ON THE ISOMORPHISM PROBLEM FOR A CLASS OF BIPARTITE GRAPHS

by

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ABSTRACT

The purpose of this thesis is to investigate the graph isomorphism problem for a special class of graphs. Each graph is characterized by its edge set, and a subgroup of its automorphism group, called the colour group. In particular, a simple, efficient algorithm for determining whether two graphs are isomorphic if at least one is a member of the class is developed.

Chapter 1 provides some basic definitions and lemmas required in the text. The concepts of reducibility and reducible bipartite graphs are introduced, and the properties of the colour groups of such graphs are investigated.

Chapter 2 establishes some results concerning the existence of reducible graphs. Conditions based on the existence of vertices with prescribed properties are shown to provide sufficient conditions for a graph to be reducible. In the special case of trees they are shown to be both necessary and sufficient. Necessary conditions for the reducibility of graphs, based on their radius and diameter are also established.

Chapter 3 describes an algorithm for determining whether a graph is completely reducible, which is applied to a test for isomorphism. An investigation of the speed of this algorithm is made and its efficiency is compared with an algorithm of D. Corneil [5], which this author considers the best for arbitrary graphs in the current literature.

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CHAPTER 1

The Bipartite Complement and Colour Group of a Graph.

1.1 INTRODUCTION

An undirected graph, G, is a non-empty set $V=V(G)$ of p vertices together with a set $E=E(G)$ of q unordered pairs of vertices called edges. If $(i,j) \in E$, then the vertices i and j are said to be adjacent. Two graphs are isomorphic if there exists a one-one correspondence between their vertices which preserves adjacency. An automorphism of a graph is a one-one mapping of the vertices onto themselves which preserves adjacency, and the set of all automorphisms constitutes a group called the group of the graph.

An n-colouring of a graph G is a function $f: V(G) \rightarrow \{c_1, c_2, \ldots, c_n\}$, the set of colours. We consider the set of all $\varphi \in \Gamma(G)$, the automorphism group of G, such that $f(\varphi(v)) = f(v)$ for all $v \in V(G)$. It will be shown later that this set forms a group called the colour group of the graph.

Let G_1 and G_2 be two graphs. Their sum, $G_1 U G_2$, is the graph with $V(G_1 U G_2) = V(G_1) U V(G_2)$ and $E(G_1 U G_2) = E(G_1) U E(G_2)$. Their cartesian product, $G_1 X G_2$, is the graph with $V(G_1 X G_2) = V(G_1) X V(G_2)$ and $E(G_1 X G_2) = \{[v,w] = [(v_1,v_2), (w_1, w_2)]$ where $v_1, w_1 \in V(G_1)$, $v_2, w_2 \in V(G_2)$ and either $v_1=w_1$ and $(v_2,w_2) \in E(G_2)$ or $v_2=w_2$ and $(v_1,w_1) \in E(G_1)\}$.

Denote by $\overline{G}$ the graph with $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{(v_i,v_j) \notin E(G)$, for $v_i \neq v_j\}$.

The graph isomorphism problem is one of finding efficient ways of determining when two graphs are isomorphic. By "efficient" is usually meant computable after p^k units of time where p is the number

of vertices, and k is a constant. The search for solutions has resulted in the generation of several related problems, some of which have been solved, and whose solution has suggested their inapplicability to the original problem.

Among these is the result of Frucht [7] that there exist infinitely many non-isomorphic graphs with a given group, and hence graphs are not uniquely characterized by their group. An extension of this idea is that of finding some suitable finite set of invariant properties of a graph by which it can be completely characterized. Partial results of a negative nature have been obtained by Sabidussi [22] who showed that there are infinitely many non-isomorphic graphs with given group and property P_i where P_i is one of:

P_1 : connectivity of G is K, $K \geq 1$.

P_2 : chromatic number of G is K, $K \geq 2$.

P_3 : G is regular of degree K, $K \geq 3$.

P_4 : G is spanned by a subgraph homeomorphic to a given graph.

Izbicki [15] extended this result to show that there exist infinitely many non-isomorphic regular graphs with given chromatic number. Even the calculation of the automorphism group itself is difficult, although Kagno [17] has determined the groups of graphs with less than seven vertices.

The most fruitful analytical approach to the problem has been made by Corneil [4,5] who has determined a procedure for constructing a "representative graph" with the property that two graphs are isomorphic if and only if they have the same "representative graph". Although the procedure is not yet proved to be deterministic, since it is based on a conjecture that the representative graphs exhibit the automorphism partitioning of the given graph, no counter-examples have yet been found.

This thesis defines and investigates a class of graphs which is shown later to have property that its members are uniquely characterized by their colour group, number of edges, and number of vertices. Finally, an algorithm is provided which determines members of the class, determines whether two members are isomorphic, and provides sufficient information to construct the colour group of the graph.

1.2 THE BIPARTITE COMPLEMENT AND GRAPH REDUCIBILITY

A graph is bipartite if its vertices can be partitioned into two disjoint sets $V_1 \cup V_2$ such that every edge is incident to a vertex in V_1 and to one in V_2. Denote by $G(m,n)$ a bipartite graph with m vertices in V_1, and n vertices in V_2. The bipartite complement of $G(m,n)$, denoted by $\hat{G}(m,n)$ is a bipartite graph with $V(\hat{G}) = V_1 \cup V_2$ and $E(\hat{G}) = \left\{(v_i,w_j),\ v_i \varepsilon V_1, w_j \varepsilon V_2 \middle| (v_i,w_j) \notin E(G)\right\}$ [6]. The adjacency matrix, $A(G)$, of a graph G with n vertices is an nxn (0,1)-matrix such that:

$$a_{ij} = \begin{cases} 0 & \text{if} \quad (i,j) \notin E(G) \\ 1 & \text{if} \quad (i,j) \varepsilon E(G) \end{cases}$$

Since G is undirected $(i,j)\varepsilon E(G)$ implies $(j,i)\varepsilon E(G)$ and hence $a_{ij}=a_{ji}$. Therefore $A(G)$ is symmetric. Further if the graph is bipartite, and the vertices in V_1 are labelled $1,2,\ldots m$, and those in V_2 are labelled $m+1,m+2,\ldots m+n$, then the adjacency matrix $A(G)$ has the form

$$A(G) = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$$

Hence the bipartite graph $G(m,n)$ can be characterized by the smaller mxn vertex matrix B, called the bipartite matrix of the graph.

Let G_1 and G_2 be two graphs, and suppose there exists a one-one mapping φ of $V(G_1)$ onto $V(G_2)$. Assuming G_1, G_2 have the same vertex sets, if φ is an isomorphism, and the vertices are labelled $1,2,\ldots n$, φ can be interpreted as a permutation P_φ of the elements $1,2,\ldots, n$. As a consequence of a LEMMA by Chao [3, p. 489], we observe:

<u>1.1 LEMMA</u> : G_1 is isomorphic to G_2 if and only if there exists a permutation matrix P such that

$$P^\tau A(G_1)P = A(G_2).$$

If G is bipartite, then the bipartite matrices of all possible isomorphic images of G can be interpreted as all possible permutations of the rows and columns of the bipartite matrix of G. These considerations lead to the following:

<u>1.2 LEMMA</u> : If $G_1(m,n)$ and $G_2(m,n)$ are connected bipartite graphs and B_1 and B_2 are their respective bipartite matrices, then $G_1(m,n)$ is isomorphic to $G_2(m,n)$ if and only if there exist permutation matrices $P_1^{(mxm)}$ and $P_2^{(nxn)}$ such that $P_1^\tau B_1 P_2 = B_2$, or $P_1^\tau B_1^\tau P_2 = B_2$.

PROOF : $G_1(m,n)$ isomorphic to $G_2(m,n)$ implies by LEMMA 1.1 the existence of a permutation matrix P such that $P^T \begin{bmatrix} 0 & B_1 \\ B_1^\tau & 0 \end{bmatrix} P = \begin{bmatrix} 0 & B_2 \\ B_2^\tau & 0 \end{bmatrix}$

Let $P = \begin{bmatrix} X_1 X_2 \\ X_3 X_4 \end{bmatrix}$, then $\begin{bmatrix} X_1^\tau X_3^\tau \\ X_2^\tau X_4^\tau \end{bmatrix} \begin{bmatrix} 0 & B_1 \\ B_1^\tau & 0 \end{bmatrix} \begin{bmatrix} X_1 X_2 \\ X_3 X_4 \end{bmatrix} =$

$$\begin{bmatrix} X_3^\tau B_1^\tau X_1 + X_1^\tau B_1 X_2 , & X_3^\tau B_1^\tau X_2 + X_1^\tau B_1 X_4 \\ X_4^\tau B_1^\tau X_1 + X_2^\tau B_1 X_3 , & X_4^\tau B_1^\tau X_2 + X_2^\tau B_1 X_4 \end{bmatrix} = \begin{bmatrix} 0 & B_2 \\ B_2^\tau & 0 \end{bmatrix}$$

Hence 1) $X_3^\tau B_1^\tau X_1 + (X_3^\tau B_1^\tau X_1)^\tau = 0,$

2) $X_4^\tau B_1^\tau X_2 + (X_4^\tau B_1^\tau X_2)^\tau = 0,$

3) $X_3^\tau B_1^\tau X_2 + X_1^\tau B_1 X_4 = B_2$

Now since X_1, X_2, X_3, X_4 and B_1, B_2 are non-negative matrices and every row and column of B has at least one non-zero entry, equations 1 and 2 imply X_1 and/or X_3 are 0 and X_2 and/or X_4 are 0. However 3 implies X_1, X_2 not both 0, X_1, X_3 not both 0, X_2 , X_4 not both 0 and X_3 , X_4 not both 0. Hence we have the solutions X_1 and X_4 are 0 or X_2 and X_3 are 0. Since P is a permutation matrix, X_1 , X_2 , X_3 , X_4 must be permutation matrices

since $P = \begin{bmatrix} X_1 & 0 \\ 0 & X_4 \end{bmatrix}$ or $P = \begin{bmatrix} 0 & X_2 \\ X_3 & 0 \end{bmatrix}$, and

in the latter case X_2 and X_3 must therefore be square and hence m=n. Hence $X_1^\tau B_1 X_4 = B_2$ or $X_2^\tau B_1^\tau X_3 = B_2$.

Conversely $P_1^\tau B_1 P_2 = B_2$ implies the existence of a permutation P such that $P^\tau A(G_1)P = A(G_2)$ and hence again by LEMMA 1.1, G_1 is isomorphic to G_2.

Since the bipartite matrix of the complete bipartite graph, $K_{m,n}$, is $J^{(mxn)}$, a matrix with all entries one, the bipartite matrix of $\hat{G}(m,n)$ is given by J-B where B is the bipartite matrix of G(m,n). An obvious but important result is:

<u>1.3 LEMMA</u> : $\hat{G}_1(m,n)$ is ismorphic to $\hat{G}_2(m,n)$ if and only if $G_1(m,n)$ is isomorphic to $G_2(m,n)$.

PROOF : From LEMMA 1.2 G_1 is isomorphic to G_2 if and only if there exist P_1, P_2 such that $P_1^\tau B_1 P_2 = B_2$. But $P_1^\tau B_1 P_2 = B_2$ if and only if $J - P_1^\tau B_1 P_2 = P_1^\tau (J-B_1) P_2 = J - B_2$, and $P_1^\tau (J-B_1) P_2 = J-B_2$ if and only if G_1 is isomorphic to G_2.

As a consequence of LEMMA 1.3 is the COROLLARY

<u>1.4 COROLLARY</u> : A graph and its bipartite complement have the same colour group.

PROOF : From LEMMA 1.3 and letting $B_1 = B_2$,

$P_1 \oplus P_2 = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} = P$ corresponds to an automorphism $\varphi \in \Gamma(\hat{G})$ if and only if $\varphi \in \Gamma(G)$ where $\Gamma(G)$ is the automorphism group of the bipartite graph G. But $\left\{ \varphi \;\middle|\; P_\varphi = P_1 \oplus P_2 \right\}$ is just the colour group of the graph G.

The concept of reducibility is now defined. A bipartite graph G(m,n) is said to be <u>reducible</u> if its bipartite complement is a graph with K components, $K > 1$. It is immediately obvious that all bipartite graphs whose complements (not to be confused with bipartite complement) are disconnected constitute reducible graphs.

The concept of complete reducibility is defined inductively. Let G be a given bipartite graph.

If G is complete bipartite, then G is said to be <u>completely reduced</u>.

The <u>first stage of reduction</u> is defined to be the operation of taking the bipartite complement of G, provided G is not completely reduced. G is said to be <u>1-reducible</u> if it is reducible.

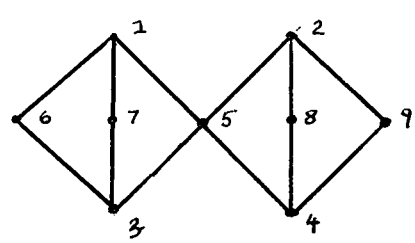
The <u>K-th stage of reduction</u> consists of determining the bipartite complement of all components not completely reduced at the K-1 stage of reduction.

A bipartite graph G is <u>K-reducible</u> if the components obtained at the K-th stage of reduction are reducible.

A bipartite graph G is <u>completely reducible</u> if for some K all components obtained at the Kth stage of reduction are complete bipartite and hence completely reduced.

Some examples serve to illustrate the idea.

G : =



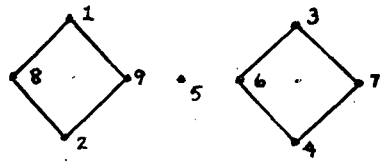
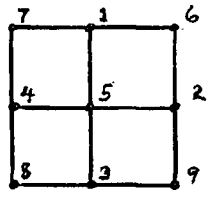
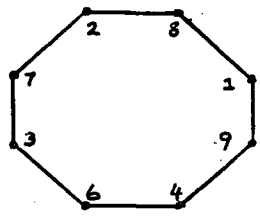
$\hat{G} :=$ 

fig. 1.1

Hence each component of G is completely reducible since each is $K_{m,n}$ for some m,n , namely m=n=2 or m=0, n=1. For completeness, an isolated vertex v of $\overline{G}$ will be defined to be $K_{1,0}$ if $v \in V_1$, and $K_{0,1}$ if $v \in V_2$.

e.g.

$G :=$ 

$\hat{G} :=$ 

.5

fig. 1.2

Although G is reducible, it is not completely reducible since the connected component of $\hat{G}$ with eight vertices is not reducible, its bipartite complement being given by :

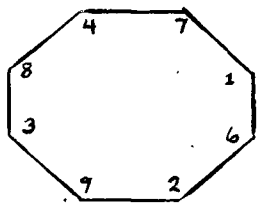
$H :=$ 

fig. 1.3

1.3 THE COLOUR GROUP OF A BIPARTITE GRAPH

As indicated in the introduction, the colour group, $\Gamma^*(G)$, of a bipartite graph is the maximal set of automorphisms with the property

that $f(\varphi(v)) = f(v)$ where is a colouring of the vertices of the graph. We show that $\Gamma^*(G)$ is in fact a group.

Obviously $I \in \Gamma^*(G)$, since $f(v) = f(Iv)$.

Now since $f(v) = f(\varphi\varphi^{-1}(v)) = f(\varphi^{-1}(v))$ it follows that $\varphi^{-1} \in \Gamma^*(G)$ for any $\varphi \in \Gamma^*(G)$.

Let $\varphi_1, \varphi_2 \in \Gamma^*(G)$. Then $f(\varphi_1\varphi_2(v)) = f(\varphi_2(v)) = f(v)$ so that $\varphi_1\varphi_2 \in \Gamma^*(G)$.

Finally, the associativity of composition in $\Gamma^*(G)$ follows immediately from the the associativity of the group operation in $\Gamma(G)$.

Since for each $\varphi \in \Gamma(G)$, there is a corresponding permutation P defined on the vertices of G, we shall henceforth consider the permutation group of the graph, and utilize some definitions from Harary [12] concerning such groups.

Let A and B be two permutation groups acting on sets X and Y. Their sum, A+B, is a group acting on the disjoint union $X \cup Y$ and whose elements are ordered pairs of permutations $\alpha \in A$, $\beta \in B$, written $\alpha+\beta$, where $z \in X \cup Y$ is permuted according to:

$$(\alpha+\beta)(z) = \alpha z \quad \text{if} \quad z \in X$$
$$= \beta z \quad \text{if} \quad z \in Y \qquad [12, \text{p. } 163]$$

Their composition A[B] is a group acting on XxY where for $\alpha \in A$ and sequence $(\beta_1, \beta_2, \ldots \beta_d)$ of d permutations in B there is a unique permutation denoted $(\alpha; \beta_1, \ldots \beta_d)$ in A[B] such that for $(x_i, y_j) \in X \times Y$ $(d; \beta_1, \ldots \beta_d) = (\alpha x_i, \beta_i Y_j)$. [12, p. 164]

We shall make use of the following theorem (see for example, Harary [12, p. 166]) concerning the group of a graph with more than one component, where $n_i G_i$ denotes n_i copies of the graph G_i :

<u>1.5 THEOREM</u> : The group of the graph G where $G=n_1G_1 \cup n_2G_2 \cup \ldots \cup n_rG_r$ is given by $\Gamma(G) \equiv S_{n_1}[\Gamma(G_1)] + S_{n_2}[\Gamma(G_2)] + \ldots + S_{n_r}[\Gamma(G_r)]$. where S_{n_i} denotes the symmetric group of degree n_i.

For the complete bipartite graph, $K_{m,n}$, the automorphism group is easily calculated.

Since $K_{m,n} = \overline{K}_m + \overline{K}_n$, where $\overline{K}_m$ is the null graph with m points, and since $\overline{K}_m + \overline{K}_n = \overline{K_m \cup K_n}$ [12,p.166] we have:

$$\begin{aligned}
\Gamma(K_{m,n}) &= \Gamma(\overline{K_m \cup K_n}) \\
&= \Gamma(K_m \cup K_n) \quad \text{since } \Gamma(G) = \Gamma(\overline{G}) \text{ [12, p. 165]} \\
&= \begin{cases} \Gamma(K_m) + \Gamma(K_n) , & m \neq n \\ \Gamma(2K_m) , & m = n \end{cases} \\
&= \begin{cases} S_m + S_n , & m \neq n \\ S_2[S_m] , & m = n \end{cases}
\end{aligned}$$

Denote by $\Gamma^*(G)$ the colour group of G. Then (i) $\Gamma^*(G) = \Gamma^*(\overline{G})$ since $\Gamma(G) = \Gamma(\overline{G})$, and (ii) $\Gamma^*(G_1 \cup G_2) = \Gamma^*(G_1) + \Gamma^*(G_2)$ where G_1 and G_2 are non-isomorphic, since $\Gamma^*(G_1) + \Gamma^*(G_2)$ is the maximal subgroup of $\Gamma(G_1) + \Gamma(G_2)$ which is colour preserving. Hence the colour group of $K_{m,n}$ is easily determined.

$$\begin{aligned}
\Gamma^*(K_{m,n}) &= \Gamma^*(\overline{K_m \cup K_n}) \\
&= \Gamma^*(K_m \cup K_n) \\
&= \Gamma^*(K_m) + \Gamma^*(K_n) \\
&= S_m + S_n
\end{aligned}$$

since $\Gamma(K_m) \equiv \Gamma^*(K_m)$ where all vertices of K_m are of the same colour. If m=n, the result still holds since for a particular colouring of $K_{m,n}$, all vertices in K_m are of colour opposite to those of K_n.

Graphs whose automorphisms result only in an interchange of vertices of the same colour have colour group identical with auto-

morphism group. Such graphs as the following have this property:

$K_{m,n}$ where $m \neq n$

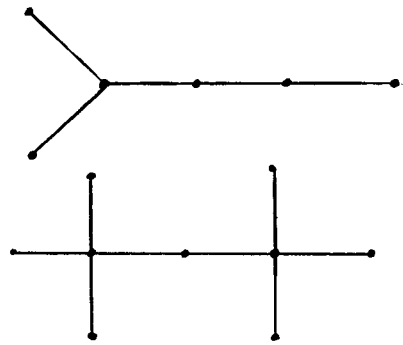


fig. 1.4

while the following do not:

$K_{m,m}$

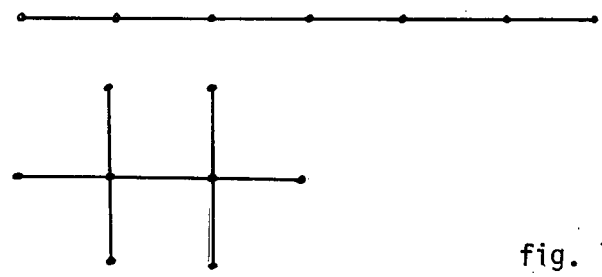


fig. 1.5

Clearly the colour group of any bipartite graph G can be written as the direct sum of two subgroups of $\Gamma(G)$, namely $\Gamma_1(G)$ consisting of all automorphisms of vertices of one colour, and $\Gamma_2(G)$, all automorphisms of vertices of the opposite colour.

We now state a result for colour groups analogous to that for automorphism groups as given in THEOREM 1.5.

<u>1.6 THEOREM</u> : The colour group of the graph $G \equiv n_1G_1 \cup n_2G_2 \cup \ldots \cup n_rG_r$ is given by $\Gamma^*(G) \equiv S_{n_1}[\Gamma^*(G_1)] + S_{n_2}[\Gamma^*(G_2)] + \ldots + S_{n_r}[\Gamma^*(G_r)]$.

The proof is a direct consequence of the previous discussion. Noting the fact that $\Gamma^*(G_i) \equiv \Gamma_1(G_i) + \Gamma_2(G_i)$ where Γ_1 and Γ_2 are as defined above, we obtain:

1.7 COROLLARY : If $G \equiv n_1G_1 \cup \ldots \cup n_rG_r$ then $\Gamma^*(G) \equiv S_{n_1}[\Gamma_1(G_1) + \Gamma_2(G_1)] + \ldots + S_{n_r}[\Gamma_1(G_r) + \Gamma_2(G_r)]$

We now establish the following lemma concerning the existence of a bipartite graph with given group:

1.8 LEMMA : Let Γ^* be the group defined as follows:

$\Gamma^* \equiv S_{k_1}[\Gamma_{m_1} + \Gamma_{n_1}] + S_{k_2}[\Gamma_{m_2} + \Gamma_{n_2}] + \ldots + S_{k_r}[\Gamma_{m_r} + \Gamma_{n_r}]$ where m_i, n_i denotes the degrees of the groups m_i and n_i. Then there exists a bipartite graph with $\sum_{i=1}^{r} k_i(m_i+n_i)$ vertices and at most $\sum_{i=1}^{r} k_i m_i n_i$ edges with colour group Γ^*.

PROOF : Let S_{k_i} act on the set $C_i = \{1,2,\ldots K_i\}$, let Γ_{m_i} act on the set V_{m_i} and Γ_{n_i} on the set V_{n_i} of m_i and n_i elements respectively. Then Γ^* acts on the set:

$$S = \{C_1 x \{V_{m_1} \cup V_{n_1}\} \cup \ldots \cup C_r x \{V_{m_r} \cup V_{n_r}\}\}$$

Clearly, this set contains $\sum_{i=1}^{r} k_i(m_i+n_i)$ ordered pairs of elements of the form $<i_j, v_j>$ where $i_j \varepsilon c_j$, $v_j \varepsilon \{V_{m_j} \cup V_{n_j}\}$.
Let these elements denote the vertices of a graph with $V(G) = S$. We now define the edge set $E(G)$ to be any subset of the set $\{(<i_j, v_{k_j}> <i_j, v_{e_j}>) \mid i_j \varepsilon C$ for some j and $v_{k_j} \varepsilon V_{m_j}$, $v_{e_j} \varepsilon V_{n_j}$, or $v_{k_j} \varepsilon V_{n_j}$ and $v_{e_j} \varepsilon V_{m_j}\}$. The maximal number of edges is thus $\sum_{i=1}^{r} k_i m_i n_i$ and if G has this number of edges then $G \equiv \bigcup_{i=1}^{r} k_i K_{m_i n_i}$.

A further lemma concerning the nature of the graphs at each stage of reduction is required before relating the concepts of isomorphism and colour groups in THEOREM 1.10.

1.9 LEMMA : Let $G^{(i)}$ denote the union of all complete bipartite graphs obtained at stage i of the reduction of a completely reducible bipartite graph H. Then $\Gamma^*(H) \equiv \Gamma^*(G^{(2)}) + \ldots \Gamma^*(G^{(t)})$, where H is completely reduced after t stages.

PROOF : For each stage of reduction

$\hat{H}^{(i)} \equiv G^{(i+1)} \cup H^{(i+1)}$, $H^{(o)} = H$

where $H^{(i)}$ denotes the union of all graphs not complete bipartite obtained at stage i. Now since H is completely reduced after t stages

$\hat{H}^{(t-1)} \equiv G^{(t)}$

Further $\Gamma^*(H) = \Gamma^*(\hat{H})$ by COROLLARY 1.4

and $\Gamma^*(\hat{H}) = \Gamma^*(G^{(1)} \cup H^{(1)})$

$= \Gamma^*(G^{(1)} + \Gamma^*(H^{(1)})$,

$\Gamma^*(\hat{H}^{(i)}) = \Gamma^*(G^{(i+1)}) + \Gamma^*(H^{(i+1)})$,

and $\Gamma^*(\hat{H}^{(t-1)}) = \Gamma^*(G^{(t)})$

hence $\Gamma^*(H) = \Gamma^*(G^{(1)}) + \Gamma^*(G^{(2)}) + \ldots + \Gamma^*(G^{(t)})$.

<u>1.10 THEOREM</u> : Let G,H be two completely reducible graphs, and denote by $G^{(i)}, H^{(i)}$ the union of all complete bipartite graphs obtained at stage i of the reduction. Then G is isomorphic to H if and only if $\Gamma^*(G^{(i)})$ is permutationally isomorphic to $\Gamma^*(H^{(i)})$ and $|V(G^{(i)})| = |V(H^{(i)})|$, $|E(G^{(i)})| = |E(H^{(i)})|$ for all i=1,2,...t, t being the index of the final stage of reduction.

PROOF : If G is isomorphic to H then obviously G and H have the same number of edges and the same decomposition.

Conversely suppose $\Gamma^*(G^{(i)})$ is permutationally isomorphic to $\Gamma^*(H^{(i)})$ for each i; since $G^{(i)}$ is the union of complete bipartite graphs, say $G^{(i)} \equiv \cup\, k_i K_{m_i n_i}$, $\Gamma^*(G)$ is given by $S_{k_1}[S_{m_1} + S_{n_1}] + S_{k_2}[S_{m_2} + S_{n_2}] + \ldots + S_{k_r}[S_{m_r} + S_{n_r}]$. Applying LEMMA 1.8 to construct a vertex set with $\sum_{i=1}^{r} k_i(m_i+n_i)$ vertices, $\Gamma^*(G^{(i)})$ and consequently $\Gamma^*(H^{(i)})$ act on vertex set $\{ (C_1 X \{V_{m_1} \cup V_{n_1}\} \cup (C_2 X \{V_{m_2} \cup V_{n_2}\}) \cup \ldots \cup (C_r X V_{m_r} \cup V_{n_r}) \}$. Now from LEMMA 1.8 $G^{(i)}$ has the maximal number of edges, and hence $E(H^{(i)}) \subseteq E(G^{(i)})$. But by hypothesis $|E(H^{(i)})| = |E(G^{(i)})|$, hence $G^{(i)}$ and $H^{(i)}$ have the same vertex and edges sets

and are hence isomorphic.

Now G can be constructed from $G^{(1)}$, $G^{(2)}$,...,$G^{(t)}$ as follows: Let $\widetilde{G}_t = G^{(t)}$ and define $\widetilde{G}_{i-1}$ to be the bipartite complement of $G^{(i-1)} \cup \widetilde{G}_i$. Then by construction G is given by $\widetilde{G}_1$ since each $\widetilde{G}_i$ corresponds to the union of all graphs not complete bipartite prior to reduction at stage i.

Utilizing a well known result concerning the sum of isomorphic groups (see for example [1] p. 146), namely, if S_i and T_i are isomorphic groups for i=1,2,...n then T is isomorphic to S where $S=S_1+S_2+\ldots+S_n$ and $T=T_1+T_2+\ldots+T_n$, we can apply THEOREM 1.10 to construct the colour group of any completely reducible graph, since all that is required is the calculation of the colour groups of complete bipartite graphs at each stage of the reduction.

CHAPTER 2

Reducible Trees and Graphs

2.1 REDUCIBLE TREES

The importance of the completely reducible graphs was established in THEOREM 1.10. Necessary and/or sufficient conditions are now determined for graphs to be reducible. In the sequel the following notation shall be adopted. If $V_i=\{v_1,\ldots v_m\}$ is a set of m identically coloured vertices, then for any $w\varepsilon V_1$, w is said to have colour m. Since any bipartite graph G(m,n) can be partitioned into m vertices of one colour, n vertices of another, every vertex in G(m,n) has colour m or n.

2.1 LEMMA : If there exists a vertex in G(m,n) of degree m and colour n, then G is reducible.

PROOF : Let v be such a vertex. Then the degree of v in $\hat{G}(m,n)$ is 0 since its adjacency to all vertices of opposite colour in G implies its non-adjacency to all vertices of opposite colour in $\hat{G}$. Hence $\hat{G}(m,n) = H(m,n-1) \cup K_{0,1}$, and G is reducible.

A second simple LEMMA providing a sufficient condition for reducibility is:

2.2 LEMMA : If there exist non-adjacent vertices of colour m, degree n-1 and colour n, degree m-1 respectively, then G(m,n) is reducible.

PROOF : Let v be a vertex of degree m-1, colour n and w be a vertex of degree n-1, colour m in G(m,n). The degree of v = degree of w =1 in $\hat{G}(m,n)$ and since $(v,w) \notin E(G)$ we have $(v,w)\varepsilon E(\hat{G})$. Further $(v,u) \notin E(\hat{G})$ and $(u,w) \notin E(\hat{G})$ for any other $u \varepsilon V(\hat{G})$. Hence $\hat{G}(m,n) = H(m-1,n-1) \cup K_{1,1}$ and G is reducible. Although the previous

two lemmas provide rather elementary sufficient conditions for the reducibility of a graph, they are both necessary and sufficient if $G(m,n)$ is a tree. This observation is embodied in THEOREM 2.4 which requires the following result in its proof.

2.3 LEMMA : If $T(m,n)$ is a reducible (bipartite) tree, then its bipartite complement $\hat{T}(m,n)$ has a component $K_{1,1}$ or $K_{o,1}$.

PROOF : Since $T(m,n)$ is reducible, $\hat{T}(m,n) = G_1(m_1,n_1) \cup G_2(m-m_1, n-n_1)$ where G_2 is not necessarily connected, and G_1 is the smallest connected component of $\hat{T}$.

Assume $m_1+n_1 \geqslant 3$.

CASE 1 : $(m-m_1) + (n-n_1) = 3$.

Then $\hat{T}(3,3) = K_{1,2} \cup K_{1,2}$ and by inspection T is disconnected.

CASE 2 : $(m-m_1) + (n-n_1) > 3$.

Then G_1 contains two vertices of one colour, G_2 contains two vertices of the opposite colour and hence T has a four cycle.

In either case $m_1+n_1 \geqslant 3$ implies T cannot be a tree. Hence $m_1+n_1 < 3$ for which either $m_1=n_1 = 1$ in which case $G_1 \equiv K_{1,1}$, or $m_1=o$, $n_1 = 1$ and therefore $G_1 \equiv K_{o,1}$.

As a direct consequence of the previous lemmas we now have:

2.4 THEOREM : A tree $T(m,n)$ is reducible if and only if it has a vertex of degree m or n or non-adjacent vertices of opposite colours m,n and degrees n-1 and m-1 respectively.

PROOF : Clearly sufficiency follows from LEMMA 2.2.

Conversely if T is reducible then by LEMMA 2.3 1.) $\hat{T} = K_{1,1} \cup G(m-1,n-1)$ or 2.) $\hat{T} = K_{o,1} \cup G(m,n-1)$.

If T is as in 2.), there exists a pair of non-adjacent vertices of opposite colours of degrees m-1, and n-1.

If T is as in 2.), there exists a vertex of degree m.

By inspection of all trees with up to six vertices (in a list of trees through 10 points, see for example Harary [12]) we have the

following corollary :

2.5 COROLLARY : All trees with less than seven vertices are reducible.

As examples of irreducible trees, we give all irreducible trees with seven vertices :

T_1 = :

T_2 = :

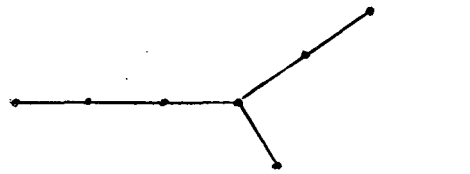


fig. 2.1

We can use THEOREM 2.4 to construct the general form of all reducible trees :

TYPE 1 : non-adjacent vertices of colours m,n and degrees n-1, m-1.

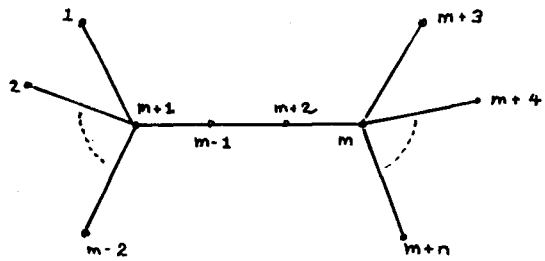


fig. 2.2

TYPE 2 : vertex of degree m and colour n

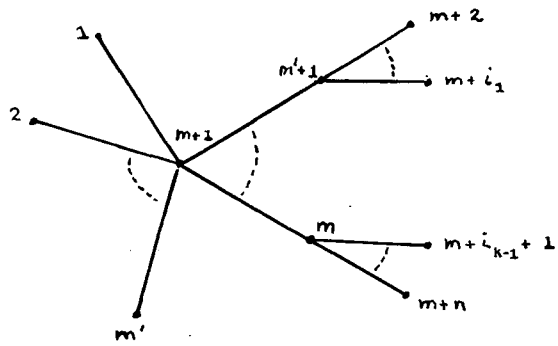


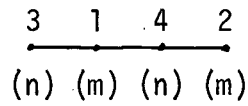
fig. 2.3

That these diagrams do in fact characterize all trees with the particular properties specified as TYPE1 or TYPE 2 can be exhibited

by examining the minimal trees with these properties together with the allowable extensions of the graph, namely those preseving the property.

For TYPE 1 trees, the minimal tree is:

$T_1 :=$



the allowable extension is the introduction of new vertices adjacent to vertices 2 or 3.

Introduction of vertices adjacent to 1 or 4 results in a TYPE 2 tree, while any vertex not adjacent to any of 1,2,3 or 4 destroys the characteristic property of TYPE 1 trees.

For TYPE 2 trees, the minimal tree is a single vertex V and the allowable extensions are any which result in the distance of the new vertex from V being less than or equal to 2. Thus all vertices of colour opposite to V are adjacent to V.

Using the bipartite matrix of the TYPE 1 and TYPE 2 trees given in the previous diagram it can now be shown that the reducible trees are in fact completely reducible.

<u>2.6 THEOREM</u> : If T(m,n) is a reducible tree then T(m,n) is completely reducible.

PROOF : The bipartite matrix for a TYPE 1 tree is given by :

$$\begin{array}{c} \\ 1 \\ 2 \\ \\ \\ \\ m-1 \\ m \end{array} \begin{array}{c} \begin{array}{ccccccc} m+1 & m+2 & & & & & \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 0 & \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & 1 & \cdot & \cdot & \cdot & 1 \end{bmatrix} \end{array}$$

and for a TYPE 2 tree is given by :

$$
\begin{array}{c}
\\ 1 \\ \\ m' \\ m'+1 \\ m'+2 \\ \\ m
\end{array}
\begin{array}{c}
\begin{array}{ccccccccccccc}
m+1, & m+2 & & m+i_1, & m+i_1+1 & & m+i_2, & m+i_2+1 & & m+i_{K-1}, & m+i_{K-1}+1 & & m+n
\end{array} \\
\begin{bmatrix}
1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
1 & 0 & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\
1 & 1 & \cdots & 1 & 0 & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & 0 \\
1 & 0 & \cdots & 0 & 1 & \cdots & 1 & 0 & \cdots & \cdot & \cdot & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
1 & 0 & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & 0 & 1 & \cdots & 1
\end{bmatrix}
\end{array}
$$

where the labelling corresponds to figures 2.2 and 2.3. Now by definition since T is reducible, its bipartite complement has more than one component. The bipartite matrices for $\hat{T}$ if T is of TYPE 1 are :

$$
B_1 = [1] \text{ and } B_2 = \begin{bmatrix}
1 & \cdots & \cdots & 1 \\
\vdots & & & \\
1 & & & \\
0 & 1 & \cdots & 1
\end{bmatrix}
$$

Now T is completely reducible if and only if G_1 and G_2 are completely reducible where $\hat{T} = G_1 U G_2$ and the bipartite matrix of G_1 is B_1, of G_2 is B_2. But B_1 is the bipartite matrix of $K_{1,1}$ hence G_1 is completely reducible. The bipartite complement of G_2 is

$$
\begin{bmatrix}
0 & \cdots & \cdots & 0 \\
\vdots & & & \vdots \\
0 & & & \vdots \\
1 & 0 & \cdots & 0
\end{bmatrix}
$$

hence $\hat{G}_2 = K_{1,1} \cup (m-1) K_{1,o} \cup (n-1) K_{o,1}$ and therefore G_2 is completely reducible.

If T is of TYPE 2 then $\hat{T} = K_{o,1} \cup G_2$ where the bipartite matrix of G_2 is given by :

$$\begin{bmatrix} 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ \vdots & & & & & & & & \\ 0 & \cdots & 0 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ 1 & \cdots & 1 & 0 & \cdots & 0 & 1 & \cdots & 1 \\ \vdots & & & & & & & & \\ 1 & \cdots & & & & & & 0 & \cdots & 0 \end{bmatrix}$$

and the bipartite matrix of $\hat{G}_2$ is hence

$$\begin{bmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & & & & & & & \\ 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ \vdots & & & & & & & & \\ 0 & \cdots & & & & & & 1 & \cdots & 1 \end{bmatrix}$$

and therefore $G_2 = K_{1i_1} \cup K_{1i_2} \cup \ldots K_{1i_{K-1}}$ and is completely reducible.

The particular nature of the bipartite matrices for reducible trees together with the result of THEOREM 2.4 allows the construction of an algorithm for isomorphism determination between reducible trees. The algorithm re-orders the labels of the vertices of a reducible tree as given by its bipartite matrix to correspond to the labelling of figure 2.2 or 2.3. The rows of the bipartite matrix are re-arranged in order of increasing degree, and the columns are then re-arranged until the matrix is in one of the forms as given on pages 17 and 18. Two reducible trees are then isomorphic if and only if they have identical labellings, and hence identical bipartite matrices.

We now state the algorithm. Note that if a tree is not reducible then the algorithm stops at step 3. The algorithm is effective for determining if an isomorphism exists between two trees provided at least one of them is reducible.

2.7 ALGORITHM :

1) Calculate all row and column sums of the bipartite matrix $B^{(mxn)}$. Assume $m < n$.

2) If $\sum_{j=1}^{n} b_{ij} = n$ or $\sum_{i=1}^{m} b_{ij} = m$ go to 4.

3) If $\sum_{j=1}^{n} b_{ij} = n-1$, $b_{iK} = 0$ and $\sum_{i=1}^{m} b_{iK} = m-1$ go to 4, else STOP B not reducible.

4) Re-order rows of B so that $\sum_{j=1}^{n} b_{i,j} \leqslant \sum_{j=1}^{n} b_{i+K,j}$ where $K > 1$.

5) set I=J=1

6) scan row I until $a_{Ij} = 1$. Interchange columns j and J. Set J=J+1.

7) If $J \leqslant n$ go to 6, else set I=I+1

8) If $I \leqslant m$ go to 6.

9) Repeat steps 1 through 8 for the bipartite matrix $C^{(mxn)}$ of the second graph.

10) If the two resulting matrices are identical then the two graphs are isomorphic.

2.2 REDUCIBLE GRAPHS

Unfortunately, reducible bipartite graphs in general are not so easily characterized as in the case of trees. Later results will show that the conditions of THEOREM 2.4 are sufficient although the following example illustrates that they are not necessary.

Let G : =

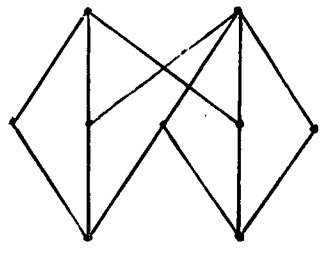


fig. 2.4

The bipartite complement is given by :

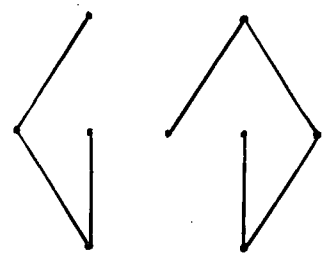


fig. 2.5

Now since the components of $\hat{G}$ are both trees with less than seven vertices we can apply COROLLARY 2.5 and hence G is completely reducible. However G satisfies none of the conditions which characterized reducible trees.

2.8 LEMMA : If G(m,n) has a reducible spanning tree then G(m,n) is reducible.

PROOF : If G has a reducible spanning tree then G contains vertices of degree m or n or non-adjacent vertices of opposite colours m,n and degrees n-1 and m-1 respectively and hence by LEMMA 2.1 or LEMMA 2.2 G is reducible.

That not all reducible graphs have reducible spanning trees was illustrated by example. Necessary and sufficient conditions for the existence of a reducible spanning tree in G are given by :

2.9 THEOREM : A connected bipartite graph G(m,n) has a reducible spanning tree if and only if there exist non-adjacent vertices of opposite colours m,n and degrees n-1 and m-1 respectively or there exists a vertex of degree m or n.

PROOF : If G(m,n) has a reducible spanning tree then from LEMMA 2.8 G has vertices which are of degree m or n or are non-adjacent, of opposite colour and degree m-1 and n-1.

Conversely suppose G has non-adjacent vertices v_1, v_{m+1} of degree n-1, m-1 and colour m,n. Then v_1 is adjacent to all vertices of colour n except v_{m+1}, v_{m+1} is adjacent to all vertices of colour m

except v_1. Since G is assumed to be connected there exists a path from v_1 to v_{m+1} whose length is 3. Otherwise there would exist a vertex of opposite colour to v_1 not adjacent to v_1, other than v_{m+1} and hence degree $v_1 < n-1$. Hence G has a reducible spanning tree of TYPE 1 consisting of all $(u,v_1) \varepsilon E(G)$, all $(w,v_{m+1}) \varepsilon E(G)$ and an edge (u,w) where u is adjacent to v_1 and w is adjacent to v_{m+1}.

If G has a vertex v of degree m, the spanning tree is given by the following:

1) set of edges $(v,u_i) \varepsilon E(G)$ together with

2) set of all edges such that $(u_i,w_j) \varepsilon E(T)$ implies w_j is adjacent to exactly one vertex $u_i \varepsilon V(T)$.

3) $V(T) = \{v, u_1,\ldots u_m, w_1,\ldots w_{n-1}\}$

By construction T is a reducible graph of TYPE 2 spanning V(G).

The spanning tree of a reducible graph is not necessarily unique, nor are all spanning trees of a reducible graph reducible as the following examples show.

Let G be the following reducible graph:

G: =

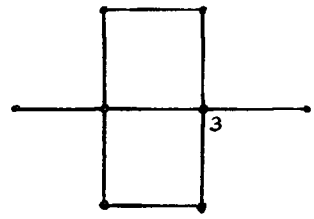
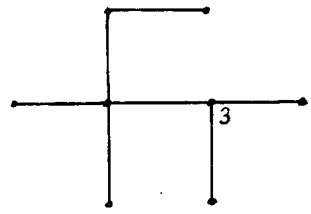


fig. 2.6

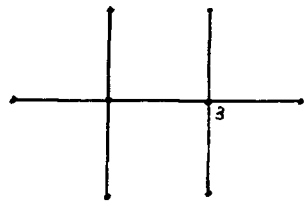
By LEMMA 2.1, since degree (3) = m = 4, G is reducible and by THEOREM 2.0 has a reducible spanning tree T_1 :

T_1: =



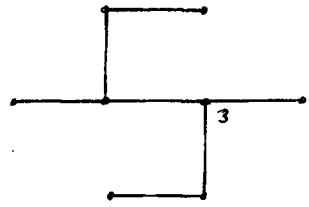
T_1 is a reducible spanning tree of TYPE 2. However T_2 is also a reducible spanning tree of TYPE 2 not isomorphic to T_1:

$T_2 :=$



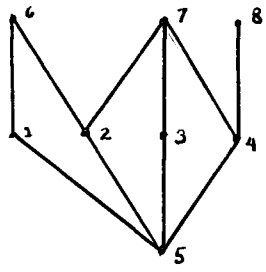
Finally T_3 is also a spanning tree but does not satisfy THEOREM 2.4 and hence is not reducible :

$T_3 :=$

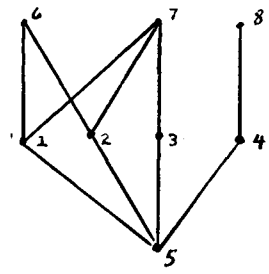


Two non-isomorphic graphs may have isomorphic reducible spanning trees as the following example illustrates :

$G_1 :=$



$G_2 :=$



$T :=$

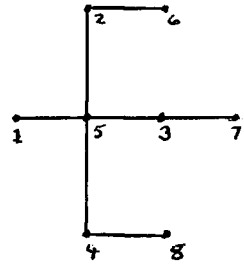


fig. 2.7

In addition to isomorphic spanning trees, G_1 and G_2 also have vertices with the same degrees, and same number of edges. A

previous example, fig. 1.2, gives two non-isomorphic graphs, with isomorphic reducible spanning trees, vertices with the same degrees and eccentricities, the eccentricity being the maximum distance of a vertex from any other in the graph.

The eccentricity of the vertex of a graph is introduced in order to relate some results concerning the radius and diameter of a reducible graph. The radius $r(G)$ is defined to be the minimum eccentricity, the diameter $d(G)$, the maximum eccentricity of the vertices.

2.10 THEOREM : All bipartite graphs of radius less than 3 are reducible.

PROOF : Let v be a point of minimum eccentricity in a bipartite graph $G(m,n)$ where $r(G) < 3$. Then all vertices are a distance at most 2 from v and hence all vertices of colour opposite to v are adjacent to v. Hence G contains a point of degree m, and is therefore reducible by LEMMA 2.1.

As a consequence, we have the following

2.11 COROLLARY : If G is a graph of radius less than 3, then the diameter of G is less than 5.

PROOF : $r(G) \leqslant 2$ implies G is reducible and has a vertex of degree m and hence a reducible spanning tree T. Now by inspection of figures 2.2 and 2.3 all reducible spanning trees have diameter $d(T) \leqslant 4$ and since $d(G) \leqslant d(T)$ for any graph, $d(G) \leqslant 4$.

2.12 LEMMA : Every circuit of length 2p is irreducible for $p \geqslant 4$.

PROOF : if p=2, then the circuit is of length 4, isomorphic to $K_{2,2}$ and consequently reducible. If p=3, the bipartite complement of the circuit of length 6 is isomorphic to $3K_{1,1}$ and hence reducible. For $p \geqslant 4$ the proof is a consequence of a theorem by Posa [20] :

Let G have $p \geqslant 3$ points. If for every n, $1 \leqslant n \leqslant (p-1)/2$, the

number of points of degree not exceeding n is less than n and if, for odd p, the number of points of degree $\frac{(p-1)}{2}$ does not exceed $(p-1)/2$ then G is Hamiltonian.

The bipartite complement of a circuit of length 2p has 2p vertices each of degree p-2. Hence the bipartite complement of the circuit with $p \geqslant 4$ satisfies Posa's condition, is Hamiltonian and consequently connected.

We use this result to establish an upper bound on the radius of a reducible graph.

<u>2.13 THEOREM</u> : All bipartite graphs of radius greater than 3 are irreducible.

PROOF : Let G(m,n) be a bipartite graph with $r(G)>3$.

If G is a tree then G is not reducible since by inspection of figures 2.1 and 2.2, the radius of all reducible trees is $\leqslant 3$.

If G has no circuits of length $\geqslant 8$ then G has a vertex v with $e(v)\geqslant 8$. Hence there exists two vertices v,w with a path P between them of length not less than 8. If the length of the path l is odd then P together with the edge (v,w) forms a circuit of length $\geqslant 2p$ where $l = 2p-1$ and by LEMMA 2.12 is irreducible. If the path length l is even, let w' be a vertex on the path adjacent to w and consider the graph P together with the edge (v,w'). This graph contains a circuit of length 2(p-1) and by LEMMA 2.12 is irreducible. The vertex w is adjacent to every vertex of opposite colour in the bipartite complement of the circuit.

Hence G is not reducible since any vertex not on the path P is adjacent to at least one vertex in $\hat{G}$.

If G has a single circuit C of length $\geqslant 8$, then again applying LEMMA 2.12 C is irreducible, and all vertices not on C must be adjacent to at least one vertex in C, and hence G is not reducible.

If G has more than one circuit of length ≥ 8, let C_1 and C_2 be any two such circuits. Then there exist vertices $v \in C_1$ and $w \in C_2$ such that (v,w) is not an edge of G. Hence $(v,w) \in E(\hat{G})$ and since $\hat{C}_1$ and $\hat{C}_2$ are connected subgraphs by LEMMA 2.12, G is again connected and hence irreducible.

Let $A=A(G)$ be the adjacency matrix of the bipartite graph G. A well known property of the adjacency matrix is :

<u>2.14 THEOREM</u> [2, p.110] : The i,j entry of A^n is the number of walks of length n from v_i to v_j.

From this result it is easily established that the radius of G is the smallest value of n such that $A + A^2 + \ldots + A^n$ contains a row which is strictly positive, while the diameter of G is the smallest value of n such that $A + A^2 + \ldots + A^n$ is a strictly positive matrix.

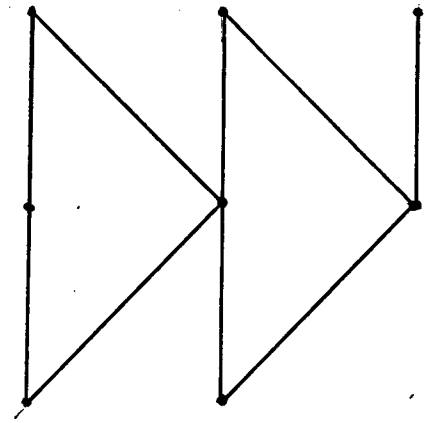
Using these results and THEOREM 2.10 yields the following corollary :

<u>2.15 COROLLARY</u> : The diameter of any connected reducible bipartite graph is less than seven.

PROOF : If G is reducible then by THEOREM 2.10 $r(G) \leq 3$. Hence there exists a row of $X = A + A^2 + A^3$ which is strictly positive, where A is the adjacency matrix of G. Now since A is symmetric and no row contains all zeros, X is symmetric and no row contains all zeros, and therefore $XX^T = X^2$ is a strictly positive matrix. But $X^2 = [A + A^2 + A^3]^2 = A^2 + 2A^3 + 3A^4 + 2A^5 + A^6 > 0$ implies $A + A^2 + A^3 + A^4 + A^5 + A^6 > 0$ and hence $d(G) \leq 6$.

Summarizing these results, the reducible graphs are included in the set of connected graphs with radius up to three and diameter up to six. THEOREM 2.10 has shown that all the graphs with radius less than three are reducible. Two examples are now given of graphs of radius three and diameter less than six which are and are not reducible.

G:=



$\hat{G}$:=

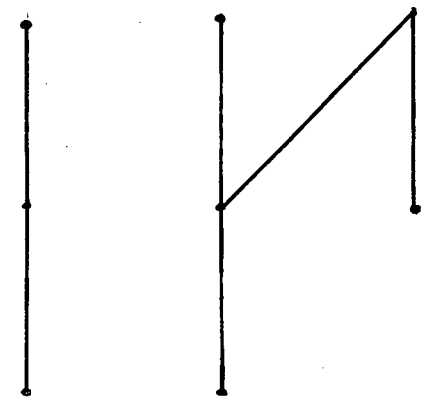


fig. 2.8

G is a reducible graph with diameter 5 and radius 3.

The following graph G has diameter and radius equal to 3, yet is not reducible :

G:=

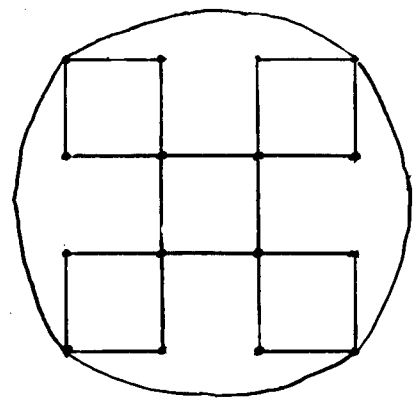
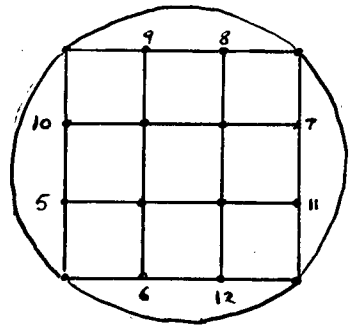
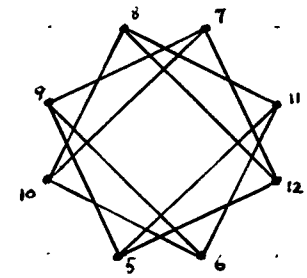


fig. 2.9

The bipartite complement of G is given by the union of the two subgraphs below as labelled.





Hence G is connected and consequently G is irreducible.

The examples in figures 2.8 and 2.9 serve to illustrate that reducible graphs are not completely characterized by their radius and diameter.

CHAPTER 3

An Isomorphism Testing Algorithm for Completely Reducible Graphs

3.1 COMPLETELY REDUCIBLE GRAPHS

THEOREM 1.10 gives necessary and sufficient conditions for two completely reducible bipartite graphs to be isomorphic. Moreover, THEOREM 2.6 provides a characterization of completely reducible trees. Before implementing THEOREM 1.10 in an isomorphism testing algorithm (section 3.2), we shall discuss other classes of completely reducible bipartite graphs.

3.1 THEOREM : The graph $K_{m,n} \times K_{1,1}$ is completely reducible.

PROOF : Since the vertices of the graph $G \equiv K_{m,n} \times K_{1,1}$ can be partitioned into two sets of m+n vertices such that $K_{m,n}$ is defined as a subgraph on each set, the bipartite matrix is given by $\begin{bmatrix} X_1 & J \\ J^T & X_2 \end{bmatrix}$ where J is an mxn matrix with all entries one. Hence G is reducible since its bipartite complement is given by $\begin{bmatrix} \overline{X}_1 & 0 \\ 0 & \overline{X}_2 \end{bmatrix}$. Now by the definition of cartesian product of graphs, there exists a one-one correspondence between the vertices of each subgraph $K_{m,n}$. Hence each vertex in one subgraph $K_{m,n}$ is adjacent to exactly one vertex in the other subgraph $K_{m,n}$. Consequently, the matrices X_1 and X_2 have exactly one non-zero entry in each row and column. Interpreting $\overline{X}_1$ and $\overline{X}_2$ as the bipartite matrices of two graphs G_1, G_2, their bipartite complements have bipartite matrices X_1 and X_2 and therefore $\hat{G}_1 \equiv mK_{1,1}$ and $\hat{G}_2 \equiv nK_{1,1}$ and thus G is completely reducible.

Let G be a graph with disjoint subgraphs $K_{m_1n_1}$ and $K_{m_2n_2}$ such that if $(v_1,v_2) \in E(G)$ for $v_1 \in V(K_{m_1n_1})$, $v_2 \in V(K_{m_2n_2})$, then (v_1,v_i)

and $(v_j, v_2) \notin E(G)$ for $j \neq 1$, $i \neq 2$.

Assume $m_1 \leqslant n_1$, $m_2 \leqslant n_2$ and $m_1 \leqslant m_2$. Then the bipartite matrix of the graph G so constructed has the form :

$$\begin{bmatrix} X_1^{(m_1 x n_2)} & J^{(m_1 x n_1)} \\ J^{(m_2 x n_2)} & X_2^{(m_2 x n_1)} \end{bmatrix}$$

The bipartite complement $\hat{G} \equiv G_1 U G_2$ where $\hat{G}_1$ has bipartite matrix X_1, $\hat{G}_2$ has bipartite matrix X_2. Now by construction each row and column of X_1 and X_2 has none or exactly one non-zero element. Hence G_1 and G_2 are completely reducible and therefore so is G.

These results are generalized further in the following THEOREM:

<u>3.2 THEOREM</u> : Let G be a bipartite graph whose vertices can be partitioned into two sets such that each set has a complete bipartite subgraph of G defined on it. Then G is completely reducible if and only if the cut-set of G consisting of all edges between the two sets of points is completely reducible.

PROOF : Since all such graphs have a bipartite matrix as previously defined to be of the form :

$$\begin{bmatrix} X_1^{(m_1 x n_2)} & J^{(m_1 x n_1)} \\ J^{(m_2 x n_2)} & X_2^{(m_2 x n_1)} \end{bmatrix}$$, the

particular cut-set referred to corresponds to the bipartite matrix

$$\begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}$$

and by definition G is completely reducible if and only if $\hat{G} = G_1 U G_2$ is completely reducible, if and only if $G_1 U G_2$ is completely reducible. But $G_1 U G_2$ corresponds to the cut-set of G.

The existence of completely reducible graphs not yet classified is illustrated by a previous example, the graph G_2 in fig. 2.7, whose bipartite complement is given by :

$\hat{G}$:=

$\hat{G}$ contains a TYPE 1 tree as a component together with an isolated vertex. Hence G is completely reducible.

3.2 GRAPH REDUCIBILITY ALGORITHM

Let G(m,n) be a bipartite graph with m+n vertices. The purpose of the algorithm is to determine whether G is completely reducible, and if so, to construct a px3 array, $p \leq m+n$, called the characteristic array of the graph.

Let (s_i, m_i, n_i) be the ith row vector of the characteristic array C. Let p be the row pointer of C, and s be the stage parameter. Assume G is connected, B is its bipartite matrix and $B_1^{(s)}, B_2^{(s)}, \ldots, B_{ks}^{(s)}$ are the matrix blocks corresponding to the bipartite matrix of each component to be treated at stage s. Initially set s to 1 j to 1 and $B = B_1^{(o)}$, $\hat{B} = B_1^{(1)} \oplus B_2^{(1)} \oplus \ldots \oplus B_{K'}^{(1)}$ where $\oplus$ devotes the direct sum of matrices.

3.3 ALGORITHM

1) Calculate $\hat{B}^{(s-1)} = B_1^{(s)} + B_2^{(s)} + \ldots + B_{Ks}^{(s)}$. If $K_s=1$ and there are no rows or columns containing only zero entries, STOP, B not reducible.

2) If $\hat{B}^{(s-1)}$ has a column with entries all zeros save (s,1,o) in the pth row of C, increment p by one and let $\hat{B}^{(s-1)}$ be the matrix

obtained after deletion of that column.

3) If $\hat{B}^{(s-1)}$ has a row with entries all zeros save (s,o,1) in the pth row of C, increment p by one and let $\hat{B}^{(s-1)}$ be the matrix obtained after deletion of that row.

4) Repeat steps 2 and 3 until $\hat{B}^{(s-1)}$ contains no rows or columns having only zero entries.

5) If $B_j^{(s)} = J^{(m_j x n_j)}$, save (s,m_j,n_j) in the pth row of C, increment p by 1, delete $B_j^{(s)}$.

6) Increment j by 1 and if $j \leqslant K_s$ go to step 5.

7) Increment s by 1 and go to step 1, unless there are no blocks left to treat.

Each row of the characteristic array corresponds to a complete bipartite subgraph obtained during the sth stage of reduction. The procedure involves treating each component obtained by taking the bipartite complement of the graph as a graph which is either complete bipartite or must be further reduced. Each stage of reduction corresponds to treating all sub-blocks not yet completely reduced. If the algorithm is successfully terminated it is possible to construct a unique bipartite matrix and the colour group for the graph.

We first illustrate the operation of the algorithm in an example. Let G be the graph G_2 given in figure 2.7. Its bipartite matrix is given by :

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$\hat{B}^{(o)} = \hat{B}$ is given by

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

and contains a column of all zeros. Hence the first entry into the characteristic array is (1,0,1). The modified matrix $\hat{B}^{(o)}$ is then given by :

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

This matrix satisfies none of the conditions in steps 1,2, hence proceeding with step 5, the new matrix is $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ with blocks

$$B_1^{(2)} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B_2^{(2)} = [1]$$

Hence the second entry into the characteristic array is (2,1,1) corresponding to $K_{1,2}$ given by $B_2^{(2)}$. Finally $B_1^{(2)}$ is not complete bipartite, so again applying step 1 we have :

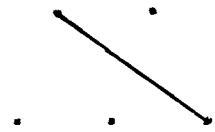
$$\hat{B}_1^{(2)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

followed by steps 2 and 3 which result in the three entries (3,0,1), (3,1,0) and (3,1,0) and finally the entry (3,1,1) corresponding to the block [1] to which $\hat{B}_1^{(2)}$ is reduced after deleting rows and columns of zeros.

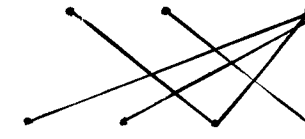
Hence the characteristic array is given by:

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 3 & 0 & 1 \\ 3 & 1 & 0 \\ 3 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix}$$

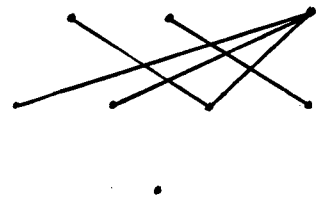
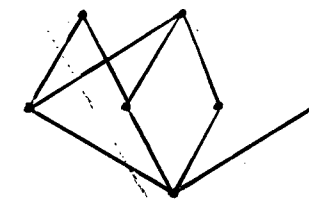
The original graph can be constructed from its characteristic array as follows. Let G_1 be the graph consisting of the union of the four complete bipartite graphs obtained at stage 3. Thus :

$G_1 :=$  $\hat{G}_1 :=$ 

Let G_2 be the union of G_1 and all complete bipartite graphs obtained at stage 2 :

$G_2 :=$  $\hat{G}_2 :=$ 

Let G_3 be the union of G_2 and all complete bipartite graphs obtained at stage 1 :

$G_3 :=$  $\hat{G}_3 :=$ 

Comparison of $\hat{G}_3$ with G_2 in figure 2.7 shows that $\hat{G}_3$ is indeed isomorphic to the original graph. The colour group of the graph is given by the entries in the array in the following manner :

Each row (s_i, m_i, n_i) corresponds, as indicated previously to a complete bipartite graph $K_{m_i n_i}$ whose colour group is $S_{m_i} + S_{n_i}$ unless one of m_i, n_i is zero in which case $K_{m_i n_i}$ corresponds to an isolated vertex whose colour group is S_1. If the characteristic array contains K identical rows then the colour group for those entries is given by

$S_K[S_{m_i} + S_{n_i}]$. The colour group of the graph is then given by the group sum of the groups represented by each row. Thus in the previous example $\Gamma^*(G_3)$ is $S_1 + [S_1 + S_1] + S_2\,[S_1] + S_1 + [S_1 + S_1] \equiv S_2$. Examination of figure 2.7 shows that an interchange of vertices 1 and 2 is the only automorphism of G.

Since the characteristic array is independent of the labelling of the vertices of G, the reduction of any completely reducible bipartite graph isomorphic to G must result in the same characteristic array up to a re-ordering of rows determined during the same stage s_i, provided $m \neq n$. If m=n, and two graphs $G_1(m,m)$ and $G_2(n,n)$ are isomorphic, and there exists P_1, P_2 such that $P_1^T B_1^T P_2 = B_2$ (see LEMMA 1.2), this is equivalent to the vertices in G_1 being coloured oppositely to those in G_2. Hence an interchange of columns 2 and 3 of the characteristic array of G_2 again yields a characteristic array identical to that of G_1 up to a re-ordering of the rows determined during each stage.

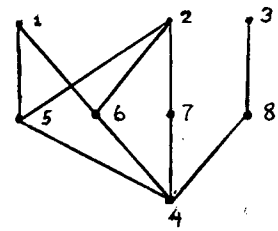
In order to use ALGORITHM 3.3 to construct a unique characteristic array for each completely reducible bipartite graph, the order of treatment of the blocks $B_i^{(m_i x n_i)}$ at each stage is to treat the block with minimum n_i among all blocks with minimum m_i among all blocks not yet treated. The rows of the characteristic array are then in ascending order by n_i, m_i and finally s_i.

The determination of isomorphism between completely reducible graphs is now just a test of whether the respective characteristic arrays are identical. The only remaining case to be considered is the possibility that with m=n, the equivalent vertex sets of two graphs under consideration are of opposite colour. If such is the case for two isomorphic graphs then $P_1^T B_1^T P_2 = B_2$ for some P_1, P_2 (see LEMMA 1.2). Rather than repeat the reduction procedure with

B_1^T as bipartite matrix rather than B_1, we need only interchange the second and third columns of the characteristic array of one of the graphs, and re-order all rows is ascending order as previously described.

As an example, we consider a re-labelling of the graph G_2 of figure 2.7 :

$G^1 :=$



Applying algorithm 3.3 as before yields (1,1,0),(2,1,1),(3,0,1), (3,0,1), (3,1,0). Finally the modified matrix is [1] hence the final entry is (3,1,1). The characteristic array is

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 3 & 0 & 1 \\ 3 & 0 & 1 \\ 3 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix}$$

Comparing this characteristic array with that of the original labelling of G, we note that it is not identical. However since m=n, an interchange of the last two columns and a re-ordering of the rows in ascending order as previously described gives :

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 3 & 0 & 1 \\ 3 & 1 & 0 \\ 3 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix}$$ which is identical

to the characteristic array for the original labelling.

Finally we observe that the algorithm yields only partial results for determining isomorphism between two arbitrary bipartite

graphs. The algorithm behaves thus :

1) If both G_1 and G_2 are completely reducible, then they are isomorphic if and only if they produce identical characteristic arrays.

2) If only one of G_1 and G_2 is completely reducible, then they are obviously not isomorphic.

3) If neither G_1 nor G_2 is completely reducible then the isomorphism problem remains unsolved.

3.3 TIMING CONSIDERATIONS

The number of operations required at each stage of the algorithm is examined in an effort to provide some estimate of its efficiency for comparison with other methods for determining graph isomorphism. Let G(m,n) be the bipartite graph under consideration. The nature and number of executions of each operation is :

STEP 1 : s bipartite complementations where s is the stage parameter.

STEP 2 : m summations of n numbers for comparison with zero.

STEP 3 : n summations of m numbers for comparison with zero.

STEP 4 : K_s tests for a complete bipartite subgraph, involving $m_i n_i$ comparisons with one, where $\sum_{i=1}^{K_s} m_i \leqslant m$, $\sum_{i=1}^{K_s} n_i \leqslant n$.

STEP 5 : K_s incrementations

STEP 6 : s incrementations

3.4 LEMMA : If $m = \sum_{i=1}^{k} Pi$ for $Pi \geqslant 1$ $(1 \leqslant i \leqslant k)$ then $(m-k+1)^2 + k-1 \geqslant \sum_{i=1}^{k} Pi$.

PROOF : For $k=2$, $m = p_1 + p_2$. Then $m^2 = p_1^2 + p_2^2 = 2p_1p_2$ and consequently $p_1^2 + p_2^2$ is maximized by maximizing $m^2 - 2p_1p_2$. Now the function $f(x) = x(m-x)$ is monotonically increasing for $0 \leqslant x \leqslant \frac{m}{2}$. Hence $p_1p_2 = p_1(m-p_1)$ is minimized for positive integer values of p_1 at $p_1 = 1$ and as a result $m^2 = 2p_1p_2 < (m-1)^2 + 1$.

For $k > 2$ assume without loss of generality that $p_1 \geqslant p_2 \geqslant \ldots \geqslant p_k$ where the p_i maximize $\sum_{i=1}^{k} p_i^2$. If $p_k > 1$ let $N = p_{k-1} + p_k$. Then by the above argument $\sum_{i=1}^{k-2} p_i^2 + (N-1) + 1 \geqslant \sum_{i=1}^{k} p_i^2$ and hence taking $p_k=1$, $p_{k-1}= n-1$ increases the value of p_i^2. Iterating $k-2$ times we obtain $p_k=p_{k-1} =\ldots= p_2=1$ and $p_1= m-k+1$ as required.

This lemma allows for the calculation of an upper bound on the total number of comparisons required in STEP 5 to reduce a graph. We assume that at each stage of the reduction of a graph there are $k=K+1$ blocks to be treated and we assume $m_i = n_i$ where the worst possible case occurs at each stage i, with the result that there is one block of size $m_i \times m_i$ and K blocks of size 1x1.

<u>3.5 THEOREM</u> : The number of comparisons required in the reduction of a complete bipartite matrix by ALGORITHM 3.3 is at most of order p^3 where p is the number of vertices.

PROOF : Under the assumption that at each stage i there is an $(m_{i-1} = K) \times (m_{i-1} - K)$ matrix to be treated whose bipartite complement consists of a block diagonal matrix with K blocks then by LEMMA 3.4 the number of comparisons required is less than or equal to $(m_i-K)^2$. Now if s stages are required until there exists a block with fewer K^2 elements to be tested, then the total number of comparisons required is $\sum_{i=1}^{s} (m_i - K)^2$. If m_i-1 is partitioned into parts m_i, and K ones then $m_{i-1} = m = iK$ with $m=m_0$.

$$\sum_{i=1}^{s} (m_i-K)^2 = \sum_{i=1}^{s} (m - iK)^2$$

$$= m^2 s - 2K(m) \frac{s(s+1)}{2} + \frac{s(s+1)(2s+1)}{6} K^2$$

Now $m_s < K$ implies $m < (s+1) K$, hence

$$Ks < m < (s+1) K$$

$$\text{or} \quad s < \frac{m}{K} < s+1$$

Therefore $$(m_i-K)^2 < \frac{m^3}{K} - m^2 \frac{(m+1)}{K} + \frac{mK}{6} \frac{(m+1)}{K} \frac{(2m+1)}{K}$$

$$= \frac{m^3}{3K} + O(m^2).$$

For non-square blocks at each stage, redefine m_i to be the maximum of m_i and n_i. Then $(m_i-K)(n_i-K) < (m_i-K)^2$ and the previous results are true for $m \neq n$.

Determination of the expected value of K is extremely difficult since it involves computations with unordered partitions [9]. The expected value of K, E(K), is given by:

$$E(K) = \sum_{i=1}^{m} i \frac{Pi(m)}{P(m)}$$

$$= \frac{1}{p(m)} \sum_{i=1}^{m} i\, p_i(m)$$

where p(m) denotes the number of unordered partitions of m; $p_i(m)$, the number of unordered partitions of m into i elements. The values for E(K) with $m \leq 10$ are given in table I.

TABLE I

m	p(m)	E(K)
1	1	1
2	2	1.5
3	3	2
4	5	2.4
5	7	2.86
6	11	3.18
7	15	3.60
8	22	3.72
9	30	3.67
10	42	4.62

The Number and Expected Value of the Unordered Partitions of the Numbers one to ten.

Since there exists only one partition of m into K parts such that $m_1 = m-K$, $m_2 = \ldots = m_K = 1$ the probability of achieving the upper bound given in THEOREM 3.5 is exceedingly small, being given by $\prod_{i=1}^{s} p(m_i)^{-1}$ if it is assumed that any partition is equally likely.

Under an alternative assumption, that at stage i the $m_i x n_i$ block has a bipartite complement with block diagonal form consisting of K equal blocks of size $\frac{m_i}{K}$, then the number of blocks to be dealt with at stage i is less than or equal to K^i. Hence after s stages, $s < \log_K(m)$, the number of comparisons required at STEP 5 of the algorithm is $< K(\frac{m}{K})^2 + K^2(\frac{m}{K2})^2 + \ldots K^s(\frac{m}{K^s})^2$

$$= m^2 \left(\frac{1}{K} + \frac{1}{K^2} + \ldots + \frac{1}{K^s}\right)$$

and hence is of order m^2.

These results compare favorably with the more general algorithm of Corneil who showed that its timing was usually of order p^2 to p^3 with an upper bound of p^5, p being the number of vertices of the graph [4].

SUMMARY

The value of any solution to the isomorphism problem is dependent upon the construction of a unique representation of a graph, independent of vertex or edge labelling, in a reasonably efficient manner. This thesis has provided such a representation for the completely reducible bipartite graphs, with the time required for computing such a representation being $O(n^3)$, where n is the order of the graph. In addition this representation has been shown to have the property that the colour group of the graph may be easily constructed from it.

Criteria for the existence of reducible bipartite graphs have also been established. In particular is the fact that all reducible graphs have radius less than four.

Although the method is a useful procedure only when at least one of two graphs in question is completely reducible, the importance of finding decomposition procedures for use as a technique in the construction of solutions to the isomorphism problem is clearly emphasized.

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