Bernoulli, given a necessary restriction on the multiples of the divisor. As in the case for a single divisor, it will be useful to define a sequence of intervals that are subsets of the sequence of sets that would arise from repeatedly applying $T_{D,\boldsymbol{\alpha}}$ to an initial open interval. Unless otherwise noted, assume that the function m denotes the Lebesgue measure.

Definition 20. Given an initial open interval $U \subset [0,1)$ and $T_{D,\boldsymbol{\alpha}} \in \mathfrak{M}$, we define the infinite sequence of intervals $\mathcal{U} = \{U_i\}_{i \in \mathbb{N}}$ as

$$U_1 = U \quad \text{and}$$

$$U_{i+1} = \begin{cases} T_{D,\boldsymbol{\alpha}}(U_i) & : \quad U_i^\circ \subseteq [0, \frac{1}{2}) \quad \text{or } U_i^\circ \subseteq [\frac{1}{2}, 1) \\ T_{D,\boldsymbol{\alpha}}(U_i \cap [0, \frac{1}{2})) & : \quad U_i^\circ \not\subseteq [0, \frac{1}{2}) \text{ and } U_i^\circ \not\subseteq [\frac{1}{2}, 1) \text{ and} \\ & \qquad\qquad m(U_i \cap [0, \frac{1}{2})) \geq m(U_i \cap [\frac{1}{2}, 1)) \\ T_{D,\boldsymbol{\alpha}}(U_i \cap [\frac{1}{2}, 1)) & : \quad U_i^\circ \not\subseteq [0, \frac{1}{2}) \text{ and } U_i^\circ \not\subseteq [\frac{1}{2}, 1) \text{ and} \\ & \qquad\qquad m(U_i \cap [0, \frac{1}{2})) < m(U_i \cap [\frac{1}{2}, 1)) \,. \end{cases}$$

Definition 21 (Critical Points). For a given $T_{D,\boldsymbol{\alpha}}$ where $\boldsymbol{\alpha} \in \mathfrak{A}_n$, define the set

$$C = \{c_i \,:\, i \in \{1, \ldots, m\}, c_i \in B \cup \{0, \tfrac{1}{2}, 1\}\}$$

where $B = \{b \,:\, \frac{1}{2} < b < 1 \text{ and } b \in \{\alpha_1 D, \ldots, \alpha_n D\} \cup \{\psi_1^x, \ldots, \psi_{n-1}^x\}\}$ and $c_1 < c_2 < \ldots < 1$. C is called the set of *critical points* for $T_{D,\boldsymbol{\alpha}}$.

Lemma 11 (Doubling). *Given $T_{D,\boldsymbol{\alpha}} \in \mathfrak{M}$, let the sequence of intervals $\mathcal{U}$ be defined as in Definition 20 and let U_i be some interval in the sequence. Furthermore, let $C = \{c_1, \ldots, c_m\}$ be the set of critical points for $T_{D,\boldsymbol{\alpha}}$. If $U_i \subseteq [c_j, c_{j+1}]$ for some $j \in \{1, \ldots, m-1\}$, then $m(U_{i+1}) = 2m(U_i)$.*

Proof. Since $U_i \subseteq [c_j, c_{j+1}]$ for some $j \in \{1, \ldots, m-1\}$, because we are in the first case of the definition of $\mathcal{U}$, either $U_i^\circ \subseteq [0, \frac{1}{2})$ or $U_i^\circ \subseteq [\frac{1}{2}, 1)$. By simple inspection of the individual cases that define $T_{D,\boldsymbol{\alpha}}$, it is apparent that all of U_i, except possibly the points c_j and c_{j+1}, fall within the same case of $T_{D,\boldsymbol{\alpha}}$. Therefore, the resulting interval U_{i+1} will be double the length of U_i. $\square$

Definition 22 (Active Valleys). Given $T_{D,\boldsymbol{\alpha}} \in \mathfrak{M}_n$, define

$$V = \{\alpha_i D \,:\, i \in \{1, \ldots, n\} \text{ and } \tfrac{1}{2} < \alpha_i D < 1\}\,.$$

V is called the set of *active valleys* for $T_{D,\boldsymbol{\alpha}}$.

Definition 23 (Active Peaks). Given $T_{D,\boldsymbol{\alpha}} \in \mathfrak{M}_n$, define

$$P = \{\psi_i^x \,:\, i \in \{1, \ldots, n-1\} \text{ and } \tfrac{1}{2} < \psi_i^x < 1\}\,.$$

P is called the set of *active peaks* for $T_{D,\boldsymbol{\alpha}}$.

Lemma 12 (Non-shrinking). *Given $T_{D,\boldsymbol{\alpha}} \in \mathfrak{M}_n$ with separation$(\boldsymbol{\alpha}) \leq \frac{5}{3}$, let the sequence of intervals $\{U_i\}_{i\in\mathbb{N}}$ be defined as above and let V denote the set of active valleys for $T_{D,\boldsymbol{\alpha}}$. For any interval $U_i \in \mathcal{U}$ such that $V \cap U_i = \varnothing$, either $m(U_{i+1}) \geq m(U_i)$ or $m(U_{i+2}) \geq m(U_i)$.*

Proof. *separation*$(\boldsymbol{\alpha}) \leq \frac{5}{3}$ implies that $\alpha_{i+1} \leq \frac{5}{3}\alpha_i$. For a given separation, the value of ψ_i^y is maximized when $\psi_i^x = 1$. This implies that $\alpha_i = \frac{3}{4D}$. We calculate the value of ψ_i^y with the assumption that $\psi_i^x = 1$ to get a bound on ψ_i^y for $D < 1$:

$$\psi_i^y \leq D(\tfrac{5}{3}\alpha_i - \alpha_i) = D(\tfrac{2}{3}\alpha_i) = D(\tfrac{2}{3}\tfrac{3}{4D}) = \tfrac{1}{2}\,.$$

Case 1: Consider when $U_i \subseteq [0, \frac{1}{2}]$. In this case, $m(U_{i+1}) = 2m(U_i)$.

Case 2: Consider when $U_i \subseteq [\frac{1}{2}, 1)$. The interval U_i can span at most one peak. Therefore, $m(U_{i+1}) \geq m(U_i)$. A further observation is that since $U_{i+1} \subseteq [0, \frac{1}{2}]$, $m(U_{i+2}) = 2m(U_i)$.

Case 3: Consider when $U_i \not\subseteq [0, \frac{1}{2}]$ and $U_i \not\subseteq [\frac{1}{2}, 1)$. In this case, U_i straddles $\frac{1}{2}$. From the definition of $\mathcal{U}$, we see that in the worst case we might throw away up to half of U_i. Call the part not thrown away $U_i{}'$ and observe that $m(U_i{}') \geq \frac{1}{2}m(U_i)$. Now, either $U_i{}' \subseteq [0, \frac{1}{2}]$ or $U_i{}' \subseteq [\frac{1}{2}, 1)$. If $U_i{}' \subseteq [0, \frac{1}{2}]$, then $m(U_{i+1}) = 2m(U_i{}') \geq m(U_i)$. If $U_i{}' \subseteq [\frac{1}{2}, 1)$, then $m(U_{i+2}) = 2m(U_i{}') \geq m(U_i)$. $\square$

Lemma 13. *A multi-divisor S-R-T division transformation $T_{D,\boldsymbol{\alpha}} \in \mathfrak{M}$ is expanding when separation$(\boldsymbol{\alpha}) \leq \frac{5}{3}$.*

Proof. Let V be the set of active valleys (as defined in Definition 22) for a $T_{D,\boldsymbol{\alpha}}$. Let P be the set of active peaks (as defined in Definition 23) for a $T_{D,\boldsymbol{\alpha}}$. Let $\mathcal{U} = \{U_i\}_{i\in\mathbb{N}}$ be the sequence of intervals associated with a $T_{D,\boldsymbol{\alpha}}$ and an initial interval U.

By way of contradiction, assume that a $T_{D,\alpha}$ is not expanding. This means that for some $T_{D,\alpha}$, there does not exist an interval U_i where any of the points in V are contained in U_i. This is true because if any of the valley points are in U_i, then $U_{i+1} = [0, \varepsilon)$ or $U_{i+1} = [0, \varepsilon]$, and after a finite number of steps, U_i will have grown to include all of $[0, 1)$.

If there is a sequence $\mathcal{U}$ that avoids all points in V, then by Lemma 12 it must be true that the intervals in the sequence can only double a finite number of times. Let $i \in \mathbb{N}$ be the first index for which there is no $j > i$ such that $m(U_j) \geq 2m(U_i)$. It now follows that U_i straddles $\frac{1}{2}$. The proof for Lemma 12 reveals that this is the only situation where it is not necessarily the case that either $m(U_{i+1}) = 2m(U_i)$ or $m(U_{i+2}) = 2m(U_i)$. In fact, U_i must straddle both $\frac{1}{2}$ and $\min P$. If $\min P$ is not straddled and $m(U_i \cap [0, \frac{1}{2})) < m(U_i \cap [\frac{1}{2}, 1))$, then either $m(U_{i+2}) \geq 2m(U_i)$ or $m(U_{i+3}) \geq 2m(U_i)$. In the other possibility where $\min P$ is not straddled and $m(U_i \cap [0, \frac{1}{2})) \geq m(U_i \cap [\frac{1}{2}, 1))$, we find that $m(U_{i+2}) \geq 2m(U_i)$.

Assuming that U_i straddles both $\frac{1}{2}$ and $\min P$, we also observe that there can be no $j > i$ such that $m(U_j \cap [0, \frac{1}{2})) \geq m(U_j \cap [\frac{1}{2}, 1))$ because this quickly leads to doubling. In other words, the right side must be larger than the left side whenever we straddle $\frac{1}{2}$. Therefore, we must be in the situation where

$$\begin{aligned}
& U_i = (\tfrac{1}{2} - \varepsilon', \tfrac{1}{2} + \varepsilon), \ \varepsilon' < \varepsilon \\
\Rightarrow \quad & U_{i+1} = (\min\{2(\tfrac{1}{2} - \alpha_i D), 2(\alpha_{i+1} D - (\tfrac{1}{2} + \varepsilon))\}, \psi_i^y) \\
\Rightarrow \quad & U_{i+2} = (2\min\{2(\tfrac{1}{2} - \alpha_i D), 2(\alpha_{i+1} D - (\tfrac{1}{2} + \varepsilon))\}, 2\psi_i^y) \\
\Rightarrow \quad & U_{i+3} = (\min\{2(\tfrac{1}{2} - \alpha_i D), 2(\alpha_{i+1} D - 2\psi_i^y)\}, \psi_i^y) \\
\Rightarrow \quad & U_{i+4} = (2\min\{2(\tfrac{1}{2} - \alpha_i D), 2(\alpha_{i+1} D - 2\psi_i^y)\}, 2\psi_i^y) \\
\Rightarrow \quad & U_{i+5} = (\min\{2(\tfrac{1}{2} - \alpha_i D), 2(\alpha_{i+1} D - 2\psi_i^y)\}, \psi_i^y) = U_{i+3}\,.
\end{aligned}$$

It is apparent that the interval represented by U_{i+4} will re-occur every other interval *ad infinitum*. We now use this interval to show that in fact such a sequence of non-expanding intervals is not possible.

Since U_{i+4} straddles $\frac{1}{2}$, we can compare the length of the left and right sides of U_{i+4}. Let $R = [\frac{1}{2}, 2\psi_i^y)$ denote the right side and let $L = (4(\frac{1}{2} - \alpha_i D), \frac{1}{2})$ and $L' = (4(\alpha_{i+1} D - 2\psi_i^y), \frac{1}{2})$ denote the two possibilities for the left side. The length of the right side is

$$m(R) = 2\psi_i^y - \tfrac{1}{2},$$

while the length of the left side is the larger of two possible lengths

$$m(L) = \tfrac{1}{2} - 4(\tfrac{1}{2} - \alpha_i D)$$

and

$$m(L') = \tfrac{1}{2} - 4(\alpha_{i+1} D - 2\psi_i^y) .$$

We then compare the differences between the right side and each of the two possible left sides. The first possibility is

$$\begin{aligned} m(R) - m(L) &= 2\psi_i^y - \tfrac{1}{2} - (\tfrac{1}{2} - 4(\tfrac{1}{2} - \alpha_i D)) \\ &= 2D(\alpha_{i+1} - \alpha_i) - 1 + 2 - 4\alpha_i D \\ &= 2\alpha_{i+1} D - 6\alpha_i D + 1 , \end{aligned}$$

while the second possibility is

$$\begin{aligned} m(R) - m(L') &= 2\psi_i^y - \tfrac{1}{2} - (\tfrac{1}{2} - 4(\alpha_{i+1} - 2\psi_i^y)) \\ &= 2D(\alpha_{i+1} - \alpha_i) - 1 + 4(\alpha_{i+1} D - 2D(\alpha_{i+1} - \alpha_i)) \\ &= -2\alpha_{i+1} D + 6\alpha_i D - 1 . \end{aligned}$$

It is now clear that

$$m(R) - m(L) = -\big(m(R) - m(L')\big) .$$

But this means that the length of the left side is always greater than or equal to the length of the right side, which contradicts our assumption that the right side must be bigger than the left side whenever the interval straddles $\frac{1}{2}$.

$\square$

Theorem 14. *$T_{D,\boldsymbol{\alpha}} \in \mathfrak{M}$ is Bernoulli when separation($\boldsymbol{\alpha}$) $\leq \frac{5}{3}$.*

Proof. Let $T = T_{D,\boldsymbol{\alpha}}$. From the definition of T, we see that $T_{D,\boldsymbol{\alpha}}$ is C^2 and that $\inf_{0\leq x\leq 1} |T'(x)| = 2 > 1$ since $|T'(x)| = 2$ for all x for which the derivative is defined. Since $\inf_{0\leq x\leq 1} |T'(x)| > 1$, by Theorem 4, there exists at least one μ such that μ is a smooth T-invariant probability measure. By Lemma 13 we see that Theorem 3 holds when *separation*($\boldsymbol{\alpha}$) $\leq \frac{5}{3}$. Hence, (T, μ) is weak-mixing and by Theorem 2, (T, μ) is Bernoulli when *separation*($\boldsymbol{\alpha}$) $\leq \frac{5}{3}$. □

3.3 Some Restrictions on $\boldsymbol{\alpha}$

In section 3.2, we showed that if all $T_{D,\boldsymbol{\alpha}} \in \mathfrak{M}$, if *separation*($\alpha$) $\leq \frac{5}{3}$, then $T_{D,\boldsymbol{\alpha}}$ is Bernoulli. In this section, we construct examples of $T \in \mathfrak{M}_n$, for every n, that fail to be Bernoulli when the restriction that *separation*($\boldsymbol{\alpha}$) $\leq \frac{5}{3}$ is relaxed.

Theorem 15. *For $T_{D,\boldsymbol{\alpha}} \in \mathfrak{M}_{n\geq 4}$, if separation($\boldsymbol{\alpha}$) $> \frac{5}{3}$, then for each $D \in [\frac{1}{2}, 1)$, there exist uncountably many $\boldsymbol{\alpha}$ for which $T_{D,\boldsymbol{\alpha}}$ is not ergodic.*

Proof. We begin this proof by considering $T \in \mathfrak{M}_{n=4}$.

Assume that we relax the restrictions on $\boldsymbol{\alpha}$ by $\varepsilon > 0$. This means that *separation*($\boldsymbol{\alpha}$) $\leq \frac{5}{3} + \varepsilon$ and that no peak can be above the line $f(x) = \frac{4+6\varepsilon x}{8+3\varepsilon}$. With this relaxation, we can define $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ with respect to a given D so that a subset of $[0, 1)$ is non-expanding. We let $\alpha_1 = \frac{30+27\varepsilon}{80D+48D\varepsilon}$, $\alpha_2 = \frac{50+57\varepsilon}{80D+48D\varepsilon}$, $\alpha_3 = \frac{30-9\varepsilon}{40D+24D\varepsilon}$, and $\alpha_4 = \frac{50+21\varepsilon}{40D+24D\varepsilon}$. For our constructed $\boldsymbol{\alpha}$ to be valid, we must be careful that conditions (a) and (b) of Definition 15 hold. Condition (a) requires that the components of $\boldsymbol{\alpha}$ remain in ascending order. This is satisfied when $\varepsilon \in (0, \frac{2}{15}]$. Since ordering is maintained, *separation*($\boldsymbol{\alpha}$) < 3, and $\min_{D\in[1/2,1),\varepsilon\in(0,2/15]} \alpha_4 = 1.\overline{2} \geq 1$, to verify that condition (b) of Definition 15 holds, it is sufficient to show (by Lemma 10) that for all values of D and ε, either α_1, α_2, or $\alpha_3 \in [\frac{1}{2}, 1]$. By maximizing and minimizing over ε and D, we find that $\alpha_1 \in [0.375, 0.\overline{7}]$ and $\alpha_2 \in [0.625, 1.\overline{3}]$. Figure

3.2 provides a visual proof that as ε is varied over $[0, \frac{2}{15}]$ and D is varied over $[\frac{1}{2}, 1]$, it is never the case that both $\alpha_1 \leq \frac{1}{2}$ and $\alpha_2 \geq 1$. Therefore, it is always the case that either α_1 or $\alpha_2 \in [\frac{1}{2}, 1]$.

Having verified that our defined $\boldsymbol{\alpha}$ satisfies Definition 15, we calculate that peak $\boldsymbol{\psi}_1 = (\frac{20+21\varepsilon}{40+24\varepsilon}, \frac{10+15\varepsilon}{40+24\varepsilon})$ and peak $\boldsymbol{\psi}_3 = (\frac{20+3\varepsilon}{20+12\varepsilon}, \frac{10+15\varepsilon}{20+12\varepsilon})$. With this definition for $\boldsymbol{\alpha}$, and our assumption that $\varepsilon \in [0, \frac{2}{15})$, the point $\boldsymbol{\psi}_3$ will always touch the line $f(x)$ while remaining above the line $g(x) = \frac{1}{2}$, and the point $\boldsymbol{\psi}_1$ will always be slightly below $f(x)$ while remaining above the line $g(x) = \frac{1}{4}$. All of the definitions have been chosen so that we are in a situation where $1 - \psi_3^x = \psi_3^y - \frac{1}{2} = 2(\psi_1^x - \frac{1}{2}) = 2(\psi_1^y - \frac{1}{4})$. Another important feature in this construction is the interval between $\alpha_2 D$ and $\alpha_3 D$. Since $\boldsymbol{\psi}_2$ is not used in our construction, it is possible to insert an arbitrary number of divisor multiples between $\alpha_2 D$ and $\alpha_3 D$. Thus, the results in this proof apply to $T \in \mathfrak{M}_n$ for arbitrarily large n. Figure 3.3 illustrates the type of transformation that we have constructed.

We are now in a position to show that there exists a set of points A with $0 < m(A) < 1$, for which $T_{D,\boldsymbol{\alpha}}(A) = A$. This is the definition of a transformation being non-ergodic [11, p. 59]. Define $A = A_1 \cup A_2 \cup A_3$ where $A_1 = [\frac{1}{4} - (\psi_1^x - \frac{1}{2}), \frac{1}{4} + (\psi_1^x - \frac{1}{2})]$, $A_2 = [\frac{1}{2} - 2(\psi_1^x - \frac{1}{2}), \frac{1}{2} + 2(\psi_1^x - \frac{1}{2})]$, and $A_3 = [1 - 2(1 - \psi_3^x), 1]$. It can be shown by calculation that $T_{D,\boldsymbol{\alpha}}(A_1) = A_2$, $T_{D,\boldsymbol{\alpha}}(A_2) = A_1 \cup A_3$, and $T_{D,\boldsymbol{\alpha}}(A_3) = A_2$. Therefore, $T_{D,\boldsymbol{\alpha}}(A) = A$, and by definition, $T_{D,\boldsymbol{\alpha}}$ is non-ergodic or non-expanding. □

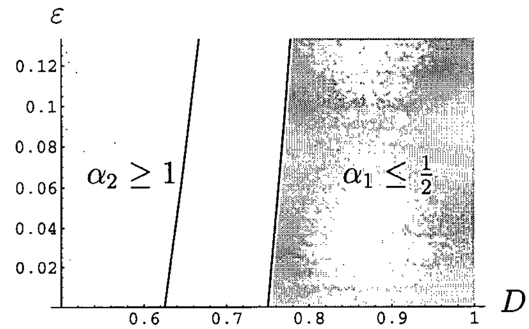

Figure 3.2: Combined plot of the regions where $\alpha_1(\varepsilon, D) \leq \frac{1}{2}$ and $\alpha_2(\varepsilon, D) \geq 1$. Over the domain $\varepsilon \in [0, \frac{2}{15}]$ and $D \in [\frac{1}{2}, 1]$, it is never true that both $\alpha_1 \leq \frac{1}{2}$ and $\alpha_2 \geq 1$.

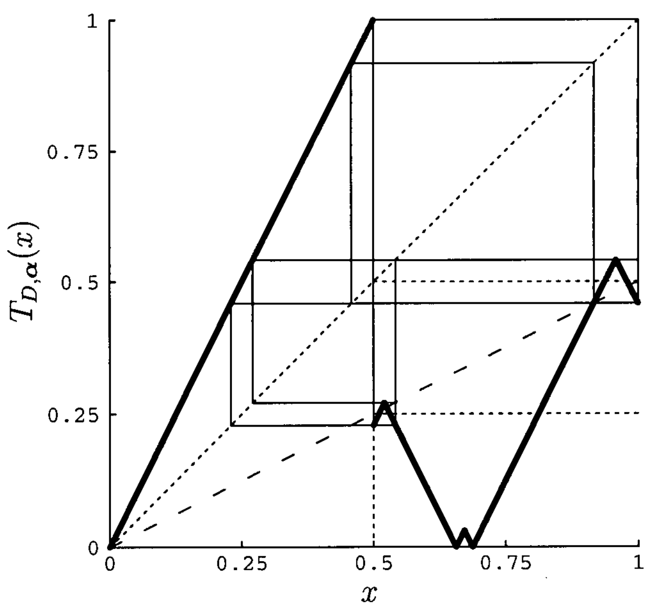

Figure 3.3: An example of a non-ergodic system for $T_{D,\boldsymbol{\alpha}} \in \mathfrak{M}_{n\geq 4}$. In this example $n = 4$, $D = \frac{11}{16}$, $\boldsymbol{\alpha} = (\frac{37}{66}, \frac{21}{22}, 1, \frac{59}{33})$, and *separation*$(\boldsymbol{\alpha}) = \frac{5}{3} + \frac{5}{51}$. The thick lines represent $T_{D,\boldsymbol{\alpha}}$. The coarse dashed line represents the necessary separation restriction on $\boldsymbol{\alpha}$ to guarantee that $T_{D,\boldsymbol{\alpha}}$ is ergodic. In this case, partial remainders in the set $A = [\frac{11}{48}, \frac{13}{48}] \cup [\frac{22}{48}, \frac{26}{48}] \cup [\frac{44}{48}, 1)$ are mapped back to A by $T_{D,\boldsymbol{\alpha}}$. This means that $T_{D,\boldsymbol{\alpha}}$ is not ergodic, and therefore not Bernoulli.

Theorem 16. *For $T_{D,\boldsymbol{\alpha}} \in \mathfrak{M}_3$, if separation($\boldsymbol{\alpha}$) $\geq \frac{9}{5}$, then for each $D \in [\frac{1}{2}, 1)$, there exists an $\boldsymbol{\alpha}$ for which $T_{D,\boldsymbol{\alpha}}$ is not ergodic.*

Proof. The proof for this theorem comes as a special case from the proof for Theorem 15. Consider $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ as defined in the proof for 15. When *separation*($\boldsymbol{\alpha}$) $= \frac{9}{5} = \frac{5}{3} + \frac{2}{15}$, we are in the special situation where $\alpha_2 = \alpha_3$. Since all of the results for the proof of Theorem 15 still hold, we now have an example transformation T with only three unique multiples of D and this T has been proven to be non-ergodic. Figure 3.4 gives an example of a non-ergodic transformation for $D = \frac{7}{12}$. □

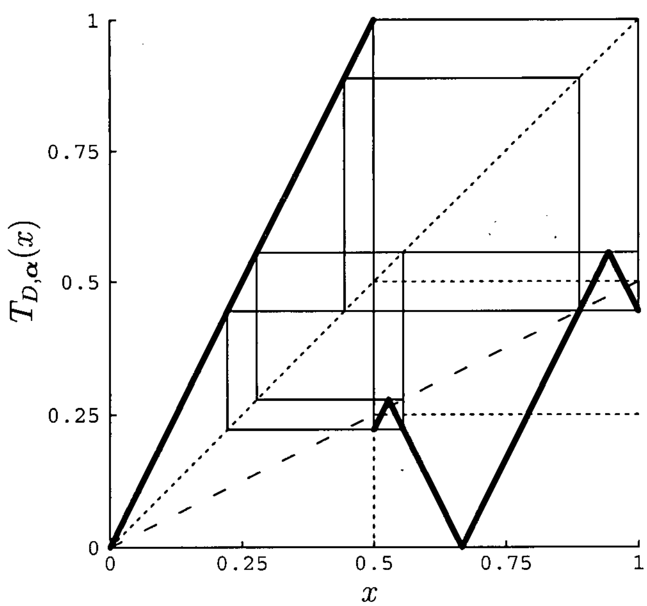

Figure 3.4: An example of a non-ergodic system for $T_{D,\boldsymbol{\alpha}} \in \mathfrak{M}_3$. In this example, $D = \frac{7}{12}$ and $\boldsymbol{\alpha} = (\frac{2}{3}, \frac{8}{7}, \frac{44}{21})$. The thick lines represent $T_{D,\boldsymbol{\alpha}}$. The coarse dashed line represents the necessary separation restriction on $\boldsymbol{\alpha}$ to guarantee that $T_{D,\boldsymbol{\alpha}}$ is ergodic. In this case, partial remainders in the set $A = [\frac{4}{18}, \frac{5}{18}] \cup [\frac{8}{18}, \frac{10}{18}] \cup [\frac{16}{18}, 1)$ are mapped back to A by $T_{D,\boldsymbol{\alpha}}$. This means that $T_{D,\boldsymbol{\alpha}}$ is not ergodic, and therefore not Bernoulli.

Theorem 17. *For $T_{D,\boldsymbol{\alpha}} \in \mathfrak{M}_2$, if separation$(\boldsymbol{\alpha}) > 3$, then for some $D \in (\frac{1}{2}, 1)$, there exist uncountably many $\boldsymbol{\alpha}$ for which $T_{D,\boldsymbol{\alpha}}$ is not ergodic.*

Proof. Assume that $separation(\boldsymbol{\alpha}) \leq 3 + \varepsilon$ and $D \in (\frac{1}{2}, \frac{2+\varepsilon}{4})$. First, we choose $\alpha_1 = \frac{1}{4D}$ so that $\alpha_1 D = \frac{1}{4}$ and $\alpha_2 = 1 + \alpha_1$. Our restriction on D in terms of ε has been chosen so that $\alpha_2/\alpha_1 < 3 + \varepsilon$ when $\alpha_2 = 1 + \alpha_1$. Since $\alpha_2 > \alpha_1$, condition (a) of Definition 15 is satisfied. Since $\alpha_1 \in (\frac{1}{4}, \frac{1}{2})$, and $\alpha_2 \in (1, 1 + \alpha_1]$, by Lemma 10, condition (b) of Definition 15 is satisfied. Thus, our defined $\boldsymbol{\alpha}$ is always valid. Define $A = [\frac{1}{2}, D]$. We now apply $T = T_{D,\boldsymbol{\alpha}}$ to A:

$$\begin{aligned}
T[\tfrac{1}{2}, D] =& [\min\{2(\tfrac{1}{2} - \alpha_1 D), 2(\alpha_2 D - D)\}, \psi_1^y] \\
=& [\min\{2(\tfrac{1}{2} - \tfrac{D}{4D}), 2(D + \tfrac{1}{4} - D)\}, D(\alpha_2 - \alpha_1)] \\
=& [\min\{\tfrac{1}{2}, \tfrac{1}{2}\}, D(1 + \tfrac{1}{4D} - \tfrac{1}{4D})] \\
=& [\tfrac{1}{2}, D] .
\end{aligned}$$

Now, since $\frac{1}{2} < D < 1$, $0 < m(A) < 1$ and $T_{D,\boldsymbol{\alpha}} A = A$, by Definition $T_{D,\boldsymbol{\alpha}}$ is not ergodic. $\square$

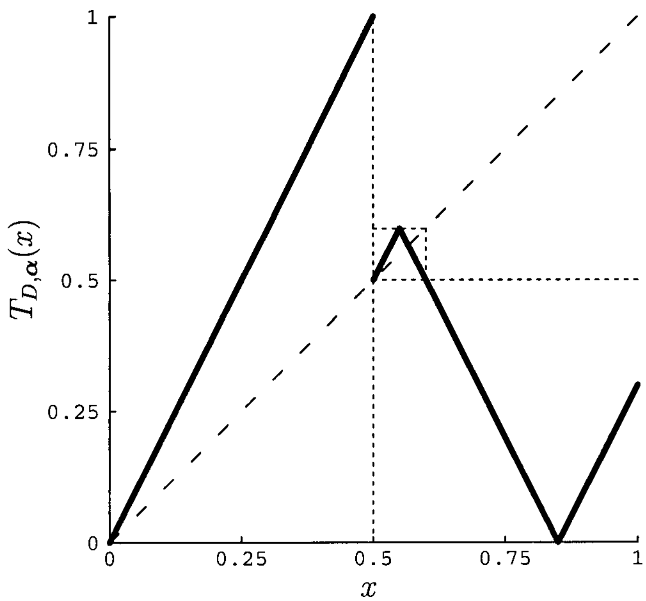

Figure 3.5: An example of a non-ergodic system for $T_{D,\boldsymbol{\alpha}} \in \mathfrak{M}_2$. In this example, $D = \frac{3}{5}$ and $\boldsymbol{\alpha} = (\frac{5}{12}, \frac{17}{12})$. The thick lines represent $T_{D,\boldsymbol{\alpha}}$. The coarse dashed line represents the necessary separation restriction on $\boldsymbol{\alpha}$ to guarantee that $T_{\boldsymbol{\alpha}}$ is ergodic. In this case, partial remainders within the interval $[\frac{1}{2}, \frac{3}{5}]$ map back to $[\frac{1}{2}, \frac{3}{5}]$ and which means $T_{D,\boldsymbol{\alpha}}$ is not ergodic, and therefore not Bernoulli.

3.4 Entropy of Multi-Divisor S-R-T Division

The calculation for entropy in multi-divisor S-R-T division follows the same method used for single divisor S-R-T division. We begin by showing that $T_{D,\boldsymbol{\alpha}}$ is a PC-map.

Lemma 18. *$T_{D,\boldsymbol{\alpha}} \in \mathfrak{M}$ is a PC-map (as defined in Definition 14).*

Proof. By inspection, each $T_{D,\boldsymbol{\alpha}}$ is a finite collection of line segments each with slope 2. Each of these line segments is a C-map by the same argument used in the proof for Theorem 8. Therefore, by definition, each $T_{D,\boldsymbol{\alpha}}$ is a PC-map. $\square$

Theorem 19. *The entropy of any $T_{D,\boldsymbol{\alpha}} \in \mathfrak{M}$ with separation$(\boldsymbol{\alpha}) \leq \frac{5}{3}$ is* $\log 2$.

Proof. By Lemma 18, all $T_{D,\boldsymbol{\alpha}} \in \mathfrak{M}$ are PC-maps. By Theorem 14, $T_{D,\boldsymbol{\alpha}}$ is Bernoulli when *separation*$(\boldsymbol{\alpha}) \leq \frac{5}{3}$ and hence there exists a unique a.c.i.m. μ. Theorem 7 says that Rohlin's formula for the entropy is true and therefore:

$$h(T_{D,\boldsymbol{\alpha}}) = \int \log \left| T_{D,\boldsymbol{\alpha}}' \right| \mathrm{d}\mu = \log 2 \int \mathrm{d}\mu = \log 2.$$

$\square$

Chapter 4

Future Work

The original question that inspired this thesis was "Is simple S-R-T division ergodic for all real divisors?" In pursuing the answer to this problem, we discovered that not only is simple S-R-T division ergodic for all divisors, but it is also Bernoulli. Having established a Bernoulli property, and having calculated the entropy for our transformations, we were able to use the Kolmogorov-Ornstein theorem to conclude that our transformations are isomorphic to each other. In proving these important properties for simple S-R-T division, we made extensive use of more general results from dynamical systems theory. Consequently, our results were shown to be easily extensible to more general division systems. In general, it is difficult to prove that a particular class of transformations are ergodic or Bernoulli. Our results have provided an effective means of proving both of these properties for a large class of S-R-T-like division algorithms.

From the standpoint of understanding an algorithm's expected performance, it is necessary to know that when a stationary distribution is found, it is unique. Having established the uniqueness of stationary distributions, the next step is to find the actual stationary distribution for as wide a class of transformations as possible. In section 1.3, we verified a known expression for the stationary distribution function for T_D where $D \in [\frac{3}{4}, 1)$. In addition, many of the stationary distribution functions have been classified by Shively and Freiman for $D \in [\frac{3}{5}, \frac{3}{4}]$, although the derivations are not nearly as simple as for $D \in [\frac{3}{4}, 1)$. It turns out that things become very complicated when $D \in [\frac{1}{2}, \frac{3}{5}]$. In his thesis [22], Shively shows many interesting properties for the stationary distribution functions in this region. For example, he shows that there are many different intervals of D where there are an infinite number of different stationary distribution equations. As such, the graph of the shift average for $D \in [\frac{1}{2}, \frac{3}{5}]$ is far from complete and appears to have a complex pattern (from the few points that have been plotted in this region). This is surprising considering the simplicity of the underlying transformation. A better understanding of this final region of simple S-R-T division would be an interesting goal to pursue.

In the work of Freiman [6], it was first shown that the shift average for $D \in [\frac{3}{5}, \frac{3}{4}]$ is constantly 3, which can be easily shown to be the maximum possible shift average. This property was then used by Metze [15] to obtain a version of S-R-T division that has an expected shift average of 3 for all divisors. Another area to pursue would be to explore shift averages for multi-divisor S-R-T division and, if other plateaus are found, they could possibly be used to obtain higher expected shift averages for all possible divisors. Undoubtedly, obtaining a complete understanding of the stationary distribution functions for multi-divisor division would be even more difficult than it is for simple S-R-T division. It is possible that such results in this area could lead to improvements in modern S-R-T division. Related to this, it would be interesting to attempt to extend the results of this thesis to modern S-R-T division.

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