BENDING OF SPHERICAL SHELLS

by

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Dipl.Ing., Technical University of Istanbul, Turkey, 1954

A thesis submitted in partial fulfillment of

the requirements for the degree of

MASTER OF APPLIED SCIENCE

in the Department of

CIVIL ENGINEERING

We accept this thesis as conforming to the standard

required from candidates for the degree of

Master of Applied Science

Members of the Department of Civil Engineering

THE UNIVERSITY OF BRITISH COLUMBIA

July 1957

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This study of the bending of a spherical shell of constant thickness consists of five chapters. In the first chapter, general differential equations of the problem are derived and presented with the variables in dimensionless form.

The second chapter deals with the solution of the two second order differential equations in terms of Bessel Functions. The solution is applicable for shells with a colatitude of about 25 degrees or less. Expressions are given for the internal stresses and displacements in terms of these values at the boundary.

The third chapter gives a numerical solution of the problem. Solutions are found for three cases of radius to thickness ratio, namely 30, 100 and 500, and for colatitudes from 8 to 90 degrees. In the preparation of tables the Alwac III-E Electronic Digital Computer was used. The flow diagram and programme for the Alwac III-E are given.

The fourth chapter compares the numerical method with one of Timoshenko's approximate solutions. The wave lengths, deflections, rotations and moments are calculated by two methods and compared. As the conclusion to these comparisons a graph is given, which shows the approximate error in the damped harmonic solution.

In the last chapter, a numerical example is solved using tables from the digital computer. The ratio of radius to thickness is 500; radius being 125 feet and the colatitude 30 degrees.The edges are considered as being on rollers.

ACKNOWLEDGMENT

The author wishes to express his indebtedness to the members of the Department of Civil Engineering at the University of British Columbia. It was a great experience and extensive pleasure to work under his supervisor, Professor Dr. R.F. Hooley, who spent many discussion hours on this thesis and gave the most valuable suggestions.

This post-graduate study was made possible through the financial support of an assistantship in the Department of Civil Engineering. The author wishes also to make particular acknowledgment to the University of British Columbia through Professor J.F. Muir, Head of the Department of Civil Engineering.

July 1957
Vancouver, British Columbia

CONTENTS

CHAPTER I

I N T R O D U C T I O N

Since 1910 the investigation of the bending of shells was the object of many papers written by curious investigators. But the complication of the solutions of the necessary differential equations for this investigation created a great deal of difficulties. After almost half a century, today one who wishes to look into a shell solution still finds a laborious work, impossibilities or great number of assumptions behind his particular problem.

The form of the shell, discontinuity of surface and variable thickness cause different problems in the solution of the differential equations. Even in the case of the spherical shell of constant thickness the difficulty of the problem forced investigators to make certain assumptions. The theory developed to date is limited to special cases. For instance, the approximate method for the solution of a spherical shell, the so-called method of asymptotic integration is valuable for very thin shells. Another approximate investigation of the bending of a spherical shell gives fairly accurate values for non-flat spherical shells.

The object of this paper was to investigate an exact solution of bending of the spherical shells by a numerical method. To analyze the problem it was necessary to evaluate some boundary values by tabulated functions. For this purpose the solution of differential equations was made by the application of Bessel Functions around the top of the shell. The solution by Bessel Functions gives almost exact values for ϕ ranging from 0° and 20°-25°.

GENERAL EQUATIONS OF A SHELL OF REVOLUTION AND INTRODUCTION OF THE DIMENSIONLESS FORMS

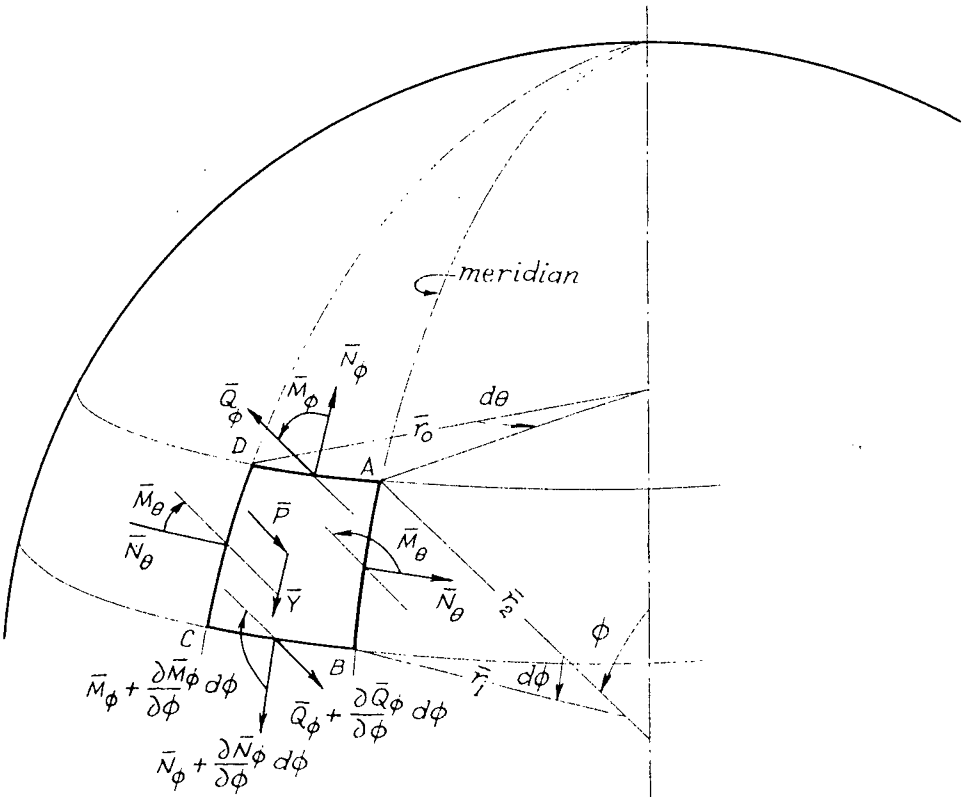


Fig. I. 1.

1. NOTATION :

θ : Angle for position of a meridian.

ϕ : Colatitude.

$\bar{P}$: External force per unit area, perpendicular to the surface.

$\bar{Y}$: External force per unit area, tangent to the meridian.

$\bar{N}_\phi, \bar{N}_\theta$: Magnitude of normal forces per unit length.

$\bar{M}_\phi, \bar{M}_\theta$: Magnitude of bending moments per unit length.

$\bar{Q}_\phi$: Magnitude of shearing force per unit length.

$\bar{T}$: Magnitude of horizontal force per unit length.

$\bar{r}_o$: Radius of horizontal circle at $\phi = \phi$.

$\bar{r}_1$: Radius of curvature at $\phi = \phi$.

$\bar{r}_2$: Distance between shell and axis of rotation on the projection of radius of curvature.

$\bar{r}_{1c}$: Radius of curvature at the top of the shell.

$\bar{\sigma}_\phi, \bar{\sigma}_\theta$: Magnitude of stresses.

$\bar{\Delta}$: Horizontal displacement.

$\bar{v}$: Tangential displacement.

$\bar{w}$: Radial displacement.

2. EQUATIONS OF EQUILIBRIUM

Let us take an element on the shell bordered by two adjacent meridians and two adjacent parallels, i.e., meridians defined by θ and $\theta + d\theta$; and parallels defined by ϕ and $\phi + d\phi$, as shown in Fig. I.1.

In writing the equations of equilibrium we can obtain three equations which relate the normal forces $\overline{N}_\phi, \overline{N}_\theta$, the shearing force $\overline{Q}_\phi$, and the bending moments $\overline{M}_\phi$, $\overline{M}_\theta$ to each other. The following equations represent these relations,

Equilibrium of forces parallel to the meridians:

$$\frac{d}{d\phi}(\overline{N}_\phi \overline{r}_o) - \overline{N}_\theta \overline{r}_1 \operatorname{Cos}\phi - \overline{Q}_\phi \overline{r}_o = - \overline{r}_o \overline{r}_1 \overline{Y} \qquad (1.1.1.)$$

Equilibrium of forces perpendicular to the element:

$$\overline{N}_\phi \overline{r}_o + \overline{N}_\theta \overline{r}_1 \operatorname{Sin}\phi + \frac{d}{d\phi} (\overline{r}_o \overline{Q}_\phi) = - P\, \overline{r}_1 \overline{r}_o \qquad (1.2.1.)$$

Equilibrium of moments of all the forces with respect to the tangent to the parallel circle:

$$\frac{d}{d\phi}(\overline{M}_\phi \overline{r}_o) - \overline{M}_\theta \overline{r}_1 \operatorname{Cos}\phi - \overline{Q}_\phi \overline{r}_o \overline{r}_1 = 0 \qquad (1.3.1.)$$

If there is no external forces, i.e., $\overline{Y} = 0$, $\overline{P} = 0$ the above formulae can be written as follows:

$$\frac{d}{d\phi}(\overline{N}_\phi \overline{r}_o) - \overline{N}_\theta \overline{r}_1 \operatorname{Cos}\phi - \overline{Q}_\phi \overline{r}_o = 0 \qquad (1.1.2.)$$

$$\overline{N}_\phi \overline{r}_o + \overline{N}_\theta \overline{r}_1 \operatorname{Sin}\phi + \frac{d}{d\phi} (\overline{Q}_\phi \overline{r}_o) = 0 \qquad (1.2.2.)$$

For the non-loading case we can express another set of two equations instead of the formulae (1.1.2.) and (1.2.2.). From Fig. I.2., since there is no load, the sum of the vertical projections of internal forces should be equal to zero,

$$\bar{N}_\phi = - \bar{Q}_\phi \operatorname{Cot}\phi \qquad (1.1.3.)$$

By substituting this relation in equation (1.1.2.) we can obtain the following relation between $\bar{N}_\theta$ and $\bar{Q}_\phi$:

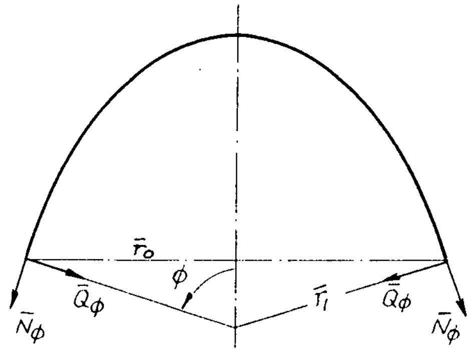


Fig. I. 2.

$$\bar{N}_\theta = - \frac{1}{\bar{r}_1} \frac{d}{d\phi} (\bar{Q}_\phi \bar{r}_2) \qquad (1.2.3.)$$

To solve for five unknowns, i.e., $\bar{N}_\phi, \bar{N}_\theta, \bar{M}_\phi, \bar{M}_\theta, \bar{Q}_\phi$, the above three relations are not sufficient. The problem becomes indeterminate and we have to consider the geometry of the displaced shell. By this way, we will introduce six more unknowns, i.e., two displacements $\bar{w}$, $\bar{v}$, two strains, $\bar{\epsilon}_\phi$, $\bar{\epsilon}_\theta$, and two curvatures $\bar{X}_\phi$, $\bar{X}_\theta$. But eleven different relations among these eleven unknowns can be established and problem can be solved. In the next articles these relations will be shown.

3. GEOMETRY OF THE SHELL OF REVOLUTION

Let us assume that in the original position, before deformations, the shell has the form of AB Fig. I.3, after deformation A'B'. Therefore $\overline{AA'}$ and $\overline{BB'}$ are the displacements of A and B respectively. Each of these displacements can be resolved into two components, i.e., $\overline{w}$ in the direction of the normal of the element, $\overline{v}$ in the direction of the tangent to the meridian.

These displacements are shown on the Fig. I.3. The relations between strains and displacements can be established by geometry. The element AB will deform into shape of A'B', therefore the change in the length of AB is:

$$\frac{d\overline{v}}{d\phi} d\phi - \overline{w}\, d\phi \qquad \text{(a)}$$

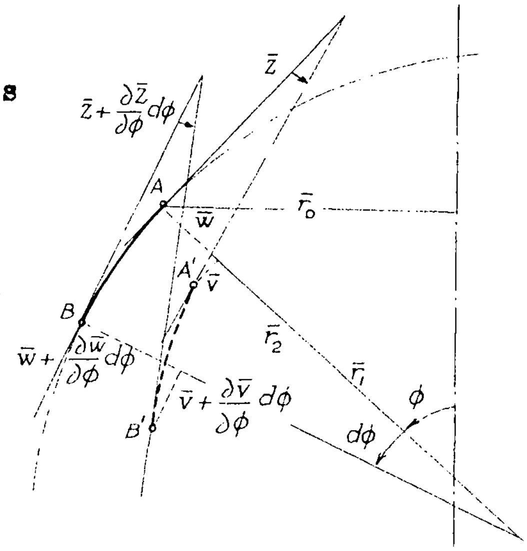


Fig. I.3.

The portion $\frac{d\overline{v}}{d\phi} d\phi$ is the extension of the element because of $\overline{v}$, and $\overline{w}\, d\phi$ is the shortening of the element because of $\overline{w}$.

The ratio between the change and the original length of the element, $\overline{r}_1 d\phi$, will give us the strain of the shell in the meridional direction.

$$\bar{\epsilon}_{\phi} = \frac{1}{\bar{r}_1} \frac{d\bar{v}}{d\phi} - \frac{\bar{w}}{\bar{r}_1} \tag{1.4.1.}$$

The strain in the parallel direction is the ratio between the increase of the circumference of the parallel circle and the original length of this circumference. This ratio is simply equal to the ratio between the increase of radius and the original radius. Since this increase is,

$$\bar{v} \text{ Cos}\,\phi \; - \; \bar{w} \text{ Sin}\,\phi \tag{b}$$

and the original length of radius is;

$$\bar{r}_o = \bar{r}_2 \text{ Sin}\,\phi \tag{c}$$

the strain in the parallel direction becomes:

$$\bar{\epsilon}_{\theta} = \frac{1}{\bar{r}_2} \left(\bar{v} \text{ Cot}\,\phi \; - \bar{w} \right) \tag{1.5.1.}$$

The changes in curvature of meridian and parallel circle can be expressed in terms of the $\bar{v}$ and $\bar{w}$ displacements. The total rotation of upper side of the element shown in Fig.I.1. is,

$$\frac{\bar{v}}{\bar{r}_1} + \frac{d\bar{w}}{d\phi} \frac{1}{\bar{r}_1} \tag{d}$$

For the lower side rotation is,

$$\frac{\bar{v}}{\bar{r}_1} + \frac{d\bar{w}}{d\phi} \frac{1}{\bar{r}_1} + \frac{d}{d\phi} \left(\frac{\bar{v}}{\bar{r}_1} + \frac{d\bar{w}}{d\phi} \frac{1}{\bar{r}_1} \right) d\phi \tag{e}$$

The change of curvature of the meridian is simply the rotation per unit length, or:

$$\bar{X}_{\phi} = \frac{1}{\bar{r}_1}\frac{d}{d\phi}\left(\frac{\bar{v}}{\bar{r}_1} + \frac{d\bar{w}}{d\phi}\frac{1}{\bar{r}_1}\right) \qquad (1.6.1.)$$

On the other hand, the change of curvature of the parallel circle is the ratio between the projection parallel to the meridian of rotation (d) and the original length $\bar{r}_o d\theta$. Since the projection parallel to the meridian is,

$$\left(\frac{\bar{v}}{\bar{r}_1} + \frac{d\bar{w}}{d\phi}\frac{1}{\bar{r}_1}\right) \operatorname{Cos}\phi \, d\theta \qquad (f)$$

and substituting (c) instead of $\bar{r}_o$, the change of curvature of parallel circle is obtained as follows:

$$\bar{X}_{\theta} = \left(\frac{\bar{v}}{\bar{r}_1} + \frac{d\bar{w}}{d\phi}\frac{1}{\bar{r}_1}\right) \frac{\operatorname{Cot}\phi}{\bar{r}_2} \qquad (1.7.1.)$$

So far, we have seven equations. The other four are relations between force components, strains and changes of curvature. In the next article these last four equations will be given.

4. RELATIONS BETWEEN FORCE COMPONENTS AND GEOMETRY

Hooke's Law will give us relations between normal forces and strains as follows:

$$\bar{\epsilon}_{\phi} = \frac{1}{E\,\bar{h}} \left(\bar{N}_{\phi} - \nu \bar{N}_{\theta} \right) \tag{g}$$

$$\bar{\epsilon}_{\theta} = \frac{1}{E\,\bar{h}} \left(\bar{N}_{\theta} - \nu \bar{N}_{\phi} \right) \tag{h}$$

From these relations $\bar{N}_{\phi}$ and $\bar{N}_{\theta}$ can be drawn out as functions of strains:

$$\bar{N}_{\phi} = \frac{E\bar{h}}{1-\nu^2} \left(\bar{\epsilon}_{\phi} + \nu \bar{\epsilon}_{\theta} \right) \tag{1.8.1.}$$

$$\bar{N}_{\theta} = \frac{E\bar{h}}{1-\nu^2} \left(\bar{\epsilon}_{\theta} + \nu \bar{\epsilon}_{\phi} \right) \tag{1.9.1.}$$

On the other hand the relations between moments and changes of curvatures are as follows:

$$\bar{M}_{\phi} = - \frac{E\bar{h}^3}{12(1-\nu^2)} \left(\bar{X}_{\phi} + \nu \bar{X}_{\theta} \right) \tag{1.10.1.}$$

$$\bar{M}_{\theta} = - \frac{E\bar{h}^3}{12(1-\nu^2)} \left(\bar{X}_{\theta} + \nu \bar{X}_{\phi} \right) \tag{1.11.1.}$$

If we substitute the values of $\bar{\epsilon}_{\phi}$, $\bar{\epsilon}_{\theta}$, $\bar{X}_{\phi}$, $\bar{X}_{\theta}$, which were given previously, in the above formulae, we can obtain the following relations between forces, moments and displacements:

$$\bar{N}_\phi = \frac{E\bar{h}}{(1-\nu^2)}\left[\frac{1}{\bar{r}_1}\left(\frac{d\bar{v}}{d\phi} - \frac{\bar{w}}{\bar{r}_1}\right) + \frac{\nu}{\bar{r}_2}\left(\bar{v}\,\mathrm{Cot}\phi - \bar{w}\right)\right] \tag{1.8.2.}$$

$$\bar{N}_\theta = \frac{E\bar{h}}{(1-\nu^2)}\left[\frac{1}{\bar{r}_2}\left(\bar{v}\,\mathrm{Cot}\phi - \bar{w}\right) - \frac{\nu}{\bar{r}_1}\left(\frac{d\bar{v}}{d\phi} - \frac{\bar{w}}{r_1}\right)\right] \tag{1.9.2.}$$

$$\bar{M}_\phi = \frac{-E\bar{h}^3}{12(1-\nu^2)}\left[\frac{1}{\bar{r}_1}\frac{d}{d\phi}\left(\frac{\bar{v}}{\bar{r}_1} + \frac{1}{\bar{r}_1}\frac{d\bar{w}}{d\phi}\right) + \frac{\nu}{\bar{r}_2}\mathrm{Cot}\phi\left(\frac{\bar{v}}{\bar{r}_1} + \frac{1}{\bar{r}_1}\frac{d\bar{w}}{d\phi}\right)\right] \tag{1.10.2.}$$

$$\bar{M}_\theta = \frac{-E\bar{h}^3}{12(1-\nu^2)}\left[\left(\frac{\bar{v}}{\bar{r}_1} + \frac{1}{\bar{r}_1}\frac{d\bar{w}}{d\phi}\right)\frac{\mathrm{Cot}\phi}{\bar{r}_2} + \frac{\nu}{\bar{r}_1}\frac{d}{d\phi}\left(\frac{\bar{v}}{\bar{r}_1} + \frac{1}{\bar{r}_1}\frac{d\bar{w}}{d\phi}\right)\right] \tag{1.11.2.}$$

Now we are to solve the eleven unknowns from the above eleven equations, numbered from (1.1.1.) to (1.11.1.). To deal with this problem is rather laborious work. To reduce this to an easier approach we will introduce two new unknowns and eliminate the rest of the unknowns in the following article.

5. DIFFERENTIAL EQUATIONS OF THE PROBLEM

By introducing $\bar{Z}$ and $\bar{U}$ as the new unknowns we can reduce the problem to a solution of two differential equations. $\bar{Z}$ is the angle of rotation of a tangent to a meridian and equal to (d) of the previous article.

$$\bar{Z} = \frac{1}{\bar{r}_1} \left(\bar{v} + \frac{d\bar{w}}{d\phi} \right) \qquad (i)$$

The other unknown is as following:

$$\bar{U} = \bar{r}_2 \, \bar{Q}_\phi \qquad (j)$$

Substituting these variables into equations (1.1.3.) (1.2.3.) (1.10.2.) (1.11.2.), and defining the first and second derivatives of any function, F as follows,

$$\dot{F} = \frac{dF}{d\phi}, \quad \ddot{F} = \frac{d^2F}{d\phi^2} \qquad (k), (l)$$

we obtain the following relations:

$$\bar{N}_\phi = -\frac{1}{\bar{r}_2} \bar{U} \operatorname{Cot}\phi \qquad (1.1.4.)$$

$$\bar{N}_\theta = -\frac{1}{\bar{r}_1} \dot{\bar{U}} \qquad (1.2.4.)$$

$$\bar{M}_\phi = -\frac{E\bar{h}^3}{12(1-\nu^2)} \left[\frac{1}{\bar{r}_1} \dot{\bar{Z}} + \frac{\nu}{\bar{r}_2} \operatorname{Cot}\phi \, \bar{Z} \right] \qquad (1.10.3.)$$

$$M_\theta = -\frac{E\bar{h}^3}{12(1-\nu^2)} \left[\bar{Z} \frac{\operatorname{Cot}\phi}{\bar{r}_2} + \frac{\nu}{r_1} \dot{\bar{Z}} \right] \qquad (1.11.3.)$$

The first equation for $\bar{U}$ and $\bar{Z}$ can be readily obtained from Eqs. (1.8.2.) and (1.9.2.) by forming $\bar{v} - \bar{w}$,

$$\ddot{\bar{U}} + \frac{\bar{r}_1}{\bar{r}_2}\left[\left(\frac{\bar{r}_2}{\bar{r}_1}\right)^{\cdot} + \frac{\bar{r}_2}{\bar{r}_1} \operatorname{Cot}\phi - \frac{\bar{r}_2}{\bar{r}_1} \frac{\dot{\bar{h}}}{\bar{h}}\right]\dot{\bar{U}} - \frac{\bar{r}_1}{\bar{r}_2}\left[\frac{\bar{r}_1}{\bar{r}_2} \operatorname{Cot}^2\phi - \nu - \nu\frac{\dot{\bar{h}}}{\bar{h}} \operatorname{Cot}\phi\right]\bar{U}$$

$$= \frac{\bar{r}_1^2}{\bar{r}_2} E \, \bar{h} \, \bar{Z} \qquad (1.12.1.$$

The second equation for $\bar{U}$ and $\bar{Z}$ is obtained from the substitution of $\bar{M}_\phi$ and $\bar{M}_\theta$ into formula (1.3.1.) by their equivalents shown as Eqs. (1.10.3.) and (1.11.3.)

$$\ddot{\bar{Z}} + \frac{\bar{r}_1}{\bar{r}_2}\left[\left(\frac{\bar{r}_2}{\bar{r}_1}\right)^{\cdot} + \frac{\bar{r}_2}{\bar{r}_1}\,\mathrm{Cot}\,\phi + \frac{3\bar{r}_2}{\bar{r}_1\,\bar{h}}\,\dot{\bar{h}}\right]\dot{\bar{Z}} - \frac{\bar{r}_1}{\bar{r}_2}\left[\nu - \frac{3\nu\,\mathrm{Cot}\,\phi}{\bar{h}}\,\dot{\bar{h}} + \frac{\bar{r}_1}{\bar{r}_2}\mathrm{Cot}^2\phi\right]\bar{Z}$$

$$= -\,\bar{U}\,\frac{12(1-\nu^2)}{E\,\bar{h}^3}\,\frac{\bar{r}_1^{\,2}}{\bar{r}_2} \qquad (1.13.1.)$$

A solution must now be obtained from the above two simultaneous second order differential equations for $\bar{U}$ and $\bar{Z}$. With $\bar{U}$ and $\bar{Z}$ known as a function of ϕ all the force components and displacements are found from the preceding equations.

Up to now all the variables used were with dimension. To make the analysis of problem easier we will introduce dimensionless forms to our variables. In the next article this aspect will be discussed.

6. DIMENSIONLESS FORMS

So far we used the bar (-) above a quantity to indicate that it has a dimension. For the dimensionless quantities we will introduce the same symbol without the bar. The tables which are given in the next few pages will link the two systems. The first table is divided into two parts. The part (a) contains the variables which were defined arbitrarily in dimensionless form. In order to get the dimensionless relations which are shown in Table I.2., it was necessary to define the rest of the variables as they are shown in Table I.1.b. The last table which is numbered as Table I.3. gives all variables in terms of $\bar{U}$ and $\bar{Z}$.

Before we presented the dimensionless forms, the two simultaneous differential equations, Eqs. (1.12.1.) and (1.13.1.) were obtained with the dimensions. After having the given tables we are able to transform these equations into the dimensionless forms. For the general case these equations are

$$\ddot{U} + \frac{r_1}{r_2}\left[\left(\frac{r_2}{r_1}\right)^{\cdot} + \frac{r_2}{r_1}\operatorname{Cot}\phi - \frac{r_2}{r_1}\left(\frac{\dot{h}}{h}\right)\right]\dot{U} - \frac{r_1}{r_2}\left[\frac{r_1}{r_2}\operatorname{Cot}^2\phi - \nu - \nu\left(\frac{\dot{h}}{h}\right)\operatorname{Cot}\phi\right]U = \frac{r_1}{r_2}(1-\nu^2)\frac{Z}{h} \tag{1.14.1.}$$

$$\ddot{Z} + \frac{r_1}{r_2}\left[\left(\frac{r_2}{r_1}\right)^{\cdot} + \frac{r_2}{r_1}\operatorname{Cot}\phi + \frac{3r_2}{r_1}\frac{\dot{h}}{h}\right]\dot{Z} - \frac{r_1}{r_2}\left[\nu - 3\nu\operatorname{Cot}\phi\frac{\dot{h}}{h} + \frac{r_1}{r_2}\operatorname{Cot}^2\phi\right]Z = -12\frac{r_1^2}{r_2}\frac{U}{h} \tag{1.15.1.}$$

For the case of the spherical shell of constant thickness these equations reduce to the following:

$$\ddot{U} + \operatorname{Cot}\phi\,\dot{U} - (\operatorname{Cot}^2\phi - \nu)\,U = \frac{(1-\nu^2)}{h}Z \tag{1.16.1}$$

$$\ddot{Z} + \operatorname{Cot}\phi\,\dot{Z} - (\operatorname{Cot}^2\phi + \nu)\,Z = -\frac{12}{h}U \tag{1.17.1.}$$

TABLE: I.1.a. Relations between with dimension and dimensionless quantities

	Dimensionless quantities	
	General case	Spherical shell case
$\bar{h}$	$h = \dfrac{\bar{h}}{\bar{r}_{1c}}$	$h = \dfrac{\bar{h}}{\bar{a}}$
$\bar{r}_o$	$r_o = \dfrac{\bar{r}_o}{\bar{r}_{1c}}$	$r_o = \dfrac{\bar{r}_o}{\bar{a}}$
$\bar{r}_1$	$r_1 = \dfrac{\bar{r}_1}{\bar{r}_{1c}}$	$a = \dfrac{\bar{a}}{\bar{a}} = 1$
$\bar{r}_2$	$r2 = \dfrac{\bar{r}_2}{\bar{r}_{1c}}$	$a = \dfrac{\bar{a}}{\bar{a}} = 1$
$\bar{v}$	$v = \dfrac{\bar{v}}{\bar{r}_{1c}}$	$v = \dfrac{\bar{v}}{\bar{a}}$
$\bar{w}$	$w = \dfrac{\bar{w}}{\bar{r}_{1c}}$	$w = \dfrac{\bar{w}}{\bar{a}}$
$\bar{\epsilon}_\phi$	$\epsilon_\phi = \bar{\epsilon}_\phi \ h$	$\epsilon_\phi = \bar{\epsilon}_\phi \ h$
$\bar{\epsilon}_\theta$	$\epsilon_\theta = \bar{\epsilon}_\theta \ h$	$\epsilon_\theta = \bar{\epsilon}_\theta \ h$
$\bar{Z}$	$Z = \bar{Z} \ h^2$	$Z = \bar{Z} \ h^2$

TABLE: I.1.b. Relations between with dimension and dimensionless quantities

Dimensionless quantities		
	General case	Spherical shell case
$\bar{N}_\phi$	$N_\phi = \bar{N}_\phi \dfrac{1-\nu^2}{E\bar{r}_{1c}}$	$N_\phi = \bar{N}_\phi \dfrac{1-\nu^2}{E\ \bar{a}}$
$\bar{N}_\theta$	$N_\theta = \bar{N}_\theta \dfrac{1-\nu^2}{E\ \bar{r}_{1c}}$	$N_\theta = \bar{N}_\theta \dfrac{1-\nu^2}{E\ \bar{a}}$
$\bar{M}_\phi$	$M_\phi = \bar{M}_\phi \dfrac{12\ (1-\nu^2)}{h\ E\ \bar{r}_{1c}^2}$	$M_\phi = \bar{M}_\phi \dfrac{12(1-\nu^2)}{hE\ \bar{a}^2}$
$\bar{M}_\theta$	$M_\theta = \bar{M}_\theta \dfrac{12\ (1-\nu^2)}{h\ E\ \bar{r}_{1c}^2}$	$M_\theta = \bar{M}_\theta \dfrac{12(1-\nu^2)}{hE\ \bar{a}^2}$
$\bar{Q}_\phi$	$Q_\phi = \bar{Q}_\phi \dfrac{1-\nu^2}{E\ \bar{r}_{1c}}$	$Q_\phi = \bar{Q}_\phi \dfrac{1-\nu^2}{E\ \bar{a}}$
$\bar{X}_\phi$	$X_\phi = \bar{X}_\phi r_{1c}\ h^2$	$X_\phi = \bar{X}_\phi \bar{a}\ h^2$
$\bar{X}_\theta$	$X_\theta = \bar{X}_\theta\ r_{1c}\ h^2$	$X_\theta = \bar{X}_\theta\ \bar{a}\ h^2$
$\bar{U}$	$U = \bar{U} \dfrac{1-\nu^2}{E\ \bar{r}_{1c}}$	$U = \bar{U} \dfrac{1-\nu^2}{E\ \bar{a}^2}$
$\bar{\sigma}_\phi$	$\sigma_\phi = \bar{\sigma}_\phi \dfrac{1-\nu^2}{E}\ h$	$\sigma_\phi = \bar{\sigma}_\phi \dfrac{1-\nu^2}{E}\ h$
$\bar{\sigma}_\theta$	$\sigma_\theta = \bar{\sigma}_\theta \dfrac{1-\nu^2}{E}\ h$	$\sigma_\theta = \bar{\sigma}_\theta \dfrac{1-\nu^2\ h}{E}$
$\bar{\Delta}$	$\Delta = \bar{\Delta} \dfrac{h(1-\nu^2)}{\bar{r}_{1c}\ r_2}$	$\Delta = \bar{\Delta} \dfrac{h(1-\nu^2)}{\bar{a}}$

TABLE: I. 2. Dimensionless Relations

General case	Spherical shell case
$N_\phi = - Q_\phi \cot\phi$	$N_\phi = - Q_\phi \mathrm{Cot}\phi$
$N_\theta = - \frac{1}{r_1} \frac{d}{d\phi}(Q_\phi r_2)$	$N_\theta = - \frac{d}{d\phi}(Q_\phi)$
$T = N_\phi \mathrm{Cos}\phi - Q_\phi \mathrm{Sin}\phi$	$T = N_\phi \mathrm{Cos}\phi - Q_\phi \mathrm{Sin}\phi$
$\frac{d}{d\phi}(M_\phi r_o) = r_1 M_\theta \mathrm{Cos}\phi + 12\, r_o r_1 Q_\phi$	$\frac{d}{d\phi}(M\ \mathrm{Sin}\phi) = M_\theta \mathrm{Cos}\phi + 12 \mathrm{Sin}\phi Q_\phi$
$\epsilon_\phi = \frac{h}{r_1}\left(\frac{dv}{d\phi} - w\right)$	$\epsilon_\phi = h\left(\frac{dv}{d\phi} - w\right)$
$\epsilon_\theta = \frac{h}{r_1}(v\ \mathrm{Cot}\phi - w)$	$\epsilon_\theta = h(v\ \mathrm{Cot}\phi - w)$
$X_\phi = \frac{h^2}{r_1} \frac{d}{d\phi}\left(\frac{v}{r_1} + \frac{dw}{d\phi}\frac{1}{r_1}\right)$	$X_\phi = h^2 \frac{d}{d\phi}\left(v + \frac{dw}{d\phi}\right)$
$X_\theta = \frac{h^2}{r_1} \frac{d}{d\phi}\left(\frac{v}{r_1} + \frac{dw}{d\phi}\frac{1}{r_1}\right) \mathrm{Cot}\phi$	$X_\theta = h^2 \frac{d}{d\phi}\left(v + \frac{dw}{d\phi}\right) \mathrm{Cot}\phi$
$N_\phi = \epsilon_\phi + \nu \epsilon_\theta$	$N_\phi = \epsilon_\phi + \nu \epsilon_\theta$
$N_\theta = \epsilon_\theta + \nu \epsilon_\phi$	$N_\theta = \epsilon_\theta + \nu \epsilon_\phi$
$M_\phi = -(X_\phi + \nu X_\theta)$	$M_\phi = -(X_\phi + \nu X_\theta)$
$M_\theta = -(X_\theta + \nu X_\phi)$	$M_\theta = -(X_\theta + \nu X_\phi)$
$\sigma_\phi = N_\phi \pm \frac{M_\phi}{2}$	$\sigma_\phi = N_\phi \pm \frac{M_\phi}{2}$
$\sigma_\theta = N_\theta \pm \frac{M_\theta}{2}$	$\sigma_\theta = N_\theta \pm \frac{M_\theta}{2}$
$\Delta = (N_\theta - \nu N_\phi) \mathrm{Sin}\phi$	$\Delta = (N_\theta - \nu N_\phi) \mathrm{Sin}\phi$
$Z = \frac{h^2}{r_1}\left(v + \frac{dw}{d\phi}\right)$	$Z = h^2\left(v + \frac{dw}{d\phi}\right)$
$U = r_2 Q_\phi$	$U = Q_\phi$

TABLE: I. 3. Dimensionless forms in terms of U and Z

General case	Spherical shell case
$N_\phi = -\frac{U}{r_2}\,\mathrm{Cot}\phi$	$N_\phi = -U\,\mathrm{Cot}\phi$
$N_\theta = -\frac{1}{r_1}\dot{U}$	$N_\theta = -\dot{U}$
$T = -\frac{U}{r_2\,\mathrm{Sin}\phi}$	$T = -\frac{U}{\mathrm{Sin}\phi}$
$Q_\phi = \frac{U}{r_2}$	$Q_\phi = U$
$M_\phi = -\frac{1}{r_1}(\dot{Z} + \frac{r_1}{r_2}\nu Z\,\mathrm{Cot}\phi)$	$M_\phi = -(\dot{Z} + \nu Z\,\mathrm{Cot}\phi)$
$M_\theta = -\frac{1}{r_1}(\frac{r_1}{r_2} Z\,\mathrm{Cot}\phi + \nu\dot{Z})$	$M = -(Z\,\mathrm{Cot}\phi + \nu\dot{Z})$
$\sigma_\phi = -\frac{U}{r_2}\mathrm{Cot}\,\phi \pm \frac{1}{2r_1}(\dot{Z} + \frac{r_1}{r_2}\nu Z\mathrm{Cot}\phi)$	$\sigma_\phi = -U\,\mathrm{Cot}\phi \pm \frac{1}{2}(\dot{Z} + \nu Z\,\mathrm{Cot}\phi)$
$\sigma_\theta = -\frac{\dot{U}}{r_1} \pm \frac{1}{2r_1}(\frac{r_1}{r_2}Z\,\mathrm{Cot}\phi + \nu\dot{Z})$	$\sigma_\theta = -\dot{U} \pm \frac{1}{2}(Z\,\mathrm{Cot}\phi + \nu\dot{Z})$
$\Delta = -\mathrm{Sin}\phi(\dot{U}\frac{1}{r_1} - \nu\frac{U}{r_2}\,\mathrm{Cot}\phi)$	$\Delta = -\mathrm{Sin}\phi(\dot{U} - \nu U\,\mathrm{Cot}\phi)$

As it can be noticed, these two equations involve two parameters; i.e., ν and h. Poisson's Ratio ν is constant for a given material and changes very little from one material to another. In Chapter III ν will be given an average value of 0.20. On the other hand h which is the ratio between the thickness and the radius of shell can vary within large limits. For each value of h, the solution of the above yields different functions for U and Z. In the next two chapters this problem will be solved by two different methods. Firstly by Bessel Functions, and secondly by a numerical method.

CHAPTER II

APPLICATION OF BESSEL FUNCTIONS TO THE SOLUTION OF SPHERICAL SHELLS

Last chapter, after presenting the dimensionless symbols we were able to simplify two simultaneous differential equations for a spherical shell of constant thickness to the following:

$$\ddot{U} + \text{Cot}\phi\, \dot{U} - (\text{Cot}^2\phi - \nu)\, U = \frac{1-\nu^2}{h} Z \qquad (2.1.1.)$$

$$\ddot{Z} + \text{Cot}\phi\, \dot{Z} - (\text{Cot}^2\phi + \nu)\, Z = -\frac{12}{h} U \qquad (2.2.1.)$$

In this chapter we will show that for small ϕ a solution to the above equations can be obtained in terms of Bessel functions.

If the angle ϕ is small, say less than 10^o or 20^o, then it is permissible to replace $\cot\phi$ by $\frac{1}{\phi}$ with fair accuracy.

For example,

$\phi = 10^\circ$	$1/\phi = 5.72$	$\cot\phi = 5.67$	Diff. = 0.88%
$\phi = 20^\circ$	$1/\phi = 2.86$	$\cot\phi = 2.75$	" = 4.00%
$\phi = 30^\circ$	$1/\phi = 1.87$	$\cot\phi = 1.73$	" = 8.09%

This approximation to $\cot\phi$ will convert our differential equations to the following:

$$\ddot{U} + \frac{\dot{U}}{\phi} - \frac{U}{\phi^2} + \nu U = \frac{(1-\nu^2)}{h} Z \qquad (2.1.2.)$$

$$\ddot{Z} + \frac{\dot{Z}}{\phi} - \frac{Z}{\phi^2} - \nu Z = -\frac{12}{h} U \qquad (2.2.2.)$$

By introducing the notation $L(F)$

$$L(F) = \ddot{F} + \frac{1}{\phi}\dot{F} - \frac{1}{\phi^2} F \qquad (a)$$

where F is any function, we can rewrite our equations in the following simplified forms:

$$L(U) + \nu U = \frac{(1-\nu^2)}{h} Z \qquad (2.1.3.)$$

$$L(Z) - \nu Z = -\frac{12}{h} U \qquad (2.2.3.)$$

From these two simultaneous differential equations of the second order we can obtain for each unknown, an equation of the fourth order by operating with L on each of the two equations and eliminating one unknown each time. At the end of this operation we obtain

$$LL(U) + \xi^4 U = 0 \qquad (2.3.1.)$$

$$LL(Z) + \xi^4 Z = 0 \qquad (2.4.1.)$$

where

$$\xi^4 = \frac{12(1-\nu^2)}{h^2} - \nu^2 \tag{b}$$

As a first step, let us solve for Z from Eq. (2.4.1.). This equation can be written in one of the following forms:

$$L\left[L(Z) + i\xi^2 Z\right] - i\xi^2\left[L(Z) + i\xi^2 Z\right] = 0 \qquad (2.4.2.)$$

$$L\left[L(Z) - i\xi^2 Z\right] + i\xi^2\left[L(Z) - i\xi^2 Z\right] = 0 \qquad (2.4.3.)$$

Therefore the solutions of the following two differential equations

$$L(Z) - i\xi^2 Z = 0 \qquad (2.5.1.)$$

$$L(Z) + i\xi^2 Z = 0 \qquad (2.5.2.)$$

will be the solutions of the fourth order differential equation (2.4.1.). Let us name the solution of Eq. (2.5.1.) Z_1 and the solution of Eq. (2.5.2.) Z_2 and rewrite the equations by opening the L operator.

$$\ddot{Z}_1 + \frac{\dot{Z}_1}{\phi} - \left(i\xi^2 + \frac{1}{\phi^2}\right) Z_1 = 0 \qquad (2.5.3.)$$

$$Z_2 + \frac{\dot{Z}_2}{\phi} + \left(i\xi^2 - \frac{1}{\phi^2}\right) Z_2 = 0 \qquad (2.5.4.)$$

The above equations are Bessel's Differential equation of first order and complex argument.

The well known general Bessel Differential equation is

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(k^2 - \frac{n^2}{x^2}\right) y = 0 \qquad (2.6.1.)$$

and its general solution is as below,

$$y = A_1^{*} \; J_n \; (kx) + B_1^{*} \; Y_n \; (kx) \qquad (2.7.1.)$$

In this solution $J_n(kx)$ and Y_n (kx) represent First and Second Kind Bessel Functions respectively. A^{*} and B_1^{*} are arbitrary constants. In this equation if n = 1 our solutions are called - Bessel Functions of First Order - i.e.,

$$y = A_1^{*} \; J_1 \; (kx) + B_1^{*} \; Y_1 \; (kx) \qquad (2.7.2.)$$

Modified Bessel Functions of First and Second Kind are $I_1(kx)$ and $K_1(kx)$ respectively. The relations between non-modified and modified functions are as follows:

$$J_1(kxi) = i \; I_1 \; (kx) \qquad (c)$$

$$Y_1(kxi) = \frac{2}{\pi} i \; K_1(kx) - I_1 \; (kx) \qquad (d)$$

These modified Bessel Functions are applicable in cases where the argument is imaginary.

By knowing this information we can obtain the solutions of our differential equations. In the next two pages the solutions for both equations are shown in a tabulated form to make easy the comparison between them. After following these tables step by step two solutions can be found such as:

$$Z_1 = A_1 I_1 \; (\xi \emptyset \; i^{\frac{1}{2}}) \; + \; B_1 K_1(\xi \emptyset \; i^{\frac{1}{2}}) \qquad (2.8.1.)$$

$$Z_2 = A_2 I_1 \; (\xi \emptyset \; i^{-\frac{1}{2}}) + \; B_2 K_1(\xi \emptyset \; i^{-\frac{1}{2}}) \qquad (2.9.1.)$$

These solutions involve I_1 and K_1 Modified Bessel Functions. For the Imaginary Argument, these functions can be expressed in terms of Thomson Functions which are tabulated in most of the books on Bessel Functions.

The relations between Thomson and Bessel Functions are

$$i^n\, I_n(kxi^{\frac{1}{2}}) = \mathrm{ber}_n(kx) + i\, \mathrm{bei}_n(kx) \qquad (e)$$

$$i^{-n}\, K_n(kxi^{\frac{1}{2}}) = \mathrm{ker}_n(kx) + i\, \mathrm{kei}_n(kx) \qquad (f)$$

where ber, bei, ker, kei are called the Thomson Functions. Recurrence formulae exist giving the n + 1 order Bessel functions in terms of the preceding orders. Since we need the first order, and the zero order Thomson functions are extensively tabulated for the small kx intervals, we will obtain the first order directly from these tabulated zero orders by the following recurrence formulae:

$$I_o'\ (kxi^{\frac{1}{2}}) = \frac{dI_o(kxi^{\frac{1}{2}})}{d(kxi^{\frac{1}{2}})} = \frac{1}{i^{\frac{1}{2}}}\ \frac{dI_o(kxi)}{d(kx)} = I_1(kxi^{\frac{1}{2}}) \qquad (g)$$

$$K_o'\ (kxi^{\frac{1}{2}}) = \frac{dK_o(kxi^{\frac{1}{2}})}{d(kxi^{\frac{1}{2}})} = \frac{1}{i^{\frac{1}{2}}}\ \frac{dK_o(kxi^{\frac{1}{2}})}{d(kx)} = -K_1(kxi^{\frac{1}{2}}) \qquad (h)$$

For the case of n = 0 which is zero order, (e) and (f) are

$$I_o(kxi^{\frac{1}{2}}) = \mathrm{ber}(kx) + i\, \mathrm{bei}(kx) \qquad (i)$$

$$K_o(kxi^{\frac{1}{2}}) = \mathrm{ker}(kx) + i\, \mathrm{kei}(kx) \qquad (j)$$

It is customary to omit the zero subscript in zero order function. If we like to express I_1 and K_1 in terms of zero order Thomson functions, from (g) and (e)

$$I_1(kxi^{\frac{1}{2}}) = i^{-\frac{1}{2}}\ (\mathrm{ber}'kx + i\ \mathrm{bei}'kx)$$

Equations for Z_1	Equations for Z_2
$\ddot{Z}_1 + \frac{1}{\phi}\dot{Z}_1 - \left(i\,\xi^2 + \frac{1}{\phi^2} \right) Z_1 = 0$	$\ddot{Z}_2 + \frac{1}{\phi}\dot{Z}_2 + \left(i\,\xi^2 - \frac{1}{\phi^2} \right) Z_2 = 0$
$\ddot{Z}_1 + \frac{1}{\phi}\dot{Z}_1 + \left(i^3\,\xi^2 - \frac{1}{\phi^2} \right) Z_1 = 0$	
since $i^2 = -1$	
Therefore solutions are:	
$Z_1 = A_1^* J_1 (\xi\phi\, i^{\frac{3}{2}}) + B_1^* Y_1 (\xi\phi\, i^{\frac{3}{2}})$	$Z_2 = A_2^* J_1 (\xi\phi\, i^{\frac{1}{2}}) + B_2^* Y_1 (\xi\phi\, i^{\frac{1}{2}})$
In terms of modified Bessel Functions:	
$J_1(\xi\phi\, i^{\frac{3}{2}}) = J_1(\xi\phi\, i^{\frac{1}{2}}\, i) = i\, I_1(\xi\phi\, i^{\frac{1}{2}})$	$J_1(\xi\phi\, i^{\frac{1}{2}}) = J_1(\xi\phi\, i^{-\frac{1}{2}}\, i) = i\, I_1(\xi\phi\, i^{-\frac{1}{2}})$
$Y_1(\xi\phi\, i^{\frac{3}{2}}) = Y_1(\xi\phi\, i^{\frac{1}{2}}\, i)$	$Y_1(\xi\phi\, i^{\frac{1}{2}}) = Y_1(\xi\phi\, i^{-\frac{1}{2}}\, i)$
$= \frac{2}{\pi}\, i\, K_1(\xi\phi\, i^{\frac{1}{2}}) - I_1(\xi\phi\, i^{\frac{1}{2}})$	$= \frac{2}{\pi}\, i\, K_1(\xi\phi\, i^{-\frac{1}{2}}) - I_1(\xi\phi\, i^{-\frac{1}{2}})$

Equations for Z_1 (Cont'd.)

therefore if we substitute these values

$$Z_1 = A_1^* i \, I_1(\xi \phi \, i^{\frac{1}{2}}) + B_1^* i \frac{2}{\pi} K_1(\xi \phi \, i^{\frac{1}{2}})$$

$$- B_1^* \, I_1 \, (\xi \phi \, i^{\frac{1}{2}})$$

$$Z_1 = (A_1^* i - B_1^*) I_1(\xi \phi \, i^{\frac{1}{2}}) + \frac{2}{\pi} B_1^* i \, K_1(\xi \phi \, i^{\frac{1}{2}})$$

$$Z_1 = A_1 I_1(\xi \phi \, i^{\frac{1}{2}}) + B_1 K_1(\xi \phi \, i^{\frac{1}{2}})$$

where

$$A_1 = (A_1^* i - B_1^*)$$

$$B_1 = \frac{2}{\pi} B_1^* \, i$$

Equations for Z_2 (Cont'd.)

$$Z_2 = A_2^* i \, I_1(\xi \phi \, i^{-\frac{1}{2}}) + B_2^* \frac{2}{\pi} i \, K_1(\xi \phi \, i^{-\frac{1}{2}})$$

$$- B_2^* I_1(\xi \phi \, i^{-\frac{1}{2}})$$

$$Z_2 = (A_2^* i - B_2^*) \, I_1(\xi \phi \, i^{-\frac{1}{2}}) + \frac{2}{\pi} B_2^* i \, K_1(\xi \phi \, i^{-\frac{1}{2}})$$

$$Z_2 = A_2 I_1 \, (\xi \phi \, i^{-\frac{1}{2}}) + B_2 K_1 \, (\xi \phi \, i^{-\frac{1}{2}})$$

$$A_2 = (A_2^* i - B_2^*)$$

$$B_2 = \frac{2}{\pi} B_2^* \, i$$

where $ber'kx = \dfrac{d\ ber(kx)}{d(kx)}$. In the above formula $i^{-\frac{1}{2}}$ can be taken into the bracket. Finally the first order Bessel functions will be expressed in terms of zero order Thomson functions as follow:

$$I_1(kxi^{\frac{1}{2}}) = i^{-\frac{1}{2}}\ ber'kx + i^{\frac{1}{2}}bei'kx \qquad (k)$$

On the other hand the expression for K_1 can be obtained by the very same way, from (f) and (h)

$$-K_1(kxi^{\frac{1}{2}}) = i^{-\frac{1}{2}}ker'kx + i^{\frac{1}{2}}\ kei'kx \qquad (l)$$

Therefore the first solution Z_1 can be written in following form by substituting (k) and (l) into Eq. (2.8.1.)

$$Z_1=A_1(i^{-\frac{1}{2}}ber'\xi\phi\ +i^{\frac{1}{2}}bei'\xi\phi\ ) - B_1(i^{-\frac{1}{2}}ker'\xi\phi\ +i^{\frac{1}{2}}kei'\xi\phi\ ) \quad 2.8.2.)$$

For the second solution Z_2 we can follow the same system.

$$i^{-n}I_n(kxi^{-\frac{1}{2}}) = ber_nkx - i\ bei_nkx \qquad (m)$$

$$i^nK_n(kxi^{-\frac{1}{2}}) = ker_nkx - i\ kei_nkx \qquad (n)$$

by replacing $n = 0$,

$$I_o(kxi^{-\frac{1}{2}}) = ber\ kx - i\ bei\ kx \qquad (o)$$

$$K_o(kxi^{-\frac{1}{2}}) = ker\ kx - i\ kei\ kx \qquad (p)$$

$$\frac{d\ I_o(kxi^{-\frac{1}{2}})}{d(kxi^{-\frac{1}{2}})} = i^{\frac{1}{2}}\ \frac{d\ I_o(kxi^{-\frac{1}{2}}\ )}{d(kx)} = i^{\frac{1}{2}}\ (\ ber'kx - bei'kx\ )$$

$$I_1(kxi^{-\frac{1}{2}}) = i^{\frac{1}{2}}\ ber'kx + i^{-\frac{1}{2}}bei'kx \qquad (q)$$

$$\frac{d\ K_o(kxi^{-\frac{1}{2}})}{d(kxi^{-\frac{1}{2}})} = i^{\frac{1}{2}}\ \frac{d\ K_o(kxi^{-\frac{1}{2}})}{d(kx)} = i^{\frac{1}{2}}(ker\ kx - i\ kei\ kx)$$

$$-K_1(kxi^{-\frac{1}{2}}) = i^{\frac{1}{2}}ker\ kx + i^{-\frac{1}{2}}kei\ kx \qquad (r)$$

Therefore, by substituting (q) and (r) into Eq. (2.9.1.) the second solution Z_2 can be obtained.

$$Z_2=A_2(i^{\frac{1}{2}}\mathrm{ber}'\xi\phi + i^{-\frac{1}{2}}\mathrm{bei}'\xi\phi\ ) - B_2(i^{\frac{1}{2}}\mathrm{ker}'\xi\phi + i^{-\frac{1}{2}}\mathrm{kei}'\xi\phi\ ) \quad (2.9.2.)$$

These two solutions together represent the complete system of independent solutions of Eq.(2.4.1.). By adding up Z_1 and Z_2 we can write the solution as in following form:

$$Z = C_1\ \mathrm{ber}'\xi\phi + C_2\ \mathrm{bei}'\xi\phi + C_3\ \mathrm{ker}'\xi\phi + C_4\ \mathrm{kei}'\xi\phi \qquad (2.10.1.)$$

where
$$C_1 = A_1\ i^{-\frac{1}{2}} + A_2 i^{\frac{1}{2}}$$

$$C_2 = A_1\ i^{\frac{1}{2}} + A_2\ i^{-\frac{1}{2}}$$

$$C_3 = -\ (B_1\ i^{-\frac{1}{2}} + B_2\ i^{\frac{1}{2}})$$

$$C_4 = -\ (B_1\ I^{\frac{1}{2}} + B_2\ i^{-\frac{1}{2}})$$

arbitrary constants.

The solution for U is easy when the Z_1 and Z_2 solutions are known. For this we will use Eqs. (2.2.2.) with (2.5.3.) and (2.5.4.). For the first solution, by rearranging Eq. (2.5.3.) we obtain

$$\ddot{Z}_1 + \frac{\dot{Z}_1}{\phi} - \frac{1}{\phi^2} Z_1 = i\ \xi^2 Z_1$$

By use of equation (2.2.2.) this becomes;

$$(i\xi^2 - \nu)\ Z_1 = -\frac{12}{h}\ U_1$$

from which

$$U_1 = \frac{h}{12}\ (\nu - i\ \xi^2)\ Z_1 \qquad (2.11.1.)$$

For the second solution, the same replacement can be done between Eqs. (2.2.2.) and (2.5.4.). By this way the relation between U_2 and Z_2 can be found:

$$U_2 = \frac{h}{12} (\nu + i\, \xi^2) \quad Z_2 \tag{2.12.1.}$$

Substituting Z_1 and Z_2 with their equivalents in terms of Bessel Functions in the above relations we can obtain the first and second solutions for U:

$$U_1 = \frac{h}{12} (\nu - i\, \xi^2) \Big[A_1 (i^{-\frac{1}{2}} \mathrm{ber}' \xi\phi + i^{\frac{1}{2}} \mathrm{bei}' \xi\phi) - B_1 (i^{-\frac{1}{2}} \mathrm{ker}' \xi\phi + i^{\frac{1}{2}} \mathrm{kei}' \xi\phi) \Big] \tag{2.11.2.}$$

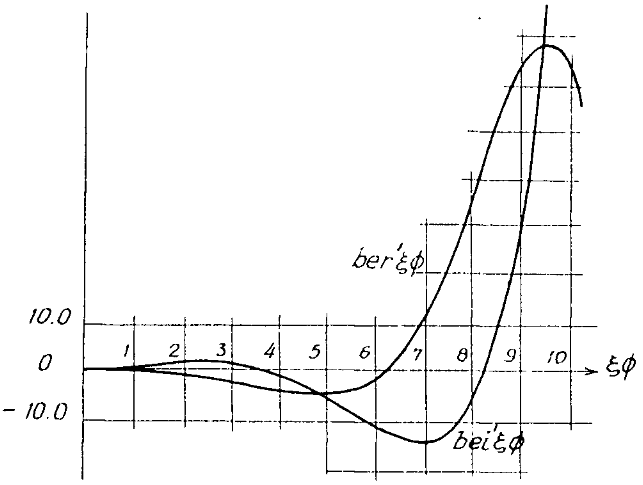
$$U_2 = \frac{h}{12} (\nu + i\, \xi^2) \Big[A_2 (i^{\frac{1}{2}} \mathrm{ber}' \xi\phi + i^{-\frac{1}{2}} \mathrm{bei}' \xi\phi) - B_2 (i^{\frac{1}{2}} \mathrm{ker}' \xi\phi + i^{-\frac{1}{2}} \mathrm{kei}' \xi\phi) \Big] \tag{2.12.2.}$$

By adding two solutions together and combining the coefficients the general solution for U can be shown as follow:

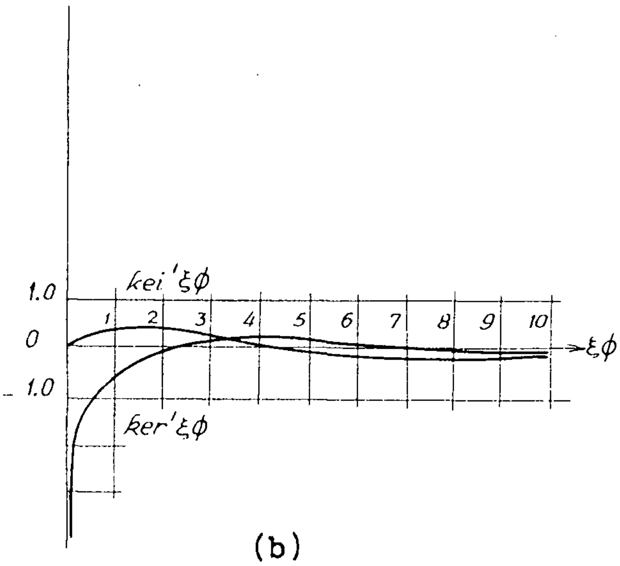
$$U = \frac{h}{12} \Big[(C_1 \nu - C_2 \xi^2)\, \mathrm{ber}' \xi\phi + (C_2 \nu + C_1 \xi^2)\, \mathrm{bei}' \xi\phi + (C_3 \nu - C_4 \xi^2)\, \mathrm{ker}' \xi\phi + (C_4 \nu + C_3 \xi^2)\, \mathrm{kei}' \xi\phi \Big] \tag{2.13.1.}$$

The Thomson Functions ber', bei', ker', kei' are shown on Fig. II.1.a and Fig. II.1.b.

U and Z involve zero order Thomson Functions. To evaluate $\dot{U}$ and $\dot{Z}$ we have to know the first derivatives of ber', bei', ker', and kei'. The following relations are



(a)



(b)

Fig. II.1.

given for this reason:

$$\frac{d}{dx}\left(x \; ber' kx \right) = - kx \; bei \; kx$$

$$\frac{d}{dx}\left(x \; bei' kx \right) = kx \; ber \; kx$$

$$\frac{d}{dx}\left(x \; ker' kx \right) = - kx \; kei \; kx \qquad (8)$$

$$\frac{d}{dx}\left(x \; kei' kx \right) = kx \; ker \; kx$$

From the above relations we can derive

$$\mathrm{ber}'kx + x\frac{d(\mathrm{ber}'kx)}{dx} = -kx\,\mathrm{bei}\,kx$$

$$\mathrm{bei}'kx + x\frac{d(\mathrm{bei}'kx)}{dx} = kx\,\mathrm{ber}\,kx$$

$$\mathrm{ker}'kx + x\frac{d(\mathrm{ker}'kx)}{dx} = -kx\,\mathrm{kei}\,kx \qquad (t)$$

$$\mathrm{kei}'kx + x\frac{d(\mathrm{kei}'kx)}{dx} = kx\,\mathrm{ker}\,kx$$

therefore, in our case substituting $k = \xi$, $x = \phi$

$$\frac{d(\mathrm{ber}'\xi\phi)}{d\phi} = -\xi\,\mathrm{bei}\,\xi\phi - \frac{1}{\phi}\,\mathrm{ber}'\xi\phi$$

$$\frac{d(\mathrm{bei}'\xi\phi)}{d\phi} = \xi\,\mathrm{ber}\,\xi\phi - \frac{1}{\phi}\,\mathrm{bei}'\xi\phi$$

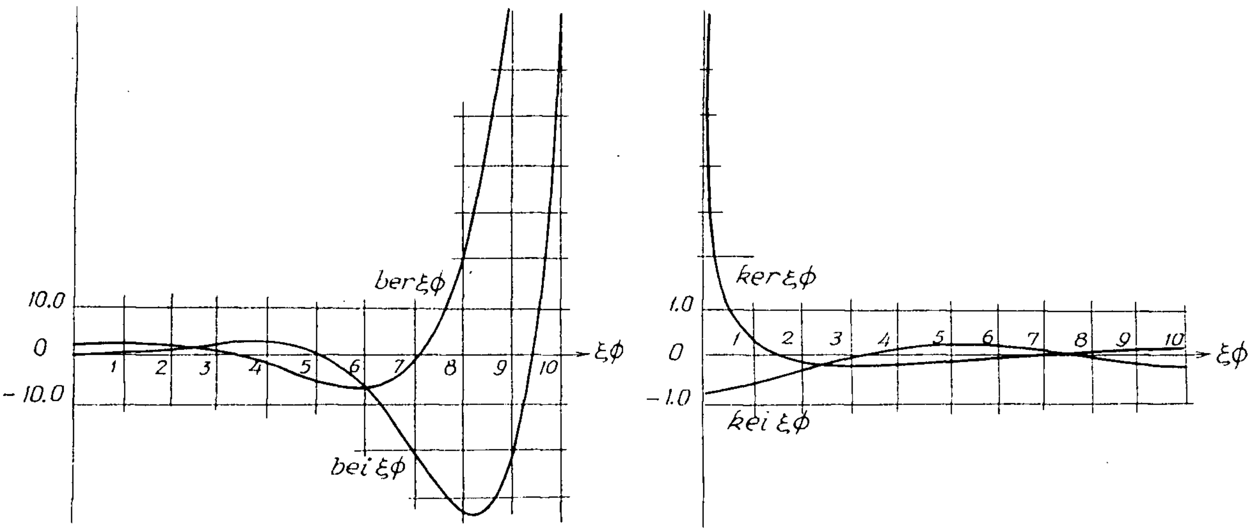
$$\frac{d(\mathrm{ker}'\xi\phi)}{d\phi} = -\xi\,\mathrm{kei}\,\xi\phi - \frac{1}{\phi}\,\mathrm{ker}'\xi\phi \qquad (u)$$

$$\frac{d(\mathrm{kei}'\xi\phi)}{d\phi} = \xi\,\mathrm{ker}\,\xi\phi - \frac{1}{\phi}\,\mathrm{kei}'\xi\phi$$

The functions ber bei ker kei are shown on Fig. II.2.a. and Fig. II.2.b.

Thomson functions are tabulated for the values of $\xi\phi$ between 0 and 10. For $\xi\phi$ bigger than 10 there are "Asymptotic Expansions" given in reference (4).

Since we know Z, $\dot{Z}$, U, $\dot{U}$ it is easy to express M_ϕ, M_θ, N_ϕ, N_θ, T , Q_ϕ in terms of Thomson Functions by using the relations given in Ch. I. i.e.,



(a) (b)

Fig. II.2.

$$N_\phi = - U \frac{1}{\phi} \tag{2.14.1.}$$

$$N_\theta = - \dot{U} \tag{2.15.1.}$$

$$T = - U \frac{1}{\phi} \tag{2.16.1.}$$

$$Q_\phi = U \tag{2.17.1.}$$

$$M_\phi = - \left(\dot{Z} + \nu Z \frac{1}{\phi} \right) \tag{2.18.1.}$$

$$M_\theta = - \left(Z \frac{1}{\phi} + \nu \dot{Z} \right) \tag{2.19.1.}$$

In the above relations $\text{Cot}\,\phi = \frac{1}{\phi}$ and $\text{Sin}\,\phi = \phi$ were used.

Up to this point the solutions were given in a general form. The coefficients C_1 C_2 C_3 C_4 are to be evaluated for the different problems. It is obvious that to obtain these we have to have four boundary conditions. Two of these conditions belong to the top of the shell and the other two are given around the periphery. For instance, let us take the case of a shell with a hole at the crown. The boundary conditions, the values of M_ϕ and T at the top and at the periphery, would give us values of the four coefficients. As an other example a concentrated load placed at the crown would yield four different coefficients.

In our case since there is neither a hole nor a concentrated load at the top the evaluation of the coefficients will be considerably easier than the above two examples. By the symmetry, there can be no rotation at the top. Therefore $Z = 0$ at $\phi = 0$ will be the first boundary condition. On the other hand at $\phi = 0$ the shearing force is equal to zero. Since $U = Q_\phi$ U is zero also.

From these conditions we can conclude that the two coefficients of ker' must be zero since ker' approaches infinity at the origin.

From $Z = 0$ $\qquad C_3 = 0$

From $U = 0$ $\qquad C_3 \nu - C_4 \xi^2 = 0$

therefore both C_3 and C_4 coefficients should be zero. As a result we can see that the solution for U and Z does not contain either ker' or kei' at all. Finally our functions are as shown below:

$$Z = C_1 \text{ ber}'\xi\phi + C_2 \text{ bei}'\xi\phi \tag{2.10.2.}$$

$$U = \frac{h}{12}\left[(C_1\nu - C_2 \xi^2) \text{ ber}'\xi\phi + (C_2\nu + C_1 \xi^2) \text{ bei}'\xi\phi\right] \tag{2.13.2.}$$

The first derivatives of the above functions are

$$\dot{Z} = -C_1\left(\xi\,\mathrm{bei}\,\xi\phi + \frac{1}{\phi}\mathrm{ber}'\xi\phi\right) + C_2\left(\xi\,\mathrm{ber}\,\xi\phi - \frac{1}{\phi}\mathrm{bei}'\xi\phi\right) \tag{2.20.1.}$$

$$\dot{U} = \frac{h}{12}\Big[-(C_1\nu - C_2\xi^2)\left(\xi\,\mathrm{bei}\,\xi\phi + \frac{1}{\phi}\mathrm{ber}'\xi\phi\right) + (C_2\nu + C_1\xi^2)\left(\xi\,\mathrm{ber}\,\xi\phi - \frac{1}{\phi}\mathrm{bei}'\xi\phi\right)\Big] \tag{2.21.1.}$$

The coefficients C_1 and C_2 are found by considering the boundary conditions at the support at $\phi = \alpha$. If M_α and T_α are prescribed at $\phi = \alpha$ then the following equations will be necessary:

$$M_\phi = C_1\left(\xi\,\mathrm{bei}\,\xi\phi + \frac{1-\nu}{\phi}\mathrm{ber}'\xi\phi\right) + C_2\left(\frac{1-\nu}{\phi}\mathrm{bei}'\xi\phi - \xi\,\mathrm{ber}\,\xi\phi\right) \tag{2.18.2.}$$

$$T = -\frac{h}{12}\,C_1(\nu\,\mathrm{ber}'\xi\phi + \xi^2\mathrm{bei}'\xi\phi) + C_2(\nu\,\mathrm{bei}'\xi\phi - \xi^2\mathrm{ber}'\xi\phi) \tag{2.16.2.}$$

From these C_1 and C_2 become,

$$C_1 = \frac{M_\alpha(\nu\,\mathrm{bei}'\xi\alpha - \xi^2\mathrm{ber}'\xi\alpha) + \frac{12\alpha T_\alpha}{h}\left(\frac{1-\nu}{\alpha}\mathrm{bei}'\xi\alpha - \xi\,\mathrm{ber}\,\xi\alpha\right)}{\Delta}$$

$$C_2 = \frac{M_\alpha(\nu\,\mathrm{ber}'\xi\alpha + \xi^2\mathrm{bei}'\xi\alpha) + \frac{12\alpha T_\alpha}{h}\left(\frac{1-\nu}{\alpha}\mathrm{ber}'\xi\alpha + \xi\,\mathrm{bei}\,\xi\alpha\right)}{\Delta}$$

where

$$\Delta = \left(\frac{1-\nu}{\alpha}\mathrm{ber}'\xi\alpha + \xi\,\mathrm{bei}\,\xi\alpha\right)(\nu\,\mathrm{bei}'\xi\alpha - \xi^2\mathrm{ber}'\xi\alpha) - \left(\frac{1-\nu}{\alpha}\mathrm{bei}'\xi\alpha - \xi\,\mathrm{ber}\,\xi\alpha\right)(\nu\,\mathrm{ber}'\xi\alpha + \xi^2\mathrm{bei}'\xi\alpha)$$

CHAPTER III

NUMERICAL SOLUTION OF THE DIFFERENTIAL EQUATIONS

Last chapter the solution of two simultaneous differential equations was made in terms of the Bessel Functions. As we have seen already, this solution is applicable for small values of ϕ . To extend the solution from the Bessel Function zone to 90°, we will use a numerical method. Let us rewrite our equations.

$$\ddot{U} + \dot{U} \cot\phi - (\cot^2\phi - \nu^2)\, U = \frac{1-\nu^2}{h} Z \qquad (3.1.1.)$$

$$\ddot{Z} + \dot{Z} \cot\phi - (\cot^2\phi + \nu^2)\, Z = - \frac{12}{h} U \qquad (3.2.1.)$$

The solution of these equations shows an oscillating and exponential behavior. This exponential build up increases very fast and exceeds the working limits of digital computers. Since we will be using a digital computer for our computations, it is more convenient to define two new functions u and z as follows:

$$u = U\,e^{-\lambda\phi} \qquad \text{(a)}$$

$$z = Z\,e^{-\lambda\phi}$$

where

$$\lambda = \sqrt[4]{\frac{3(1-\nu^2)}{h^2}} \qquad \text{(b)}$$

As it will be shown in Ch. IV, this λ appears in the approximate solution. These u and z functions have a small exponential build up and an oscillatory behavior. If λ were some more complicated function of h then u and z would have no exponential build up. It was not considered necessary to use this though. Two differential equations for u and z are obtained by substituting these expressions into the original equations for U and Z.

$$\ddot{u} + (2\lambda + \cot\phi)\,\dot{u} + (\lambda^2 + \lambda\cot\phi - \cot^2\phi + \nu)\,u = \frac{1-\nu^2}{h}\,z \qquad (3.1.2.)$$

$$\ddot{z} + (2\lambda + \cot\phi)\,\dot{z} + (\lambda^2 + \lambda\cot\phi - \cot^2\phi - \nu)\,z = -\frac{12}{h}\,u \qquad (3.2.2.)$$

For further rearrangement we will present a new variable x instead of ϕ. The relation between them is

$$\lambda\phi = x \qquad \text{(c)}$$

If we differentiate this once, then,

$$\lambda\, d\phi = dx \qquad \text{(d)}$$

As a dot ($\dot{F}$) represents derivative with respect to $\emptyset$. Then we will let a prime (F') represent derivative with respect to x. If we replace $\emptyset$ by x in Eqs. (3.1.2.) and (3.2.2.) and divide by λ^2 we obtain:

$$u'' + \left(2 + \frac{\cot\frac{x}{\lambda}}{\lambda}\right) u' + \left(1 + \frac{\cot\frac{x}{\lambda}}{\lambda} - \frac{\cot^2\frac{x}{\lambda}}{\lambda^2} + \frac{\nu}{\lambda^2}\right) u = \sqrt{\frac{1-\nu^2}{3}}\, z \tag{3.1.4.}$$

$$z'' + \left(2 + \frac{\cot\frac{x}{\lambda}}{\lambda}\right) z' + \left(1 + \frac{\cot\frac{x}{\lambda}}{\lambda} - \frac{\cot^2\frac{x}{\lambda}}{\lambda^2} - \frac{\nu}{\lambda^2}\right) z = -4\sqrt{\frac{3}{1-\nu^2}}\, u \tag{3.2.4.}$$

To simplify the above equations in writing, let us define a new shorthand such that:

$$u'' + a\, u' + \left(b + \frac{\nu}{\lambda^2} \right) u = Cz \tag{3.1.5.}$$

$$z'' + a\, z' + \left(b - \frac{\nu}{\lambda^2} \right) z = -D\, u \tag{3.2.5.}$$

where

$$\left.\begin{aligned} a &= 2 + \frac{\cot\emptyset}{\lambda} \\ b &= 1 + \frac{\cot\emptyset}{\lambda} - \frac{\cot^2\emptyset}{\lambda^2} \\ C &= \sqrt{\frac{1-\nu^2}{3}} \\ D &= 4\sqrt{\frac{3}{1-\nu^2}} \end{aligned}\right\} \tag{e}$$

In this chapter up to now we have been reforming our two simultaneous differential equations of second order. Finally we obtained the forms shown in Eqs. (3.1.5.) and (3.2.5.)

To solve these equations we will replace the first and second derivatives by approximate expressions derived from finite difference equations.

Rather than deal with the continuous function u we will consider only the values u_{n-1} , u_n , u_{n+1} et cetera evaluated at a finite number of points n-1, n, n+1 ... on the curve u = f(x). With reference to Fig. III.1. it is seen that at point n the function and its first derivative are given by u_n and u_n' while at point n+1 by u_{n+1} and u_{n+1}'. Each point is dx apart.

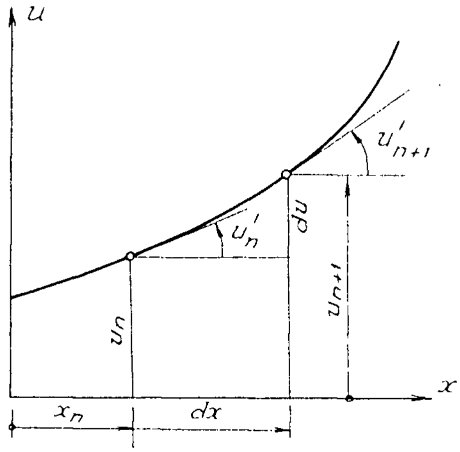


Fig. III.1.

To replace the first derivative by an approximate expression we will argue that the average of the slopes at points n and n+1 is du / dx, or:

$$\frac{u_{n+1}' + u_n'}{2} = \frac{u_{n+1} - u_n}{dx} \qquad \text{(f)}$$

Similarly we can replace the second derivative by arguing that the average of the second derivatives at point n, and n+1 is the change in the first derivatives / dx, or:

$$\frac{u_{n+1}'' + u_n''}{2} = \frac{u_{n+1}' - u_n'}{dx} \qquad \text{(g)}$$

Since this is also true for z we have:

$$(z'_{n+1} + z'_n) / 2 = (z_{n+1} - z_n) / dx \qquad \text{(h)}$$

$$(z''_{n+1} + z''_n) / 2 = (z'_{n+1} - z_n) / dx \qquad \text{(i)}$$

Two more equations are known if we write our differential equations for point n+1.

$$u''_{n+1} + a_{n+1} u'_{n+1} + (b_{n+1} + \frac{\nu}{\lambda^2}) u_{n+1} = C z_{n+1} \qquad \text{(j)}$$

$$z''_{n+1} + a_{n+1} z'_{n+1} + (b_{n+1} - \frac{\nu}{\lambda^2}) z_{n+1} = - D u_{n+1} \qquad \text{(k)}$$

If the function and its derivatives are known at point n, then these six equations (f to k) will enable us to solve for six unknowns at point n+1. These six unknowns are u_{n+1} u'_{n+1} u''_{n+1} z_{n+1} z'_{n+1} and z''_{n+1}. With the function known at point n+1 it may be again evaluated at point n+2. These six equations are shown solved for the values at point n+1 below:

$$u''_{n+1} = \frac{g_{n+1} \left[e_{n+1} - \frac{\nu}{\lambda^2}(dx/2)^2 \right] + F\ k_{n+1}}{\left[e^2_{n+1} - \frac{\nu^2}{\lambda^4}(dx/2)^4 \right] + H\ F} \qquad (3.3.1.)$$

$$z''_{n+1} = \frac{k_{n+1} \left[e_{n+1} + \frac{\nu}{\lambda^2}(dx/2)^2 \right] - H\ g_{n+1}}{\left[e^2_{n+1} - \frac{\nu^2}{\lambda^4}(dx/2)^4 \right] + H\ F} \qquad (3.4.1.)$$

$$u'_{n+1} = u''_{n+1}\ (dx/2) + (u'_n + (dx/2)\ u''_n\ ) \qquad (3.5.1.)$$

$$z'_{n+1} = z''_{n+1}\ (dx/2) + (z'_n + (dx/2)\ z''_n\ ) \qquad (3.6.1.)$$

$$u_{n+1} = u_n\ + (dx/2)\ u'_{n+1}\ + (dx/2)\ u'_n \qquad (3.7.1.)$$

$$z_{n+1} = z_n\ + (dx/2)\ z'_{n+1}\ + (dx/2)\ z'_n \qquad (3.8.1.)$$

The factors and coefficients used in this solution are shown below:

$$\left.\begin{aligned}
e_{n+1} &= 1 + a_{n+1}\ (dx/2) + b_{n+1}\ (dx/2)^2 \\
F &= C\ (dx/2)^2 \\
H &= D\ (dx/2)^2 \\
K &= u'_n + (dx/2)\ u''_n \\
L &= z'_n + (dx/2)\ z''_n \\
M &= u_n + (dx)\ u'_n + (dx/2)^2\ u''_n \\
N &= z_n + (dx)\ z'_n + (dx/2)^2\ z''_n \\
g_{n-1} &= C\ N - a_{n+1}\ K - (b_{n+1} + \frac{\nu}{\lambda^2})\ M \\
k_{n-1} &= -D\ M - a_{n+1}\ L - (b_{n+1} - \frac{\nu}{\lambda^2})\ N
\end{aligned}\right\} \qquad (1)$$

In the above solution the main problem arises in choosing the right interval dx. If we take a large interval the error in accuracy is great. On the other hand very small intervals cause quite appreciable round off errors which make the results unreliable. Of course, the range of the optimum intervals varies with different h ratios. In the numerical analysis, for each h value this interval was investigated and then by using these optimum intervals the rest of the problem was solved with the above equations. Once u z and their first and second derivatives are found, the normal forces, bending moments and deformations can be evaluated. In order to get these values the relations should be known. The Table III.1. is given for this reason. It can be noticed that after finding the quantities shown by the small lettering symbols we can easily obtain the quantities shown by the capital letters which are the products of the previous values times $\lambda e^{\lambda \phi}$, the magnification factor.

It was shown that to find the next point values we have to know the values for the previous point. At the very first step we need to know the values for u, z, u', z', u'', z''. These starting values taken at $\phi = \phi_0$ are calculated by Bessel Functions. To obtain a fairly good result these starting values are supposed to be as close as possible to the exact values. For the small ϕ values the Bessel Solutions give us an almost exact answer; therefore if we find the values for a small ϕ, say $5^\circ - 10^\circ$ by Bessel functions we can use the numerical solutions for the rest of the problem. In Ch. II, $U, Z, \dot{U}, \dot{Z}$ were found for small ϕ. Since the relation (a) is known between U's and u's, substituting these into Eqs. (2.10.2.) (2.13.2.) (2.20.1.) (2.21.1.) we obtain the

Table III.1. The relations between the functions and non-exponential values

Functions		Non-exponential values
$N_\phi = -U \text{Cot}\,\phi$	$N_\phi = \lambda e^{\lambda\phi} n_\phi$	$n_\phi = -\frac{u}{\lambda} \text{Cot}\,\phi$
$N_\theta = -\dot{U}$	$N_\theta = \lambda e^{\lambda\phi} n_\theta$	$n_\theta = -(u + u')$
$Q_\phi = U$	$Q_\phi = \lambda e^{\lambda\phi} q_\phi$	$q_\phi = \frac{u}{\lambda}$
$T = -U \frac{1}{\text{Sin}\,\phi}$	$T = \lambda e^{\lambda\phi} t$	$t = -\frac{u}{\lambda} \frac{1}{\text{Sin}\,\phi}$
$M_\phi = -(\dot{Z} + \nu Z \text{Cot}\,\phi)$	$M_\phi = \lambda e^{\lambda\phi} m_\phi$	$m_\phi = -\left[(z'+z) + \nu z \frac{\text{Cot}\,\phi}{\lambda}\right]$
$M_\theta = -(\nu \dot{Z} + Z \text{Cot}\,\phi)$	$M_\theta = \lambda e^{\lambda\phi} m_\theta$	$m_\theta = -\left[(z'+z)\nu + z \frac{\text{Cot}\,\phi}{\lambda}\right]$
$\Delta = \text{Sin}\,\phi(\nu U \text{Cot}\phi - \dot{U})$	$\Delta = \lambda e^{\lambda\phi}$	$\delta = \text{Sin}\,\phi\left[u(\frac{\text{Cot}\,\phi}{\lambda}\nu - 1) - u'\right]$
$Z = Z$	$Z = \lambda e^{\lambda\phi} z^*$	$z^* = \frac{z}{\lambda}$

following:

$$z = \frac{C_1}{e^{\lambda\phi}} \text{ber}'\xi\phi + \frac{C_2}{e^{\lambda\phi}} \text{bei}'\xi\phi \tag{3.9.1.}$$

$$z' = -\frac{C_1}{e^{\lambda\phi}}\left[\frac{\xi}{\lambda}\text{bei}\,\xi\phi + \frac{1}{\lambda\phi}\text{ber}'\xi\phi\right] + \frac{C_2}{e^{\lambda\phi}}\left[\frac{\xi}{\lambda}\text{ber}\,\xi\phi - \frac{1}{\lambda\phi}\text{bei}'\xi\phi\right] - z \tag{3.10.1.}$$

$$u = \left[C_1\frac{h\nu}{12e^{\lambda\phi}} - C_2\frac{\xi^2 h}{12e^{\lambda\phi}}\right]\text{ber}'\xi\phi + \left[C_2\frac{\nu h}{12e^{\lambda\phi}} + C_1\frac{\xi^2 h}{12e^{\lambda\phi}}\right]\text{bei}'\xi\phi \tag{3.11.1.}$$

$$u' = -\left[C_1 \frac{h\nu}{12e^{\lambda\phi}} - C_2 \frac{\xi^2 h}{12e^{\lambda\phi}}\right]\left[\frac{\xi}{\lambda}\,\mathrm{bei}\,\xi\phi + \frac{1}{\lambda\phi}\,\mathrm{ber}'\,\xi\phi\right]$$

$$+\left[C_2 \frac{h\nu}{12e^{\lambda\phi}} + C_1 \frac{\xi^2 h}{12e^{\lambda\phi}}\right]\left[\frac{\xi}{\lambda}\,\mathrm{ber}\,\xi\phi - \frac{1}{\lambda\phi}\,\mathrm{bei}'\,\xi\phi\right] - u \quad (3.12.1.)$$

On the other hand the Eqs. (3.1.5.) and (3.3.5.) can be rewritten by rearranging so that the second derivatives should be separated at the left of the equations.

$$z'' = -D\,u - a\,z' - \left(b - \frac{\nu}{\lambda^2}\right) z \quad (3.1.6.)$$

$$u'' = C\,z - a\,u' - \left(b + \frac{\nu}{\lambda^2}\right) u \quad (3.2.6.)$$

In the above equations the coefficients C_1 and C_2 should be determined. As we indicated in the Ch.II, U and Z equal zero at the top of a spherical shell. But U' and Z' can have any values at all. However, each of these possibilities can be expressed as a linear combination of the following two cases:

Case I : at $\phi = 0$ $u' = K$, a constant
$z' = 0$

Case II : at $\phi = 0$ $u' = 0$
$z' = K$, a constant.

namely,

A. (Case I) + B. (Case II) = Actual case

Therefore if we obtain the solutions for the Case I and Case II, it is a matter of finding the constants A and B for any particular case.

Case I. Since at $\phi = 0$ the Thomson Functions are zero except that ber (0) = 1. Therefore,

$z' = 0$ will give us $C_2 = 0$

$u' = K$ $\quad C_1 = \dfrac{12\ K}{h\xi^2} \cdot \dfrac{\lambda}{\xi}$

If we choose K as such that $C_1 = 12$, the Eqs. from (3.9.1.) to (3.12.1.) will be as follows:

$$z = 12\ \mathrm{ber}'\,\xi\phi \tag{3.9.2.}$$

$$z' = -12\left[\frac{\xi}{\lambda}\mathrm{bei}\,\xi\phi + \frac{1}{\lambda\phi}\mathrm{ber}'\,\xi\phi\right] - z \tag{3.10.2.}$$

$$u = h\nu\,\mathrm{ber}'\,\xi\phi + \xi^2 h\ \mathrm{bei}'\,\xi\phi \tag{3.11.2.}$$

$$u' = -h\nu\left[\frac{\xi}{\lambda}\mathrm{bei}\,\xi\phi + \frac{1}{\lambda\phi}\mathrm{ber}'\,\xi\phi\right] + \xi^2 h\left[\frac{\xi}{\lambda}\mathrm{ber}\,\xi\phi - \frac{1}{\lambda\phi}\mathrm{bei}'\,\xi\phi\right] - u \tag{3.12.2.}$$

The equations (3.1.6.) and (3.2.6.) will remain as they are.

Case II.

$z' = K$ will give us $\quad C_2 = \dfrac{K\lambda}{\xi}$

$u' = 0$ $\quad C_2 \dfrac{\nu}{\xi^2} = -C_1$

If we choose K such that $C_2 = 12$ the above equations will appear as shown below.

$$z = -\frac{12\nu}{\xi^2}\mathrm{ber}'\,\xi\phi + 12\ \mathrm{bei}'\,\xi\phi \tag{3.9.3.}$$

$$z' = \frac{12\nu}{\xi^2}\left[\frac{\xi}{\lambda}\mathrm{bei}\,\xi\phi + \frac{1}{\lambda\phi}\mathrm{ber}'\,\xi\phi\right] + 12\left[\frac{\xi}{\lambda}\mathrm{ber}\,\xi\phi - \frac{1}{\lambda\phi}\mathrm{bei}'\,\xi\phi\right] - z \tag{3.10.3.}$$

$$u = -\left[\frac{\nu^2 h}{\xi^2} + \xi^2 h\right] \text{ber}' \xi \phi \qquad (3.11.3.)$$

$$u' = \left[\frac{\nu^2 h}{\xi^2} + \xi^2 h\right]\left[\frac{\xi}{\lambda} \text{bei}\, \xi\phi + \frac{1}{\lambda\phi} \text{ber}' \xi \phi\right] - u \qquad (3.12.3.)$$

In these formulae ξ and λ depend upon the parameter h, assuming ν, Poisson's Ratio is constant. In the Table III.2., ξ and λ are tabulated for different h values, by taking $\nu = 0.2$ and the starting angle as $\phi_o = 6^\circ$. The values of the Thomson Functions at this angle are obtained from Ref.(4) and shown on Table III.3. The starting values for the Case I. and Case II. are calculated and shown on the Table III.4. for the same values of the parameter h. Tables III.2. and 4. were calculated by Alwac III-E Electronic Digital Computer.

TABLE III. 2.

$\nu = 0.2 \qquad \phi_o = 6^\circ$

1/h	λ	ξ	$\lambda\phi_o$	$\xi\phi_o$
30	7.135243	10.090747	.747200	1.056699
40	8.239069	11.651796	.862792	1.220172
50	9.211559	13.027106	.964631	1.364194
100	13.027111	18.423115	1.364194	1.929262
150	15.954888	22.563617	1.670790	2.362854
200	18.423117	26.054221	1.929262	2.728389
250	20.597671	29.129504	2.156981	3.050431
300	22.563618	31.909775	2.362854	3.341580
400	26.054223	36.846235	2.728389	3.858524
500	29.129506	41.195343	3.050431	4.313961

TABLE III. 3.

$\nu = 0.2 \quad \phi_o = 6^o$

1/h	$\xi\phi_o$	ber$\xi\phi_o$	bei$\xi\phi_o$	ber'$\xi\phi_o$	bei'$\xi\phi_o$
30	1.0567	.9805	.2786	- .0737	.5249
40	1.2202	.9654	.3708	- .1133	.6031
50	1.3642	.9460	.4625	- .1582	.6698
100	1.9293	.7848	.9082	- .4434	.8953
150	2.3629	.5195	1.3206	- .8022	.9912
200	2.7284	.1549	1.6835	-1.2090	.9761
250	3.0504	- .3024	1.9813	-1.6420	.8528
300	3.3416	- .8437	2.1990	-2.0830	.6208
400	3.8585	-2.1353	2.3389	-2.9160	- .1709
500	4.3140	-3.6176	2.0045	-3.5630	-1.3890

TABLE III. 4. $\nu = 0.2$ $\phi = 6^\circ$

Case I / Case II / 1/h	u / z / u / z	u' / z' / u' / z'	u" / z" / u" / z"
30	1.781075	.539149	- 3.293490
	- .884040	- 2.660035	- 3.239540
	.250045	.752373	.916281
	6.300536	1.916634	-11.631543
40	2.046420	.213008	- 3.127757
	- 1.359600	- 3.356808	- 2.767788
	.384553	.949449	.782851
	7.239202	.762987	-11.050151
50	2.272741	- .090621	- 2.999983
	- 1.898400	- 3.370957	- 4.017557
	.536949	.953451	1.136339
	8.039838	- .312449	-10.597082
100	3.037860	- 1.500082	- 2.553856
	- 5.320800	- 6.191232	1.785724
	1.504949	1.751144	- .505073
	10.746736	- 5.296298	- 9.031352
150	3.363172	- 2.884882	- 2.130806
	- 9.626400	- 7.023405	6.389965
	2.722755	1.986517	- 1.807346
	11.898181	-10.194088	- 7.538544

TABLE III. 4. (Continued)

Case I	u / z	u' / z'	u" / z"
Case II / 1/h	u / z	u' / z'	u" / z"
200	3.311783	- 4.287341	- 1.558924
	-14.508000	- 6.542474	11.168408
	4.103481	1.850491	- 3.158901
	11.717474	-15.154191	- 5.518204
250	2.893183	- 5.688075	- .756350
	-19.704000	- 4.784883	15.914215
	5.573126	1.353368	- 4.501204
	10.238244	-20.108150	- 2.681536
300	2.105674	- 7.048501	.309267
	-24.995999	- 1.743280	20.414091
	7.069928	.493073	- 5.773962
	7.454510	-24.919558	1.085492
400	- .581511	- 9.456595	3.288541
	-34.992000	8.124331	27.990059
	9.897225	- 2.297906	- 7.916779
	- 2.045645	-33.436529	11.618575
500	- 4.715846	-11.103581	7.401575
	-42.756000	22.754273	32.555154
	12.093220	- 6.435879	- 9.207986
	-16.662962	-39.262461	26.160855

In this chapter, so far, we obtained the mathematical relations between the known values u_n, z_n, their first and second derivatives at the point (n) and the unknown values u_{n+1}, z_{n+1}, their first and second derivatives at the next point (n+1). To obtain the next point values from the above shown formulae we will use the Alwac III-E Digital computer. Since the accuracy of a numerical method depends upon the interval dx, we face the problem of investigating the most acceptable interval. In the following pages three different cases were examined, i.e., $1/h = 30$, 100, 500. To investigate the interval for each case the starting values of Case I were used and the corresponding values were typed out for every 20° - i.e., 26°, 46°, 66°, 86° - . The curves shown on Fig. III. 2.a.b.c. are for the z values at different h's and every 20°. From these figures, it is easy to see that the thicker shell ($1/h = 30$) gives the larger intervals; the thinner shell ($1/h = 500$) requires the shorter intervals. The curves for the thick shell go almost in horizontal shape and even for the larger intervals the obtained values come close to the smaller interval results. But for the thin shell, say $1/h = 500$, the very same curves show a large curvature and get stable and horizontal for only the quite small intervals. For instance, for 500 $d\phi = 0.05$ degrees was taken as the closest interval to the best, and 0.25 for 30. The reason for this is that the thin shell has many more nodes than the thick shell. It requires a smaller interval to approximate these many waves.

After the investigation of the intervals the stress resultants were calculated with these intervals for the different cases. The results are tabulated on the Tables III.5.6. and 7.

The flow diagram of the Programme is shown on pages 54 and 55. This seven channel programme was performed on Working Channel I. On the other hand the Working Channels II and III were used as the scratch pads for temporary storage, and are shown on page 56. The words which are indicated by an asterisk contain several intermediate computation steps. The value $\bar{M}$ shown in Channel II defines the type out interval. That is the stress resultants were typed out at an angular interval of $\bar{M}.d\phi$.

During the calculations B which places the binary point, was taken as 23 in decimal system which means 17 in the hexadecimal system. The Main Memory Channels, numbered from 98 to 9e were used for the storage of the programme. The entrance word is the first word of Ch. 98. After the operations are done, the programme stops at the second word from the end of Ch. 9e.

It is very interesting to compare the wave lengths for the different h values, i.e., 30 100, and 500. The thickest shell gives a wave length more or less four times bigger than the thinnest shell does. The intermediate shell gives a wave length which is between the other two.

In the Figs. III.3.4.5. the m_ϕ curves are shown for the different cases.

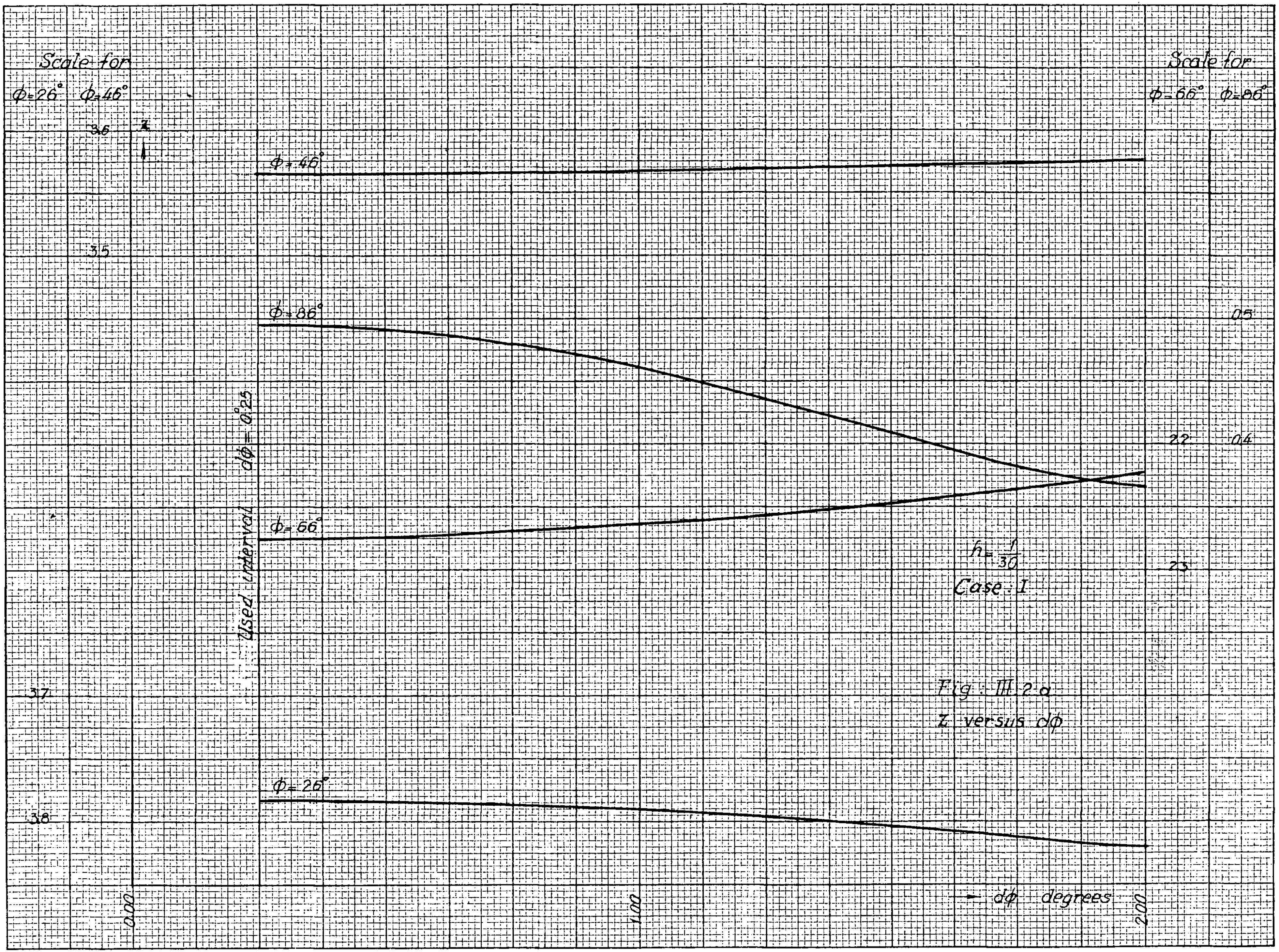
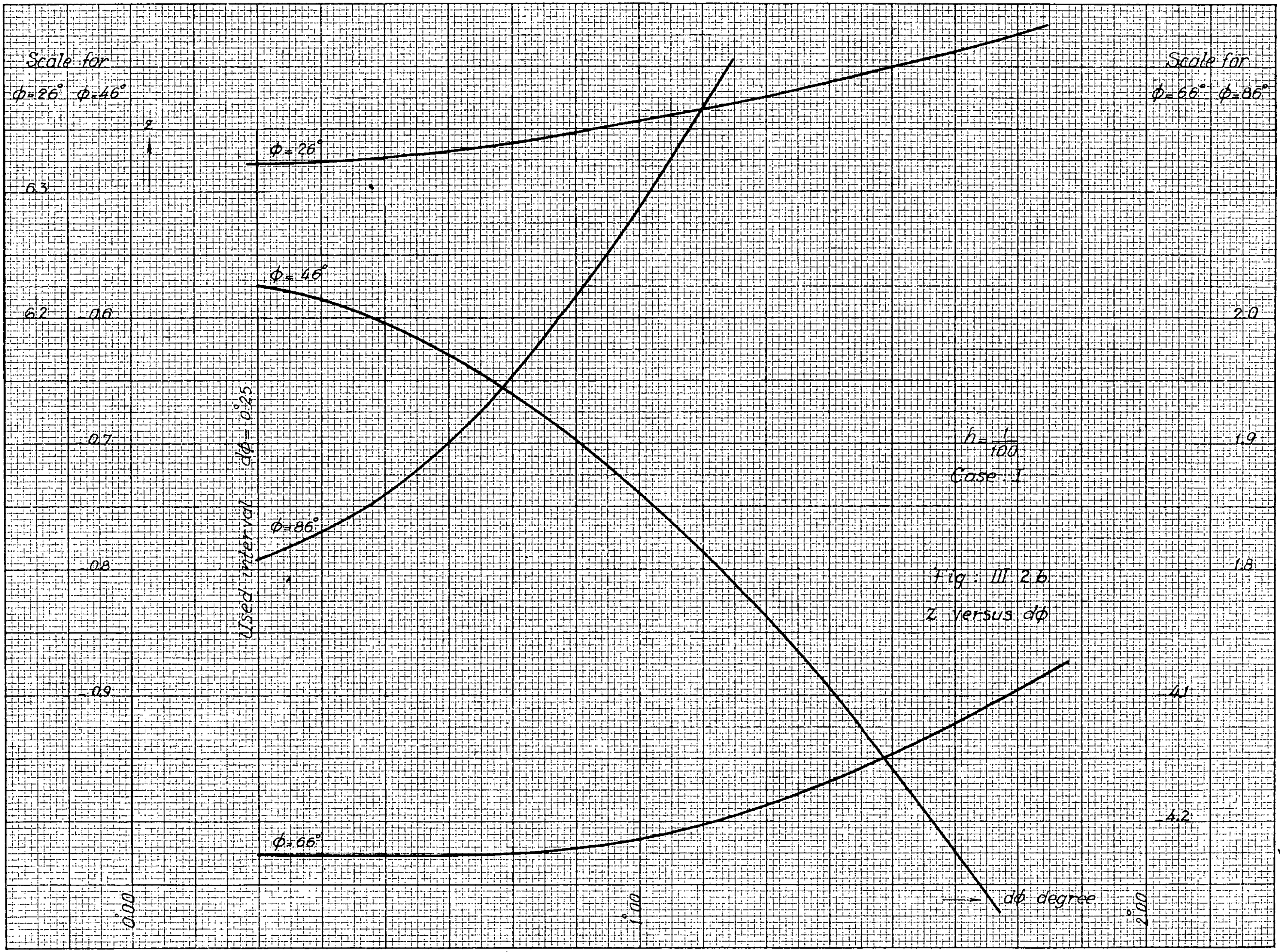


Fig: III 2 a
z versus dφ



Scale for
φ=26° φ=46°
Scale for
φ=66° φ=86°
z
6.3
6.2
0.6
0.7
0.8
0.9
2.0
1.9
1.8
4.1
4.2
φ=26°
φ=46°
φ=86°
φ=66°
Used interval dφ=0.25°
h=1/100
Case : I
Fig : III 2.b
z versus dφ
dφ degree
0.00°
1.00°
2.00°

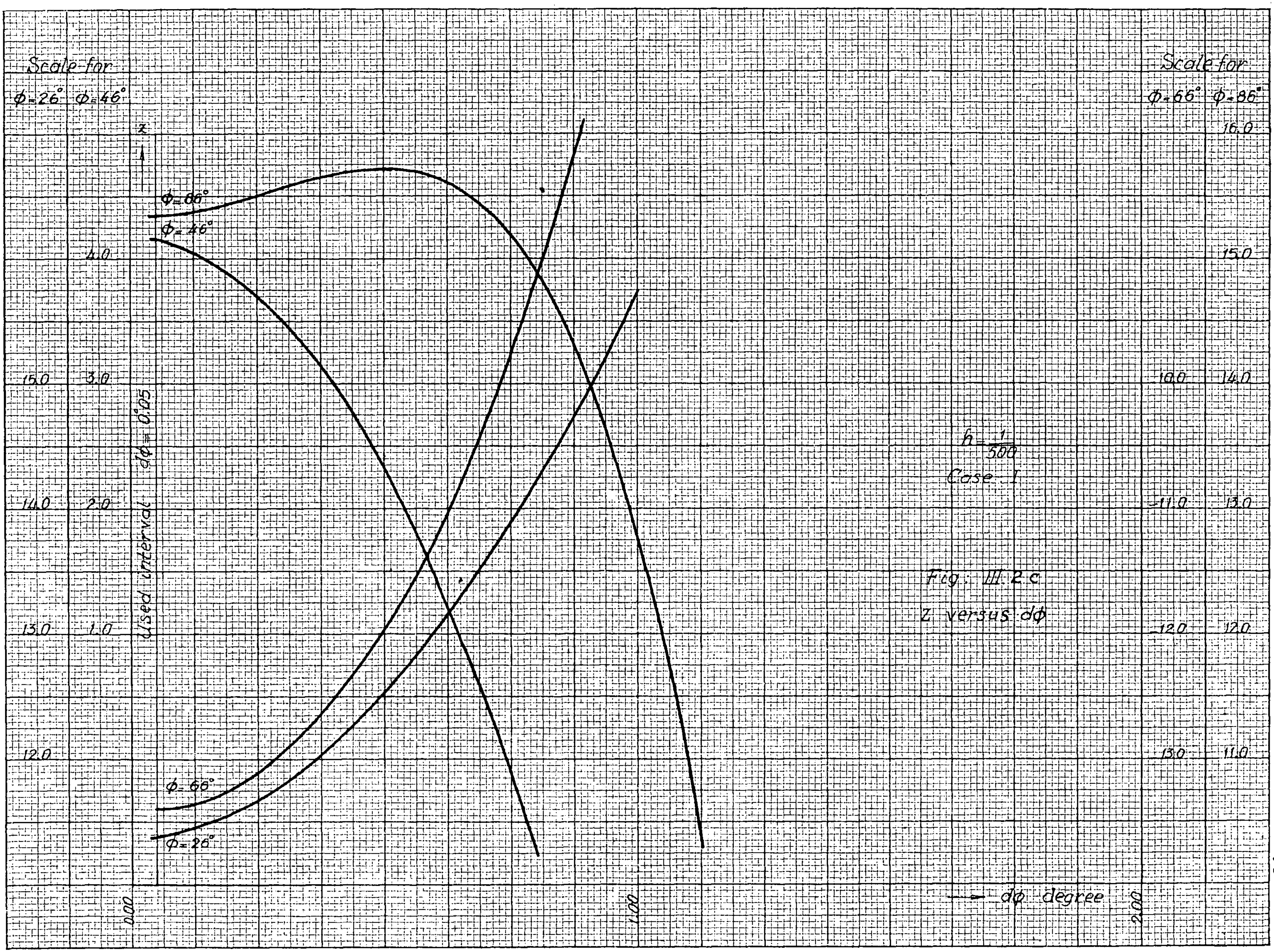


Fig. III.2 c

Z versus $d\phi$

FLOW DIAGRAM OF PROGRAMME

Decimal Input: $\phi_o^o, u_n, u_n', u_o'', z_n, z_n', z_n'', \nu, \lambda, d\phi^o, \overline{M}$

Store the above input values in Working Channel II as noted.

Main Mem.

Ch : 98

Calculate the following values and store in the indicated places in Ch. II or III.

C, D, $d\phi^{rad}$, dx, $(\frac{dx}{2})^2$, F,H. FH,

$\frac{\nu}{\lambda^2}(\frac{dx}{2})^2$,

Main Mem.

Ch : 99

Put $\overline{M}$ in the storage word (45)

Add $d\phi^o$ to ϕ_n^o and put ϕ_{n+1}^o in (20)

Calculate: $\phi_{n+1}^{rad}\text{Sin}\phi_{n+1}\text{Cox}\phi_{n+1}, \text{Cot}\phi_{n+1},$

$\frac{\text{Cot}\phi_{n+1}}{\lambda}, \frac{\text{Cot}^2\phi_{n+1}}{\lambda^2}$

Calculate: $a_{n+1}, b_{n+1}, K, L,$

Main Mem.

Ch : 9a

Calculate: M, N, $e_{n+1}, g_{n+1}, k_{n+1}$

(2)

(1)

Main Mem.

Ch : 9b

Calculate: $u'' \; z'' \; u' \; z' \; u \; z$ at $\phi = \phi_{n+1}$

Main Mem.

Ch : 9c

Transfer $u \; u' \; u'' \; z \; z' \; z''$ to the old places.

Copy ϕ present to (5f)

Subtract 1 from $\overline{M}$ at (45). If,

to (2) — $A \neq 0$

$A = 0$

Calculate: $n_\phi \; n_\theta \; m_\phi \; m_\theta \; q_\phi \; t \; z^* \delta$

Main Mem.

Ch : 9d

Decimal Output: $\phi \quad m_\phi \; m_\theta \; n_\phi \; n_\theta$

$t \; q_\phi \; z^* \delta$

Check $90 - \phi_{present}$. If,

to (1) — $A > 0$

$A < 0$

STOP

Main Mem.

Ch : 9e

The following charts show the working channels which were used as the scratch pads for the programme.

WORKING CHANNEL II

ϕ_0^0 latter ϕ_{n+1}^0	u_n	u'_n	u''_n
z_n	z'_n	z''_n	ν
λ	$d\phi^0$	$\overline{M}$	u_{n+1}
u'_{n+1}	u''_{n+1}	z_{n+1}	z'_{n+1}
z''_{n+1}	m_ϕ	m_θ	n_ϕ
n_θ	t	q_ϕ	z^*
δ	C	D	F
H	FH	$d\phi$(radians)	dx

WORKING CHANNEL III

$(\frac{dx}{2})^2$	$\frac{\nu}{\lambda^2}\ (\frac{dx}{2})^2$	$\pi/2$	90^0
Unity	$\overline{M}$ (transferred)	ϕ_{n+1}(radians)	$\mathrm{Sin}\phi_{n+1}$
$\mathrm{Cot}\phi_{n+1}$	$\mathrm{Cot}\phi_{n+1}/\lambda$	$(\mathrm{Cot}\phi_{n+1}/\lambda)^2$	a_{n+1}
b_{n+1}	K	L	M
N	e_{n+1}	$(\frac{\nu}{\lambda^2})$	g_{n+1}
k_{n+1}	$(e_{n+1}-\frac{\nu}{\lambda^2}(\frac{dx}{2})^2)$	$(e_{n+1}+\frac{\nu}{\lambda^2}(\frac{dx}{2})^2)$	$(e^2_{n+1}-\frac{\nu^2}{\lambda^4}(\frac{dx}{2})+HF$
*	*	*	*
*	*	*	*

Note: In the above tables all the values are based on B = 17 (Hexadecimal) decimal point location.

The complete programme which is to be stored in the Main Memory drum is as follows:

9804

START	571f2800	00000000	00000000	00000000
	48601704	00000000	00000000	00000000
(9800)	871e571b	00000000	4127e727	00000000
	5b171160	00000000	a309651a	00000000
	482b170c	00000000	81991100	00000000
	790ff782	00000000	00000000	00170010
	11060000	00000000	00800000	000b0000
	00000000	00000000	03170619	00400000

9904

3128a317	e713eb17	a309493c	4943791b
eb1f2800	c53ee728	4127e509	a7034944
a3098705	a117c53f	eb282800	819a1100
5b1b1160	e73fa307	a317eb28	00000000
a3174939	49403a00	c541413b	00c90fda
2800411b	4139e518	e73ca309	2d000000
a31ceb39	a309493b	493d7913	00100000
c53a4129	413ae500	49427917	01800000

9a04

792a4945	a109c547	a309494a	4126e506
79206129	28004146	7944a701	a3086125
49203128	a3095b0d	6149494b	494e4122
e742eb43	11600011	6744674a	e510a309
c5462800	a30eeb47	494c0000	6121495e
a3098706	c5482800	5b3f4123	4125e500
5b1c11e4	a317eb28	e51ba308	a3096124
00000001	c549e749	6122494d	819b1100

9b04

495d4140	e74ba308	e74fa309	e750a309
e723a309	6144615c	495b414b	4959414b
615e494f	49514127	e74da309	e74ea309
4140e726	2800a317	495a4139	4958413a
a309615d	eb282800	e750a309	e74fa309
49504140	a317eb28	675a675b	63586759
e74ca309	c5523000	4953794c	49547951
495c413f	614c3128	67523000	819c1100

9c04

67414955	a309495e	eb57c530	792f6125
79516141	413be754	79306126	3000e73f
49563000	a309615d	3000e73f	a3086124
e755a309	3128a317	a3086125	492e792c
613d4957	eb57c52d	492f792d	61223000
4155e753	413ce753	61233000	e73fa308
a309495d	a309655e	e73fa308	6121492b
4156e754	3128a317	6122592c	819d1100

9d04

151f571b	a3092e00	49354124	4931415c
78314827	49337921	2800a317	e727a309
17047920	61222e00	eb28c537	495b4124
495f7945	49344121	79246125	e749a309
67444934	2800a317	495c4124	615b2e00
1918111c	eb28c536	e727a117	819e1100
819a1104	2800a317	e749a309	00060000
4121e749	eb47312e	615c2e00	1b000000

9e04

49324149	79201160	11601719	00000000
e727a309	57175b13	790ff781	00000000
67443000	791bf781	79436720	03170682
e721a309	78351160	1d1e0000	5b6d0000
67223000	1785790f	819a1100	03170611
e747a309	f7815717	00000000	00040000
49380000	5b0b791b	00000000	58000000
871f5b1f	f7817839	1b000000	03170005

STOP

TABLE III.5.a.

h = 1/30 Case: I

ϕ	m_ϕ	m_θ	n_ϕ	n_θ
Degrees	t	q_ϕ	z^*	δ
8	5.210935 -1.837284	2.602285 .255700	-1.819404 -.228393	-1.674068 -.182343
10	6.275797 -1.390701	3.139213 .241493	-1.369573 -.346051	-1.087089 -.141206
12	6.899218 -1.024379	3.466403 .212980	-1.001994 -.461991	-.542312 -.071088
14	7.067032 -.721506	3.582645 .174548	-.700074 -.563387	-.037007 .024920
16	6.802714 -.470505	3.504568 .129689	-.452278 -.640405	.422434 .141371
18	6.153980 -.263445	3.258890 .081409	-.250551 -.686424	.824826 .270370
20	5.184844 -.094813	2.877774 .032428	-.089095 -.697915	1.157594 .402015
22	3.970350 .039397	2.395949 -.014758	.036528 -.674168	1.409300 .525196
24	2.592412 .142339	1.848732 -.057895	.130033 -.616943	1.571403 .628569
26	1.136021 .216760	1.270569 -.095021	.194822 -.530086	1.639378 .701575
28	-.314516 .265285	.693808 -.124544	.234233 -.419115	1.613306 .735408
30	-1.679284 .290596	.147643 -.145298	.251664 -.290780	1.498016 .723841
32	-2.886329 .295518	-.342795 -.156601	.250614 -.152619	1.302866 .663852
34	-3.874884 .283039	-.757193 -.158273	.234650 -.012495	1.041217 .555998
36	-4.598075 .256291	-1.080607 -.150644	.207343 .121841	.729667 .404513

TABLE III.5.a. (Continued)

h = 1/30 Case: I

ϕ Degrees	m_ϕ / t	m_θ / q_ϕ	n_ϕ / z^*	n_θ / δ
38	-5.024940 .218498	-1.303797 -.134521	.172179 .243183	.387109 .217127
40	-5.141629 .172904	-1.423307 -.111141	.132452 .345238	.033665 .004611
42	-4.951722 .122687	-1.441292 -.082093	.091174 .422953	-.310424 -.219916
44	-4.475627 .070872	-1.365096 -.049232	.050981 .472755	-.625909 -.441876
46	-3.749109 .020245	-1.206601 -.014563	.014063 .492718	-.895567 -.646240
48	-2.821010 -.026723	-.981377 .019859	-.017881 .482625	-1.105114 -.818602
50	-1.750309 -.067948	-.707681 .052051	-.043676 .443952	-1.243939 -.946221
52	-.602675 -.101782	-.405354 .080206	-.062664 .379742	-1.305634 -1.018977
54	.553298 -.127051	-.094687 .102787	-.074679 .294412	-1.288279 -1.030156
56	1.649958 -.143071	.204709 .118611	-.080004 .193479	-1.194480 -.977003
58	2.624355 -.149646	.474944 .126907	-.079300 .083230	-1.031156 -.861019
60	3.421704 -.147052	.700770 .127351	-.073526 -.029642	-.809087 -.687955
62	3.998332 -.135994	.870323 .120076	-.063845 -.138423	-.542258 -.467511
64	4.323986 -.117556	.975653 .105659	-.051533 -.236758	-.247023 -.212759
66	4.383361 -.093132	1.013019 .085081	-.037880 -.319001	.058841 .060675
68	4.176784 -.064350	.982952 .059664	-.024106 -.380532	.357166 .335628

TABLE III.5.a. (Continued)

$h = 1/30$ Case: I

ϕ	m_ϕ	m_θ	n_ϕ	n_θ
Degrees	t	q_ϕ	z^*	δ
70	3.720003	.890055	-.011281	.630424
	-.032982	.030993	-.418001	.594525
72	3.043125	.742598	-.000267	.862723
	-.000863	.000821	-.429508	.820549
74	2.188750	.551904	.008325	1.040697
	.030203	-.029033	-.414691	.998782
76	1.209417	.331577	.014159	1.154249
	.058528	-.056789	-.374731	1.117215
78	.164517	.096624	.017176	1.197107
	.082614	-.080809	-.312273	1.167587
80	-.883173	-.137489	.017577	1.167162
	.101223	-.099685	-.231257	1.145968
82	-1.871113	-.355782	.015787	1.066562
	.113431	-.112327	-.136680	1.053056
84	-2.740599	-.544658	.012404	.901574
	.118667	-.118017	-.034308	.894168
86	-3.440123	-.692702	.008143	.682207
	.116735	-.116450	.069673	.678920
88	-3.928315	-.791329	.003763	.421627
	.107816	-.107750	.169006	.420618
90	-4.176297	-.835260	.000000	.135398
	.092462	-.092462	.257733	.135398

TABLE III.5.b.

$h = 1/30$ Case: II

ϕ Degrees	m_ϕ / t	m_θ / q_ϕ	n_ϕ / z^*	n_θ / δ
8	-7.225722 -.464166	-7.626549 .064599	-.459649 .904936	-1.381947 -.179535
10	-4.836535 -.563658	-5.623202 .097878	-.555095 .855166	-1.664046 -.269680
12	-2.652985 -.628492	-3.939680 .130671	-.614759 .754815	-1.828444 -.354591
14	-.653630 -.658684	-2.515382 .159350	-.639118 .619335	-1.871037 -.421721
16	1.147000 -.657146	-1.314108 .181134	-.631690 .461035	-1.797762 -.460707
18	2.714864 -.628283	-.315393 .194150	-.597533 .290521	-1.621103 -.464018
20	4.009352 -.577159	.492239 .197400	-.542353 .117392	-1.358026 -.427373
22	4.992867 -.509023	1.116262 .190683	-.471958 -.049531	-1.028594 -.349958
24	5.637518 -.429019	1.563628 .174498	-.391929 -.202266	-.654860 -.234473
26	5.929381 -.342019	1.843029 .149931	-.307405 -.333869	-.259835 -.086953
28	5.870768 -.252505	1.966195 .118544	-.222948 -.438683	.133548 .083630
30	5.480844 -.164490	1.948448 .082245	-.142453 -.512566	.503463 .265977
32	4.794885 -.081460	1.808670 .043167	-.069082 -.553069	.830193 .447257
34	3.862411 -.006321	1.568842 .003534	-.005240 -.559533	1.097031 .614038

TABLE III.5.b. (Continued)

h = 1/30 Case: II

ϕ Degrees	m_ϕ / t	m_θ / q	n_ϕ / z^*	n_θ / δ
36	2.744448 .058630	1.253271 -.034462	.047432 -.533086	1.291047 .753282
38	1.510130 .111721	.887595 -.068782	.088037 -.476559	1.403662 .853340
40	.232883 .151913	.497687 -.097648	.116372 -.394298	1.431000 .904868
42	-1.013592 .178783	.108509 -.119629	.132861 -.291907	1.373987 .901596
44	-2.159293 .192490	-.256977 -.133715	.138466 -.175918	1.238206 .840893
46	-3.141638 .193735	-.578800 -.139362	.134580 -.053424	1.033493 .724071
48	-3.908732 .183688	-.840799 -.136507	.122911 .068317	.773321 .556421
50	-4.422000 .163918	-1.031238 -.125569	.105364 .182286	.473990 .346955
52	-4.658060 .136302	-1.143181 -.107407	.083916 .282079	.153680 .107876
54	-4.609739 .102930	-1.174610 -.083272	.060501 .362250	-.168595 -.146186
56	-4.286185 .066009	-1.128289 -.054724	.036912 .418594	-.474060 -.399134
58	-3.712077 .027759	-1.011374 -.023541	.014710 .448358	-.745221 -.634478
60	-2.925995 -.009681	-.834819 .008384	-.004840 .450370	-.966836 -.836466
62	-1.978030 -.044342	-.612582 .039152	-.020817 .425076	-1.126736 -.991173

TABLE III.5.b. (Continued)

h = 1/30 Case: II

ϕ Degrees	m_ϕ t	m_θ q_ϕ	n_ϕ z^*	n_θ δ
64	-.926790 -.074505	-.360704 .066965	-.032661 .374491	-1.216476 -1.087490
66	.164026 -.098765	-.096301 .090227	-.040172 .302060	-1.231769 -1.117937
68	1.229318 -.116083	.163466 .107630	-.043485 .212439	-1.172677 -1.079223
70	2.206297 -.125816	.402398 .118228	-.043032 .111219	-1.043571 -.972548
72	3.038046 -.127735	.606178 .121483	-.039472 .004590	-.852832 -.803584
74	3.676709 -.122019	.763150 .117292	-.033633 -.101019	-.612346 -.582159
76	4.086148 -.109235	.864935 .105990	-.026426 -.199309	-.336791 -.321658
78	4.243915 -.090297	.906831 .088324	-.018774 -.284475	-.042779 -.038171
80	4.142444 -.066418	.887994 .065409	-.011533 -.351533	.252105 .250546
82	3.789386 -.039039	.811386 .038659	-.005433 -.396601	.530316 .526231
84	3.207088 -.009757	.683505 .009704	-.001020 -.417120	.775362 .771317
86	2.431243 .019754	.513905 -.019706	.001378 -.411988	.972738 .970093
88	1.508775 .047831	.314548 -.047802	.001669 -.381620	1.110762 1.109751
90	.495113 .072898	.099023 -.072898	.000000 -.327917	1.181236 1.181236

TABLE III.6.a.

h = 1/100 Case: I

ϕ	m_ϕ	m_θ	n_ϕ	n_θ
Degrees	t	q_ϕ	z^*	δ
8	13.041071 -1.182708	6.695467 .164601	-1.171198 -.598359	.182697 .058026
10	11.230198 -.444361	6.058271 .077162	-.437610 -.700207	1.596820 .292483
12	7.514625 .056304	4.603750 -.011706	.055075 -.686563	2.573510 .532772
14	2.775996 .361491	2.715496 -.087453	.350753 -.561065	3.005121 .710033
16	-2.038789 .504017	.767108 -.138926	.484492 -.350924	2.862312 .762251
18	-6.059520 .516657	-.921849 -.159656	.491370 -.098171	2.210673 .652767
20	-8.621853 .434765	-2.119093 -.148698	.408545 .149653	1.200274 .382572
22	-9.364172 .295206	-2.701062 -.110586	.273710 .348568	.036941 -.006668
24	-8.273068 .133548	-2.659161 -.054319	.122002 .465889	-1.056603 -.439683
26	-5.668062 -.019198	-2.089280 .008416	-.017255 .485532	-1.881576 -.823316
28	-2.128400 -.139404	-1.166153 .065446	-.123086 .410121	-2.298293 -1.067426
30	1.623985 -.212668	-.106909 .106334	-.184176 .259631	-2.249344 -1.106254
32	4.863163 -.234373	.869792 .124199	-.198760 .066940	-1.766908 -.915253
34	6.993694 -.208917	1.582065 .116825	-.173200 -.128807	-.963302 -.519301

TABLE III.6.a. (Continued)

h = 1/100 Case: I

ϕ	m_ϕ	m_θ	n_ϕ	n_θ
Degrees	t	q_ϕ	z^*	δ
36	7.656354	1.914117	-.119570	-.006984
	-.147797	.086873	-.289744	.009951
38	6.786403	1.831967	-.052705	.911155
	-.066883	.041178	-.386319	.567453
40	4.614913	1.383519	.012754	1.614645
	.016649	-.010702	-.402537	1.036234
42	1.614482	.683784	.064859	1.974484
	.087277	-.058400	-.338484	1.312507
44	-1.598986	-.111303	.096038	1.932244
	.133509	-.092743	-.209730	1.328906
46	-4.388784	-.837121	.103783	1.509143
	.149402	-.107471	-.043833	1.070655
48	-6.220419	-1.352435	.090348	.799691
	.135024	-.100342	.125350	.580858
50	-6.761783	-1.565320	.061628	-.048521
	.095876	-.073446	.264375	-.046611
52	-5.941950	-1.448408	.025526	-.864725
	.041461	-.032672	..346674	-.685436
54	-3.958298	-1.041027	-.009832	-1.488656
	-.016726	.013532	.357526	-1.202757
56	-1.231704	-.438408	-.037685	-1.801464
	-.067392	.055871	.296615	-1.487233
58	1.680526	.229475	-.053760	-1.747994
	-.101450	.086034	.177753	-1.473264
60	4.196398	.824947	-.056754	-1.346271
	-.113508	.098300	.025859	-1.156075
62	5.823633	1.230141	-.048173	-.682611
	-.102611	.090600	-.128151	-.594203

TABLE III.6.a. (Continued)

h = 1/100 Case: I

ϕ	m_ϕ	m_θ	n_ϕ	n_θ
Degrees	t	q_ϕ	z^*	δ
64	6.254542 -.072156	1.369626 .064853	-.031631 -.253550	.106494 .101402
66	5.423312 -.029027	1.223929 .026518	-.011806 -.325831	.861968 .789604
68	3.515150 .017844	.831581 -.016545	.006684 -.331432	1.433767 1.328126
70	.926157 .059225	.279654 -.055653	.020256 -.270235	1.710284 1.603334
72	-1.817581 .087311	-.315043 -.083038	.026981 -.155400	1.639868 1.554475
74	-4.165443 .097120	-.830173 -.093357	.026770 -.010591	1.240336 1.187141
76	-5.651392 .087327	-1.162551 -.084733	.021126 .134832	.594750 .572984
78	-5.985374 .060401	-1.248433 -.059081	.012558 .251689	-.165603 -.164441
80	-5.109281 .022032	-1.075476 -.021698	.003826 .316761	-.887569 -.874838
82	-3.206766 -.020015	-.684170 .019820	-.002786 .317353	-1.426620 -1.412184
84	-.665302 -.057413	-.158662 .057099	-.006001 .253739	-1.675460 -1.665087
86	2.001788 -.082860	.391024 .082658	-.005780 .139034	-1.585144 -1.580129
88	4.257826 -.091451	.851682 .091395	-.003191 -.003480	-1.174605 -1.173251
90	5.649567 -.081590	1.129914 .081590	-.000000 -.145021	-.526737 -.526737

TABLE III.6.b.

h = 1/100 Case: II

ϕ Degrees	m_ϕ / t	m_θ / q_ϕ	n_ϕ / z^*	n_θ / δ
8	-.197601 -1.216052	-4.019519 .169242	-1.204218 .582658	-3.447728 -.446311
10	5.322942 -1.140515	-.425201 .198048	-1.123189 .273636	-2.951741 -.473556
12	9.128823 -.933999	2.009041 .194189	-.913589 -.040580	-1.942739 -.365929
14	10.869460 -.655968	3.361844 .158693	-.636483 -.308531	-.657874 -.128358
16	10.464796 -.360098	3.735997 .099256	-.346148 -.490764	.645885 .197112
18	8.170506 -.089856	3.301526 .027767	-.085458 -.564353	1.730982 .540184
20	4.542661 .123760	2.295648 -.042328	.116296 -.525905	2.415369 .818149
22	.335185 .263183	.997017 -.098590	.244019 -.391392	2.599784 .955614
24	-3.639638 .323977	-.312656 -.131773	.295967 -.192595	2.280783 .903602
26	-6.657900 .313111	-1.389020 -.137329	.281566 .029183	1.546856 .653411
28	-8.210224 .247086	-2.058941 -.116000	.218164 .230904	.558370 .241654
30	-8.084782 .146869	-2.241562 -.073435	.127193 .375642	-.484771 -.255104
32	-6.393242 .035729	-1.953141 -.018933	.030300 .439031	-1.381570 -.735332
34	-3.536503 -.065151	-1.295379 .036432	-.054013 .413191	-1.967313 -1.094067
36	-.118262 -.139425	-.429939 .081952	-.112797 .307484	-2.142985 -1.246355

TABLE III.6.b. (Continued)

h = 1/100 Case: II

ϕ Degrees	m_ϕ / t	m_θ / q_ϕ	n_ϕ / z^*	n_θ / δ
38	3.176157 -.177480	.455786 .109267	-.139856 .146040	-1.891513 -1.147311
40	5.712217 -.177126	1.185191 .113855	-.135687 -.037364	-1.278157 -.804139
42	7.024818 -.143078	1.624679 .095738	-.106327 -.206075	-.435379 -.277096
44	6.901310 -.085395	1.705981 .059321	-.061428 -.327649	.464547 .331236
46	5.414187 -.017235	1.435043 .012398	-.011972 -.379916	1.243730 .896387
48	2.898553 .047708	.886492 -.035454	.031923 -.354912	1.753016 1.298000
50	-.120003 .097614	.185423 -.074776	.062745 -.259981	1.899972 1.445850
52	-3.032218 .124432	-.519500 -.098054	.076608 -.115920	1.665315 1.300212
54	-5.265484 .124996	-1.086173 -.101124	.073471 .047422	1.104881 .881980
56	-6.394337 .101196	-1.406550 -.083895	.056588 .197184	.337060 .270053
58	-6.220089 .059285	-1.426361 -.050276	.031416 .303969	-.481608 -.413755
60	-4.804866 .008446	-1.153585 -.007314	.004223 .347515	-1.187765 -1.029366
62	-2.454319 -.041052	-.654446 .036247	-.019273 .320472	-1.643318 -1.447560

TABLE III.6.b. (Continued)

h = 1/100 Case: II

ϕ Degrees	m_ϕ / t	m_θ / q_ϕ	n_ϕ / z^*	n_θ / δ
64	.346777 -.079790	-.038144 .071715	-.034978 .229590	-1.762056 -1.577438
66	3.032781 -.100880	.566320 .092159	-.041032 .094137	-1.525738 -1.386334
68	5.069719 -.101105	1.036480 .093743	-.037874 -.058104	-.986660 -.907791
70	6.060002 -.081339	1.280641 .076434	-.027820 -.196447	-.256393 -.235702
72	5.819033 -.046216	1.255325 .043954	-.014281 -.293400	.516946 .494361
74	4.409093 -.003116	.972675 .002996	-.000859 -.330056	1.178337 1.132855
76	2.124359 .039304	.496615 -.038136	.009508 -.299736	1.596554 1.547284
78	-.569562 .072779	-.071228 -.071188	.015131 -.209180	1.689893 1.650005
80	-3.129306 .090976	-.612813 -.089594	.015798 -.077086	1.441963 1.416945
82	-5.042058 .090643	-1.017816 -.089761	.012615 .069702	.904487 .893186
84	-5.927106 .072164	-1.205760 -.071768	.007543 .201577	.186667 .184144
86	-5.610777 .039421	-1.141762 -.039325	.002750 .292078	-.566741 -.565909
88	-4.160131 -.000985	-.842859 .000984	-.000034 .323134	-1.204288 -1.203547
90	-1.868956 -.041018	-.373791 .041018	-.000000 .288637	-1.597939 -1.597939

TABLE III.7.a.

h = 1/500 Case: I

ϕ Degrees	m_ϕ / t	m_θ / q_ϕ	n_ϕ / z^*	n_θ / δ
7	-6.525208 2.606849	6.098029 -.317695	2.587418 -.946856	15.756419 1.857157
8	-31.448894 2.755683	-4.413855 -.383516	2.728864 -.274629	12.162363 1.616715
9	-46.794640 2.263577	-11.724693 -.354101	2.235709 .390312	6.103398 .884832
10	-49.887868 1.409638	-14.884017 -.244781	1.388223 .901185	-.877380 -.200568
11	-40.973178 .456842	-13.900183 -.087169	.448448 1.155255	-7.123819 -1.376401
12	-22.987845 -.378152	-9.630182 .078622	-.369889 1.114285	-11.244416 -2.322463
13	-.752762 -.948503	-3.516666 .213367	-.924192 .809509	-12.414977 -2.751179
14	20.223431 -1.188437	2.771313 .287509	-1.153136 .330716	-10.539430 -2.493923
15	35.043821 -1.110348	7.716131 .287379	-1.072514 -.197435	-6.236938 -1.558720
16	40.525771 -.789069	10.267951 .217497	-.758502 -.646011	-.664435 -.141329
17	35.859836 -.337193	10.035653 .098586	-.322460 -.911995	4.778519 1.415958
18	22.688492 .123239	7.318370 -.038083	.117207 -.941140	8.789742 2.708933
19	4.607351 .487042	2.980272 -.158565	.460507 -.738440	10.468242 3.378138
20	-13.773610 .685384	-1.794011 -.234415	.644051 -.364240	9.511718 3.209140

TABLE III.7.a. (Continued)

h = 1/500 Case: I

ϕ Degrees	m_ϕ / t	m_θ / q_ϕ	n_ϕ / z^*	n_θ / δ
21	-27.972286 .695439	-5.800948 -.249223	.649248 .082566	6.264759 2.198553
22	-34.703046 .539335	-8.104486 -.202038	.500062 .489829	1.611296 .566136
23	-32.620488 .273775	-8.242075 -.106972	.252011 .759622	-3.258037 -1.292709
24	-22.571906 -.026516	-6.306839 .010785	-.024224 .831304	-7.144121 -2.903802
25	-7.313947 -.287424	-2.893189 .121471	-.260495 .694797	-9.129733 -3.836370
26	9.221728 -.451968	1.074309 .198129	-.406226 .391221	-8.792606 -3.818806
27	22.923286 -.491355	4.582518 .223070	-.437800 .001136	-6.292161 -2.816828
28	30.506373 -.408694	6.780844 .191870	-.360856 -.376388	-2.312616 -1.051824
29	30.290483 -.235209	7.181317 .114032	-.205719 -.648553	2.119764 1.047628
30	22.562391 -.020460	5.761776 .010230	-.017719 -.751334	5.897504 2.950521
31	9.449568 .180727	2.951306 -.093081	.154913 -.664322	8.105461 4.158659
32	-5.652695 .321424	-.493767 -.170328	.272582 -.414479	8.240114 4.337702
33	-18.949366 .372867	-3.689031 -.203078	.312713 -.068217	6.322537 3.409434
34	-27.189675 .329729	-5.844985 -.184382	.273358 .285998	2.879987 1.579895

TABLE III.7.a. (Continued)

h = 1/500 Case: I

ϕ Degrees	m_ϕ / t	m_θ / q_ϕ	n_ϕ / z^*	n_θ / δ
35	-28.454307 .209557	-6.458643 -.120197	.171659 .560006	-1.196280 -.705849
36	-22.596676 .046943	-5.427997 -.027592	.037977 .687689	-4.879317 -2.872453
37	-11.237868 -.115861	-3.063894 .069727	-.092531 .640773	-7.263042 -4.359867
38	2.673714 -.239475	.000313 .147435	-.188709 .434940	-7.783414 -4.768707
39	15.620487 -.296540	2.975955 .186618	-.230454 .124963	-6.350616 -3.967562
40	24.398444 -.277445	5.119602 .178338	-.212535 -.209698	-3.361493 -2.133401
41	26.904572 -.191508	5.916644 .125641	-.144533 -.485106	.408474 .286947
42	22.636573 -.063602	5.203015 .042558	-.047265 -.633753	4.004716 2.686001
43	12.786982 .072713	3.196696 -.049590	.053179 -.620996	6.533725 4.448732
44	-.084210 .183728	.433523 -.127628	.132163 -.453034	7.383747 5.110816
45	-12.710000 .243696	-2.374362 -.172319	.172319 -.174624	6.371798 4.481169
46	-21.940791 .240605	-4.520483 -.173077	.167138 .142735	3.781867 2.696399
47	-25.525890 .178374	-5.480316 -.130454	.121651 .419047	.286615 .191823
48	-22.659122 .075112	-5.038253 -.055819	.050259 .585880	-3.226184 -2.404990

TABLE III.7.a. (Continued)

h = 1/500 Case: I

ϕ	m_ϕ	m_θ	n_ϕ	n_θ
Degrees	t	q_ϕ	z^*	δ
49	-14.156984	-3.334795	-.027514	-5.877979
	-.041939	.031652	.603222	-4.432011
50	-2.230649	-.823936	-.092192	-7.018143
	-.143425	.109869	.469011	-5.362080
51	10.085625	1.846810	-.129070	-6.382836
	-.205094	.159388	.219085	-4.940331
52	19.702830	4.002306	-.131423	-4.155415
	-.213466	.168213	-.082315	-3.253796
53	24.248633	5.109411	-.101500	-.914670
	-.168656	.134695	-.358970	-.714275
54	22.650014	4.907951	-.049255	2.514745
	-.083799	.067794	-.541875	2.042439
55	15.384601	3.471084	.010705	5.271271
	.018663	-.015288	-.586379	4.316215
56	4.342233	1.181225	.063078	6.672062
	.112803	-.093518	-.483036	5.520928
57	-7.664103	-1.371020	.095678	6.381495
	.175672	-.147331	-.259532	5.335920
58	-17.612051	-3.538411	.102169	4.491134
	.192802	-.163505	.026672	3.791366
59	-23.027995	-4.780408	.083159	1.492194
	.161462	-.138400	.303050	1.264803
60	-22.599957	-4.797302	.045471	-1.851512
	.090942	-.078758	.500327	-1.611331
61	-16.493962	-3.601994	-.000050	-4.697406
	-.000103	.000090	.569779	-4.108432
62	-6.297868	-1.512357	-.041554	-6.335778
	-.088511	.078151	.495225	-5.586819

TABLE III.7.a. (Continued)

$h = 1/500$ Case: I

ϕ Degrees	m_ϕ / t	m_θ / q_ϕ	n_ϕ / z^*	n_θ / δ
63	5.389725 -.152435	.932786 .135820	-.069204 .296762	-6.366208 -5.659998
64	15.619525 -.176674	3.112018 .158794	-.077448 .025388	-4.795070 -4.295855
65	21.833507 -.156238	4.478638 .141600	-.066029 -.250051	-2.030438 -1.828232
66	22.502635 -.097310	4.697259 .088897	-.039579 -.460277	1.223471 1.124927
67	17.501864 -.015450	3.725700 .014222	-.006037 -.552953	4.145110 3.816703
68	8.130695 .068460	1.822270 -.063475	.025646 -.505667	6.002442 5.560609
69	-3.223185 .133496	-.522536 -.124629	.047841 -.331338	6.335841 5.906081
70	-13.690464 .163848	-2.711989 -.153967	.056039 -.074711	5.071511 4.755127
71	-20.643355 .152631	-4.194481 -.144315	.049692 .199089	2.537485 2.389842
72	-22.353539 .103411	-4.602040 -.098349	.031955 .421036	-.621172 -.596847
73	-18.420628 .029129	-3.841315 -.027856	.008516 .535563	-3.606113 -3.450170
74	-9.865164 -.051262	-2.114644 .049277	-.014130 .514433	-5.666974 -5.444726
75	1.135263 -.117595	.133502 .113588	-.030436 .363683	-6.289523 -6.069331
76	11.798799 -.153502	2.330576 .148942	-.037135 .121928	-5.323660 -5.158316

TABLE III.7.a. (Continued)

$\dot{h}$ = 1/500 Case: I

ϕ Degrees	m_ϕ t	m_θ q_ϕ	n_ϕ z^*	n_θ δ
77	19.441061 -.150421	3.921347 .146565	-.033837 -.149502	-3.019456 -2.935473
78	22.149255 -.109615	4.507818 .107220	-.022790 -.382085	.037382 .041024
79	19.259770 -.041736	3.948496 .040969	-.007964 -.517357	3.074013 3.019097
80	11.520304 .035934	2.392351 -.035388	.006240 -.521579	5.325387 5.243252
81	.896993 .103846	.239325 -.102567	.016245 -.394127	6.226541 6.146671
82	-9.923844 .145068	-1.962165 -.143656	.020189 -.167542	5.554042 5.495990
83	-18.213411 .149479	-3.654560 -.148365	.018217 .100762	3.481253 3.451688
84	-21.886995 .116230	-4.412009 -.115593	.012149 .343005	.533772 .528431
85	-20.027048 .053895	-4.047249 -.053690	.004697 .498131	-2.543538 -2.534794
86	-13.111830 -.021727	-2.657754 .021674	-.001516 .527157	-4.974343 -4.961922
87	-2.892775 -.091584	-.599834 .091458	-.004793 .422949	-6.146256 -6.136875
88	8.048088 -.138141	1.602512 .138057	-.004821 .211972	-5.764760 -5.760284
89	16.949055 -.149750	3.390690 .149727	-.002614 -.052419	-3.927065 -3.925944
90	21.564290 -.123544	4.312859 .123544	-.000000 -.303440	-1.097392 -1.097392

TABLE III.7.b.

h = 1/500 Case: II

ϕ	m_ϕ	m_θ	n_ϕ	n_θ
Degrees	t	q_ϕ	z^*	δ
7	57.538487	20.287947	-2.181150	2.281837
	-2.197530	.267811	-1.122998	.331249
8	44.937471	18.249131	-.552700	9.005629
	-.558131	.077677	-1.355871	1.268724
9	23.170692	12.222947	.697019	13.096118
	.705707	-.110397	-1.252028	2.026873
10	-2.108630	4.291216	1.445574	13.821304
	1.467874	-.254893	-.865644	2.349838
11	-24.859754	-3.448523	1.681012	11.252760
	1.712475	-.326755	-.308463	2.082975
12	-40.011156	-9.256492	1.482747	6.205393
	1.515873	-.315167	.277709	1.228517
13	-44.546905	-12.045404	.991751	.014563
	1.017838	-.228963	.754174	-.041343
14	-38.082677	-11.530111	.375171	-5.795084
	.386656	-.093541	1.016419	-1.420109
15	-22.817555	-8.203913	-.208409	-9.870207
	-.215761	.055843	1.016085	-2.543808
16	-2.895027	-3.153959	-.637219	-11.334974
	-.662898	.182719	.769120	-3.089210
17	16.658156	2.236493	-.843721	-9.973947
	-.882272	.257951	.348768	-2.866761
18	31.153966	6.627955	-.819263	-6.253420
	-.861425	.266195	-.134422	-1.881778
19	37.335371	9.029604	-.606580	-1.181839
	-.641532	.208862	-.560439	-.345272
20	34.087667	9.003289	-.283052	3.952375
	-.301218	.103023	-.828697	1.371152

TABLE III.7.b. (Continued)

$h = 1/500$ Case: II

ϕ Degrees	m_ϕ t	m_θ q_ϕ	n_ϕ $z^{\#}$	n_θ δ
21	22.614955 .065166	6.726664 -.023353	.060837 -.881156	7.899589 2.826597
22	6.058570 .369840	2.909257 -.138544	.342910 -.714428	9.746920 3.625565
23	-11.333017 .549876	-1.410846 -.214853	.506163 -.378383	9.125233 3.525954
24	-25.270095 .578085	-5.135816 -.235128	.528107 .037935	6.278676 2.510805
25	-32.460958 .465002	-7.376006 -.196518	.421435 .429300	1.984409 .803025
26	-31.375980 .252421	-7.653792 -.110654	.226874 .700401	-2.653673 -1.183183
27	-22.561131 .000708	-5.998169 -.000321	.000631 .788672	-6.483809 -2.943642
28	-8.438686 -.226762	-2.912683 .106458	-.200219 .678453	-8.588461 -4.013234
29	7.341896 -.378372	.769883 .183438	-.330931 .403316	-8.501255 -4.089399
30	20.832986 -.425019	4.106162 .212509	-.368077 .036346	-6.307991 -3.117185
31	28.764449 -.364825	6.278433 .187899	-.312716 -.328935	-2.610197 -1.312138
32	29.327286 -.221227	6.790484 .117232	-.187611 -.602105	1.636346 .887014
33	22.579137 -.035426	5.577188 .019295	-.029711 -.717974	5.365632 2.925566
34	10.381992 .144659	3.004302 -.080892	.119928 -.651957	7.666416 4.273589

TABLE III.7.b. (Continued)

h = 1/500 Case: II

ϕ	m_ϕ	m_θ	n_ϕ	n_θ
Degrees	t	q_ϕ	z^*	δ
35	-4.101407	-.237471	.226209	8.002850
	.276151	-.158394	-.425092	4.564292
36	-17.218819	-3.314651	.267717	6.337759
	.330917	-.194508	-.097715	3.693766
37	-25.741515	-5.462172	.240511	3.130445
	.301152	-.181238	.246371	1.854999
38	-27.652594	-6.170893	.157458	-.787733
	.199817	-.123019	.521161	-.504364
39	-22.619458	-5.306046	.043647	-4.426870
	.056163	-.035345	.659766	-2.791411
40	-12.040706	-3.129567	-.070685	-6.886785
	-.092272	.059312	.630570	-4.417649
41	1.335636	-.223561	-.157840	-7.578193
	-.209141	.137209	.444321	-4.951027
42	14.120057	2.663429	-.199080	-6.362772
	-.267889	.179252	.150614	-4.230880
43	23.134800	4.807307	-.188355	-3.579032
	-.257543	.175644	-.175182	-2.415200
44	26.198967	5.688265	-.132690	.050356
	-.184461	.128137	-.451128	.053415
45	22.652559	5.115343	-.049391	3.604808
	-.069849	.049391	-.609199	2.555967
46	13.494279	3.266174	.038986	6.197996
	.056123	-.040371	-.611954	4.452853
47	1.105371	.634059	.110526	7.197707
	.162062	-.118524	-.461325	5.247898
48	-11.365411	-2.102377	.149207	6.379125
	.222987	-.165712	-.197487	4.718434

TABLE III.7.b. (Continued)

h = 1/500 Case: II

ϕ Degrees	m_ϕ / t	m_θ / q_ϕ	n_ϕ / z^*	n_θ / δ
49	-20.797920 .226070	-4.252854 -.170617	.148315 .111764	3.974536 2.977231
50	-24.877550 .173171	-5.288331 -.132656	.111312 .388337	.608660 .449206
51	-22.660383 .079736	-4.970116 -.061967	.050180 .563470	-2.862681 -2.232519
52	-14.789187 -.029545	-3.403916 .023282	-.018190 .594743	-5.569163 -4.385690
53	-3.311331 -.127132	-1.006829 .101532	-.076510 .476302	-6.843247 -5.453036
54	8.850804 -.189446	1.602893 .153265	-.111354 .239817	-6.384119 -5.146840
55	18.640703 -.202469	3.764381 .165853	-.116131 -.053912	-4.328195 -3.526421
56	23.632882 -.164797	4.940599 .136623	-.092153 -.330521	-1.209738 -.987638
57	22.631455 -.087527	4.850995 .073407	-.047671 -.520831	2.177268 1.834005
58	15.954960 .008895	3.537770 -.007544	.004714 -.578085	4.980496 4.222898
59	5.339936 .099998	1.350281 -.085715	.051503 -.489390	6.502999 5.565325
60	-6.508605 .163406	-1.147323 -.141514	.081703 -.278569	6.375892 5.507529
61	-16.603989 .184260	-3.320897 -.161158	.089331 .000185	4.647330 4.049017
62	-22.428258 .158640	-4.626631 -.140071	.074477 .276188	1.766411 1.546496

TABLE III.7.b. (Continued)

h = 1/500 Case: II

ϕ Degrees	m_ϕ / t	m_θ / q_ϕ	n_ϕ / z^*	n_θ / δ
63	-22.558153 .094204	-4.746483 -.083937	.042768 .480128	-1.532997 -1.373531
64	-17.011584 .007989	-3.665184 -.007181	.003502 .561415	-4.418568 -3.972009
65	-7.230521 -.078036	-1.670240 .070725	-.032979 .500689	-6.168851 -5.584897
66	4.292330 -.142506	.724083 .130186	-.057962 .314407	-6.353113 -5.793264
67	14.646785 -.169905	2.908814 .156398	-.066387 .050414	-4.936996 -4.532305
68	21.238055 -.154257	4.334610 .143024	-.057786 -.224301	-2.288151 -2.110820
69	22.435171 -.100384	4.649383 .093717	-.035975 -.440552	.918848 .864535
70	17.973357 -.022488	3.784870 .021131	-.007691 -.544337	3.873785 3.641611
71	9.011372 .059555	1.970950 -.056311	.019389 -.510279	5.834944 5.513379
72	-2.168307 .125215	-.325170 -.119087	.038694 -.347817	6.314797 5.998366
73	-12.739175 .158401	-2.518893 -.151480	.046312 -.098612	5.200878 4.964764
74	-20.043449 .151367	-4.056616 -.145504	.041722 .174099	2.781946 2.666155
75	-22.258618 .106494	-4.555005 -.102865	.027562 .401510	-.326613 -.320808
76	-18.851029 .035542	-3.896241 -.034487	.008598 .526561	-3.338923 -3.241410

TABLE III.7.b. (Continued)

h = 1/500 Case: II

ϕ Degrees	m_ϕ / t	m_θ / q_ϕ	n_ϕ / z^*	n_θ / δ
77	-10.703903 -.043398	-2.255635 .042286	-.009762 .518223	-5.496810 -5.354023
78	.110830 -.110484	-.055205 .108070	-.022971 .379169	-6.260161 -6.118865
79	10.858111 -.149069	2.144571 .146331	-.028444 .144969	-5.441797 -5.336229
80	18.829788 -.149801	3.787117 .147525	-.026013 -.124994	-3.253231 -3.198683
81	22.025426 -.112866	4.460210 .111476	-.017656 -.362538	-.250177 -.243609
82	19.653125 -.047853	3.999146 .047388	-.006660 -.507863	2.808218 2.782207
83	12.325270 .028714	2.526888 -.028500	.003499 -.524573	5.150830 5.111741
84	1.900920 .097551	.421428 -.097016	.010197 -.408766	6.188537 6.152607
85	-8.984723 .141431	-1.780992 -.140893	.012327 -.189939	5.662039 5.638036
86	-17.584942 .149466	-3.522125 -.149102	.010426 .076505	3.706500 3.695390
87	-21.733008 .119792	-4.362866 -.119628	.006269 .323254	.816947 .814575
88	-20.386817 .059991	-4.093725 -.059955	.002094 .488055	-2.276761 -2.275792
89	-13.890120 -.014828	-2.786896 .014826	-.000259 .529378	-4.793864 -4.793082
90	-3.884954 -.085826	-.776991 .085826	-.000000 .436866	-6.099302 -6.099302

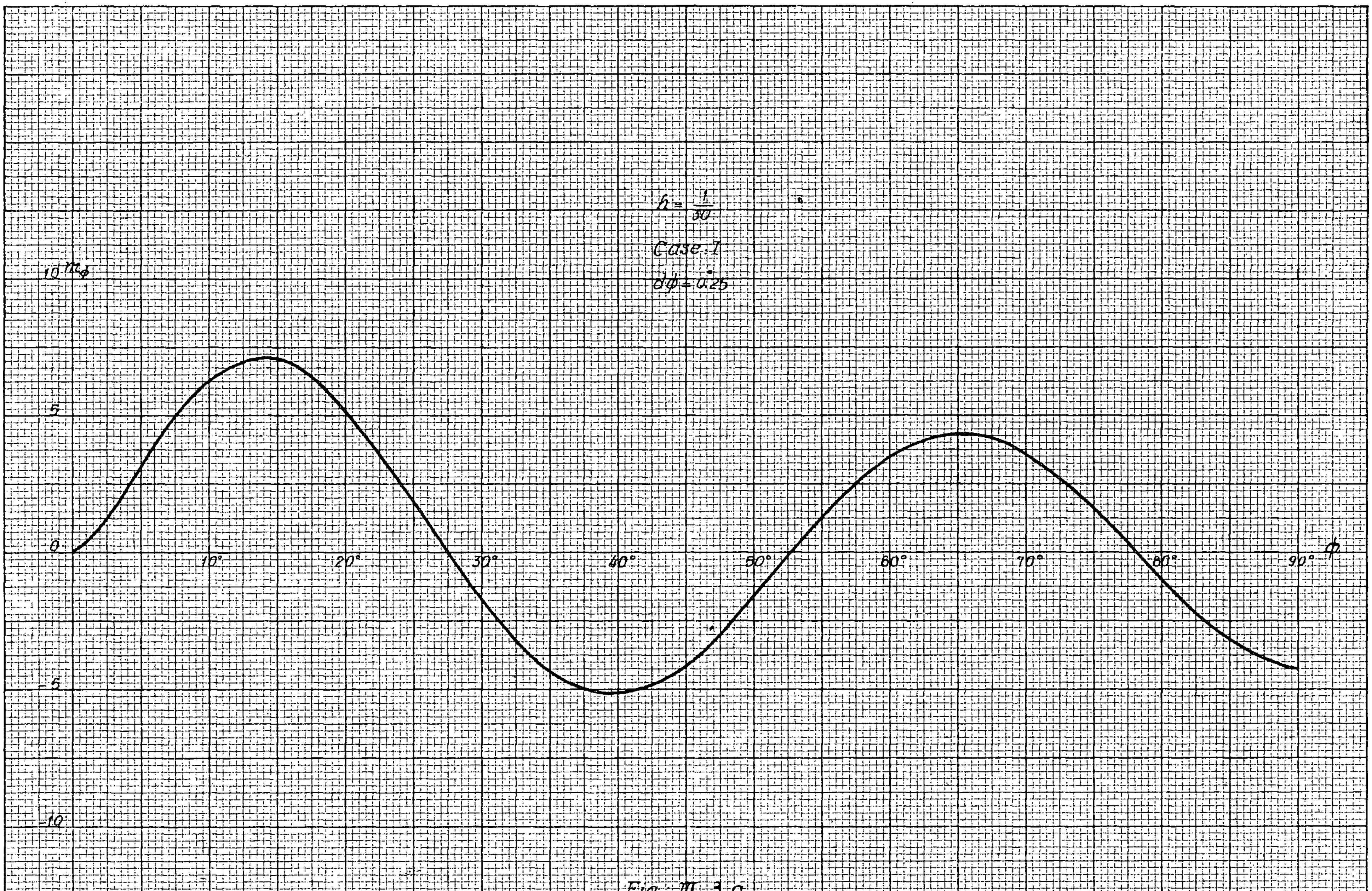


Fig: III 3 a

m_ϕ versus ϕ curve

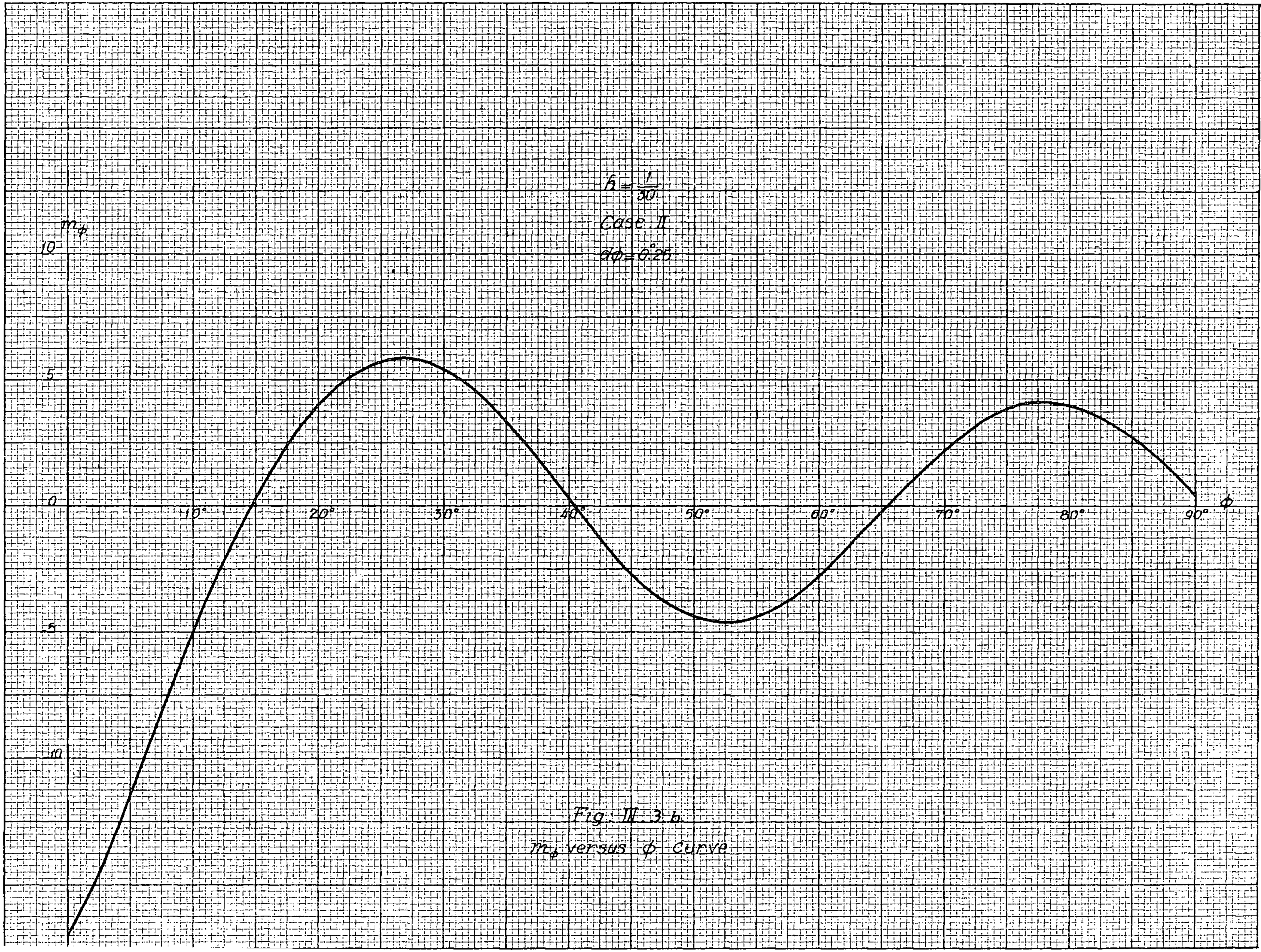


Fig: IV.3 b.
m_ϕ versus ϕ curve

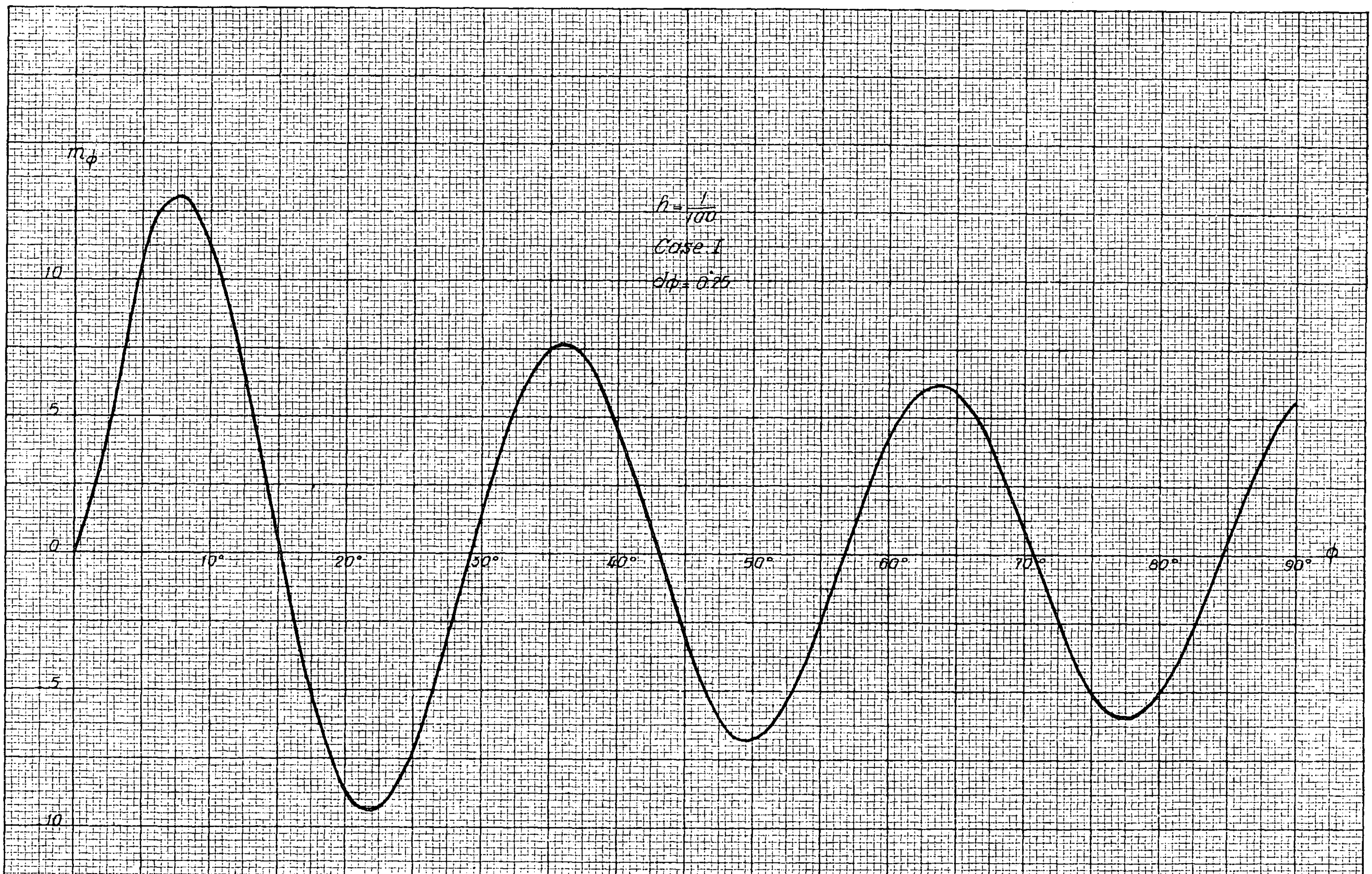


Fig. III 4a

m_ϕ versus ϕ curve

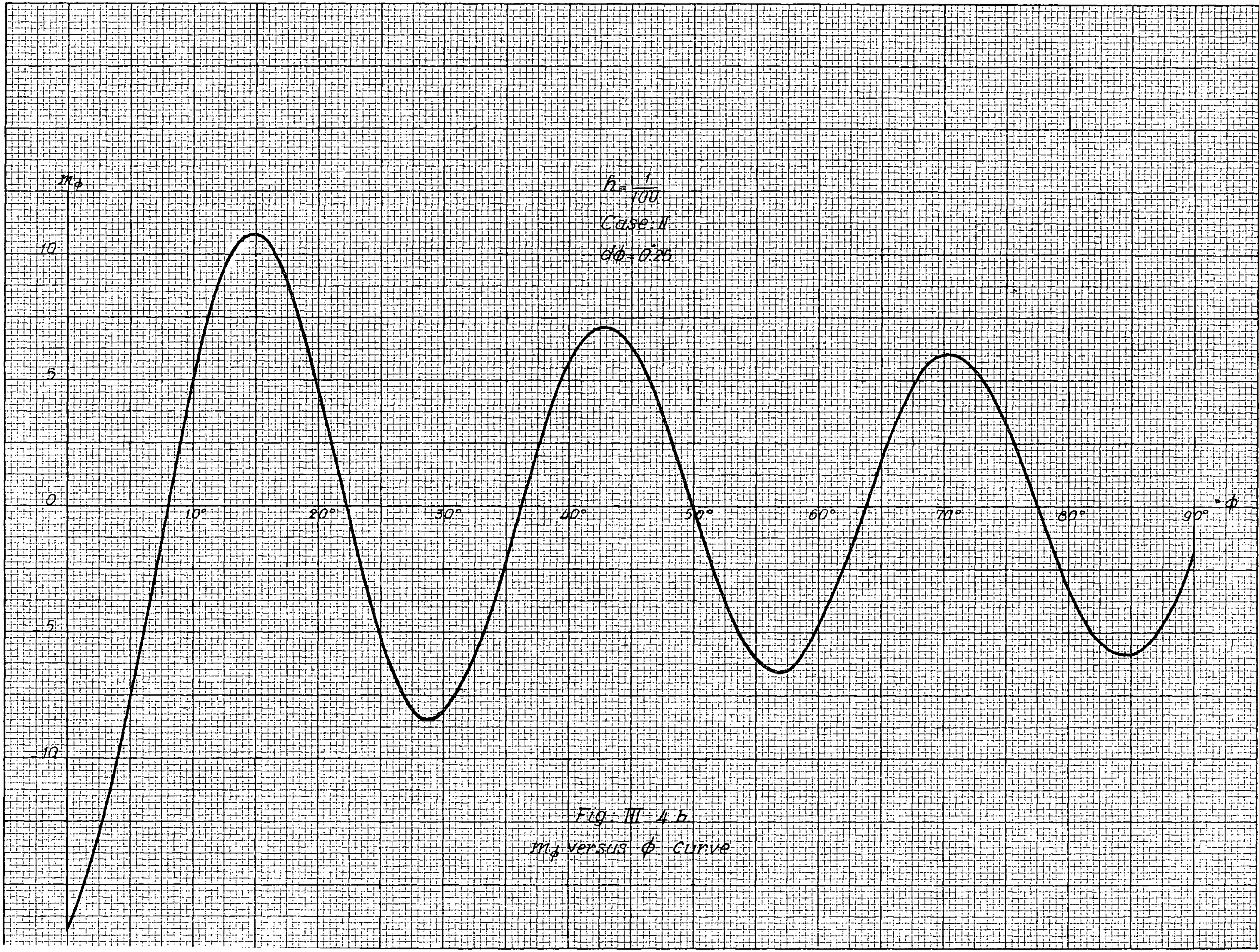


Fig: III 4 b
m_ϕ versus ϕ curve

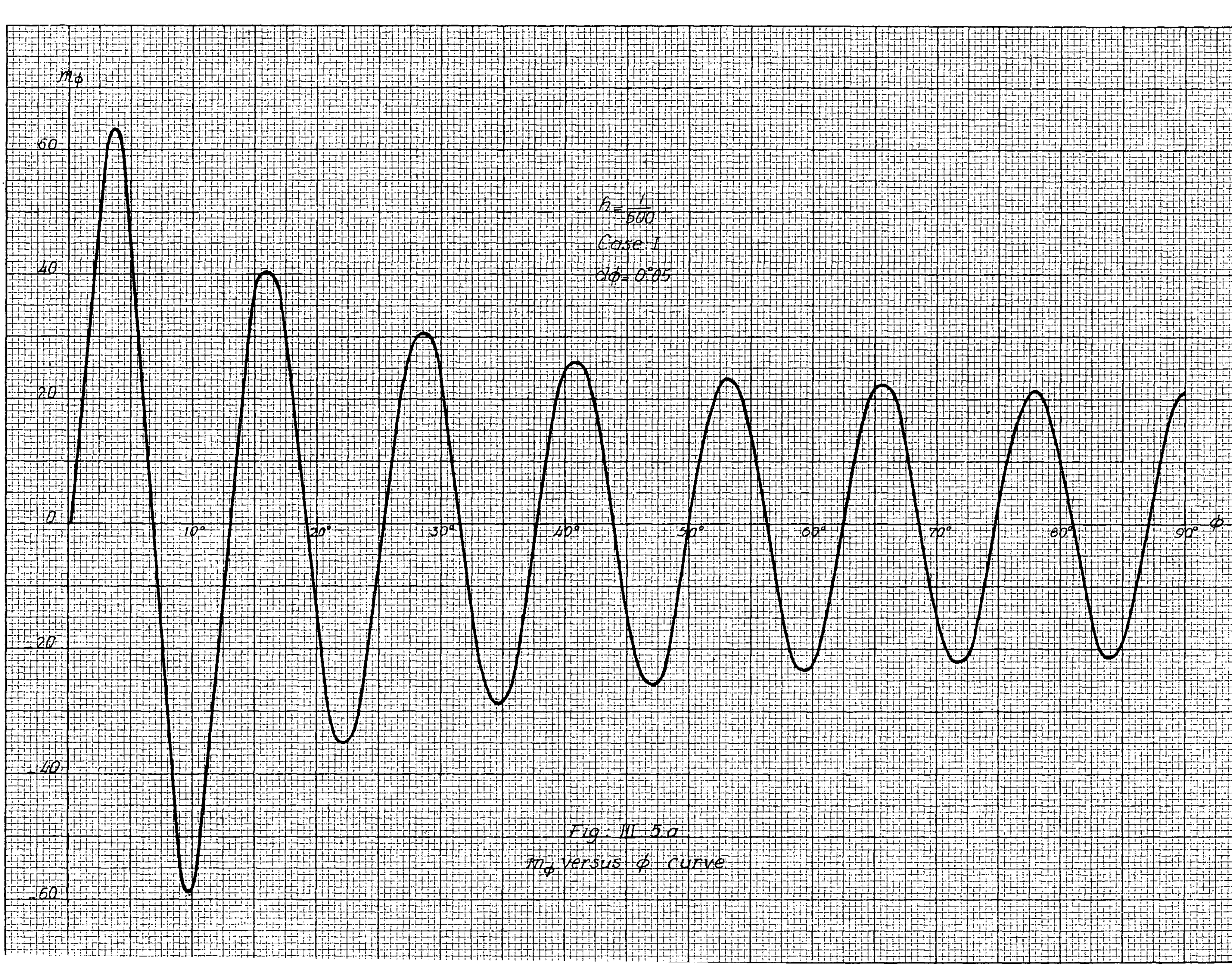


Fig: III 5.a
m_ϕ versus ϕ curve

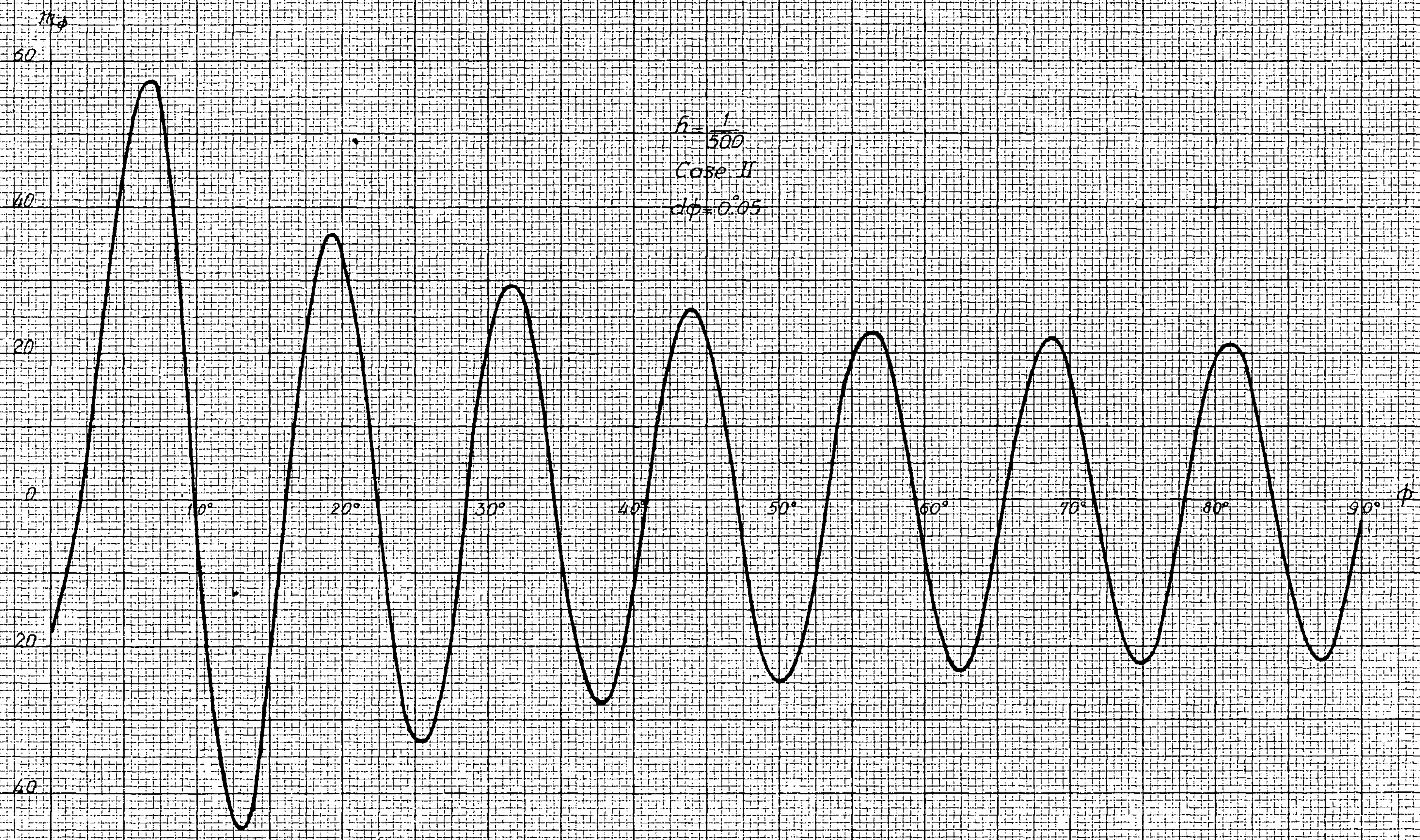


Fig. III. 5 b
m_ϕ versus ϕ curve

CHAPTER IV

COMPARISON OF THE NUMERICAL METHOD WITH AN APPROXIMATE SOLUTION

The approximate solutions of the differential equations are given in many published papers. In this chapter we will define Timoshenko's solution (Reference 1, p. 469) in terms of the dimensionless forms and compare the numerical results with this approximate solution. The two differential equations of dimensionless form are as follows:

$$\ddot{U} + \mathrm{Cot}\phi\,\dot{U} - (\mathrm{Cot}^2\phi - \nu)\,U = \frac{1-\nu^2}{h}\,Z \qquad (4.1.1.)$$

$$\ddot{Z} + \mathrm{Cot}\phi\,Z - (\mathrm{Cot}^2\phi + \nu)\,Z = -\frac{12}{h}\,U \qquad (4.2.1.)$$

For the large values of ϕ and a very thin shell, U and Z functions are damped out very fast from the edge of shell to the crown. The functions U and Z and their first derivatives are comparatively much smaller than the second derivatives besides for large ϕ value of $\cot\phi$ goes to zero. Therefore as an approximation we can neglect them in the left side of our equations and obtain the following relations.

$$\ddot{U} = \frac{1-\nu^2}{h} Z \qquad (4.1.2.)$$

$$\ddot{Z} = -\frac{12}{h} U \qquad (4.2.2.)$$

Eliminating Z from these equations,

$$\ddot{\ddot{U}} + 4 \lambda^4 U = 0 \qquad (4.3.1.)$$

can be obtained, where

$$\lambda^4 = 3(1-\nu^2) \left(\frac{1}{h}\right)^2 \qquad (a)$$

The general solution of this equation is

$$U = C_1 e^{\lambda\phi} \operatorname{Cos} \lambda\phi + C_2 e^{\lambda\phi} \operatorname{Sin} \lambda\phi + C_3 e^{-\lambda\phi} \operatorname{Cos} \lambda\phi + C_4 e^{-\lambda\phi} \operatorname{Sin} \lambda\phi \qquad (4.4.1.)$$

Since at the crown there is no hole and the function should go to zero, C_3 and C_4 coefficients must be zero. Therefore the solution will be reduced to

$$U = e^{\lambda\phi} (C_1 \operatorname{Cos} \lambda\phi + C_2 \operatorname{Sin} \lambda\phi) \qquad (4.4.2.)$$

On the other hand the relation between U and u was given in the last chapter as being

$$u = e^{-\lambda\phi} U \qquad (b)$$

therefore the solution for u will be

$$u = C_1 \operatorname{Cos} \lambda\phi + C_2 \operatorname{Sin} \lambda\phi \qquad (4.5.1.)$$

Considering Eq. (4.2.2.) solution for z can be written.

$$z = 2\sqrt{\frac{3}{1-\nu^2}}\ (-C_1 \text{Sin}\,\lambda\phi + C_2\ \text{Cos}\,\lambda\phi) \qquad (4.6.1.)$$

The relations between u, z and the stress functions are given in the Table III.1.

We will compare the two ways by taking the example which is shown on Fig. IV.1. At the $\phi=\alpha$, the angle at the support, only the horizontal force exists and has a value of unity. In the two methods the coefficients C_1,C_2 and A, B can be determined by equating t = 1 and m_ϕ = 0. These coefficients are calculated for the three h values, 30, 100 and 500. In each case the angles at the support are taken as 90^o, 60^o, 30^o. Now having these coefficients we can compare our two methods. The comparison will be made in three ways.

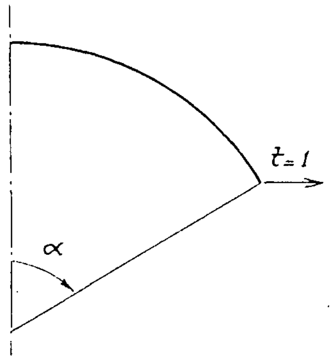


Fig. IV. 1.

For the first comparison δ, horizontal displacement and z^*, the angle of rotation were calculated for both the methods. These results are shown on the Table IV.1. The curves for δ and z^* versus α are drawn on the Fig. IV.2.a. and b.

Comparing the two methods on these figures, we can easily conclude that the two methods give more or less the same results for the stiffnesses at the small values of

TABLE: IV. 1.

1/h	α	Approximate method				Numerical method			
		C_1	C_2	δ	z^*	A	B	δ	z^*
30	90	-8.47	5.52	14.3	-3.55	1.41	11.9	14.3	-3.55
	80	2.34	9.64	13.8	-3.47	11.5	2.44	13.7	-3.47
	60	3.30	-8.00	10.4	-3.02	-6.31	-7.39	10.5	-3.17
	30	2.14	9.63	6.82	-3.38	4.17	1.27	3.36	-1.87
100	90	13.5	-12.5	27.0	-3.79	-4.87	-14.7	26.1	-3.50
	60	4.7	-15.2	19.4	-3.05	-9.43	-8.25	19.4	-3.11
	30	-2.33	-8.80	6.35	-1.73	-5.46	-1.10	6.30	-1.83
500	90	32.7	-25.1	58.2	-3.54	-1.67	-9.26	58.2	-3.50
	60	-35.3	4.71	43.3	-3.04	-2.11	7.30	42.6	-3.09
	30	19.4	6.48	14.4	-1.74	2.28	-2.47	14.4	-1.80

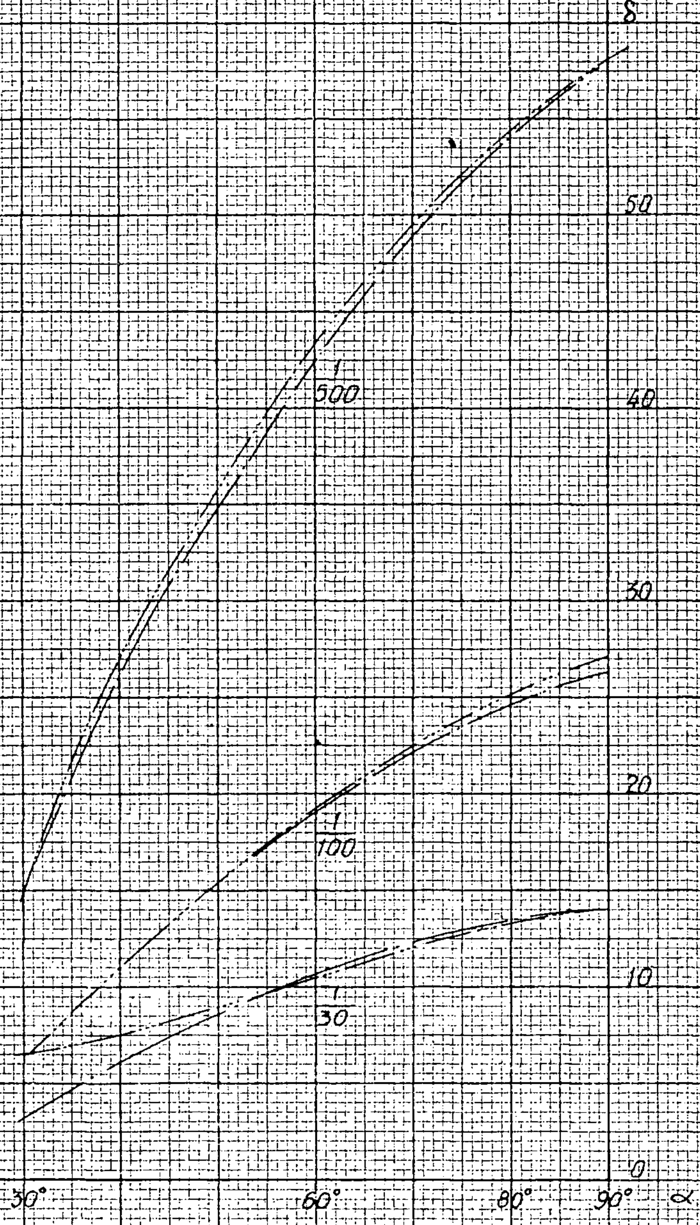


Fig. IV 2 a

δ at φ=α versus α

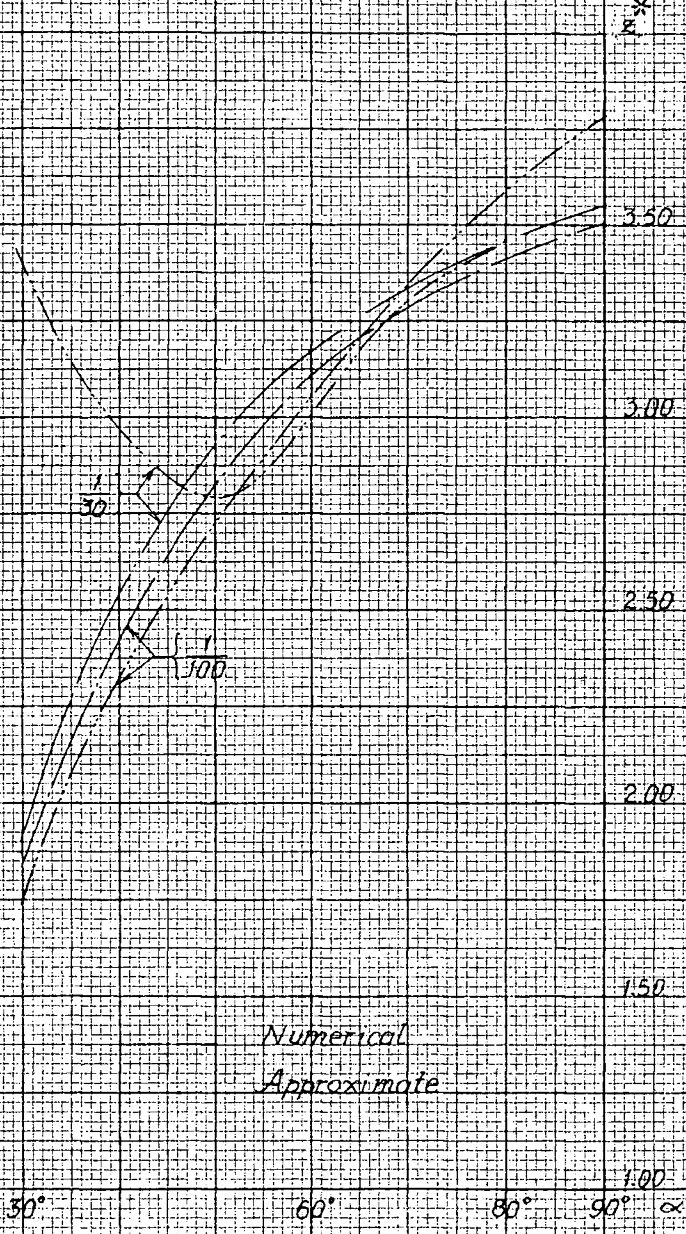


Fig. IV 2 b

z* at φ=α versus α

shell thickness, but for the thick shells two methods differ quite appreciably at the small values of the support angle .

The second comparison may be based on the wave lengths for the two methods. The wave lengths from numerical method were measured from the Figs. III.3.4.5. and found 25°, 14°, 6° approximately, for l/h = 30, 100, 500 respectively. On the other hand, the wave lengths for the approximate method were determined by equating Eq. (4.5.1.) to zero.

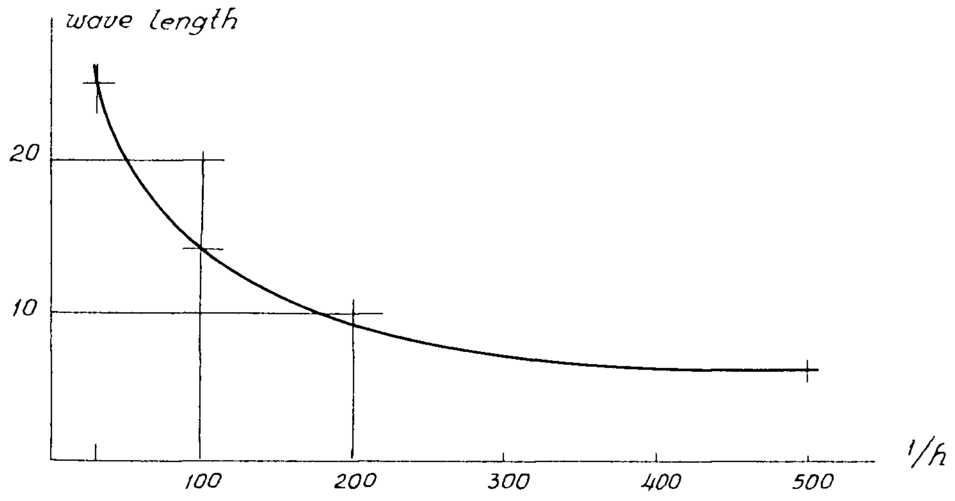


Fig. IV. 3.

$$C_1 \text{ Cox } \lambda \phi + C_2 \text{ Sin } \lambda \phi = 0$$

from this equation it can be concluded that:

$$\tan \lambda \phi = - \frac{C_1}{C_2} \qquad \text{(c)}$$

therefore

$$\lambda \phi = \pi$$

From this relation, the wave lengths for the different h values can be determined as the ratio $\frac{\pi}{\lambda}$. The results are 25.2, 13.8 and 6.2 degrees for $\frac{1}{h}$ = 30, 100, 500 respectively.

From the Fig. IV.3. the curve of the wave length vs. the values of h, can be observed. It is obvious that the two methods give almost the same wave lengths for each case.

As the third comparison m_ϕ curves were drawn for the two ways. Fiv. IV.4. These computations were based on t = 1 and $m_\phi = 0$ at the support $\alpha = 60^\circ$. The tables which contain the values of m_ϕ's for the different points are given as Table: IV.2.

From here, again it can be easily seen that the values for the two methods are close to each other in the case of the thin shell. But for the thick shell, the difference between the two ways is out of the tolerance limits. At the peak points the percentage of error gets larger from the thin shells to the thick ones. For instant, 1.6% for the case of $h = \frac{1}{500}$ and about 3.0% for $h = \frac{1}{100}$.

From the above three comparisons we can draw a general conclusion. The Figs. Iv.4.a. and b. show that the percentage of error in approximate method gets larger for smaller support angle and thicker shell. This conclusion is shown in the error curves of Fig. IV. 5. Therefore in the case of shell which plots in Zone - II the approximate method can be used with an error of not more than 6%. As we concluded in

TABLE: IV.2.a.

	ϕ Dgr.	Appx. m_ϕ	Numr. m_ϕ		ϕ Dgr.	Appx. m_ϕ	Numr. m_ϕ
$h = \frac{1}{30}$	60	0.00	0.00	$h = \frac{1}{100}$	60	0.00	0.00
$\lambda = 7.135$	56	21.0	21.3	$\lambda = 13.03$	58	36.1	35.5
$\alpha = 60^\circ$	52	36.8	38.2	$\alpha = 60^\circ$	56	63.8	64.3
$C_1 = -3.98$	48	43.8	46.7	$C_1 = 4.70$	54	78.4	80.8
$C_2 = 9.67$	44	40.4	44.2	$C_2 = -15.2$	52	77.2	81.0
$A = -6.31$	40	26.8	30.7	$A = -9.43$	50	60.2	64.8
$B = -7.39$	36	7.7	8.73	$B = -8.25$	48	30.9	34.7
	32		-14.9		46	-4.68	-3.28

	ϕ Degr.	Appx. m_ϕ	Numr. m_ϕ
$h = \frac{1}{500}$	60	0.00	0.00
$\lambda = 29.13$	59	86.0	87.5
$\alpha = 60^\circ$	58	151.	153.
$C_1 = -35.3$	57	178.	181.
$C_2 = 4.71$	56	160.	163.
$A = -2.11$	55	101.	104.
$B = 7.30$	54	17.3	16.9
	53	-71.7	-75.2

TABLE: IV.2.b.

	ϕ Degree	Apprx. m_ϕ	Numr. m_ϕ
$h = \frac{1}{30}$	30	0.00	0.00
$\lambda = 7.135$	26	24.2	12.3
$\alpha = 30^\circ$	22	43.0	22.9
$C_1 = 2.14$	18	51.4	29.1
$C_2 = 9.63$	14	47.0	28.6
$A = 4.17$	10	30.3	20.0
$B = 1.27$	8	18.1	12.5

	ϕ Degree	Apprx. m_ϕ	Numr. m_ϕ
$h = \frac{1}{100}$	30	0.00	0.00
$\lambda = 13.03$	28	20.3	20.6
$\alpha = 30^\circ$	26	36.6	38.2
$C_1 = -2.33$	24	45.3	49.2
$C_2 = -8.80$	22	44.9	50.8
$A = -5.46$	20	35.0	42.1
$B = -1.10$	18	18.3	24.1
	16	-2.65	-0.33

	ϕ Degree	Apprx. m_ϕ	Numr. m_ϕ
$h = \frac{1}{500}$	30	0.00	0.00
$\lambda = 29.13$	29	50.0	50.8
$\alpha = 30^\circ$	28	87.3	90.2
$C_1 = 19.4$	27	103.	108.
$C_2 = 6.48$	26	92.1	98.3
$A = 2.28$	25	53.9	63.4
$B = -2.47$	24	9.46	10.9
	23	-41.7	-46.3

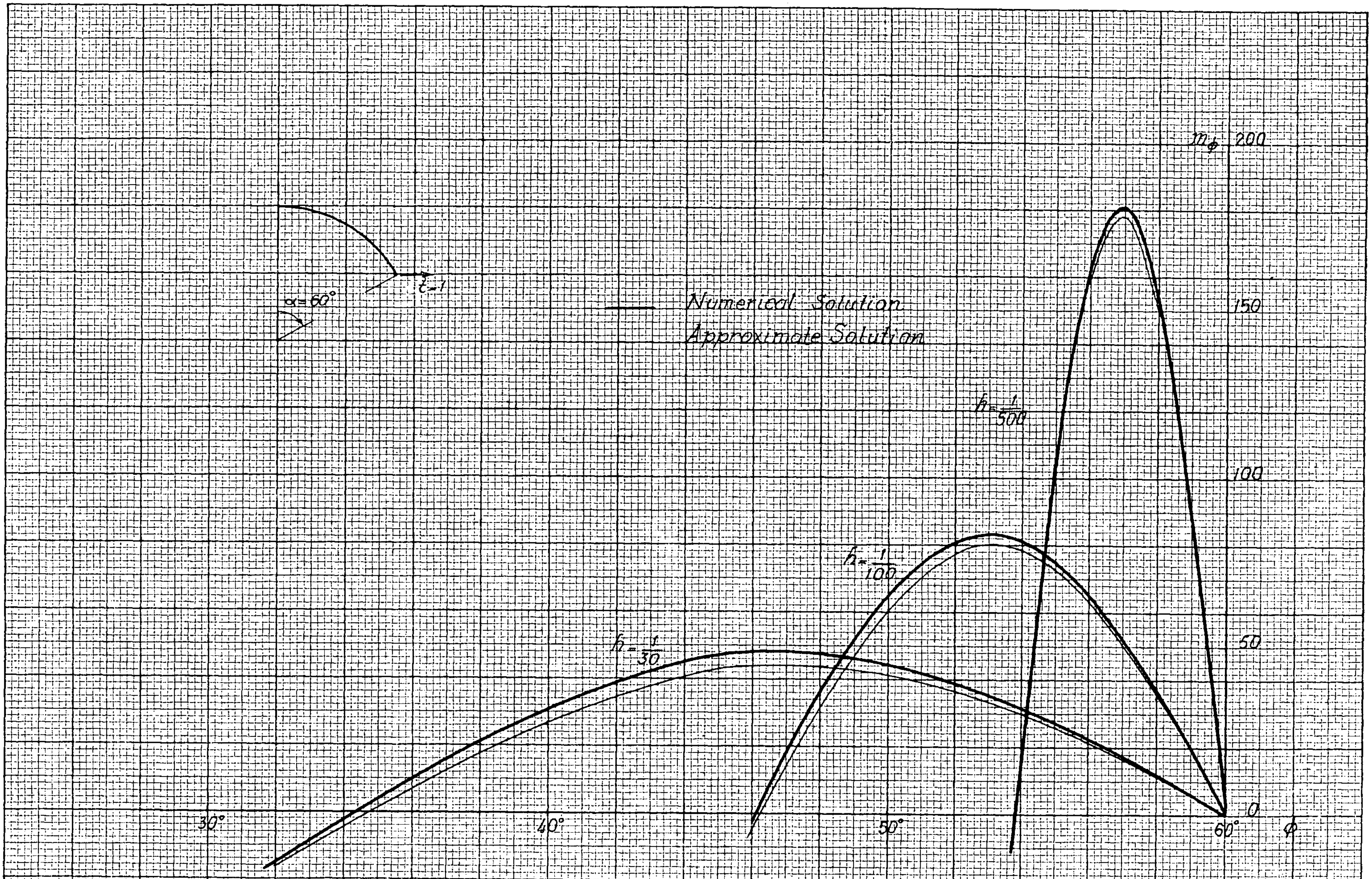


Fig. IV.4.a
m_ϕ versus ϕ

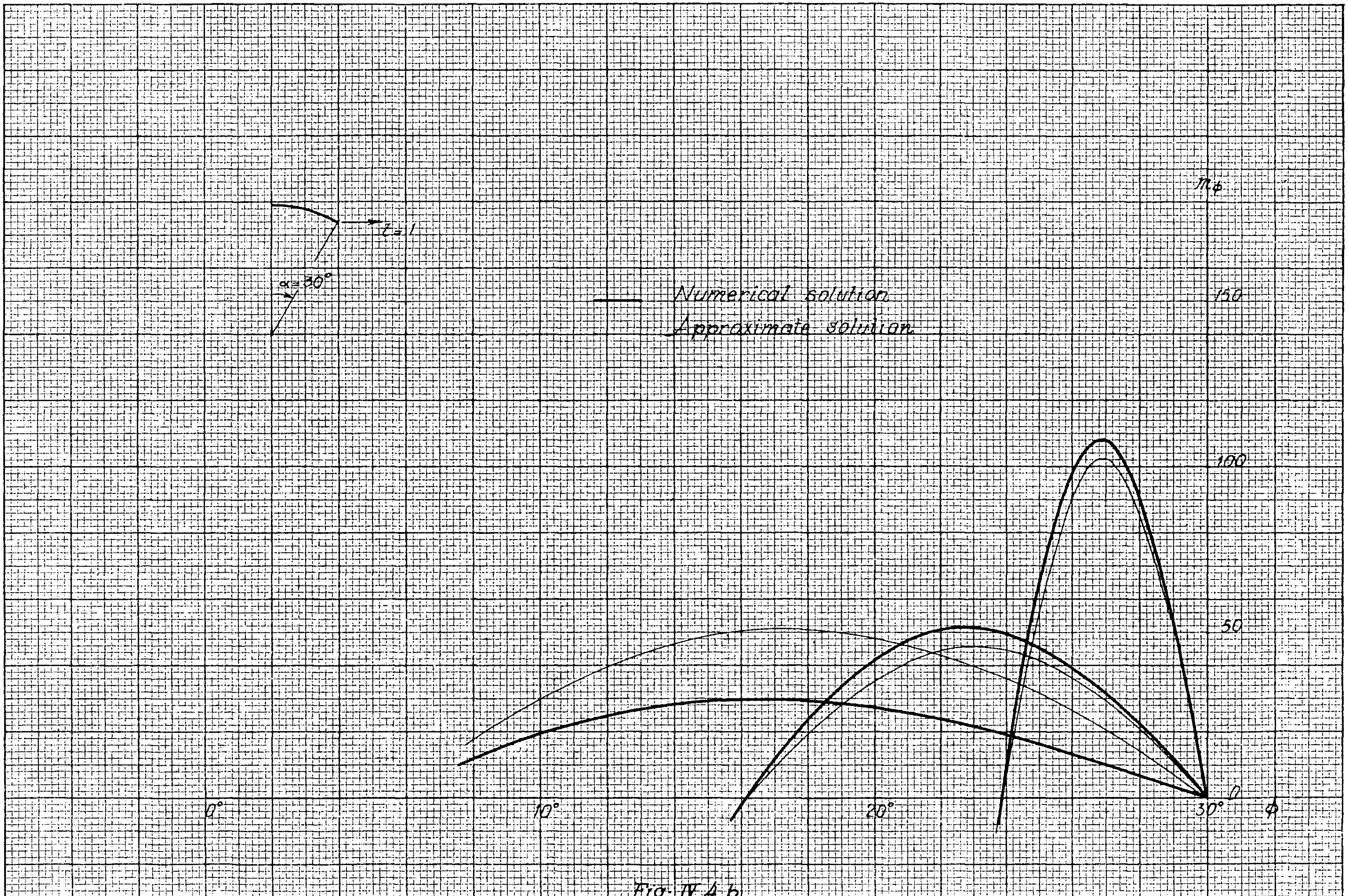


Fig. IV 4 b
m_ϕ versus ϕ

Ch. II, the Bessel solution was accurate enough for a shell which is located on Zone - I. On the other hand numerical method can be applied over entire surface. That is to say, numerical method is good for any thickness and support angle.

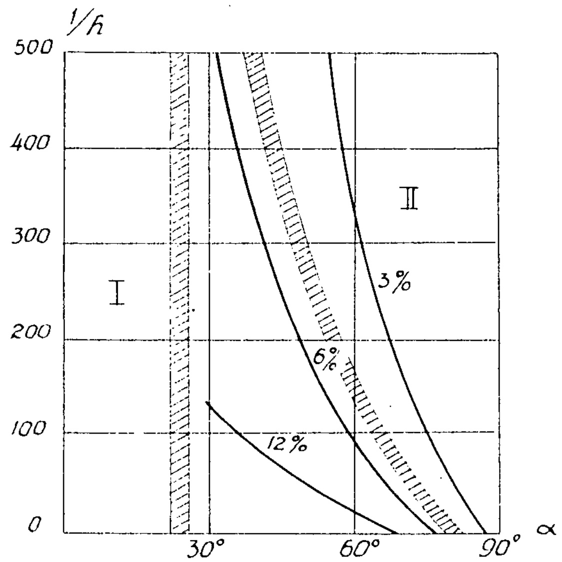


Fig. IV.5.

CHAPTER V

NUMERICAL EXAMPLE

For a numerical example we will consider a spherical shell of 125 feet radius and 30 degrees support angle as shown on Fig. V.I. The loading consists of the dead load and vertical live load, 40 psf and 20 psf respectively. The ratio of the thickness to radius is 3 inch./125 (12) = 1/500. The edge is considered to be on rollers so that only a vertical reaction is provided.

The results of membrane analysis can be obtained easily from the statics and are shown as follows:

$$\bar{N}_\phi = - \bar{a} \left(\frac{\bar{w}_{LL}}{2} + \bar{w}_{DL} \frac{1- \cos\phi}{\sin \phi} \right) \qquad (4.1.1.)$$

$$\frac{\bar{N}_\phi}{\bar{a}} + \frac{\bar{N}_\theta}{\bar{a}} = - \bar{P} \qquad (4.2.1.)$$

therefore $$\bar{N}_\theta = - \left(\bar{P} + \frac{\bar{N}_\phi}{\bar{a}}\right) \bar{a} \qquad (4.2.2.)$$

where w_{LL} = live load, 20 psf. of horizontal plan.

w_{DL} = dead load, 40 psf. of shell.

Substituting these values into the formulae, $\overline{N}_\phi$ and $\overline{N}_\theta$ can be found. At the periphery, $\alpha = 30^\circ$.

$\overline{N}_\alpha$ = -3930 lb/ft.

$\overline{N}_\theta$ = -2280 lb/ft.

On the other hand the horizontal force, $\overline{T}$ can be evaluated as the horizontal projection of $\overline{N}_\phi$. Therefore,

$\overline{T} = \overline{N}_\alpha \cos\alpha$ = -3400 lb/ft.

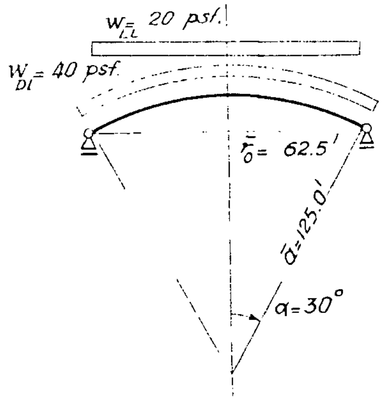


Fig. V.1.

Since there is only the vertical reaction around the periphery, then $\overline{T}$ and $\overline{M}_\phi$ must be equal to zero at the supports. In order to have these conditions, $\overline{T}$, caused by membrane forces, should be eliminated by an opposite force of the same magnitude.

Knowing the boundary conditions at the $\alpha = 30^\circ$ as $\overline{T} = -\overline{T}_{mem}$ and $\overline{M}_\phi = 0$ we can easily obtain the coefficients A and B for the numerical method from the following relations:

$At_1 + B\,t_2 = t$ (4.3.1.)

$A\,m_{\phi 1} + B\,m_{\phi 2} = 0$ (4.4.1.)

where t_1, t_2, $m_{\phi 1}$, $m_{\phi 2}$ are taken from the Tables III.7. a and b.

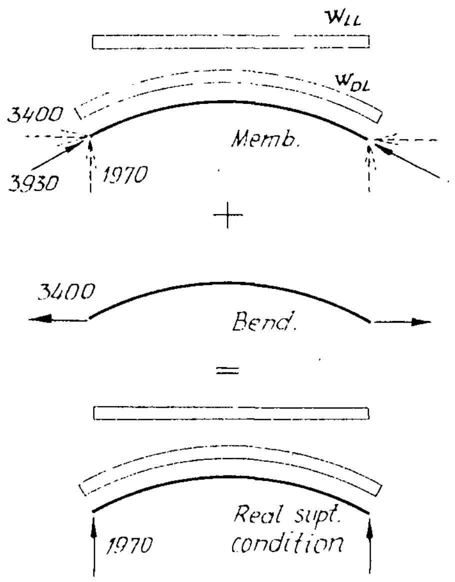


Fig. V.2.

In these relations, t is evaluated by means of the known relations:

$$T = \overline{T} \frac{1-\nu^2}{E\,\overline{a}}$$

$$T = t\, e^{\lambda\phi}\lambda$$

therefore, $$t = \overline{T} \frac{1-\nu^2}{E\,\overline{a}} \quad \frac{1}{e^{\lambda\phi}\lambda}$$

Since $\overline{T}$ = - (-3400) lb/ft., t will be,

$$t = 0.2140 \frac{1}{10^6 E}$$

Substituting the values which are taken from Tables III.7.a.b. into Eqs. (4.3.1.) (4.4.1.),

$$A\,(-0.0205) + B\,(-0.425) + 0.2140 \frac{1}{10^6 E}$$

$$A\,(22.56) \quad + B\,(20.83) \quad = 0$$

and solving A and B from the above equations, the following cam be obtained:

$$A = 0.486 \frac{1}{10^6 E}$$

$$B = -0.527 \frac{1}{10^6 E}$$

Having A and B the stress functions and deformations can be calculated easily.

$$m_\phi = A\, m_{\phi 1} + B\, m_{\phi 2}$$
$$m_\theta = A\, m_{\theta 1} + B\, m_{\theta 2}$$

$$n_\phi = A\, n_{\phi 1} + B\, n_{\phi 2}$$
$$n_\theta = A\, n_{\theta 1} + B\, n_{\theta 2}$$
$$q_\phi = A\, q_{\phi 1} + B\, q_{\phi 2}$$
$$z = A\, z_1 + B\, z_2$$
$$\delta = A\, \delta_1 + B\, \delta_2$$

Multiplying the above values by ($e^{\lambda\phi}\lambda$), $M_\phi M_\theta N_\phi Q_\phi$ Z Δ can be obtained. To transform to the values with dimension, these dimensionless quantities should be multiplied by the following factors:

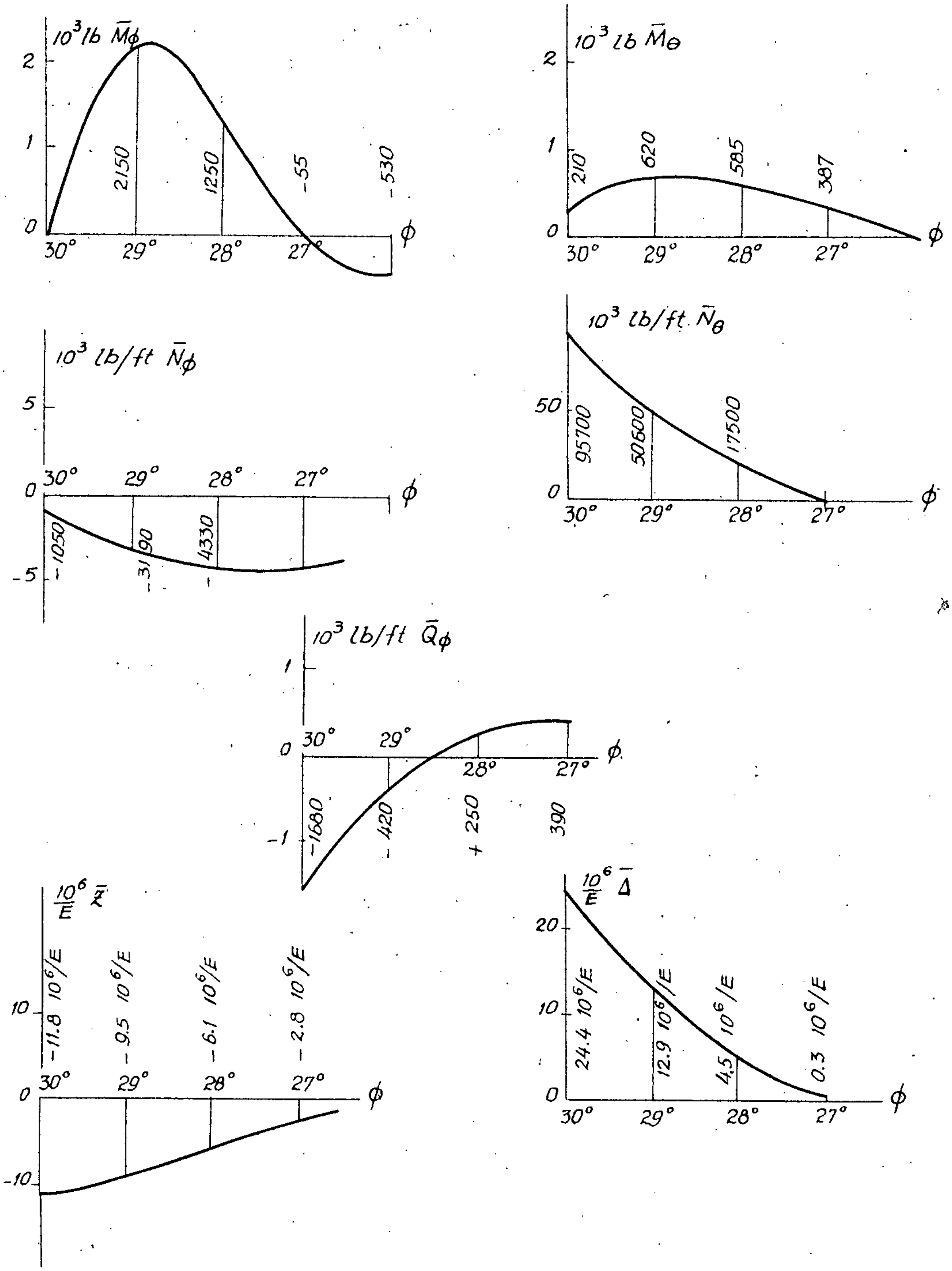
$$\overline{M} = M \frac{hE\,\bar{a}^2}{12(1-\nu^2)} = M \quad 2.71\ E \qquad \text{for moments.}$$

$$\overline{N} = N \frac{\bar{a}\ E}{1-\nu^2} = N\ 130.2\ E \qquad \text{for forces.}$$

$$\overline{Z} = Z \frac{1}{h^2} = Z \quad 25.10^4$$

$$\overline{\Delta} = \Delta \frac{\bar{a}}{h\ (1-\nu^2)} = \Delta \quad 6.51\ 10^4$$

Final values which are sum of membrane and bending solutions are computed and shown on Fig.V.3.



10³lb M̄φ
2150
1250
-55
-530
10³ lb M̄θ
210
620
585
387
10³ lb/ft N̄φ
-1050
-3190
4330
10³ lb/ft N̄θ
95700
50600
17500
10³ lb/ft Q̄φ
-1680
-420
+250
390
10⁶/E χ̄
-11.8 10⁶/E
-9.5 10⁶/E
-6.1 10⁶/E
-2.8 10⁶/E
10⁶/E Δ̄
24.4 10⁶/E
12.9 10⁶/E
4.5 10⁶/E
0.3 10⁶/E
30°
29°
28°
27°
φ

Fig.V.3.

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