

THE PERFORMANCE OF ROBOT MANIPULATION

By

Cornelis Pieter van den Doel

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE

in

THE FACULTY OF GRADUATE STUDIES
DEPARTMENT OF COMPUTER SCIENCE

We accept this thesis as conforming
to the required standard

THE UNIVERSITY OF BRITISH COLUMBIA

March 1994

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Department of Computer Science

The University of British Columbia
Vancouver, Canada

Date April 28 1994

Abstract

A theory of the performance of robot manipulation is presented. The theory deals with general robotic manipulation, which is viewed as the physical interaction between a generalized manipulator and a generalized object. The generalized manipulator can be arbitrarily complex, i.e. it can consist of any number of robotic devices. Similarly, the generalized object represents the set of all the material bodies manipulated by the robot in the environment. The study of the performance of such systems is somewhat analogous to the study of the performance of algorithms and complexity theory in computer science. A digital computer produces certain desired changes in information, it transforms input into output, and complexity theory deals with the efficiency of this process. Similarly, a robotic device transforms the state of the objects it manipulates. The manipulation we are concerned with here is only different in that material objects are affected and the measure for the resources needed to perform certain tasks is not runtime or memory usage. How performance should then be measured instead is the subject of this thesis.

The theory regards the manipulation as a mapping from the space of all manipulator configurations to the space of all object configurations. By endowing these manifolds with Riemannian metrics we can quantify motion in the manifolds. Performance measures, which quantify how well the robot is executing its manipulation task, arise naturally as geometrical objects associated with the mapping and the metric structure of the manifolds.

Generalizations of known measures (agreeing with those previously defined for the simplest kinds of systems) arise naturally in this formalism and we construct several new types of measures. The new types of measures presented here are the effective inertia on

work space, the kinematic and dynamic anisotropy measures for redundant manipulators, non-linearity measures, and redundancy measures.

An extension of the theory broadens its domain of application to situations in which the relation between the robot and the object it is manipulating can not be given in explicit form, but by a set of constraint equations (possibly non-holonomic) that define it implicitly.

The main contribution of this work is the theory of manipulation, which provides a theoretical foundation for robot manipulation. Computational aspects of the theory are described briefly and the theory is applied to some simple planar manipulating devices, for which several of the measures have been constructed and optimized. These applications are given here for illustrative purpose.

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Acknowledgement

This work was done in collaboration with Dinesh K. Pai. I thank Alan Boulton for useful discussions and help with L^AT_EX.

Chapter 1

Introduction

What is the best way to swat a fly? To hammer a nail? To carry a heavy object? One of the goals of robotics is to understand and automate the manipulation of the physical world by computer. The agents that interact physically with material objects are the robot manipulators. For a given task (a desired change in the state of the objects) one can ask questions such as: “What configuration of the manipulator is optimal?”, “What is the best manipulator for this task?”, and “What class of tasks can be performed by a particular robot?”. The answers to these questions are critical to the design, selection, and programming of robots.

A theory of performance is presented, which assigns a numerical value (“performance measure”) to an elementary interaction between the manipulator and the object, within the context of a task. The questions mentioned above can then be translated into optimization and feasibility problems. This theory is a step towards a complexity theory of robots and robot tasks.

Performance measures are already widely used for design and posture optimization for robot arms. The theory presented here has the following novel features:

- A unification of measures proposed before in one theoretical framework, based on differential geometry.
- A “toolkit” to make your own performance measure suited for the specific manipulator and task at hand.

- The construction of promising new performance measures for redundancy and non-linearity.
- The measures can be used for arbitrarily complex manipulators and objects, unlike existing measures.

As a simple illustration of these ideas, consider a robot arm with seven revolute joints and a gripper (end-effector) attached to the end of the arm. The position and orientation of the end-effector has six degrees of freedom, whereas the arm has seven. Such a manipulator is called a redundant manipulator. See for example [Nak91]. Therefore, if the task consists of grasping an object, there are infinitely many postures to do this. If we have a local performance measure, i.e. a number μ associated with every posture (in the context of a specific task), we can resolve this redundancy by finding the posture with optimal μ .

From a design perspective we can find the optimal design for a set of tasks, by selecting the design with the largest average value of some relevant local performance measure. If this involves a continuous infinity of tasks that the manipulator is expected to perform, we obtain such a design measure by the integration of a local measure over some finite region in space.

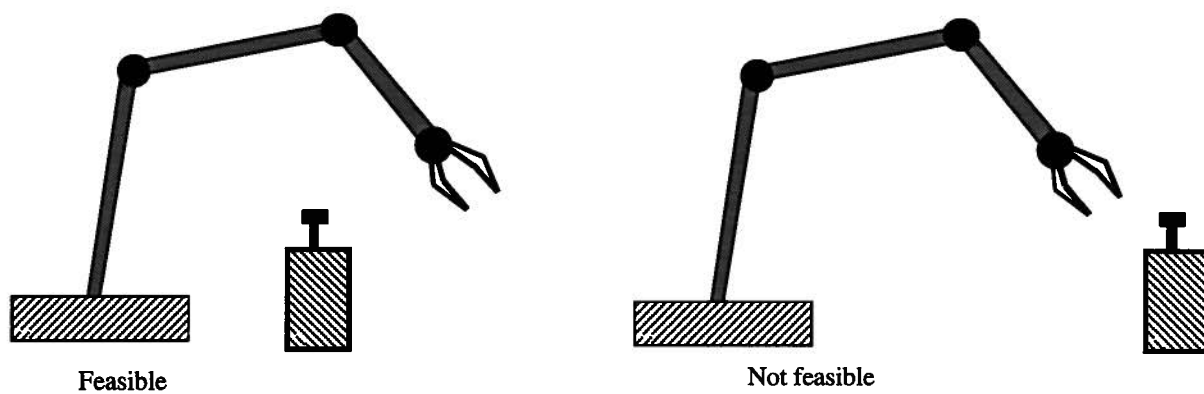


Figure 1.1: Feasible and not feasible tasks

Finally, for a given task it is often possible to determine if the manipulator can perform it by examining a performance measure. For example, suppose the task is to pull a nail from a wooden structure with a robot arm. Depending on the location of the nail with respect to the base of the arm, the arm may or may not be strong enough to perform the task. One of the performance measures we shall investigate later can be interpreted as “the minimal sum of the squares of the joint torques needed to apply a (generalized) force with the end-effector”, μ for short. A necessary condition on the feasibility of such a task is that $\mu \leq \mu_{\max}$, where $\mu_{\max}$ is the sum of the squares of the maximum joint torques that the motors can provide. So μ acts as a complexity measure of the manipulation in this example. In figure 1.1 we depict two situations. In the leftmost picture there exists a posture that grabs the nail and has $\mu \leq \mu_{\max}$, but in the rightmost picture all postures have $\mu > \mu_{\max}$ and the task is not feasible.

A complete task will consist of a sequence of elementary tasks. A task planner could utilize local performance measures by examining the feasibility of these subtasks in advance.

In this thesis we will deal with the construction of local performance measures for various tasks, for generic robotic devices.

Several local measures have been proposed in the past. Yoshikawa [Yos85] proposed the scalar $\sqrt{\det \mathbf{J}\mathbf{J}^T}$, where $\mathbf{J}$ is the Jacobian matrix of the manipulator. This measures how much the end-effector moves for a given infinitesimal movement of joint angles, averaged over all directions. It is the generalization to redundant manipulators of the determinant of the Jacobian. The related matrix $(\mathbf{J}\mathbf{J}^T)^{-1}$, when sandwiched between the work space direction vector and its transpose, measures the analogous manipulability in a particular direction. The inverse, $\mathbf{J}\mathbf{J}^T$, then measures “pushability” in a given direction [Chi88].

From the singular value decomposition representation [KL80, Nak91] some closely related measures can be derived. Salisbury and Craig [SC82] proposed to use the condition number of $\mathbf{J}$, which is the ratio of the largest and smallest singular values of $\mathbf{J}$. Klein and Blaho [KB87] use the minimum singular value as a measure. The distance of the joint angles from their central positions was used by Liegeois [Lie77]. Asada [Asa83] computed the effective inertia matrix for a non-redundant manipulator and proposed to minimize the inhomogeneity of the moment of inertia.

Angeles and Ma [MA90, MA93] introduced the dynamical conditioning index, which is a measure for the anisotropy of the moment of inertia of the arm, which was used to design isotropic manipulators. The kinematic isotropy was investigated in [ALC92].

All the measures mentioned above arise naturally in the framework presented here and several new measures are introduced. Care has to be taken to ensure that a constructed measure corresponds to a physical property of the device and is not just a mathematical construct. This is achieved here by imposing invariance under general coordinate transformations in configuration space, i.e. we consider measures that do not depend on the actual coordinates (such as the joint angles for revolute joint arms for example) used to describe the configuration of the manipulator. This criterion is sufficient, but not necessary in general, as the coordinates used usually have a direct physical meaning.

The approach taken here is to define various metrics on the configuration space, i.e. to define a “distance” between configurations, by a metric tensor. From the forward kinematics of the system we then construct an “induced” metric tensor on the work space, which has its own metric structure. This permits the use of standard differential geometry (see, for example, [DP90]) to construct geometrical tensor fields which are interpreted and used as performance measures.

The physical interpretation of these measures will depend on the initial choice of the metric on configuration space. For example, the Euclidean metric leads to kinematic

measures, while using the inertia matrix as the metric tensor leads to dynamic measures.

A mathematical formulation of these ideas are presented in chapter 2. Chapter 3 deals with an extension of the theory to incorporate systems with constraints (such as closed loop kinematic chains, for example). Chapter 4 presents applications of the theory to the planar two and three link manipulators, for which we compute and optimize some of the measures. In chapter 5 we illustrate the use of performance measures for constrained systems for a planar closed loop linkage. Conclusions and directions for future work are presented in chapter 6. In appendix A some computational issues regarding the measure are discussed.

Chapter 2

The Theory of Manipulation

2.1 Model of Robot Manipulation

We model the interaction between a generalized manipulator (any collection of robotic manipulating devices) and a generalized object (the collection of material bodies that are manipulated) as follows. The set of all possible manipulator configurations is described by the *configuration space*, which we denote by $\mathcal{C}$, and its dimension is n . A point in $\mathcal{C}$ represents a configuration of the manipulator, and all points in $\mathcal{C}$ represent different configurations. Similarly, all object configurations are described by the *work space*, which we denote by $\mathcal{W}$. We denote its dimension by m . A task can now be seen as a motion from one point in $\mathcal{W}$ (the present state of affairs) to another point. This motion is to be achieved by the manipulator, and we describe the coupling between the two by a mapping

$$\kappa : \mathcal{C} \longrightarrow \mathcal{W}$$

which associates a point in $\mathcal{W}$ with every point in $\mathcal{C}$. For typical robot arms with a manipulating device attached to the end of the arm, this mapping is the forward kinematics of the system. See figure 2.2.

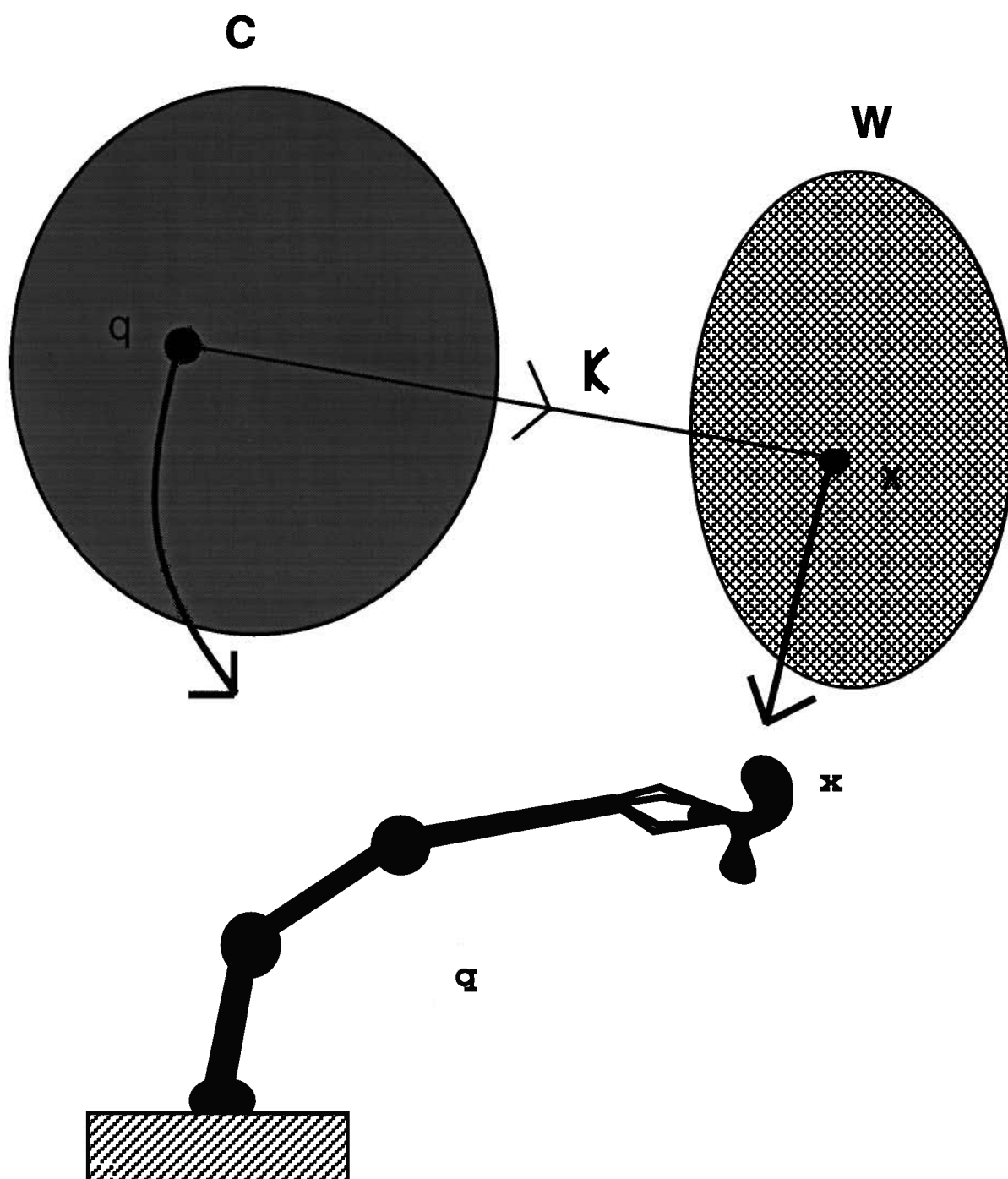


Figure 2.2: Relation between robot configurations and manifolds

In this section we shall assume that κ is given in explicit form. The generalization to constrained systems, where κ is defined implicitly, will be deferred to the next chapter, mainly because this is technically a little more involved, although there is no conceptual difference.

With $\mathcal{C}$ and $\mathcal{W}$ we associate 1-to-1 mappings $\mathcal{C} \rightarrow \mathbb{R}^n$ and $\mathcal{W} \rightarrow \mathbb{R}^m$, which define charts or coordinate systems on the manifolds. The coordinate system on a manifold is considered here as an arbitrary labeling of the points in the manifold. A point in $\mathcal{C}$ is denoted by the coordinates

$$\mathbf{q} = \begin{pmatrix} q^1 \\ \vdots \\ q^n \end{pmatrix}$$

and a point in $\mathcal{W}$ by

$$\mathbf{x} = \begin{pmatrix} x^1 \\ \vdots \\ x^m \end{pmatrix}.$$

The preimage of a point $\mathbf{x}$ in $\mathcal{W}$ is the self-motion manifold $\mathcal{N}_{\mathbf{x}} \subset \mathcal{C}$, which is the set of all configurations that map to this point in $\mathcal{W}$. The tangent space to $\mathcal{N}$ at $\mathbf{q}$ is the null-space of the Jacobian of the mapping κ . Where it is clear which point $\mathbf{x}$ is being referred to, we will drop the subscript $\mathbf{x}$. The self-motion manifolds are important for constructing redundancy measures, as discussed below.

To quantify displacements in the manifolds, we need to define a distance. As the manifolds are non-Euclidean in general, we define a vector norm on the tangent bundles $T\mathcal{C}$ and $T\mathcal{W}$. This corresponds intuitively to defining the length of infinitely small line segments everywhere in the manifolds. These infinitesimal line segments expand to

finite vectors in the tangent space associated with the manifold at the location of the infinitesimal line segment. Distance on $\mathcal{C}$ represents the amount of “work” the manipulator has to do for the corresponding motion. It can be defined in many way, which depends on the application at hand. Distance on $\mathcal{W}$ represents how much is achieved with the corresponding movement.

For example, consider the configuration space of a planar two link arm, with two revolute joints. The configuration space $\mathcal{C}$ is isomorph to the two-torus T^2 , or to a section of it if there are limits on the joint angles. A natural coordinate system is provided by the two joint angles, i.e. $\mathbf{q} = (q^1, q^2)^T$.

An example of a work space is the space of all configurations of a rigid body, $SE(3) \sim \mathbb{R}^3 \times SO(3)$. However, the object that is manipulated may itself be some mechanical device, such as a mechanical arm.

The performance can now be described by an “induced” distance on $\mathcal{W}$, that describes how much the manipulator has to do in order to achieve a given infinitesimal displacement in $\mathcal{W}$. Taking the example of a robot arm with an end-effector attached to the tip, the distance on $\mathcal{C}$ measures movements of the arm, the distance on $\mathcal{W}$ measures movements of the end-effector and the induced distance on $\mathcal{W}$ measures the minimal movement of the arm needed to achieve this motion in $\mathcal{W}$.

The construction of the induced norm on $\mathcal{W}$ proceeds as follows. Let the manipulator be in posture $\mathbf{q}$, with the end-effector at $\mathbf{x}$. See figure 2.3. We define the induced distance $\|\mathbf{dx}\|_{\text{induced}}$ from $\mathbf{x}$ to $\mathbf{x} + \mathbf{dx}$ as the length of the shortest path in $\mathcal{C}$ (with respect to the chosen metric $\mathbf{h}$) that generates a motion in $\mathcal{W}$ from $\mathbf{x}$ to $\mathbf{x} + \mathbf{dx}$.

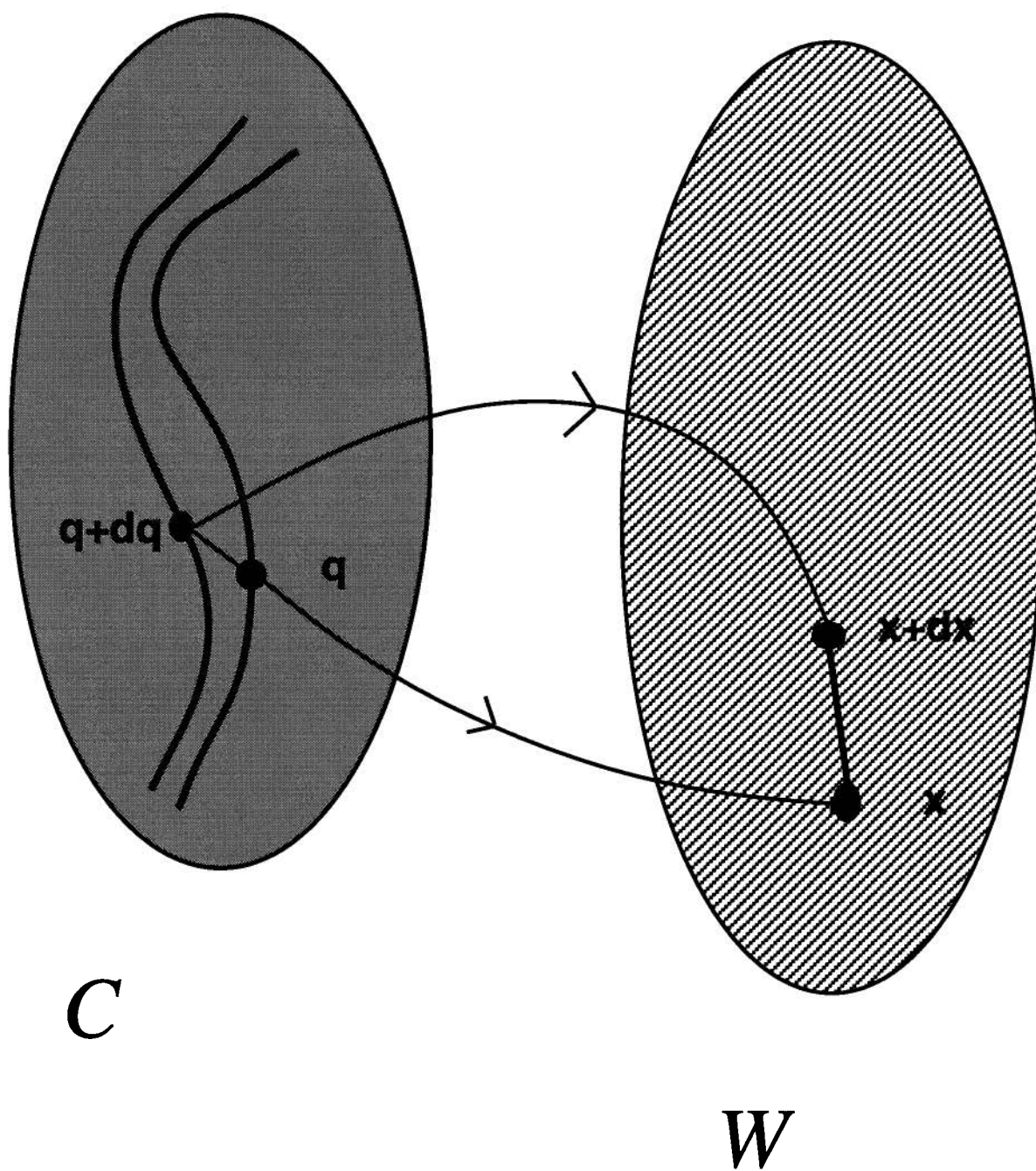


Figure 2.3: Construction of induced distance

A vector norm $\|\cdot\|$ has the following properties:

- $\|\mathbf{x}\| > 0$ iff $\mathbf{x} \neq 0$
- $\|\alpha\mathbf{x}\| = \alpha\|\mathbf{x}\|$
- $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.

It is not difficult to prove that the induced norm on $T\mathcal{W}$, as defined above, satisfies these conditions if the norm on $T\mathcal{C}$ does.

We now make the additional assumption that the norms on $T\mathcal{C}$ and $T\mathcal{W}$ are *quadratic* norms, i.e. they can be given by $\|\mathbf{x}\| = \sqrt{\mathbf{x}^T \mathbf{A} \mathbf{x}}$, where $\mathbf{A}$ is a symmetric, nondegenerate, positive definite matrix. The matrix $\mathbf{A}$ is the metric tensor of the manifold. A manifold with such a quadratic norm is a Riemannian manifold. (Relaxing the condition of positive definiteness on the metric tensor leads to a pseudo-Riemannian manifold.) A non-trivial result is that if the norm on $\mathcal{C}$ is quadratic, then the induced norm is also quadratic. This will be proved by explicit construction below.

Tensor indices in $\mathcal{C}$ are denoted by lowercase latin letters ranging from 1 to n , in $\mathcal{W}$ by greek letters ranging from 1 to m and in $\mathcal{N}$ by uppercase latin letters ranging from 1 to $n - m$.¹

The components of tensors in $\mathcal{C}$ are w.r.t. the coordinate basis $\mathbf{e}_i = \frac{\partial}{\partial q^i}$ and the dual 1-form basis $\omega^i = dq^i$. We use the Einstein summation convention for tensor indices throughout. That is, any repeated tensor index is implicitly summed over its appropriate range. See [DP90] (or any book on differential geometry) for a detailed description of the tensor formalism.

The metric tensor on $\mathcal{C}$ is written as $\mathbf{h}$ with components h_{ij} . The (length)² of an

¹Generically, i.e. if the mapping κ is non-singular. See also [PL92].

infinitesimal vector $dq^i \mathbf{e}_i$ is given by the “line-element”

$$ds^2 = h_{ij} dq^i dq^j.$$

In vector notation this reads

$$ds^2 = \mathbf{d}\mathbf{q}^T \mathbf{h} \mathbf{d}\mathbf{q}.$$

h_{ij} is the covariant (lower indices) form of the metric tensor. The inverse of $\mathbf{h}$ has components h^{ij} and satisfies

$$h_{ij} h^{jk} = \delta_i^k$$

where δ is given by

$$\delta_j^i = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

The δ_j^i are the components of the identity matrix. On $\mathcal{W}$ we have a separate “task” metric tensor $\boldsymbol{\eta}$ with components $\eta_{\mu\nu}$ such that

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

is the square of the “task” length of the infinitesimal vector with components dx^μ . For a typical robot arm, this will be some combination of the spatial distance between points and their difference in orientation in the work space. Note that this metric is non-trivial in general as $\mathcal{W}$ is generally non-Euclidean. For example if $\mathcal{W}$ is the set of positions and orientations of the end-effector of a three dimensional manipulator $\mathcal{W} = \mathbb{R}^3 \otimes SO(3)$, which is not Euclidean.

Besides the task metric we construct an “induced” metric tensor on $\mathcal{W}$ (induced by the metric $\mathbf{h}$ on $\mathcal{C}$), written as $\mathbf{g}$, with components $g_{\mu\nu}$. We shall call this the “performance” metric on $\mathcal{W}$.

2.2 Construction of Performance Metric and Performance Measures

2.2.1 Outline

The construction of the measures proceeds as follows.

- Define an appropriate configuration space $\mathcal{C}$ and a metric tensor $\mathbf{h}$ on $\mathcal{C}$. This describes the robots and quantifies their motions.
- Define a task space $\mathcal{W}$ and a metric tensor $\boldsymbol{\eta}$ on $\mathcal{W}$. This describes the objects to be manipulated, with an appropriate metric for their motions.
- Define a mapping $\kappa : \mathcal{C} \rightarrow \mathcal{W}$. This describes the manipulation of the objects by the manipulator.
- From κ and $\mathbf{h}$ we construct an induced metric on $\mathcal{W}$, the performance metric $\mathbf{g}$. This is a tensor valued performance field from which other work space performance measures can be derived.
- From $\mathbf{g}$ and $\boldsymbol{\eta}$ we construct performance measures. Note that, in general, a measure is defined by the tuple $\{\mathcal{C}, \mathcal{W}, \kappa, \mathbf{h}, \boldsymbol{\eta}\}$. The measures we consider here are the following.
 - A relative directional performance field $U_{\mathbf{u}}$. This field depends on a direction specified by the vector $\mathbf{u}$ in $\mathcal{W}$.
 - A relative average performance field, which we call the generalized Yoshikawa measure Y .
 - A measure of the performance anisotropy of $\mathbf{g}$, the anisotropy measure A .
 - A measure R for the non-linearity of the motion of the manipulator. This is the double contraction of the Riemann tensor of the manifold.

- A class of measures for the reconfigurability of the manipulator (i.e. its ability to move without moving the objects), which we call the redundancy measures.

2.2.2 The Performance Metric

The map $\kappa : \mathcal{C} \rightarrow \mathcal{W}$ is defined by

$$\mathbf{x}^\mu = \kappa^\mu(\mathbf{q}) \quad (2.1)$$

and the Jacobian $\mathbf{J}$ of the map is written as

$$J_i^\mu(\mathbf{q}) = \frac{\partial \kappa^\mu(\mathbf{q})}{\partial q^i},$$

where $J_i^\mu(\mathbf{q})$ are the components of the Jacobian $\mathbf{J}$. The construction of an induced metric on $\mathcal{W}$ proceeds as follows. Let the manipulator be in posture $\mathbf{q}$, with the end-effector at $\mathbf{x}$. We define the induced distance $d_{\mathbf{x}}(\mathbf{y})$ from $\mathbf{x}$ to $\mathbf{y}$ as the length of the shortest path in $\mathcal{C}$ (with respect to the chosen metric $\mathbf{h}$) that moves the end-effector from $\mathbf{x}$ to $\mathbf{y}$.

The metric tensor on $\mathcal{W}$ can now be derived by considering the shortest path in $\mathcal{C}$, from $\mathbf{q}$ to $\mathbf{q} + d\mathbf{q}$, that moves the end-effector from $\mathbf{x}$ to $\mathbf{x} + d\mathbf{x}$. From eq. (2.1) we obtain

$$d\mathbf{x} = \mathbf{J}d\mathbf{q}. \quad (2.2)$$

The minimum norm solution of eq. (2.2) is given by

$$d\mathbf{q} = \mathbf{J}^+ d\mathbf{x} \quad (2.3)$$

where the pseudo-inverse $\mathbf{J}^+$ is given by

$$\mathbf{J}^+ = \mathbf{h}^{-1} \mathbf{J}^T (\mathbf{J} \mathbf{h}^{-1} \mathbf{J}^T)^{-1}.$$

We assume here that the configuration is non-singular, i.e. the inverse of $\mathbf{J} \mathbf{h}^{-1} \mathbf{J}^T$ exists. Note that this pseudo-inverse depends on the metric $\mathbf{h}$, as this defines the norm. So the

induced norm of $\mathbf{dx}$ is given by

$$ds^2 = h_{ij} dq^i dq^j = \mathbf{dx}^T \mathbf{J}^{+T} \mathbf{h} \mathbf{J}^+ \mathbf{dx}$$

and we define

$$\mathbf{g} = \mathbf{J}^{+T} \mathbf{h} \mathbf{J}^+$$

as the covariant metric tensor on $\mathcal{W}$. The field $\mathbf{g}$ is a tensor valued performance field.

$\mathbf{g}$ has a simpler contravariant form

$$g^{\mu\nu} = J_i^\mu h^{ij} J_j^\nu$$

or, in matrix notation (as $g^{\mu\nu}$ are the components of $\mathbf{g}^{-1}$),

$$\mathbf{g}^{-1} = \mathbf{J} \mathbf{h}^{-1} \mathbf{J}^T.$$

The minimum norm solution to eq. (2.2) can now be written as

$$dq^i = J_\mu^i dx^\mu$$

where J_μ^i are the components of $\mathbf{J}^+$ and are obtained from J_i^μ by raising and lowering its indices with the appropriate metric:

$$J_\mu^i = g_{\mu\nu} h^{ij} J_j^\nu.$$

This provides an elegant tensor representation of the pseudo-inverse of the Jacobian.

Note that $g_{\mu\nu}$ does not carry any $\mathcal{C}$ space indices and is therefore invariant under general coordinate transformations in $\mathcal{C}$. It transforms as a tensor under general coordinate transformations in $\mathcal{W}$. Scalar directional performance fields can be obtained from $g_{\mu\nu}$ by contraction with an appropriate vector $\mathbf{u}$ in $\mathcal{W}$, i.e. $g_{\mu\nu} u^\mu u^\nu$.

For robot arms, some natural choices for $\mathbf{h}$ are the Euclidean metric, which measures (distance)² as the sum of the squares of the joint angle differences, and the inertia tensor, i.e. we take

$$\mathbf{h} = \mathbf{H},$$

where $\mathbf{H}$ is the inertia matrix of the arm. We shall refer to the former as the kinematic metric and to the latter as the inertial metric. With the inertial metric, the geodesics on $\mathcal{C}$ are precisely the solutions to the equations of motion of the system in the absence of torques, friction, and external effects such as gravity.

Another possible metric is the joint-compliance matrix, which measures the “stiffness” of the arm (e.g., [PL91]). This can be considered as a generalization of the Euclidean metric with different weights for joint distances. We note that the kinematic measures are the same as the measures constructed from a compliance matrix with equal joint compliances.

The choice of the metric $\boldsymbol{\eta}$ on $\mathcal{W}$ is a non-trivial matter in general. For the simplest kind of object, a dimensionless point in space, the obvious choice for $\boldsymbol{\eta}$ is the identity matrix (in a Cartesian coordinate frame). But for a rigid body there is no unique metric. The reason is that we want to define the distance between configurations that differ in position *and* orientation. Position is measured by coordinates, in meters, but orientation is measured by angles, which are dimensionless. To define a combined distance we need some dimensional weight factor that converts between the two. From a mathematical point of view, there exists no bi-invariant metric on $SE(3)$, the group of rigid body motions [Lon85]. A natural metric for a rigid body is constructed as follows. Let $\mathbf{dx}$ be the difference in position and let $d\theta$ be the angle of rotation (around some axis). The metric is defined by

$$ds^2 = \mathbf{dx}^T \mathbf{dx} + \lambda^2 d\theta^2,$$

with λ a length scale. This combines rotation and translation through the length scale λ .

Another natural choice would be to take the inertia matrix of the rigid body as the metric tensor. In that case the inertia matrix would provide the necessary dimensional

conversion between angles and positions. Indeed, for a sphere with mass M and moment of inertia I , the inertial metric would coincide with the metric mentioned above, with $\lambda = I/M$.

Which is the metric to choose depends on the application at hand.

2.2.3 Work Space Measures

The scalar

$$U_{\mathbf{u}} = \frac{\eta_{\mu\nu} u^\mu u^\nu}{g_{\mu\nu} u^\mu u^\nu} \quad (2.4)$$

with $\eta_{\mu\nu}$ the task metric on $\mathcal{W}$, measures task space length per $\mathcal{C}$ space length for movements in direction $\mathbf{u}$. It is a directional performance field.

From $\mathbf{g}$ and $\boldsymbol{\eta}$ we derive a scalar measure that we call the generalized Yoshikawa measure. Consider an infinitesimal parallelepiped $dx^1 \dots dx^m$ located at the point $\mathbf{x}$ in $\mathcal{W}$ and suppose the manipulator is in configuration $\mathbf{q}$. The m -volume of the smallest region in $\mathcal{C}$ that will cover the parallelepiped in $\mathcal{W}$ is given by

$$\sqrt{\det(\mathbf{g})} dx^1 \dots dx^m.$$

It measures the volume of the minimal region in which the manipulator has to move to cover the given infinitesimal parallelepiped in work space. The task volume of this region is

$$\sqrt{\det(\boldsymbol{\eta})} dx^1 \dots dx^m.$$

We define the generalized Yoshikawa measure as the $\mathcal{W}$ space volume per $\mathcal{C}$ space volume of the parallelepiped,

$$Y = \sqrt{\frac{\det(\boldsymbol{\eta})}{\det(\mathbf{g})}}. \quad (2.5)$$

It is a measure of the performance of the manipulator, averaged over all directions. Note that Y is invariant under general coordinate transformations in $\mathcal{C}$ as well as in $\mathcal{W}$

space and therefore measures a true physical property of the system, independent of the coordinate systems.² If $\mathbf{h}$ is the Euclidean metric in “joint-angle” coordinates *and* the task metric on $\mathcal{W}$ is Euclidean, it reduces to the original Yoshikawa measure $\sqrt{\det(\mathbf{J}\mathbf{J}^T)}$ as defined in [Yos85].

A second scalar derived from the metric tensor $\mathbf{g}$ is the anisotropy. It measures how isotropic the manipulator behaves locally. There is a slight complication in defining this measure, as for a general metric $\boldsymbol{\eta}$ there is already an inherent anisotropy present in the work space, which depends on the coordinate system used. We first eliminate this by finding a local coordinate transformation that brings $\boldsymbol{\eta}$ into Euclidean form, ie.

$$\mathbf{T}\boldsymbol{\eta}\mathbf{T}^T = \mathbf{1},$$

where $\mathbf{T}$ is the matrix associated with the coordinate transformation. In these new coordinates the induced metric becomes

$$\hat{\mathbf{g}} = \mathbf{T}\mathbf{g}\mathbf{T}^T.$$

The equation

$$\mathbf{x}^T \hat{\mathbf{g}} \mathbf{x} = 1$$

defines an ellipsoid, with the lengths of the principal axes corresponding to the eigenvalues of $\hat{\mathbf{g}}$. The elongation of the ellipsoid (i.e. how much it differs from a sphere) in this special coordinate system is a physical measure of the anisotropy of the manipulator. It is easy to show that the eigenvalues of $\hat{\mathbf{g}}$ are the same as the eigenvalues of $\boldsymbol{\eta}^{-1}\mathbf{g}$, in any coordinate system. The same information is contained in the inverse of this matrix, i.e. $\mathbf{g}^{-1}\boldsymbol{\eta}$. Since $\mathbf{g}$ may have singularities (if the manipulator is in a singular configuration such that it cannot move in some direction in $\mathcal{W}$, the metric tensor $\mathbf{g}$ will become infinite in that direction) it is better to work with this matrix instead.

²The field $\sqrt{\det(\mathbf{g})}$ itself is not invariant under general coordinate transformations in $\mathcal{W}$. It is called a scalar density.

We define the invariant measure of the anisotropy A as

$$A = 1 - \frac{\mu_-}{\mu_+}, \quad (2.6)$$

where μ_+ and μ_- are the largest and smallest eigenvalues of $\mathbf{g}^{-1}\boldsymbol{\eta}$.

The measure given in eq. (2.6) measures the anisotropy. If isotropy is desired for a task, this measure should be minimized. For positive definite metrics $\mathbf{h}$ we have $0 \leq A \leq 1$ and isotropy is obtained if $A = 0$.

We note that the anisotropy measure as defined by eq. (2.6) may depend on the length units used if we consider a work space which incorporates both positioning and orientation information. In this case, the measure A will depend on an arbitrary length scale that corresponds to the relative weight we assign to orientation (described by dimensionless coordinates, e.g. Euler angles) relative to positioning (described by coordinates with the dimension *length*). The role of such a length scale was investigated in [TAR92]. It emerges here in the form of the factor $\boldsymbol{\eta}$, i.e. the work space metric. For systems that incorporate both positioning and orientation information it is not always possible to find a unique metric $\boldsymbol{\eta}$. This arises because we somehow have to combine angles and lengths and we need a dimensional parameter to convert between the two.

Other performance measures can be derived as needed from the above by covariant differentiation and contraction. For example, the local variation of the generalized Yoshikawa measure could be defined by

$$g^{\mu\nu} J_\mu^i J_\nu^j \frac{\partial^2 Y}{\partial q^i \partial q^j}.$$

2.3 Redundancy Measures

An important class of measures is related to the ability of the manipulator to move in the self-motion manifold $\mathcal{N}_{\mathbf{x}}$ associated with every point $\mathbf{x}$ in $\mathcal{W}$.

Consider a redundant robot arm that performs some assembly task inside some enclosed region by reaching inside, through a small opening. It will be “easier” to reconfigure the arm (change its posture without moving the end-effector) if the segment of the arm that enters through the opening does not move very much during the reconfiguration. A possible measure for the “goodness” of the posture for a task like this is the ratio of the $\mathcal{C}$ space distance and the displacement of the segment that reaches through the hole (as measured in the plane of the surface with the opening).

Redundancy measures quantify the mobility of the manipulator in the implicitly defined self motion manifold $\mathcal{N}_{\mathbf{x}}$. For the case of a robot arm, this means that we are effectively considering a linkage with a closed loop, constructed out of the original arm by fixing the end-effector at $\mathbf{x}$.

The redundancy measures are obtained by a similar construction as the work space measures. We take the $\mathbf{x}$ -dependent tuple

$$\mathcal{T}_{\mathbf{x}} = \{\mathcal{C}_{\mathbf{x}}, \mathcal{W}_{\mathbf{x}}, \kappa_{\mathbf{x}}, \mathbf{h}_{\mathbf{x}}, \boldsymbol{\eta}_{\mathbf{x}}\}$$

as the generator of the measures.

The tuple $\mathcal{T}_{\mathbf{x}}$ is related to $\mathcal{C}$, $\mathcal{W}$ and κ as follows. We take

$$\mathcal{C}_{\mathbf{x}} = \mathcal{W}_{\mathbf{x}} = \mathcal{N}_{\mathbf{x}}$$

so that $\kappa_{\mathbf{x}}$ is the identity map. (The domain and range of the map depend on $\mathbf{x}$.) However, we measure “distance” in $\mathcal{C}_{\mathbf{x}}$ and $\mathcal{W}_{\mathbf{x}}$ using different metrics. The metric $\mathbf{h}_{\mathbf{x}}$ on $\mathcal{C}_{\mathbf{x}}$ is just $\mathbf{h}$ restricted to $\mathcal{N}_{\mathbf{x}}$. We denote it by $\mathbf{n}$, dropping the subscript $\mathbf{x}$. The task metric $\boldsymbol{\eta}_{\mathbf{x}}$ on $\mathcal{W}_{\mathbf{x}}$ is obtained by restricting a “secondary” metric $\hat{\mathbf{h}}$ to $\mathcal{N}_{\mathbf{x}}$. With these choices for $\mathcal{T}_{\mathbf{x}}$ it is possible to construct the measures without explicitly constructing $\mathcal{N}_{\mathbf{x}}$ as follows.

An infinitesimal displacement $d\mathbf{q}$ in $\mathcal{N}_{\mathbf{x}}$ can be written as

$$dq^i = P_j^i d\psi^j$$

for arbitrary $d\psi^j$. The projection operator $\mathbf{P}$ is given by

$$P_j^i = \delta_j^i - J_\mu^i J_j^\mu.$$

The Jacobian J_i^μ defines $n - m$ independent vectors $\mathbf{w}_A$, ($A = 1, \dots, n - m$) that generate motions in $T\mathcal{N}_{\mathbf{x}}$ (except at singular points). The $\mathbf{w}_A$ satisfy

$$J_i^\mu w_A^i = 0.$$

An arbitrary vector in $T\mathcal{N}_{\mathbf{x}}$ can be written as

$$v^i = w_A^i v^A.$$

Its $\mathbf{h}$ -length is

$$h_{ij} v^i v^j = h_{ij} w_A^i w_B^j v^A v^B$$

and we obtain the components of the metric tensor $\mathbf{n}$ on $\mathcal{C}_{\mathbf{x}}$ as

$$n_{AB} = h_{ij} w_A^i w_B^j.$$

Similarly, the task metric $\eta_{\mathbf{x}}$ is obtained by restricting $\hat{\mathbf{h}}$ to $T\mathcal{N}_{\mathbf{x}}$, i.e.

$$\eta_{\mathbf{x}AB} = \hat{h}_{ij} w_A^i w_B^j.$$

As the mapping $\kappa_{\mathbf{x}}$ is just the identity, the Jacobian is just the identity matrix and the induced performance metric $\mathbf{g}_{\mathbf{x}}$ on $\mathcal{W}_{\mathbf{x}}$ is just the metric $\mathbf{n}$. With these definitions we can construct all the measures derived previously for the work space $\mathcal{W}_{\mathbf{x}}$, such as the directional performance U , the generalized Yoshikawa measure Y , and the anisotropy A . By considering all points $\mathbf{x}$ and noting that the $\mathcal{N}_{\mathbf{x}}$ are all disjoint, we finally arrive at

measures defined on the entire configuration space $\mathcal{C}$. We call these the “redundancy” measures.

The precise interpretation of a measure will depend on the secondary task metric $\hat{\mathbf{h}}$ chosen, which is very task-specific in general. In the examples given in chapter 4, we make some choices for $\hat{\mathbf{h}}$ and interpret the secondary generalized Yoshikawa measures. Other redundancy measures, such as the anisotropy or non-linearity (discussed below) can only be defined if the dimension of $\mathcal{N}_{\mathbf{x}}$ is at least 2.

A more general set of redundancy measures (which are just the work space measures of the self-motion manifold) can be obtained by considering non-trivial mappings $\kappa_{\mathbf{x}}$, to a work space $\mathcal{W}_{\mathbf{x}}$ of possibly lower dimension than $\mathcal{N}_{\mathbf{x}}$. We shall not consider those measures here, but they can be constructed easily by using the extension of the theory to constrained systems, which is the subject of chapter 3.

2.4 Non-linearity Measures

The purpose of the measures introduced in this section is to quantify a posture in terms of the “complexity” of the dynamics of the manipulator in a small region in configuration space around this posture. Robot manipulators obey highly non-linear equations of motion. One can expect that a control scheme that incorporates some form of numerical integration of the equations of motion will require less resources if the non-linearity of the system can be minimized in some sense. This issue is complicated due to the fact that the equations of motion depend on the coordinate system that is used on configuration space. As explained previously, the equations of motion of a manipulator (in the absence of torques, friction and gravity) are identical to the equations for the geodesics on $\mathcal{C}$ with the inertia matrix as the metric tensor $\mathbf{h}$ on $\mathcal{C}$.

The equations of motion can be written as

$$\dot{q}^i = \Gamma^i_{jk}(\mathbf{q}) \dot{q}^j \dot{q}^k \quad (2.7)$$

where $\mathbf{q}$ are the coordinates on $\mathcal{C}$. The Christoffel symbols Γ are given by

$$\Gamma_{ijk} = \frac{1}{2} \left(\frac{\partial h_{ij}}{\partial q^k} + \frac{\partial h_{ik}}{\partial q^j} - \frac{\partial h_{jk}}{\partial q^i} \right)$$

and the Γ^i_{jk} are obtained by raising the first index, i.e.

$$\Gamma^i_{jk} = h^{il} \Gamma_{ljk}.$$

The non-linearity of eq. (2.7) can be seen as a result of the presence of the Γ^i_{jk} . However, as this is not a tensor (see for example [DP90]), the Christoffel symbols themselves cannot be directly used to obtain a physical (i.e. coordinate system independent) measure of the non-linearity.

We propose to measure the non-linearity of the manipulator with respect to the metric $\mathbf{h}$ on $\mathcal{C}$ by the Riemann tensor, which is a rank 4 tensor field on $\mathcal{C}$, with components R^i_{jkl} . The Riemann tensor has $\frac{n^2}{12}(n^2 - 1)$ independent components. In a coordinate basis it is given by

$$R^i_{jkl} = \frac{\partial \Gamma^i_{jl}}{\partial q^k} - \frac{\partial \Gamma^i_{jk}}{\partial q^l} + \Gamma^i_{mk} \Gamma^m_{jl} - \Gamma^i_{ml} \Gamma^m_{jk}.$$

It is well known that a global coordinate transformation can be found that linearizes the equations of geodesic motion if and only if the Riemann tensor vanishes everywhere. This point was stressed for mechanical systems in [Spo92, Bed92].

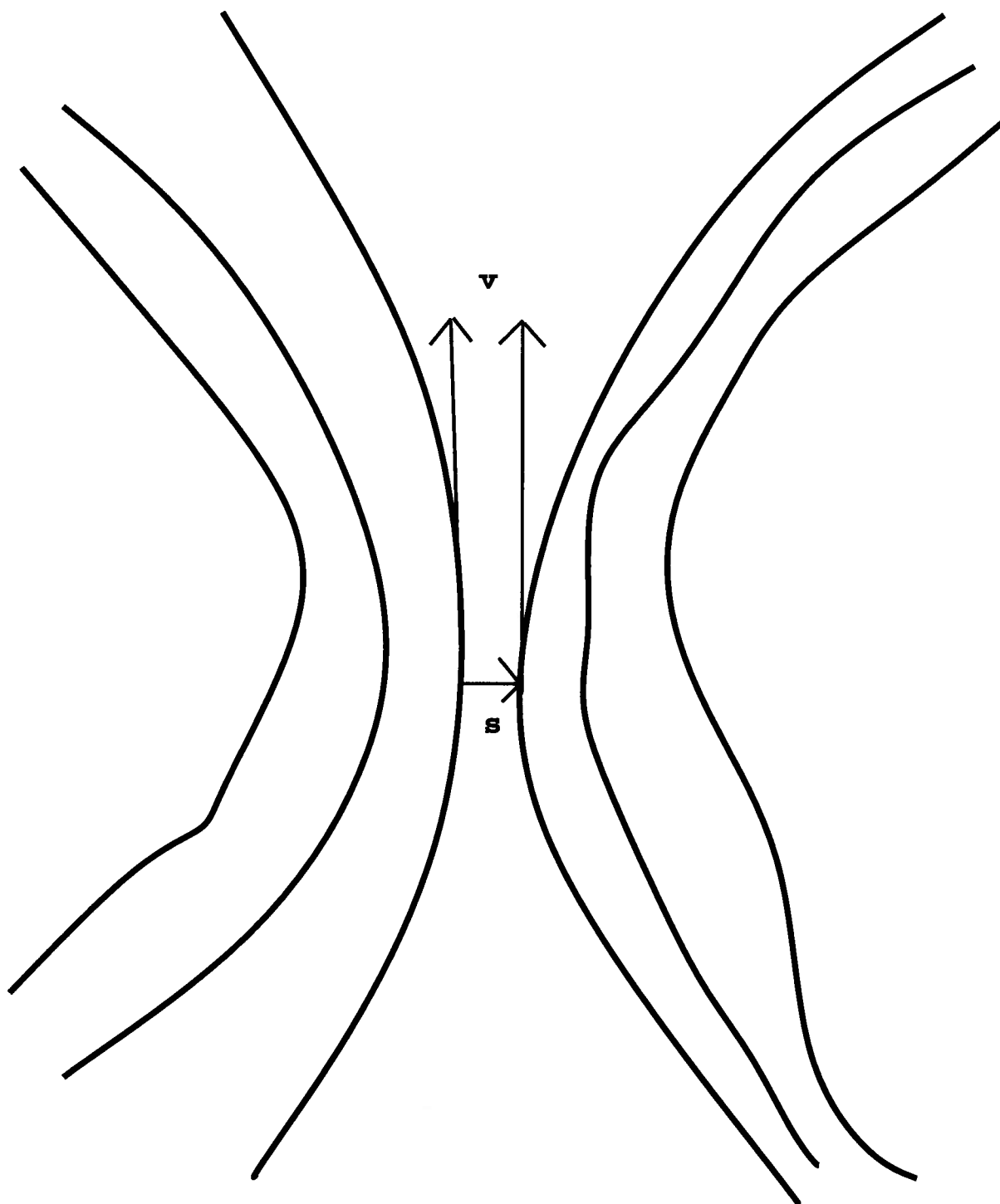


Figure 2.4: Relative acceleration of nearby geodesics

The Riemann tensor has the following interpretation. Suppose we have two nearby points in configuration space with (small) separation vector $\mathbf{s}$ and these points move in the same direction (i.e. on parallel trajectories), described by a velocity vector $\mathbf{v}$, on geodesic trajectories (see figure 2.4). If the space is non-Euclidean, the trajectories will not remain parallel. They will diverge or converge (at least for generic $\mathbf{s}$ and $\mathbf{v}$) and the relative acceleration between the points is given by

$$a^i = R^i_{jkl} v^j s^k v^l. \quad (2.8)$$

Therefore, this tensor measures directional non-linearity of the arm. If the tensor vanishes, an error in the position of the arm (in configuration space) remains constant over time, as nearby geodesics remain parallel. The relative acceleration as given by eq. (2.8) is independent of the particular coordinate system used and therefore measures an intrinsic property of the manipulator. There is also an interesting relation between curvature and stability. If the curvature is negative, the geodesics are exponentially unstable (because the effective repelling force is proportional to the separation), as nearby geodesics repel each other, whereas if the curvature is positive we obtain stable orbits. See [Arn89], appendix 1, for details. The Riemann tensor is a generalization of the Gaussian curvature of a surface. (If the configuration space is two-dimensional, $\mathbf{R}$ has only one independent component, due to its symmetries, which is the Gaussian curvature of the manifold.)

The associated curvature scalar,

$$R = h^{ij} R^k_{ikj} \quad (2.9)$$

gives an overall measure of the non-linearity. The curvature R is a scalar field, i.e. it is invariant under general coordinate transformations in $\mathcal{C}$ and therefore measures an intrinsic property of the system. Note that this measure does not depend on the forward kinematics. This measure (or R^2) could be used to minimize the error of linear approximations used in controlling the manipulator.

If $R = 0$ at some point, the configuration space is flat at this point. An interesting observation is that if there are points in $\mathcal{C}$ with $R > 0$ and with $R < 0$, there are $(n - 1)$ -dimensional subspaces of $\mathcal{C}$ with vanishing curvature. If $m < n$ these subspaces will map onto m -dimensional regions in $\mathcal{W}$, so these are regions in $\mathcal{W}$ that can be reached with an optimal posture with respect to the curvature measure. We shall call such postures “flat”.

A curvature tensor and its contractions can also be defined on $\mathcal{W}$, using the performance metric $\mathbf{g}$. This measures the non-linearity of the motion of the manipulated objects. Note that for a non-redundant manipulator, i.e. $m = n$, we can view the forward kinematics at non-singular points as a change of coordinate system on $\mathcal{C}$. Because R is independent of the coordinate system used, the curvature scalar R will therefore be the same whether constructed from h_{ij} on $\mathcal{C}$ or from $g_{\mu\nu}$ on $\mathcal{W}$, for non-redundant manipulators.

Chapter 3

Extending the Theory to Constrained systems

3.1 Description of Constrained Systems

Many systems are most easily described in terms of a more general system upon which constraints are placed. For example, a walking machine could be most easily described by the configuration of all its joints and the position and orientation of its main body. If the machine is walking, we have the constraint that its feet should be on the ground. For such systems, the geometric ideas presented in the previous chapter can be used to quantify performance, but the details are a little different.

The configurations of the constrained systems we are interested in have a natural description in the form of a set of n coordinates

$$\mathbf{q} = \begin{pmatrix} q^1 \\ \vdots \\ q^n \end{pmatrix}$$

upon which some constraint equations are imposed. The coordinates $\mathbf{q}$ can be thought of as representing configurations that may violate the constraints, which usually has a physical interpretation as the opening up of some closed kinematic loop. Following [Lue77], we call the unconstrained coordinates together with the work space coordinates *descriptors*. The unconstrained coordinates $\mathbf{q}$ by themselves are called the manipulator descriptors. The space of all configurations including the ones that violate the constraints is called the manipulator descriptor space, denoted by $\mathcal{C}^+$, of dimension n . The points in

$\mathcal{C} \subset \mathcal{C}^+$ are labeled by those coordinates $\mathbf{q}$ that satisfy the s constraints. The dimension of $\mathcal{C}$ is taken to be l , and the work space $\mathcal{W}$ has dimension m . The descriptor space is $\mathcal{C}^+ \times \mathcal{W}$, of dimension $n + m$. Furthermore, we have

$$n + m - s = l.$$

For an illustrative example, consider the two link planar manipulator depicted in figure 3.5. The base and elbow joints, with angles θ^1 and θ^2 , are actuated and the tip of the arm is constrained to slide along a vertical line, at a distance 1.0 to the right of the base of the arm. The two links are of unit length.

In this case the manipulator descriptor space $\mathcal{C}^+$ is the 2-torus

$$\mathcal{C}^+ = \{(\theta^1, \theta^2) : -\pi < \theta^1, \theta^2 \leq \pi\}$$

depicted as the shaded area in figure 3.6, and the configuration space $\mathcal{C} \subset \mathcal{C}^+$ is the one dimensional region

$$\mathcal{C} = \{(\theta^1, \theta^2) : -\pi/2 < \theta^1 \leq \pi/2, \cos \theta^1 + \cos(\theta^1 + \theta^2) = 1.0\},$$

which is drawn in figure 3.6. The work space $\mathcal{W}$ is a line segment of length $2\sqrt{3}$.

Tensor indices in TC^+ (the tangent space to $\mathcal{C}^+$) are denoted by lowercase Latin letters, tensor indices in TC by uppercase Latin letters and tensor indices in TW by lowercase Greek letters.

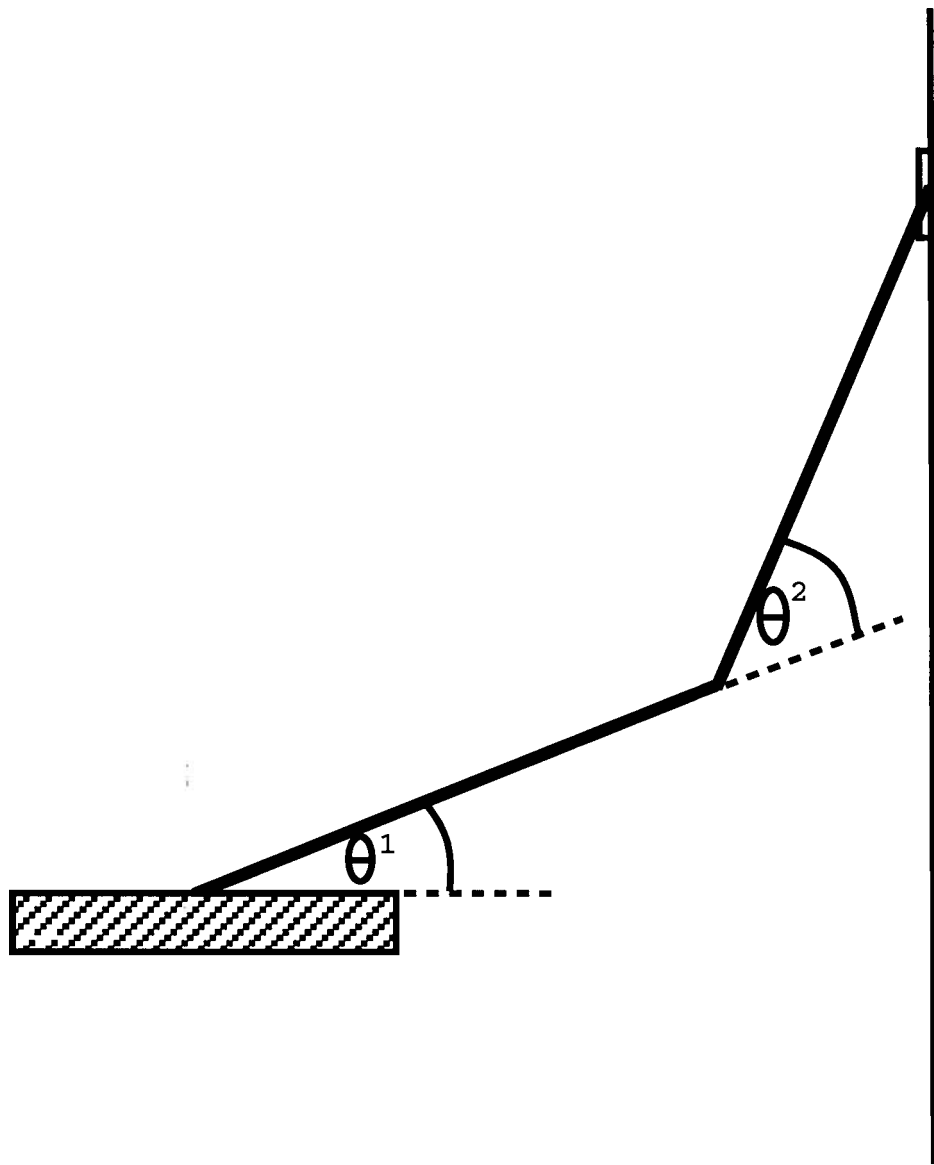


Figure 3.5: Constrained two link planar arm. The end-effector is constrained to remain on the vertical line.

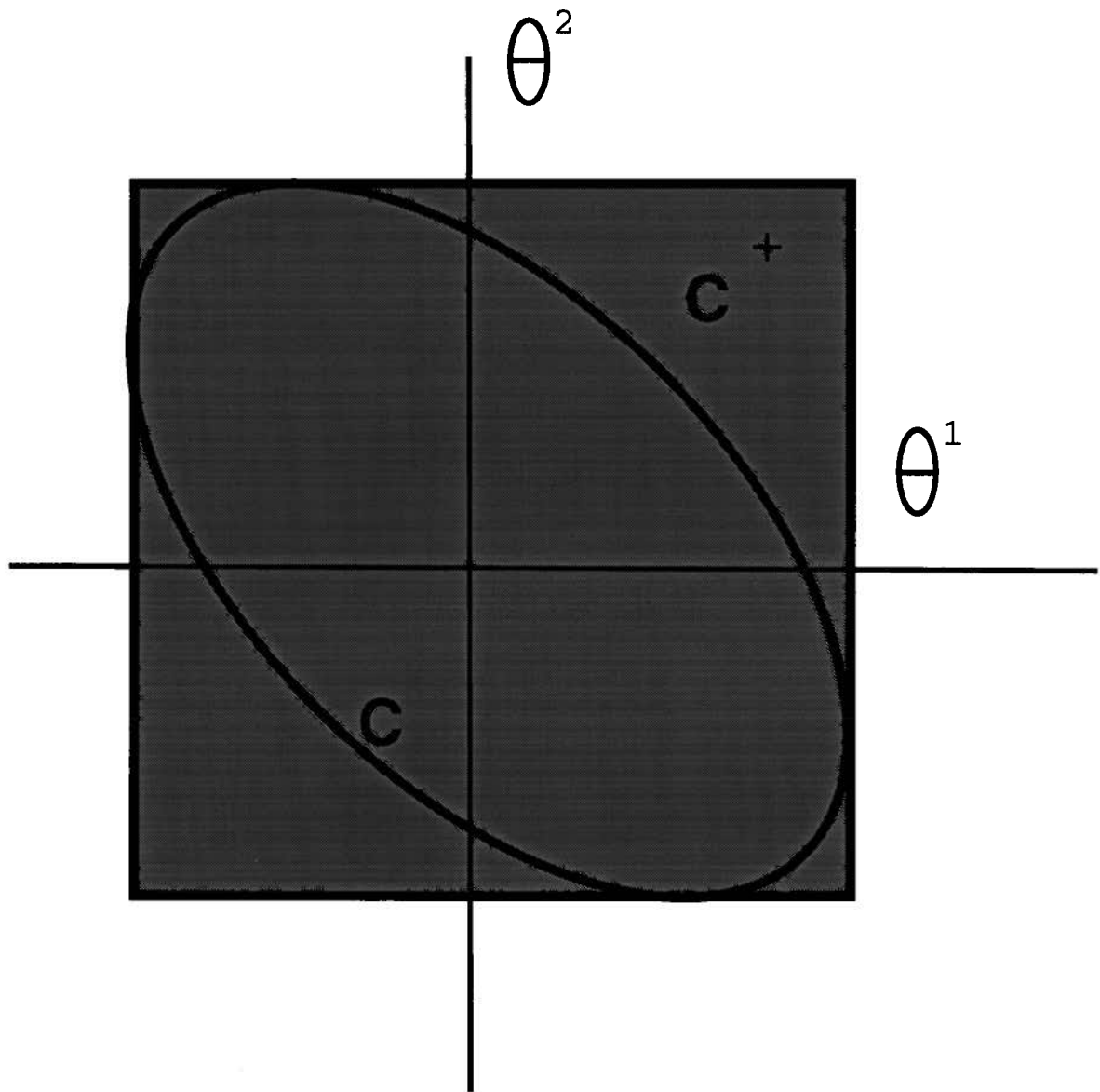


Figure 3.6: Manipulator descriptor space $\mathcal{C}^+$ and configuration space $\mathcal{C}$ of constrained two-link planar arm.

The forward kinematics as well as the constraints on the coordinates $\mathbf{q}$ are implicitly given by s constraint equations

$$f^\Lambda(\mathbf{x}, \mathbf{q}) = 0, \quad (3.10)$$

where $\Lambda = 1, \dots, s$, which can not be solved explicitly in general. If eq. (3.10) is linear in the work space coordinates $\mathbf{x}$, the forward kinematics of the system can be found explicitly.

Note that in the following we don't use eq. (3.10) directly, but only its differential as given in eq. (3.11) below. By allowing arbitrary factors f_μ^Λ and f_i^Λ the entire construction carries over to non-holonomic systems as well.

On $\mathcal{W}$ we have a “task” metric tensor $\boldsymbol{\eta}$, as before. Besides the task metric we construct an “induced” metric on $\mathcal{W}$ (induced by the metric $\mathbf{h}$ on $\mathcal{C}$), written as $\mathbf{g}$, with components $g_{\mu\nu}$. We shall call this the “performance” metric on $\mathcal{W}$. This construction is similar to the one presented in the previous chapter.

3.2 Construction of the Performance Metric

On $\mathcal{C}^+$ we define the distance between $\mathbf{q}$ and $\mathbf{q} + d\mathbf{q}$ by a metric tensor $\mathbf{h}$, i.e. with the line element

$$ds^2 = h_{ij} dq^i dq^j.$$

The metric on $\mathcal{C} \subset \mathcal{C}^+$ then follows from the above for infinitesimal vectors $d\mathbf{q}$ tangent to $\mathcal{C}$.

Suppose the manipulator is in a configuration labeled by $\mathbf{q}$ (satisfying the constraints), with the end-effector at a point labeled by $\mathbf{x}$ in $\mathcal{W}$ and that we have defined an appropriate metric on $\mathcal{C}^+$. The induced distance to a point $\mathbf{x} + d\mathbf{x}$ in $\mathcal{W}$ is defined as the distance of the shortest path in $\mathcal{C}$, starting at $\mathbf{q}$, that takes the end-effector to $\mathbf{x} + d\mathbf{x}$, where distance is measured by the metric tensor $\mathbf{h}$ on $\mathcal{C}^+$.

From the metric on $\mathcal{C}^+$ we thus derive an “induced” metric on $\mathcal{W}$, as before. We construct it explicitly as follows.

Consider an infinitesimal motion $\mathbf{dq}$ in $\mathcal{C}$ with the induced displacement $\mathbf{dx}$ in $\mathcal{W}$. From the constraint equations we obtain

$$t^\Lambda = f_\mu^\Lambda dx^\mu + f_i^\Lambda dq^i = 0, \quad (3.11)$$

where

$$f_\mu^\Lambda = \frac{\partial f^\Lambda}{\partial x^\mu}$$

and

$$f_i^\Lambda = \frac{\partial f^\Lambda}{\partial q^i}.$$

For general $\mathbf{dq}$ lying in $T\mathcal{C}^+$ (the tangent space to $\mathcal{C}^+$ at the point under consideration), there is no vector $\mathbf{dx}$ that satisfies eq. (3.11). However, we can find the $\mathbf{dx}$ that minimizes the quadratic norm

$$t^\Lambda t_\Lambda, \quad (3.12)$$

(where there is an implicit sum over Λ) using the pseudo-inverse of f_μ^Λ viewed as an $s \times m$ matrix. We thus obtain

$$dx^\mu = J_i^\mu dq^i \quad (3.13)$$

where the “Jacobian” is given by

$$J_i^\mu = -f^{+\mu}_\Lambda f_i^\Lambda \quad (3.14)$$

and $f^{+\mu}_\Lambda$ are the components of the pseudo-inverse of f_μ^Λ , which is given by

$$f^{+\mu}_\Lambda = \rho^{\mu\nu} f_\nu^\Lambda,$$

where $\rho^{\mu\nu}$ is the inverse of $f_\mu^\Lambda f_\nu^\Lambda$, i.e.

$$\rho^{\mu\lambda} f_\lambda^\Lambda f_\nu^\Lambda = \delta_\nu^\mu, \quad (3.15)$$

provided that the inverse exists. Eq. (3.15) can be written in matrix form as follows. Let us write $\mathbf{F}$ for the $s \times m$ matrix with components f_μ^Λ , and $\mathbf{F}^+$ for its pseudo-inverse. With this notation, eq. (3.15) reads

$$\mathbf{F}^+ = \mathbf{F}^T (\mathbf{F} \mathbf{F}^T)^{-1}.$$

We can now obtain an explicit condition on $\mathbf{d}\mathbf{q}$ by demanding that eq. (3.13) is a solution to eq. (3.11). By substituting eq. (3.13) into eq. (3.11) we obtain the following explicit constraint on $\mathbf{d}\mathbf{q}$

$$P_i^\Lambda dq^i = 0, \quad (3.16)$$

where

$$P_i^\Lambda = f_\mu^\Lambda J_i^\mu + f_i^\Lambda.$$

Eq. (3.16) defines $s - m$ independent constraints on $\mathbf{d}\mathbf{q}$, leaving $l = m + n - s$ actual degrees of freedom.

Let us choose a basis for the tangent space TC to $\mathcal{C}$ by choosing l vectors in TC , $\mathbf{w}_A$, $A = 1, \dots, l$, such that

$$P_i^\Lambda w_A^i = 0.$$

In this basis, the $\mathcal{C}$ -space metric tensor is

$$h_{AB} = h_{ij} w_A^i w_B^j.$$

The Jacobian on $\mathcal{C}$ is the restriction of the Jacobian given in eq. (3.14) to TC . It is given by

$$J_A^\mu = J_i^\mu w_A^i. \quad (3.17)$$

The induced performance work space metric can now be obtained. Note that this metric is a field on $\mathcal{C}^+ \times \mathcal{W}$ with its components living in $T\mathcal{W}$. For a given posture, it quantifies the metric in the work space locally. We define the induced distance $d_{\mathbf{x}}(\mathbf{y})$

from $\mathbf{x}$ to $\mathbf{y}$ as the length of the shortest path in $\mathcal{C}$ (with respect to the metric $\mathbf{h}$) that moves the end-effector from $\mathbf{x}$ to $\mathbf{y}$. For kinematically redundant manipulators, this distance depends on the posture with which the point $\mathbf{x}$ is reached. By considering infinitesimal motions we derive the induced performance metric as follows. An infinitesimal displacement $d\mathbf{q}$ in TC can be written as

$$d\mathbf{q} = \mathbf{w}_A dr^A,$$

where the dr^A are the components of $d\mathbf{q}$ in the basis for TC . From eq. (3.14) and eq. (3.17) we obtain

$$dx^\mu = J_A^\mu dr^A. \quad (3.18)$$

The induced length of the vector $d\mathbf{x}$ is now defined as the length of the shortest (under the metric $\mathbf{h}$) vector $d\mathbf{r}$ that satisfies eq. (3.18). Eq. (3.18) reads in matrix form

$$d\mathbf{x} = \mathbf{J} d\mathbf{r} \quad (3.19)$$

and the weighted pseudo-inverse $\mathbf{J}^+$,

$$\mathbf{J}^+ = \mathbf{h}^{-1} \mathbf{J}^T (\mathbf{J} \mathbf{h}^{-1} \mathbf{J}^T)^{-1}$$

gives the minimum norm solution to eq. (3.19) in the form

$$d\mathbf{r} = \mathbf{J}^+ d\mathbf{x}.$$

The norm of $d\mathbf{r}$ now gives us the line element

$$ds^2 = h_{AB} dr^A dr^B = d\mathbf{x}^T \mathbf{J}^{+T} \mathbf{h} \mathbf{J}^+ d\mathbf{x}$$

from which we read off the performance metric tensor in covariant form as

$$\mathbf{g} = \mathbf{J}^{+T} \mathbf{h} \mathbf{J}^+.$$

The inverse of $\mathbf{g}$, corresponding to the contravariant form of $\mathbf{g}$ (i.e. $g^{\mu\nu}$) has the simpler form

$$\mathbf{g}^{-1} = \mathbf{J}\mathbf{h}^{-1}\mathbf{J}^T.$$

In component notation this reads

$$g^{\mu\nu} = J_A^\mu h^{AB} J_B^\nu.$$

It defines distance in $\mathcal{W}$ as derived from minimal motion in $\mathcal{C}$.

All the performance measures can now be constructed as for the unconstrained systems.

Chapter 4

Applications to Planar Arms

In this section we shall construct and interpret some of the measures for the two and three link planar manipulators. We shall consider two metrics on $\mathcal{C}$, the Euclidean metric which measures (distance)² as the sum of the squares of the joint angle differences, and the inertia tensor, i.e. we take

$$\mathbf{h} = \mathbf{H} \quad (4.20)$$

where $\mathbf{H}$ is the inertia matrix of the arm. We shall refer to the Euclidean metric as the kinematic metric and to the metric defined by eq. (4.20) as the inertial metric.

The two link arm consists of two links of length l_1 and l_2 . We take all mass to be concentrated in the middle joint and in the tip of the second link, with masses m_1 and m_2 respectively. The configuration of the arm is described by the two joint angles q^1 and q^2 , with $-\pi < q^i \leq \pi$.

The three link arm consists of three links of length l_i ($i = 1, 2, 3$) which we take to be thin rods of uniform mass density m_i . The configuration of the arm is described by the three joint angles q^i , with $-\pi < q^i \leq \pi$.

Note that the non-directional measures do not depend on the first joint angle q^1 because of rotation invariance, hence it is not shown in plots of the values of the measures.

Unless given explicitly in a formula, we take all lengths to be equal to one and all masses to be equal to 0.5 in our calculations.

4.1 Kinematic Measures

In this case the induced metric on $\mathcal{W}$, $\mathbf{g}$, leads to Yoshikawa's [Yos85] manipulating force ellipses, defined as the set of all forces f_μ satisfying

$$g^{\mu\nu} f_\mu f_\nu \leq 1$$

and the associated manipulability ellipses, defined as the set of all end-effector velocities v^μ that satisfy

$$g_{\mu\nu} v^\mu v^\nu \leq 1.$$

For a desired end-effector force f_μ , the measure

$$g^{\mu\nu} f_\mu f_\nu$$

equals the Euclidean norm of the generalized joint torque needed to apply a prescribed force $\mathbf{f}$ with the end-effector. Denoting the joint torque by τ_i , the condition of static equilibrium is

$$\tau_i = f_\mu J_i^\mu$$

and it follows that

$$\sum_i \tau_i^2 = h^{ij} \tau_i \tau_j = f_\mu f_\nu J_i^\mu J_j^\nu h^{ij} = f_\mu f_\nu g^{\mu\nu}.$$

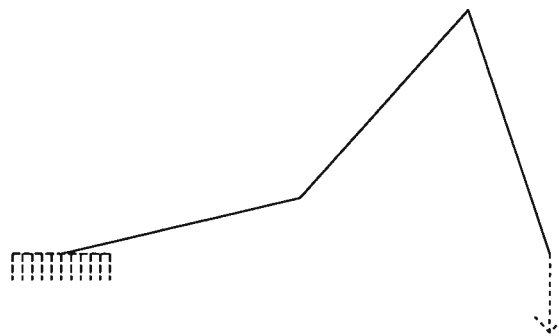


Figure 4.7: Good posture for applying vertical force

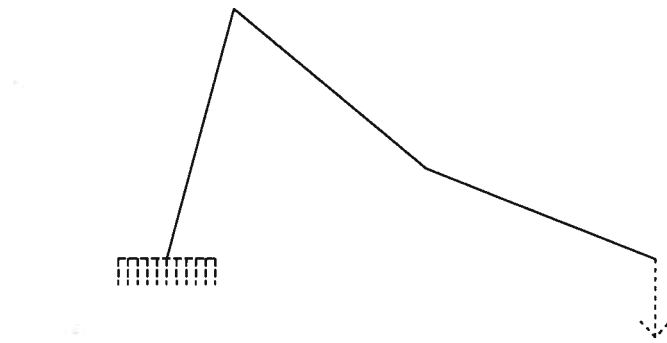


Figure 4.8: Bad posture for applying vertical force

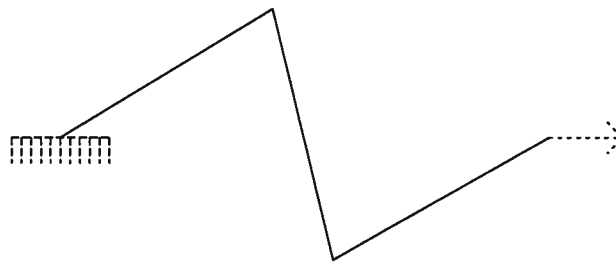


Figure 4.9: Good posture for applying horizontal force

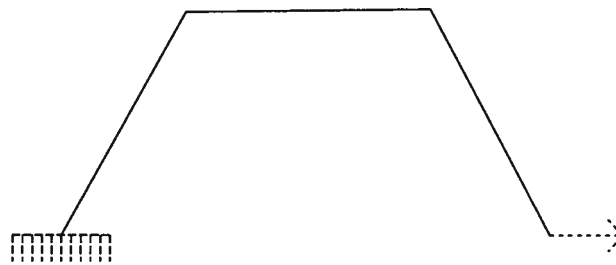


Figure 4.10: Bad posture for applying horizontal force

In figures 4.7, 4.8, 4.9, and 4.10 we show the best and worst postures of the three link arm for applying a vertical and horizontal force with the end-effector at a distance of 2.0 from the base of the arm.

The generalized Yoshikawa measure, as defined in eq. (2.5), reduces to the Yoshikawa measure [Yos85] if the factor $\sqrt{\det(\boldsymbol{\eta})}$ is constant and equal to one as is the case for the planar positioning manipulators we are considering. For easy reference we reproduce here some of the results from [Yos85] for the planar manipulators. For the two link arm we have

$$Y = l_1 l_2 |\sin(q^2)|$$

and for the three link arm we have

$$\begin{aligned} Y^2 = & \frac{l_1^2 l_2^2}{2} + l_1^2 l_3^2 + l_2^2 l_3^2 + l_1 l_2 l_3^2 \cos(q^2) - \frac{l_1^2 l_2^2 \cos(2q^2)}{2} + \\ & l_1^2 l_2 l_3 \cos(q^3) - l_2^2 l_3^2 \cos(2q^3) - l_1^2 l_3^2 \cos(2(q^2 + q^3)) - \\ & l_1^2 l_2 l_3 \cos(2q^2 + q^3) - l_1 l_2 l_3^2 \cos(q^2 + 2q^3). \end{aligned}$$

In figure 4.11 we have plotted Y as a function of the joint angles q^2 and q^3 . Figure 4.12 shows the optimal postures for reaching a point at a distance of $d = \{0.5, 1.0, 1.5, 2.0, 2.5\}$ from the base of the arm and figure 4.13 shows the values of Y at these optimal postures as a function of the reach d .

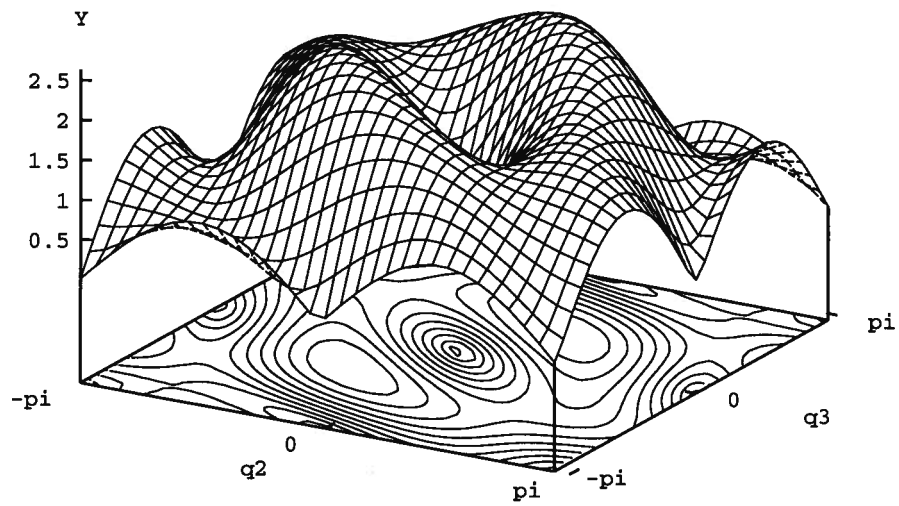


Figure 4.11: Yoshikawa measure for 3-joint redundant arm

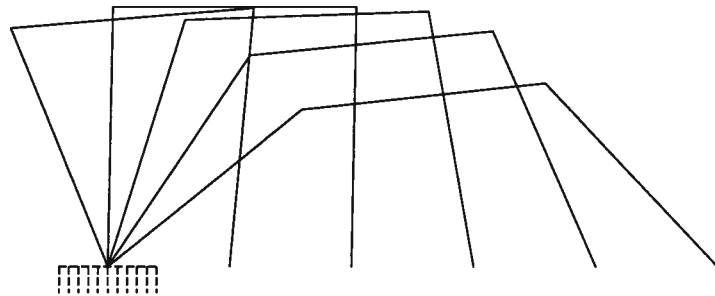


Figure 4.12: Optimal postures for Yoshikawa measure for given end-effector position

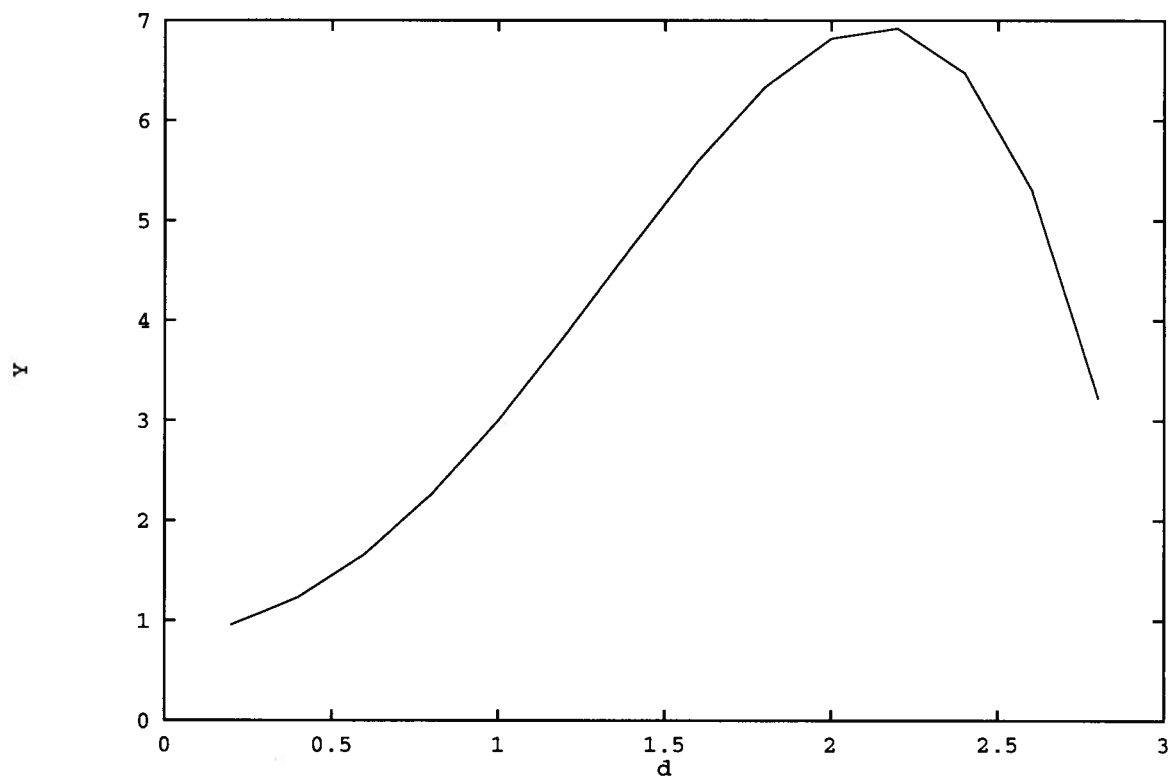


Figure 4.13: Optimal values for Yoshikawa measure

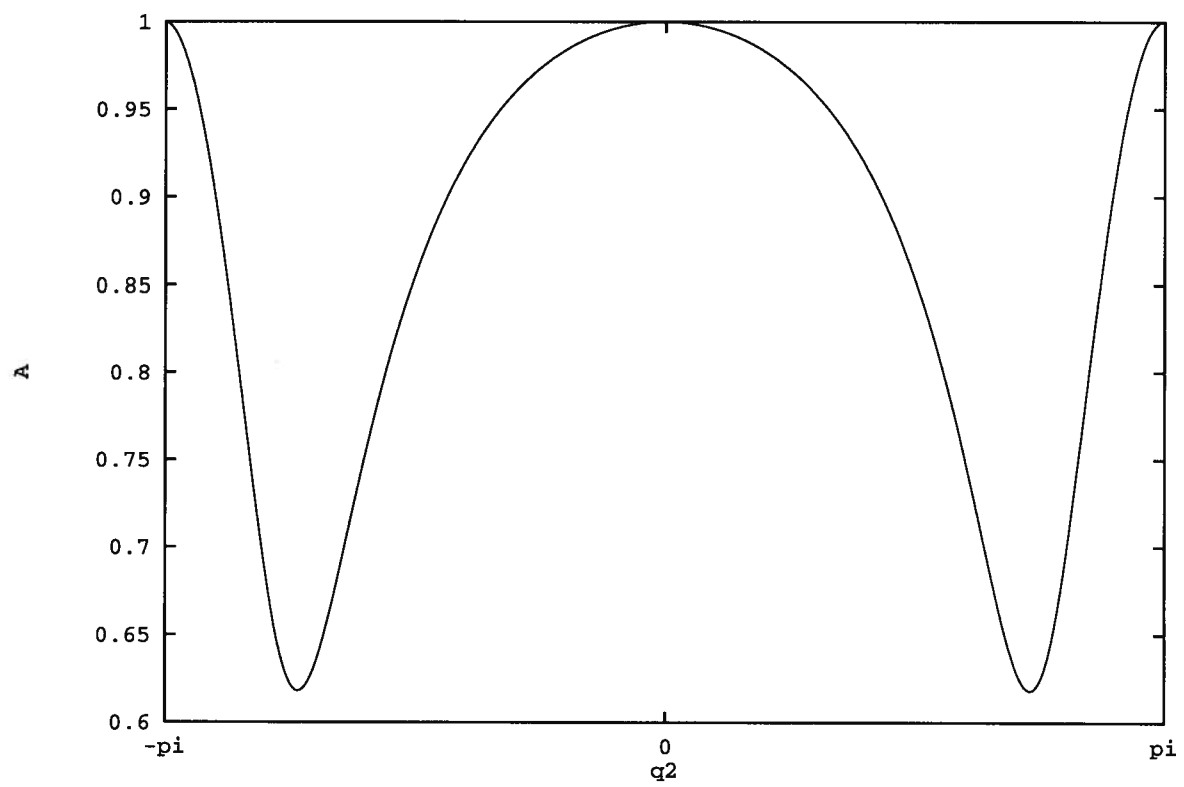


Figure 4.14: Kinematic anisotropy measure for two link arm

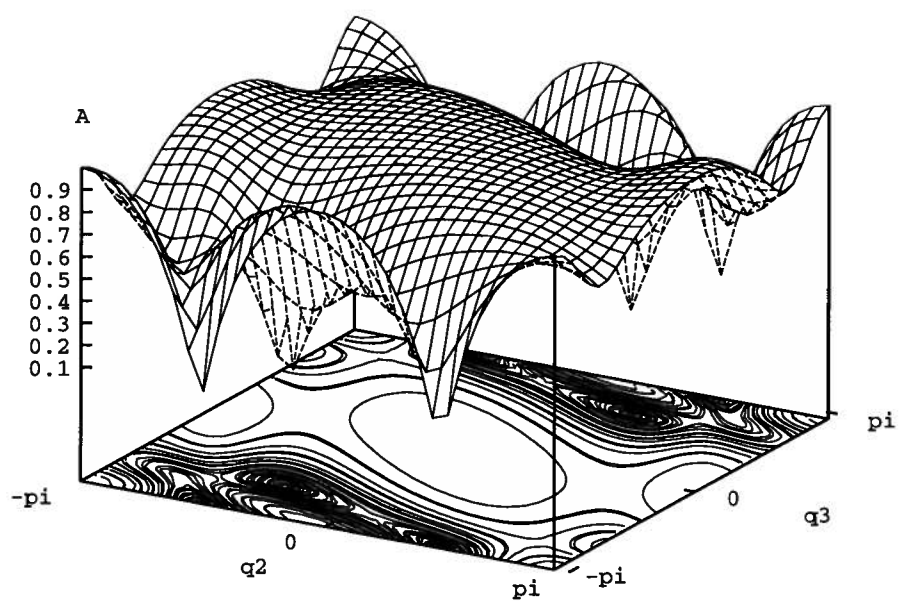


Figure 4.15: Kinematic anisotropy measure A for 3-joint redundant arm

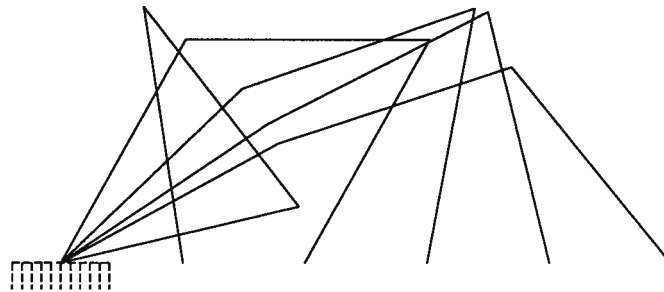
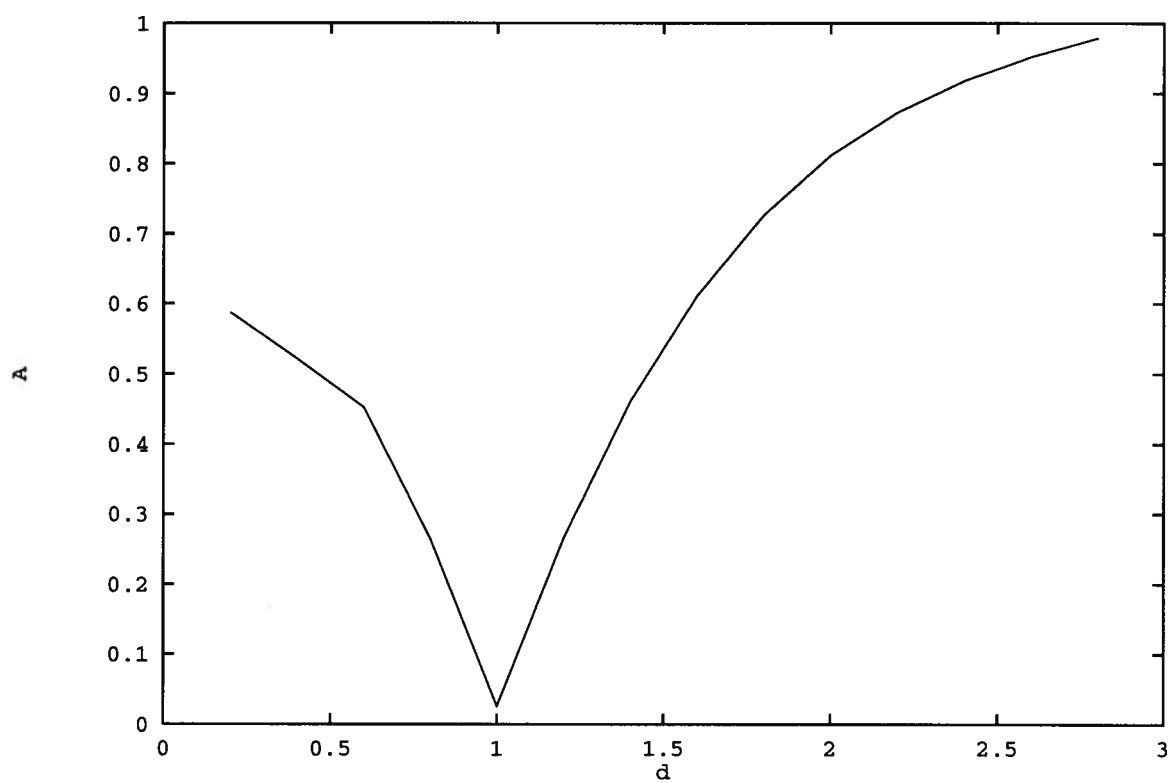


Figure 4.16: Optimal postures for A

Figure 4.17: Optimal values for A

The anisotropy measure A , as defined in eq. (2.6), measures the deviation from kinematic isotropy. It is zero if the manipulability and force ellipses become circles. In figure 4.14 we plot A as a function of the second joint angle q^2 for the two link arm. Figures 4.15, 4.16, and 4.17 show the graph of A for the three link arm, its optimal postures and the values of A for these postures. There is a completely isotropic posture ($A = 0$) at a reach of $d = 1$.

Note that if the arm moves inward from a stretched position, there is a discontinuity in the optimal arm posture between $d = 1.0$ and $d = 0.5$. A more detailed investigation (a more dense set of postures) shows that the *critical distance* d_c is located at $d_c \approx 0.5$. The discontinuity arises as follows. For a given position $\mathbf{x}$ of the end-effector, the possible postures form the self-motion manifold $\mathcal{N}_{\mathbf{x}}$. The optimal posture is found as the global minimum (or maximum) of the measure on $\mathcal{N}_{\mathbf{x}}$. If for a certain $\mathbf{x}$ there are *two* global minima¹, an infinitesimal displacement $d\mathbf{x}$ of $\mathbf{x}$ can resolve the degeneracy. It will depend on the direction of $d\mathbf{x}$ in general which of the two minima will become the new global minimum. It follows that a continuous motion that optimizes the measure through $\mathbf{x}$ is not possible in general. An interesting observation is that since $\mathcal{N}_2$ is not necessarily connected, a motion may be required from one connected component to the other, which is not possible. In that case there is *no* self-motion possible at all that optimizes the measure. For the three link arm, the connected components of $\mathcal{N}_{\mathbf{x}}$, which appear for $d < 1$, are related by the symmetry $q^i \rightarrow -q^i$. As the measures we consider all observe this symmetry, there will always be two symmetry-related global minima. By picking the appropriate one, the arm is thus never forced to move from one connected component to the other, so that self-motion optimizing the measure is guaranteed to be possible. This is true for the $\mathcal{W}$ space measures, but not for the redundancy measures in general, for

¹We refer here to “accidentally” degenerate minima that are not related to each other by a symmetry of the system.

the task space and therefore the task metric $\hat{\mathbf{h}}$ need not have the same symmetries as the manipulator (though it does in all our examples).

The Riemann tensor and therefore the curvature measure R , as defined in eq. (2.9), vanishes for the kinematic metric on $\mathcal{C}$, as the h_{ij} are constant.

4.2 Inertial Measures

The generalized inertia matrix $\mathbf{H}$ of the system is given by

$$H_{ij} = \frac{\partial E_k(\mathbf{q}, \dot{\mathbf{q}})}{\partial \dot{q}^i \partial \dot{q}^j}$$

where E_k is the kinetic energy of the arm. We take the $\mathcal{C}$ space metric as

$$h_{ij} = H_{ij}.$$

The line-element

$$ds^2 = h_{ij} dq^i dq^j$$

measures the displaced mass over the distance $d\mathbf{q}$ in $\mathcal{C}$. If the arm moves by $d\mathbf{q}$ in a time dt with constant velocity, the kinetic energy of this motion is given by

$$E_k = \frac{ds^2}{dt^2}.$$

The geodesics in $\mathcal{C}$ are now precisely the motions of the arm in the absence of torques, friction and gravity.

The induced metric $\mathbf{g}$ can be interpreted as an effective inertia matrix on $\mathcal{W}$. That is, if the end-effector moves with velocity $\dot{\mathbf{x}}$ in such a way as to minimize the total kinetic energy of the arm, this kinetic energy is given by

$$E_k = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu.$$

The tensor $g_{\mu\nu}$ can be measured, in principle, by applying forces to the end-effector, with the arm at rest, and recording the resulting accelerations of the end-effector. For if a force f_μ is applied to the end-effector, with the arm at rest, the $\mathcal{W}$ space acceleration $\ddot{x}^\mu$ satisfies

$$\ddot{x}^\mu = J_i^\mu \ddot{q}^i + \frac{\partial J_i^\mu}{\partial q^j} \dot{q}^i \dot{q}^j = J_i^\mu \ddot{q}^i. \quad (4.21)$$

The equations of motion give

$$\frac{d}{dt}(h_{ij}\dot{q}^j) = h_{ij}\ddot{q}^j = f_\mu J_i^\mu. \quad (4.22)$$

Combining eq. (4.21) and eq. (4.22) gives

$$\ddot{x}^\mu = h^{ij} J_i^\mu J_j^\nu f_\nu$$

or

$$f_\mu = g_{\mu\nu} \ddot{x}^\nu.$$

Contracting $g_{\mu\nu}$ with a velocity vector v^μ gives the required kinetic energy of the arm to move the end-effector with this velocity.

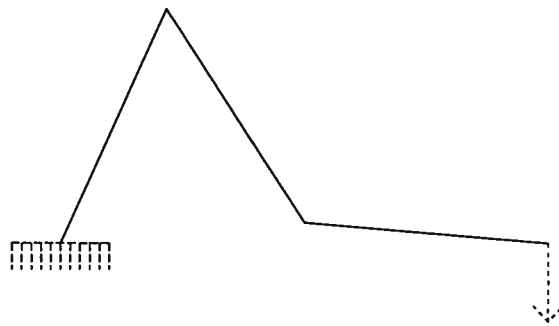


Figure 4.18: Good posture for vertical swatting

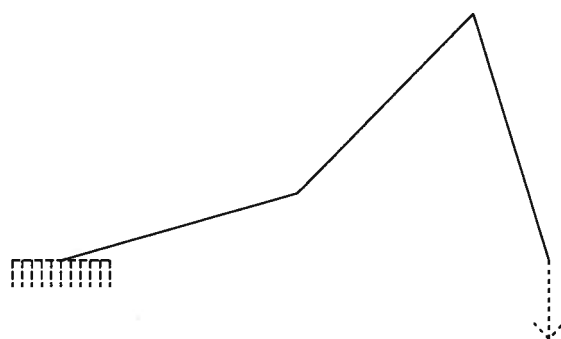


Figure 4.19: Bad posture for vertical swatting

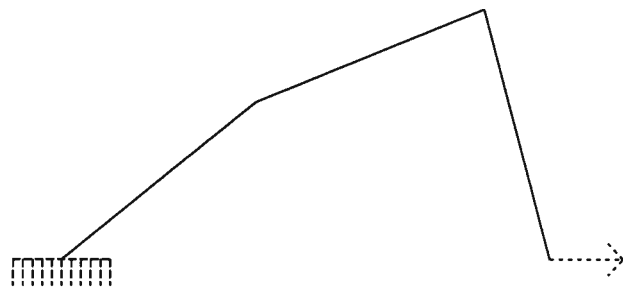


Figure 4.20: Good posture for horizontal swatting

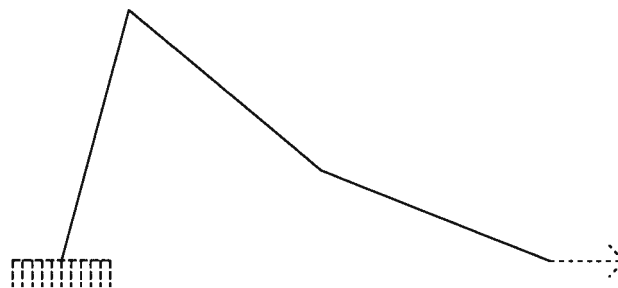


Figure 4.21: Bad posture for horizontal swatting

In figures 4.18, 4.19, 4.20, and 4.21 we show the best and worst postures for achieving a high vertical and horizontal velocity of the end-effector (“fly-swatting”) at a distance $d = 2.0$ from the base of the arm.

The generalized Yoshikawa measure as given in eq. (2.5) is now the inverse of the square root of the determinant of the effective inertia matrix; it measures the “lightness” of the arm. For the two link arm it is given by

$$\frac{|\sin(q^2)|}{\sqrt{m_2(m_1 + m_2\sin^2(q^2))}}.$$

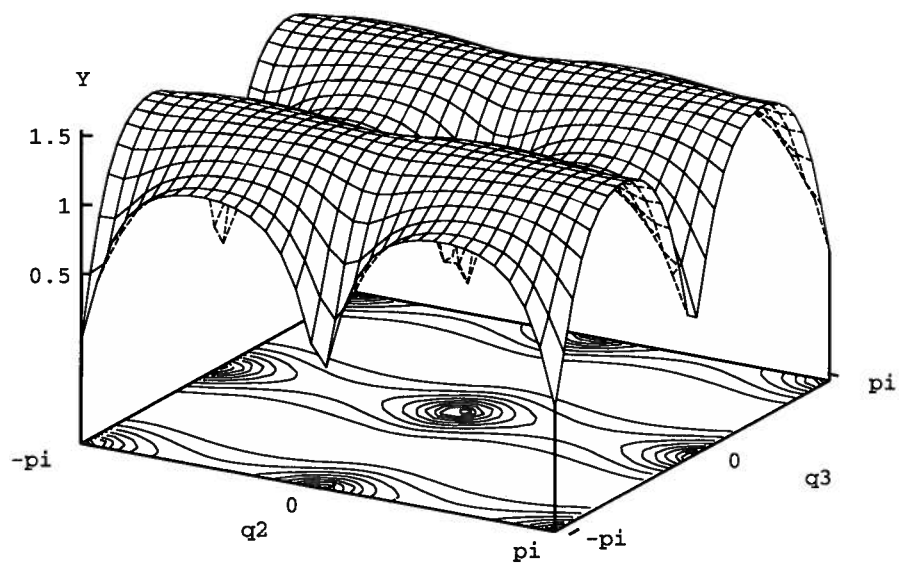


Figure 4.22: Inertial Yoshikawa measure Y for 3-joint redundant arm

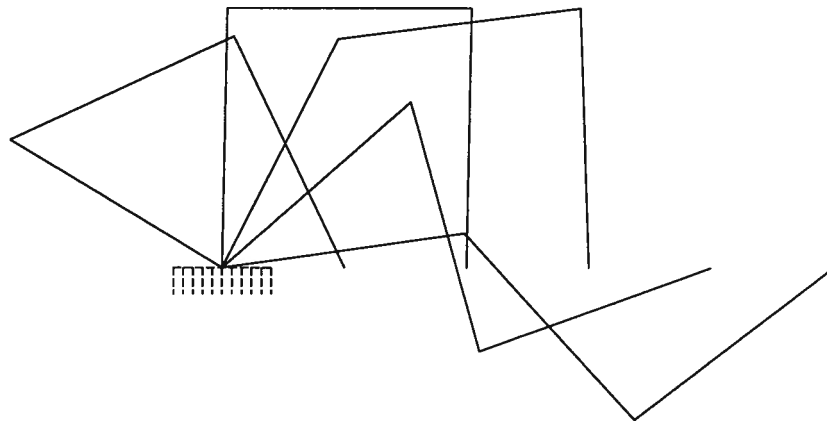
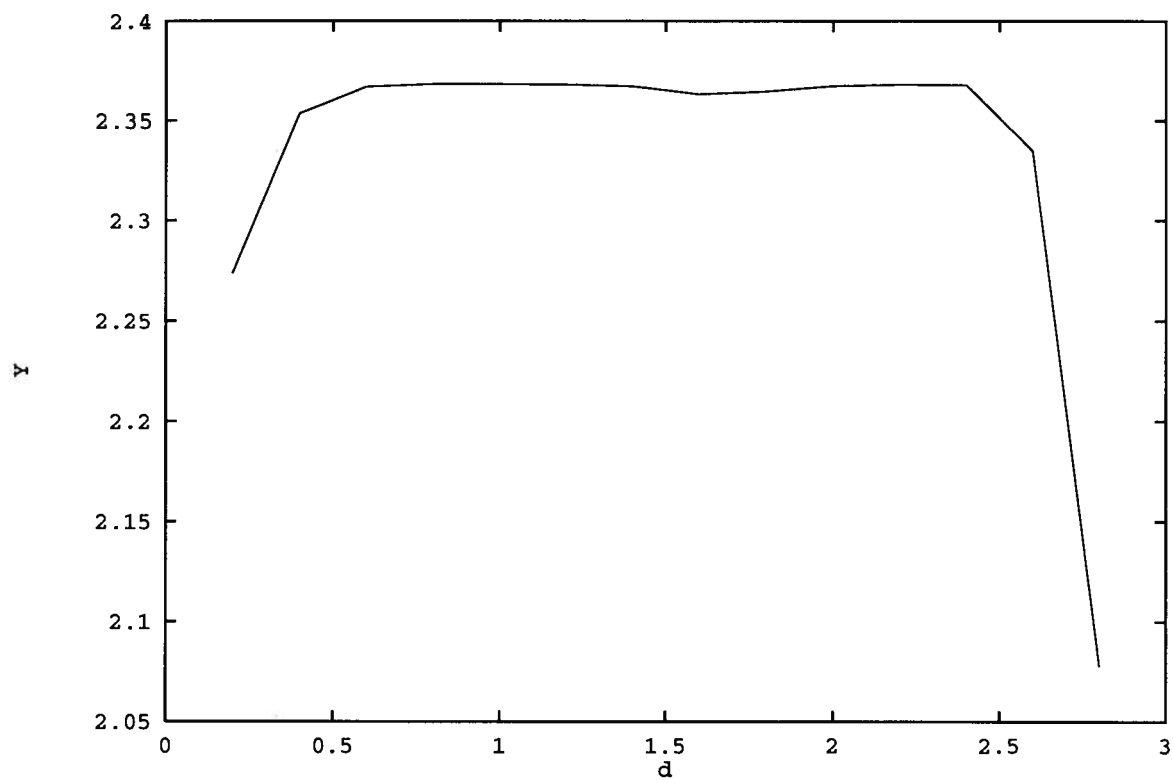


Figure 4.23: Optimal postures for Y

Figure 4.24: Optimal values for Y

In figure 4.22 we plot this measure for the three link arm. Due to the relatively flat ridges at $q^3 = \pm\pi/2$, there is a large range of postures with good values of this measure as can be seen in figures 4.23 and 4.24. There is a critical distance $d_c \approx 1.5$ where there is a discontinuity in posture as a function of the reach.

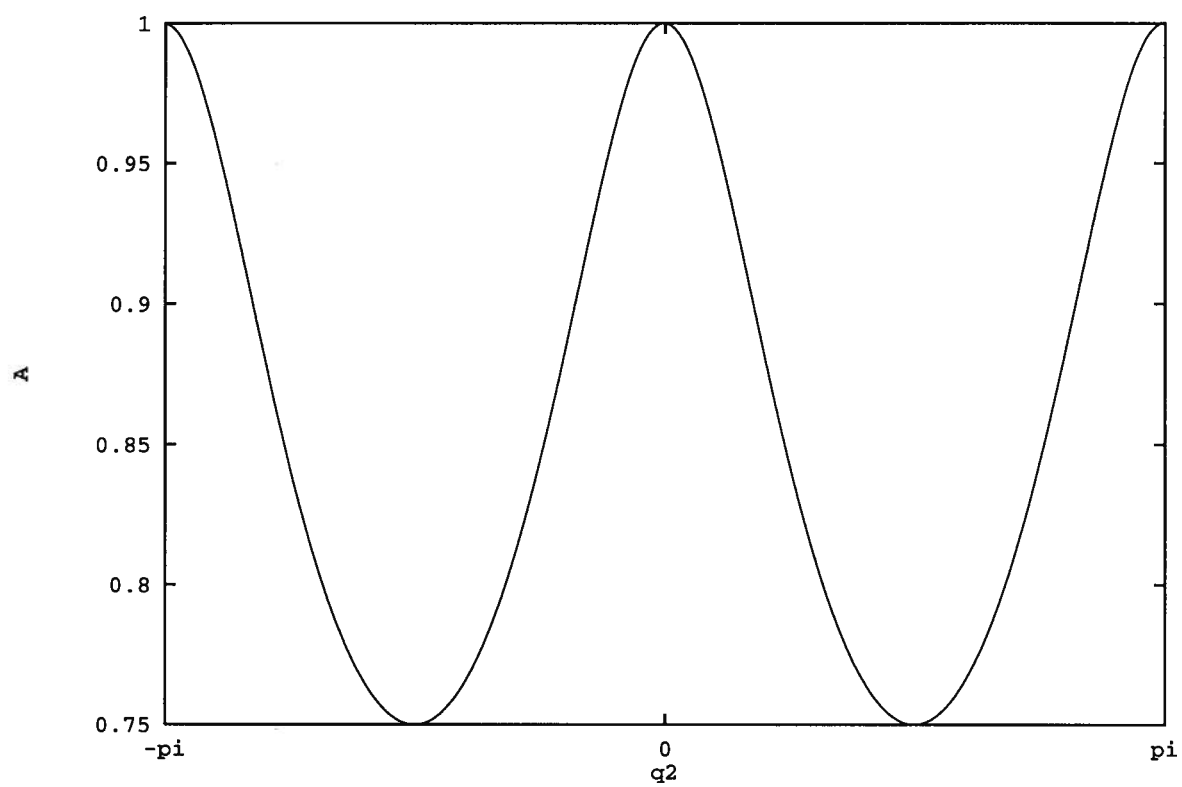


Figure 4.25: Inertial anisotropy measure for 2-joint arm

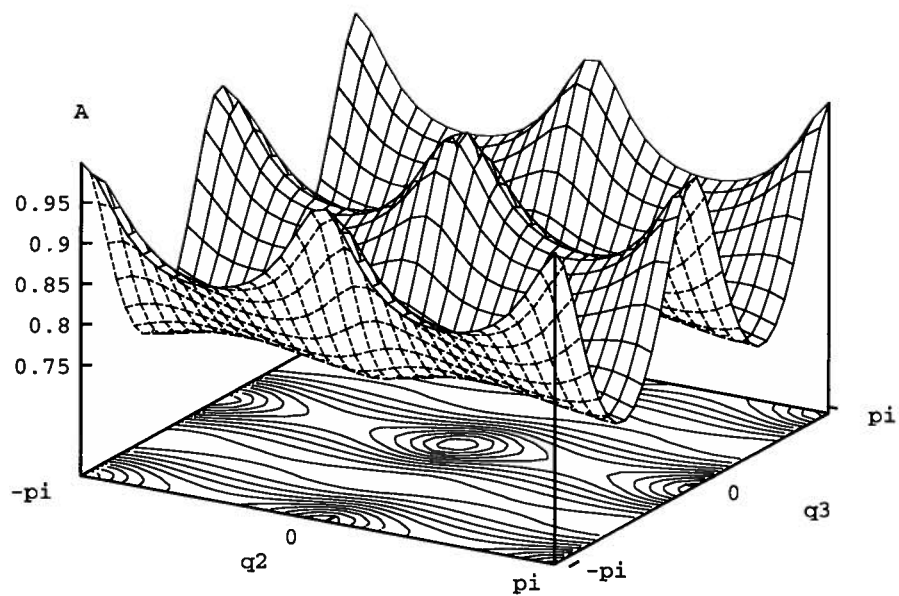


Figure 4.26: Inertial anisotropy measure for 3-joint redundant arm

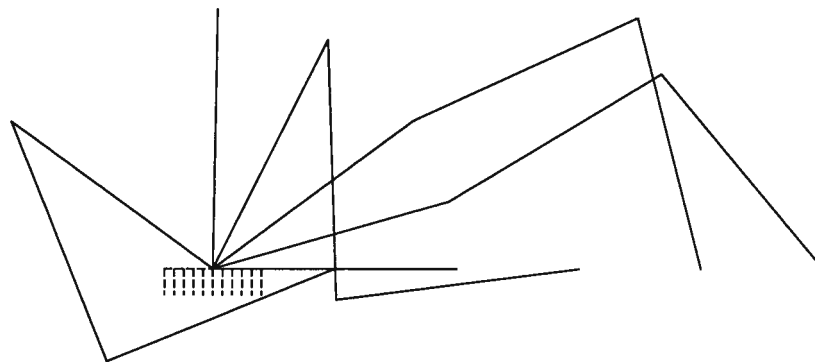
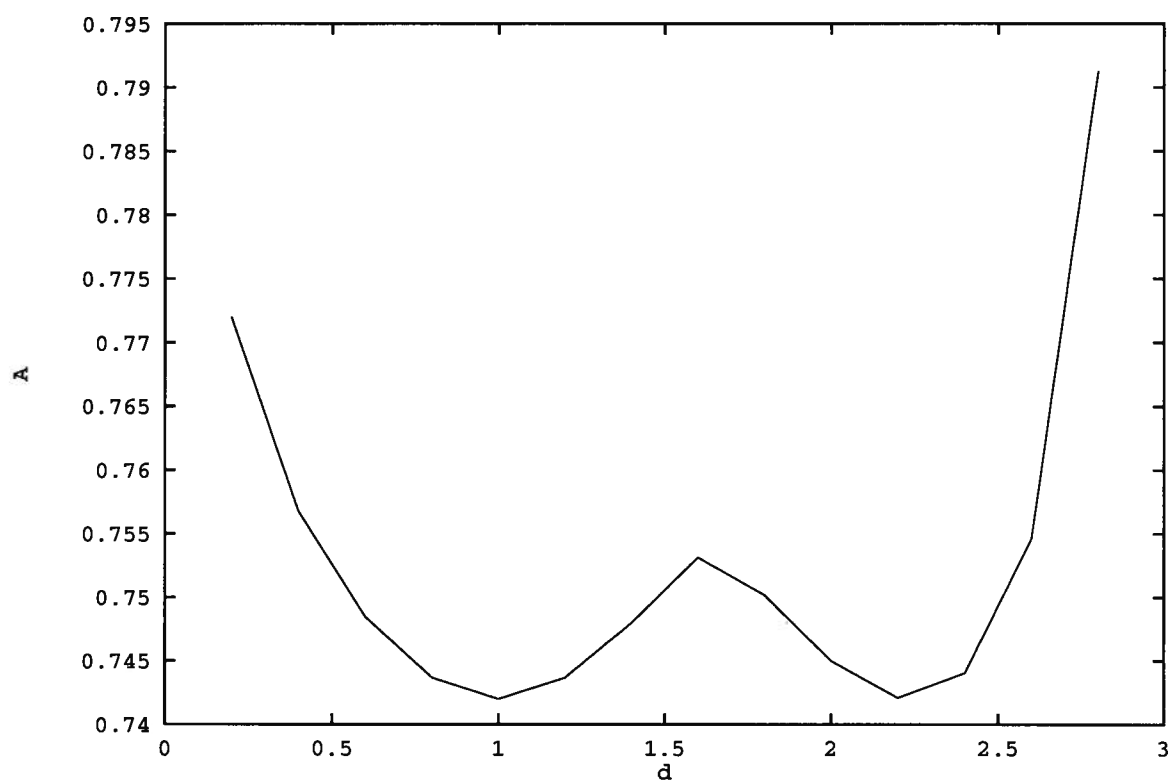


Figure 4.27: Optimal postures for A

Figure 4.28: Optimal values for A

In figure 4.25 we show the inertial anisotropy, as defined in eq. (2.6) for the two link arm.

Figures 4.26, 4.27, and 4.28 show this measure for the three link arm. It also has two valleys at $q^3 = \pm\pi/2$, indicating the existence of a range of “good” postures for inertial isotropy. As the shape of the valleys is different than that of the the ridges for Y , the optimal postures are quite different though. Note that there is no isotropic point in $\mathcal{W}$. The critical distance is at $d_c \approx 1.7$.

The curvature scalar R as given in eq. (2.9) is now a measure of the geodesic deviation of orbits in $\mathcal{C}$ space and is a measure of the non-linearity of the motion of the arm. For the two link arm it is given by

$$R = \frac{2m_1 \cos(q^2)}{l_1 l_2 (m_1 + m_2 \sin^2(q^2))^2}.$$

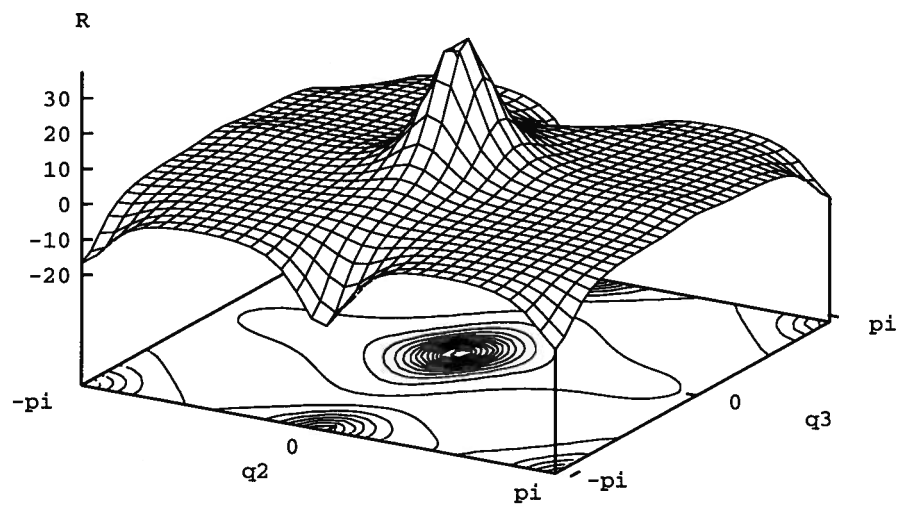
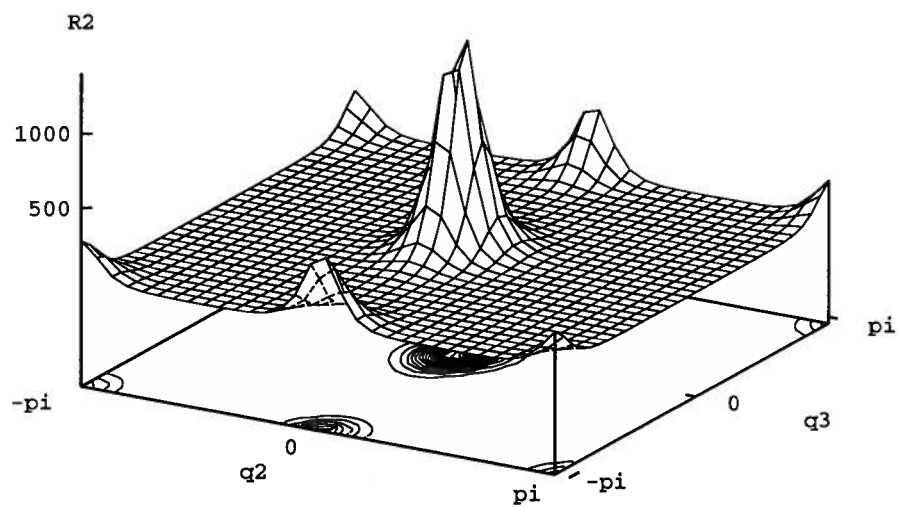


Figure 4.29: Curvature scalar R for 3-joint redundant arm

Figure 4.30: R^2 for 3-joint redundant arm

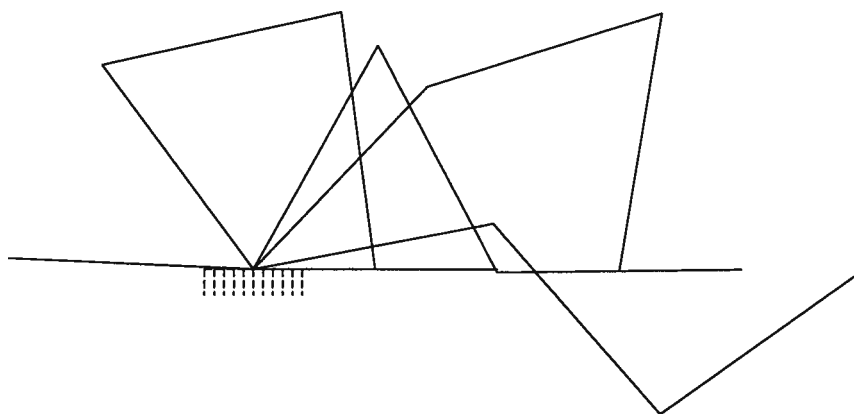
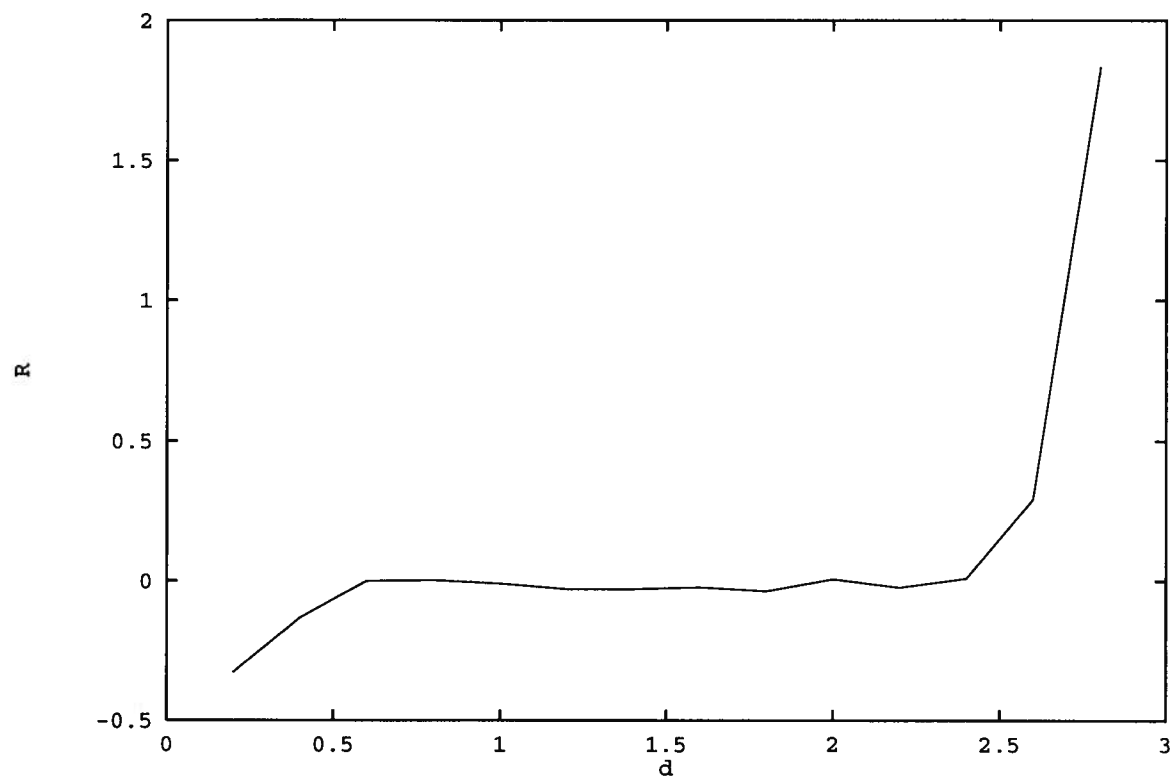


Figure 4.31: Optimal postures for R

Figure 4.32: Optimal values for R

The curvature scalar is plotted for the three link arm in figure 4.29. In figure 4.30 we show R^2 which is the actual measure to minimize for maximal linearity of motion. As R is positive near the origin of the (q^2, q^3) plane and negative at the edges, R^2 is zero on a two-dimensional surface in $\mathcal{C}$, where R changes sign. So there is a two-dimensional region in $\mathcal{W}$ that can be reached with “linear” postures. R has local extrema at the singular points $(0, 0)$ ($R = 38.1$), $(0, \pi)$ ($R = -21.5$) and (π, π) ($R = -16.3$), but not at $(\pi, 0)$. It is interesting to note that this gives a classification of the singular points according to their “non-linearity”, as the values of the extrema are distinct. The optimal postures and values of R are shown in figures 4.31 and 4.32. The critical distance is $d_c \approx 1.5$.

4.3 Redundancy Measures

Let us choose the inertial metric as the metric $\mathbf{h}$ on $\mathcal{C}$, and the kinematic metric as the “secondary” task metric $\hat{\mathbf{h}}$. The redundant generalized Yoshikawa measure measures the Euclidean distance per mass moved in $\mathcal{C}$ for self-motions. That is, if Y is small, a “large amount of matter” has to be moved to change the posture of the arm. Y should thus be maximized for optimal mobility in this sense. As the inverse $D = 1/Y$ behaves better than Y itself we have shown data for D instead of Y itself.

The data for this measure for the three link arm is given in figures 4.33, 4.34, and 4.35. The optimal values of D as a function of d (i.e. the distance of the end-effector from the base of the arm) is relatively constant for $d > 1.0$ but increases fast below that value. In figure 4.36 we have plotted d as a function of (q^2, q^3) , from which we see that at small d the arm will come near the large peaks in D , which explains this behavior. The critical distance is at $d_c \approx 1.1$.

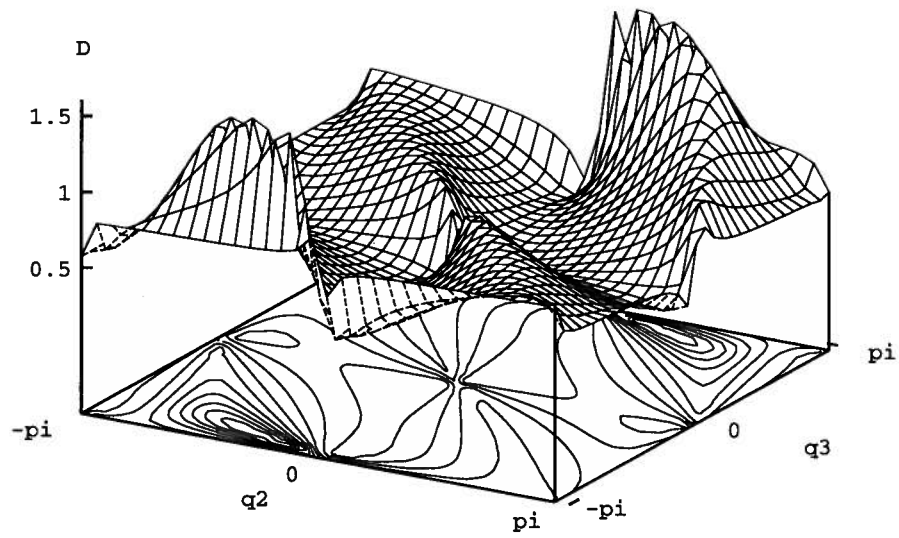


Figure 4.33: Redundancy measure D for 3-joint redundant arm

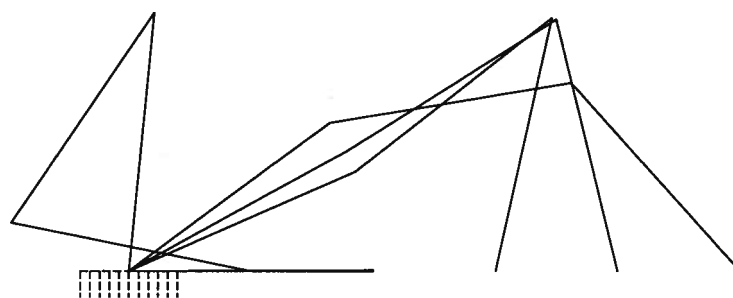
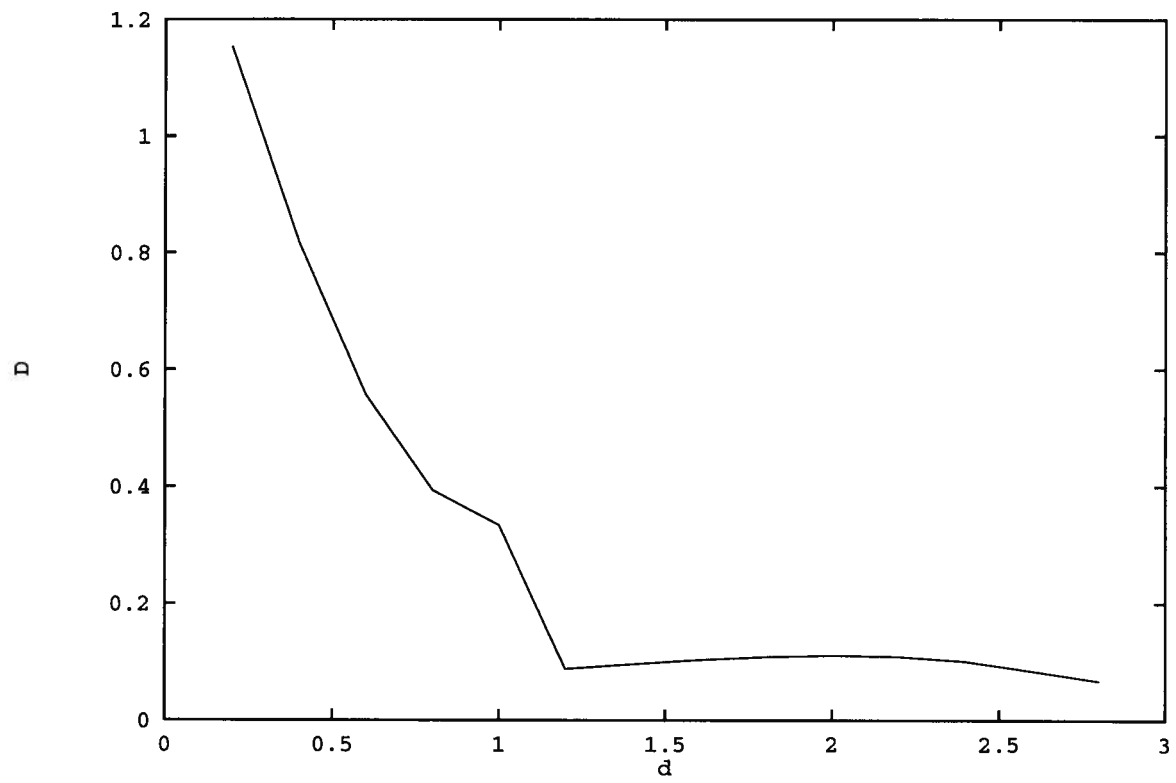
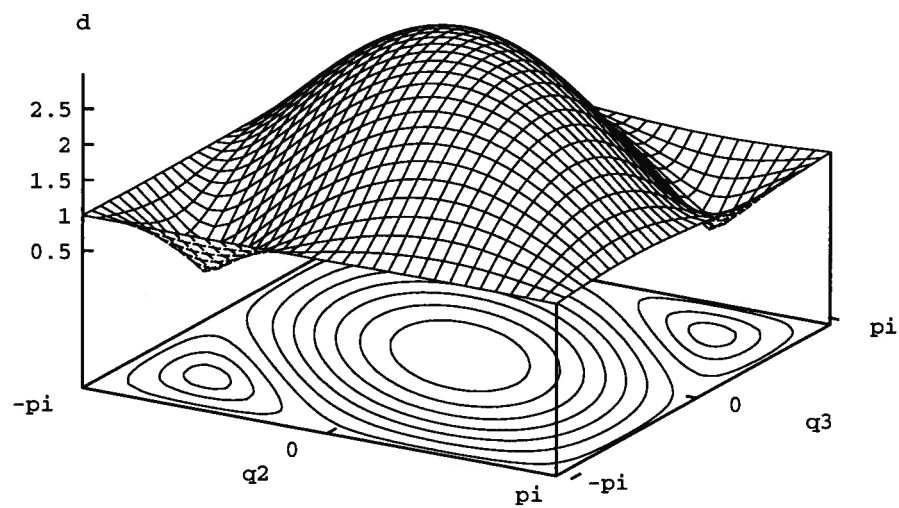


Figure 4.34: Optimal postures for D

Figure 4.35: Optimal values for D

Figure 4.36: Distance of end-effector from base d

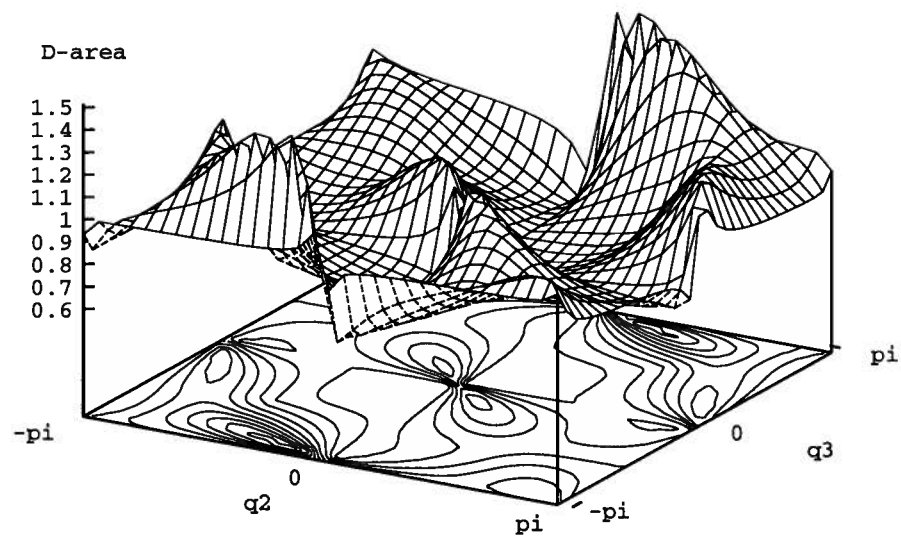


Figure 4.37: Area redundancy measure D for 3-joint redundant arm

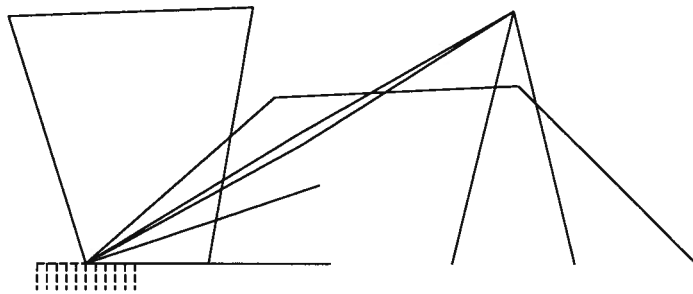
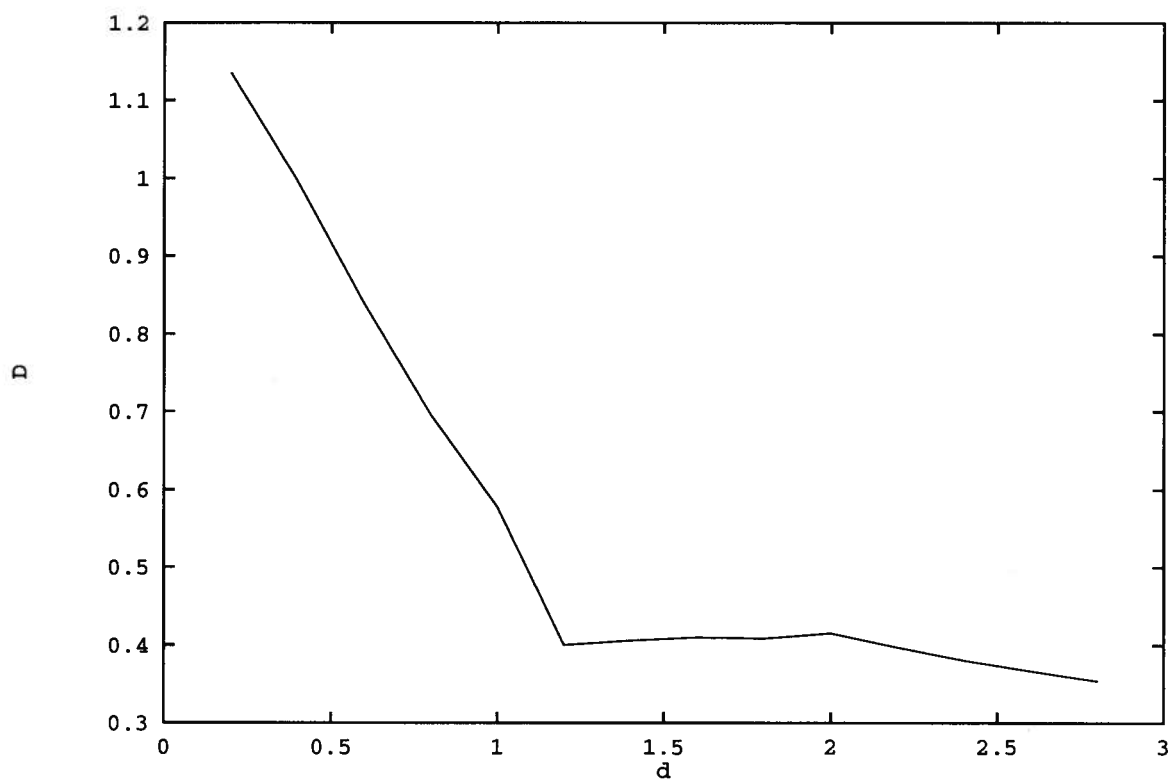


Figure 4.38: Optimal postures for D

Figure 4.39: Optimal values for D

We would like to interpret this measure as a measure of how much the arm “swings” when reconfiguring. A better measure for this would be the area that the arm sweeps out per Euclidean distance in C . Unfortunately, the “swept-area” (or volume for three-dimensional manipulators) metric cannot be described by a quadratic form, so this type of measure falls outside our framework. For comparison we have computed this “swept-area” version of the redundancy measure D . It is given in figures 4.37, 4.38, and 4.39. We observe that this measure behaves qualitatively similar, which supports the interpretation of D given above.

A different class of redundant generalized Yoshikawa measures can be obtained by choosing the “secondary” task metric on C as

$$\hat{h}_{ij}^{(e)} = \begin{cases} 1, & \text{if } i = j = e, \\ 0, & \text{otherwise.} \end{cases}$$

This metric just measures the movement of one particular joint, e and the corresponding redundancy measure $D = 1/Y$ will measure the amount of mass moved per movement of joint e . This could be useful if we want to find an arm position that allows reconfiguration with minimal movement of a particular joint. We call these measures (there are n of them for an n link arm) the joint redundancy measures

$$E^{(e)} = Y^{(e)}.$$

$E^{(e)}$ measures how much joint e has to move to move a specified amount of mass. It should be minimized if we want to reconfigure the arm with as little movement as possible of this particular joint.

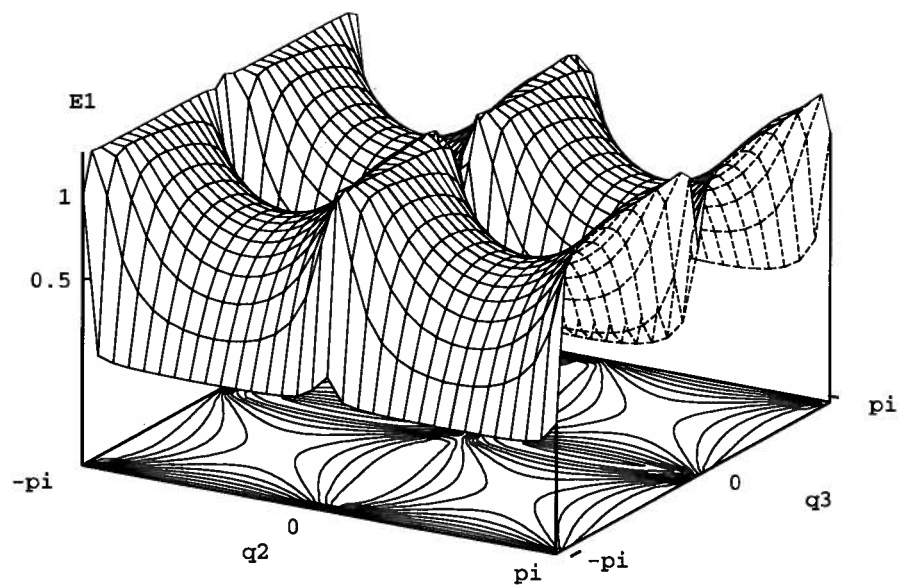
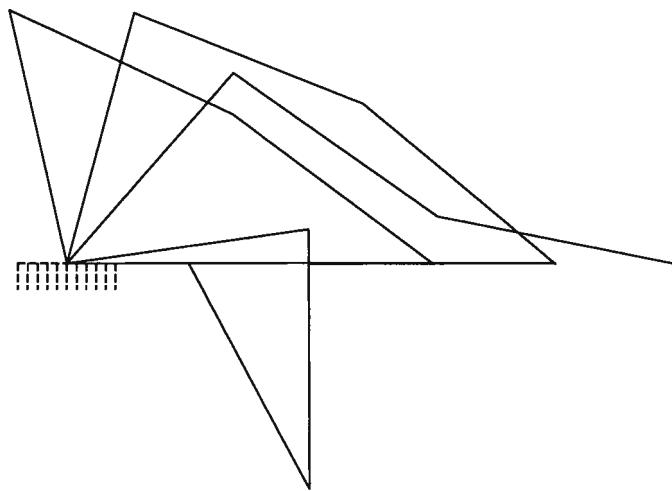
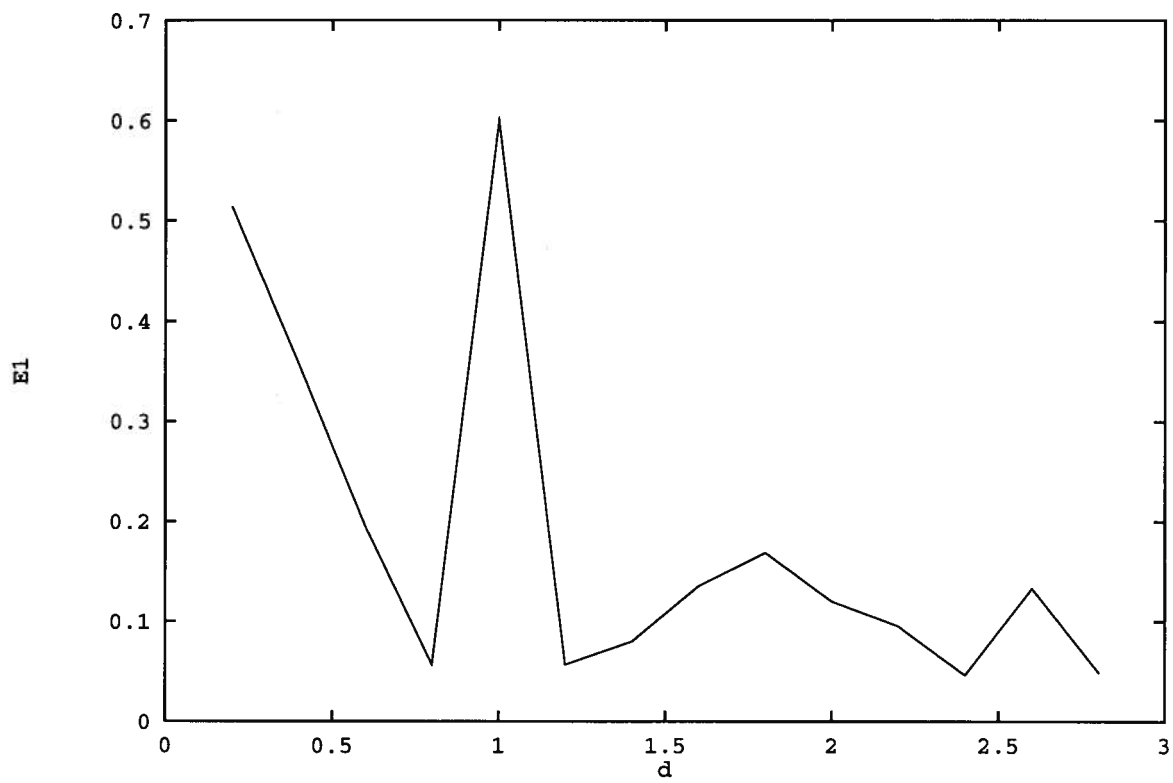


Figure 4.40: Joint redundancy measure $E^{(1)}$ for 3-joint redundant arm

Figure 4.41: Optimal postures for $E^{(1)}$

Figure 4.42: Optimal values for $E^{(1)}$

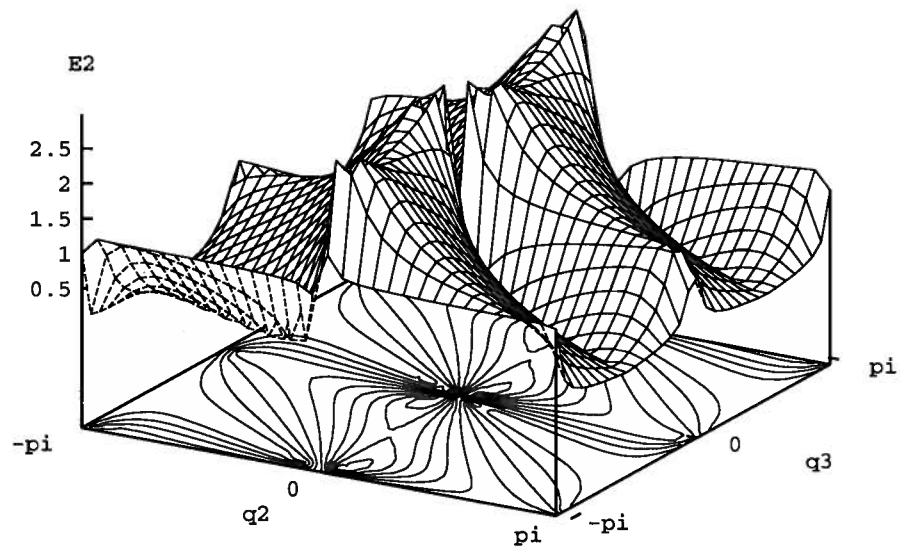
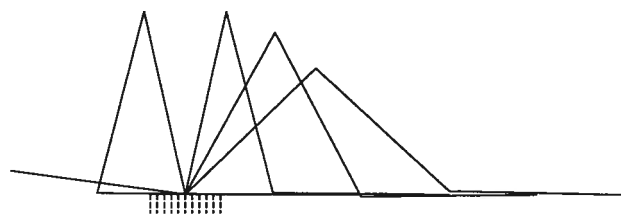
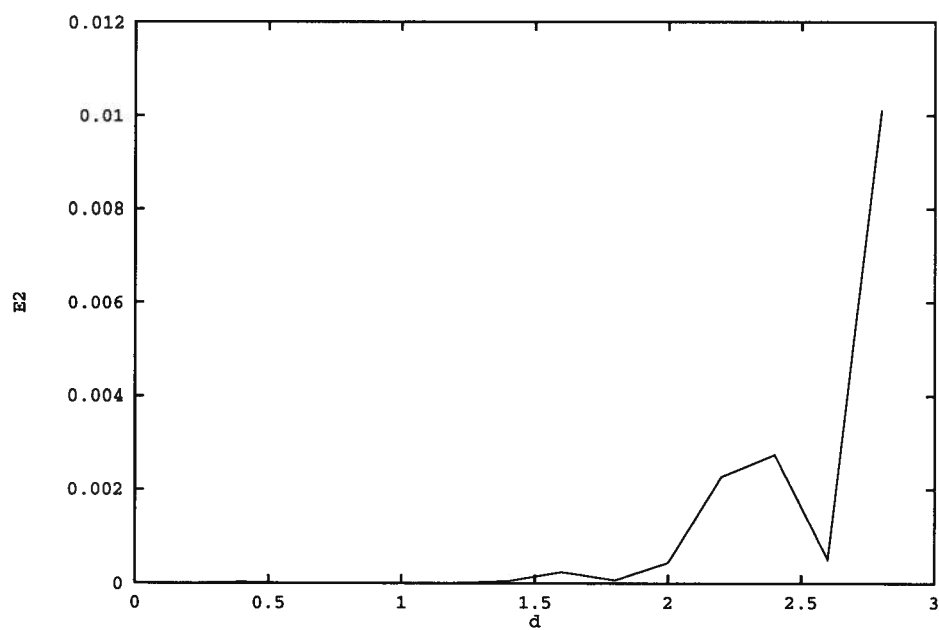


Figure 4.43: Joint redundancy measure $E^{(2)}$ for 3-joint redundant arm

Figure 4.44: Optimal postures for $E^{(2)}$ Figure 4.45: Optimal values for $E^{(2)}$

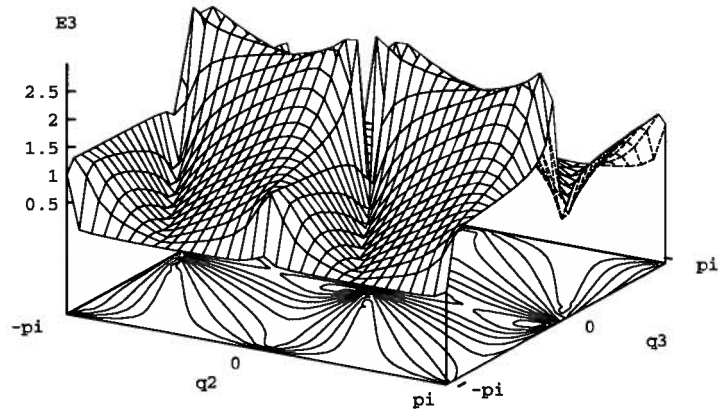


Figure 4.46: Joint redundancy measure $E^{(3)}$ for 3-joint redundant arm

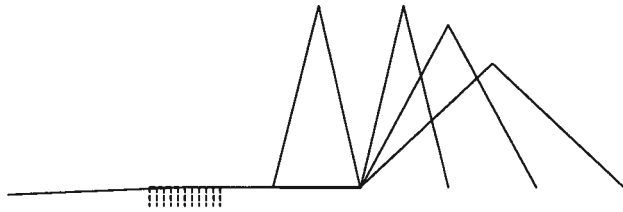
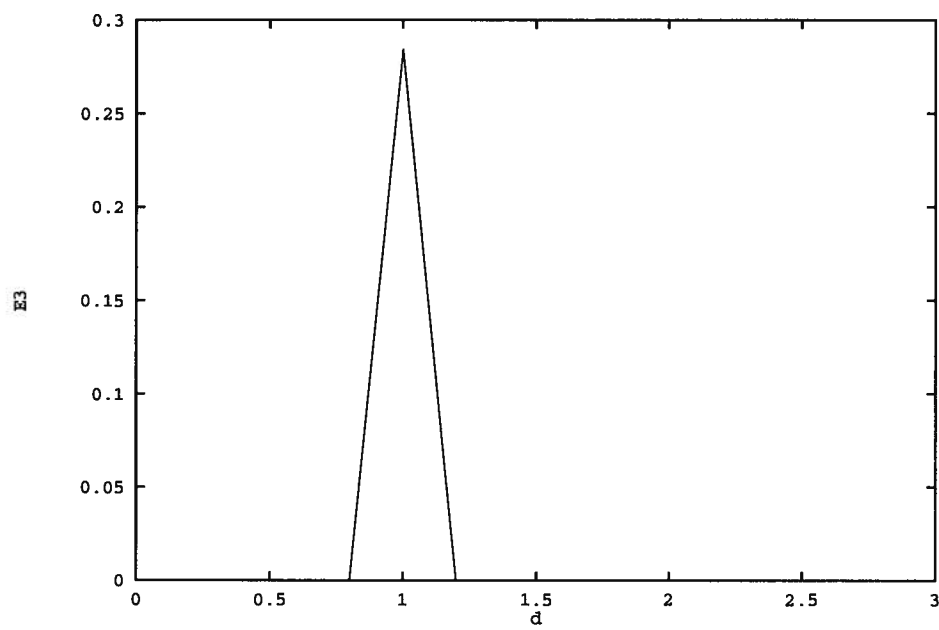


Figure 4.47: Optimal postures for $E^{(3)}$

Figure 4.48: Optimal values for $E^{(3)}$

Plots, optimal postures and optimal values of the joint redundancy measures are given in figures 4.40 - 4.48. The critical distances are $d_c \approx 1.1$, $d_c \approx 0.9$ and $d_c \approx 1.0$ for $E^{(1)}$, $E^{(2)}$ and $E^{(3)}$ respectively.

The last example of a secondary generalized Yoshikawa measures can be obtained by choosing the “secondary” metric on $\mathcal{C}$ according to

$$ds^2 = \text{Euclidean displacement of a test point on a link.}$$

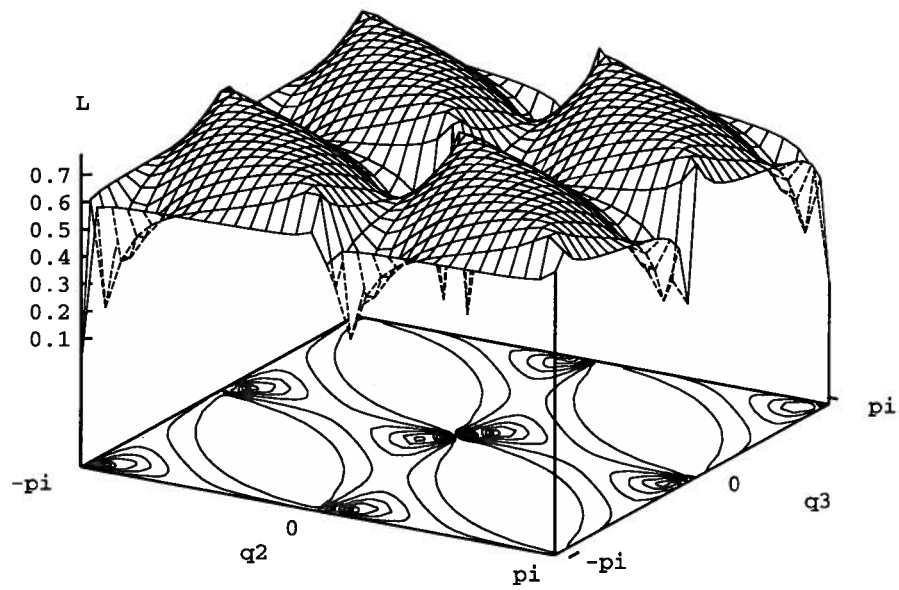
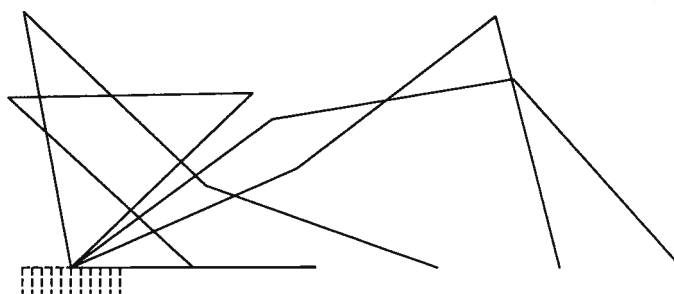
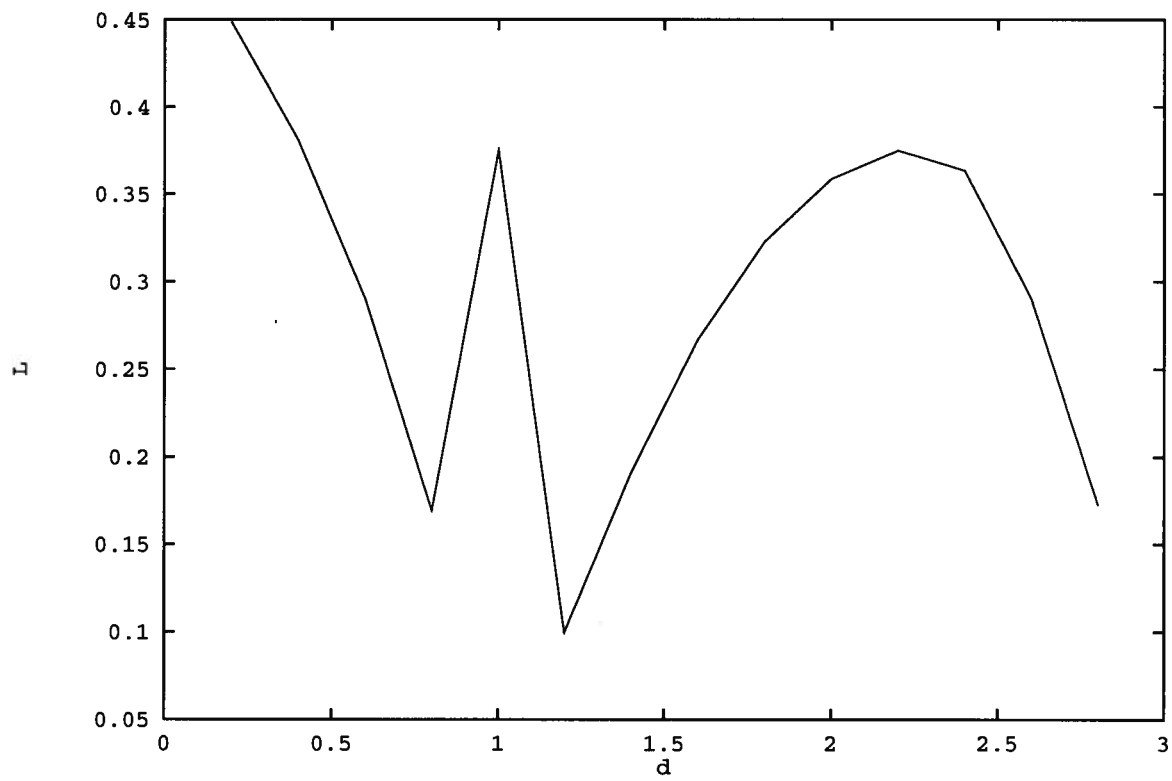


Figure 4.49: Link redundancy measure L for 3-joint redundant arm

Figure 4.50: Optimal postures for L

Figure 4.51: Optimal values for L

For concreteness, we take the test point to be the mid-point of link two. This metric measures the movement of the center of the middle link and the corresponding redundancy measure $D = 1/Y$ will measure the amount of mass moved per movement of this link. This could be useful if we want to find an arm position that allows reconfiguration with minimal movement of this link. We call this measure the link redundancy measure

$$L = Y.$$

L measures how much the middle link has to move in order to move a specified amount of mass. It should be minimized if we want to reconfigure the arm with as little movement as possible of this particular link.

Plots, optimal postures and optimal values of the link redundancy measure are given in figures 4.49, 4.50, and 4.51. The critical distance is $d_c \approx 1.5$.

Chapter 5

Applications To A Closed Loop Linkage

The closed loop planar kinematic chain we shall investigate here is depicted in figure 5.52. It consists of five links, four actuated joints $(\theta_L^1, \theta_L^2, \theta_R^1, \theta_R^2)$ and two passive joints (θ_L^3, θ_R^3) . The end-effector is attached to the center of the middle link. We denote its position and orientation by the work space coordinates (x, y, α) . The distance between joints L^1 and R^1 is d and the length of the middle link is b . The other four links are taken to have unit length.

The descriptor space of the device is labeled by the four joint angles $(\theta_L^1, \theta_L^2, \theta_R^1, \theta_R^2)$ which must satisfy one constraint on C^+ (or equivalently four constraints on $C^+ \times \mathcal{W}$).

We shall take the C^+ -space metric to be given by the

line element

$$ds^2 = (d\theta_L^1)^2 + (d\theta_L^2)^2 + (d\theta_R^1)^2 + (d\theta_R^2)^2,$$

i.e. the kinematic metric. This metric defines the distance between configurations in terms of the actuated joint movements.

The chain can be interpreted as an object with configuration (x, y, α) manipulated by two planar fingers with joint angles (θ_L^1, θ_L^2) and (θ_R^1, θ_R^2) with contacts at $\mathbf{p}_L$ and $\mathbf{p}_R$.

We note that the device has a dual interpretation as a standing posture of a walking machine. The joints θ_R^1 and θ_R^2 will be passive in this case, representing the feet of the machine and the other four joints will be actuated. In this case an appropriate choice of metric will involve those actuated joints.

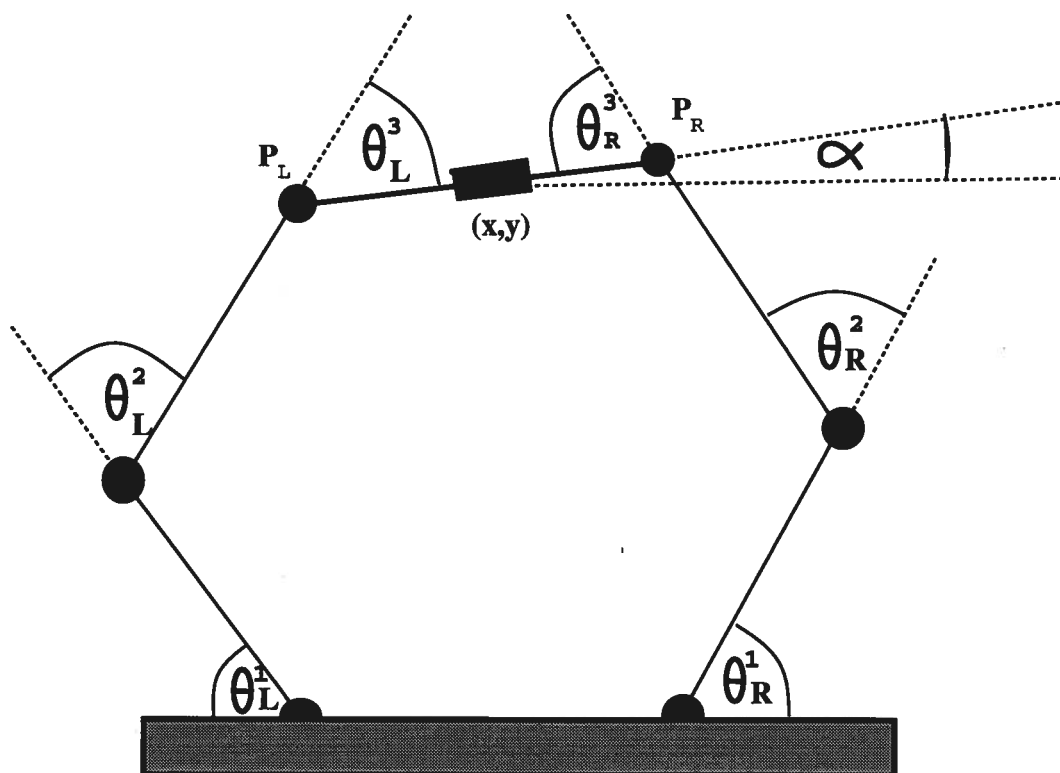


Figure 5.52: Closed loop manipulator.

The coordinates of the passive joints $\mathbf{p}_L$ and $\mathbf{p}_R$ are given by

$$\begin{aligned} p_L^x &= -d/2 - c_L^1 - c_L^{12}, \\ p_L^y &= s_L^1 + s_L^{12}, \\ p_R^x &= d/2 + c_R^1 + c_R^{12}, \\ p_R^y &= s_R^1 + s_R^{12}, \end{aligned}$$

where we have written

$$\begin{aligned} c_L^1 &= \cos \theta_L^1, \\ c_R^1 &= \cos \theta_R^1, \\ c_L^{12} &= \cos(\theta_L^1 + \theta_L^2), \\ c_R^{12} &= \cos(\theta_R^1 + \theta_R^2), \end{aligned}$$

and similarly for the sine functions. The constraints and forward kinematics of the system are given by the four equations

$$f^\Lambda = 0,$$

where $\Lambda = 1, \dots, 4$, and

$$\begin{aligned} f^1 &= (p_R^x - p_L^x)^2 + (p_R^y - p_L^y)^2 - b^2, \\ f^2 &= (p_R^y - p_L^y) \tan \alpha - (p_R^x - p_L^x), \\ f^3 &= x - (c_R^1 - c_L^1 + c_R^{12} - c_L^{12})/2, \\ f^4 &= y - (s_R^1 + s_L^1 + s_R^{12} + s_L^{12})/2. \end{aligned}$$

For this simple system, the constraint on the four joint angles is given by $f^1 = 0$, whereas the forward kinematics is defined by the other three equations.

The three vectors $\mathbf{w}_A$, $A = 1, \dots, 3$, are chosen as

$$\mathbf{w}_1 = (1, 0, 0, -\frac{\partial f^1}{\partial \theta_L^1} / \frac{\partial f^1}{\partial \theta_R^2}),$$

$$\mathbf{w}_2 = (0, 1, 0, -\frac{\partial f^1}{\partial \theta_L^2} / \frac{\partial f^1}{\partial \theta_R^2}),$$

and

$$\mathbf{w}_3 = (1, 0, 0, -\frac{\partial f^1}{\partial \theta_R^1} / \frac{\partial f^1}{\partial \theta_R^2}).$$

The performance metric, as given in the previous section can now be computed with straightforward algebra. To compute the generalized Yoshikawa measure as given in eq. (2.5), we need to define a task metric on $\mathcal{W}$. Note that since we are considering both the position and the orientation of the end-effector, the definition of a nondegenerate metric on $\mathcal{W}$ requires the introduction of a parameter λ which is a characteristic length scale. This length scale needs to be introduced as the Euclidean group does not have a natural positive definite metric, see for example [Lon85]. A similar length scale was arrived at via a different route in [TAR92]. We take the task metric to be defined by the line element

$$ds^2 = dx^2 + dy^2 + \lambda^2 d\alpha^2.$$

The value of λ determines the relative weight we attach to orientation versus position in defining a distance between end-effector positions.

From eq. (2.5) we see that the factor

$$\sqrt{\det(\boldsymbol{\eta})} = \lambda$$

just multiplies the inverse of the square root of the determinant of the performance metric $\mathbf{g}$. The generalized Yoshikawa measure is thus only defined up to a constant multiplicative factor.

We have computed Y explicitly as a function of the four actuated joint angles using Mathematica [Wol91]. Since the resulting expression is too complicated to be given here, we have reduced it for illustration to configurations with $x = 0$ and $\alpha = 0$, corresponding to symmetric postures, with the center of the middle link above the midpoint between the two base joints. There is one remaining degree of freedom, say θ , and if we also take $b = d$, we have

$$\begin{aligned}\theta_L^1 &= \theta_R^1 = \theta, \\ \theta_L^2 &= \theta_R^2 = \pi - 2\theta,\end{aligned}$$

and for Y we obtain

$$Y = \frac{\lambda}{d} \frac{\sin^2(2\theta)}{\sqrt{5(1 - \cos(2\theta))}}.$$

The measure has a maximum value for $\theta = 35.3^\circ$, which represents the optimal posture under the given constraints.

In figure 5.53 we show optimal postures for configurations where the center of the middle link is above the midpoint between the base joints (i.e. at $x = 0$) and the angle α is restricted to have the fixed values $\alpha = 0., 0.5, 1.0, 1.5$.

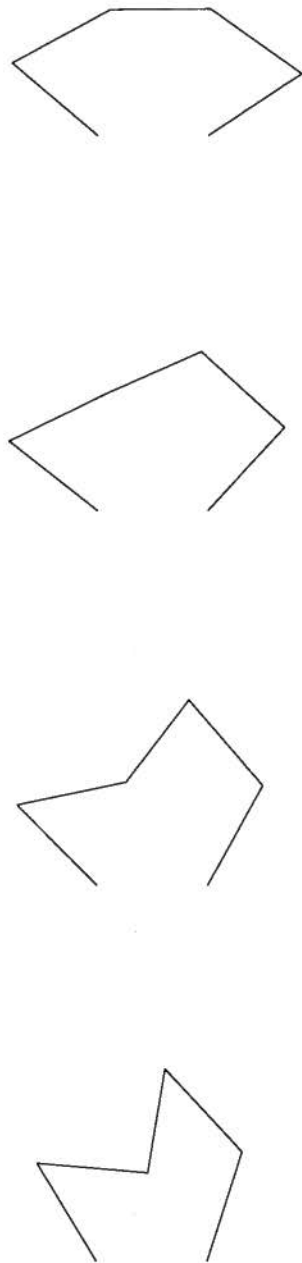


Figure 5.53: Optimal postures for various orientations of the middle link.

Chapter 6

Conclusions

The principal merit of the unified formalism for the construction of performance measures for robot manipulators presented here, is that it allows the systematic construction of invariant measures appropriate for a given application.

Well known measures such as the Yoshikawa measure arise naturally in this framework. In addition, new performance measures have been constructed using this formalism, to measure non-linearity and redundancy (or self-motion). The formalism was presented for systems in which the relation between the manipulator and the object can be given explicitly. The extension to constrained systems was presented in a separate chapter.

Examples of the measures were computed for some manipulators, the planar two and three link arms, and a planar five link closed linkage chain and we computed some optimal postures for these devices.

An interesting property of all measures of the three link arm except the Yoshikawa measure is that there is no smooth motion possible from a stretched position (with the end-effector far from the base) to a position with the end-effector close to the base, that optimizes the measure at each point on the trajectory. And for redundancy measures that break the symmetry relating the connected components of the self-motion manifold, it is possible to have end-effector trajectories for which *no* such motion is possible at all.

The curvature scalar measure R that we introduced seems particularly interesting for redundant manipulators, as it identifies an m -dimensional region in work space where

the measure vanishes and the arm behaves locally “linear”. This measure also identifies singular points and classifies them according to their curvature. We note that a singular configuration with a large positive curvature is stable in the sense that geodesics in $\mathcal{C}$ attract each other (as the manifold has positive curvature). Unstable configurations arise for negative R , where nearby orbits repel each other and diverge.

The measures that were constructed here are by no means the only ones. For example, one could also choose the joint-compliance matrix [PL91] as the metric on $\mathcal{C}$, with a set of associated measures. And the secondary metric on $\mathcal{C}$ for the redundancy measure could be any task specific function.

While this work was being completed, an article appeared [PB94] in which a similar geometric approach to performance manipulation was taken. The measures examined there were different however, as the authors concentrated on global performance criteria rather than on local ones.

Future work will focus on the implementation of the theoretical framework into an automated computation of performance measures from a specification of the system at hand. An important problem that needs to be solved is an examination of the computational complexities of the performance measures. In particular the curvature scalar is difficult to compute, as it seems to require $O(n^4)$ operations (with n the dimension of the space), at least in a naive implementation where all the $O(n^4)$ components of the Riemann tensor are computed.

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Appendix A

Computational Aspects

In chapters 4 and 5, applications of the formalism to some simple planar manipulators were presented. For such low dimensional systems, one need not worry too much about numerical stability and efficiency of the computations. Most of the results presented there were calculated by symbolic calculations in Mathematica ([Wol91]), with the exception of the non-linearity measure, which was computed by numerical differentiation from the configuration space metric.

In the following, we will briefly describe a different method by which the measures for the unconstrained system could be computed, based on the singular value decomposition. The method described here is preferable when some of the matrices involved in the computation become ill-conditioned.

The inverse of the performance metric $\mathbf{g}^{-1} = \mathbf{J}\mathbf{h}^{-1}\mathbf{J}^T$ could be computed by explicitly inverting $\mathbf{h}$, but problems arise with numerical stability if $\mathbf{h}$ is ill-conditioned. This will happen for a manipulator arm if one of the links is much lighter than the others, for example. An alternative method, more suited to this case, is to find a coordinate system on $\mathcal{C}$ that brings the metric into Euclidean form (a local Cartesian basis) and evaluate the metric in this frame. In the following it will be shown how this relates to the singular value decomposition.

Let a general coordinate transformation in $\mathcal{W}$ be described by the $m \times m$ matrix $\mathbf{F} \in GL(m)$, and one in $\mathcal{C}$ by the $n \times n$ matrix $\mathbf{G} \in GL(n)$. $GL(n)$ is the group of

non-singular $n \times n$ matrices. The configuration space metric transforms as

$$\mathbf{h} \rightarrow \mathbf{G}^T \mathbf{h} \mathbf{G}.$$

Similarly, the work space metric transforms as

$$\boldsymbol{\eta} \rightarrow \mathbf{F}^T \boldsymbol{\eta} \mathbf{F},$$

and the Jacobian transforms as

$$\mathbf{J} \rightarrow \mathbf{F}^{-1} \mathbf{J} \mathbf{G}.$$

It follows that

$$\mathbf{g}^{-1} \rightarrow \mathbf{F}^{-1} \mathbf{g}^{-1} \mathbf{F}^{T-1}$$

which does not depend on $\mathbf{G}$. We can therefore find a $\mathbf{G}$, such that $\mathbf{h}$ transforms into the identity matrix. Computing the singular value decomposition [KL80] of $\mathbf{h}$, i.e.

$$\mathbf{h} = \mathbf{U} \boldsymbol{\Sigma} \mathbf{U}^T$$

with $\mathbf{U}$ orthogonal and $\boldsymbol{\Sigma}$ diagonal, gives the desired transformation matrix $\mathbf{G}$ as

$$\mathbf{G} = \mathbf{U} \boldsymbol{\Sigma}^{-1/2}$$

and $\mathbf{g}^{-1}$ is given by

$$\mathbf{g}^{-1} = \mathbf{J} \mathbf{U} \boldsymbol{\Sigma}^{-1} \mathbf{U}^T \mathbf{J}^T.$$

A similar method can be used for the generalized Yoshikawa measure (eq. 2.5) and the anisotropy measure (eq. 2.6). Define a matrix $\mathbf{A}$ by

$$\mathbf{A} = \mathbf{g}^{-1} \boldsymbol{\eta}.$$

The eigenvalues of $\mathbf{A}$ determine the anisotropy measure and the generalized Yoshikawa measure. $\mathbf{A}$ is invariant under $\mathcal{C}$ space coordinate transformations. It transforms as

$$\mathbf{A} \rightarrow \mathbf{F}^{-1} \mathbf{A} \mathbf{F}$$

from which it follows that the eigenvalues do not depend on $\mathbf{F}$. From the singular value decomposition of the work space metric $\boldsymbol{\eta}$,

$$\boldsymbol{\eta} = \mathbf{V}\boldsymbol{\Xi}\mathbf{V}^T$$

we obtain a transformation $\mathbf{F}$ that brings $\boldsymbol{\eta}$ in Euclidean form, namely

$$\mathbf{F} = \mathbf{V}\boldsymbol{\Xi}^{-1/2}.$$

we can now express $\mathbf{A}$ as

$$\mathbf{A} = \mathbf{B}\mathbf{B}^T,$$

with

$$\mathbf{B} = \boldsymbol{\Xi}^{1/2}\mathbf{V}^T\mathbf{J}\mathbf{U}\boldsymbol{\Sigma}^{-1/2}.$$

The eigenvalues of $\mathbf{A}$ can now be computed as the squares of the singular values of $\mathbf{B}$.

If one of the matrices (i.e. the Jacobian, the work space and the configuration space metric) is ill-conditioned this will give more accurate results than computing the eigenvalues of $\mathbf{A}$ directly. In this case some of the elements of $\mathbf{B}$ will be very small or very large, and computing $\mathbf{A}$ will therefore effectively square these entries with a resultant loss of accuracy. However, this method is slower than a straightforward computation of the eigenvalues of $\mathbf{A}$.

The above construction could be adapted to a computation of the measures over a path by using a smooth singular value decomposition, [MK89, BGMBN90]. This will be faster than a recalculation of the singular value decomposition at each point on the path. If all quantities are smooth, the singular value decomposition at nearby points can be computed by small adjustments, in very few iterations, as described in [MK89].

The scalar non-linearity measure as given in eq. 2.9 seems to require the explicit computation of $\mathbf{h}^{-1}$ as well as the computation of all the $\frac{n^2}{12}(n^2 - 1)$ components of the

Riemann curvature tensor, which makes it quite demanding to compute. If there is a more efficient way of computing this quantity seems to be an open problem.