SIMPLIFIED CALCULATION OF CABLE TENSION IN SUSPENSION BRIDGES

by

KENNETH MARVIN RICHMOND

B.A.Sc. (Civil Eng.)

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We accept this thesis as conforming to the required standard

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Department of Civil Engineering

The University of British Columbia,
Vancouver 8, Canada.

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This thesis presents a method which facilitates rapid determination of the cable tension in suspension bridges. A set of tables and curves is included for use in the application of the method. The method is valid for suspension bridges with stiffening girders or trusses either hinged at the supports or continuous.

A modified superposition method is discussed and the use of influence lines for cable tension in non-linear suspension bridges is demonstrated.

A derivation of the suspension bridge equations is included and various refinements in the theory are discussed.

A computer program to analyse suspension bridges was written as an aid in the research and for the purpose of testing the manual method proposed. A description of the program is included along with its Fortran listing.
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K. M. R.

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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>CHAPTER</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>CHAPTER 1.</strong> INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td><strong>CHAPTER 2.</strong> THEORY AND REFINEMENTS</td>
<td>5</td>
</tr>
<tr>
<td>General</td>
<td>5</td>
</tr>
<tr>
<td>Cable Equation</td>
<td>8</td>
</tr>
<tr>
<td>Girder Equation</td>
<td>12</td>
</tr>
<tr>
<td>Solution of Equations (11) and (35)</td>
<td>19</td>
</tr>
<tr>
<td>Effect of Refinements in Theory on Accuracy</td>
<td>21</td>
</tr>
<tr>
<td><strong>CHAPTER 3.</strong> COMPUTER PROGRAM</td>
<td>25</td>
</tr>
<tr>
<td>Solution of the Girder Equation</td>
<td>27</td>
</tr>
<tr>
<td>Integration of the Cable Equation</td>
<td>32</td>
</tr>
<tr>
<td>Program Linkage</td>
<td>33</td>
</tr>
<tr>
<td>Input Data for the Program</td>
<td>34</td>
</tr>
<tr>
<td>Final Notes on the Computer Program</td>
<td>36</td>
</tr>
<tr>
<td><strong>CHAPTER 4.</strong> DETERMINATION OF H</td>
<td>37</td>
</tr>
<tr>
<td>General</td>
<td>37</td>
</tr>
<tr>
<td>Superposition of Partial Loading Cases</td>
<td>38</td>
</tr>
<tr>
<td>Single Span</td>
<td>39</td>
</tr>
<tr>
<td>Three-Span Bridge with Hinged Supports</td>
<td>44</td>
</tr>
<tr>
<td>Three-Span Bridge with Continuous Girder</td>
<td>45</td>
</tr>
<tr>
<td>Variable EI</td>
<td>50</td>
</tr>
<tr>
<td><strong>CHAPTER 5.</strong> CONCLUSIONS</td>
<td>52</td>
</tr>
<tr>
<td><strong>APPENDIX 1.</strong> BLOCK DIAGRAM AND FORTRAN LISTING FOR COMPUTER PROGRAM</td>
<td>55</td>
</tr>
<tr>
<td><strong>APPENDIX 2.</strong> TABLES OF CONSTANTS</td>
<td>60</td>
</tr>
<tr>
<td><strong>APPENDIX 3.</strong> NUMERICAL EXAMPLES OF CALCULATION OF H</td>
<td>68</td>
</tr>
<tr>
<td><strong>BIBLIOGRAPHY</strong></td>
<td>81</td>
</tr>
</tbody>
</table>
TABLE OF SYMBOLS

Geometry

L = Length of span
B = Difference in elevation of cable supports
x = Abscissa of undeflected cable
y = Ordinate of undeflected cable measured from chord joining undeflected cable supports
dx = Increment in x
dy = Increment in y
ds = Incremental length of cable corresponding to dx and dy

\[ L_t = \sum \int_0^L \left( \frac{ds}{dx} \right)^2 \, dx \] for all spans

\[ L_e = \sum \int_0^L \left( \frac{ds}{dx} \right)^3 \, dx \] for all spans

Deflections

v = Vertical deflection of cable and girder
h = Horizontal deflection of cable
h_A = Horizontal deflection of left cable support
h_B = Horizontal deflection of right cable support
\( \Delta \) = Equivalent support displacement for inextensible cable
(includes effect of temperature and stress elongation of cable)
Forces
\[ w = \text{Uniformly distributed dead load of bridge} \]
\[ p = \text{Distributed live load on bridge} \]
\[ q = \text{Distributed load equivalent to suspender forces} \]
\[ R_A = \text{Girder support reaction at left end of span} \]
\[ R_B = \text{Girder support reaction at right end of span} \]
\[ H = \text{Total horizontal component of cable tension} \]
\[ H_D = \text{Horizontal component of dead load cable tension} \]
\[ H_L = \text{Horizontal component of cable tension due to live load, temperature change and support displacement} \]
\[ H_L' = \text{Horizontal component of cable tension due to live load on equivalent bridge with inextensible cable and immovable supports} \]
\[ \delta H = \text{Correction to } H_L' \text{ to account for extension of cable and support movement} \]

Bending Moments
\[ M = \text{Bending moment in girder} \]
\[ M_A = \text{Bending moment in girder at left support} \]
\[ M_B = \text{Bending moment in girder at right support} \]
\[ M' = \text{Bending moment in equivalent girder with no cable} \]

Elastic and Thermal Properties
\[ \epsilon = \text{Coefficient of thermal expansion for cable} \]
\[ t = \text{Temperature rise} \]
\[ A = \text{Cross-sectional area of cable} \]
\[ E = \text{Young's Modulus} \]
\[ I = \text{Moment of inertia of girder} \]
\[ m = \text{Coefficient of shear distortion for girder or truss} \]
A_w = Cross-sectional area of girder web
G = Shear modulus
A_d = Cross-sectional area of truss diagonal(s)
\( \phi \) = Angle measured from truss vertical to diagonal(s)

**Computer Program**

\( a_{ij} \) = Coefficients of difference equations approximating girder equation

D = 1 for Deflection Theory solution
    = 0 for Elastic Theory solution

\( F_h \) = 1 to include effect of horizontal deflection
    = 0 to delete effect of horizontal deflection

\( F_s \) = 1 to include change in cable slope in cable equation
    = 0 to delete effect of cable slope change

**Miscellaneous**

c = \sqrt{\frac{H}{EI}}

a = Ratio of side span length \( L_s \) to main span length \( L \)

b = \left( \frac{L_s}{L} \right)^2 \frac{EI}{EI_s}
This thesis adds a few new words, and perhaps a few new thoughts to an area of study which has already been the subject of a considerable amount of study and literature. The analysis of suspension bridges is a problem somewhat different from the usual problems encountered by the structural engineer and somewhat more difficult to solve. It is because of the differences and the difficulties that so much work has been done both to explore extensively the problems involved and to overcome the difficulties in analysing and designing suspension bridges.

The problem in analysis of suspension bridges is a result of their relative flexibility and their desirable ability to deflect in such a manner as to minimize the bending stresses in the stiffening girder. Doubling the load applied to a suspension bridge does not necessarily double the bending moments. Therefore, suspension bridges are said to be non-linear structures. That is, there is a non-linear relationship between load
and resultant stresses. A direct result of this non-linearity is that superposition of results of partial loadings, and methods of analysis dependent on superposition are not applicable in the analysis of suspension bridges. It will be shown here that a modified superposition method can be adapted to the solution of suspension bridge problems.

Investigation of suspension bridges has been inspired by two objectives and at least two main theories have been developed. One goal of investigators has been the development of an exact theory of analysis. As in most engineering problems, so in the analysis of suspension bridges, a completely exact theory is virtually impossible to develop and would be extremely cumbersome to use for design purposes. However, reasonably accurate solutions can be obtained by the use of the Deflection Theory, which takes account of the non-linear behavior of suspension bridges. Another goal of investigators has been the simplification of suspension bridge theory in order to reduce the labor required for analysis and design. A result has been the Elastic Theory, which ignores the changes in geometry resulting from deflection of a suspension bridge under live load and temperature changes. Thus, the Elastic Theory is a linear relationship between load and stress and the usual methods of superposition can be used. As might be expected, the two theories give different results which can vary widely depending on the flexibility of the bridge.

Chapter 2 is devoted to a development of the Deflection Theory or forms of it. There seems to be no universally accepted standard Deflection Theory. Each of the many experts in the field favors a slightly different version. Various
refinements in the theory may be included to improve the accuracy of the calculated results and thus the Deflection Theory equations may take different forms depending on the accuracy desired. Some of these refinements are discussed and the effect on the equations is shown. The Elastic Theory is shown as a simplified version of the Deflection Theory. Also included is a quantitative indication of the effects on accuracy which might be expected as a result of inclusion or neglect of some of the refinements. No new theory is to be found in Chapter 2 but the development has been included here to provide a framework of reference for the following chapters.

It should be noted that throughout this work consideration is confined to static conditions and static loadings. No attention is given here to the more complex considerations of dynamic loadings on suspension bridges.

It will be seen in development of the theory that solution of a suspension bridge problem involves the simultaneous solution of a differential equation and an integral equation. In the more general and more exact solutions, it is necessary to resort to numerical methods for the solution of each of these equations. The simultaneous solution is found by a cut and try method. Hence, solution of a numerical example can become an extremely lengthy and tedious procedure by hand calculations.

Fortunately, because of the existence of computers it is no longer necessary to perform all calculations by hand. The problem of suspension bridge analysis is well suited for solution on a computer. Chapter 3 describes a program which was written for the IBM 1620 digital computer in order to investigate suspension bridge analysis. It is believed that
the methods employed in the program are well suited for computer analysis. For that reason, a listing of the Fortran program has been included in the hopes that it may serve as a guide in the preparation of suspension bridge programs.

The key to a simplified solution to suspension bridge problems is a rapid determination of the value of the cable tension. In the more exact methods, cable tension is found by a cut and try method. Chapter 4 describes a method, believed to be new, whereby $H$, the horizontal component of cable tension can be found extremely quickly. The principles upon which the method depends are shown and the method is developed. Use of the method requires the use of tables or curves relating certain dimensionless ratios. These are included, along with numerical examples illustrating the application of the method. The method employs a form of superposition, which is shown to be valid, providing the total value of $H$ is known. Since $H$ is the value to be determined and is therefore unknown, an estimate is required to initiate calculations. The initial estimate of $H$ is improved by a rapidly converging iterative procedure to give an accurate value of $H$. The method may be applied to suspension bridges either with continuous stiffening girders or with girders hinged at the supports.
CHAPTER 2
THEORY AND REFINEMENTS

General

The following derivation is concerned with the case of a loaded girder, or equivalent plane truss, of known rigidity, suspended by vertical suspenders from a perfectly cable, which is anchored at tower tops or anchorages. In the analysis, the following simplifying assumptions are made:

1. The suspenders are inextensible.
2. The suspenders are so close together that they may be replaced by a continuous fastening.
3. The dead load of the bridge is distributed along the girders.
4. The girder is initially straight under the action of dead load alone, and carries no bending moment.
5. The dead load is constant for each span, and hence the cable is initially parabolic.

The above assumptions are usual for the so-called "Deflection Theory" or more exact theory of suspension bridge analysis. Less exact forms of the Deflection Theory in common use usually make the following additional assumptions:

6. The horizontal deflections of the cable are very small compared with the vertical deflections, and can be neglected.
7. Deflections of the cable are very small compared with cable ordinates, and their effect on cable slope can be neglected in calculation of cable extension.

8. Shear deflections in the girders are very small compared with bending deflection and can be neglected.

Assumptions 6, 7 and 8 may be excluded with little difficulty in the derivation, and may even be excluded in an analysis by digital computer. Therefore, the effects of horizontal deflections, cable slope change, and shear deflection are included here and discussed briefly. It is not to be thought that their inclusion results in a complete theory, but perhaps these are some of the more important refinements which can be made. Others* have discussed the effect of the above refinements, and in addition have introduced, or at least mentioned, other refinements such as tower horizontal force, tower shortening cable lock at midspan, effect of loads between hangers, temperature differentials between girder flanges, finite hanger spacing, weight of cable and hangers, variation of horizontal component of cable tension with hanger inclination, and so forth.

The Deflection Theory of suspension bridge analysis results in a non-linear relationship between forces and deflections and hence the principle of superposition and methods dependent on superposition are not applicable in the usual manner. In order to simplify the force-deflection relationship into a linear one, it is necessary to make a further simplifying

* Reference (12)
assumption. It is this further assumption which is the basis for the "Elastic Theory". It may be stated as follows:

9. The deflections of the cable and girder are so small as to have a negligible effect on the geometry of the cable and hence on the moment arm of the cable force.

It is well known that the Elastic Theory results in errors which are too large to satisfy economy of design. Were it not for the lengthiness of Deflection Theory calculations, the Elastic Theory would have long since passed out of usefulness. Much energy has been expended in attempts to simplify the Deflection Theory to yield results of high accuracy with an ease approaching that of the Elastic Theory; and it is to that end that this thesis is devoted.

Solution by the Deflection Theory consists of the simultaneous solution of two equations. The first is referred to here as the cable equation, and relates cable deflections to cable loads. The second equation is the differential equation relating girder deflection to girder loads and cable tension. This second equation is referred to here as the girder equation.

Figure 1 shows a single span suspension bridge with applied loads. All distances, forces and deflections are positive as shown. Both cable and girder are initially supported at A and B separated by a distance equal to the span length L. The girder is loaded with a constant dead load w, a live load p, end reactions R_A and R_B, and end moments M_A and M_B, which may be end moments applied to a hinged girder or the result of continuity at the support. In addition, the girder is subject to the distributed load q equivalent to the suspender forces. The
Figure 1.

Figure 2.
cable is connected to the girder by vertical suspenders and carries the distributed load \( q \). The cable is in tension, the horizontal component of which is constant and is equal to \( H \). At the supports, the vertical components of the cable tension are \( V_A' \) and \( V_B' \). Under the action of live load and temperature changes, the cable and girder deflect from the positions shown in solid lines to the positions indicated by dashed lines. The cable supports deflect horizontally the distances \( h_A \) and \( h_B \). The original cable position is given by co-ordinates \( x \) and \( y \) measured horizontally from \( A \) and vertically from the chord joining the undeflected cable supports at \( A \) and \( B \). A point \( P \) on the cable deflects from its initial position to a point \( P' \) horizontally a distance \( h \) and vertically a distance \( v \). A point \( Q \) on the girder deflects from its initial position vertically below \( P \) to a position \( Q' \) vertically a distance \( v \).

**Cable Equation**

Figure 2 shows an elemental length of the cable at point \( P \). Its undeflected position is shown as a solid line, while its deflected position is shown as a dashed line. The length of the element in the undeflected position is given by

\[
(ds)^2 = (dx)^2 + \left( dy - \frac{Bdx}{L} \right)^2
\]  

... (1)

Under the action of live loads the cable deflects as shown and the length of the same element of cable in the deflected position is given by

\[
(ds + 6ds)^2 = (dx + dh)^2 + \left( dy - \frac{Bdx}{L} + dv \right)^2
\]  

... (2)
Subtracting (1) from (2) and rearranging terms, it is found that
\[ \frac{d}{dx} \left( \delta ds \frac{1}{dx} \right) = \frac{dh}{dx} \left( \frac{1}{dx} \frac{dh}{dx} \right) + \frac{dv}{dx} \left( \frac{dy}{dx} \frac{B}{L} + \frac{1}{dx} \right) \ldots (3) \]

Since \( \frac{\delta ds}{dx} \) and \( \frac{dh}{dx} \) are both extremely small compared with unity, they may be dropped from the terms \( 1 + \frac{1}{2} \frac{ds}{dx} \) and \( 1 + \frac{1}{2} \frac{dh}{dx} \).

The term \( \frac{1}{2} \frac{dv}{dx} \) is generally small compared with \( \frac{dy}{dx} \frac{B}{L} \) over most of the span, but may be significant, especially in very flat cables. Expression (3) then reduces to

\[ \frac{d}{dx} \delta ds = \frac{dh}{dx} + \frac{dv}{dx} \left( \frac{dy}{dx} \frac{B}{L} + \frac{1}{dx} \right) \ldots (4) \]

The extension of the cable \( \delta ds \) as caused by temperature expansion and stress is given by

\[ \frac{\delta ds}{dx} = \frac{\epsilon t ds}{dx} + \frac{H_L ds}{AE dx} \left[ 1 + \frac{(\frac{dy}{dx} \frac{B}{L} + \frac{dv}{dx})^2}{\left(1 + \frac{dh}{dx}\right)^2} \right] \frac{1}{2} \ldots (5) \]

where: \( \epsilon \) = coefficient of thermal expansion
\( t \) = temperature rise
\( H_L \) = change in horizontal component of cable tension due to application of live load, temperature changes, support movement, etc.
\( A \) = cross-sectional area of cable
\( E \) = Young's Modulus for cable material

Again, since \( \frac{dh}{dx} \) is extremely small compared with unity, it may be deleted from the term \( \left(1 + \frac{dh}{dx}\right)^2 \). Then, the binomial theorem can be applied to expand the bracketed term. All but the first two terms can be neglected, giving

\[ \frac{\delta ds}{dx} = \frac{\epsilon t ds}{dx} + \frac{H_L ds}{AE dx} \left[ 1 + \frac{1}{2} \left( \frac{dy}{dx} \frac{B}{L} \right)^2 + \frac{dv}{dx} \left( \frac{dy}{dx} \frac{B}{L} + \frac{1}{dx} \right) \right] \ldots (6) \]
or, since
\[
\frac{ds}{dx} \approx 1 + \frac{1}{2} \left( \frac{dy}{dx} - \frac{B}{L} \right)^2
\] ... (7)

then
\[
\delta ds = \epsilon \frac{ds}{dx} + \frac{H_L}{AE} \frac{ds}{dx} \left[ \frac{ds}{dx} + \frac{dv}{dx} \left( \frac{dy}{dx} - \frac{B}{L} + \frac{1}{2} \frac{dv}{dx} \right) \right]
\] ... (8)

A combination of equation (4) representing cable geometry and equation (8) representing Hooke's Law gives the cable equation as
\[
\frac{dh}{dx} = \epsilon \left( \frac{ds}{dx} \right)^2 + \frac{H_L}{AE} \left( \frac{ds}{dx} \right)^3 - \left[ 1 - \frac{H_L}{AE} \left( \frac{ds}{dx} \right)^2 \right] \left[ \frac{dv}{dx} \left( \frac{dy}{dx} - \frac{B}{L} + \frac{1}{2} \frac{dv}{dx} \right) \right]
\] ... (9)

The above cable equation may be simplified significantly if it is observed that the term \( \frac{dv}{dx} \left( \frac{dy}{dx} - \frac{B}{L} \right) \) in expression (5) is normally less than .2, and \( \frac{dv}{dx} \) is generally small compared with \( \frac{dy}{dx} - \frac{B}{L} \). Hence, \( \frac{dv}{dx} \) is not very significant in the total expression and can reasonably be neglected. Since \( \frac{dh}{dx} \) has already been neglected compared with unity in the same expression, this amounts to neglect of the effect of deflections on cable slope and expression (5) becomes
\[
\delta ds = \epsilon \frac{ds}{dx} + \frac{H_L}{AE} \left( \frac{ds}{dx} \right)^2
\] ... (5a)

When (5a) is combined with (4), the simplified cable equation is
\[
\frac{dh}{dx} = \epsilon \left( \frac{ds}{dx} \right)^2 + \frac{H_L}{AE} \left( \frac{ds}{dx} \right)^3 - \frac{dv}{dx} \left( \frac{dy}{dx} - \frac{B}{L} + \frac{1}{2} \frac{dv}{dx} \right)
\] ... (9a)

It can be seen that neglect of the change of cable slope is reflected in expression (9) by neglect of the term \( \frac{H_L}{AE} \left( \frac{ds}{dx} \right)^2 \)
compared with unity. $\frac{H_L}{AE}$ is usually of the order .001 and $\frac{ds}{dx}$ is normally not much larger than 1, so a term of order .001 has been neglected compared with 1. On this basis, Timoshenko argues that it is negligible. However, it is not difficult to see that a given percentage error in one term of expression (9) could be magnified by subtracting that term from another of similar magnitude to give a larger percentage error in $\frac{dh}{dx}$.

Expression (9) can be further simplified if $\frac{1}{2} \frac{dv}{dx}$ is neglected compared with $\frac{dv}{dx} - \frac{B}{L}$ in expression (4). Then the cable equation becomes

$$\frac{dh}{dx} = \epsilon t \left( \frac{ds}{dx} \right)^2 + \frac{H_L}{AE} \left( \frac{ds}{dx} \right)^3 - \frac{dv}{dx} \left( \frac{dy}{dx} - \frac{B}{L} \right)$$  \hspace{1cm} (9b)

This final expression gives a linear relationship between horizontal and vertical deflections.

It should be noted that the above linear relationship between horizontal and vertical deflections does not imply that the structure is linear. The cable equation has been reduced to a linear equation, but a non-linear relationship can and does still exist between stresses and applied loads.

If the cable equation is integrated over the span length and the horizontal displacements of the supports are inserted as constants of integration, the following expression results:

$$h_B - h_A = \int_0^L \epsilon t \left( \frac{ds}{dx} \right)^2 \, dx + \int_0^L \frac{H_L}{AE} \left( \frac{ds}{dx} \right)^3 \, dx$$

$$- \int_0^L \left[ 1 - \frac{H_L}{AE} \left( \frac{ds}{dx} \right)^2 \right] \left[ \frac{dv}{dx} \left( \frac{dy}{dx} - \frac{B}{L} + \frac{1}{2} \frac{dv}{dx} \right) \right] \, dx$$  \hspace{1cm} (10)
or, if \( \int_0^L \left( \frac{ds}{dx} \right)^2 \, dx \) is denoted by \( L_t \) and \( \int_0^L \left( \frac{ds}{dx} \right)^3 \, dx \) is denoted by \( L_e \), then

\[
h_B - h_A = \frac{\epsilon}{t} L_t + \frac{H_L}{AE} L_e
\]

\[
- \int_0^L \left[ 1 - \frac{H_L}{AE} \left( \frac{ds}{dx} \right)^2 \right] \left[ \frac{dv}{dx} \left( \frac{dy}{dx} - \frac{B}{L} + \frac{1}{2} \frac{dv}{dx} \right) \right] \, dx
\]

... (11)

If the change in cable slope can be neglected, then

\[
h_B - h_A = \frac{\epsilon}{t} L_t + \frac{H_L}{AE} L_e - \int_0^L \frac{dv}{dx} \left( \frac{dy}{dx} - \frac{B}{L} + \frac{1}{2} \frac{dv}{dx} \right) \, dx
\]

... (11a)

If the term \( \frac{1}{2} \frac{dv}{dx} \) can be neglected compared with \( \frac{dy}{dx} - \frac{B}{L} \), then

\[
h_B - h_A = \frac{\epsilon}{t} L_t + \frac{H_L}{AE} L_e - \int_0^L \frac{dv}{dx} \left( \frac{dy}{dx} - \frac{B}{L} \right) \, dx
\]

... (11b)

The cable equation (11) or the simplified forms (11a) and (11b) relate cable displacements to cable loads. It will be seen that the refinement represented in equation (11) is difficult to justify, considering the additional accuracy attained and the effort required to solve the equation. Equation (11a) requires considerably less effort to solve, but is also difficult to justify. Equation (11b) will be found to be sufficiently accurate for most purposes.

The cable equation is one of the two equations to be solved in analysis of suspension bridges. Cable deflections must be consistent with support movements and with girder deflections as given by the girder equation.

**Girder Equation**

In order to derive the girder equation, it is necessary to consider separately the two main components of the bridge,
Figure 3.

Figure 4.
the cable and the girder. Figure 3 shows a free-body diagram of the girder under the action of applied loads. The reaction $R_A$ can be found from

$$R_A = \frac{M_B - M_A}{L} + \int_0^L \frac{(p + w - q)(L - a)}{L} da \ldots (12)$$

Then, the bending moment in the girder at $x$, denoted by $M$ is given by

$$M = M_A + \frac{(M_B - M_A)x}{L} + \int_0^L \frac{(p+w-q)(L-a)da}{L} \ldots (13)$$

For simplicity define a quantity $M'$ equal to the bending moment produced by all loads except those applied by the cable. This is given by

$$M' = M_A + \frac{(M_B - M_A)x}{L} + \int_0^L \frac{(p+w)(L-a)da}{L} \ldots (14)$$

Then

$$M = M' + \int_0^L q(L-a) da + \int_0^X q(x-a) da \ldots (15)$$

Now, it is necessary to consider the static equilibrium of the cable under the applied loads. Figure 4 shows the forces acting on the deflected cable, indicated by a dashed line. The cable tensions at the supports are resolved into components in the direction of the chord joining the deflected points of support of the cable, and vertical components $V_A$ and $V_B$. The vertical component $V_A$ is given by

$$V_A = \int_0^L q \left( \frac{L + h_B - (a+h)}{L + h_B - h_A} \right) da \ldots (16)$$

It will be noted that the forces in the direction of the closing chord have horizontal components equal to $H$, the horizontal component of cable tension. Then, for the equilibrium of the
cable
\[ H(y-\delta_1+\delta_2) = V_A(x-h_A) - \int_0^x q(x-(a+h)) \, da \] ...

(17)

where \( y - \delta_1 + \delta_2 \) is the vertical distance from the deflected position of the closing chord to the deflected cable. It is clear from geometry that

\[ \delta_1 = \frac{B}{L} \left( \frac{h_A}{L} + \frac{(h_B-h_A)x}{L} \right) \] ...

(18)

Reference to Figure 2 will show that

\[ \delta_2 = v \left( \frac{dy}{dx} - \frac{B}{L} \right) \] ...

(19)

The term \( V_A(x-h_A) \) in equation (17) now becomes

\[ V_A(x-h_A) = x \left[ \int_0^L \frac{q(L-a) \, da}{L+h_B-h_A} + \int_0^L \frac{q(h_B-h) \, da}{L+h_B-h_A} \right] - V_A h_A \] ...

(20)

When it is observed that the horizontal deflections are very small compared with the dimensions \( L, a \) and \( x \), it can be seen that the terms \( x \int_0^L \frac{q(h_B-h) \, da}{L+h_B-h_A} \) and \( V_A h_A \) are very small compared with \( x \int_0^L \frac{q(L-a) \, da}{L+h_B-h_A} \) and can be approximated with negligible error in the total term \( V_A(x-h_A) \) as follows:

\[ x \int_0^L \frac{q(L-a)}{L+h_B-h_A} \, da \approx x \int_0^L \frac{q(L-a)}{L} \, da \] ...

(21)

\[ V_A h_A = h_A \left( \frac{dy}{dx} \right) \] ...

(22)

Then, the term \( x \int_0^L \frac{q(L-a)}{L+h_B-h_A} \, da \) of equation (20) can be expanded, neglecting all but the first two terms to give

\[ x \int_0^L \frac{q(L-a)}{L} \, da = x \int_0^L \frac{q(L-a)}{L} \, da + \frac{x(h_A-h_B)}{L} \int_0^L \frac{q(L-a)}{L} \, da \] ...

(23)

Note that the second term on the right hand side of expression (23) is much smaller than the first term. It is therefore per-
possible to approximate it as follows:

\[
\frac{x(h_A-h_B)}{L} \int_0^L \frac{q(L-a)}{L} \, da + \frac{x(h_A-h_B)}{L} \frac{H}{dx} \Bigg|_A 
= \frac{x(h_A-h_B)}{L} \int_0^L \frac{q(L-a)}{L} \, da + \frac{x(h_A-h_B)}{L} \frac{H}{dx} \Bigg|_A
\]

... (24)

If substitutions from expressions (21), (22), (23) and (24) are made in expression (20), it is found that

\[
V_A(x-h_A) = \frac{x}{L} \int_0^L q(L-a) \, da + \frac{x(h_A-h_B)}{L} \frac{H}{dx} \Bigg|_A
+ \frac{x}{L} \int_0^L q(h_B-h) \, da + \frac{h_A}{L} \frac{H}{dx} \Bigg|_A
\]

... (25)

If substitutions from expressions (18), (19) and (25) are made in expression (17) and the result is combined with expression (15), the following expression results:

\[
M = \frac{M''}{H} \left[ y \left( \frac{h_A}{L} + \frac{(h_B-h_A)x}{L} \right) + \frac{1}{L} \int_0^x q \, da \right]
- \frac{h_A}{L} \frac{H}{dx} \Bigg|_A + x \left[ \frac{H}{dx} \frac{h_A-h_B}{L} + \frac{L}{L} \int_0^L q(h_B-h) \, da \right]
\]

... (26)

Since terms involving the horizontal deflections are small, it is permissible to make some approximations in these terms. Specifically, it is permissible to approximate the suspender forces \(q\) by

\[
q = - \frac{d^2y}{dx^2}
\]

... (27)

If the above is substituted in (26) the expression for bending moment in the girder becomes

\[
M = \frac{M''}{H} \left[ y \left( \frac{h_A}{L} + \frac{(h_B-h_A)x}{L} \right) - \frac{h}{L} \left( \frac{dy}{dx} \frac{B}{L} \right) + \frac{1}{L} \int_0^x q \, da \right]
- \frac{h_A}{L} \frac{H}{dx} \Bigg|_A + x \left[ \frac{H}{dx} \frac{h_A-h_B}{L} - \frac{L}{L} \int_0^L \frac{d^2y(h_B-h)}{dx^2} \, da \right]
\]

... (28)
It can be shown that, if all horizontal displacements are increased by a constant amount $h_0$, the bending moment in the girder at all points will be unchanged. Hence, if $h_0$ is set equal to $-h_A$, $h_A$ may be replaced by zero in expression (28), $h_B$ may be replaced by $h_B - h_A$, and $h$ may be replaced by $h - h_A$.

Expression (28) may be differentiated twice to give

$$\frac{d^2M}{dx^2} = -p - w - H \left[ \frac{d^2y}{dx^2} + \frac{d^2v}{dx^2} - \frac{d^2h}{dx^2} \left( \frac{dy}{dx} - \frac{B}{L} \right) - \frac{dh}{dx} \frac{d^2y}{dx^2} \right]$$

... (29)

If horizontal deflections are neglected in the girder equation, expressions (28) and (29) reduce to

$$M = M'' - H(y + v)$$

... (28a)

$$\frac{d^2M}{dx^2} = -p - w - H \left( \frac{d^2y}{dx^2} + \frac{d^2v}{dx^2} \right)$$

... (29a)

If vertical deflections are neglected in the girder equation, expressions (28) and (29) reduce to the Elastic Theory expressions

$$M = M'' - Hy$$

... (28b)

$$\frac{d^2M}{dx^2} = \frac{d^2y}{dx^2}$$

... (29b)

Expression (13) can be differentiated twice to give

$$\frac{d^2M}{dx^2} = q - p - w$$

... (30)

Then equations (29b) and (30) can be combined to give expression (27), an approximate relationship which was used earlier.

From elementary strength of materials, the basic differential equation relating deflection to bending moment and shear in a girder is

$$\frac{EI d^2 v}{dx^2} = -M + EI \frac{d^2 M}{dx^2}$$

... (31)
where:  

\( E \) = Young's Modulus for the girder material  

\( I \) = moment of inertia of the girder  

\( m \) = coefficient of shear distortion  

\[ m = \frac{1}{A_w G} \text{ for a girder} \]  

\[ m = \frac{1}{A_D E \sin \phi \cos^2 \phi} \text{ for a plane truss} \]  

\( A_w \) = cross-sectional area of girder web  

\( G \) = shear modulus  

\( A_D \) = cross-sectional area of the diagonal member(s)  

\( \phi \) = angle measured from vertical to diagonal member(s)  

The deflections due to shear are small compared with the deflections due to bending. Therefore, negligible error results if \( \frac{\partial^2 M}{\partial x^2} \) is represented by the approximate expression (29a). If expressions (28) and (29a) are substituted in expression (30), and the resulting equation is differentiated twice, the following fourth-order differential equation is found:  

\[
(1 + Hm) \left[ E I d^4 v + \frac{2d(EI)}{dx} \frac{d^3 v}{dx^3} + \frac{d^2(EI)}{dx^2} \frac{d^2 v}{dx^2} \right] = p + w
\]  

\[ \frac{H}{dx^2} \left[ \frac{d^2 y}{dx^2} + \frac{d^2 v}{dx^2} \right] - \left[ \frac{d^2 h}{dx^2} \left( \frac{dy}{dx} \frac{dy}{L} - \frac{dh}{dx} \frac{d^2 y}{dx^2} \right) \right] 
\]  

\[- \frac{d^2(EI)}{dx^2} m \left[ p + w + \frac{Hd^2 y}{dx^2} \right] \]  

\[
... (32)
\]

The above differential equation reduces in the Elastic Theory to  

\[
\left[ E I d^4 v + \frac{2d(EI)}{dx} \frac{d^3 v}{dx^3} + \frac{d^2(EI)}{dx^2} \frac{d^2 v}{dx^2} \right] = p + w + \frac{Hd^2 y}{dx^2}
\]  

\[- \frac{md^2(EI)}{dx^2} \left[ p + w + \frac{Hd^2 y}{dx^2} \right] \]  

\[
... (33)
\]

Substitution from the cable equation can be made in expression
(32) to give the horizontal deflections in terms of the vertical deflections. Cable equation (9b) is sufficiently accurate here. Remember that horizontal deflections have a very small effect on bending moments. It is a simple matter to prove that

\[
\frac{d^2h}{dx^2} \left( \frac{dy}{dx} - \frac{B}{L} \right) - \frac{dh}{dx} \frac{d^2y}{dx^2} = \frac{Et^2y}{dx^2} \left[ 1 + 3 \left( \frac{dy}{dx} - \frac{B}{L} \right)^2 \right]
\]

\[
+ \frac{H_L}{AE} \frac{d^2y}{dx^2} \frac{ds}{dx} \left[ 1 + 4 \left( \frac{dy}{dx} - \frac{B}{L} \right)^2 \right] - \frac{2dv}{dx} \frac{d^2y}{dx^2} \left( \frac{dy}{dx} - \frac{B}{L} \right)
\]

\[
- \frac{d^2v}{dx^2} \left( \frac{dy}{dx} - \frac{B}{L} \right)^2
\]

... (34)

Expression (34) can be substituted in equation (32) and terms may be collected to give the following fourth-order linear differential equation

\[
\frac{d^4v}{dx^4} \left[ EI(1 + H_m) \right] + \frac{d^3v}{dx^3} \left[ 2d(EI)(1 + H_m) \right] + \frac{d^2v}{dx^2} \left[ 2d(EI)(1 + H_m) \right]
\]

\[
- \frac{H}{AE} \frac{d^2y}{dx^2} \left( \frac{dy}{dx} - \frac{B}{L} \right)^2 + \frac{dv}{dx} \left[ -2Hd^2y \left( \frac{dy}{dx} - \frac{B}{L} \right) \right] = p + W + \frac{Hd^2y}{dx^2}
\]

\[
- \frac{H\epsilon t^2y}{dx^2} \left[ 1 + 3 \left( \frac{dy}{dx} - \frac{B}{L} \right)^2 \right] - \frac{HHL}{AE} \frac{d^2y}{dx^2} \frac{ds}{dx} \left[ 1 + 4 \left( \frac{dy}{dx} - \frac{B}{L} \right)^2 \right]
\]

\[
- \frac{md^2(EI)}{dx^2} \left[ \frac{p + W + \frac{Hd^2y}{dx^2}}{dx^2} \right]
\]

... (35)

Equation (35) is the girder equation in a general form. It includes the effect of variable girder stiffness, horizontal cable deflection and shear deflection. Unknowns present in the equation are H and v. The cable equation (11) also relates unknowns H and v. Therefore, simultaneous solution of equations
(11) and (35) will yield the values of $H$ and $v$.

The ordinates of the cable in the undeflected position can be found by considering the equilibrium of the cable and girder under the action of dead load alone. No bending moment is present anywhere in the girder and the applied moment $M'$ is given by

$$M' = \frac{wx(L - x)}{2} \quad \ldots (36)$$

From equation (28)

$$H_Dy = \frac{wx(L - x)}{2} \quad \ldots (37)$$

If the cable ordinate at mid-span is denoted by $f$, then

$$H_D = \frac{wL^2}{8f} \quad \ldots (39)$$

and

$$y = \frac{4f x(L - x)}{L^2} \quad \ldots (40)$$

$$\frac{dy}{dx} = \frac{4f(L - 2x)}{L^2} \quad \ldots (41)$$

$$\frac{d^2y}{dx^2} = -\frac{8f}{L^2} \quad \ldots (42)$$

Solution of Equations (11) and (35)

Solutions to equations (11) and (35) individually cannot be evaluated in terms of simple functions except in a few very special cases. Therefore, some method of numerical analysis must be resorted to in most cases.

Whether the evaluation is by algebraic or numerical methods, the general approach is the same. First, some estimate
must be made of H. Then, equation (35) must be solved to give values of v corresponding to the estimated value of H. Then, equation (11) can be solved to give a value of \( h_B - h_A \) which must be equal to a value consistent with external conditions (for example, zero for fixed cable anchorages). In general, the first estimate of H will result in an error in the computed value of \( h_B - h_A \); and so a second estimate of H must be made, and the procedure repeated to give a second error in the computed value of \( h_B - h_A \). Subsequent estimates of H can be made by interpolation between previous estimates to give finally a tolerable error in \( h_B - h_A \), and a sufficiently accurate value of H. The last solution of equation (35) thus obtained can be used to evaluate bending moments at all points, and shearing forces at all points.

In multiple span bridges, the procedure for analysis is similar to that for a single span. Equation (35) must be solved to give values of v for all spans, corresponding to an estimated value of H. Then equation (11) must be evaluated for all spans to give the total value of \( h_B - h_A \) where \( h_B \) and \( h_A \) here represent the horizontal displacements of the cable anchorages at the ends of the bridge. In doing this, \( L_e \) and \( L_t \) represent the sums of values of \( \int_0^L (\frac{ds}{dx})^3 \, dx \) and \( \int_0^L (\frac{ds}{dx})^2 \, dx \), respectively, for all spans.

If numerical analysis is resorted to in the solution of equations (11) and (35), no mathematical difficulties are encountered due to inclusion of the effects of cable slope change, horizontal deflection of the cable, and shear deflection in the girder. Except that the calculations are a little longer, it is as convenient to include these refinements as to leave them
out. Equations (11) and (35) are suitable for solution by numerical analysis on a digital computer, and in the program outlined in the next chapter, the refinements mentioned above may be included or left out at will.

It is not to be thought that the reduction in labor due to simplification of equations (11) and (35) is insignificant. In fact, if the error introduced is tolerable, considerable saving in labor is possible by the use of equation (11b) in place of (11) and by the omission of horizontal deflection and shear deflection from the girder equation. The girder equation may then be used in the form

$$\frac{EId^2v}{dx^2} - Hv - Hy - M' = \ldots \text{(43)}$$

Equation (43) is usually considered to be the basic differential equation for suspension bridges. Solutions to equation (43) are to be found tabulated in Steinman's text* and elsewhere for loading conditions commonly encountered in design. Numerical analysis is unnecessary except for programming a digital computer.

**Effect of Refinements in Theory on Accuracy**

It may be of some value to indicate here briefly the order of magnitude of error introduced by neglect of the effects of change in slope of the cable, horizontal deflection of the cable and shear deflection in the girder.

It has been shown that the error resulting from neglect of the change of cable slope in the calculation of cable extension is largely dependent on the ratio $\frac{H_l}{AE}$. Test calculations were performed with the aid of the computer program outlined in the next chapter. In the test case, the value of $\frac{H_l}{AE}$ was .0231,

* Reference (1)
considerably larger than the usual range of values of $\frac{H_L}{AE}$. The test bridge was a three-span bridge, loaded over the entire main span. The resultant error in $H_L$ was 0.7%, and the maximum error in bending moment was 0.11%. Therefore, it is reasonable to expect that neglect of the effect of cable slope change will never result in errors larger than 1% in cable tension and bending moments.

Shear deflections have been investigated by C. D. Crosthwaite.* It was found that the largest errors in bending moments and shears occur near the supports and that the effect of shear deformation diminishes toward mid-span. Neglect of shear deformation was shown to account for errors as large as 9.5% in shears and 6.5% in bending moments. The writer has compared the results of calculations for the case of a three-span bridge loaded over one half of the main span. The magnitude of the percentage error depends on the ratio of shear flexibility to bending flexibility in the girder, and in addition depends to some extent on the cable tension and cable area. Figure 5 indicates the relationship between percentage error and given dimension ratios. The computed quantity under consideration is the bending moment at the quarter point on the loaded portion of the main span. Within the normal range of values of $\frac{EIm}{L^2}$, errors in computed values of the bending moment can be as high as 15%. For all values of the dimensionless ratios within the range of normal values shown in Figure 5, it was found that the errors in the value of $H_L$ determined by neglecting shear deflection were less than 1%.

* Reference (18)
The effect of longitudinal deflection has been investigated by H. H. Rode * and he has reported that the reduction in bending moment will usually be between 5% and 2%. The writer has examined a single case of a three-span bridge loaded over half the main span and the adjacent side span. The bridge under consideration was very flexible, having a ratio $\frac{HDL^2}{EI}$ of 100. The errors in bending moments due to neglect of the horizontal deflection were found to be as high as 8.5% of the maximum bending moment in the girder. The error in $H_L$ was 0.32%.

The writer has compared also the effect of using cable equation (11b) for a particular example calculation. The bridge examined was a three-span bridge having ratio $\frac{HDL^2}{EI}$ of 100 for the main span. It carried a live load of 0.4 times the dead load over one half of the main span. It was found that approximation of the cable equation by equation (11b) instead of equation (11a) resulted in an error in $H_L$ of 3.7%. This was reflected in the girder moments by an error of 1.1% of the maximum moment.

It was observed that any approximation of the cable equation resulted in an error in $H_L$ which was reflected by an error of reduced significance in the girder bending moments. On the other hand, approximation of the girder equation by neglecting shear deflection or horizontal cable deflection results in small errors in $H_L$ but larger errors in bending moments. These larger errors in bending moment are due not to the inaccuracy in $H$ but to the omission of terms in the girder equation. Hence, it would seem reasonable to determine $H_L$ by

---

* Reference (16)
using approximate equations (43) and either (1la) or (1lb). Then, the value of the $H$ found by these approximate equations can be used with the more exact girder equation (35) or a simplified version of it to yield bending moments of required accuracy. Usually, the value of $H_L$ found by the approximate equations will be in error by less than 1% due to the neglect of cable slope change, shear deformation and horizontal deflection. Choice of equation (1la) or (1lb) will depend on the flexibility of the girder and the ratio of live load to dead load; and on the accuracy desired. It is suggested that equation (1lb) will be satisfactory for all but very flexible girders and heavy loads.
CHAPTER 3
COMPUTER PROGRAM

The analysis of suspension bridges is a problem especially suitable for solution on a digital computer. Each of the equations (11) and (35) may be readily programmed for numerical solution, and their simultaneous solution by a cut and try method reduces simply to an iteration procedure. It is a simple matter to delete terms in either equation to test the effect of neglecting refinements in the theory. Also the elastic theory solution can be readily obtained from the deflection theory equations merely by deleting terms which apply only to the deflection theory. Therefore, the program has been written to give, for each loading case, both a deflection theory solution and an elastic theory solution.

The computer used is an IBM 1620 desk-size digital computer, at the University of British Columbia Computing Centre. The core-storage memory holds up to 40,000 digits of data and instructions. Available input devices are the console typewriter and a card-reader. Output is by console typewriter, card punch or on-line printer.

It was found to be convenient to write the program using Fortran, the simplified scientific language available for IBM computers. The particular version of Fortran used is designated by the U.B.C. Computing Centre as Fortran 1A.
All input data required for the program, and all output are in dimensionless form. It is convenient to think of the unit of length as being \( L \), the length of the main span, and the unit of load or force is \( wL \), the total uniformly distributed dead load on the main span.

The program was written originally for the purpose of investigating the relationship between elastic theory bending moments and deflection theory bending moments. It analyses a three-span suspension bridge using first the elastic theory, and then the deflection theory, and compares the computed bending moments. In both cases, a cut and try method is used to determine \( H \), the horizontal component of cable tension. In general, each trial value of \( H \) produced an error in the computed value of the relative support movement. Each trial value of \( H_L \) and its accompanying value of error is printed out. The actual values printed out are \( \frac{H_L}{wL} \) and \( \frac{\Delta \text{error}}{L} \), in dimensionless form. When the error is practically zero, the values of \( \frac{HDL^2}{EI} \), \( \frac{HLL^2}{EI} \), and \( \frac{HL^2}{EI} \) are printed and the bending moments at several points on the girder are computed. When the bending moment has been computed by both methods, the program prints the deflection theory bending moment, the elastic theory bending moment, the ratio between the two, and the deflection at points along the girder.

The program analyses a three-span suspension bridge with constant or variable girder stiffness. The girder may be either continuous at the towers or hinged. In the numerical procedures used, the main span is divided into twenty equal partial lengths equal in length to the main span sections. The bending moments, deflections and so forth are computed for the points separating these partial lengths of girder. The points are
numbered consecutively from left to right on the girder with points at the left and right tower numbered 17 and 37, respectively. In its present form, the program requires that the side spans be divided into an even number of parts, and storage limitations further limit the ratio of side span length to main span length to a maximum of .7. Therefore, acceptable ratios would be .7, .6, .5, .3, .2 or .1.

The method of solution is illustrated in the general block diagram for the program shown in Figure 6. It can be seen that the heart of the program is the solution of the differential equation for the girder and the integral equation for the cable. These equations are included in the program in as exact a form as it is deemed practical, including such refinements as shear deflection, horizontal deflection of cable and change in slope of cable. Terms which are found in the deflection theory equations but not in the elastic theory equations are multiplied by a factor D, which is made zero to give an elastic theory solution, or unity to give a deflection theory solution. The above-mentioned refinements in the deflection theory are included or deleted by the same technique.

Solution of the Girder Equation

Two approaches to numerical solution of the girder equation were considered. The equation could be treated as an "initial value" problem, and some assumption as to unknown initial conditions could be made in order to find a solution by the Runge-Kutta method. Then, for each unknown initial condition, a different value of the initial condition could be assumed to give an additional solution to the equation. The
Note: $D = 0$ for Elastic Theory. $D = 1$ for Deflection Theory.

Figure 6.
solutions thus found could then be superimposed so as to satisfy all boundary equations. Another approach is to deal with the equation directly as a "boundary condition" problem, by substituting for the differential equation a system of difference equations, which may be solved to give the deflections at points along the girder. The latter method was thought to be most suitable for this problem and was adopted.

At a point \( i \) on the girder, the derivatives in the girder equation may be approximated by finite differences as given below:

\[
\frac{d^4 v}{dx^4} = \frac{1}{d^4} \left( v_{i-2} - 4v_{i-1} + 6v_i - 4v_{i+1} + v_{i+2} \right)
\]

\[
\frac{d^3 v}{dx^3} = \frac{1}{d^3} \left( \frac{1v_{i-2} + v_{i-1} - v_{i+1} + \frac{1v_{i+2}}{2}}{2} \right)
\]...

\[
\frac{d^2 v}{dx^2} = \frac{1}{d^2} \left( v_{i-1} - 2v_i + v_{i+1} \right)
\]

\[
\frac{dv}{dx} = \frac{1}{d} \left( \frac{1v_{i-1} + v_{i+1}}{2} \right)
\]

where \( v_{i-2}, v_{i-1}, v_i, v_{i+1}, v_{i+2} \) are deflections at points separated by a small distance \( d \).

If substitution from equations (44) is made in equation (35), given in the preceding chapter, it is possible at each point on the girder to write a difference equation of the form:

\[
a_{11}v_{i-2} + a_{12}v_{i-1} + a_{13}v_i + a_{14}v_{i+1} + a_{15}v_{i+2} = b_i \]...

(45)
where

\[ a_{11} = \left[ \frac{EI}{d^4} - \frac{1}{d^3} \frac{dE}{dx} \right] (1 + DHm) \]

\[ a_{12} = \left[ \frac{4EI}{d^4} + \frac{2}{d^3} \frac{dE}{dx} + \frac{1}{d^2} \frac{d^2E}{dx^2} \right] (1 + DHm) \]

\[ + \frac{DH}{d} \left[ \frac{1}{d^2} - \frac{F_h}{d^2} \left( \frac{dy}{dx} - \frac{B}{L} \right)^2 + \frac{F_h}{d^2} \left( \frac{d^2y}{dx^2} \left( \frac{dy}{dx} - \frac{B}{L} \right) \right) \right] \]

\[ a_{13} = \left[ \frac{6EI}{d^4} - \frac{2}{d^2} \frac{d^2E}{dx^2} \right] (1 + DHm) \]

\[ + \frac{DH}{d^2} \left[ \frac{2}{d^2} + \frac{2F_h}{d^2} \left( \frac{dy}{dx} - \frac{B}{L} \right)^2 \right] \]

\[ a_{14} = \left[ \frac{4EI}{d^4} - \frac{2}{d^3} \frac{dE}{dx} + \frac{1}{d^2} \frac{d^2E}{dx^2} \right] (1 + DHm) \]

\[ + \frac{DH}{d^2} \left[ \frac{1}{d^2} - \frac{F_h}{d^2} \left( \frac{dy}{dx} - \frac{B}{L} \right)^2 - \frac{F_h}{d^2} \left( \frac{d^2y}{dx^2} \left( \frac{dy}{dx} - \frac{B}{L} \right) \right) \right] \]

\[ a_{15} = \left[ \frac{EI}{d^4} - \frac{1}{d^3} \frac{dE}{dx} \right] (1 + DHm) \]

\[ b_1 = (p + w) \left[ 1 - \frac{md^2(EI)}{dx^2} \right] + H \left[ \frac{d^2y}{dx^2} \left( 1 - \frac{md^2(EI)}{dx^2} \right) \right] \]

\[ + HF_h \left[ \frac{6t}{dx^2} = \left( 1 + 3 \left( \frac{dy}{dx} - \frac{B}{L} \right)^2 \right) \right. \]

\[ - \frac{H_L}{AE} \left( \frac{1}{dx} \frac{d^2y}{dx^2} \left( 1 + 4 \left( \frac{dy}{dx} - \frac{B}{L} \right)^2 \right) \right) \]

In equations (46) above, D may be either one or zero, for deflection theory or elastic theory, as previously described. In addition \( F_h \) may be one or zero to include or delete the effect
of horizontal deflection of the cable. The effect of shear deformation may be deleted simply by assuming a zero value for \( m \), the shear flexibility of the girder. Derivatives of \( EI \), the flexural rigidity of the girder, may be replaced by finite differences, as in equations (44). In the case of a girder of constant \( EI \), derivatives of \( EI \) vanish.

In order to obtain a solution to the differential equation, it is necessary to express the boundary conditions in similar form to equation (45). The case of zero deflection at the supports is obviously expressed in the form

\[
v_i = 0
\]  
(47)

For the case of zero moment at a point, use is made of the following equation, derived from equations (29a) and (31).

\[
M = \frac{d^2v}{dx^2} \left( EI(1 + DHm) \right) - EI \left( \frac{p + w + Hd^2y}{dx^2} \right) \]

(48)

If the bending moment is zero at a point \( i \), then the difference equation becomes

\[
\frac{1}{d^2} v_{i-1} - \frac{2}{d^2} v_i + \frac{1}{d^2} v_{i+1} = \frac{-m}{1 + DHm} \left( \frac{p + w + Hd^2y}{dx^2} \right) i
\]

(49)

If the girder is divided into a finite number \( N \) of equal partial lengths, at each of the \( N - 1 \) interior points on the girder, it is possible to write an equation of the form (45) involving the deflection at the point under consideration and at two adjacent points on each side of the point under consideration. Note that the equation for the interior points nearest the exterior supports involve the deflections at the supports and at fictitious points exterior to the supports. Therefore, it is possible to write \( N - 1 \) typical interior equations involving \( N + 3 \) unknown deflections. The four additional equations
required are provided according to the boundary conditions; that is, zero deflection at each exterior support, and zero bending moment at each exterior support. In the case of a continuous girder, at one or more of the interior points there is a support. In that case, the typical interior equation at that point is simply replaced by an equation for zero deflection at that point. An array of coefficients for a set of equations to be written for a girder with no interior supports is indicated below. Coefficients different from zero are indicated by an x.

<table>
<thead>
<tr>
<th></th>
<th>v₁</th>
<th>v₂</th>
<th>v₃</th>
<th>v₄</th>
<th>v_N+3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>x</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td></td>
</tr>
<tr>
<td>N+1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>x</td>
</tr>
<tr>
<td>N+2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>N+3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>x</td>
</tr>
</tbody>
</table>

A systematic method of solving the equations is evident from the form of the array. The first coefficient of the third equation can be reduced to zero by subtracting a multiple of the first equation. Then, the first non-zero coefficients in the third and fourth equations can be reduced to zero by subtracting a multiple of the second equation. This process of elimination of coefficients can be continued until the last equation is reduced to an equation involving a single unknown deflection which is readily determined. Then, all other deflections can be quickly determined by substitution of values as they are determined into preceding equations. The procedure described above is known as triangularization and back substitution. It is relatively simple and fast here since there is a band width
of only five non-zero coefficients. It is important to note that the values of the coefficients in the shaded part of the array remain zero throughout the entire solution. In programming, use is made of this knowledge, and no memory space is allotted in the computer for these zero coefficients. For storage in the computer, the array is condensed to the form shown to the right of the full array.

In the solution of a suspension bridge problem, it is necessary to solve the girder equation several times, once for each each new trial value of \( H \). The equations (46) for \( a_{ij} \) represent time-consuming calculations, even on a computer. Therefore, it is worthwhile to factor out terms involving \( H \) in each of the coefficients \( a_{ij} \) and \( b_i \). Then it is possible to compute and store values of constants \( a_{i,j} \), such that for each new trial value of \( H \), the coefficients \( a_{i,j} \) and \( b_i \) are computed as follows:

\[
\begin{align*}
a_{i1} &= a_{i1} (1 + DHm) \\
a_{i2} &= a_{i3} + DHa_{i4} \\
a_{i3} &= a_{i5} + DHa_{i6} \\
a_{i4} &= a_{i7} + DHa_{i8} \\
a_{i5} &= a_{i9} (1 + DHm) \\
b_{i} &= a_{i10} + Hob_{i11} + DHa_{i12} + DHHa_{i13}
\end{align*}
\]

The composition of the terms \( a_{i,j} \) is obvious from an examination of equations (46) and need not be written here.

Integration of the Cable Equation

Integration of the cable equation is performed by applying Simpson's Rule, which may be stated as follows:
\[ \int_{A}^{B} f(x) \, dx = \sum_{i=2,4,6}^{N} \frac{d}{6} \left[ f_{i-1} + 4f_i + f_{i+1} \right] \] ... (51)

where, in the case of the cable equation, \( f \) is \( \frac{dh}{dx} \) and the limits of integration are the two ends of the span under consideration.

In the program, the equation for \( \frac{dh}{dx} \) is

\[ \frac{dh}{dx} = \frac{H_L}{AE} \left( \frac{ds}{dx} \right)^3 + \epsilon \left( \frac{ds}{dx} \right)^2 - \frac{dy}{dx} - \frac{dv}{dx} \]

\[ + D \left[ \frac{F_s H_L}{AE} \left( \frac{ds}{dx} \right)^2 \left( \frac{dv}{dx} + \frac{1}{2} \left( \frac{dv}{dx} \right)^2 \right) - \frac{1}{2} \left( \frac{dv}{dx} \right)^2 \right] \] ... (52)

where \( F_s \) is one or zero to include or delete the effect of change of slope in the cable. Equation (52) is equivalent to equation (9) when the factors \( F_s \) and \( D \) are made unity. It should be noted that Simpson's Rule requires that each span be divided into an even number of equal partial lengths. Since the length \( d \) has been made equal to one-twentieth of the main span, it is required that the ratio of side span length to main span length be .1, .2, .3, etc.

Program Linkage

Four main parts of the program which are used more than once are written in the form of subroutines and they are linked up largely by the use of "computed GO TO" statements. One of these subroutines computes the array \( a_{1j} \) for a single span, from the left support at II to the right support at IE. Another subroutine integrates the cable equation for a single span from II to IE. A third subroutine solves the girder equation, either for a single span from II to IE, or for a
continuous girder from II to IE with intermediate supports at 
I=37 and I=17. Each of the above subroutines requires the 
location of II, the initial point I of the calculation and IE, 
the end point I of the calculation. Two of the subroutines 
also require the span length L of the span under consideration 
and the difference in elevation of the anchorages B. Therefore, 
there is a subroutine which specifies the location of II and IE 
for successive spans as well as the values of B and L for each 
span for use in the other subroutines described above. Before 
entering a subroutine, it is necessary to specify the value of 
a constant which is to be used in a "computed GO TO" statement 
to determine the route of the program upon exit from the sub­ 
routine. Figure 7 shows how the linkage is effected to solve 
the girder equation for either a continuous girder or for hinges 
at the towers.

Input Data for the Program

It has been stated that certain refinements in the 
theory may be deleted or included at will by setting the values 
of $F_h$ and $F_s$ equal to zero or unity. Further, the girder may 
be either hinged at the towers or continuous, of constant 
rigidity or variable. In order to specify any combination of 
the above choices, a single code number is read in from the 
first two columns of the first data card. The meaning of the 
code number is shown by the following table:
PORTION OF MAIN LINE PROGRAM

A
CONTINUOUS

CONTINUOUS

HINGED

HINGES

N4 = 2
(CONTINUOUS
GIRDER)

N4 = 1
HINGES

SPECIFY II
AND IE FOR
CONTINUOUS
GIRDER

N2 = 3
(GIRDER
CALC.)

400
24
200

CONTINUATION OF
MAIN PROGRAM

SUBROUTINE 4

400

SOLVE
GIRDER
EQUATION
II TO IE

N4

Determine
v at all
points

N1

1

1

2

2

3

3

201
202
203
204

SUBROUTINE 2

200

SPECIFY
II,IE,L,B
LEFT SIDE
SPAN

N1 = 1

201

SPECIFY
II,IE,L,B
MAIN
SPAN

N1 = 2

202

SPECIFY
II,IE,L,B
RIGHT SIDE
SPAN

N1 = 3

N2

1

2

3

400

24

Figure 7.
<table>
<thead>
<tr>
<th>CODE</th>
<th>Constant EI</th>
<th>Variable EI</th>
<th>$F_h$</th>
<th>$F_S$</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>11</td>
<td>01</td>
<td>1</td>
<td>1</td>
<td>Continuous</td>
</tr>
<tr>
<td>12</td>
<td>02</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>&quot;</td>
</tr>
<tr>
<td>13</td>
<td>03</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>&quot;</td>
</tr>
<tr>
<td>14</td>
<td>04</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>&quot;</td>
</tr>
<tr>
<td>15</td>
<td>05</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>Hinged</td>
</tr>
<tr>
<td>16</td>
<td>06</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>&quot;</td>
</tr>
<tr>
<td>17</td>
<td>07</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>&quot;</td>
</tr>
<tr>
<td>18</td>
<td>08</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>&quot;</td>
</tr>
</tbody>
</table>

The next three data cards each contain the value of a constant related to the elastic properties of the bridge. They are $\frac{H D L^2}{\sqrt{EI}}$, $\frac{E I m}{L^2}$, where $I$ is the girder moment of inertia at mid-span. The above values each occupy the first 14 columns of a data card and are in E14.7 format. There follow three data cards specifying the geometry of the bridge. They are $\frac{F}{L}$ for the main span, the length of the side spans as a ratio of the main span length, and the rise of the side span as a ratio of the main span length. They are in F6.4 format and occupy the first six columns of the cards. Following are two data cards specifying, in E14.7 format, the values of $\epsilon t$ and the anchorage slip as a ratio of the main span length. If the girder rigidity is variable, there is required a set of data cards giving the value of the girder stiffness $EI$ at each point on the girder from the left anchorage to the right anchorage. The values given are in F7.4 format, one to a card, and are the ratio of the girder rigidity at that point to the girder rigidity at the centre of the main span. If the girder has constant $EI$, this set of data cards is omitted. Finally, a set of cards is read
giving the value of the live load on the girder as a ratio \( \frac{P}{w} \). One card is read for each point on the girder from the left anchorage to the right anchorage. The ratios are given in F6.4 format.

Final Notes

In Appendix 1 there is a complete block diagram and a listing of the program as it was written and used in investigations for this thesis. No claim is made that the program is the best one that could have been written for the studies undertaken. However, it did give satisfactory results. The accuracy was good and the amount of information yielded by the program was entirely adequate. It may be that the program will be of little further value in its present form, and therefore it was not considered worthwhile to revise it to the more sophisticated form usually required of a library program. However, it may have some value as a guide to others in the preparation of similar programs, and for that reason it has been preserved here.

The major portion of the computing time was taken to solve the set of equations representing the girder equation. Running time for each trial solution was approximately two minutes. Since three trials were required for an Elastic Theory solution and four or five trials were required for a Deflection Theory solution, the total computing time was approximately fourteen to sixteen minutes.
CHAPTER 4
DETERMINATION OF H

General

It has been shown that analysis of suspension bridges involves the simultaneous solution of two equations in H and v. Since the method of solution must be a trial and error procedure, the calculations are lengthy. However, once the value of H is determined, it is a straightforward matter to compute all deflections, bending moments and shears in the girder. It has further been shown that, for most purposes, it is sufficiently accurate to use equations (11b) and (43) in the determination of H. The equations are repeated here for convenience of reference.

\[ h_B - h_A = \frac{\varepsilon t L_t}{AE} + \frac{H_L L_e}{AE} \int_0^L \frac{dv}{dx} \left( \frac{dy}{dx} - \frac{B}{L} \right) dx \]  \hspace{1cm} \ldots (11b)

\[ \frac{E I d^2 v}{dx^2} - H v - H y = M' \]  \hspace{1cm} \ldots (43)

It will be noted immediately that extension of the cable due to temperature rise and stress elongation has an effect exactly equivalent to a small relative support movement. If the term \( \Delta \) is defined as

\[ \Delta = h_B - h_A - \frac{\varepsilon t L_t}{AE} - \frac{H_L L_e}{AE} \]  \hspace{1cm} \ldots (53)

then \( \Delta \) may be thought of as the equivalent support displacements
of an inextensible cable. Equation (11b) reduces to

\[ \Delta = - \int_0^L \frac{dv}{dx} \left( \frac{dy}{dx} - \frac{B}{L} \right) dx \]  \hspace{1cm} \cdots (54) \]

A method based on equations (43) and (54) is presented here, whereby a designer can determine very quickly the value of \( H \) for a single span or for a multiple span bridge either hinged at the supports or continuous.

**Superposition of Partial Loading Conditions**

Since equation (43) is a linear differential equation, it is permissible to superimpose a number of solutions to (43) for various right hand side expressions. In particular, it is permissible to replace \( H \) on the right hand side of the equation by \( H_0 + H_1 \); then equation (43) may be written

\[ \frac{EId^2v}{dx^2} - Hv = H_0y + H_1y + M' \]  \hspace{1cm} \cdots (55) \]

The solution to (55) may be given by

\[ v = v_0 + v_1 \]

where

\[ \frac{EId^2v_0}{dx^2} - Hv_0 = H_0y + M' \]  \hspace{1cm} \cdots (57) \]

\[ \frac{EId^2v_1}{dx^2} - Hv_1 = H_1y \]  \hspace{1cm} \cdots (58) \]

Then \( \Delta \) may be replaced by \( \Delta_0 + \Delta_1 \) in equation (54) to give

\[ \Delta_0 = - \int_0^L \frac{dv_0}{dx} \left( \frac{dy}{dx} - \frac{B}{L} \right) dx \]  \hspace{1cm} \cdots (59) \]

\[ \Delta_1 = - \int_0^L \frac{dv_1}{dx} \left( \frac{dy}{dx} - \frac{B}{L} \right) dx \]  \hspace{1cm} \cdots (60) \]
The physical significance of equations (57) to (60) is difficult to describe since the superposition is really only a superposition of mathematical solutions and not the sum of two physical states. However, it might be convenient to think of a bridge with movable anchorages restrained by a constant force H. Then the deflections \( v_0 \) and \( \Delta_0 \) are a result of the applied load moments \( M' \) and a portion of the cable tension represented by \( H_0 \). The deflections \( v_1 \) and \( \Delta_1 \) can be attributed to the action of the partial cable tension represented by \( H_1 \). When \( H_0 \) and \( H_1 \) total H and when \( \Delta_0 \) and \( \Delta_1 \) total \( \Delta \), then compatibility exists and the solution for \( v \) is given by the sum of \( v_0 \) and \( v_1 \) for the two partial loading cases.

It will be shown that it is advantageous to make \( \Delta_0 \) equal to zero. Then \( H_0 \) is the portion of cable tension resulting from the applied load acting on a bridge with inextensible cable and immovable supports. Then, \( H_1 \) is the portion of cable tension resulting from \( \Delta \), which, it will be remembered, is the effect of cable stretch and support displacement. Both \( H_0 \) and \( H_1 \) are still functions of the total value of H, but it will be shown that determination of \( H_0 \) is not sensitive to error in an estimated value of H.

**Single Span**

Since superposition of results has been shown to be valid if applied in the manner outlined, it is permissible to make use of the Reciprocal Theorem in determination of cable tension. Figure 8 shows the two cases required for application of the theorem. Case 1 illustrates the deflections \( v_1 \) and \( \Delta_1 \) attributable to the partial cable tension \( H_1 \). This corresponds
Figure 8.
to the solution of equations (58) and (60). Case 2 shows the live load tension $H_L$ due to a unit load on the span. In case 2 the cable is assumed to be inextensible and suspended from immovable supports. This corresponds to a solution of equations (57) and (59) for the case of a single unit load on the span, where $H_0$ is the sum of $H_L$ and the dead load tension $H_D$.

According to the reciprocal theorem, the equation of the influence line for $H_L$ is given by

$$H_L' = -\frac{v_1}{\Delta_1}$$  \hspace{1cm} ... (61)

Equation (58) can be readily solved to give

$$v_1 = \frac{H_1 f L^2}{EI (CL)^2} \left[ \left( \frac{x}{L} \right)^2 - \frac{x}{L} \frac{2}{(CL)^2} \frac{2((1-e^{-CL})e^{CL}+(eCL-1)e^{-CL})}{(CL)^2(eCL-e-CL)} \right]$$  \hspace{1cm} ... (62)

where

$$C = \sqrt{\frac{H}{EI}}$$  \hspace{1cm} ... (63)

Equation (62) can be written in a simpler form as

$$v_1 = \frac{H_1 f L^2}{EI} v_1$$  \hspace{1cm} ... (64)

where

$$v_1 = \frac{4}{(CL)^2} \left[ \left( \frac{x}{L} \right)^2 - \frac{x}{L} \frac{2}{(CL)^2} \frac{2((1-e^{-CL})e^{CL}+(eCL-1)e^{-CL})}{(CL)^2(eCL-e-CL)} \right]$$  \hspace{1cm} ... (65)

When the expression for $v_1$ is differentiated and substitution is made in equation (60), it is found that

$$\Delta_1 = \frac{H_1 f^2 L}{EI} \Delta_1$$  \hspace{1cm} ... (66)
where

\[
\bar{\Delta}_1 = \frac{64}{(CL)^2} \left[ \frac{1}{12} - \frac{4 + e^{CL(CL-2)} - e^{-CL(CL+2)}}{(CL)^3(e^{CL} - e^{-CL})} \right] \quad \ldots (67)
\]

Then the influence line for \( H_L' \) is given by

\[
H_L' = \frac{L}{\bar{v}_1} \left( \frac{\bar{v}_1}{\bar{\Delta}_1} \right) \quad \ldots (68)
\]

Note that \( \bar{v}_1 \) and \( \bar{\Delta}_1 \) are dimensionless and are functions of the dimensionless quantity CL, where C is defined by equation (63). \( \bar{v}_1 \) is, of course, also a function of the relative position on the span. Values of \( \frac{\bar{v}_1}{\bar{\Delta}_1} \), representing the influence line for \( H_L' \) on a single span in dimensionless form, are tabulated in Table 1 of Appendix 2. Also, Figure 10 shows influence lines for values of \((CL)^2 = 1\) and \((CL)^2 = 100\). To facilitate determination of \( H \) for distributed loads, the area under one of the influence curves in Figure 10 has been plotted in the same figure. The curve shows the partial area \( A_{1L} \) to the left of point x, where

\[
A_{1L} = \int_0^x \frac{\bar{v}_1}{\bar{\Delta}_1} \, dx \quad \ldots (69)
\]

Tabulated values are also included in Table 1.

By the use of the curves or tables described above it is possible to determine \( H_L' \) and hence \( H_0 \), the tension that would exist in an inextensible cable with immovable supports. Then a correction \( \delta H \) must be added for the effect of cable stretch and support displacement. Figure 13 shows a plot of \( \bar{\Delta}_1 \) against \((CL)^2\) which can be used to find the correction \( \delta H \). Tabulated values are also included in Table 1.

Equation (53) shows that \( \Delta \) is a function of \( H_L' \), the
unknown live load tension. It is possible to avoid estimating this value initially by rewriting (53) in the form,

\[ \Delta = \Delta - \frac{\delta H L_e}{AE} \]  \hspace{1cm} \ldots (70) 

where

\[ \Delta = \frac{h_B - h_A - e t L_e - h_L L_e}{AE} \] \hspace{1cm} \ldots (71) 

The correction \( \delta H \) is, in the case of a single span

\[ \delta H = \frac{\Delta H_1}{\Delta_1} \] \hspace{1cm} \ldots (72) 

where it has been shown in equation (66) that

\[ \frac{H_1}{\Delta_1} = \frac{1}{f^2 L \frac{\Delta_1}{EI}} \] \hspace{1cm} \ldots (73) 

Equations (73) and (71) can be substituted in equation (72) to give

\[ \delta H = \frac{\Delta - \frac{\delta H L_e}{AE}}{f^2 L \frac{\Delta_1}{EI}} \] \hspace{1cm} \ldots (74) 

Equation (74) can then be solved for \( \delta H \) to give

\[ \delta H = \frac{\Delta^1}{f^2 L \frac{\Delta_1}{EI} + \frac{L_e}{AE}} \] \hspace{1cm} \ldots (75) 

In order to determine \( \delta H \), it is necessary to compute \( \Delta^1 \) once for \( H_1 \) and estimate the value of \( H \) to find \( \Delta_1 \) from Figure 13. Then \( \delta H \) and hence \( H \) can be computed from equation (75). If the computed value of \( H \) does not agree with the estimated value, another computation must be made with a different value of
Numerical examples in Appendix 3 show that the iteration converges rapidly.

Determination of $H_L'$ depended on an initial estimate of $H$ and so an iterative procedure is implied for the determination of $H_L'$. However, iteration is usually unnecessary because the influence lines are relatively independent of the value of CL. This is illustrated by a study of the extreme values of CL. At one extreme as CL approaches zero, the elastic theory becomes valid and the influence line is given by

$$H_L' = \frac{L}{f} \left[ \frac{x}{L} - \frac{2}{3} \left( \frac{x}{L} \right)^3 + \frac{1}{4} \left( \frac{x}{L} \right)^4 \right]$$ ... (76)

At the other extreme as CL becomes infinitely large, the girder is so flexible as to offer no resistance to deflection, and it can be shown that the influence line equation is

$$H_L' = \frac{L}{f} \left[ \frac{x}{L} - \left( \frac{x}{L} \right)^2 \right]$$ ... (77)

Figure 9 shows influence lines for $H_L'$ for the extreme values of CL. Further investigation shows that the influence line ordinates for all values of CL lie within the range defined by the extreme values except in the region where the curves intersect. It is apparent that no significant error in $H_L'$ will be introduced by inaccurate estimates of the value of H. For all values of CL the area under the curve must be $\frac{L^2}{8f}$. This point becomes clearer when it is realized that a uniform load $p$ covering the entire span introduced no girder bending moment and results in a cable tension with a horizontal component equal to $\frac{pL^2}{8f}$. 
Figure 9.
Three-Span Bridge with Hinged Supports

It is a simple matter to extend the application of the method described above to the case of a three-span bridge with hinges at the girder supports. Equations (58) and (60) must be solved to find a total $\Delta_1$ for all the spans which is

$$\Delta_1 = \frac{H_{1r}^2L^2\Delta_1}{EI} \left[1 + \frac{2a^3b(\Delta_1)_s}{\Delta_1}\right] \quad \ldots (78)$$

where: $a = \text{ratio of side span length } L_s \text{ to main span length } L$

$$b = \frac{L_s^2}{L^2} \frac{EI}{EIS}$$

$$(\Delta_1)_s = \frac{\Delta_1}{(bCL)}$$

Figure 14(a) shows curves of $b(\Delta_1)_s$ against $(CL)^2$ for selected values of $b$.

The influence line for $H_L'$ in the main span is then

$$H_L' = \frac{L}{f} \left(\frac{\bar{v}_1}{\Delta_1}\right) X \quad \ldots (79)$$

where

$$X = \frac{1}{1 + \frac{2a^3b(\Delta_1)_s}{\Delta_1}} \quad \ldots (80)$$

The influence line for $H_L'$ in the side span is

$$H_L' = \frac{L}{f} \left(\frac{\bar{v}_1}{\Delta_1}\right)_s X_s \quad \ldots (81)$$

The subscript $s$ in the term $\left(\frac{\bar{v}_1}{\Delta_1}\right)_s$ indicates that the influence curve for the side span is generally different from that for the main span, due to the different value of $CL$. The multiplier $X_s$ for the side span is given by
\[ X_s = \frac{a^3 b (\Delta_1)_s}{\frac{\Delta_1}{1 + 2a^3 b (\Delta_1)_s}} = \frac{(1 - X)}{2} \quad \ldots \quad (82) \]

In equations (79) and (81), \( L \) and \( f \) are the values of span length and sag for the main span. Figure 14(b) shows \( X \) and \( X_s \) plotted against \( \frac{b (\Delta_1)_s}{\Delta_1} \) for selected values of \( a \). It is intended that Figures 14(a) and 14(b) be used together in order to determine \( X_s \) and \( X \), the multipliers for the influence line ordinates from Figure 10.

In the case of the multiple span bridge, it can be shown that the equation for \( \delta H \) is given by

\[ \delta H = \Delta \left( \frac{\frac{H_1}{\Delta_1}}{X} \right) \quad \ldots \quad (83) \]

As in the case of the single span, substitution can be made from equations (71) and (73) and the resulting equation can be solved for \( \delta H \) to give

\[ \delta H = \frac{\Delta'}{f^2 L \frac{\Delta_1}{\Delta_1} + \frac{L_e}{EI X} \frac{A_b}{AE}} \quad \ldots \quad (84) \]

Values of \( \frac{b (\Delta_1)_s}{\Delta_1} \) are tabulated in Table 2, Appendix 2, and Appendix 3 contains a numerical example showing how the tables or Figures 10 and 14 can be used to determine \( H \) for a three-span bridge with hinged supports. It is clear that the general procedure is the same as that for a single span.

**Three-Span Bridge with Continuous Girder**

Up to this point, consideration has been restricted
to single and multiple span bridges with hinges at all supports. The problem becomes somewhat more complicated if the girder is continuous at the supports, but it may be solved in a similar manner. Equation (43) can be replaced by equation (57) and by equations (85) to (87) below:

\[ \frac{EId^2v_1}{dx^2} - Hv_1 - H_1y = 0 \] \hfill (85)

\[ \frac{EId^2v_2}{dx^2} - Hv_2 - \frac{M_2x}{L} = 0 \] \hfill (86)

\[ \frac{EId^2v_3}{dx^2} - Hv_3 - \frac{M_3(L-x)}{L} = 0 \] \hfill (87)

Equations (85) to (87) represent a single span with no applied load moments but with end moments \( M_2 \) and \( M_3 \) resulting from the action of the partial tension \( H_1 \) acting on a continuous girder. The three equations together are equivalent to the single equation (58) for a single span. In analysis of a continuous structure, it is necessary to equalize end slopes where continuity exists. Equation (85) has been solved for the case of a single span and it can be shown that the slopes are

\[ \frac{dv_1}{dx} = \frac{H_1fL}{EI} \left( \frac{dv_1}{dx} \right) \] \hfill (88)

where

\[ \left( \frac{dv_1}{dx} \right)_c = - \left( \frac{dv_1}{dx} \right)_L = \frac{4}{(CL)^3} \left[ \frac{4 + e^{CL}(CL-2) - e^{-CL}(CL+2)}{e^{CL} - e^{-CL}} \right] \] \hfill (89)

Equation (86) can be solved to give

\[ v_2 = \frac{M_2 L^2}{EI} \] \hfill (90)

where
\[ \bar{v}_2 = \frac{1}{(CL)^2} \left[ \frac{x}{L} - \frac{e^{Cx} - e^{-Cx}}{e^{CL} - e^{-CL}} \right] \] ... (91)

Then the end slopes can be found from
\[ \frac{dv_2}{dx} = \frac{M_2 L}{EI} \frac{d\bar{v}_2}{dx} \] ... (92)

where
\[
\begin{align*}
\left( \frac{d\bar{v}_2}{dx} \right)_o &= \frac{1}{CL} \left[ \frac{1}{CL} - \frac{2}{e^{CL} - e^{-CL}} \right] \\
\left( \frac{d\bar{v}_2}{dx} \right)_L &= \frac{1}{CL} \left[ \frac{1}{CL} - \frac{e^{CL} + e^{-CL}}{e^{CL} - e^{-CL}} \right]
\end{align*}
\] ... (93) ... (94)

From symmetry of the girder, it is clear that
\[ (v_3)_x = (v_2)_L-x \] ... (95)

\[
\begin{align*}
\left( \frac{d\bar{v}_3}{dx} \right)_o &= - \left( \frac{d\bar{v}_2}{dx} \right)_L \\
\left( \frac{d\bar{v}_3}{dx} \right)_L &= - \left( \frac{d\bar{v}_2}{dx} \right)_o
\end{align*}
\] ... (96) ... (97)

In the case of a symmetrical three-span bridge, with no applied load moment, the bending moments at the towers are equal. The unknown moment \( M \) can be found by equalizing end slopes at the towers and is given by
\[
M = \frac{H_1 f \int_a^b \left( \frac{d\bar{v}_1}{dx} \right) L \left( \frac{d\bar{v}_1}{dx} \right)_o}{\left[ \left( \frac{d\bar{v}_2}{dx} \right)_o + \left( \frac{d\bar{v}_3}{dx} \right)_L \right] - \frac{b}{a} \left( \frac{d\bar{v}_2}{dx} \right)_L} \] ... (98)

It can be shown that in the case of an extremely stiff girder where the elastic theory is valid, equation (98) can be reduced to
In the iterative procedure required to compute \( H \), it will be necessary to compute \( M \) a number of times, and it will be shown that it is advantageous to compute \( M_e \) which is independent of \( H \) and correct by means of a multiplier \( K \) which must be determined for each new trial value of \( H \). Values of \( K \) are tabulated in Appendix 2 for selected values of \( a, b \) and \((CL)^2\). \( K \) is also plotted against these parameters in Figure 16. When \( M_e \) is computed and \( K \) is found from the tables or curves, \( M \) is found from

\[
M = K M_e
\]

... (100)

From the solutions to equations (86) and (87), it is found that

\[
\Delta_2 = \frac{M_2 fL}{EI} \bar{\Delta}_2
\]

... (101)

\[
\Delta_3 = \frac{M_3 fL}{EI} \bar{\Delta}_3
\]

... (102)

where

\[
\bar{\Delta}_2 = \bar{\Delta}_3 = -\frac{4}{(CL)^3} \left[ \frac{4 + e^{CL}(CL-2) - e^{-CL}(CL+2)}{e^{CL} - e^{-CL}} \right]
\]

... (103)

The total value of the support movement corresponding to the solution of equations (85) to (87) for all three spans is given by

\[
\Delta_t = \frac{H_1 f^2 L T}{EI} \bar{\Delta}_1
\]

... (104)

where

\[
T = 1 + \frac{2a^3 b(\bar{\Delta}_1)_s}{\bar{\Delta}_1} + \bar{M} \left[ \frac{2\bar{\Delta}_2}{\bar{\Delta}_1} + \frac{2ab(\bar{\Delta}_2)_s}{\bar{\Delta}_1} \right]
\]

... (105)
where
\[ M = K \begin{bmatrix} 1 + ab & \frac{3}{\Delta_1} & \frac{-b}{2} & \frac{a}{\Delta_1} \end{bmatrix} \] ... (106)

Values of \( \frac{b(\Delta_1)}{\Delta_1} \) are the same as those used for the case of a three-span bridge with hinged supports and are found in Appendix 2 and Figure 14(a). Values of \( \frac{\Delta_2}{\Delta_1} \) and \( \frac{b(\Delta_2)}{\Delta_1} \) are plotted against \( (CL)^2 \) in Figure 15 for selected values of \( b \) and are also included in Appendix 2.

Influence lines for \( H_L' \) in a continuous suspension bridge must be found by superimposing two influence lines. The first is the same influence line used for hinged girders. The second is a correction for continuity. In the case of the main span the influence line is given by
\[ H_L' = \frac{L}{f} \left[ \frac{X}{\Delta_1} \frac{\bar{v}_1}{\Delta_1} + \frac{Y}{\Delta_1} \frac{\bar{v}_2 + \bar{v}_3}{\Delta_2 + \Delta_3} \right] \] ... (107)

where
\[ X = \frac{1}{T} \] ... (108)
\[ Y = \frac{M}{T} \frac{2\Delta_2}{\Delta_1} \] ... (109)

Curves of \( \frac{\bar{v}_2 + \bar{v}_3}{\Delta_2 + \Delta_3} \) are shown plotted in Figure 12 and tabulated values are found in Appendix 2.

For the left side span, the influence line for \( H_L' \) is given by
\[ H_L' = \frac{L}{f} \left[ X_s \left( \frac{\bar{v}_1}{\Delta_1} \right)_s + Y_s \left( \frac{\bar{v}_2}{\Delta_2} \right)_s \right] \] ... (110)
where
\[ X_s = \frac{a^3b}{T} \frac{(\Delta_1)_s}{\Delta_1} \] ...
\[ (111) \]
\[ Y_s = \frac{\bar{V}_{ab}}{T} \frac{(\Delta_2)_s}{\Delta_1} \] ...
\[ (112) \]

Figure 11 shows curves of \( \frac{\bar{V}_2}{\Delta_2} \) and Appendix 2 has tabulated values for selected values of \((CL)^2\). The influence line for the right side span is of course similar to that for the left side span but opposite hand.

The numerical example in Appendix 3 shows that the iteration procedure for determination of \( H \) converges rapidly and use of the tables or curves makes the calculations simple.

Variable \( EI \)

It can be seen that the solutions to equations (58), (86) and (87) are given for the special case in which the girder rigidity \( EI \) within the span is constant. Therefore, use of the constants tabulated in Appendix 2 depends on the assumption of constant \( EI \) within each span.

It is possible to solve the equations, at least numerically, for other particular variations in girder rigidity and tabulate similar data for use in analysis. A suitable approach might be to determine a typical mode of variation of girder stiffness such that most or all suspension bridge girders have a stiffness variation which lies within a range defined by two sets of tabulated data. Analysis constants might then be determined by interpolation between the tabulated values.

However, it is suggested that the assumption of constant
girder rigidity is a reasonable and desirable one for hand calculations. Some numerical examples were solved using the computer program described in the preceding chapter. A three-span bridge was analysed as a continuous girder and with hinges at the supports. The main span girder stiffness varied from a minimum of .5 times the midspan stiffness at the towers to a maximum of 1.5 times the mid-span stiffness at the quarter points. The same bridge was analysed assuming a constant girder rigidity equal to the average value. The results of the study indicated that a reasonably accurate value of $H_L$ can be determined by assuming an average value for the girder stiffness. Errors in $H_L$ encountered were less than 2 per cent. Therefore, it is recommended that for all hand calculations, a constant girder rigidity $EI$ equal to the average value be assumed for each span in order to determine the value for $H$. 
CHAPTER 5
CONCLUSIONS

It is doubtful that Deflection Theory analysis of suspension bridges by hand calculations will ever be a simple procedure or one that the structural engineer will approach enthusiastically. However, it has been found that Elastic Theory solutions are too inaccurate even for preliminary design in many cases. Therefore, the need for Deflection Theory solutions does exist. It would be expedient to turn the task over to a computer and avoid all hand calculations, but that is not always a practical procedure. During the early stages of a design, there will always be a necessity for some hand calculations, and it is hoped that these will be made somewhat easier with the methods presented here.

It was apparent in the chapter on Theory and Refinements that the simpler Deflection Theory represented by equations (43) and (1lb) gives results of high accuracy. It is questionable whether the additional accuracy attained by further refinement is justified in hand calculations, and it is recommended that the more refined versions of the Deflection Theory be reserved for computer analysis. Equation (43) is relatively attractive for use in hand calculations since solutions are to be found tabulated in Steinman's text* on suspension bridges.

* Reference (1)
Once the cable tension for a total loading case is known it is possible to superimpose solutions for partial loading conditions as given in Steinman's text.

Equations (43) and (11b) form the basic theory for the method presented of determining the total value of cable tension. No approximation not inherent in the above equations is made for the method given and hence the value of cable tension calculated by this method has a relatively high accuracy.

It can be seen in the sample calculations given in Appendix 3 that the method is extremely easy to apply, especially in the case of a bridge with girders hinged at the supports. A continuous girder presents some additional difficulty but no more than should be expected. Continuity in any structure is partly paid for by effort in analysis.

The first step toward an accurate, simplified method of analysing suspension bridges must be an accurate, simple method of determining the cable tension. It is believed that the method developed in Chapter 3 and illustrated in sample calculations in Appendix 3 meets the objectives of accuracy and simplicity. Therefore, the method should be useful as part of a total method of analysis.

One approach to a simplified method of analysis might be a determination and tabulation of information on amplification factors in a manner similar to that described by A. Franklin in his thesis on non-linear arches.

Whatever methods are used to complete the analysis, it is important to recognize that methods of superposition are valid in suspension bridge analysis, despite the non-linear behavior of suspension bridges. So long as the total value of the cable
tension is known and applied in the equations for the partial loadings, the bending moments and deflections for the partial loading cases may be superimposed to give the total values. The key, then, is the determination of the total cable tension, and a simple, accurate method of determining the cable tension has been presented in this work.
APPENDIX 1
COMPUTER PROGRAM

KEN RICHMOND
CIVIL ENGINEERING
THESIS

PROGRAM TO ANALYSE SUSPENSION BRIDGES AND COMPUTE MAGNIFICATION FACTORS.

MAIN LINE PROGRAM

DIMENSION EI(53), P(53), AB(13,53), A(53,5)
DIMENSION E(53), B(53), BM(2,53)

43 READ 1, KODE, C, R, S
1 FORMAT (12 / (E14.7))
READ 2, F, SIDE, RISE, T, SLIP
2 FORMAT (F6.4 / F6.4 / F6.4 / (E14.7))
PRINT 3, KODE, R, S
3 FORMAT(/6H KODE= 14,3H R= E14.7 ,3H S= E14.7 /)
PRINT 4, F, SIDE, RISE
4 FORMAT(3H F= F7.4, 6H SIDE= F11.4 , 6H RISE= F11.4 /)
PRINT 5, T, SLIP
5 FORMAT(3H T= E17.7 , 6H SLIP= E17.7 )
EIO =1.0 / (8.0 * F * C )
1A= 17.0 - SIDE * 20.0
ID= 37 + 17 - 1A
IF ( KODE - 10 ) 8, 8, 6
6 KODE = KODE - 10
DO 7 I = IA , ID , 1
7 EI(I) = EIO
GO TO 11
8 DO 9 I = IA , ID , 1
9 READ 10, EI(I)
10 FORMAT (F7.4)
9 EI(I) = EIO * EI(I)
11 PRINT 12
12 FORMAT(/23H I P EI /)
DO 13 I = IA, ID, 1
13 READ 14, P(I)
14 FORMAT (F8.4)
13 PRINT 15 , I , P(I) , EI(I)
15 FORMAT (13 , F9.4 , E17.7 )
DF = 0.0
SF = 0.0
GO TO ( 16,17,18,19,16,17,18,19 ), KODE
16 DF = 1.0
17 SF = 1.0
18 GO TO 19
19 N2 = 1
20 D = 0.0
21 HT = HL + HD
22 N4 = 2
23 N4 = 1
24 ERROR = 0.0
25 ERROR = ERROR + SLIP
26 FORMAT (/#HL= E17.7 , 7H ERROR= E17.7 )
27 HL1 = HL
28 IF(ABS(ERROR)-1.0E-5) 30,30,29
29 DELH = ERROR *(HL - HL1)/(ERROR - ERR)
30 E(IA-1) = -E(IA+1)
31 BM(N,1) = (400.0*(2.0*E(1)-E(1-1)-E(1+1)))*(1.0+D*HT*SM)
32 BM(N,17) = 0.0
33 SUM = 0
34 Q = ID - IA + 1
35 BM(N,IA) = 0.0
36 BM(N,ID) = 0.0
37 IF( (KODE -4 ) ) 33,33,32
38 BM(N,37) = 0.0
39 SUM = SUM + EI(1)
AVG = SUM / Q
CDL = HD / AVG
CLL = HL / AVG
CTOT = HT / AVG
PRINT 35, CDL, CLL, CTOT
35 FORMAT ( / 5H CDL= El17.7 , 5H CLL= El17.7 , 6H CTOT= El17.7 )
36 D = 1.0
PRINT 41
41 FORMAT ( / 5H DEFLECTION BM ELASTIC BM PHI / )
DO 37 I = IA, ID, 1
IF ( BM(1,I) ) 38,39,38
38 PHI = 1.0
GO TO 37
39 PHI = BM(2,I) / BM(1,I)
GO TO 37
37 PRINT 42, I, BM(2,I), BM(1,I), PHI, E(I)
42 FORMAT ( I3#F14.8, F17.8, F17.8, E20.7 )
GO TO 43

C
C SUBROUTINE 1
C
100 EI(IA-1) = EI(IA+1)
EI(ID+1) = EI(1D-1)
SM = S / EI0
D2Y = -8.0 * F
RAE = R * R / EI0
DO 101 I = IA, IE, 1
DEI = 10.0 * (EI(I+1) - EI(I-1))
D2EI = 400.0 * (EI(I-1) - 2.0 * EI(I) + EI(I+1))
X = 1 - I
X = .05 * X
DY = 4.0 * F * (AL - 2.0 * X ) - BL / AL
DS = SQR(1.0 + DY * DY)
AB(1,I) = 1.6E5 * EI(I) - 8.0E3 * DEI
AB(2,I) = -6.4E5 * EI(I) + 1.6E4 * DEI + 400.0 * D2EI
AB(3,I) = SM*AB(2,I) + 400.0*(-1.0-DF*DY*DY)+20.0*DF*DY*D2Y
AB(4,I) = 9.6E5 * EI(I) - 800.0 * D2EI
AB(5,I) = -6.4E5 * EI(I) + 1.6E4 * DEI + 400.0 * D2EI
AB(6,I) = SM*AB(5,I) + 800.0 * (1.0 + DF*DY*DY)
AB(7,I) = -6.4E5 * EI(I) - 1.6E4 * DEI + 400.0 * D2EI
AB(8,I) = SM*AB(7,I) + 400.0 * (-1.0-DF*DY*DY)-20.0*DF*DY*D2Y
AB(9,I) = 1.6E5 * EI(I) + 8.0E3 * DEI
AB(10,I) = (P(I)+1.0) * (1.0 - SM * D2EI)
AB(11,I) = D2Y * (1.0 - SM * D2EI)
AB(12,I) = -DF*T*D2Y*(-3.0*DY*DY + 1.0)
101 AB(13,I) = -DF*RAE*D2Y*DS*(4.0*DY*DY + 1.0)
GO TO (201,202,203), N1

C
C SUBROUTINE 2
C
200 I1 = IA
IE = 17
AL = SIDE
BL = RISE
N1 = 1
GO TO 204

201 II = 17
IE = 37
AL = 1.0
BL = 0.0
N1 = 2
GO TO 204

202 II = 37
IE = 1D
AL = SIDE
BL = -RISE
N1 = 3

204 GO TO (100, 300, 400), N2
203 GO TO (20, 25, 24), N2

C SUBROUTINE 3

300 DO 301 I = II, IE, 1
  X = I - II
  X = .05 * X
  DY = 4.0 * F * (AL - 2.0 * X) - BL / AL
  DS = SQRT(1.0 + DY * DY)
  RAE = R * R / EIO
  IF (I - II) 302, 302, 303
  302 DE = 20.0 * E(I+1)
  GO TO 307
  303 IF (I - IE) 305, 304, 304
  304 DE = -20.0 * E(I-1)
  GO TO 307
  305 DE = 10.0 * (E(I+1) - E(I-1))
  307 B(I) = HL*RAE*SF*DS*DS*(DY*DE + .5*DE*DE) -.5*DE*DE
  B(I) = 2*B(I) + RAE*HL*DS**3 + T*DS*DS - DY*DE
  IS = IE -2
  DO 306 I = II, IS, 2
  306 ERROR = ERROR + (.05/3.0) * (B(I) + 4.0 * B(I+1) + B(I+2))
  GO TO (201, 202, 203), N1

C SUBROUTINE 4

400 DO 401 I = II, IE, 1
  A(I,1) = AB(1,1) * (1.0 + SM*D*HT)
  A(I,2) = AB(3,1) + D*HT*AB(4,1)
  A(I,3) = AB(5,1) + D*HT*AB(6,1)
  A(I,4) = AB(7,1) + D*HT*AB(8,1)
  A(I,5) = AB(9,1) * (1.0 + SM*D*HT)
  401 B(I) = AB(10,1) + HT*AB(11,1) + D*HT*AB(12,1) + D*HT*HL*AB(13,1)
  IM = II -1
  IN = IE +1
  B(IM) = -(P(I1+1) + 1.0 + HT*D2Y) * (SM / (1.0 + HT * SM))
B(11) = 0.0
B(17) = 0.0
B(37) = 0.0
B(1E) = 0.0
B(IN) = -(P(1E-1) + 1.0 + HT * D2Y) * (SM / (1.0 + HT * SM))
DO 402 J = 1, 5, 1
A(11, J) = 0.0
A(17, J) = 0.0
A(37, J) = 0.0
A(1E, J) = 0.0
A(11, 3) = 1.0
A(17, 3) = 1.0
A(37, 3) = 1.0
A(1E, 3) = 1.0
A(IN, 3) = 400.0
A(IN, 4) = -800.0
A(IN, 5) = 400.0
A(IN, 1) = 400.0
A(IN, 2) = -800.0
A(IN, 3) = 400.0
IK = IN - 2
DO 403 I = IM, IK, 1
R1 = A(I+1, 2)/A(I, 3)
R2 = A(I+2, 1)/A(I, 3)
DO 404 J = 3, 4, 1
A(I+1, J) = A(I+1, J) - R1 * A(I, J+1)
B(I+1) = B(I+1) - R1 * B(I)
DO 405 J = 2, 3, 1
A(I+2, J) = A(I+2, J) - R2 * A(I, J+2)
B(I+2) = B(I+2) - R2 * B(I)
R1 = A(IN, 2) / A(1E, 3)
E(IN) = (B(IN) - R1 * B(1E)) / (A(IN, 3) - R1 * A(1E, 4))
E(1E) = (B(1E) - E(IN) * A(1E, 4)) / A(1E, 3)
I = IK
DO 406 E(I) = (B(I) - E(I+1) * A(I, 4) - E(I+2) * A(I, 5)) / A(I, 3)
I = I - 1
IF (I - IM) 407, 407, 406
407 GO TO (408, 24), N4
408 GO TO (201, 202, 203), N1
END
MAIN LINE PROGRAM

READ: KODE, C, R, S, F, SIDE, RISE, T, SLIP

PRINT: KODE, R, S, F, SIDE, RISE, T, SLIP

COMPUTE EIO, IA, ID

KODE > 10

KODE = KODE - 10

I = IA, ID

EIO

I = IA, ID

READ EI(I)

COMPUTE EI(I)

I = IA, ID

READ P(I)

PRINT I, P(I), EI(I)

A

SHEET 1 OF 6
A

SF = 0
DF = 0

NOTE: SF = F_s
DF = F_h

KODE

1
2
5
6

DF = 1

SF = 1

3
4
7
8

DF = 1

N2 = 1

200

20

COMPUTE
H_D

ESTIMATE
H_L = .1wL

N = 1, 2

B

SWITCH
K = 0

(Continous) KODE ≤ 4

KODE

KODE > 4 (Hinged)

II = IA
IE = ID

N4 = 2

400

N4 = 1
N2 = 3

200

SHEET 2 OF 6
TO FOLLOW PAGE 59

```
ERROR = 0

N2 = 2

ERROR = ERROR + SLIP

PRINT: HL, ERROR

SWITCH

K = 0

SWITCH

K = 1

ERROR < 10^-6

|ERROR| < 10^-6

COMPUTE BM (N, IA) TO BM (N, ID)

INTERPOLATE NEW ESTIMATE

HL

NEW ESTIMATE

HL = HL + H_D

COMPUTE AVERAGE EI

CDL, CLL, CTOT

PRINT CDL, CLL, CTOT

D = 1

N = 1, 2

B

C

SHEET 3 OF 6
```
SUBROUTINE I.

Compute and store constants AB(I,I) to AB(13,I)
SUBROUTINE 2

SUBROUTINE 3

Integrate cable equation by Simpson's Rule
SUBROUTINE 4

Compute deflections at all points II-1 to IE+1 and store in E(I)

COMPUTE
A(I,1) TO A(I,5) & B(I)

COMPUTE BOUNDARY CONDITIONS

REDUCE
A(I+1,2)
& A(I+2,1)
TO ZERO

REDUCE
A(IE+1,2)
TO ZERO

COMPUTE
E(IE+1), E(IE)

I = IE - 1

COMPUTE
E(I)

I = I - 1

I ≠ II - 1

I: II - 1

N4

N1

201

202

203

24

1

2

3
### TABLE 1. INFLUENCE CURVES

\[
F_1 = -\frac{\bar{v}_1}{\Delta_1} \\
F_s = -\frac{\bar{v}_2}{\Delta_2} \\
F_m = -\frac{\bar{v}_2 + \bar{v}_3}{\Delta_2 + \Delta_3} \\
A_1 = \int_0^L \frac{\bar{v}_1}{\Delta_1} \, dx \\
A_s = \int_0^x \frac{\bar{v}_2}{\Delta_2} \, dx \\
A_m = \int_0^x \frac{\bar{v}_2 + \bar{v}_3}{\Delta_2 + \Delta_3} \, dx
\]

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<th>\bar{\Delta}_4</th>
<th>\frac{\Delta}{L}</th>
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<th>F_s</th>
<th>F_m</th>
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## TABLE 4. \( K = \frac{M}{M_c} \) TOWER MOMENTS

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Example 1. Single Span

Given:

Length of span \( L = 1,000 \) ft.
Sag of cable \( f = 100 \) ft.
\( EI = 1.5 \times 10^8 \) K ft.\(^2\)/girder
\( AE \) of cable = \( 7.0 \times 10^5 \) K
\( *L_e = 1,082 \) ft.
\( *L_t = 1,054 \) ft.
Dead load \( w = 1.0 \) K/ft.
Live load \( p = .4 \) K/ft, distributed where shown
+ 25 K at quarter point as shown
\( \epsilon_t = 3.25 \times 10^{-4} \)
Support displacement \( h_B - h_A = .5 \) ft.

* Formulae given in References (1) and (6)
Step 1. Compute $H_D$ and design constants.

$$H_D = \frac{wL^2}{8f} = \frac{1.0 \times (1000)^2}{8 \times (100)} = 1250 \text{ K}$$

$$L^2 = \frac{(1000)^2}{EI} = 0.00667$$

$$\frac{L^2}{EI} = \frac{1.5 \times (10^8)}{1250 \times (0.00667)} =$$

$$\frac{L_e}{AE} = \frac{1082}{7 \times (10^5)} = 0.0015$$

$$\frac{r^2L}{EI} = \frac{(100)^2 \times (1000)}{1.5 \times (10^8)} = 0.0667$$

Step 2. Compute $H_L'$.

Here, it is necessary to estimate $H$ in order to select an influence line. It is sufficiently accurate to use $H = H_D$. Then

$$HL^2 = 1250 \times (0.00667) = 8.3$$

$$\frac{HL^2}{EI}$$

$H_L'$ is found from the influence lines plotted in Figure 10. The influence line ordinate at the location of the point load is $0.139 L$. The area under the curve from $0.25$ to $0.50$ is $(0.0625 - 0.0185) \times L^2$. Therefore

$$H_L' = \frac{25 \times (0.139) \times (1000)}{100} + \frac{0.4 \times (0.0625 - 0.0185) \times (1000)^2}{100}$$

$$= 35 + 176 = 211 \text{ K}$$

Step 3. Compute $\Delta'$.

$$\Delta' = h_B - h_A - \frac{tL_t}{\nu} - H_L' \frac{L_e}{\nu}$$

$$= -0.5 - 3.25 \times (10^{-4}) \times (1054) - 211 \times (0.0015)$$

$$= -1.17 \text{ ft.}$$
Step 4. Compute $\delta H$.

Here, another estimate of $H$ must be made in order to determine $\Delta_1$. It is sufficiently accurate to estimate $H = H_D + H_L'$. Then

$$HL^2 = (1250 + 211) \cdot 0.00667 = 9.75$$

$\frac{EL}{EI}$

Figure 13 is used to determine $\Delta_1$ for $\frac{HL^2}{EI} = 9.75$ and it is found that $\Delta_1 = 0.277$. Then $\delta H$ can be found from

$$\delta H = \frac{\Delta_1}{\frac{L_d^2}{EI} + \frac{L_e}{AE}} = \frac{-1.17}{0.277 \cdot (0.0667) + 0.0015} = -59K$$

Step 5. Compute $H$.

$$H = H_D + H_L' + \delta H$$

$$= 1250 + 211 - 59 = 1402 \text{ K}$$

Step 6. Compare the value of $H$ computed in step 5 with the value estimated in step 4, and repeat steps 4 and 5 until they are the same.

$$HL^2 = 1402 \cdot (0.00667) = 9.37$$

$\frac{EL}{EI}$

From Figure 13, $\Delta_1 = 0.284$

$$\delta H = \frac{-1.17}{0.284 \cdot (0.0667) + 0.0015} = -58 \text{ K}$$

$$H = 1403 \text{ K}$$

In the case of a single span, one repetition of steps 4 and 5 should give a sufficiently accurate value of $H$. 
Example 2. Three-Span Bridge with Hinged Supports

Given:

Main span length $L = 1000$ ft.
Main span sag $f = 100$ ft.
Side span length $L_s = 500$ ft.
$EI$ main span $= 5.0 \times 10^7$ K ft.$^2$/girder
$EI$ side span $= 2.5 \times 10^7$ K ft.$^2$/girder
AE of cable $= 7.5 \times 10^5$ K
Side span rise $= 112.1$ ft.
$L_e = 2181$ ft.
$L_t = 2116$ ft.
Dead load $1$ K/ft.
Live load $.4$ K/ft. on main span as shown
$+ 25$ K on side span where shown
$\epsilon t = 3.25 \times 10^{-4}$
Support displacement $H_B - h_A = 0$
Step 1. Compute $H_D$ and design constants.

From equation (39)

$$H_D = \frac{1.0 \, (1000)^2}{8 \, (100)} \cdot 1250 \, K$$

$$L^2 = \frac{(1000)^2}{EI} = 0.020$$

$$a = \frac{500}{1000} = 0.5$$

$$b = \frac{500 \, 2 \, 5.0 \, (10^7)}{1000 \, 2.5 \, (10^7)} = 0.5$$

$$f^2L = \frac{(100)^2 \, (1000)}{EI} = 0.20$$

Step 2. Compute $H_L'$

Estimate $H = H_D = 1250 \, K$

$$HL^2 = 1250 \, (0.020) = 25.0$$

$$EI$$

Figure 14(a) shows values of $\frac{b(\Delta L)}{\Delta L}$ plotted against $\frac{HL^2}{EI}$ for selected values of b. For $HL^2 = 25.0$ and $b = 0.5$, the ordinate $\frac{b(\Delta L)}{\Delta L}$ is 0.780. Figure 14(b) shows values of the multipliers $X$ and $X_s$ plotted as abscissae against the ordinate $\frac{b(\Delta L)}{\Delta L}$ for selected values of a. For $a = 0.5$ and $\frac{b(\Delta L)}{\Delta L} = 0.780$, the multipliers are:

$$X = 0.836$$

$$X_s = 0.082$$

The main span influence line area from 0.00 to 0.75 is found from Figure 10 to be $0.1063 \frac{L^2}{f} X$. Therefore the contribution to $H_L'$ from the main span is
The influence line ordinate for the point load on the side span is \(0.194 \frac{L}{f} x_g\). Therefore, the contribution to \(H_L'\) from the side span is

\[
H_L' = 0.194 \left(1000\right) \left(0.082\right) \left(25\ K\right) \frac{4\ K}{100} = 4\ K
\]

The total \(H_L' = 356 + 4 = 360\ K\).

Step 3. Compute \(\Delta'\).

From equation (71)

\[
\Delta' = 0 - 3.25 \left(10^{-4}\right) \left(2116\right) - 360 \left(0.0029\right) = -1.72\ ft.
\]

Step 4. Compute \(\delta H\).

Estimate \(H = H_D + H_L' = 1250 + 360 = 1610\ K\)

\[
HL^2 = 1610 \left(0.020\right) = 32.2\ 
\]

\[
EI
\]

Figure 13 shows \(\overline{\Delta_1}\) plotted against \(\frac{HL^2}{EI}\). For \(\frac{HL^2}{EI} = 32.2\)

\[
\overline{\Delta_1} = 0.126
\]

From equation (75)

\[
\delta H = \frac{-1.72}{0.126 \left(0.20\right) + 0.0029} = 52\ K
\]

Step 5. Compute \(H\).

\[
H = 1250 + 360 - 52 = 1558\ K
\]

Step 6. Compare the value of \(H\) computed in step 5 with the value estimated in step 4. Repeat steps 4 and 5 until convergence.

\[
HL^2 = 1558 \left(0.020\right) = 31.2\ 
\]

\[
EI
\]
From Figure 13, $\Delta_1 = .130$

$$8H = \frac{-1.72}{.130 (.20) + .0029} = -50 \text{ K}$$

$$H = 1250 + 360 - 50 = 1560 \text{ K}$$

Step 7. Compare the value of $H$ computed at the end of step 6 with the value estimated in step 2. Repeat steps 2 and 5 to convergence.

From Figure 14

$$X = .833$$

$$X_s = .083$$

$$H_L = \frac{.1063 (1000)^2 (.833) (.4) + .194 (1000) (.083) (25)}{100} = 354 + 4 = 358 \text{ K}$$

$$H = 1250 + 358 - 50 = 1558 \text{ K}$$

Step 7 will seldom produce any significant improvement in the accuracy of $H$. For most suspension bridges $a$ is usually less than .5 and $b$ is usually less than .5. Therefore, $X_s$ is small and not sensitive to changes in $H$. Since $X$ is equal to $1 - 2X_s$, it also is not sensitive to changes in $X$. 
Example 3. Continuous Suspension Bridge

Given:

- Main span length $L = 1000$ ft.
- Main span sag $f = 100$ ft.
- Side span length $L_s = 500$ ft.
- Main span $EI = 1.5 \times 10^8$ K ft.$^2$
- Side span $EI = 7.5 \times 10^7$ K ft.$^2$
- AE of cable = $7.5 \times 10^5$ K
- Side span rise = 112.1 ft.
- $L_e = 2181$ ft.
- $L_t = 2116$ ft.
- Dead load = 1.0 K/ft.
- Live load = .4 K/ft. distributed as shown
  - + 25 K at main span quarter point

$t = 3.25 \times 10^{-4}$

Support displacement $h_B - h_A = 0$
Step 1. Compute $H_D$ and design constants.

\[
H_D = \frac{(1000)^2 (1.0)}{8 (100)} = 1250 \text{ K}
\]

\[
L^2 = \frac{(1000)^2}{1.5 \times 10^8} = .00667
\]

\[
L_e = \frac{2181}{7.5 \times 10^5} = .0029
\]

\[
f^2L = \frac{(100)^2 (1000)}{1.5 \times 10^8} = .0667
\]

\[
a = \frac{500}{1000} = .5
\]

\[
b = \frac{500}{1000} \times \frac{1.5 \times 10^8}{7.5 \times 10^7} = .5
\]

\[
M_e = \frac{1 + (.5) (.5)}{1.5 + .5/5} = .500
\]

Step 2. Compute $H_L$.

Estimate $HL^2 = 1250 \times .00667 = 8.33$

\[
EI
\]

Figure 16 shows $K$ plotted against $HL^2$ for selected values of $a$ and $b$. For $a = .5$ and $b = .5$ (Figure 16(d)) and $HL^2 = 8.33$, $K$ is found to be .836. Then from equation (106)

\[
\bar{M} = .836 \times .500 = .418
\]

From Figure 14, $\frac{b(\Delta_1) s}{\Delta_1} = .623$

From Figure 15, $\frac{\Delta_2}{\Delta_1} = -.630$

\[
\frac{b(\Delta_2) s}{\Delta_1} = -.405
\]
By equation (105)
\[ T = 1 + 2 (.53) (.623) + .418 2 (-.630) + 2 (.5) (-.405) \]
\[ = 1 + 2 (.078) - .526 - 2 (.085) \]
\[ T = .460 \]

The multipliers X and Y for the main span influence line ordinates are found from equations (108) and (109)

\[ X = \frac{1}{.460} = 2.17 \]

\[ Y = -\frac{.526}{.460} = -1.14 \]

The multipliers \( X_s \) and \( Y_s \) for the side span influence line ordinates are found from equations (111) and (112)

\[ X_s = \frac{.078}{.460} = .170 \]

\[ Y_s = -\frac{.085}{.460} = -.185 \]

The main span influence line is made up by superimposing curves from Figure 10 and Figure 12 in accordance with equation (107). The contribution to \( H_L' \) from the main span is

\[ H_L' = \frac{2.17 (1000) .4 (1000) (.106) + .139 (25)}{100} \]

\[ - \frac{1.14 (1000) .4 (1000) (.106) + .142 (25)}{100} \]

\[ H_L' = 997 - .524 = 473 \text{ K} \]

The side span influence line is made up by superimposing curves from Figure 10 and Figure 11 in accordance with equation (110). The contribution to \( H_L' \) from the side span is
\[
\begin{align*}
H_L^' & = \frac{0.170 \times (1000)^2 \times 0.4 \times (0.125 - 0.019)}{100} \\
& - \frac{0.185 \times (1000)^2 \times 0.4 \times (0.125 - 0.014)}{100} \\
H_L^' & = 77 - 82 = -5 \text{ K}
\end{align*}
\]

The total value of \(H_L^'\) from main and side spans is
\[
H_L^' = 473 - 5 = 468 \text{ K}
\]

Step 3. Compute \(\Delta^'\).

From equation (71)
\[
\Delta^' = 0 - 3.25 \times (10^{-4}) \times (2116) - 468 \times (0.0029)
\]
\[
= -2.03 \text{ ft.}
\]

Step 4. Compute \(6H\).

Estimate \(H = 1250 + 468 = 1718 \text{ K}\)
\[
H_L^2 = 1718 \times (0.00667) = 11.4
\]

From Figure 13, \(\overline{\Delta^1} = 0.252\)

By equation (104)
\[
6H = \frac{-2.03}{0.252 \times (0.460) \times (0.0667) + 0.0029} = -191 \text{ K}
\]

Step 5. Compute \(H\).

\(H = 1250 + 468 - 191 = 1527 \text{ K}\)

Step 6. Compare the value of \(H\) computed in step 5 with the

value estimated in step 4 and repeat steps 4 and 5 until

convergence.

\[
H_L^2 = 1527 \times (0.00667) = 10.1
\]

From Figure 13, \(\overline{\Delta^1} = 0.270\)

\[
6H = \frac{-2.03}{0.270 \times (0.460) \times (0.0667) + 0.0029} = -181 \text{ K}
\]
\[ H = 1250 + 468 - 181 = 1537 \text{ K} \]

Step 7. Compare the value of \( H \) computed at the end of step 6 with the value estimated at the beginning of step 2 and repeat steps 2 to 6 until convergence.

\[
H L^2 = 1537 (.00667) = 10.2
\]

From Figure 16(d), \( K = .812 \)

\[
\bar{M} = .812 (.500) = .406
\]

From Figure 14, \( \frac{b(\Delta_1)}{\Delta_1} = .636 \)

From Figure 15, \( \frac{\Delta_2}{\Delta_1} = -.632 \)

\[
\frac{b(\Delta_2)}{\Delta_1} = -.420
\]

\[
T = 1 + 2 (.5)^3 (.636) + .403 2 (-.632) + 2 (.5) (-.420)
= 1 + 2 (.080) - .509 - 2 (.085)
\]

\( T = .481 \)

\[
X = \frac{1}{.481} = 2.08
\]

\[
Y = \frac{.509}{.481} = -1.06
\]

\[
X_s = \frac{.080}{.481} = .166
\]

\[
Y_s = \frac{.085}{.481} = -.177
\]

\[
H_L' = \frac{2.08 (1000) .4 (1000) (.106) + .139 (25)}{100}
- \frac{1.06 (1000) .4 (1000) (.106) + .142 (25)}{100}
\]
\[
\begin{align*}
\frac{.166 \cdot (1000)^2 \cdot (.4) \cdot (.111)}{100} \\
- \frac{.177 \cdot (1000)^2 \cdot (.4) \cdot (.111)}{100}
\end{align*}
\]

\[= 945 - 487 + 74 - 69 = 473 \text{ K}\]

\[\Delta' = 0 - 3.25 \cdot (10^{-4}) \cdot (2116) - 473 \cdot (.0029)\]

\[= -2.05 \text{ ft.}\]

From Figure 13 \(\bar{A}_1 = .270\)

\[8H = -2.05\]

\[= \frac{.270 \cdot (.481) \cdot (.0667) + .0029}{.178 \text{ K}}\]

\[H = 1250 + 473 - 178 = 1545 \text{ K}\]

Here, the improvement in accuracy from 1537 K to 1545 K represents a relatively large improvement compared with what might usually be expected. For shorter or more rigid side spans, the improvement will be less significant and step 7 might reasonably be omitted in many cases.
BIBLIOGRAPHY


