COMPUTER ANALYSIS OF PLANAR AND SPATIAL GRID FRAMEWORKS

by

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ABSTRACT

The application of the stiffness approach to the exact analysis of both planar and spatial grid frameworks of any complexity and high degree of statical indeterminacy is presented. Ordinarily, even the simplest planar grid is such a highly redundant structure that it cannot be analyzed rigorously by manual methods without recourse to some simplifying assumptions at the expense of accuracy. In general, most authors neglect the effect of member torsional rigidities in order to reduce the size of the problem and make use of the plate theory for the purpose of evaluating deflections. Matrix methods of analysis, however, remove the necessity for resorting to any such approximations and prove extremely convenient for computer application.

The fundamentals of the stiffness approach are explained in complete detail and applied to the analysis of rectangular planar grids. For the purpose of comparison, example grids given by Ewell, Okubo & Abrams[1] and Woinowsky-Krieger[2] have been analyzed by the stiffness method and the comparative results are tabulated.

The principle of orthogonal transformation, which is an essential part of the analysis of diagrids and spatial grids is fully described and its application demonstrated by various numerical examples including a skew bridge, a cantilever diagrid and a hyperbolic paraboloid space grid. The application of stiffness analysis has been further extended to problems involving temperature changes and support settlements and, also, the procedure to reduce the size of symmetrical structures is described. A special successive elimination and matrix partitioning technique has also been introduced in order to enable the solution of extremely large numbers of simultaneous equations within the limited core memory capacity of digital computers, by taking

advantage of the band form of the stiffness matrices of structures. A complete Fortran II computer program for the IBM 1620 and a 1405 disk file is given, as well as, sample inputs and outputs of the IBM 1620 and 7090.

After the first attempts by Engessers in 1889 and Zschetzsche in 1893, a great variety of hand calculation methods have been developed for the analysis of planar grid frameworks. Among these, Hendry & Jaeger's[3] harmonic analysis and C. Massonet's[4] anisotropic plate theory methods are the most convenient and easily applicable, in the opinion of the author. The basic assumptions and underlying principles of both these methods are outlined and the procedure of analysis is illustrated by means of a numerical example in each case. Furthermore, in order to obtain an idea of their accuracy several planar grids with 2 to 6 longitudinals have been analyzed by the stiffness method, harmonic analysis and anisotropic plate theory. In every case, two solutions have been performed, assuming the constituent members of the grid to possess first zero and then maximum values of torsional rigidity. The comparative values of the load distribution factors for the longitudinal and transversal bending moments have been tabulated.

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Dedicated to my father Karam Narain Kinra

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INTRODUCTION

A grid framework, which is basically defined as a system of beams interconnected at the nodal points either in a plane or in space is an extremely redundant structure. The degree of redundancy may be expressed as

$$N = 3J - R$$

for planar grids, and, as

$$N = 6J - R$$

for spatial grids, where J represents the number of joints and R the number of restraints at the supports. This follows from the fact that, in general, each joint of a planar grid undergoes two rotations and a translation, while that of a spatial grid is subject to three rotations and three translations, upon the application of external loads.

Prior to the advent of digital computers, an exact analysis of even a simple grid was, understandably, considered impractical. Several approximate methods, however, based on one or more simplifying assumptions, have been developed over the past 60 years. A brief survey of these methods is described below.

M. Hetenyi[5] outlined a method for determining the deflections in a gridwork, assuming that the individual beams deflect without rotation at their intersections. The simultaneous differential equations written for the deflections, on this assumption, are solved to yield the joint displacements.

Neglecting the torsional rotations of the members, Melan ε. Schindler[6] used a procedure of equating the beam deflections at the joints to arrive at a system of homogeneous linear equations which involve complicated eigen value problems. Based on an analogous assumption that no bending or twisting moment is transmitted at beam intersections, S. Timoshenko[7] employed trigonometric series to define the elastic curves developed by the individual grid beams.

The plate theory has been widely used in connection with grid framework analysis. Neglecting torsional stiffnesses, the theory was successfully applied first by Guyon[8] and then by Greenberg[9]. Subsequently, C. Massonet[4] and Taraporewalla[10] used anisotropic plate theory to include the effects of torsional rigidities. Massonet has compiled a fairly detailed set of coefficients for the evaluation of longitudinal and transversal bending moments for torsional and non torsional cases.

Finite difference equations have been applied to rectangular and skew slabs divided into a network of points by Henri Marcus[11] and Vernon P. Jensen[12]. By dividing the plate theory fourth-order differential equation into two parts, two sets of linear equations, containing moments and deflections as unknowns respectively, are obtained and solved simultaneously to yield the required moments and deflections. If the finite difference equivalents are determined directly from the fourth-order equation, a solution is required for only one set of linear equations. The latter procedure, however, is not recommended by Marcus.

Semih S. Tezcan[13] approached the problem in a different manner. First, neglecting torsional stiffnesses, the reactions at each joint are determined by equating the deflections at beam intersections. The resulting statically determinate system is solved to obtain the bending moments and shears. Next, anisotropic plate theory is applied to determine the effect of torsional rigidities alone and these values added to the results of the first stage to yield the required exact stress resultants.

Ewell, Okubo and Abrams[2] employed an auxiliary force system for controlling the vertical displacements of the joints and a moment and torque distribution process for transmission of the displacement effects. After the bending and torsional moments have been distributed over the grid, only one series of linear equations need be solved to define the deflection pattern as the equations are written in terms of unknown deflections produced by auxiliary forces at the grid joints. Once the deflected surface is found, it is a simple matter to evaluate the bending and torsional moments. The above moment distribution or relaxation method is applicable to any kind of framework, but, it's general usage is limited as very lengthy computational work is entailed.

The unique method of harmonic analysis developed by Hendry & Jaeger[3] has proved to be very efficient for practical design purposes. In their approach, Hendry & Jaeger replace the transverse grid members by a uniform spread medium, which may or may not cover the full span. Differential equations are written for the loading on each longitudinal, including terms due to rotation and twist, and solved by harmonic analysis to obtain the amplitude of the harmonics of the deflection or bending moment curve for each girder.

Plastic theory has been recently applied to planar grid analysis for the first time by Shaw[14]. It is independent of the support conditions and configuration of the structure, but, requires that the torsional stiffnesses of it's constituent members be negligible.

Without exception, all the hand calculation methods of gridwork analysis outlined above, involve a great deal of tedious arithmetical calculations. Moreover, in view of the simplifying assumptions indicated, the results obtained can at best be approximate, except when using the relaxation procedure. The common assumption made by most authors is to neglect the torsional stiffness of members. For steel grids this is not a serious approximation; however, when working with reinforced concrete, considerable errors ranging from 10 to 40% may occur. Further, most of the methods are directly applicable only to grids with a regular beam layout and particular support conditions.

When matrix analysis is used, however, a complete and exact solution is possible for grids of any complexity and support conditions. The effect of torsional rigidities, length changes, shear deformations, temperature changes and support settlements may be easily taken into account. Moreover, while the analysis of spatial grids is impossible by any one of the hand calculation methods mentioned above, the mathematical formulation of the stiffness analysis of spatial grids remains the same as that of planar grids except that an orthogonal transformation is required and the size of the individual member matrices is increased from 6 by 6 to 12 by 12. Above all, the stiffness approach allows complete automation, so that the analysis of planar or spatial grids of any degree of redundancy is reduced merely to the clerical job of preparing the input data for the computer and absolutely no calculations are required from the engineer.

CHAPTER 1. PLANAR RECTANGULAR GRIDS

1. Basic Assumptions and Definitions

The following assumptions are made in the stiffness matrix analysis method presented:

(a) the structure is stable,

(b) the structural material has a linear stress strain diagram,

(c) the displacements of the structure vary linearly with the applied forces,

(d) the effect of the displacements on the forces and moments in the structure are negligible, and

(e) static forces are applied.

A structural member undergoes an elastic deformation δ, when subjected to an axial force p. The assumed linear force deformation relation can be expressed as

$$p = k\,\delta$$

where k is the axial force required to produce a unit axial deformation.

By the same principle, if a member has end deformations in more than one direction, the total force p_i along direction i due to deformations δ_j in each direction is given by

$$p_i = \sum_{j=1}^{n} k_{ij}\,\delta_j \tag{1}$$

in which k_{ij} is defined as the force required along direction i to maintain a unit deformation along direction j only, while deformations along all the other specified directions are prevented. The factor k_{ij} is termed a "stiffness influence coefficient" and Eq. 1 is the basic "stiffness equation" in the stiffness method of linear structural analysis.

Sign Convention

Clockwise rotations and downward translations acting on a member are assumed to be positive. Similarly, clockwise moments and downward shears are positive.

2. Stiffness Equation of a Straight Structural Member

The possible end rotations and displacements of a straight member ij are indicated by arrows numbered 1, 2, 3 and 4 along the assumed positive directions in Fig. 1.

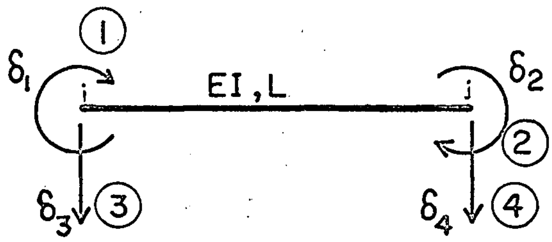

Fig. 1 End deformations of a straight member

The stiffness influence coefficients of a member are evaluated by considering a unit deformation along each one of the specified directions and calculating the forces required along all the specified directions. From the stiffness coefficients, the stiffness equation of the individual member may be derived in 5 stages as fully illustrated in Fig. 2.

Stage 1

The member is considered to be clamped at both ends and the forces along the assumed directions i.e. the fixed end moments and reactions are computed.

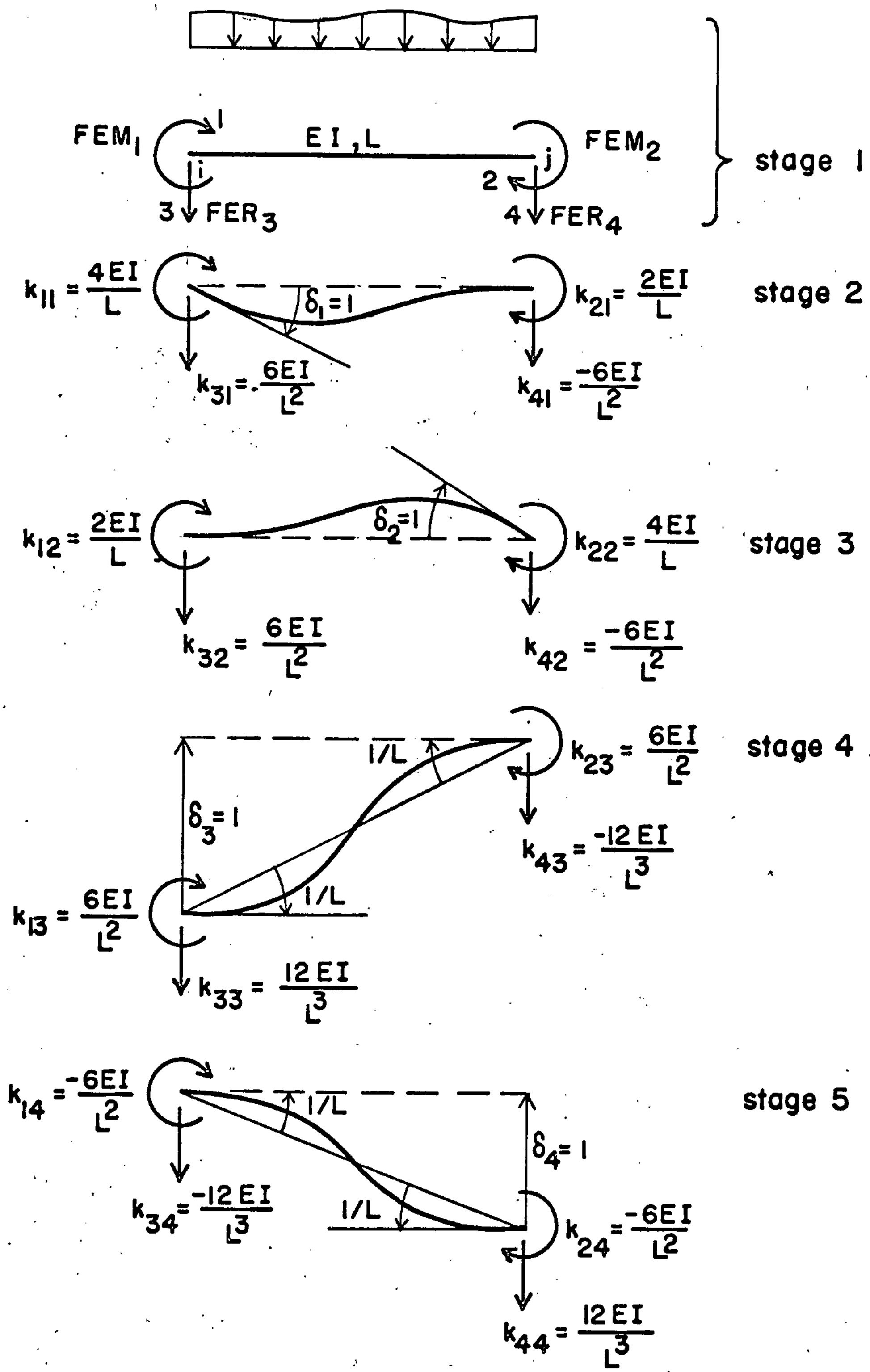

Fig. 2 Stiffness coefficients of a straight member

Stage 2

The forces required along the specified directions, to maintain a unit deformation along direction 1 only, are k_{11}, k_{21}, k_{31} and k_{41} as shown in Fig. 3.

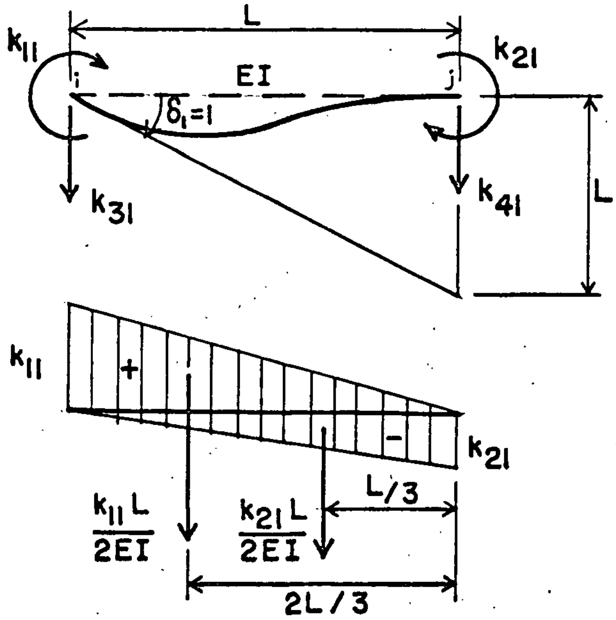

Fig. 3 Unit deformation along direction 1

By the First Moment Area Theorem, the change in slope between the tangents at the ends of the elastic curve is equal to $\frac{1}{EI}$ times the area of the bending moment diagram. Therefore, for $\delta_1 = 1$

$$\frac{k_{11}.L}{2EI} - \frac{k_{21}.L}{2EI} = 1 \tag{2}$$

By the Second Moment Area Theorem, the tangential deviation between the ends i, j is equal to $\frac{1}{EI}$ times the moment of the bending moment diagram about j. Consequently,

$$\frac{k_{11} L}{2} \cdot \frac{2L}{3} - \frac{k_{21} \cdot L}{2} \cdot \frac{L}{3} = L \tag{3}$$

Solving Eqs. 2 and 3,

$$k_{11} = \frac{4\ EI}{L} \quad \text{and} \quad k_{21} = \frac{2\ EI}{L}$$

For equilibrium, the required shears at the ends of the member are obtained by dividing the sum of the end moments by the length of the member. Therefore,

$$k_{31} = \frac{k_{11} + k_{21}}{L} = \frac{6\ EI}{L^2}$$

$$k_{41} = -\frac{k_{11} + k_{21}}{L} = \frac{-6\ EI}{L^2}$$

Stage 3

For a unit deformation along direction 2, from symmetry we obtain

$$k_{12} = \frac{2\ EI}{L}; \quad k_{22} = \frac{4\ EI}{L} \quad \text{and } k_{32} = -k_{42} = \frac{6\ EI}{L^2}$$

Stage 4

A unit deformation along direction 3 may be considered as a rotation of $\frac{1}{L}$ units along directions 1 and 2. Therefore, using the results of the first two stages,

$$k_{13} = \frac{4\ EI}{L} \cdot \frac{1}{L} + \frac{2\ EI}{L} \cdot \frac{1}{L} = \frac{6\ EI}{L^2}.$$

Similarly,

$$k_{23} = k_{13} = \frac{6\ EI}{L^2}$$

$$k_{33} = \frac{k_{13} + k_{23}}{L} = \frac{12\ EI}{L^3}$$

$$k_{43} = -\frac{k_{13} + k_{23}}{L} = -\frac{12\ EI}{L^3}$$

Stage 5

For a unit deformation along direction 4, from symmetry we obtain

$$k_{14} = -\frac{6\ EI}{L^2};\quad k_{24} = -\frac{6\ EI}{L^2} \quad \text{and} \quad k_{34} = -k_{44} = \frac{12\ EI}{L^3}$$

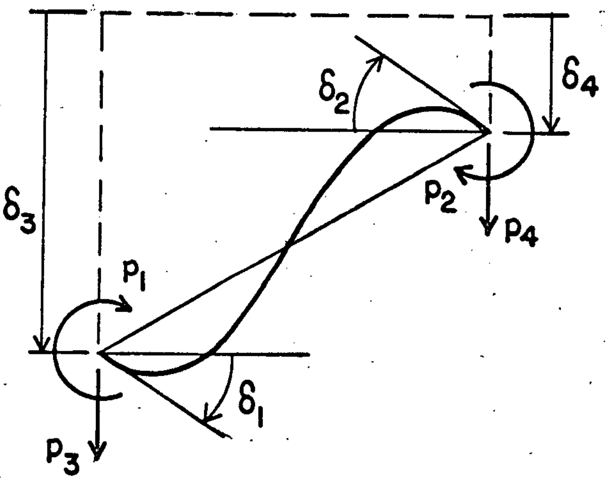

Fig. 4 Final end reactions and deformations

Now, superimposing the results of each stage, the final end moments and shears at the ends of the member, shown in Fig. 4, are obtained as

$$p_1 = FEM_1 + \frac{4\ EI}{L}\delta_1 + \frac{2\ EI}{L}\delta_2 + \frac{6\ EI}{L^2}\delta_3 - \frac{6\ EI}{L^2}\delta_4$$

$$p_2 = FEM_2 + \frac{2\ EI}{L}\delta_1 + \frac{4\ EI}{L}\delta_2 + \frac{6\ EI}{L^2}\delta_3 - \frac{6\ EI}{L^2}\delta_4$$

$$p_3 = FER_3 + \frac{6\ EI}{L^2}\delta_1 + \frac{6\ EI}{L^2}\delta_2 + \frac{12\ EI}{L^3}\delta_3 - \frac{12\ EI}{L^3}\delta_4$$

$$p_4 = FER_4 - \frac{6\ EI}{L^2}\delta_1 - \frac{6\ EI}{L^2}\delta_2 - \frac{12\ EI}{L^3}\delta_3 + \frac{12\ EI}{L^3}\delta_4$$

in which FEM_1, FEM_2, FER_3 and FER_4 are the fixed end moments and reactions due to the external loading.

The matrix form of the above equations, called the "stiffness equation", is given by

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix} = \frac{EI}{L} \begin{bmatrix} 4 & 2 & \frac{6}{L} & -\frac{6}{L} \\ 2 & 4 & \frac{6}{L} & -\frac{6}{L} \\ \frac{6}{L} & \frac{6}{L} & \frac{12}{L^2} & -\frac{12}{L^2} \\ -\frac{6}{L} & -\frac{6}{L} & -\frac{12}{L^2} & \frac{12}{L^2} \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{Bmatrix} + \begin{Bmatrix} FEM_1 \\ FEM_2 \\ FER_3 \\ FER_4 \end{Bmatrix} \qquad (4)$$

In short matrix notation

$$\{p\} = [k]\{\delta\} + \{F\} \qquad (5)$$

where, p = column vector of final stress resultants,

k = stiffness matrix of the member,

δ = column vector of deformations,

F = column vector of fixed end moments and reactions.

The stiffness matrix $[k]$ is symmetrical, as is the case with all structural members, due to "Maxwell's Theorem of Reciprocity".

3. Stiffness Matrix of a Straight Member with Variable Moment of Inertia

The elastic behaviour of a member with variable moment of inertia can be conveniently characterized by the rotational stiffness factors of its ends. Essentially, a member ij has three basic stiffness factors, namely a_{ij}, a_{ji} and b_{ij}, which are defined as below.

$a_{ij} \frac{EI_o}{L}$ = the moment required at end i, to develop a unit rotation at i only. (I_o is the minimum moment of inertia of the member).

$a_{ji} \frac{EI_o}{L}$ = the moment required at end j, to develop a unit rotation at j only.

$b_{ij} \frac{EI_o}{L}$ = the moment required at end j, when a unit rotation is developed at i only. ($b_{ij} = b_{ji}$ by Maxwell's Reciprocity Theorem).

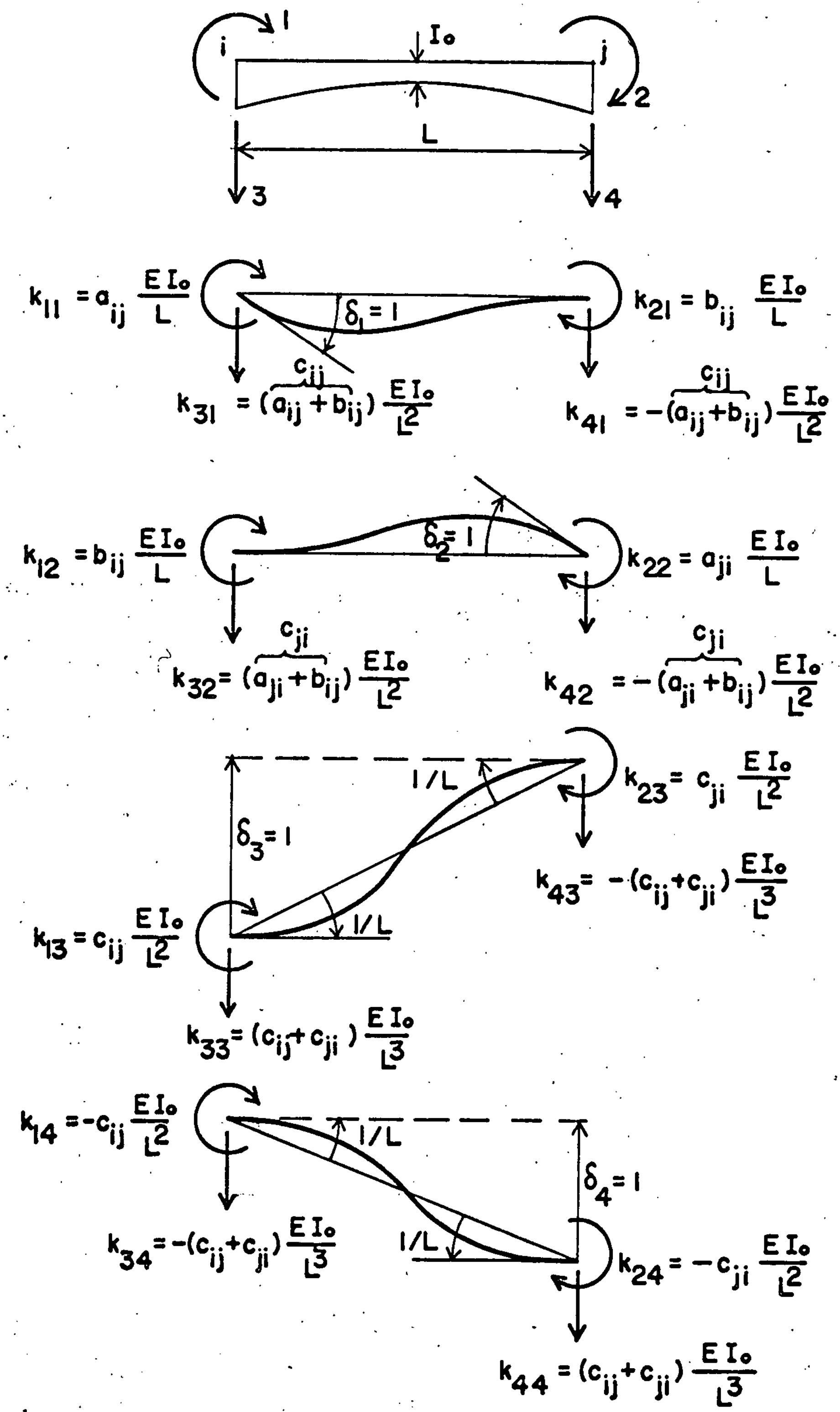

Fig. 5 Stiffness coefficients of a variable moment of inertia member

For the customary forms of variation, the basic stiffness factors of a member are available in standard tables.[(a), (b)]

Now, proceeding exactly in the same manner as described above i.e. by giving unit deformations separately along each of the 4 specified directions and calculating the forces required to maintain them, the stiffness matrix of a member with variable moment of inertia, as illustrated in Fig. 5, becomes

$$[k] = \frac{EI_o}{L}\begin{bmatrix} a_{ij} & & & \text{symmetrical} \\ b_{ij} & a_{ji} & & \\ \frac{c_{ij}}{L} & \frac{c_{ji}}{L} & \frac{c_{ij}+c_{ji}}{L^2} & \\ -\frac{c_{ij}}{L} & -\frac{c_{ji}}{L} & -\frac{c_{ij}+c_{ji}}{L^2} & \frac{c_{ij}+c_{ji}}{L^2} \end{bmatrix} \qquad (6)$$

in which

$$c_{ij} = a_{ij} + b_{ij}$$

$$c_{ji} = a_{ji} + b_{ij}$$

For a member with constant moment of inertia

$$a_{ij} = a_{ji} = 4.0; \quad b_{ij} = 2.0 \quad \text{and} \quad c_{ij} = c_{ji} = 6.0$$

4. Stiffness Matrix of a Straight Member including Torsional Rotations

The possible end rotations and displacements of a typical member are indicated by numbered arrows as shown in Fig. 6.

Evidently the first 4 x 4 portion of the required stiffness matrix is identical with the stiffness matrix of the straight member given by Eq. 4 ,

(a) R. Guldan, "Rahmentragwerke und Durchlauftrager". Spring ε Verlag, Wien 1943, pp. 278 - 351.
(b) "Handbook of Frame Constants", Publication of Portland Cement Association, 1958.

because the specified deformations 1 to 4 are the same. Further, when a unit torsional rotation is given along direction 5, no forces are required in directions 1 to 4. Therefore,

$$k_{15} = k_{25} = k_{35} = k_{45} = 0$$

The force required along direction 5 is

$$k_{55} = \frac{GJ}{L}$$

where GJ is the torsional rigidity of the member.[a]

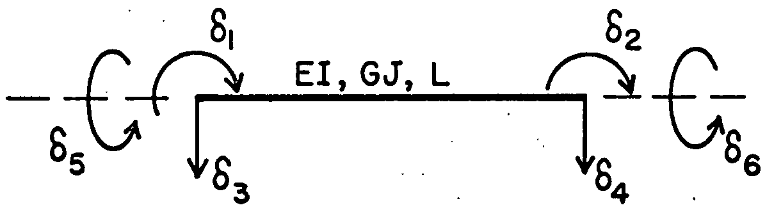

Fig. 6 Flexural member with torsion

For equilibrium, the force required along direction 6 is

$$k_{65} = -k_{55} = -\frac{GJ}{L}$$

Similarly, for a unit torsional rotation along direction 6

$$k_{16} = k_{26} = k_{36} = k_{46} = 0$$

$$k_{56} = -\frac{GJ}{L} \quad \text{and} \quad k_{66} = \frac{GJ}{L}$$

Assembling the values obtained for the stiffness influence coefficients, the stiffness matrix of the member when torsional rotations are considered, is

(a) Torsional rigidity constants for various cross sections are given in Appendix B.

$$[k] = \frac{EI}{L}\begin{bmatrix} 4 & & & & & \\ 2 & 4 & & & \text{Symmetrical} & \\ \frac{6}{L} & \frac{6}{L} & \frac{12}{L^2} & & & \\ -\frac{6}{L} & -\frac{6}{L} & -\frac{12}{L^2} & \frac{12}{L^2} & & \\ 0 & 0 & 0 & 0 & \frac{GJ}{EI} & \\ 0 & 0 & 0 & 0 & -\frac{GJ}{EI} & \frac{GJ}{EI} \end{bmatrix} \tag{7}$$

5. Stiffness Matrix of a Straight Member Considering Length Changes

The possible rotations, vertical deflections and axial deformations of a typical member are shown by numbered arrows in the positive direction in Fig. 7.

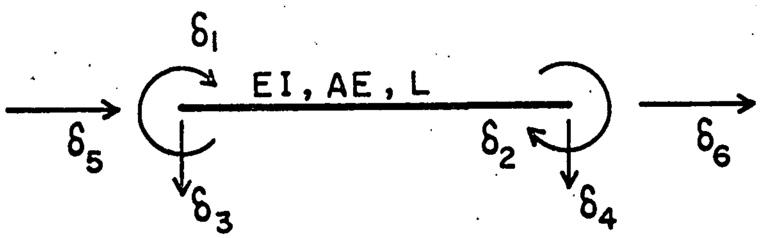

Fig. 7 Flexural member subject to length changes

Again, the first four specified deformations are the same as those of the straight member in section 2 and, therefore, the first 4 x 4 portion of the required stiffness matrix is as in Eq. 4.

For a unit axial deformation along direction 5 the forces required in each of the specified directions are,

$$k_{15} = k_{25} = k_{35} = k_{45} = 0$$

$$k_{55} = \frac{AE}{L} \quad \text{and} \quad k_{65} = -\frac{AE}{L}$$

where A is the cross sectional area of the member.

Similarly

$$k_{16} = k_{26} = k_{36} = k_{46} = 0$$

$$k_{56} = -\frac{AE}{L} \quad \text{and} \quad k_{66} = \frac{AE}{L}$$

Arranging these coefficients in matrix form, the stiffness matrix of a member, taking it's length changes into account, becomes

$$[k] = \frac{EI}{L}\begin{bmatrix} 4 & & & & & \\ 2 & 4 & & & \text{Symmetrical} & \\ \frac{6}{L} & \frac{6}{L} & \frac{12}{L^2} & & & \\ -\frac{6}{L} & -\frac{6}{L} & -\frac{12}{L^2} & \frac{12}{L^2} & & \\ 0 & 0 & 0 & 0 & \frac{A}{I} & \\ 0 & 0 & 0 & 0 & -\frac{A}{I} & \frac{A}{I} \end{bmatrix} \tag{8}$$

6. <u>Stiffness Matrix of a Member Considering the Effect of Shear Deflections</u>

Though the overall effect of shearing strains is generally small, it is sometimes desirable to take them into account.

The typical member for this case remains the straight member of section 2, shown in Fig. 1.

If, instead of the Moment Area Theorems, the unit load theorem[(a)] is used to evaluate the member stiffness influence coefficients and the strain energy due to shear forces is considered, the required stiffness matrix is

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(a) Hall, S.A. and Woodhead, R.W., "Frame Analysis", John Wiley & Sons, Inc. 1961, pp 31 - 33 and pp 150 - 152.

obtained as

$$[k] = \frac{EI}{L}\begin{bmatrix} 3\varepsilon + 1 & & & \\ 3\varepsilon - 1 & 3\varepsilon + 1 & \text{Symmetrical} & \\ \frac{6\varepsilon}{L} & \frac{6\varepsilon}{L} & \frac{12\varepsilon}{L^2} & \\ -\frac{6\varepsilon}{L} & -\frac{6\varepsilon}{L} & -\frac{12\varepsilon}{L^2} & \frac{12\varepsilon}{L^2} \end{bmatrix} \qquad (9)$$

in which the shearing strain parameter ε is

$$\varepsilon = \frac{1}{1 + 12\lambda \dfrac{EI}{L^2 AG}} \qquad (10)$$

At any member cross section, the numerical factor λ is used to multiply the average shearing stress to obtain the shearing stress at the centroid. For rectangular sections $\lambda = 1.5$, while for circular sections $\lambda = 4/3$.

When shear deflections are neglected it is assumed that $AG = \infty$, so that ε equals 1. On substituting unity for ε in Eq. 9, the resulting matrix is identical with that of Eq. 4 obtained for a similar member without taking into account the effect of shear deformations.

7. Stiffness Equation of a Structural System

As for a single member, the loads acting on a structure are related to its resultant deformed shape by the basic stiffness equation

$$\{P\} = [K]\{D\} \qquad (11)$$

in which,

$\{P\}$ = the column vector of loads acting on the structure,

$[K]$ = the stiffness matrix of the system,

$\{D\}$ = the column vector of the deformations of the system.

8. Evaluation of the Stiffness Matrix of a System

Though the main stiffness matrix of a structure K may be generated following the fundamental definition as illstrated for a single member, this becomes a cumbersome procedure. Several simplified methods have been developed using reduced stiffness matrices[15], external strain energy[16] and the code number approach[17]. The last mentioned, which is a special application of the strain energy method, is extremely suitable for efficient computer application and has been used for all the problems presented in this thesis.

External Strain Energy Method

a. Transformation Matrices

When a structure deforms under the influence of external loads, the deformations of its constituent members, $\{\delta\}$, must conform with those of the system, $\{D\}$. This relation is expressed by the following compatibility equations.

$$\{\delta\} = [T] \{D\} \tag{12}$$

wherein the coefficient matrix $[T]$ is called a transformation matrix. Each member has its own transformation matrix, which depends on its location and interconnection in the structure and can be easily expressed by visual inspection.

The compatibility equations for members 1, 2 and 3 of the portal frame in Fig. 8, with respect to the assumed deformation directions of a typical member are:

For member 1, from an examination of the figure,

$$\begin{aligned}
\delta_1 &= 0.\,D_1 + 0.\,D_2 + 0.\,D_3\\
\delta_2 &= 1.\,D_1 + 0.\,D_2 + 0.\,D_3\\
\delta_3 &= 0.\,D_1 + 0.\,D_2 + 0.\,D_3\\
\delta_4 &= 0.\,D_1 + 0.\,D_2 + 1.\,D_3
\end{aligned}$$

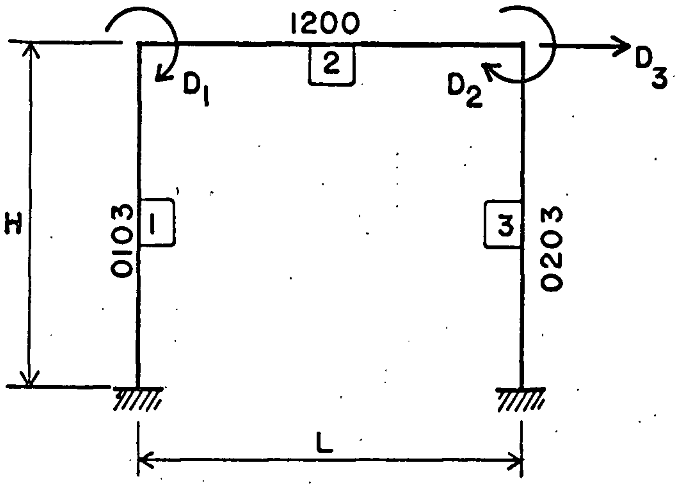


Fig. 8 Example portal frame

In matrix form

$$\{\delta\}_1 = [T]_1 \{D\}$$

in which

$$[T]_1 = \begin{bmatrix} 0. & 0. & 0. \\ 1. & 0. & 0. \\ 0. & 0. & 0. \\ 0. & 0. & 1. \end{bmatrix}$$

is the transformation matrix of member 1.

Similarly, for members 2 and 3

$$[T]_2 = \begin{bmatrix} 1. & 0. & 0. \\ 0. & 1. & 0. \\ 0. & 0. & 0. \\ 0. & 0. & 0. \end{bmatrix} \quad \text{and} \quad [T]_3 = \begin{bmatrix} 0. & 0. & 0. \\ 0. & 1. & 0. \\ 0. & 0. & 0. \\ 0. & 0. & 1. \end{bmatrix}$$

The generation of the stiffness matrix of a structure from the transformation and stiffness matrices of its individual members is based on the principle of conservation of strain energy, which states that

"The total strain energy of a system is equal to the sum of the strain energies stored in each constituent member".

In other words the sum of the work done by the loads, $\{P\}$, of a system along its possible joint deformations, $\{D\}$, must equal the work done by the stress resultants $\{p\}$ of the individual members along their respective end deformations $\{\delta\}$. Therefore,

$$\tfrac{1}{2}(p_1 D_1 + p_2 D_2 + \text{---} \; p_i D_i) = \tfrac{1}{2}\sum_{m=1}^{M} (p_{1m}\,\delta_{1m} + p_{2m}\,\delta_{2m} + \text{---})$$

where M is the number of members.

In matrix notation, after cancelling $\frac{1}{2}$ from both sides,

$$\{D\}^T \{P\} = \sum_{m=1}^{M} \{\delta\}_m \{p\}_m \quad (m = 1 \text{ ---} M, \text{ for all members})$$

Substituting for $\{P\}$, $\{\delta\}$ and $\{p\}$ from Eqs. 11, 12 and 1 respectively,

$$\{D\}^T \underbrace{[K]}_{\text{must equal}} \{D\} = \sum_{m=1}^{M} \{D\}^T \underbrace{[T]_m^T [k]_m [T]_m}_{\text{must equal}} \{D\}$$

It follows that

$$[K] = \sum_{m=1}^{M} [T]_m^T [k]_m [T]_m \quad (m = 1 \text{ ---} M, \text{ for all members}) \tag{13}$$

wherein $[K]$ is the required stiffness matrix of the system.

b. <u>The Code Number Approach</u>

When the directions of the deformations of a system coincide with those of the members, the transformation matrices consist only of zeros and ones. In such a case, the main stiffness matrix can be generated directly without carrying out the triple matrix products indicated above.

Each member is given a code number which characterizes it's location in the structure and is composed of those deformations of the system which coincide with the specified deformations of the member in question. These numbers correspond to the non-zero terms of the transformation matrices and are written down by visual inspection. In writing a code number, it is necessary to follow the same sequence as assumed for numbering the end deformations of the typical member. One end deformation is taken at a time and the number of the deformation of the system coinciding with that particular member end deformation is entered into the code number. If there is no coinciding system deformation, a zero is inserted.

As an example, consider member (1) of the frame in Fig. 8. The left and right end rotations of the member coincide with deformations 0 and 1 of the system, whereas the left and right end translations coincide with deformations 0 and 3 of the system. Therefore, its code number is 0103.

To generate the main stiffness matrix using code numbers, each number is taken at a time and coupled first with itself and then with the remaining numbers in the code number. The sequential order in the code number of the number and of that with which it is coupled constitute the row and column numbers of the elements to be taken from the individual member stiffness matrix, whereas the corresponding numbers themselves indicate the row and column numbers of the general stiffness matrix into which these elements must be placed. This operation must be repeated for all the constituent members of a structure to obtain its general matrix. Note that zero code numbers do not contribute to the general stiffness matrix.

As an example, consider member (1) of Fig. 8 again. Its code number 0103 indicates the following information:

Sequence Number	1	2	3	4
Code Number	$\boxed{0}$	$\boxed{1}$	$\boxed{0}$	$\boxed{3}$

Element 22 of $[k]_1$ is placed into 11 of $[K]_{system}$

" 24 " " 13 "

" 44 " " 33 "

" 42 " " 31 "

Besides being used to generate the main stiffness matrix of a structure, code numbers also indicate the appropriate deformation of the system to be used during the back substitution process to determine the final end moments and shears of the individual members as subsequently indicated in the procedure of analysis.

9. Concept of Joint Loads

To facilitate the application of matrix methods, all the external loads applied on a structure are replaced by equivalent joint loads acting along the specified directions of deformations. Joint loads may be regarded as the resultant forces transferred to the joints from the ends of a member and, therefore, are obtained for each joint in the structure by the sum of the fixed end moments and reactions of all the members meeting at each joint, with their directions reversed. The fixed end reactions are calculated by assuming the members to be clamped at both ends. Note that no special consideration or modification factor is necessary in the case of members with hinged ends. The condition of zero moment at a hinged support is automatically satisfied in the stiffness method by specifying a rotation at the hinge.

When determining the final stress resultants of the individual members in accordance with Eq. 5, the fixed end reactions are substituted with their proper signs.

10. Procedure of Analysis for Planar Rectangular Grids

Step 1 The members and possible joint deformations are numbered on a sketch of the structure to be analyzed.

Step 2 The code numbers of each member are written alongside the members.

Step 3 The individual stiffness matrices $[k]$ of each member of the structure are evaluated.

Step 4 The main stiffness matrix of the system $[K]$ is generated by means of the member code numbers, from the individual stiffness matrices.

Step 5 The column vector of joint loads $\{P\}$ is determined by taking the sum of the fixed end reactions at each joint, with their signs reversed.

Step 6 The basic stiffness equation of the structure, given by Eq. 11 $\{P\} = [K]\{D\}$, is now used to yield the required deformations $\{D\}$ of the system either, from

$$\{D\} = [K]^{-1} \{P\} \tag{14}$$

or, by applying Gaussean elimination directly to the stiffness equation. In most cases the elimination process requires considerably less computer time than inversion and is, consequently, preferred.

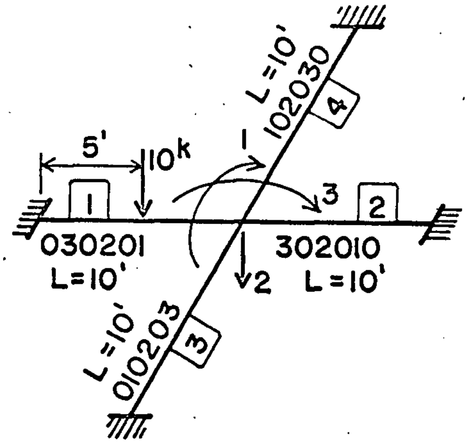
Step 7 Finally, the stress resultants $\{p\}$ of each individual member are computed by means of back substitution in the member stiffness equation (Eq. 5),

$$\{p\} = [k] \{\delta\} + \{F\}$$

The column vector of deformations $\{\delta\}$ is composed of the appropriate deformations of the system and is easily obtained using the member code numbers, by spelling out the information initially stored in them. The column vector $\{F\}$ represents the fixed end reactions of the member.

The following example planar grid with only three possible joint deformations, i.e. 3 unknowns, is manually solved to demonstrate the procedure outlined above.

11. Numerical Example



EI = 100.0 units
GJ = 30.0 units

Fig. 9 Example planar rectangular grid

Step 1 The members and joint deformations of the grid are numbered as shown in Fig. 9.

Step 2 The code numbers are written alongside each member.

Step 3 Individual member stiffness matrices are evaluated.

The typical member shown in Fig. 6, considering torsional rotations, is adopted. Therefore, substituting numerical values into Eq. 7,

$$[k] = \begin{bmatrix} 40 & & & & & \text{Symmetrical} \\ 20 & 40 & & & & \\ 6 & 6 & 1.2 & & & \\ -6 & -6 & -1.2 & 1.2 & & \\ 0 & 0 & 0 & 0 & 3.0 & \\ 0 & 0 & 0 & 0 & -3.0 & 3.0 \end{bmatrix}$$

which is the same for all the four members.

Step 4 Using the member code numbers, the stiffness matrix of the structure is obtained as

$$[K]_{system} = \begin{array}{c} 1 \\ \\ 2 \\ \\ 3 \end{array} \begin{bmatrix} \begin{pmatrix} 3+3 \\ 40+40 \end{pmatrix} & 0 & 0 \\ -6+6 & \begin{pmatrix} 1.2+1.2 \\ 1.2+1.2 \end{pmatrix} & -6+6 \\ 0 & 0 & \begin{pmatrix} 40+40 \\ 3+3 \end{pmatrix} \end{bmatrix} = \begin{bmatrix} 86 & 0 & 0 \\ 0 & 4.8 & 0 \\ 0 & 0 & 86 \end{bmatrix}$$

Step 5 Joint loads are obtained from the fixed end reactions. The fixed end reactions of member (1) are shown in Fig. 10.

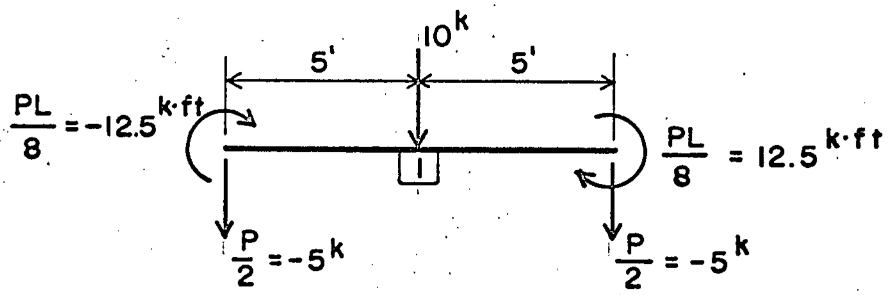


Fig. 10 Fixed end reactions of member (1)

Therefore, the joint loads are

$$P_1 = 0$$

$$P_2 = +\ 5.0\ k$$

$$P_3 = -\ 12.5\ k\ ft$$

Note that the signs of the joint loads are with respect to the assumed positive directions of the typical member.

<u>Step 6</u> Deformations of the system from Eq. 14,

$$\begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix} = \begin{bmatrix} 1/86 & 0 & 0 \\ 0 & 1/4.8 & 0 \\ 0 & 0 & 1/86 \end{bmatrix} \begin{Bmatrix} 0 \\ 5.0 \\ -\ 12.5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1.042 \\ -\ .145 \end{Bmatrix}$$

<u>Step 7</u> For each member, the final stress resultants are evaluated by back-substituting the appropriate joint deformations of the system into the member stiffness equation in accordance with Eq. 5.

For member (1),

$$\begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{Bmatrix} = \begin{bmatrix} 40 & 20 & 6 & -6 & 0 & 0 \\ 20 & 40 & 6 & -6 & 0 & 0 \\ 6 & 6 & 1.2 & -1.2 & 0 & 0 \\ -6 & -6 & -1.2 & 1.2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} 0 \\ -0.145 \\ 0 \\ 1.041 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} -12.5 \\ +12.5 \\ -\ 5.0 \\ -\ 5.0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -21.65 \\ +\ 0.45 \\ -\ 7.12 \\ -\ 2.88 \\ 0 \\ 0 \end{Bmatrix}$$

The final stress resultants of the remaining members are computed in a similar manner and shown in Fig. 11. The computer output of the above example, obtained using the program presented in Appendix A, is given on the following page.

ID NUMBER - 0663
PRINTED FOR S. TEZCAN ON MAR. 31 AT 3 HR. 21.5 MIN.

EXECUTE FORTRAN PROGRAM.
4 3

NO.	LENGTH	EI	GJ	UDL	LOAD	U	CODE NUMBER
1	10.00	100.00	30.00	0.00	10.00	5.00	0 3 0 2 0 1
2Δ	10.00	100.00	30.00	0.00	0.00	0.00	3 0 2 0 1 0
3	10.00	100.00	30.00	0.00	0.00	0.00	0 1 0 2 0 3
4	10.00	100.00	30.00	0.00	0.00	0.00	1 0 2 0 3 0

DEFORMATIONS OF THE SYSTEM
1 0.0000 2 1.0416Δ 3 -.1453

FINAL END MOMENTS AND REACTIONS

	M1 (K.ft)	M2 (K.ft)	R1 (Kips)	R2 (Kips)	T1 (K.ft)	T2 (K.ft)
1Δ	-21.656	.436	-7.122	-2.877	0.000	0.000‡
2	.436	3.343	.377	-.377	0.000	0.000
3	-6.249	-6.249	-1.249	1.249	.436	-.436
4	6.249	6.249	1.249	-1.249	-.436	.436

PROGRAM CAME TO NORMAL END
ALL DATA CARDS WERE READ BY THE PROGRAM

END OF THIS RUN AT 3 HR. 25.7 MIN.

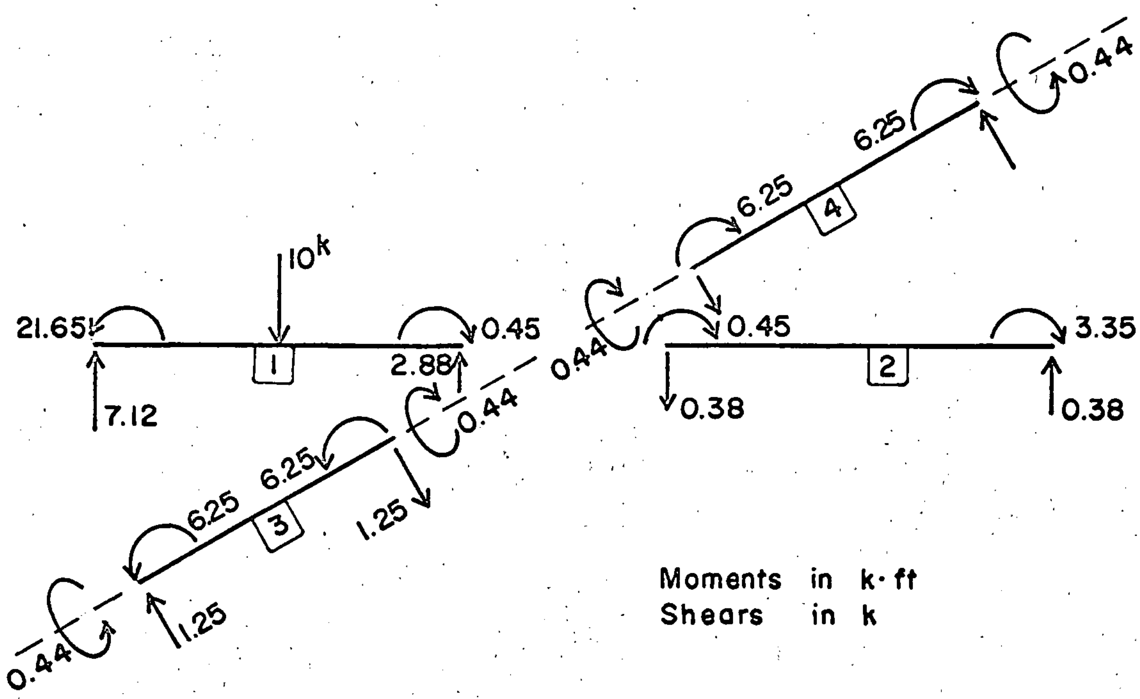


Fig. 11 Final stress resultants

12. Comparative Results with Other Methods

(a) Woinowsky - Krieger's[1] method

The four girder grid shown in Fig. 12, originally presented by Woinowsky-Krieger, was analyzed by the stiffness method strictly for the purpose of comparison. In his method, Woinowsky-Krieger neglects the torsional rigidities of the grid members and, further, replaces the bending stiffness of the cross girders by that of a continuous torsionless plate of the same length as the main girders. Based on these assumptions, the longitudinal bending moments are expressed using Clapeyron's equations in terms of unknown joint deflections. The joint deflections in turn, as well as, the external loads are expressed in Fourier series and back substituted into Clapeyron's equations to form a system of homogeneous linear equations. The eigen value solutions of these equations yield the required final moments in the form of a sine series.

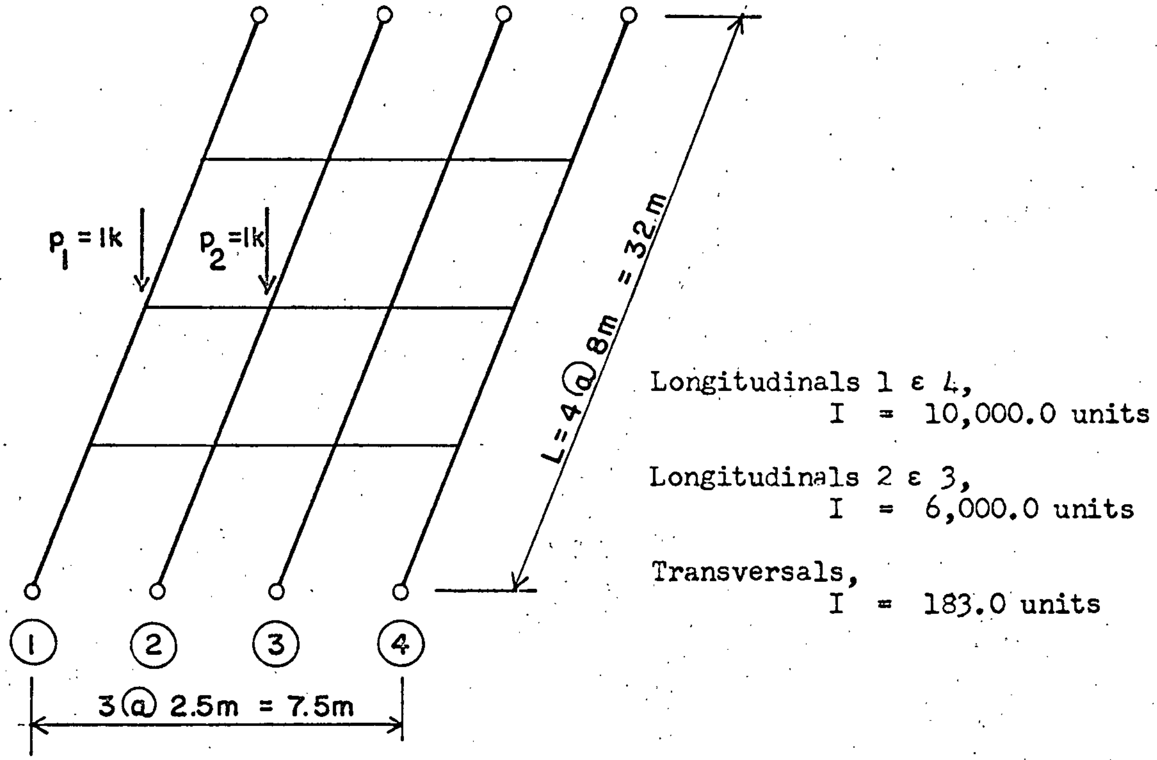


Fig. 12 Woinowsky-Krieger's grid

For the grid shown in Fig. 12, Woinowsky-Krieger gives the longitudinal bending moments at the mid spans of the main girders due to loads P_1 and P_2. These values are tabulated in Table 1 along with the corresponding stiffness analysis and Stahlbau-Kalender[(a)] results.

(a) Vgl. Stahlbau-Kalendar 1934, S. 342. Berlin 1943.

GIRDER NO.	P_1 = 1 k at midspan Girder 1			P_2 = 1 k at midspan Girder 2		
	Woinowsky Krieger	Stahlbau	Stiffness	Woinowsky Krieger	Stahlbau	Stiffness
1	6.68	6.70	6.694	2.94	2.92	2.928
2	1.76	1.74	1.757	2.82	2.84	2.830
3	0.43	0.42	0.411	1.54	1.56	1.563
4	- 0.87	- 0.86	- 0.861	0.70	0.69	0.684

Table 1. Comparative Midspan Moments

(b) Method of moment distribution by Ewell, Okubo & Abrams[2]

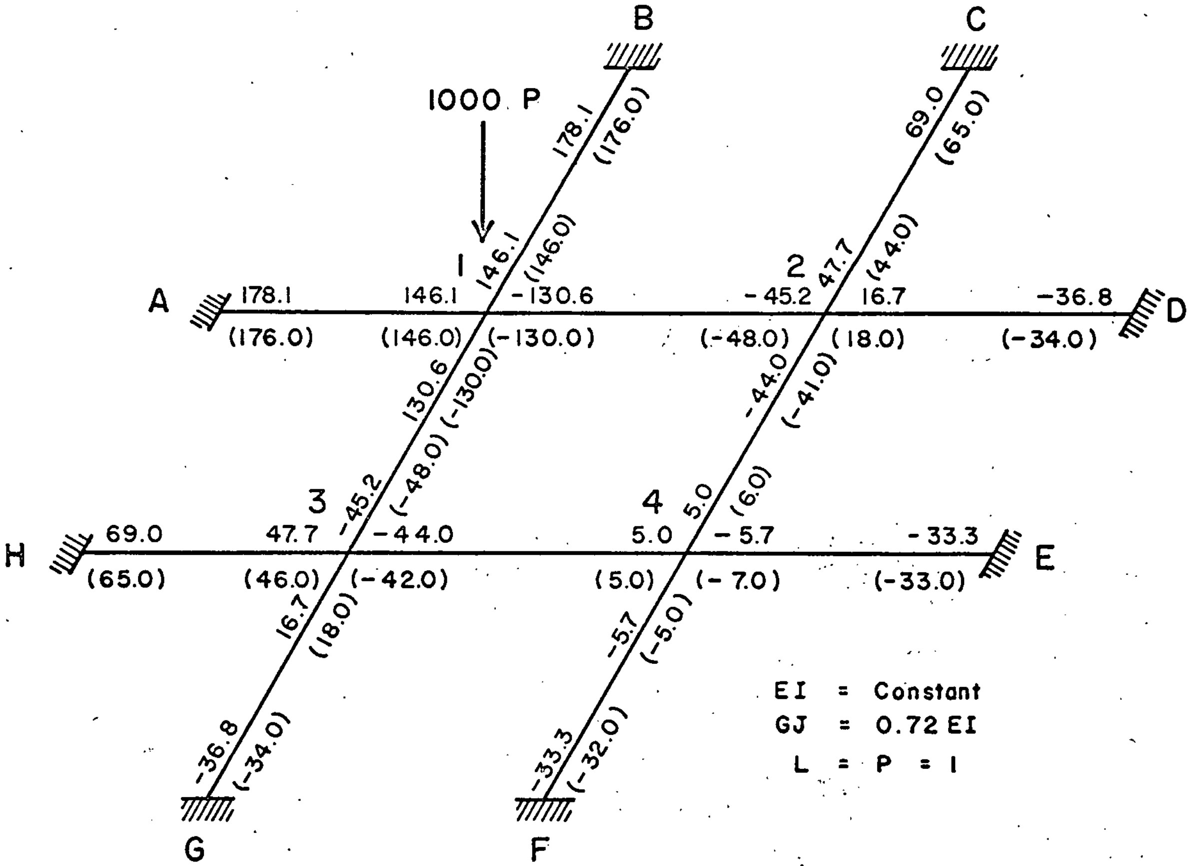
Ewell, Okubo and Abrams eliminate two of the three unknown quantities at each joint by a moment and torque distribution process. The unknown deflections are then expressed in terms of auxiliary forces at the grid joints and, therefore, only one series of linear equations has to be solved in order to define the deflection pattern. Torsional rigidities are duly considered and the procedure is applicable to grids of any configuration and support conditions. However, for large unsymmetrical grids the arithmetical calculations become tedious. A separate solution must be made for each joint displacement, followed by the corresponding reactions at each joint, and, finally, to obtain the deflections a set of simultaneous equations which satisfy the shear relations must be solved.

In some cases, the problem may be reduced by adopting the method of successive shear corrections, analogous to that used for planar frames with sidesway, as suggested by Scordelis[a].

The symmetrical grid shown in Fig. 13, presented by Ewell, Okubo & Abrams, consisting of members of equal length and cross section, has been

(a) Scordelis, A.C., Discussion of "Deflections in Gridworks and Slabs" ASCE Transactions, Vol. 117, 1952, p. 869.

analyzed for comparison by the stiffness method. The results are compared with those evaluated by Scordelis in Fig. 13, the latter values being indicated by parantheses.



<u>Fig. 13</u> Ewell, Okubo ε Abrams grid

The only value computed by Ewell, Okubo ε Abrams is

$$M_{A1} = 181.27 \text{ PL}$$

CHAPTER 2 ORTHOGONAL TRANSFORMATION OF AXES

1. Axis Transformation of Forces and Deformations at a Point

A system of forces acting at a point O, represented by components along coordinate axes x, y, z, may be transformed into an equivalent system along another set of axes x', y', z' with the same origin O, as shown in Fig. 14.

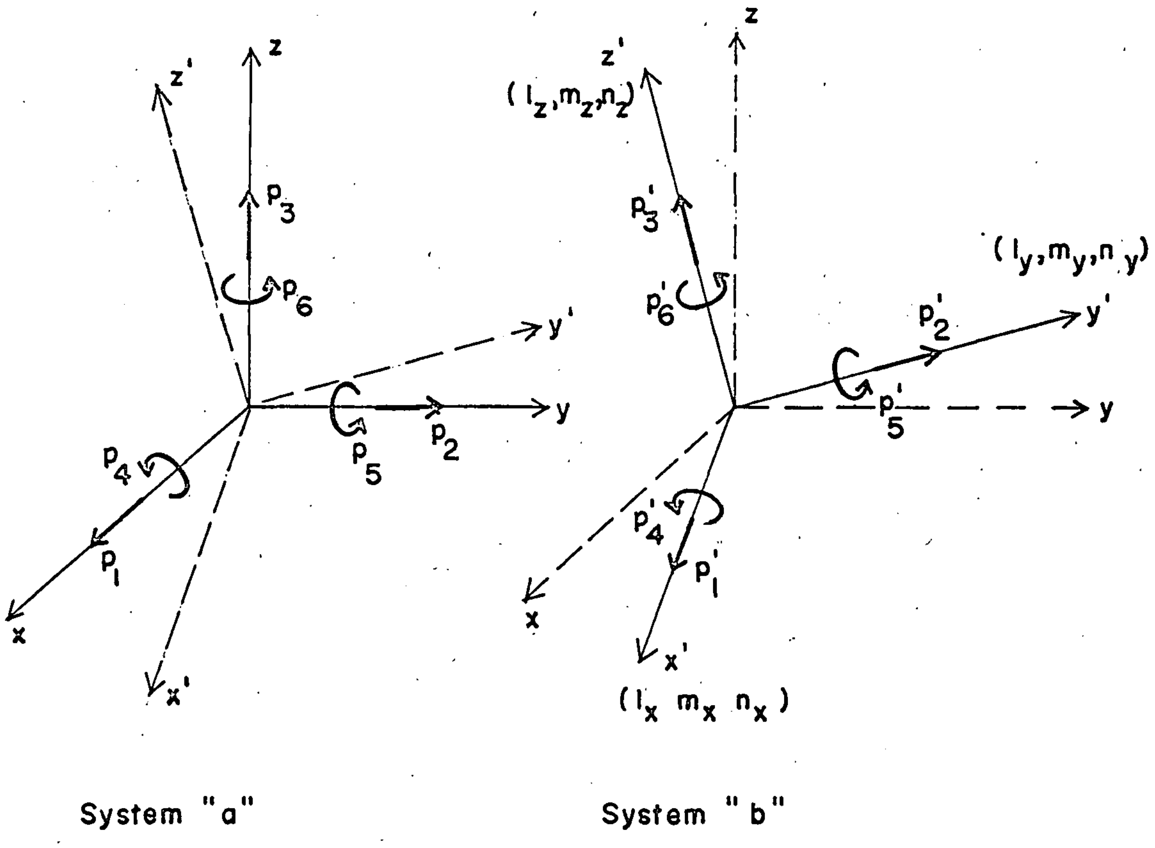


Fig. 14 Transformation from x, y, z to x', y', z'

To write the relations between the original and transformed sets of forces, referred to as "a" and "b" respectively, consider for example the transformed force p_1', which, in terms of the original forces may be expressed as

$$p_1' = l_x\, p_1 + m_x\, p_2 + n_x\, p_3$$

where l_x, m_x and n_x are evidently the direction cosines of axis x' with respect to the original axes x, y, z. Writing similar equations for the remaining forces the desired matrix relation between the two systems is obtained as

$$\begin{Bmatrix} p_1' \\ p_2' \\ p_3' \\ p_4' \\ p_5' \\ p_6' \end{Bmatrix}_b = \begin{bmatrix} \begin{bmatrix} l_x & m_x & n_x \\ l_y & m_y & n_y \\ l_z & m_z & n_z \end{bmatrix} & \begin{bmatrix} & & \\ & 0 & \\ & & \end{bmatrix} \\ \begin{bmatrix} & & \\ & 0 & \\ & & \end{bmatrix} & \begin{bmatrix} l_x & m_x & n_x \\ l_y & m_y & n_y \\ l_z & m_z & n_z \end{bmatrix} \end{bmatrix}_a \begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{Bmatrix}_a \qquad (15)$$

$$\text{i.e.} \quad \{p'\}_b = [T]_a \{p\}_a \qquad (16)$$

Note that the transformation matrix $[T]_a$ consists entirely of the direction cosines of x', y', z' with respect to x, y, z.

As forces and deformations are identically related to the coordinate system and a deformation can be expressed as a vector, a system of deformations "a" can be transformed to an equivalent system "b" by a similar equation.

$$\{\delta'\}_b = [T]_a \{\delta\}_a \qquad (17)$$

2. Transformation of Member Stiffness Matrices

To enable the analysis of diagrids and spatial grids it is necessary to obtain the stiffness matrix of the structure with respect to the chosen common axes. Consequently, the stiffness matrices of each individual member have to be transformed from their member axes to the common axes before generating the main stiffness matrix of the structure.

Let us assume that x, y, z represent the common axes of the system while x', y', z' are the member axes. The work done by the forces of the common axes along the corresponding deformations must be equal to the work done by the statically equivalent forces of the member axes, along the corresponding deformations. This may be written as

$$\frac{1}{2}\left\{\delta\right\}^{T}_{xyz}\left\{p\right\}_{xyz} = \frac{1}{2}\left\{\delta'\right\}_{x'y'z'}\left\{p'\right\}_{x'y'z'} \qquad (18)$$

From the basic stiffness equation of a member (Eq. 5),

$$\{p\}_{xyz} = [k]_{xyz}\,\{\delta\}_{xyz}$$

$$\{p'\}_{x'y'z'} \quad [k']_{x'y'z'} \quad \{\delta'\}_{x'y'z'}$$

From Eq. 16, after applying the rule of matrix transposition

$$\left\{\delta'\right\}^{T}_{x'y'z'} = \left\{\delta\right\}^{T}_{x'y'z'}\left[T\right]^{T} \qquad (19)$$

Therefore, substituting the above values of $\{p\}$, $\{p'\}$, $\{\delta'\}^{T}$ in Eq. 18 and cancelling $\frac{1}{2}$ from both sides,

$$\left\{\delta\right\}^{T}_{xyz}\underbrace{\left[k\right]_{xyz}}\left\{\delta\right\}_{xyz} = \left\{\delta\right\}^{T}_{xyz}\underbrace{\left[T\right]^{T}\left[k'\right]_{x'y'z'}\left[T\right]}\left\{\delta\right\}_{xyz} \qquad (20)$$

Comparing the left and right hand sides of the above equation, the transformed stiffness matrix k_{xyz} of the common axes is obtained from the stiffness matrix $k_{x'y'z'}$ of the member axes in the following form

$$\left[k\right]_{xyz} = \left[T\right]^{T}\left[k'\right]_{x'y'z'}\left[T\right] \qquad (21)$$

CHAPTER 3 SPATIAL GRIDS

1. Stiffness Matrix of a Spatial Grid Member

The number of possible end deformations of a spatial grid member is twelve as depicted in Fig. 15.

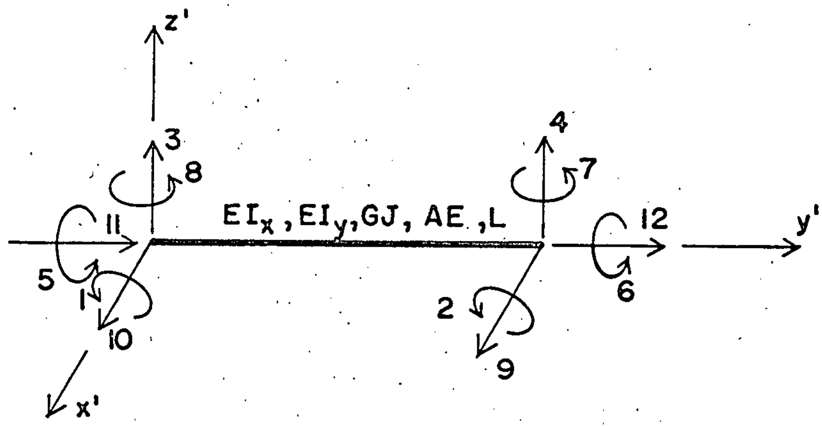


Fig. 15 End deformations of a spatial grid member

Sign convention: Translations are positive in the positive sense of the axes and rotations are positive in accordance with the right hand screw rule.

Deformations δ_1' --- δ_4' correspond to flexural rotations and deflections in relation to the major axis x'; δ_5' and δ_6' represent torsional rotations; δ_7' --- δ_{10}' are the directions of flexural rotation and deflection with respect to the minor axis z', while δ_{11}' and δ_{12}' are axial deformations. In other words, the stiffness matrix of a space member consists of the stiffness matrices of flexural members acting about the major and minor axes, a pure torque member and that of a member subject to axial deformations only.

A somewhat irregular pattern of numbering was adopted for the deformation directions in order to obtain the corresponding stiffness matrix in a very regular and convenient form as shown below.

$$
[k]_{x'y'z'} = \begin{array}{c}
\begin{array}{cccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{array} \\
\left[\begin{array}{cccc|cc|cccc|cc}
\frac{4EI_x}{L} & & \text{sym} & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{2EI_x}{L} & \frac{4EI_x}{L} & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{6EI_x}{L^2} & \frac{6EI_x}{L^2} & \frac{12EI_x}{L^3} & \frac{-12EI_x}{L^3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{-6EI_x}{L^2} & \frac{-6EI_x}{L^2} & \frac{-12EI_x}{L^3} & \frac{12EI_x}{L^3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & \frac{AE}{L} & \frac{-AE}{L} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{-AE}{L} & \frac{AE}{L} & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & \frac{4EI_z}{L} & & \text{sym} & & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{2EI_z}{L} & \frac{4EI_z}{L} & & & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{6EI_z}{L^2} & \frac{6EI_z}{L^2} & \frac{12EI_z}{L^3} & \frac{-12EI_z}{L^3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{-6EI_z}{L^2} & \frac{-6EI_z}{L^2} & \frac{-12EI_z}{L^3} & \frac{12EI_z}{L^3} & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{GI_p}{L} & -\frac{GI_p}{L} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{GI_p}{L} & \frac{GI_p}{L}
\end{array}\right]
\end{array} \qquad (22)
$$

2. Axis Transformation of a Spatial Member

A typical space member with its possible deformations marked along the member and common axes is shown in Fig. 16, with the same order of numbering as previously adopted for it's stiffness matrix.

The end forces and deformations given with respect to the member and common axes may be related to each other by means of a transformation matrix T in accordance with Eqs. 15 and 16. To correspond with the new order of numbering for the member end deformations as shown in Fig. 16, the transformation matrix is rearranged and given by

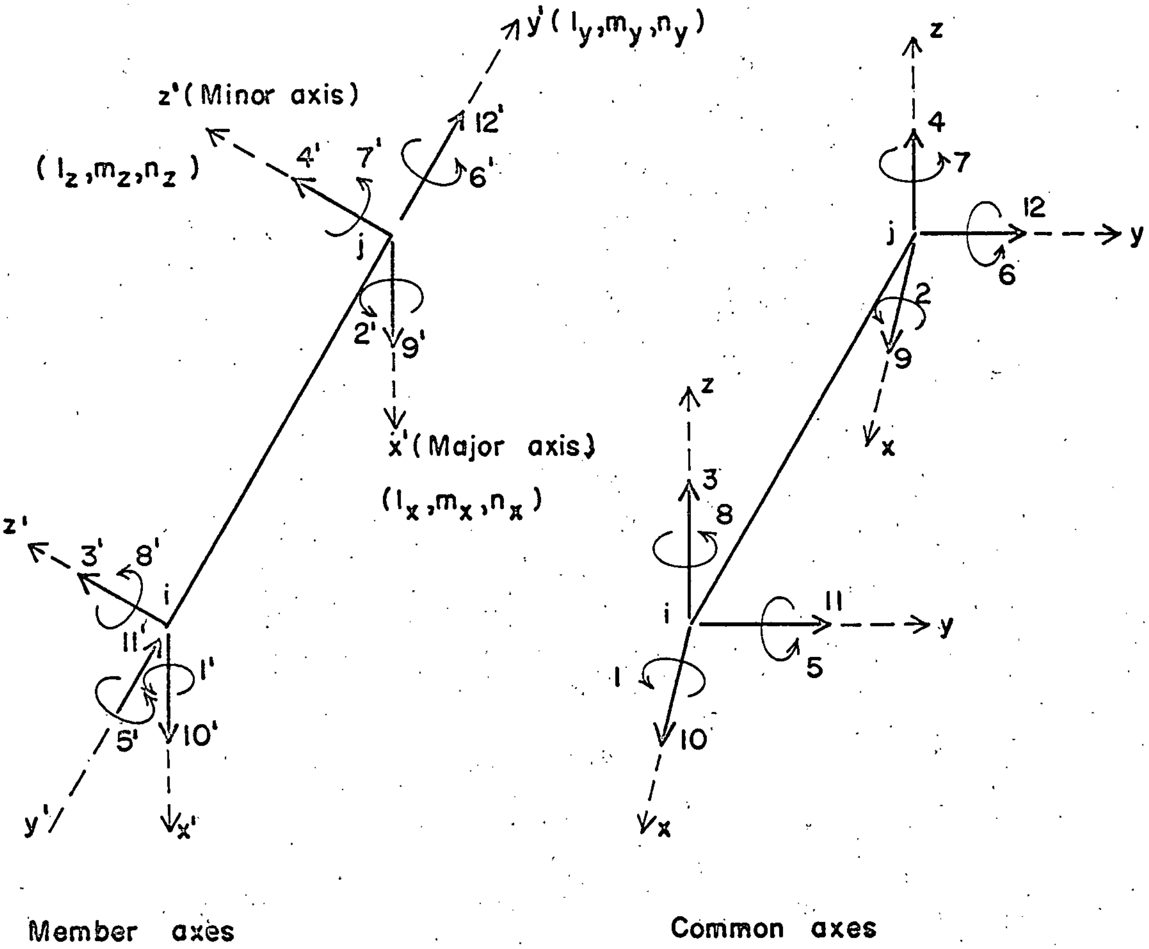


Fig. 16 A space member with respect to member and common axes

$$\begin{Bmatrix} p'_1 \\ p'_2 \\ p'_3 \\ p'_4 \\ p'_5 \\ p'_6 \\ p'_7 \\ p'_8 \\ p'_9 \\ p'_{10} \\ p'_{11} \\ p'_{12} \end{Bmatrix}_{x'y'z'} = \begin{bmatrix} l_x & 0 & 0 & 0 & m_x & 0 & 0 & n_x & 0 & 0 & 0 & 0 \\ 0 & l_x & 0 & 0 & 0 & m_x & n_x & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & n_z & 0 & 0 & 0 & 0 & 0 & 0 & l_z & m_z & 0 \\ 0 & 0 & 0 & n_z & 0 & 0 & 0 & 0 & l_z & 0 & 0 & m_z \\ l_y & 0 & 0 & 0 & m_y & 0 & 0 & n_y & 0 & 0 & 0 & 0 \\ 0 & l_y & 0 & 0 & 0 & m_y & n_y & 0 & 0 & 0 & 0 & 0 \\ 0 & l_z & 0 & 0 & 0 & m_z & n_z & 0 & 0 & 0 & 0 & 0 \\ l_z & 0 & 0 & 0 & m_z & 0 & 0 & n_z & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & n_x & 0 & 0 & 0 & 0 & l_x & 0 & 0 & m_x \\ 0 & 0 & n_x & 0 & 0 & 0 & 0 & 0 & 0 & l_x & m_x & 0 \\ 0 & 0 & n_y & 0 & 0 & 0 & 0 & 0 & 0 & l_y & m_y & 0 \\ 0 & 0 & 0 & n_y & 0 & 0 & 0 & 0 & l_y & 0 & 0 & m_y \end{bmatrix} \begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \\ p_9 \\ p_{10} \\ p_{11} \\ p_{12} \end{Bmatrix}_{xyz} \quad (23)$$

3. Direction cosines of the member axes with respect to the common axes

In general, the common axes are normally chosen for a structure to be along the transverse, longitudinal and gravity directions. These are shown in Fig. 17 by x, y and z respectively.

The direction cosines of the member axes x', y', z' with respect to the common axes x, y, z, may be obtained in terms of the coordinates of the ends of the member as follows.

First, the direction cosines of the member centre line axis y', by definition, are

$$\begin{aligned} l_y &= \frac{x_2 - x_1}{L} \\ m_y &= \frac{y_2 - y_1}{L} \\ n_y &= \frac{z_2 - z_1}{L} \end{aligned} \quad (24)$$

in which the member length L, is

$$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \tag{25}$$

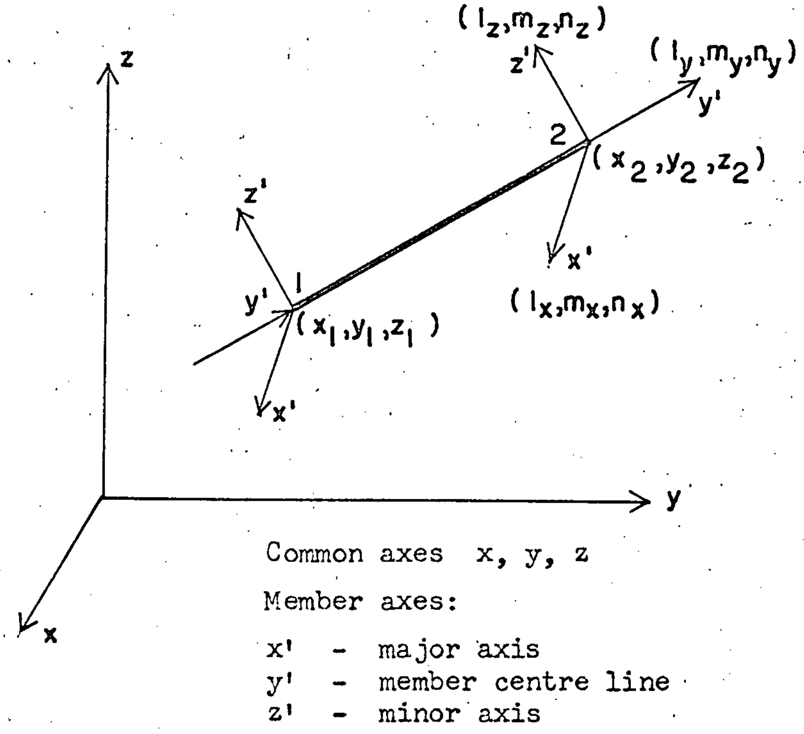


Fig. 17 Spatial grid member

Next, it is assumed that the x' axis of the member is horizontal i.e. parallel to the x, y plane, as is normally the case for most structures. Therefore,

$$n_x = 0 \tag{26}$$

The remaining direction cosines of the x' and z' axes are evaluated in terms of ly, my and ny from the conditions of normality and orthogonality.

Conditions of normality:

$$l_x^2 + m_x^2 + n_x^2 = 1 \tag{27}$$

$$l_y^2 + m_y^2 + n_y^2 = 1 \tag{28}$$

$$l_z^2 + m_z^2 + n_z^2 = 1 \tag{29}$$

Conditions of orthogonality:

$$l_x l_y + m_x m_y + n_x n_y = 0 \tag{30}$$

$$l_x l_z + m_x m_z + n_x n_z = 0 \tag{31}$$

$$l_y l_z + m_y m_z + n_y n_z = 0 \tag{32}$$

From Eq. 30,

$$m_x = -\frac{l_x l_y}{m_y} \tag{33}$$

Solving Eqs. 27 and 33,

$$l_x = \frac{m_y}{Q} \tag{34}$$

and

$$m_x = -\frac{l_y}{Q} \tag{35}$$

where $Q = \sqrt{l_y^2 + m_y^2}$

Substituting the above values into Eq. 31,

$$m_z = -\frac{l_x l_z}{m_x} \tag{36}$$

or, using Eq. 33,

$$m_z = m_y \frac{l_z}{l_y} \tag{37}$$

From Eq. 32, after replacing for m_z from Eq. 37,

$$l_y l_z + \frac{m_y^2}{l_y^2} l_z + n_y n_z = 0$$

whereby,

$$n_z = -\frac{l_z}{n_y l_y} Q^2 \qquad (38)$$

Substituting for m_z and n_z from Eqs. 37 and 38 into Eq. 29

$$l_z^2 + m_y^2 \frac{l_z^2}{l_y^2} + \frac{l_z^2}{n_y^2 l_y^2} Q^2 = 1$$

Solving this for l_z,

$$l_z = -\frac{l_y n_y}{Q} \qquad (39)$$

The negative value of the solution for l_z is used in order to conform with the directions of the assumed right-hand reference system.

Finally, from Eqs. 37 and 39,

$$m_z = -\frac{m_y n_y}{Q} \qquad (40)$$

Similarly, from Eqs. 38 and 39,

$$n_z = Q \qquad (41)$$

All the required direction cosines have now been determined and are shown in Table 2.

Major Axis x'	Centre line axis y'	Minor Axis z'
$l_x = m_y/Q$	$l_y = (x_2 - x_1)/L$	$l_z = - l_y n_y/Q$
$m_x = - l_y/Q$	$m_y = (y_2 - y_1)/L$	$m_z = - m_y n_y/Q$
$n_x = 0$	$n_y = (z_2 - z_1)/L$	$n_z = Q$
$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$;		$Q = \sqrt{l_y^2 + m_y^2}$

Table 2. Direction cosines of member axes x', y', z' with respect to common axes x, y, z.

4. Procedure of Analysis for Spatial Grids

Except for a few important differences, the procedure remains the same as described for planar grids. However, for the purpose of clarity, the steps in the analysis are fully enumerated below.

Step 1 The members, joints and possible joint deformations are numbered on a sketch of the structure. The joint deformations are indicated by arrows along the positive sense i.e., translations in the direction of the axes and rotations according to the right hand screw rule.

Step 2 The following input data is prepared:

a. Number of members and deformations.

b. Coordinates of the joints with reference to a chosen common axes.

c. Rigidities of each member, EI_x, EI_z, GJ and AE.

d. Code numbers of each member as illustrated in the following numerical example.

e. External loads acting on the structure.

Step 3 The individual member stiffness matrices $\left[k\right]_{x'y'z'}$ of Eq. 22, with respect to the member axes are evaluated numerically for each member.

Step 4 The transformation matrices of each member $\left[T\right]$ of Eq. 23, consisting of the direction cosines of its member axes are computed. The required direction cosines are defined entirely by the coordinates of the ends of the member with reference to the common axes, as shown in Table 2.

Step 5 The transformed stiffness matrix $\left[k\right]_{xyz}$ of each member is obtained by carrying out the triple matrix product of Eq. 21

Step 6 The main stiffness matrix of the system $\left[K\right]$, is generated by means of code numbers from the transformed individual stiffness matrices.

Step 7 The column vector of joint loads $\{P\}$, acting along the assumed directions of deformation of the joints, is determined by taking the sum of the fixed end reactions at each joint, with their signs reversed.

Step 8 The basic stiffness equations of the structure,

$$\{P\} = [K]\ \{D\}$$

are now used to yield the required deformations $\{D\}$ of the system along the common axes either, by inversion from

$$\{D\} = [K]^{-1}\ \{P\}$$

or by Gaussean elimination.

Step 9 The final stress resultants $\{p\}_{xyz}$ of each member along the common axes are evaluated by means of back substitution into the member stiffness equation, Eq. 5,

$$\{p\}_{xyz} = [k]_{xyz}\ \{\delta\}_{xyz} + \{F\}$$

The column vector of deformations $\{\delta\}$, is composed of the appropriate deformations of the system and is obtained by means of the member code numbers. The column vector $\{F\}$ represents the fixed end reactions of the member.

Step 10 In case the final stress resultants along the member axes $\{p'\}_{x'y'z'}$ are required, the following transformation of Eq. 16 is applied to $\{p\}_{xyz}$.

$$\{p'\}_{x'y'z'} = [T]_{xyz}\ \{p\}_{xyz}$$

5. Numerical Example (Hyperbolic Paraboloid Space Grid)

A line diagram of the grid is shown in Fig. 18. The structure is symmetrical and has 40 members and 126 joint deformations. The beams are assumed to be 18 WF 114. The torsional rigidity of this section was calculated from the corresponding formula given in Table B , Appendix B.

As an example the code number of member 3, in accordance with the order of numbering of the deformations shown in Fig. 15, is

1 13 2 14 3 15 16 4 17 5 6 18

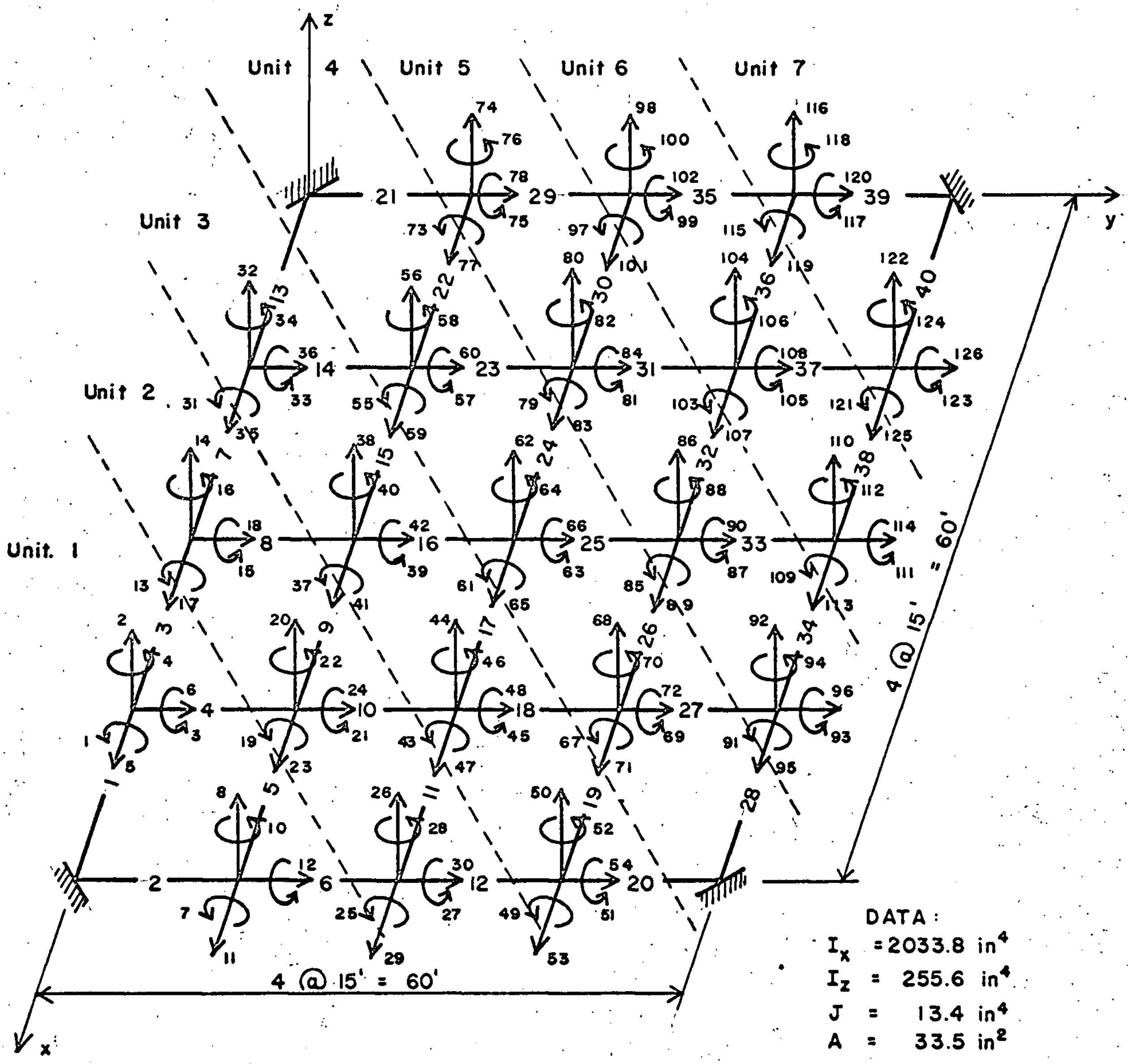


Fig. 18 Line diagram of hyperbolic paraboloid space grid

The external loads and the final bending moments along the member axes of the members for a quarter of the structure are shown in Fig. 19. The partial computer output corresponding to the final stress resultants of members 1 to 21 is also given in the following pages. The stress resultants of the remaining members may be easily obtained from symmetry.

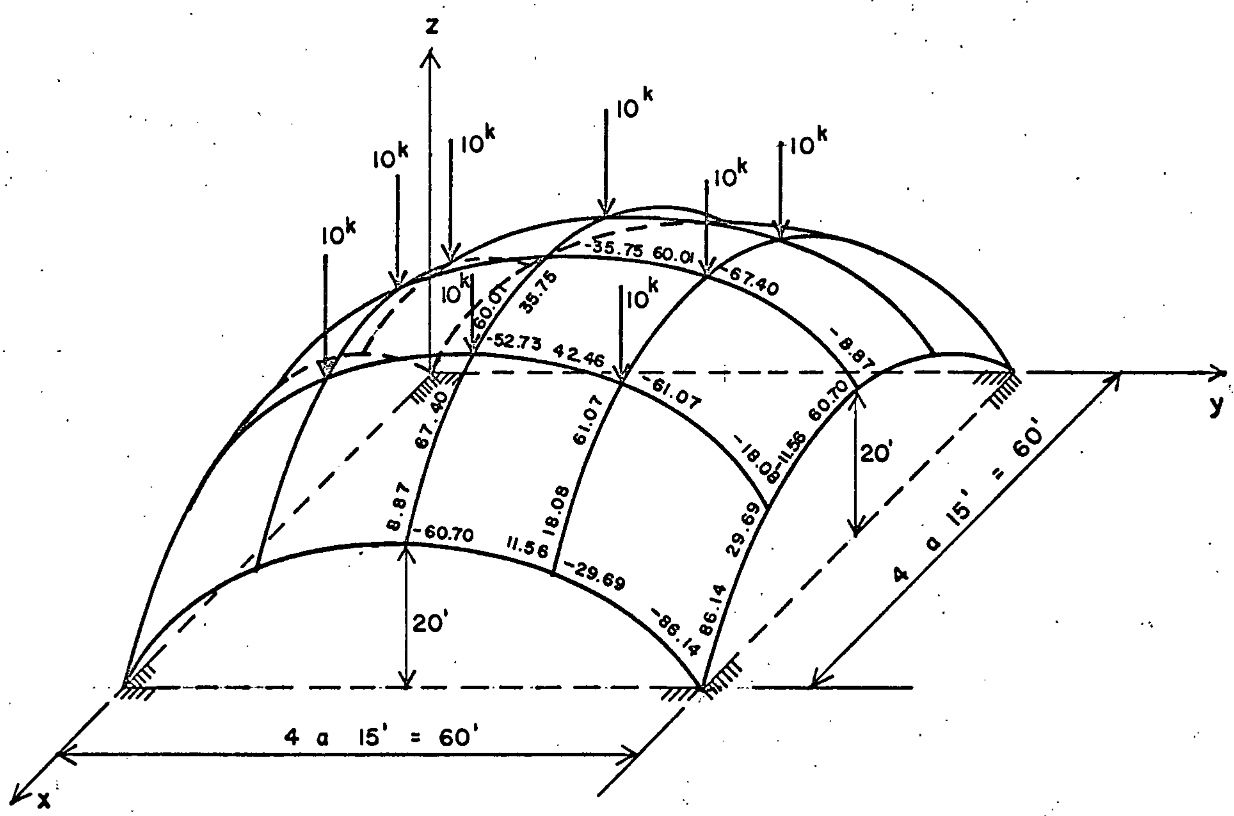


Fig. 19 Hyperbolic paraboloid space grid (Moments in k.ft.)

X 28= -.00000109
X 29= 5.09540450
X 30= -.00000140

UNIT NO. 1
LOAD NO. 1

X 1= -.59475417
X 2= -7.25365840
X 3= -.28607302
X 4= .35409652
X 5= .45614705
X 6= -5.05769620
X 7= -.28607360
X 8= -7.25367670
X 9= -.59475412
X 10= -.35409496
X 11= 5.05766090
X 12= -.45615530

FINAL STRESS RESULTANTS (IN KIPS AND KIP-FT)
FIRST LINE IS WRT COMMON, SECOND LINE IS WRT MEMBER AXES

MEMBER NUMBERS	MX1 MZ2	MX2 MZ1	Z1 X2	Z2 X1	MY1 Y1	MY2 Y2
UNIT NO. 1 LOAD NO. 1						
1	-14.949	-17.761	9.999	-9.999	86.137	29.689
	-18.586	-16.114	2.416	-2.416	2.313	-2.313
1	86.137	29.689	5.618	-5.618	-.175	.175
	-25.707	-21.980	-2.313	2.313	8.617	-8.617
2	86.137	29.689	9.999	-9.999	-14.949	-17.761
	18.586	16.114	2.313	-2.313	2.416	-2.416
2	86.137	29.689	5.618	-5.618	.175	-.175
	25.707	21.980	2.313	-2.313	8.617	-8.617

UNIT NO. 2
LOAD NO. 1

3	-.318	-4.435	3.306	-3.306	-11.557	60.700
	-11.401	-.765	.078	-.078	.8811	-.
3	-11.557	60.700	3.051	-3.051	.017	-.017
	-12.234	-.829	-.811	.811	1.276	-1.276
4	18.079	61.071	6.692	-6.692	-18.132	-14.925
	15.716	19.351	2.337	-2.337	1.502	-1.502
4	18.079	61.071	3.839	-3.839	.080	-.080
	21.674	26.519	2.337	-2.337	5.683	-5.683
5	-18.132	-14.925	6.692	-6.692	18.079	61.071
	-15.716	-19.352	1.502	-1.502	2.337	-2.337
5	18.079	61.071	3.839	-3.839	-.080	.080
	-21.674	-26.519	-2.337	2.337	5.683	-5.683
6	-11.557	60.700	3.306	-3.306	-.318	-4.435
	11.401	.765	.811	-.811	.078	-.078
6	-11.557	60.700	3.051	-3.051	-.017	.017
	12.234	.829	.811	-.811	1.276	-1.276

UNIT NO. 3
LOAD NO. 1

7	-4.435	-.318	-3.306	3.306	-60.699	11.557
	.765	11.401	.078	-.078	-.811	.811
7	-60.699	11.557	-3.051	3.051	-.017	.017
	.829	12.234	.811	-.811	1.276	-1.276
8	8.870	67.396	6.613	-6.613	0.000	0.000
	0.000	0.000	0.000	0.000	1.622	-1.622
8	8.870	67.396	3.699	-3.699	0.000	0.000
	0.000	0.000	0.000	0.000	5.717	-5.717
9	-3.682	-3.693	1.692	-1.692	-42.463	52.729

	-9.451	-9.427	2.581	-2.581	1.258	-1.258
9	-42.463	52.729	.637	-.637	0.000	0.000
	-10.147	-10.121	-1.258	1.258	3.020	-3.020
10	-42.462	52.729	1.692	-1.692	-3.682	-3.693
	9.451	9.428	1.258	-1.258	2.581	-2.581
10	-42.462	52.729	.637	-.637	0.000	0.000
	10.147	10.121	1.258	-1.258	3.020	-3.020
11	0.000	0.000	6.613	-6.613	8.870	67.396
	0.000	0.000	1.622	-1.622	0.000	0.000
11	8.870	67.396	3.699	-3.699	0.000	0.000
	0.000	0.000	0.000	0.000	5.717	-5.717
12	-60.700	11.557	-3.306	3.306	-4.435	-.318
	-.765	-11.401	-.811	.811	.078	-.078
12	-60.700	11.557	-3.051	3.051	.017	-.017
	-.829	-12.234	-.811	.811	1.276	-1.276

UNIT NO. 4
LOAD NO. 1

13	-17.761	-14.949	-9.999	9.999	-29.689	-86.137
	16.114	18.586	2.416	-2.416	-2.313	2.313
13	-29.689	-86.137	-5.618	5.618	.175	-.175
	21.980	25.707	2.313	-2.313	8.617	-8.617
14	18.079	61.071	6.692	-6.692	18.132	14.925
	-15.716	-19.352	-2.337	2.337	1.502	-1.502
14	18.079	61.071	3.839	-3.839	-.080	.080
	-21.674	-26.519	-2.337	2.337	5.683	-5.683
15	-3.693	-3.682	-1.692	1.692	-52.729	42.463
	9.428	9.451	2.581	-2.581	-1.258	1.258

15	-52.729	42.463	-.637	.637	0.000	0.000
	10.121	10.147	1.258	-1.258	3.020	-3.020
16	-60.008	35.750	0.000	0.000	0.000	0.000
	0.000	0.000	0.000	0.000	4.139	-4.139
16	-60.008	35.750	-1.506	1.506	0.000	0.000
	0.000	0.000	0.000	0.000	3.855	-3.855
17	0.000	0.000	0.000	0.000	-60.009	35.750
	0.000	0.000	4.139	-4.139	0.000	0.000
17	-60.009	35.750	-1.506	1.506	0.000	0.000
	0.000	0.000	0.000	0.000	3.855	-3.855
18	-52.729	42.463	-1.692	1.692	-3.693	-3.682
	-9.428	-9.451	-1.258	1.258	2.581	-2.581
18	-52.729	42.463	-.637	.637	0.000	0.000
	-10.121	-10.147	-1.258	1.258	3.020	-3.020
19	18.132	14.925	6.692	-6.692	18.079	61.071
	15.716	19.352	1.502	-1.502	-2.337	2.337
19	18.079	61.071	3.839	-3.839	.080	-.080
	21.674	26.519	2.337	-2.337	5.683	-5.683
20	-29.689	-86.138	-9.999	9.999	-17.761	-14.949
	-16.114	-18.586	-2.313	2.313	2.416	-2.416
20	-29.689	-86.138	-5.618	5.618	-.175	.175
	-21.980	-25.707	-2.313	2.313	8.617	-8.617

UNIT NO. 5
LOAD NO. 1

21=13	86.137	29.689	9.999	-9.999	14.949	17.761
	-18.586	-16.114	-2.313	2.313	2.416	-2.416
21=13	86.137	29.689	5.618	-5.618	-.175	.175
	-25.707	-21.980	-2.313	2.313	8.617	-8.617

CHAPTER 4 PLANAR DIAGRIDS

1. Stiffness Matrix of a Diagrid Member

A diagrid may be regarded as a special case of a space grid. Some of the possible beam outlines of planar diagrids are illustrated in Fig. 20

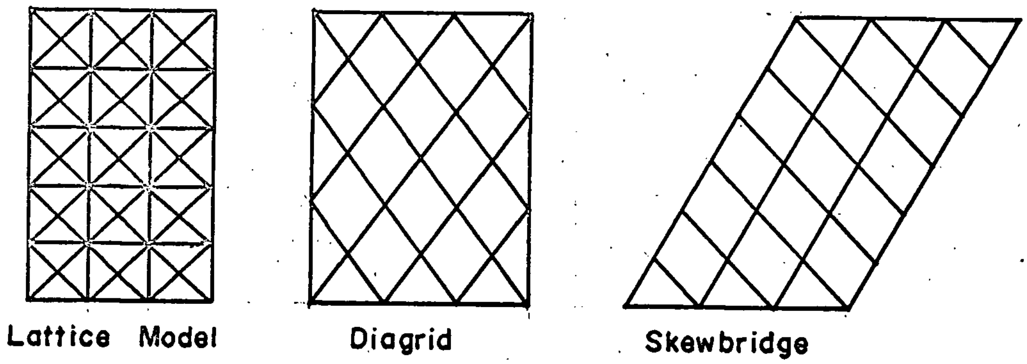


Fig. 20 Common types of planar diagrids

As for rectangular planar grids, the ends of the constituent members of a diagrid are subject to bending rotations δ_1, δ_2, vertical translations δ_3, δ_4 and torsional rotations δ_5, δ_6 as shown by numbered arrows in Fig. 21.

The directions of the end deformations are not the same as those shown in Fig. 6 for rectangular grid members, but, instead, they coincide with the directions of the corresponding deformations adopted for space members in Fig. 15. Accordingly, translations are specified along the positive sense of the coordinate axes and rotations obey the right hand screw rule.

Nevertheless, following the basic definition, the stiffness matrix $[k]_{x'y'z'}$ of a diagrid member is obtained in exactly the same form as Eq. 7 for rectangular planar grids.

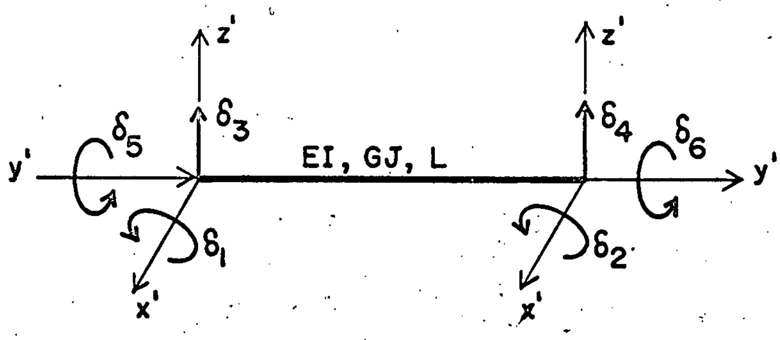


Fig. 21 End deformations of a diagrid member.

2. Axis Transformation of a Diagrid Member

The possible end deformations of a typical diagrid member are shown in Fig. 22 along the member and common axes.

Since the end deformations of a diagrid are identical with the first 6 assumed end deformations of a typical spatial grid member, the matrix $[T]_a$ required for the purpose of transforming forces from the common axes to the member axes of a diagrid is exactly the same as the first 6 by 6 submatrix of Eq. 23. That is,

$$\begin{Bmatrix} p_1' \\ p_2' \\ p_3' \\ p_4' \\ p_5' \\ p_6' \end{Bmatrix}_{x'y'z'} = \begin{bmatrix} l_x & 0 & 0 & 0 & m_x & 0 \\ 0 & l_x & 0 & 0 & 0 & m_x \\ 0 & 0 & n_z & 0 & 0 & 0 \\ 0 & 0 & 0 & n_z & 0 & 0 \\ l_y & 0 & 0 & 0 & m_y & 0 \\ 0 & l_y & 0 & 0 & 0 & m_y \end{bmatrix} \begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{Bmatrix}_{xyz} \qquad (42)$$

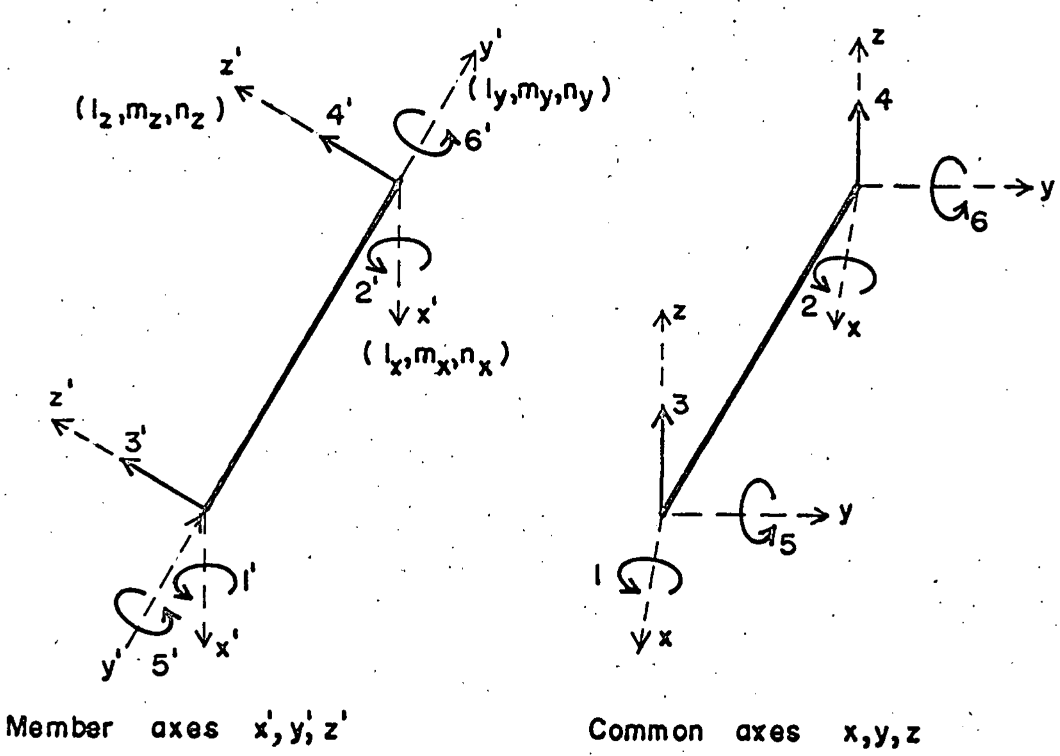


Fig. 22 Member and common axes of a diagrid

3. Direction Cosines of a Diagrid Member

The direction cosines of a diagrid member, as required by Eq. 42, may be obtained from those of a space member using Table 2. However, it is possible to further simplify them as follows.

The direction cosines of the member centre line axis, which always lies in the xy plane, are

$$l_y = \frac{x_2 - x_1}{L}$$

$$m_y = \frac{y_2 - y_1}{L}$$

$$n_y = 0$$

in which $$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Since n_y is zero, in order to satisfy the condition of normality of Eq. 28,

$$Q = \sqrt{l_y^{\,2} + m_y^{\,2}} = 1$$

Using this value of Q, the remaining direction cosines reduce to the values shown in Table 3.

Major Axis x'	Centre line Axis y'	Minor Axis z'
$l_x = m$	$l_y = \frac{x_2 - x_1}{L} = l$	$l_z = 0$
$m_x = -l$	$m_y = \frac{y_2 - y_1}{L} = m$	$m_z = 0$
$n_x = 0$	$n_y = 0$	$n_z = 1$

Table 3. Direction cosines of member axes x'y'z' of a diagrid

4. Transformed Stiffness Matrix of a Diagrid Member

The transformed stiffness matrix $[k]_{xyz}$ is obtained by carrying out the triple matrix product of Eq. 21.

Representing the ratio of $\frac{GJ}{EI}$ by q, the required transformed matrix $[k]_{xyz}$ is obtained as

$$[k]_{xyz} = \frac{EI}{L} \begin{bmatrix} (4m^2 + ql^2) & & & & & \\ (2m^2 - ql^2) & (4m^2 + ql^2) & & & & \text{Symmetrical} \\ \frac{6m}{L} & \frac{6m}{L} & \frac{12}{L^2} & & & \\ -\frac{6m}{L} & -\frac{6m}{L} & -\frac{12}{L^2} & \frac{12}{L^2} & & \\ (-4ml + qml) & (-2ml - qml) & -\frac{6l}{L} & \frac{6l}{L} & (4l^2 + qm^2) & \\ (-2ml - qml) & (-4ml + qml) & -\frac{6l}{L} & \frac{6l}{L} & (2l^2 - qm^2) & (4l^2 + qm^2) \end{bmatrix} \quad (43)$$

In order to save computer time, the transformed stiffness matrix was fed into the diagrid programme directly in the above form, thus eliminating the time otherwise required to carry out a triple matrix product for each member.

5. Procedure of Analysis for Diagrids

The procedure of analysis, which remains exactly the same as for spatial grids, has been illustrated below by several numerical examples.

6. Numerical Examples

a) Two beam grid

Step 1 The members, joints and possible joint deformations of the structure are numbered as shown in Fig. 23.

Step 2 The code numbers of each member are written alongside the members.

Step 3 Individual member stiffness matrices are obtained by substituting the numerical values of EI, GJ and L into the member stiffness matrix of Eq. 7. These matrices for all the members are

$$[k]_{x'y'z'} = \begin{bmatrix} 40 & & & & & \\ 20 & 40 & & & \text{Symmetrical} & \\ 6 & 6 & 1.2 & & & \\ -6 & -6 & -1.2 & 1.2 & & \\ 0 & 0 & 0 & 0 & 3.0 & \\ 0 & 0 & 0 & 0 & -3.0 & 3.0 \end{bmatrix}$$

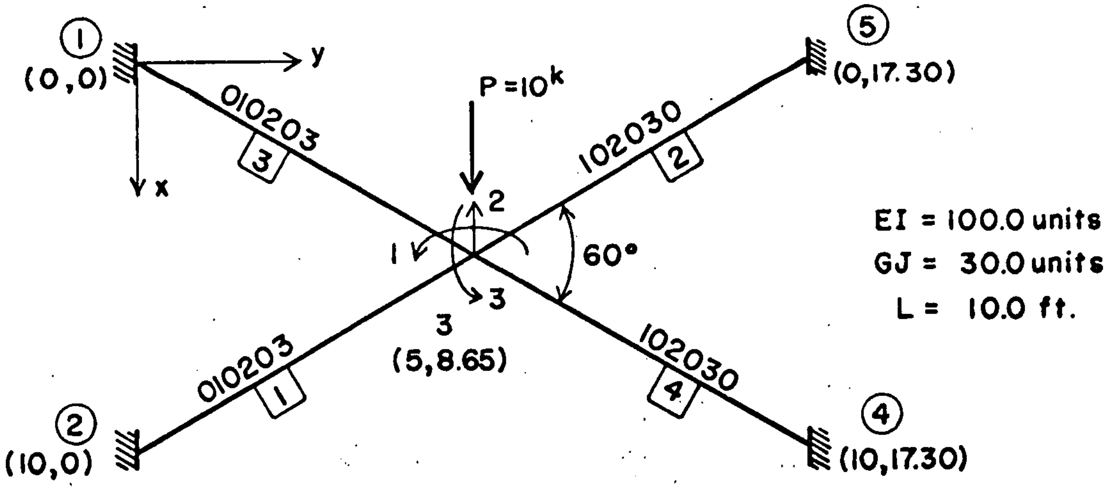


Fig. 23 Two beam diagrid

Step 4 The transformation matrix $[T]$ of Eq. 42 has to be evaluated for each member. The required direction cosines of members 1 and 2 as taken from Table 3, are

$$l = \frac{x_3 - x_2}{L} = \frac{5 - 10}{10} = -0.5$$

$$m = \frac{y_3 - y_2}{L} = \frac{8.65 - 0}{10} = 0.865$$

Therefore,

$$[T]_1 = [T]_2 = \begin{bmatrix} 0.865 & 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0.865 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -0.5 & 0 & 0 & 0 & 0.865 & 0 \\ 0 & -0.5 & 0 & 0 & 0 & 0.865 \end{bmatrix}$$

Similarly, for members 3 and 4

$$l = \frac{x_3 - x_1}{L} = \frac{5 - 0}{10} = 0.5$$

$$m = \frac{y_3 - y_1}{L} = \frac{8.65 - 0}{0} = 0.865$$

$$[T]_3 = [T]_4 = \begin{bmatrix} 0.865 & 0 & 0 & 0 & -0.5 & 0 \\ 0 & 0.865 & 0 & 0 & 0 & -0.5 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0.865 & 0 \\ 0 & 0.5 & 0 & 0 & 0 & 0.865 \end{bmatrix}$$

Step 5 The original member stiffness matrices $[k]_{x'y'z'}$ of step 3 are transformed to the common axes by carrying out the triple matrix product of Eq. 21. Alternatively, the transformed stiffness matrix of Eq. 43 yields the same results as follows,

$$[k]_1 = [k]_2 = \begin{bmatrix} 30.76 & & & & & \text{symmetrical} \\ 14.25 & 30.76 & & & & \\ 5.20 & 5.20 & 1.20 & & & \\ -5.20 & -5.20 & -1.20 & 1.20 & & \\ 16.05 & 9.97 & 3.01 & -3.01 & 12.28 & \\ 9.97 & 16.05 & 3.01 & -3.01 & 2.76 & 12.28 \end{bmatrix}$$

and, similarly,

$$[k]_3 = [k]_4 = \begin{bmatrix} 30.76 & & & & & \text{symmetrical} \\ 14.25 & 30.76 & & & & \\ 5.20 & 5.20 & 1.20 & & & \\ -5.20 & -5.20 & -1.20 & 1.20 & & \\ -16.05 & -9.97 & -3.01 & 3.01 & 12.28 & \\ -9.97 & -16.05 & -3.01 & 3.01 & 2.76 & 12.28 \end{bmatrix}$$

Step 6 Using the code numbers shown in Fig. 23, the stiffness matrix of the structure is obtained from the transformed member matrices as

$$[K] = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} \quad 1 \quad & \quad 2 \quad & \quad 3 \quad \end{array} \\ \begin{bmatrix} \begin{pmatrix} 30.76 + 30.76 \\ 30.76 + 30.76 \end{pmatrix} & & \text{symmetrical} \\ \begin{pmatrix} -5.2 + 5.2 \\ -5.2 + 5.2 \end{pmatrix} & \begin{pmatrix} 1.2 + 1.2 \\ 1.2 + 1.2 \end{pmatrix} & \\ \begin{pmatrix} 9.97 + 16.05 \\ -16.05 - 9.97 \end{pmatrix} & \begin{pmatrix} -3.01 + 3.01 \\ -3.01 + 3.01 \end{pmatrix} & \begin{pmatrix} 12.28 + 12.28 \\ +12.28 + 12.28 \end{pmatrix} \end{bmatrix} \end{array} = \begin{bmatrix} 123.04 & 0 & 0 \\ 0 & 4.8 & 0 \\ 0 & 0 & 49.12 \end{bmatrix}$$

Step 7 A single concentrated load is applied in a direction opposite to that of deformation 2. Therefore the joint loads are

$$P_1 = P_3 = 0$$

$$P_2 = -10.0$$

<u>Step 8</u> Deformations of the system:

$$\begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix} = \begin{bmatrix} 1/123.04 & 0 & 0 \\ 0 & 1/4.8 & 0 \\ 0 & 0 & 1/49.12 \end{bmatrix} \begin{Bmatrix} 0 \\ -10 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -2.08 \\ 0 \end{Bmatrix}$$

<u>Step 9</u> The final stress resultants $\{p\}_{xyz}$ with respect to the common axes, using the member stiffness equation, are given for member 1 by

$$\begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{Bmatrix}_{xyz} = \begin{bmatrix} 30.76 & & & & & \\ 14.25 & 30.76 & & & \text{symmetrical} & \\ 5.20 & 5.20 & 1.20 & & & \\ -5.20 & -5.20 & -1.20 & 1.20 & & \\ 16.05 & 9.97 & 3.01 & -3.01 & 12.28 & \\ 9.97 & 16.05 & 3.01 & -3.01 & 2.76 & 12.28 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -2.08 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 10.81 \\ 10.81 \\ 2.50 \\ -2.50 \\ 6.25 \\ 6.25 \end{Bmatrix}$$

The required final stress resultants $\{p'\}_{x'y'z'}$ with respect to the member axes are given by Eq. 16.

$$\begin{Bmatrix} p'_1 \\ p'_2 \\ p'_3 \\ p'_4 \\ p'_5 \\ p'_6 \end{Bmatrix}_{x'y'z'} = \begin{bmatrix} .865 & 0 & 0 & 0 & .5 & 0 \\ 0 & .865 & 0 & 0 & 0 & .5 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -.5 & 0 & 0 & 0 & .865 & 0 \\ 0 & -.5 & 0 & 0 & 0 & .865 \end{bmatrix} \begin{Bmatrix} 10.81 \\ 10.81 \\ 2.50 \\ -2.50 \\ 6.25 \\ 6.25 \end{Bmatrix}_{xyz} = \begin{Bmatrix} 12.50 \\ 12.50 \\ 2.50 \\ -2.50 \\ 0.0 \\ 0.0 \end{Bmatrix}_{x'y'z'}$$

The stress resultants of the remaining members are similar to those of the first member because of symmetry.

b) Cantilever Diagrid

The cantilever diagrid shown in Fig. 24 was analyzed by the stiffness method, both for zero and for an assumed torsional rigidity, GJ = 0.2 EI. The final bending moment values along the member axes obtained in the two cases as well as the error percentages caused by neglecting torsion are shown in Table 4. In addition, the complete computer output for the torsional analysis is also presented in the following pages.

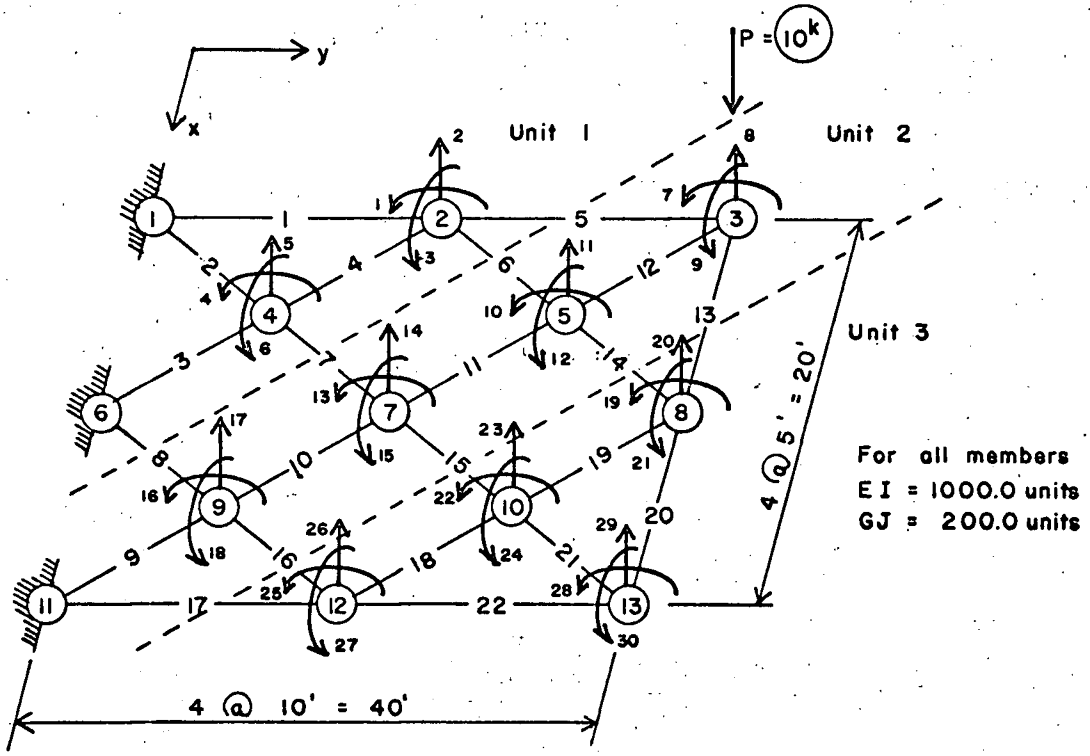


Fig. 24 Cantilever diagrid

Member No.	Bending Moment (Kip ft.)				Error %	
	Considering Torsion GJ = 0.2 EI		Neglecting Torsion GJ = 0			
	Left	Right	Left	Right	Left	Right
1	103.27	- 42.18	98.66	- 62.52	- 4.5	48.2
2	87.55	- 35.70	98.66	- 35.84	12.7	0.4
3	- 72.18	66.44	- 83.00	67.20	15.0	1.1
4	- 67.46	34.36	- 67.20	13.31	- 0.4	- 61.3
5	61.31	16.06	86.32	27.29	40.8	69.9
6	10.64	13.36	- 13.31	23.88	- 225.1	78.7
7	35.03	- 10.53	35.84	- 11.08	2.3	5.2
8	68.00	- 18.18	76.05	- 11.46	11.8	- 37.0
9	- 43.90	66.38	- 45.94	71.67	4.6	8.0
10	- 61.96	52.20	- 71.67	59.73	15.7	14.4
11	- 52.08	40.30	- 59.73	50.19	14.7	24.5
12	- 45.42	13.09	- 50.19	30.52	10.5	133.2
13	- 16.18	- 16.22	- 13.65	- 11.94	- 15.6	- 26.4
14	- 14.38	17.84	- 23.88	24.33	66.1	36.4
15	12.21	10.28	11.08	16.08	- 9.3	56.4
16	21.42	- 3.85	11.46	15.47	- 46.4	- 501.8
17	43.51	- 56.54	29.70	- 76.07	- 31.7	34.5
18	- 29.68	36.75	- 15.47	40.44	- 47.9	10.0
19	- 36.76	18.52	- 40.44	24.33	10.0	31.4
20	- 5.27	- 7.72	- 9.82	- 0.96	86.3	- 87.6
21	- 6.95	4.13	- 16.08	2.15	131.4	- 47.9
22	37.01	- 5.97	48.40	- 1.92	30.8	- 67.8

Table 4. Bending Moments of Cantilever Diagrid

```
ID NUMBER - 0663
PRINTED FOR   S. TEZCAN                 ON  MAR. 32  AT  0 HR.  38.9 MIN.

CANTILEVER DIAGRID FOR THESIS (TORSIONAL CASE)
EXECUTE FORTRAN PROGRAM.
 25
  3  1 13
   UNIT               SIZE    MEMBERS        UNCOMMON MEMBERS
    1Δ                  6        8                  4
    2                  12       13                  8
    3Δ                 12       10                 10

XΔAND Y COORDINATES OF THE JOINTS
    1     0.0    0.0    2    0.0   20.0    3     0.0   40.0    4    5.0   10.0
    5     5.0   30.0    6   10.0    0.0    7    10.0   20.0    8   10.0   40.0
    9    15.0   10.0   10   15.0   30.0Δ  11    20.0    0.0   12   20.0   20.0
   13    20.0   40.0

                                     NO.          EI                 GJ
UNIT  1.    JO=  1  2  1  4  4  6  2  4
                                      1       1000.00             200.00
                                      2       1000.00             200.00
                                      3       1000.00             200.00
                                      4       1000.00             200.00
UNIT  2.    JO=  2  3  2  5  4  7  6  9  9 11  7  9  5  7  3  5
                                      5       1000.00             200.00
                                      6       1000.00             200.00
                                      7       1000.00             200.00
                                      8       1000.00             200.00
                                      9       1000.00             200.00
                                     10       1000.00             200.00
                                     11       1000.00             200.00
                                     12       1000.00             200.00
UNIT  3.    JO=  3  8  5  8  7 10  9 12 11 12 10 12  8 10  8 13 10 13 12 13
                                     13       1000.00             200.00
                                     14       1000.00             200.00
                                     15       1000.00             200.00
                                     16       1000.00             200.00
                                     17       1000.00             200.00
                                     18       1000.00             200.00
                                     19       1000.00             200.00
```

```
                              20          1000.00          200.00
                              21          1000.00          200.00
                              22          1000.00          200.00
CODE NUMBERS
      1    0  1  0  2  0  3     2    0  4  0  5  0  6     3    4  0  5  0  6  0
      4    1  4  2  5  3  6     5    1  0  2  0  3  0     6    1  0  2  0  3  0
      7    4  0  5  0  6  0     8    0  0  0  0  0  0
      1    1  7  2  8  3  9     2    1 10  2 11  3 12     3    4 13  5 14  6 15
      4    0 16  0 17  0 18     5   16  0 17  0 18  0     6   13 16 14 17 15 18
      7   10 13 11 14 12 15     8    7 10  8 11  9 12     9    7  0  8  0  9  0
     10   10  0 11  0 12  0    11   13  0 14  0 15  0    12   16  0 17  0 18  0
     13    0  0  0  0  0  0
      1    7 19  8 20  9 21     2   10 19 11 20 12 21     3   13 22 14 23 15 24
     4Δ   16 25 17 26 18 27     5    0 25  0 26  0 27     6   22 25 23 26 24 27
      7   19 22 20 23 21 24     8   19 28 20 29 21 30     9   22 28 23 29 24 30
     10   25 28 26 29 27 30
JOINT LOADS OF UNIT  1

LOADING NO. 1
    0.000     0.000     0.000     0.000     0.000     0.000
JOINT LOADS OF UNIT  2

LOADING NO. 1
    0.000   -10.000     0.000     0.000     0.000     0.000     0.000     0.000
    0.000     0.000     0.000     0.000
JOINT LOADS OF UNIT  3

LOADING NO. 1
    0.000     0.000     0.000     0.000     0.000     0.000     0.000     0.000
    0.000     0.000     0.000     0.000


JOINT DEFORMATIONS


UNIT NO. 3
LOAD NO. 1
                        X19=      -1.62661430
                        X20=     -43.73321200
                        X21=       -.87726953
                        X22=      -1.39341410
                        X23=     -24.61836700
```

X24= -.65263376
X25= -1.00052320
X26= -9.57067070
X27= -.60801556
X28= -1.43030770
X29= -34.91345800
X30= -.86502215

UNIT NO. 2
LOAD NO. 1

X 7= -1.90705980
X 8= -52.77691500
X 9= -.87747784
X10= -1.68473800
X11= -30.81055200
X12= -.59077318
X13= -1.22888080
X14= -13.77354200
X15= -.34769403
X16= -.61392249
X17= -3.21199970
X18= -.15061574

UNIT NO. 1
LOAD NO. 1

X 1= -1.45454930
X 2= -16.58178300
X 3= -.09629858
X 4= -.81833589
X 5= -4.39166590
X 6= -.09604986

FINAL STRESS RESULTANTS
FIRST LINE IS WRT COMMON,SECOND LINE IS WRT MEMBER AXES

	M1 (K.ft)	M2 (K.ft)	Z1 (Kips)	Z2 (kips)	Y1 (K.ft)	Y2 (K.ft)
UNIT NO. 1						
LOAD NO. 1						
1	103.271	-42.183	3.054	-3.054	.962	-.962
1	103.271	-42.183	3.054	-3.054	.962	-.962

2	81.922	-35.545	4.637	-4.637	-31.923	8.735
2	87.550	-35.699	4.637	-4.637	8.083	-8.083
3	-61.662	66.802	-.514	.514	-25.229	27.799
3	66.435	-72.182	-.514	.514	-5.009	5.009
4	-33.003	62.611	-2.960	2.960	-10.815	25.619
4	34.355	-67.458	-2.960	2.960	-5.085	5.085
UNIT NO. 2						
LOAD NO. 1						
5	61.311	16.060	3.868	6.131	7.811	-7.811
5	61.311	16.060	3.868	6.131	7.811	-7.811
6	13.875	7.591	2.146	-2.146	3.966	-14.700
6	10.636	13.364Δ	2.146	-2.146	9.753	-9.753
7	34.596	-12.688	2.190	-2.190	-9.124	-1.829
7	35.025	-10.530	2.190	-2.190	7.310	-7.310
8	64.093	-19.535	4.455	-4.455	-23.861	1.582
8	67.997	-18.180	4.455	-4.455	7.321	-7.321
9	-60.488	40.383	2.010	-2.010	-27.447	17.394
9	66.377	-43.899	2.010	-2.010	-2.501	2.501
10	-47.475	56.212	-.873	.873	-21.762	26.131
10	52.195	-61.963	-.873	.873	-1.766	1.766
11	-35.936	46.476 Δ	-1.053	1.053	-18.239	23.509
11	40.299	-52.083	-1.053	1.053	.242	-.242
12	-10.452	39.366	-2.891	2.891	-8.366	22.823
12	13.090	-45.417	-2.891	2.891	2.808	-2.808
UNIT NO. 3						
LOAD NO. 1						
13	-5.608	5.608	-3.239	3.239	16.177	16.218

13	-16.177	-16.218	-3.239	3.239	-5.608	5.608
14	-11.021	14.112	.309	-.309	10.115	-11.661
14	-14.381	17.837	.309	-.309	4.118	-4.119
15	13.687	6.419	2.010	-2.010	.082	-10.136
15	12.205	10.275	2.010	-2.010	6.195	-6.195
16	23.812	-8.096	1.571	-1.571	-.265	-7.591
16	21.417	-3.847	1.571	-1.571	10.411	-10.411
17	43.507	-56.544	-.651	.651	6.080	-6.080
17	43.507	-56.544	-.651	.651	6.080	-6.080
18	-33.956	27.635	.632	-.632	-14.262	11.101
18	36.749	-29.682	.632	-.632	-2.429	2.429
19	-15.795	32.105	-1.630	1.630	-9.830	17.985
19	18.524	-36.758	-1.630	1.630	1.728	-1.728
20	-3.926	3.926	-1.299	1.299	5.273	7.722
20	-5.273	-7.722	-1.299	1.299	-3.926	3.926
21	-4.568	2.045	-.252	.252	6.413	-5.152
21	-6.954	4.133	-.252	.252	3.693	-3.693
22	37.006	-5.972	1.551	-1.551	2.570	-2.570
22	37.006	-5.972	1.551	-1.551	2.570	-2.570

PROGRAM CAME TO NORMAL END
ALL DATA CARDS WERE READ BY THE PROGRAM

END OF THIS RUN AT 1 HR. 14.7 MIN.

c) Four Girder Skew Bridge

The grid shown in Fig. 25 has been taken from Hendry ε Jaeger[3], page 43, and analyzed by the stiffness approach.

The final bending moments of all the members are shown in Fig. 25. In addition, the bending moment diagrams of girders 1 and 2 and cross girder GG, furnished by Hendry ε Jaeger are compared with those of the stiffness analysis in Fig. 26.

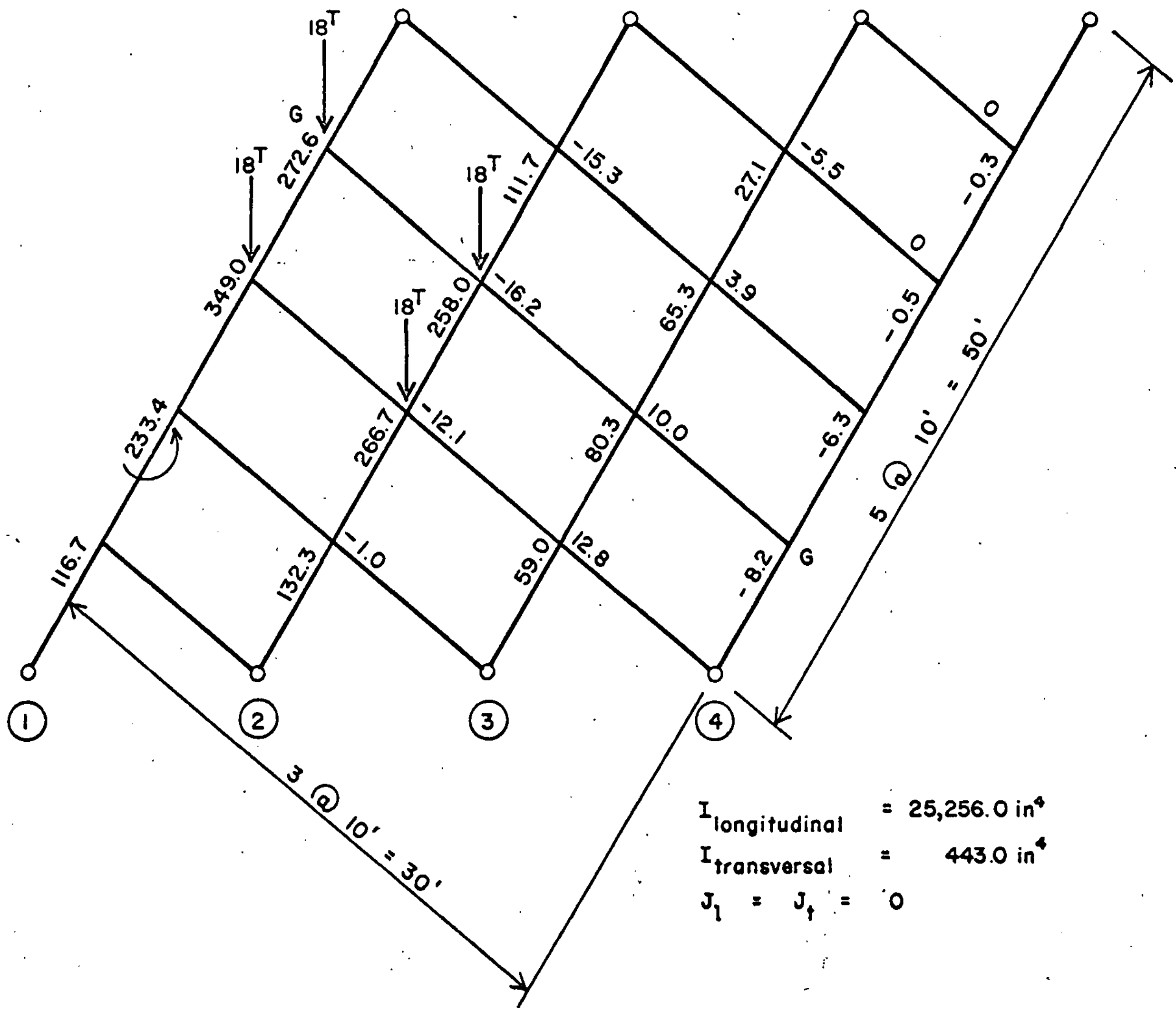


Fig. 25 Skew Bridge from Hendry ε Jaeger

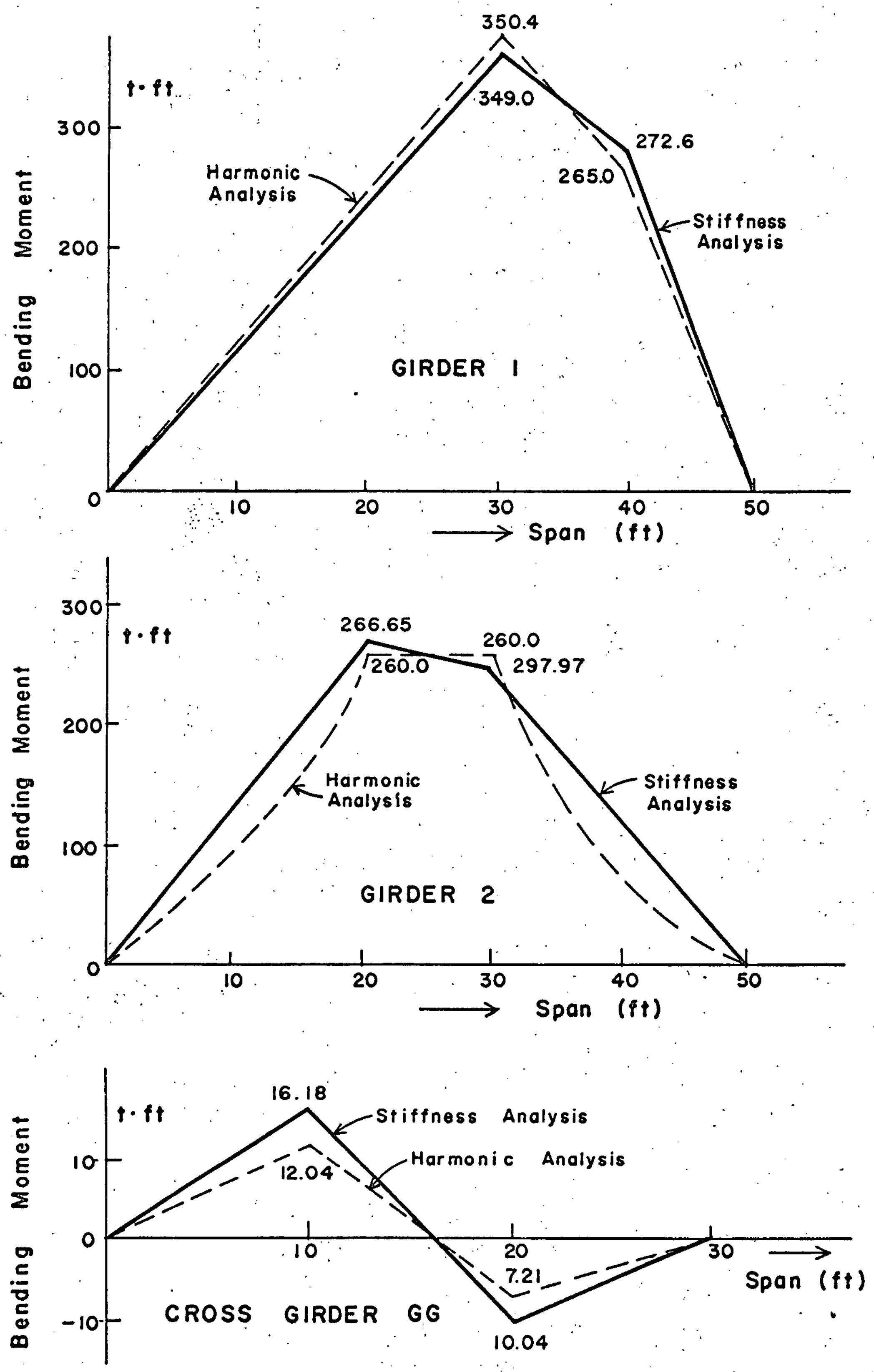


Fig. 26 Comparative results of stiffness and harmonic analysis

CHAPTER 5. SPECIAL TOPICS

1. Reduction due to Symmetry

In view of the fact that the computer time required for the solution of linear simultaneous equations, or the inversion of matrices, is approximately proportional to the cube of the number of unknowns, any reduction that is possible in their sizes is naturally desirable. For grids with one way symmetry, the number of equations can be halved as follows. Note that the loading need not be symmetrical, because any unsymmetrical irregular loading may be considered as the sum of a symmetrical and an antisymmetrical loading.

If the numbering of the joint deformations of a symmetrical structure follows a symmetrical pattern similar to that of the structure, the column vectors of joint loads and deformations can be divided into two equal portions. In case of symmetrical loading, the deformations and joint loads of the two portions are identical. Applying matrix partitioning to Eq. 11 this may be expressed as

$$\begin{Bmatrix} P \\ \hline P \end{Bmatrix} = \left[\begin{array}{c|c} K_{11} & K_{12} \\ \hline K_{21} & K_{22} \end{array}\right] \begin{Bmatrix} D \\ \hline D \end{Bmatrix} \tag{44}$$

It follows that

$$\{P\} = \left[K_{11} + K_{12}\right] \{D\} = \left[K\right]_s \{D\} \tag{45}$$

where $\left[K\right]_s$ is the effective stiffness matrix of a symmetrically loaded grid.

On the other hand, for anti-symmetrical loading, although the deformations and the joint loads of the two portions are equal, they are opposite in sign. Therefore, Eq. 11 becomes

$$\left\{\begin{array}{c} P \\ \hline -P \end{array}\right\} = \left[\begin{array}{c|c} K_{11} & K_{12} \\ \hline K_{21} & K_{22} \end{array}\right] \left\{\begin{array}{c} D \\ \hline -D \end{array}\right\} \tag{46}$$

This gives

$$\{P\} = \left[K_{11} - K_{12}\right]\{D\} = [K]_a \{D\} \tag{47}$$

in which $[K]_a$ denotes the effective stiffness matrix of an anti-symmetrically loaded grid.

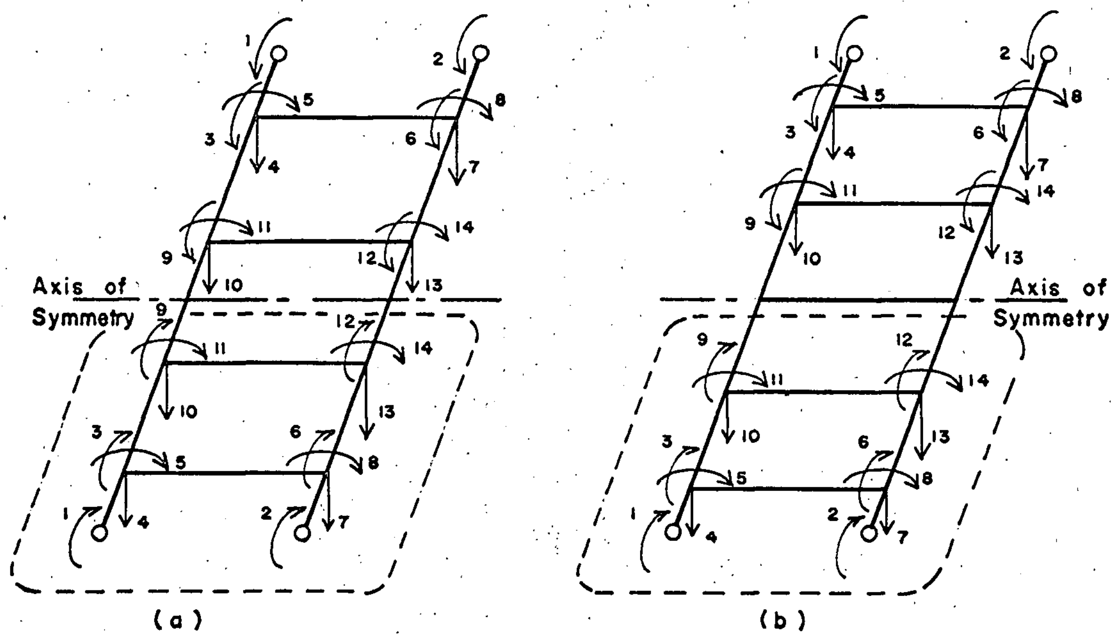


Fig. 27 Reduction due to symmetry

There is no difficulty encountered in applying Eqs. 45 & 47 to a structure such as shown in Fig. 27 (a), in which the joints are equally divided by the axis of symmetry. But, if the axis of symmetry lies along a line of joints as in Fig. 27 (b), the deformations and joint loads cannot be divided equally. It is possible, however, to make the latter case similar to the former by considering the members along the axis of symmetry to be dummy members as they remain straight upon the application of symmetrical or anti-symmetrical loads and, therefore, do not contribute to the strength of the structure.

2. Thermal Effects

The effect of temperature changes on a structure may be taken into account by the introduction of fictitious forces acting on the individual members. These forces are equal to the forces required to deform the members by an amount equal to that which would be caused by the known temperature change.

For a uniform temperature rise Δt, the elongation of a member is

$$\Delta L = \alpha_t \, \Delta t \, L \tag{48}$$

where α_t is the coefficient of thermal expansion of the material and L is the length of the member. The axial force required to produce the same elongation is

$$P = \frac{AE}{L} \, \Delta L \tag{49}$$

Substituting for ΔL from Eq. 48, the fictitious force to replace the effect of the temperature change becomes

$$P = (\alpha_t \, \Delta t) \, AE \tag{50}$$

Obviously, the force P is tensile for a temperature rise and compressive for a temperature drop.

In case the top and bottom faces of a member are subject to unequal temperature changes, Δt_1 and Δt_2 as shown in Fig. 28, the fictitious loads consist of an axial force and a moment. Assuming a linear variation of temperature between the two faces, the elongation of the centre line, from Eq. 48, is

$$\Delta L = \alpha_t \frac{\Delta t_1 + \Delta t_2}{2} L \tag{51}$$

Consequently, the fictitious axial force is obtained as

$$P = \alpha_t \frac{\Delta t_1 + \Delta t_2}{2} AE \tag{52}$$

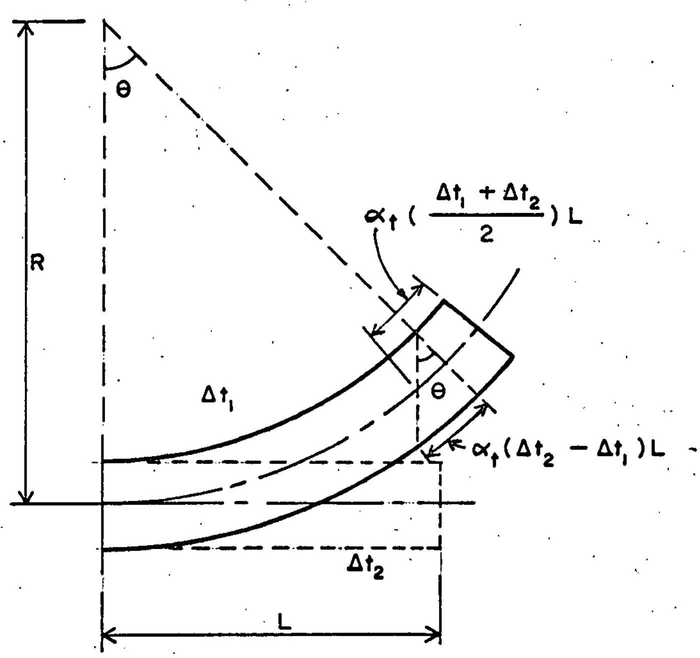


Fig. 28 Non-uniform temperature change

The rotation θ of the section due to the unequal length changes of the top and bottom fibres is

$$\theta = \frac{\alpha_t L (\Delta t_2 - \Delta t_1)}{d} \tag{53}$$

where d is the distance between the faces. The fictitious end moment required to develop the same rotation is obtained from the fundamental expression for flexural strain given by

$$\frac{1}{R} = \frac{M}{EI} \tag{54}$$

in which

$$R = \frac{L}{\theta} \tag{55}$$

Substituting Eqs. 53 and 55 into Eq. 54,

$$M = \alpha_t \frac{(\Delta t_2 - \Delta t_1)}{d} EI \tag{56}$$

To simulate the effect of thermal changes, the structure is considered to be loaded by a set of joint loads, which are calculated from the sum of the fictitious member forces given by either, Eq. 50 in the case of uniform, or, Eqs. 52 and 56 in the case of non-uniform temperature change. The rest of the analysis remains the same as described in the previous chapters. However, note that, as the assumed loads are fictitious, these must be subtracted from the calculated stress resultants of the members concerned in order to obtain the actual thermal stresses.

3. Support Settlements

A statically indeterminate structure subject to support settlements may be analyzed by specifying additional joint deformations along the directions of the settlements and enlarging the stiffness matrix of the system to include the rows and columns corresponding to the newly introduced deformations. The enlarged stiffness matrix is then multiplied by the column vector of known displacements i.e. the settlements of the structure, to yield a column vector of external loads. These are, by definition, the forces required at the joints to prevent the rest of the structure from deforming when the support settlements take place. Therefore, to simulate the actual deformed shape of the structure, that is to obtain the final joint deformations due to support settlements, the calculated external loads are reversed in sign to yield the fictitious joint loads. The analysis then proceeds in the usual manner, as for an ordinary structure under external loads. However, it is to be noted that the support settlements are considered as part of the final end deformations of the members concerned.

4. Solution of Extremely Large Structures by Band Matrices

Large numbers of equations can be solved within the limited core memory capacity of digital computers, if the coefficient matrix is obtained in a band form. This is easily achieved by adopting a regular pattern for numbering the joint deformations. The width of the band is entirely dependent on the number of joints in the transverse direction of the structure. Therefore, in order to obtain the narrowest possible band width, the numbering of deformations should start from one end of the frame and proceed consecutively along the longer dimension of the structure, covering the full width at each stage, so that the difference between any two adjacent deformation numbers is kept minimum.

When a structure is divided into substructures by transverse planes across it's width, the corresponding stiffness matrix is also divided into sub-matrices, such that for each substructure considered there is a main block located along the diagonal and two side blocks on either side consisting of terms contributed by members common with the adjacent substructures. Therefore, the stiffness matrix of any structure divided into substructures is of a triple block band matrix form. The mathematical solution of such a matrix can be achieved by applying Gaussean elimination and matrix partitioning as described below.

A group of unknowns can be eliminated by means of matrix partitioning from a set of equations, represented by Eq. 57, as follows:

$$\left[\begin{array}{c|c} K_{11} & K_{12} \\ \hline K_{21} & K_{22} \end{array}\right] \left\{\begin{array}{c} D_1 \\ \hline D_2 \end{array}\right\} = \left\{\begin{array}{c} P_1 \\ \hline P_2 \end{array}\right\} \qquad (57)$$

From Eq. 57,

$$K_{11}\, D_1 + K_{12}\, D_2 = P_1 \tag{58}$$

$$K_{21}\, D_1 + K_{22}\, D_2 = P_2 \tag{59}$$

Solving for D_1 from Eq. 58,

$$D_1 = K_{11}^{-1}\, P_1 - K_{11}^{-1}\, K_{12}\, D_2 \tag{60}$$

and, substituting into Eq. 59

$$K_{22}^{*}\, D_2 = P_2^{*} \tag{61}$$

where

$$K_{22}^{*} = (K_{22} - K_{21}\, K_{11}^{-1}\, K_{12}) \tag{62}$$

$$P_2^{*} = (P_2 - K_{21}\, K_{11}^{-1}\, P_1) \tag{63}$$

The same process of elimination may be applied successively to large band matrices. This is exemplified below for a structure divided into four units:

$$\left[\begin{array}{c:ccc} K_{11} & K_{12} & 0 & 0 \\ \hdashline K_{21} & K_{22} & K_{23} & 0 \\ 0 & K_{32} & K_{33} & K_{34} \\ 0 & 0 & K_{43} & K_{44} \end{array}\right] \left\{\begin{array}{c} D_1 \\ \hdashline D_2 \\ D_3 \\ D_4 \end{array}\right\} = \left\{\begin{array}{c} P_1 \\ \hdashline P_2 \\ P_3 \\ P_4 \end{array}\right\} \tag{64}$$

After eliminating D_1 using Eq. 61, the above equation becomes

$$\left[\begin{array}{c:cc} K_{22}^{*} & K_{23} & 0 \\ \hdashline K_{32} & K_{33} & K_{34} \\ 0 & K_{43} & K_{44} \end{array}\right] \left\{\begin{array}{c} D_2 \\ \hdashline D_3 \\ D_4 \end{array}\right\} = \left\{\begin{array}{c} P_2^{*} \\ \hdashline P_3 \\ P_4 \end{array}\right\} \tag{65}$$

in which K_{22}^{*} and P_2^{*} are the same as in Eqs. 62 and 63, respectively..

Next, eliminating D_2 from Eq. 65 in a similar manner,

$$\left[\begin{array}{c|c} K_{33}^{*} & K_{34} \\ \hline K_{43} & K_{44} \end{array}\right] \left\{\begin{array}{c} D_3 \\ \hline D_4 \end{array}\right\} = \left\{\begin{array}{c} P_3^{*} \\ \hline P_4 \end{array}\right\} \tag{66}$$

where

$$K_{33}^{*} = (K_{33} - K_{32}\, K_{22}^{*-1}\, K_{23}) \tag{67}$$

$$P_3^{*} = (P_3 - K_{32}\, K_{22}^{*-1}\, P_2^{*}) \tag{68}$$

Finally, eliminating D_3 from Eq. 66,

$$\left[K_{44}^{*}\right]\left\{D_4\right\} = \left\{P_4^{*}\right\} \tag{69}$$

in which

$$K_{44}^{*} = (K_{44} - K_{43}\, K_{33}^{*-1}\, K_{34}) \tag{70}$$

$$P_4^{*} = (P_4 - K_{43}\, K_{33}^{*-1}\, P_3^{*}) \tag{71}$$

Solving the last group of unknowns from Eq. 69,

$$\left\{D_4\right\} = \left[K_{44}^{*}\right]^{-1} \left\{P_4^{*}\right\} \tag{72}$$

These values are now back substituted step by step in accordance with Eq. 60 to obtain the remaining deformations of the structure. This gives

$$\left\{D_3\right\} = \left[K_{33}^{*}\right]^{-1} \left\{P_3^{*} - K_{34}\, D_4\right\} \tag{73}$$

$$\left\{D_2\right\} = \left[K_{22}^{*}\right]^{-1} \left\{P_2^{*} - K_{23}\, D_3\right\} \tag{74}$$

$$\left\{D_1\right\} = \left[K_{11}\right]^{-1} \left\{P_1 - K_{12}\, D_2\right\} \tag{75}$$

The chief advantage of the above technique for the solution of large numbers of unknowns is that it is extremely convenient for computer application, because the calculations follow a uniform pattern and only the standard expressions of Eqs. 60 and 61 are repeatedly employed.[a]

(a) A computer programme for the "Solution of large capacity band matrices" using triple block matrices, developed by S.S. Tezcan is available through the IBM 1620 library, Computing Centre, The University of British Columbia; Programme No. (C-FTN-P) M6-3.

CHAPTER 6. HARMONIC ANALYSIS

1. Brief Outline

Hendry & Jaeger's method of harmonic analysis is based on the assumption of a continuous transverse spread medium, in which the torsional rigidities of the transversals are neglected.

For any given loading $w = w(x)$ applied on a simple beam, Hendry & Jaeger express the load in terms of sine functions as follows,

$$w = w_1 \sin\frac{\pi x}{L} + w_2 \sin\frac{2\pi x}{L} + \text{----} + w_n \sin\frac{n\pi x}{L} + \text{---}$$

$$\therefore w = \sum_{n=1}^{n=\infty} w_n \sin\frac{n\pi x}{L} \tag{76}$$

Then, by successive integration, the shear force F, the bending moment M, the slope θ and the deflection y, are obtained as

$$F = \frac{L}{\pi} \sum_{n=1}^{n=\infty} \frac{w_n}{n} \cos\frac{n\pi x}{L} \tag{77}$$

$$M = \frac{L^2}{\pi^2} \sum_{n=1}^{n=\infty} \frac{w_n}{n^2} \sin\frac{n\pi x}{L} \tag{78}$$

$$\theta = \frac{L^3}{EI\,\pi^3} \sum_{n=1}^{n=\infty} \frac{w_n}{n^3} \cos\frac{n\pi x}{L} \tag{79}$$

$$y = \frac{L^4}{EI\,\pi^4} \sum_{n=1}^{n=\infty} \frac{w_n}{n^4} \sin\frac{n\pi x}{L} \tag{80}$$

The constants of integration are zero at each stage because of the assumed hinged ends.

Applying Fourier series to Eq. 76, the coefficients w_1, w_2 --- w_n are given by

$$w_n = \frac{2}{L}\int_0^L w \sin\frac{n\pi x}{L}\,dx \tag{81}$$

For a uniform load w per unit length, Eq. 81 reduces to

$$w_n = 0 \quad \text{if } n \text{ is even}$$

$$w_n = \frac{4w}{\pi} \quad \text{if } n \text{ is odd}$$

Substituting these values into Eq. 76,

$$w = \frac{4w}{\pi}\left(\sin\frac{\pi x}{L} + \frac{1}{3}\sin\frac{3\pi x}{L} + \text{----}\right) \tag{82}$$

For a point load W at a distance b from the left-hand support, the corresponding harmonic series for the load is

$$w = \frac{2W}{L}\left(\sin\frac{\pi b}{L}\sin\frac{\pi x}{L} + \sin\frac{2\pi b}{L}\sin\frac{2\pi x}{L} + \text{----}\right) \tag{83}$$

Once the relevant load series is determined, the shear, moment, slope and deflection curves are readily computed using Eqs. 77 - 80.

For the analysis of single span interconnected bridge girders with more than one longitudinal, Hendry & Jaeger evaluate distribution coefficients of the first and higher harmonics of the free deflection curve of one of the longitudinals, assuming that it carries the entire loading by itself. These distribution coefficients are obtained by considering the relative deflections of the longitudinals in the transverse direction, in terms of the dimensionless grid parameter α, which represents the stiffness of the frame.

$$\alpha = \frac{12}{\pi^4}\left(\frac{L}{h}\right)^3 \frac{n\,EI_T}{EI} \tag{84}$$

where, EI_T = flexural rigidity of a transversal,

EI = flexural rigidity of a longitudinal,

n = number of cross girders,

h = spacing of the longitudinals.

The distribution factors derived to distribute the free deflection curve are assumed to be equally valid for distributing the loads, shears and bending moments. Further, the same distribution factor is used for the whole length of a longitudinal, irrespective of the position of the load. This is not really the case and the accuracy of these factors is limited, as is amply illustrated by the examples in the following section.

For grids with 2 - 6 longitudinals, Hendry & Jaeger provide the values of the distribution factor ρ, for zero ($\beta = 0$) and full torsion, ($\beta = \infty$). The torsional parameter β characterizes the torsional rigidity of the grid and is given by

$$\beta = \frac{\pi^2}{2n}\left(\frac{h}{L}\right)\frac{GJ_L}{EI_T} \tag{85}$$

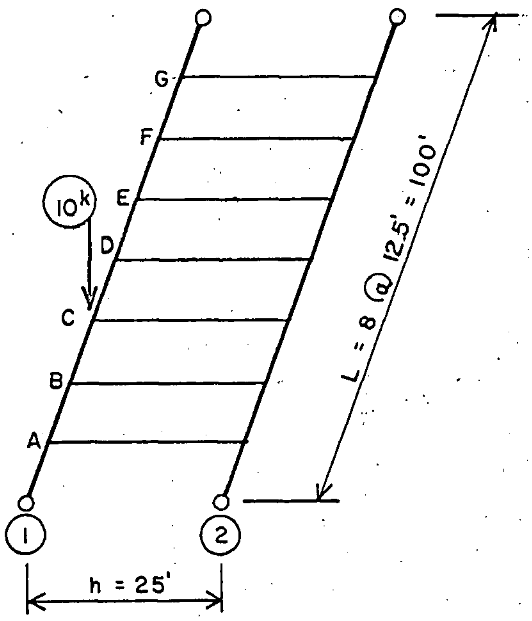
in which GJ_L is the torsional rigidity of the longitudinals. For intermediate values of β, Hendry & Jaeger recommend the following interpolation function for the distribution factors,

$$\rho_\beta = \rho_o + (\rho_\infty - \rho_o)\sqrt{\frac{\beta\sqrt{\alpha}}{3 + \beta\sqrt{\alpha}}} \tag{86}$$

To evaluate the transversal bending moments, the deflected shape of the longitudinals and, hence, the relative deflections of the ends of the transversals are determined. Slope deflection equations then yield the required moments. The transverse moment expressions, in terms of the longitudinal bending moment distribution coefficients and the free bending moment curve, are tabulated in Appendix IC.

In order to demonstrate the method of harmonic analysis and to compare the results with those of the stiffness approach, a two girder bridge is completely analyzed below.

2. Numerical Example - Two Girder Bridge



Longitudinals
EI_L = 10,000.0 units
GJ_L = 3,550.0 units
Transversals
EI_T = 500.0 units
GJ_T = 177.5 units

Fig. 29 Two girder bridge

Step 1 Using an average number of transversals i.e. $n = 8$, the dimensionless parameters are:

$$\alpha = \frac{12}{\pi^4}\left(\frac{L}{h}\right)^3 \frac{n\,EI_T}{EI_L} = \frac{12}{\pi^4}\left(\frac{100}{25}\right)^3 \frac{8 \times 500}{10,000} = 3.154$$

$$\beta = \frac{\pi^2}{2n}\left(\frac{h}{L}\right)\frac{GJ_L}{EI_T} = \frac{\pi^2}{2 \times 8}\left(\frac{25}{100}\right)\frac{3550}{500} = 1.25$$

$$\eta = \frac{\text{outer}}{\text{inner}} = \frac{EI_1}{EI_2} = 1.0$$

Step 2 By Eq. 86, the interpolation function for $\beta = 1.25$ is

$$\rho_{(1.25)} = \rho_o + (\rho_\infty - \rho_o)\sqrt{\frac{1.25\sqrt{3.154}}{3 + 1.25\sqrt{3.154}}}$$

$$\rho_{(1.25)} = \rho_o + (\rho_\infty - \rho_o)\ 0.651$$

Step 3 Referring to Appendix I-B Pg. 252, the first harmonic bending moment distribution factors for a load on girder (1) are obtained as follows:

For $\beta = o$, $\rho_{11} = 1.0$

$\rho_{21} = 0$

For $\beta = \infty$, $\rho_{11} = \dfrac{1 + \alpha_o}{1 + 2\alpha_o} = 0.728$

$\rho_{21} = \dfrac{\alpha_o}{1 + 2\alpha_o} = 0.272$

where, $\alpha_o = \alpha(1 - \frac{8}{\pi^2}) = 3.154\ (1 - \frac{8}{\pi^2}) = 0.597$

Interpolating for $\beta = 1.25$, the required distribution factors become

$$\rho_{11} = 1.0 + (0.728 - 1.0)\ 0.651 = 0.822$$

$$\rho_{21} = 0. + (0.272 - 0)\ 0.651 = 0.178$$

Step 4 The simple bending moment diagram for the grid treated as a single girder is shown in Fig. 30.

Step 5 The longitudinal bending moments are obtained simply by distributing the simple bending diagram in proportion with the calculated distribution factors. For example the bending moment at the load point, M_{1C}, is

$$M_{1C} = \quad M = 0.822 \times 234.37 \;=\; 193.0 \text{ k.ft}$$

The remaining moments at the nodal points are similarly calculated.

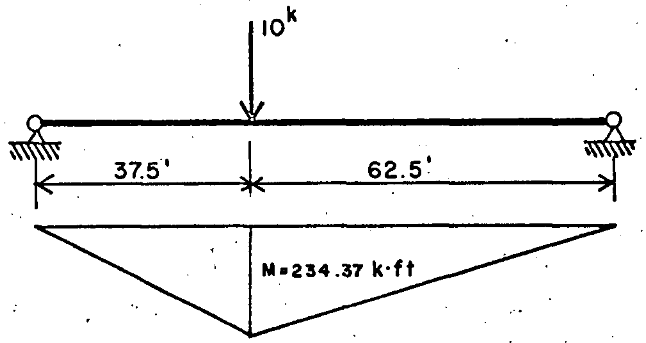


Fig. 30 Simple bending moment diagram

Step 6 From Appendix I-C pg. 260, the transversal bending moments are

$$M_{12} = M_{21} = 1.79 \frac{h}{L^2} \quad (\rho_1 - \rho_2) \;\; M \;=\; \mu M \tag{87}$$

per unit length of the transverse medium. The distribution factor μ introduced here will be referred to as the transverse bending moment coefficient. Substituting the appropriate numerical values into Eq. 87, the required transversal bending moments are obtained as

$$M_{12} = M_{21} = 1.79 \times \frac{25}{100^2} \times 3.154\ (0.822 - 0.178) \;\; M = 0.0091\ M$$

Hence, for the loaded transversal

$$M_{CC} = 0.0091 \times 234.37 \times 12.5 \;=\; 26.65 \text{ k.ft}$$

The remaining transversal bending moments are calculated in like fashion, and the complete results of harmonic analysis, as well as, the corresponding stiffness solution are shown in Fig. 31. In addition, the complete IBM-7090 computer output of the above example for several loading conditions is also presented in the following pages.

```
TWO BEAM GRID USED TO CHECK HENDRYS DISTRIBUTION FACTORS(27T)
 23 46 32 22 36  1
CODE NUMBERS,ICD(I)
   1    1  2  0 33  0  3    2    2  4 33 34  3  5    3    4  6 34 35  5  7
   4    6  8 35 36  7  9    5    8 10 36 37  9 11    6   10 12 37 38 11 13
   7   12 14 38 39 13 15    8   14 16 39  0 15  0    9   17 18  0 40  0 19
  10   18 20 40 41 19 21   11   20 22 41 42 21 23   12   22 24 42 43 23 25
  13   24 26 43 44 25 27   14   26 28 44 45 27 29   15   28 30 45 46 29 31
  16   30 32 46  0 31  0   17    3 19 33 40  2 18   18    5 21 34 41  4 20
  19    7 23 35 42  6 22   20    9 25 36 43  8 24   21   11 27 37 44 10 26
  22   13 29 38 45 12 28   23   15 31 39 46 14 30
LENGTHS OF THE MEMBERS
      12.5      12.5      12.5      12.5      12.5      12.5      12.5      12.5
      12.5      12.5      12.5      12.5      12.5      12.5      12.5      12.5
      25.0      25.0      25.0      25.0      25.0      25.0      25.0
EI VALUES OF THE MEMBERS
   10000.0   10000.0   10000.0   10000.0   10000.0   10000.0   10000.0   10000.0
   10000.0   10000.0   10000.0   10000.0   10000.0   10000.0   10000.0   10000.0
     500.0     500.0     500.0     500.0     500.0     500.0     500.0
GJ VALUES OF THE MEMBERS
    3550.0    3550.0    3550.0    3550.0    3550.0    3550.0    3550.0    3550.0
    3550.0    3550.0    3550.0    3550.0    3550.0    3550.0    3550.0    3550.0
     167.5     167.5     167.5     167.5     167.5     167.5     167.5
INVERSE OF SG FROM NS+1 (DEFLECTION INFLUENCE SURFACE)
   33
      0.2962      0.4732      0.5363      0.5159      0.4350      0.3122      0.1626
      0.1025      0.1860      0.2368      0.2491      0.2242      0.1680      0.0897
   34
      0.4732      0.8321      0.9884      0.9707      0.8276      0.5974      0.3122
      0.1860      0.3398      0.4358      0.4616      0.4176      0.3141      0.1680
   35
      0.5363      0.9884      1.2663      1.2998      1.1329      0.8276      0.4350
      0.2368      0.4358      0.5648      0.6046      0.5517      0.4176      0.2242
   36
      0.5159      0.9707      1.2998      1.4284      1.2998      0.9707      0.5159
      0.2491      0.4616      0.6046      0.6550      0.6046      0.4616      0.2491
   37
      0.4350      0.8276      1.1329      1.2998      1.2663      0.9884      0.5363
      0.2242      0.4176      0.5517      0.6046      0.5648      0.4358      0.2368
   38
      0.3122      0.5974      0.8276      0.9707      0.9884      0.8321      0.4732
      0.1680      0.3141      0.4176      0.4616      0.4358      0.3398      0.1860
   39
      0.1626      0.3122      0.4350      0.5159      0.5363      0.4732      0.2962
      0.0897      0.1680      0.2242      0.2491      0.2368      0.1860      0.1025
   40
      0.1025      0.1860      0.2368      0.2491      0.2242      0.1680      0.0897
      0.2962      0.4732      0.5363      0.5159      0.4350      0.3122      0.1626
   41
      0.1860      0.3398      0.4358      0.4616      0.4176      0.3141      0.1680
      0.4732      0.8321      0.9884      0.9707      0.8276      0.5974      0.3122
   42
      0.2368      0.4358      0.5648      0.6046      0.5517      0.4176      0.2242
      0.5363      0.9884      1.2663      1.2998      1.1329      0.8276      0.4350
   43
      0.2491      0.4616      0.6046      0.6550      0.6046      0.4616      0.2491
      0.5159      0.9707      1.2998      1.4284      1.2998      0.9707      0.5159
   44
      0.2242      0.4176      0.5517      0.6046      0.5648      0.4358      0.2368
```

	0.4350	0.8276	1.1329	1.2998	1.2663	0.9884	0.5363
45							
	0.1680	0.3141	0.4176	0.4616	0.4358	0.3398	0.1860
	0.3122	0.5974	0.8276	0.9707	0.9884	0.8321	0.4732
46							
	0.0897	0.1680	0.2242	0.2491	0.2368	0.1860	0.1025
	0.1626	0.3122	0.4350	0.5159	0.5363	0.4732	0.2962

FINAL END MOMENTS FOR 10K (IN k.ft)

1	-0.000	-96.915	-0.00035	2	96.112	-72.079	-0.00028	
3	71.835	-52.805	-0.00015	4	52.884	-38.036	-0.00010	
5	38.288	-26.396	0.00003	6	26.732	-16.708	0.00016	
7	17.080	-8.111	0.00018	8	8.498	0.000	0.00009	
9	-0.000	-12.461	-0.00006	10	13.264	-21.673	-0.00007	LOAD AT
11	21.917	-25.323	0.00002	12	25.243	-24.467	-0.00000	1A
13	24.214	-20.482	-0.00000	14	20.146	-14.544	0.00004	
15	14.171	-7.515	0.00004	16	7.128	0.000	0.00003	
17	4.052	4.052	0.00000	18	5.004	5.004	0.00000	
19	4.182	4.182	-0.00000	20	2.956	2.956	0.00000	
21	1.869	1.869	0.00000	22	1.055	1.055	-0.00000	
23	0.471	0.471						
1	-0.000	-72.954	-0.00060	2	71.560	-149.529	-0.00060	
3	148.809	-109.996	-0.00039	4	110.004	-79.128	-0.00020	
5	79.543	-54.697	0.00019	6	55.321	-34.474	0.00033	
7	35.195	-16.684	0.00038	8	17.446	0.000	0.00018	
9	-0.000	-20.799	-0.00011	10	22.192	-37.975	-0.00015	
11	38.694	-46.259	0.00001	12	46.251	-45.877	-0.00000	LOAD AT
13	45.462	-39.057	0.00003	14	38.434	-28.029	0.00008	1B
15	27.309	-14.568	0.00006	16	13.805	0.000	0.00007	
17	5.016	5.016	-0.	18	8.218	8.218	0.00000	
19	7.938	7.938	-0.00000	20	6.032	6.032	0.00000	
21	3.999	3.999	-0.00000	22	2.336	2.336	0.00000	
23	1.064	1.064						
1	-0.000	-54.014	-0.00066	2	52.491	-110.709	-0.00066	
3	109.572	-175.739	-0.00039	4	175.356	-126.578	-0.00023	
5	126.957	-87.144	0.00010	6	87.942	-54.619	0.00030	
7	55.630	-26.317	0.00048	8	27.425	0.000	0.00021	
9	-0.000	-24.114	-0.00015	10	25.636	-45.546	-0.00015	LOAD AT
11	46.682	-58.642	0.00003	12	59.025	-60.929	-0.00005	1C
13	60.550	-53.487	0.00002	14	52.689	-39.135	0.00007	
15	38.124	-20.560	0.00008	16	19.452	0.000	0.00009	
17	4.204	4.204	0.00000	18	7.950	7.950	-0.00000	
19	10.055	10.055	0.00000	20	8.967	8.967	0.00000	
21	6.491	6.491	-0.00000	22	4.010	4.010	0.00000	
23	1.887	1.887						
1	-0.000	-39.250	-0.00063	2	37.865	-80.095	-0.00061	
3	78.912	-127.191	-0.00057	4	126.425	-183.675	-0.00031	
5	183.674	-126.425	0.00017	6	127.190	-78.912	0.00034	
7	80.095	-37.864	0.00057	8	39.250	0.000	0.00031	
9	-0.000	-23.252	-0.00015	10	24.638	-44.910	-0.00021	LOAD AT
11	46.092	-60.316	-0.00001	12	61.081	-66.333	0.00000	1D
13	66.333	-61.081	0.00003	14	60.316	-46.092	0.00009	
15	44.910	-24.637	0.00009	16	23.252	0.000	0.00011	
17	2.980	2.980	0.00000	18	6.049	6.049	0.00000	
19	8.972	8.972	-0.00000	20	10.503	10.503	0.00000	
21	8.972	8.972	-0.00000	22	6.049	6.049	0.00001	
23	2.980	2.980						

UNIFORM LOADS ON THE MEMBERS

1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
0.	0.	0.	0.	0.	0.	0.	0.

0. 0. 0. 0. 0. 0. 0.

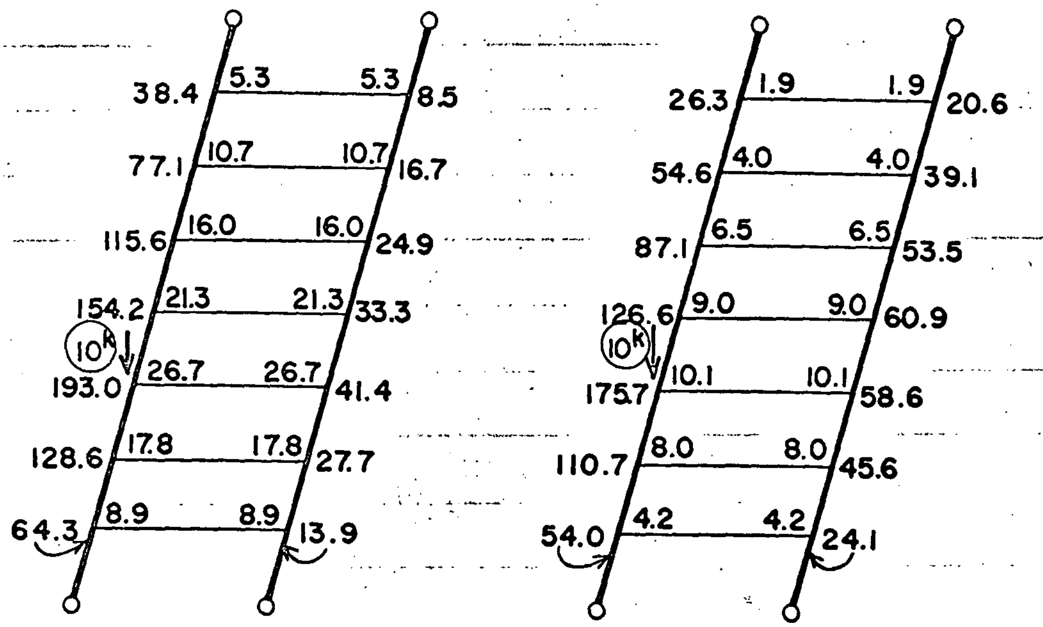
SET OF LOADING NO. 1

DEFORMATIONS OF THE SYSTEM TIMES E (INCHES)

1	2.8621	2	2.5999	3	-0.5401	4	1.9394	5	-0.9917
6	1.0296	7	-1.2886	8	-0.0000	9	-1.3919	10	-1.0296
11	-1.2886	12	-1.9394	13	-0.9917	14	-2.5999	15	-0.5401
16	-2.8621	17	1.3047	18	1.2089	19	-0.5401	20	0.9253
21	-0.9917	22	0.5004	23	-1.2886	24	-0.0000	25	-1.3919
26	-0.5004	27	-1.2886	28	-0.9253	29	-0.9917	30	-1.2089
31	-0.5401	32	-1.3047	33	34.6494	34	63.3612	35	82.1128
36	88.6115	37	82.1128	38	63.3612	39	34.6493	40	15.9098
41	29.4153	42	38.4357	43	41.6012	44	38.4356	45	29.4153
46	15.9097								

	M1	M2	CHECK		M1	M2	CHECK
1	-0.002	-393.495	-0.00299	2	384.172	-646.553	-0.00137
3	639.758	-789.814	-0.00245	4	786.266	-835.085	-0.00318
5	835.082	-786.267	0.00146	6	789.814	-639.753	0.00356
7	646.551	-384.170	0.00325	8	393.493	0.002	0.00179
9	-0.000	-153.398	0.00019	10	162.718	-290.981	-0.00083
11	297.774	-382.102	0.00050	12	385.649	-414.962	0.00001
13	414.962	-385.647	0.00037	14	382.102	-297.773	0.00052
15	290.980	-162.717	0.00066	16	153.398	0.001	0.00079
17	25.136	25.136	-0.00002	18	43.933	43.933	0.00000
19	55.012	55.012	0.00003	20	58.626	58.626	-0.00003
21	55.012	55.012	-0.00001	22	43.932	43.932	0.00003
23	25.136	25.136		0			

w = 1 k/ft on 1



HARMONIC ANALYSIS STIFFNESS ANALYSIS

Fig. 31 Final Bending Moments (k.ft)

3. Comparison of Harmonic and Stiffness Analysis Distribution Factors

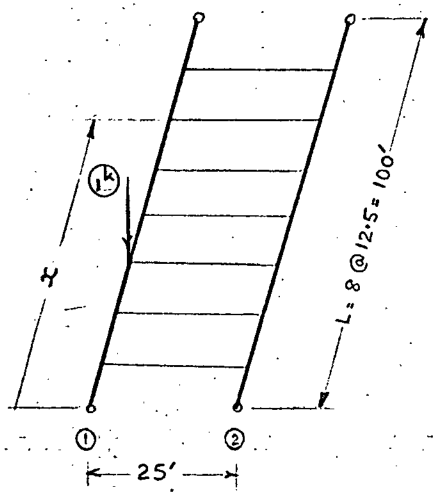
Several example grids with 2 - 6 longitudinals at a moderate spacing were selected in order to provide a relative idea of the results obtained by harmonic and stiffness analysis. In every case, seven transversals were used to simulate Hendry & Jaeger's assumption of a uniform transverse spread medium.

Constant values of $EI_L = 10,000.$ and $EI_T = 500.$ units were adopted for the elastic rigidities of the longitudinals and transversals, respectively. Each grid was analyzed first assuming the members to possess zero torsional rigidity ($\beta = 0$), and then assuming the torsional rigidity of each member to be of such a value that the harmonic analysis torsional parameter β equals 1.25, which almost corresponds to infinite torsional rigidity. In order to cover a wide range of loading conditions, a point load and a uniformly distributed load of unit intensity were non-concurrently applied at the joints and along the longitudinals respectively.

In the following tables, the harmonic analysis longitudinal and transversal bending moment distribution coefficients ρ and μ, calculated as described in the preceding numerical example, are tabulated against the corresponding values of stiffness analysis. It is clearly seen, that the harmonic analysis distribution factors are entirely independent of the location of the section and the point of application of the load along the longitudinals. This approximation gives rise to considerable inaccuracies, as the exact analysis provides different values for each different section and point of application of the load. These errors range from $\mp$ 10 to $\mp$ 200% depending upon the arrangement of the girders and loads.

GIRDER NO.	SECTION x	STIFFNESS ANALYSIS					HARMONIC ANALYSIS ρ
		Load on Girder 1					
		Point Load at				UDL	
		.125L	.25L	.375L	.5L		
1	.125L	.885 [b] 1.	.770 1.	.676 1.	.609 1.	.719 1.	
	.25L	.770 1.	.798 1.	.703 1.	.633 1.	.690 1.	
	.375L	.676 1.	.703 1.	.750 1.	.675 1.	.673 1.	
	.5L	.609 1.	.633 1.	.675 1.	.735 1.	.669 1.	ρ_{11} .822 1.
	.625L	.563 1.	.585 1.	.620 1.	.675 1.	.673 1.	
	.75L	.535 1.	.551 1.	.585 1.	.633 1.	.690 1.	
	.875L	.518 1.	.535 1.	.563 1.	.609 1.	.719 1.	
2	.125L	.115 0	.230 0	.324 0	.391 0	.281 0	
	.25L	.230 0	.202 0	.297 0	.367 0	.310 0	
	.375L	.324 0	.297 0	.250 0	.325 0	.317 0	
	.5L	.391 0	.367 0	.325 0	.265 0	.331 0	ρ_{21} .178 0
	.625L	.437 0	.415 0	.380 0	.325 0	.317 0	
	.75L	.465 0	.449 0	.415 0	.367 0	.310 0	
	.875L	.482 0	.465 0	.437 0	.391 0	.281 0	

ρ_{ij} -2 GIRDERS



DATA:

EI_L = 10,000 units
EI_T = 500.0 units
GJ_L = 3,550. units
GJ_T = 177.5 units

Table 5. DISTRIBUTION FACTORS FOR LONGITUDINALS[a]

(a) The ρ_{ij} values, when multiplied by the simple moments, yield the longitudinal moments.
(b) The lower values are for zero torsion. ($\beta = 0$)

GIRDER NO.	SECTION x	STIFFNESS ANALYSIS: Load on Girder 1: Point Load at .125L	.25L	.375L	.5L	UDL	HARMONIC ANALYSIS
		$\mu \times 10^4$					
1	.25L	42.6 / 0[b]	43.9 / 0	40.7 / 0	38.7 / 0	37.5 / 0	
	.5L	37.8 / 0	38.6 / 0	38.3 / 0	42.0 / 0	37.6 / 0	91.0 / 0
	.75L	27.0 / 0	29.8 / 0	34.2 / 0	38.7 / 0	37.5 / 0	
2	.25L	42.6 / 0	43.9 / 0	40.7 / 0	38.7 / 0	37.5 / 0	
	.5L	37.8 / 0	38.6 / 0	38.3 / 0	42.0 / 0	37.6 / 0	91.0 / 0
	.75L	27.0 / 0	29.8 / 0	34.2 / 0	38.7 / 0	37.5 / 0	

Table 6. DISTRIBUTION FACTORS FOR TRANSVERSALS[a]

(a) The μ_{ij} values, when multiplied by the simple moments, yield the transverse moments per unit length of the transverse medium.

(b) The lower values are for zero torsion ($\beta = 0$). The torsional values are given for the left hand sides of interior supports.

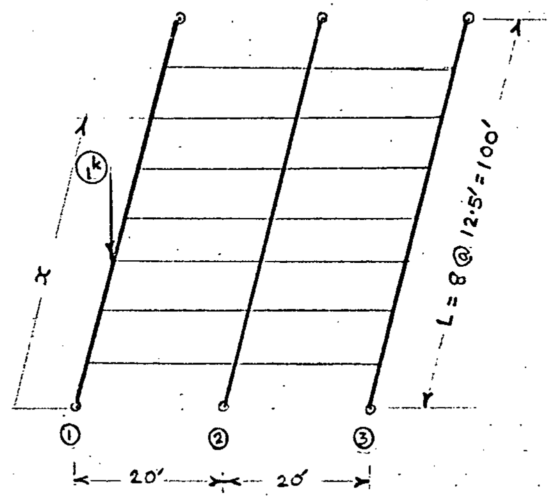
ρ_{ij} - 3 GIRDERS

STIFFNESS ANALYSIS — Load on Girder 1

GIRDER NO.	SECTION x	Point Load at .125L	.25L	.375L	.5L	UDL	HARMONIC ANALYSIS ρ
1	.125L	.831[b] .950	.670 .904	.556 .861	.469 .830	.612 .874	
	.25L	.670 .904	.721 .914	.588 .874	.499 .842	.574 .865	
	.375L	.549 .861	.588 .874	.654 .891	.554 .860	.553 .859	
	.5L	.469 .830	.499 .842	.554 .860	.635 .885	.546 .859	ρ_{11} .665 .863
	.625L	.420 .810	.442 .820	.485 .836	.554 .860	.553 .859	
	.75L	.390 .797	.408 .805	.442 .820	.499 .842	.574 .865	
	.875L	.376 .790	.390 .797	.420 .810	.469 .830	.612 .874	
2	.125L	.134 .098	.252 .195	.316 .277	.361 .338	.276 .252	
	.25L	.252 .195	.215 .172	.296 .252	.344 .314	.298 .270	
	.375L	.326 .277	.296 .252	.246 .214	.310 .278	.310 .282	
	.5L	.361 .338	.344 .314	.310 .278	.255 .227	.314 .286	ρ_{21} .287 .274
	.625L	.374 .380	.366 .361	.348 .328	.310 .278	.310 .282	
	.75L	.376 .405	.374 .390	.366 .361	.344 .314	.298 .270	
	.875L	.373 .420	.376 .405	.374 .380	.361 .338	.276 .252	

STIFFNESS ANALYSIS — Load on Girder 2

GIRDER NO.	SECTION x	Point Load at .125L	.25L	.375L	.5L	UDL	HARMONIC ANALYSIS ρ
1	.125L	.134 .098	.252 .195	.316 .277	.361 .338	.276 .252	
	.25L	.252 .195	.215 .172	.296 .252	.344 .314	.298 .270	
	.375L	.326 .277	.296 .252	.246 .214	.310 .278	.310 .282	
	.5L	.361 .338	.344 .314	.310 .278	.255 .227	.314 .286	ρ_{12} .287 .274
	.625L	.374 .380	.366 .361	.348 .328	.310 .278	.310 .282	
	.75L	.376 .405	.374 .390	.366 .361	.344 .314	.298 .270	
	.875L	.373 .420	.376 .405	.374 .380	.361 .338	.276 .252	
2	.125L	.732 .808	.496 .610	.348 .446	.278 .324	.448 .496	
	.25L	.496 .610	.570 .656	.408 .496	.312 .372	.404 .460	
	.375L	.348 .446	.408 .496	.508 .572	.380 .444	.380 .436	
	.5L	.278 .324	.312 .372	.380 .444	.490 .546	.372 .428	ρ_{22} .426 .452
	.625L	.252 .240	.268 .278	.304 .344	.380 .444	.380 .436	
	.75L	.248 .190	.252 .220	.268 .278	.312 .372	.404 .460	
	.875L	.254 .160	.248 .190	.252 .240	.278 .324	.448 .496	



DATA:

EI_L = 10,000. units
EI_T = 500.0 units
GJ_L = 4,430. units
GJ_T = 221.5 units

Table 7. DISTRIBUTION FACTORS FOR LONGITUDINALS[a]

(a) The ρ_{ij} values, when multiplied by the simple moments, yield the longitudinal moments.
(b) The lower values are for zero torsion. ($\beta = 0$)

GIRDER NO.	SECTION x	STIFFNESS ANALYSIS — Load on Girder 1 — Point Load at ·125L	·25L	·375L	·5L	UDL	HARMONIC ANALYSIS ρ
3	·125L	·035 -·038	·078 -·099	·118 -·138	·170 -·169	·112 -·126	ρ_{31} ·048 -·137
	·25L	·078 -·099	·064 -·086	·116 -·126	·157 -·156	·128 -·135	
	·375L	·125 -·138	·116 -·126	·100 -·105	·136 -·138	·137 -·141	
	·5L	·170 -·169	·157 -·156	·136 -·138	·110 -·112	·140 -·145	
	·625L	·206 -·190	·192 -·181	·167 -·164	·136 -·138	·137 -·141	
	·75L	·234 -·202	·218 -·195	·192 -·181	·157 -·156	·128 -·135	
	·875L	·251 -·210	·234 -·202	·206 -·190	·170 -·169	·112 -·126	

GIRDER NO.	SECTION x	STIFFNESS ANALYSIS — Load on Girder 2 — Point Load at ·125L	·25L	·375L	·5L	UDL	HARMONIC ANALYSIS ρ
3	·125L	·134 ·098	·252 ·195	·316 ·277	·361 ·338	·276 ·252	ρ_{32} ·287 ·274
	·25L	·252 ·195	·215 ·172	·296 ·252	·344 ·314	·298 ·270	
	·375L	·326 ·277	·296 ·252	·246 ·214	·310 ·278	·310 ·282	
	·5L	·361 ·338	·344 ·314	·310 ·278	·255 ·227	·314 ·286	
	·625L	·374 ·380	·366 ·361	·348 ·328	·310 ·278	·310 ·282	
	·75L	·376 ·405	·374 ·390	·366 ·361	·344 ·314	·298 ·270	
	·875L	·373 ·420	·376 ·405	·374 ·380	·361 ·338	·276 ·252	

Table 7. (cont'd) DISTRIBUTION FACTORS FOR LONGITUDINALS

GIRDER NO.	SECTION x	STIFFNESS ANALYSIS, Load on Girder 1, Point Load at ·125L ($\mu \times 10^4$)	·25L	·375L	·5L	UDL	HARMONIC ANALYSIS	GIRDER NO.	SECTION x	STIFFNESS ANALYSIS, Load on Girder 2, Point Load at ·125L ($\mu \times 10^4$)	·25L	·375L	·5L	UDL	HARMONIC ANALYSIS
1	·25L	52·5 0	42·7 0	47·2 0	41·6 0	42·4 0	62·6 0	1	·25L	-37·1 0	-29·3 0	-27·0 0	-16·6 0	-21·0 0	-23·0 0
	·5L	39·2 0	41·5 0	43·0 0	38·8 0	41·5 0			·5L	-12·0 0	-16·5 0	-21·3 0	-21·2 0	-19·0 0	
	·75L	23·8 0	27·8 0	34·1 0	41·6 0	42·4 0			·75L	-6·8 0	-2·2 0	-5·8 0	-16·6 0	-21·0 0	
2	·25L	-52·3 -29·0	-42·5 -23·8	-46·5 -29·0	-40·3 -28·8	-41·3 -26·8	-72·3 -20·3	2	·25L	53·7 58·0	43·0 47·6	45·2 58·0	37·0 57·6	39·1 53·6	43·3 40·6
	·5L	-37·7 -28·8	-40·2 -28·8	-42·0 -27·6	-38·0 -23·7	-40·5 -27·4			·5L	33·5 57·6	37·0 57·6	40·0 55·2	36·7 47·4	38·0 54·8	
	·75L	-21·0 -20·6	-25·4 -23·0	-32·2 -29·0	-40·3 -28·8	-41·3 -26·8			·75L	13·4 41·2	18·4 46·0	26·7 52·2	37·0 57·6	39·1 53·6	
3	·25L	-15·4 0	-13·7 0	-20·1 0	-32·8 0	-21·3 0	-39·6	3	·25L	-37·1 0	-29·3 0	-27·0 0	-16·6 0	-21·0 0	-23·0 0
	·5L	-27·2 0	-25·2 0	-21·8 0	-16·4 0	-22·6 0			·5L	-12·0 0	-16·5 0	-21·3 0	-21·2 0	-19·0 0	
	·75L	-30·6 0	-30·0 0	-28·3 0	-32·8 0	-21·3 0			·75L	-6·8 0	-2·2 0	-5·8 0	-16·6 0	-21·0 0	

Table 8. DISTRIBUTION FACTORS FOR TRANSVERSALS[a]

(a) The μ_{ij} values, when multiplied by the simple moments, yield the transverse moments per unit length of the transverse medium.

(b) The lower values are for zero torsion ($\beta = 0$). The torsional values are given for the left hand sides of interior supports.

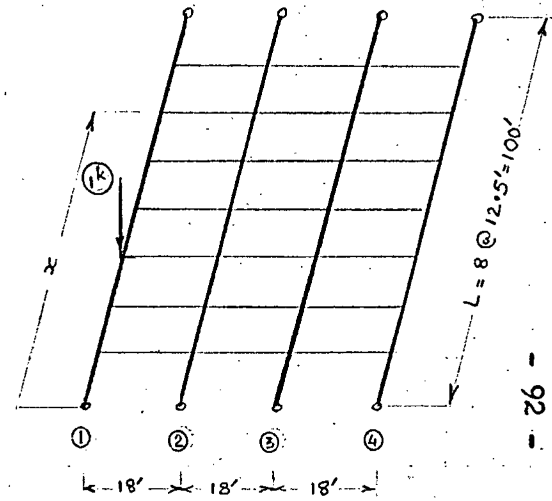
ρ_{ij} - 4 GIRDERS

STIFFNESS ANALYSIS — Load on Girder 1

GIRDER NO.	SECTION x	Point Load at .125L	Point Load at .25L	Point Load at .375L	Point Load at .5L	UDL	HARMONIC ANALYSIS ρ
1	.125L	.814 .928 (b)	.640 .858	.518 .796	.440 .749	.584 .812	
	.25L	.640 .858	.690 .875	.559 .815	.470 .765	.545 .799	
	.375L	.518 .796	.559 .815	.629 .840	.526 .792	.525 .790	
	.5L	.440 .749	.470 .765	.526 .792	.610 .830	.518 .786	ρ_{11} .596 .796
	.625L	.392 .714	.413 .728	.456 .756	.526 .792	.525 .790	
	.75L	.364 .688	.380 .702	.413 .728	.470 .765	.545 .799	
	.875L	.350 .675	.364 .688	.392 .714	.440 .749	.584 .812	
2	.125L	.141 .112	.258 .218	.321 .304	.344 .365	.270 .274	
	.25L	.258 .218	.217 .192	.293 .276	.330 .341	.290 .294	
	.375L	.321 .304	.293 .276	.242 .234	.300 .302	.300 .306	
	.5L	.344 .364	.330 .341	.300 .302	.248 .247	.303 .310	ρ_{21} .282 .297
	.625L	.347 .405	.343 .385	.332 .353	.300 .302	.300 .306	
	.75L	.340 .428	.344 .414	.343 .385	.330 .341	.290 .294	
	.875L	.334 .440	.340 .428	.347 .405	.344 .365	.270 .274	

STIFFNESS ANALYSIS — Load on Girder 2

GIRDER NO.	SECTION x	Point Load at .125L	Point Load at .25L	Point Load at .375L	Point Load at .5L	UDL	HARMONIC ANALYSIS ρ
1	.125L	.141 .112	.258 .218	.321 .304	.344 .365	.270 .274	
	.25L	.258 .218	.217 .192	.293 .276	.330 .341	.290 .294	
	.375L	.321 .304	.293 .276	.242 .234	.300 .302	.300 .306	
	.5L	.344 .364	.330 .341	.300 .302	.248 .247	.303 .310	ρ_{12} .282 .297
	.625L	.347 .405	.343 .385	.332 .353	.300 .302	.300 .306	
	.75L	.340 .428	.344 .414	.343 .385	.330 .341	.290 .294	
	.875L	.334 .440	.340 .428	.347 .405	.344 .365	.270 .274	
2	.125L	.697 .765	.445 .552	.295 .400	.226 .300	.400 .465	
	.25L	.445 .552	.529 .610	.356 .451	.262 .341	.354 .428	
	.375L	.295 .400	.356 .451	.463 .539	.331 .412	.330 .406	
	.5L	.226 .300	.262 .341	.331 .412	.447 .519	.323 .400	ρ_{22} .368 .423
	.625L	.205 .246	.219 .274	.256 .326	.331 .412	.330 .406	
	.75L	.204 .220	.206 .238	.219 .274	.262 .341	.354 .428	
	.875L	.208 .210	.204 .220	.205 .246	.226 .300	.400 .465	



DATA:

EI_L = 10,000. units

EI_T = 500.0 units

GJ_L = 3,940. units

GJ_T = 197 units

Table 9. DISTRIBUTION FACTORS FOR LONGITUDINALS[a]

(a) The ρ_{ij} values, when multiplied by the simple moments, yield the longitudinal moments

(b) The lower values are for zero torsion. (β = 0)

GIRDER NO.	x SECTION	Load on Girder 1: Point Load at .125L	.25L	.375L	.5L	UDL	HARMONIC ANALYSIS f
3	.125L	.034 -.011	.077 -.011	.121 .003	.160 .027	.109 .014	f_{31} .103 .017
	.25L	.077 -.011	.070 -.005	.110 .004	.147 .022	.122 .015	
	.375L	.121 .003	.110 .004	.095 .007	.128 .016	.129 .018	
	.5L	.160 .027	.147 .022	.128 .016	.103 .012	.132 .019	
	.625L	.194 .054	.177 .044	.156 .030	.128 .016	.129 .018	
	.75L	.210 .077	.198 .064	.177 .044	.147 .022	.122 .016	
	.875L	.221 .093	.210 .077	.194 .054	.160 .027	.109 .014	
4	.125L	.010 -.023	.025 -.065	.041 -.103	.057 -.140	.037 -.101	f_{41} .019 -.110
	.25L	.025 -.065	.023 -.061	.038 -.095	.053 -.129	.043 -.109	
	.375L	.041 -.103	.038 -.095	.033 -.081	.046 -.112	.046 -.114	
	.5L	.057 -.140	.053 -.129	.046 -.112	.038 -.090	.048 -.116	
	.625L	.073 -.171	.067 -.158	.058 -.139	.046 -.112	.046 -.114	
	.75L	.085 -.194	.078 -.180	.067 -.158	.053 -.129	.043 -.109	
	.875L	.094 -.209	.085 -.194	.073 -.171	.057 -.140	.037 -.101	

GIRDER NO.	x SECTION	Load on Girder 2: Point Load at .125L	.25L	.375L	.5L	UDL	HARMONIC ANALYSIS f
3	.125L	.127 .134	.222 .240	.263 .296	.269 .310	.222 .246	f_{32} .247 .263
	.25L	.222 .240	.184 .202	.240 .268	.262 .296	.234 .262	
	.375L	.263 .296	.240 .268	.199 .220	.241 .268	.240 .270	
	.5L	.269 .310	.262 .296	.241 .268	.201 .223	.242 .272	
	.625L	.259 .296	.261 .298	.258 .292	.241 .268	.240 .270	
	.75L	.246 .273	.253 .286	.261 .298	.262 .296	.234 .262	
	.875L	.236 .255	.246 .273	.259 .296	.269 .310	.222 .246	
4	.125L	.034 -.011	.077 -.011	.121 .003	.160 .027	.109 .014	f_{42} .103 .017
	.25L	.077 -.011	.070 -.005	.110 .004	.147 .022	.122 .016	
	.375L	.121 .003	.110 .004	.095 .007	.128 .016	.129 .018	
	.5L	.160 .027	.147 .022	.128 .016	.103 .012	.132 .019	
	.625L	.194 .054	.177 .044	.156 .030	.128 .016	.129 .018	
	.75L	.210 .077	.198 .064	.177 .044	.147 .022	.122 .016	
	.875L	.221 .093	.210 .077	.194 .054	.160 .027	.109 .014	

Table 9. (cont'd) DISTRIBUTION FACTORS FOR LONGITUDINALS

μ_{ij} -4 GIRDERS

GIRDER NO.	SECTION x	STIFFNESS ANALYSIS – Load on Girder 1 – Point Load at .125L	.25L	.375L	.5L	UDL	HARMONIC ANALYSIS
		$\mu \times 10^4$					
1	.25L	53.3 0	43.4 0	46.2 0	39.2 0	41.1 0	85.3 0
	.5L	36.0 0	39.2 0	41.9 0	38.4 0	39.9 0	
	.75L	20.7 0	24.5 0	31.0 0	39.2 0	41.1 0	
2	.25L	-53.1 -36.4	-43.0 -33.8	-45.2 -37.5	-37.3 -38.6	-39.6 -35.9	-95.0 -36.4
	.5L	-33.9 -39.0	-37.3 -38.6	-40.4 -36.8	-37.2 -31.5	-38.4 -36.6	
	.75L	-17.0 -33.6	-21.2 -35.2	-28.3 -37.3	-37.3 -38.6	-39.6 -35.9	
3	.25L	-18.7 -12.1	-16.4 -10.4	-23.9 -16.9	-29.3 -23.0	-24.8 -19.0	-17.6 -19.6
	.5L	-31.6 -25.4	-29.3 -23.0	-25.5 -19.5	-20.7 -15.5	-26.3 -20.4	
	.75L	-34.0 -32.4	-33.6 -30.8	-32.4 -27.7	-29.3 -23.0	-24.8 -19.0	
4	.25L	-6.1 0	-5.7 0	-8.9 0	-11.9 0	-10.1 0	-22.1 0
	.5L	-13.0 0	-11.9 0	-10.4 0	-8.4 0	-10.7 0	
	.75L	-18.0 0	-16.7 0	-14.6 0	-11.9 0	-10.1 0	

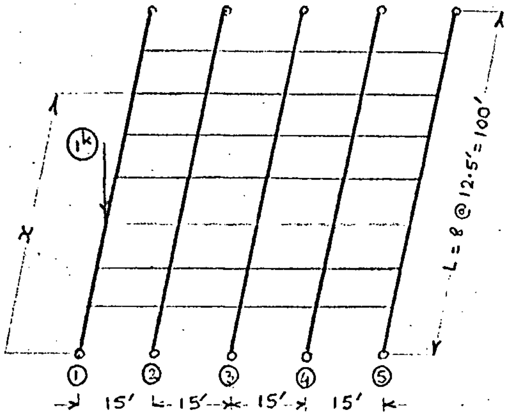
GIRDER NO.	SECTION x	STIFFNESS ANALYSIS – Load on Girder 2 – Point Load at .125L	.25L	.375L	.5L	UDL	HARMONIC ANALYSIS
		$\mu \times 10^4$					
1	.25L	-34.2 0	-26.6 0	-21.8 0	-9.2 0	-15.5 0	-23.5 0
	.5L	-3.5 0	-9.1 0	-15.7 0	-17.2 0	-12.9 0	
	.75L	14.7 0	10.4 0	2.5 0	-9.2 0	-15.5 0	
2	.25L	54.9 61.5	43.9 50.1	44.7 58.0	34.4 54.4	38.0 52.7	48.6 53.8
	.5L	29.6 52.8	34.4 54.4	38.8 54.1	36.4 4.8	36.3 52.9	
	.75L	10.2 34.7	14.9 39.6	23.2 46.8	34.4 54.4	38.0 52.7	
3	.25L	-38.4 -13.9	-30.2 -9.6	-27.0 -4.1	-15.5 7.3	-21.1 2.1	52.1 3.1
	.5L	-10.3 11.6	-15.5 7.3	-21.4 2.1	-22.0 -0.6	-18.8 4.1	
	.75L	6.3 31.3	2.4 26.4	-4.8 18.2	-15.5 7.3	-21.1 2.1	
4	.25L	-13.0 0	-11.1 0	-15.6 0	-18.2 0	-15.5 0	-39.2 0
	.5L	-19.4 0	-18.1 0	-16.0 0	-12.8 0	-16.3 0	
	.75L	-17.4 0	-18.3 0	-18.8 0	-18.2 0	-15.5 0	

Table 10. DISTRIBUTION FACTORS FOR TRANSVERSALS[a]

(a) The μ_{ij} values, when multiplied by the simple moments, yield the transverse moments per unit length of the transverse medium.

(b) The lower values are for zero torsion ($\beta = 0$). The torsional values are given for the left hand sides of interior supports.

ρ_{ij} - 5 GIRDERS



DATA:

EI_L = 10,000. units

EI_T = 500.0 units

GJ_L = 5,910 units

GJ_T = 295 units

GIRDER NO.	SECTION x	STIFFNESS ANALYSIS: Load on Girder 1 — Point Load at .125L	.25L	.375L	.5L	UDL	HARMONIC ANALYSIS ρ
1	.125L	.755[b] / .907	.538 / .817	.400 / .740	.324 / .684	.488 / .762	ρ_{11} .482 / .743
	.25L	.538 / .817	.605 / .839	.451 / .765	.357 / .705	.444 / .745	
	.375L	.400 / .740	.451 / .765	.540 / .800	.420 / .738	.420 / .735	
	.5L	.324 / .684	.357 / .705	.420 / .738	.521 / .787	.413 / .730	
	.625L	.284 / .644	.304 / .660	.344 / .693	.420 / .738	.420 / .735	
	.75L	.263 / .615	.276 / .632	.304 / .660	.357 / .705	.444 / .745	
	.875L	.254 / .600	.263 / .615	.284 / .644	.324 / .684	.488 / .762	

GIRDER NO.	SECTION x	STIFFNESS ANALYSIS: Load on Girder 2 — Point Load at .125L	.25L	.375L	.5L	UDL	HARMONIC ANALYSIS ρ
1	.125L	.159 / .130	.273 / .247	.314 / .332	.312 / .384	.260 / .294	ρ_{12} .276 / .317
	.25L	.273 / .247	.225 / .214	.288 / .296	.306 / .362	.276 / .315	
	.375L	.314 / .332	.288 / .296	.238 / .254	.286 / .323	.282 / .326	
	.5L	.312 / .384	.306 / .362	.286 / .323	.240 / .265	.284 / .330	
	.625L	.294 / .413	.299 / .399	.301 / .371	.286 / .323	.282 / .326	
	.75L	.275 / .430	.286 / .420	.299 / .399	.306 / .362	.276 / .315	
	.875L	.263 / .436	.275 / .430	.294 / .413	.312 / .384	.260 / .294	

GIRDER NO.	SECTION x	STIFFNESS ANALYSIS: Load on Girder 3 — Point Load at .125L	.25L	.375L	.5L	UDL	HARMONIC ANALYSIS ρ
1	.125L	.050 / -.003	.108 / .012	.158 / .044	.194 / .067	.135 / .053	ρ_{13} .131 / .060
	.25L	.108 / .012	.096 / .017	.143 / .041	.180 / .074	.150 / .058	
	.375L	.158 / .044	.143 / .041	.120 / .038	.157 / .060	.158 / .062	
	.5L	.194 / .067	.180 / .074	.157 / .060	.127 / .047	.161 / .064	
	.625L	.215 / .123	.204 / .108	.186 / .085	.157 / .060	.158 / .062	
	.75L	.226 / .154	.220 / .136	.204 / .108	.180 / .074	.150 / .058	
	.875L	.230 / .172	.226 / .154	.215 / .123	.194 / .067	.135 / .053	

Table 11. DISTRIBUTION FACTORS FOR LONGITUDINALS[a]

(a) The ρ_{ij} values, when multiplied by the simple moments, yield the longitudinal moments.
(b) The lower values are for zero torsion. ($\beta = 0$)

GIRDER NO.	SECTION x	STIFFNESS ANALYSIS — Load on Girder 1 — Point Load at ·125L	·25L	·375L	·5L	UDL	HARMONIC ANALYSIS f
2	·125L	·159 ·130(b)	·273 ·247	·314 ·332	·312 ·384	·260 ·294	f_{21} ·276 ·317
	·25L	·273 ·247	·225 ·214	·288 ·296	·306 ·362	·276 ·315	
	·375L	·314 ·332	·288 ·296	·238 ·254	·286 ·323	·282 ·326	
	·5L	·312 ·384	·306 ·362	·286 ·323	·240 ·265	·284 ·330	
	·625L	·294 ·413	·299 ·399	·301 ·371	·286 ·323	·282 ·326	
	·75L	·275 ·430	·286 ·420	·299 ·399	·306 ·362	·276 ·315	
	·875L	·263 ·436	·275 ·430	·294 ·413	·312 ·384	·294 ·260	
3	·125L	·050 -·003	·108 ·012	·158 ·044	·194 ·067	·135 ·053	f_{31} ·131 ·060
	·25L	·108 ·012	·096 ·017	·143 ·041	·180 ·074	·150 ·058	
	·375L	·158 ·044	·143 ·041	·120 ·038	·157 ·060	·158 ·062	
	·5L	·194 ·067	·180 ·074	·157 ·060	·127 ·047	·161 ·064	
	·625L	·215 ·123	·204 ·107	·186 ·085	·157 ·060	·158 ·062	
	·75L	·226 ·154	·220 ·136	·204 ·107	·180 ·074	·150 ·058	
	·875L	·230 ·172	·226 ·154	·215 ·123	·194 ·067	·135 ·053	

GIRDER NO.	SECTION x	STIFFNESS ANALYSIS — Load on Girder 2 — Point Load at ·125L	·25L	·375L	·5L	UDL	HARMONIC ANALYSIS f
2	·125L	·638 ·720	·361 ·486	·228 ·334	·187 ·254	·345 ·418	f_{22} ·312 ·374
	·25L	·361 ·486	·465 ·560	·290 ·392	·212 ·291	·300 ·379	
	·375L	·228 ·334	·290 ·392	·408 ·490	·224 ·362	·278 ·357	
	·5L	·187 ·254	·212 ·291	·224 ·362	·396 ·474	·272 ·350	
	·625L	·185 ·219	·189 ·238	·212 ·282	·224 ·362	·278 ·357	
	·75L	·194 ·209	·190 ·217	·189 ·238	·212 ·291	·300 ·379	
	·875L	·202 ·208	·194 ·209	·185 ·219	·187 ·254	·345 ·418	
3	·125L	·135 ·152	·222 ·256	·243 ·294	·232 ·288	·204 ·244	f_{32} ·224 ·256
	·25L	·222 ·256	·181 ·211	·226 ·268	·232 ·282	·214 ·256	
	·375L	·243 ·294	·226 ·268	·186 ·221	·220 ·264	·217 ·262	
	·5L	·232 ·288	·232 ·282	·220 ·264	·187 ·222	·218 ·264	
	·625L	·212 ·262	·220 ·270	·226 ·276	·220 ·264	·217 ·262	
	·75L	·198 ·237	·207 ·252	·220 ·270	·232 ·282	·214 ·256	
	·875L	·190 ·220	·198 ·237	·212 ·262	·232 ·288	·204 ·244	

GIRDER NO.	SECTION x	STIFFNESS ANALYSIS — Load on Girder 3 — Point Load at ·125L	·25L	·375L	·5L	UDL	HARMONIC ANALYSIS f
2	·125L	·135 ·152	·222 ·256	·243 ·294	·232 ·288	·204 ·244	f_{23} ·224 ·256
	·25L	·222 ·256	·181 ·211	·226 ·268	·232 ·282	·214 ·256	
	·375L	·243 ·294	·226 ·268	·186 ·221	·220 ·264	·217 ·262	
	·5L	·232 ·288	·232 ·282	·220 ·264	·187 ·222	·218 ·264	
	·625L	·212 ·262	·220 ·270	·226 ·276	·220 ·264	·217 ·262	
	·75L	·198 ·237	·207 ·252	·220 ·270	·232 ·282	·214 ·256	
	·875L	·190 ·220	·198 ·237	·212 ·262	·232 ·288	·204 ·244	
3	·125L	·630 ·702	·340 ·463	·199 ·323	·151 ·256	·320 ·408	f_{33} ·291 ·368
	·25L	·340 ·463	·448 ·544	·264 ·380	·179 ·288	·272 ·370	
	·375L	·199 ·323	·264 ·380	·391 ·482	·246 ·354	·250 ·351	
	·5L	·151 ·256	·179 ·288	·246 ·354	·373 ·465	·242 ·345	
	·625L	·146 ·230	·152 ·244	·176 ·280	·246 ·354	·250 ·351	
	·75L	·152 ·218	·149 ·226	·152 ·244	·179 ·288	·272 ·370	
	·875L	·159 ·216	·152 ·218	·146 ·230	·151 ·256	·320 ·408	

Table 11. (cont'd) DISTRIBUTION FACTORS FOR LONGITUDINALS

GIRDER NO.	SECTION x	STIFFNESS ANALYSIS					HARMONIC ANALYSIS ρ
		Load on Girder 1					
		Point Load at				UDL	
		·125L	·25L	·375L	·5L		
4	·125L	·023 -·015	·052 -·031	·080 -·033	·107 -·044	·073 -·039	
	·25L	·052 -·031	·047 -·028	·074 -·041	·099 -·051	·082 -·042	
	·375L	·080 -·033	·074 -·041	·064 -·034	·086 -·044	·087 -·045	
	·5L	·107 -·044	·099 -·051	·086 -·044	·070 -·035	·089 -·045	ρ_{41} ·061 -·043
	·625L	·129 -·058	·120 -·056	·105 -·051	·086 -·044	·087 -·045	
	·75L	·144 -·057	·135 -·057	·120 -·056	·099 -·051	·082 -·042	
	·875L	·154 -·054	·144 -·057	·129 -·058	·107 -·044	·073 -·039	
5	·125L	·013 -·020	·030 -·044	·047 -·070	·065 -·078	·043 -·071	
	·25L	·030 -·044	·027 -·041	·043 -·065	·060 -·090	·049 -·076	
	·375L	·047 -·070	·043 -·065	·038 -·057	·052 -·078	·052 -·080	
	·5L	·065 -·078	·060 -·090	·052 -·078	·042 -·063	·054 -·080	ρ_{51} ·051 -·077
	·625L	·080 -·122	·074 -·112	·065 -·097	·052 -·078	·052 -·080	
	·75L	·092 -·142	·085 -·130	·074 -·112	·060 -·090	·049 -·076	
	·875L	·099 -·154	·092 -·142	·080 -·122	·065 -·078	·043 -·071	

GIRDER NO.	SECTION x	STIFFNESS ANALYSIS					HARMONIC ANALYSIS ρ
		Load on Girder 2					
		Point Load at				UDL	
		·125L	·25L	·375L	·5L		
4	·125L	·044 ·011	·093 ·043	·135 ·088	·164 ·131	·116 ·084	
	·25L	·093 ·043	·082 ·043	·122 ·078	·152 ·116	·128 ·093	
	·375L	·135 ·088	·122 ·078	·102 ·067	·133 ·095	·134 ·099	
	·5L	·164 ·131	·152 ·116	·133 ·095	·108 ·075	·137 ·102	ρ_{42} ·127 ·095
	·625L	·181 ·163	·172 ·148	·157 ·124	·133 ·095	·134 ·099	
	·75L	·189 ·181	·184 ·169	·172 ·138	·152 ·116	·128 ·093	
	·875L	·192 ·190	·189 ·181	·181 ·163	·164 ·131	·116 ·084	
5	·125L	·023 -·015	·052 -·031	·080 -·033	·107 -·044	·073 -·039	
	·25L	·052 -·031	·047 -·028	·074 -·041	·099 -·051	·082 -·042	
	·375L	·080 -·033	·074 -·041	·064 -·034	·086 -·044	·087 -·045	
	·5L	·107 -·044	·099 -·051	·086 -·044	·070 -·035	·089 -·045	ρ_{52} ·061 -·043
	·625L	·129 -·058	·120 -·056	·105 -·051	·086 -·044	·087 -·045	
	·75L	·144 -·057	·135 -·057	·120 -·056	·099 -·051	·082 -·042	
	·875L	·154 -·054	·144 -·057	·129 -·058	·107 -·044	·073 -·039	

GIRDER NO.	SECTION x	STIFFNESS ANALYSIS					HARMONIC ANALYSIS ρ
		Load on Girder 3					
		Point Load at				UDL	
		·125L	·25L	·375L	·5L		
4	·125L	·135 ·152	·222 ·256	·243 ·294	·232 ·288	·204 ·244	
	·25L	·222 ·256	·181 ·211	·226 ·268	·232 ·282	·214 ·256	
	·375L	·243 ·294	·226 ·268	·186 ·221	·220 ·264	·217 ·262	
	·5L	·232 ·288	·232 ·282	·220 ·264	·187 ·222	·218 ·264	ρ_{43} ·224 ·256
	·625L	·212 ·262	·220 ·270	·226 ·276	·220 ·264	·217 ·262	
	·75L	·198 ·237	·207 ·252	·220 ·270	·232 ·282	·214 ·256	
	·875L	·190 ·220	·198 ·237	·212 ·262	·232 ·288	·204 ·244	
5	·125L	·050 -·003	·108 ·012	·158 ·044	·194 ·067	·135 ·053	
	·25L	·108 ·012	·096 ·017	·143 ·041	·180 ·074	·150 ·058	
	·375L	·158 ·044	·143 ·041	·120 ·038	·157 ·060	·158 ·062	
	·5L	·194 ·067	·180 ·074	·157 ·060	·127 ·047	·161 ·064	ρ_{53} ·131 ·060
	·625L	·215 ·123	·204 ·108	·186 ·085	·157 ·060	·158 ·062	
	·75L	·226 ·154	·220 ·136	·204 ·108	·180 ·074	·150 ·058	
	·875L	·230 ·172	·226 ·154	·215 ·123	·194 ·067	·135 ·053	

Table 11. (cont'd) DISTRIBUTION FACTORS FOR LONGITUDINALS

GIRDER NO.	SECTION x	STIFFNESS ANALYSIS, Load on Girder 1, Point Load at .125L	.25L	.375L	.5L	UDL	HARMONIC ANALYSIS ρ
4	.125L	.023 -.015	.052 -.031	.080 -.033	.107 -.044	.073 -.039	
	.25L	.052 -.031	.047 -.028	.074 -.041	.099 -.051	.082 -.042	
	.375L	.080 -.033	.074 -.041	.064 -.034	.086 -.044	.087 -.045	
	.5L	.107 -.044	.099 -.051	.086 -.044	.070 -.035	.089 -.045	ρ_{41} .061 -.043
	.625L	.129 -.058	.120 -.056	.105 -.051	.086 -.044	.087 -.045	
	.75L	.144 -.057	.135 -.057	.120 -.056	.099 -.051	.082 -.042	
	.875L	.154 -.054	.144 -.057	.129 -.058	.107 -.044	.073 -.039	
5	.125L	.013 -.020	.030 -.044	.047 -.070	.065 -.078	.043 -.071	
	.25L	.030 -.044	.027 -.041	.043 -.065	.060 -.090	.049 -.076	
	.375L	.047 -.070	.043 -.065	.038 -.057	.052 -.078	.052 -.080	
	.5L	.065 -.078	.060 -.090	.052 -.078	.042 -.063	.054 -.080	ρ_{51} .051 -.077
	.625L	.080 -.122	.074 -.112	.065 -.097	.052 -.078	.052 -.080	
	.75L	.092 -.142	.085 -.130	.074 -.112	.060 -.090	.049 -.076	
	.875L	.099 -.154	.092 -.142	.080 -.122	.065 -.078	.043 -.071	

GIRDER NO.	SECTION x	STIFFNESS ANALYSIS, Load on Girder 2, Point Load at .125L	.25L	.375L	.5L	UDL	HARMONIC ANALYSIS ρ
4	.125L	.044 .011	.093 .043	.135 .088	.164 .131	.116 .084	
	.25L	.093 .043	.082 .043	.122 .078	.152 .116	.128 .093	
	.375L	.135 .088	.122 .078	.102 .067	.133 .095	.134 .099	
	.5L	.164 .131	.152 .116	.133 .095	.108 .075	.137 .102	ρ_{42} .127 .095
	.625L	.181 .163	.172 .148	.157 .124	.133 .095	.134 .099	
	.75L	.189 .181	.184 .169	.172 .138	.152 .116	.128 .093	
	.875L	.192 .190	.189 .181	.181 .163	.164 .131	.116 .084	
5	.125L	.023 -.015	.052 -.031	.080 -.033	.107 -.044	.073 -.039	
	.25L	.052 -.031	.047 -.028	.074 -.041	.099 -.051	.082 -.042	
	.375L	.080 -.033	.074 -.041	.064 -.034	.086 -.044	.087 -.045	
	.5L	.107 -.044	.099 -.051	.086 -.044	.070 -.035	.089 -.045	ρ_{52} .061 -.043
	.625L	.129 -.058	.120 -.056	.105 -.051	.086 -.044	.087 -.045	
	.75L	.144 -.057	.135 -.057	.120 -.056	.099 -.051	.082 -.042	
	.875L	.154 -.054	.144 -.057	.129 -.058	.107 -.044	.073 -.039	

GIRDER NO.	SECTION x	STIFFNESS ANALYSIS, Load on Girder 3, Point Load at .125L	.25L	.375L	.5L	UDL	HARMONIC ANALYSIS ρ
4	.125L	.135 .152	.222 .256	.243 .294	.232 .288	.204 .244	
	.25L	.222 .256	.181 .211	.226 .268	.232 .282	.214 .256	
	.375L	.243 .294	.226 .268	.186 .221	.220 .264	.217 .262	
	.5L	.232 .288	.232 .282	.220 .264	.187 .222	.218 .264	ρ_{43} .224 .256
	.625L	.212 .262	.220 .270	.226 .276	.220 .264	.217 .262	
	.75L	.198 .237	.207 .252	.220 .270	.232 .282	.214 .256	
	.875L	.190 .220	.198 .237	.212 .262	.232 .288	.204 .244	
5	.125L	.050 -.003	.108 .012	.158 .044	.194 .067	.135 .053	
	.25L	.108 .012	.096 .017	.143 .041	.180 .074	.150 .058	
	.375L	.158 .044	.143 .041	.120 .038	.157 .060	.158 .062	
	.5L	.194 .067	.180 .074	.157 .060	.127 .047	.161 .064	ρ_{53} .131 .060
	.625L	.215 .123	.204 .108	.186 .085	.157 .060	.158 .062	
	.75L	.226 .154	.220 .136	.204 .108	.180 .074	.150 .058	
	.875L	.230 .172	.226 .154	.215 .123	.194 .067	.135 .053	

Table 11. (cont'd) DISTRIBUTION FACTORS FOR LONGITUDINALS

Stiffness analysis, load on Girder 1 ($\mu \times 10^4$)

GIRDER NO.	SECTION x	Point Load at ·125L	Point Load at ·25L	Point Load at ·375L	Point Load at ·5L	UDL	HARMONIC ANALYSIS
2	·25L	-41·0	-33·9	-40·3	-39·8	-37·9	-38·1
	·5L	-39·6	-39·8	-39·0	-23·8	-38·4	
	·75L	-31·6	-33·7	-36·8	-39·8	-37·9	
3	·25L	-18·6	-16·7	-26·0	-34·0	-28·3	-29·8
	·5L	-37·2	-34·0	-29·1	-23·2	-30·3	
	·75L	-44·4	-42·8	-39·6	-34·0	-28·3	
4	·25L	-5·8	-5·6	-9·2	-13·1	-11·1	-11·4
	·5L	-14·3	-13·1	-11·4	-9·3	-11·8	
	·75L	-22·5	-20·3	-17·0	-13·1	-11·1	

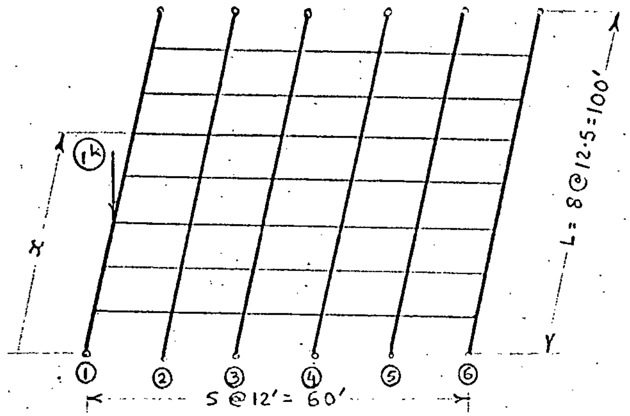
Stiffness analysis, load on Girder 2 ($\mu \times 10^4$)

GIRDER NO.	SECTION x	Point Load at ·125L	Point Load at ·25L	Point Load at ·375L	Point Load at ·5L	UDL	HARMONIC ANALYSIS
2	·25L	63·4	51·1	54·7	45·6	47·6	47·0
	·5L	41·8	45·6	48·9	44·4	46·5	
	·75L	19·1	12·1	34·2	45·6	47·6	
3	·25L	-15·1	-11·0	-4·5	6·4	0·2	1·3
	·5L	11·5	6·4	0·3	-2·9	2·6	
	·75L	24·2	10·4	15·7	6·4	0·2	
4	·25L	-7·0	-5·5	-7·7	-7·9	-6·1	-6·4
	·5L	-8·8	-7·9	-6·4	-4·7	-6·7	
	·75L	0·6	-0·1	-5·6	-7·9	-6·1	

Table 12. DISTRIBUTION FACTORS FOR TRANSVERSALS[a]

(a) The μ_{ij} values, when multiplied by the simple moments, yield the transverse moments per unit length of the transverse medium.

(b) The values given are for zero torsion only ($\beta = 0$).



DATA:

EI_L = 10,000. units

EI_T = 500.0 units

GJ_L = 7,390. units

GJ_T = 370. units

ρ_{ij} - 6 GIRDERS

STIFFNESS ANALYSIS — Load on Girder 1

GIRDER NO.	SECTION x	Point Load at .125L	.25L	.375L	.5L	UDL	HARMONIC ANALYSIS ρ
1	.125L	.685 .878[b]	.435 .765	.300 .675	.238 .613	.400 .705	
	.25L	.435 .765	.524 .793	.356 .702	.266 .635	.355 .685	
	.375L	.300 .675	.356 .702	.460 .748	.329 .676	.332 .672	
	.5L	.238 .613	.266 .635	.329 .676	.443 .735	.325 .670	ρ_{11} 0.405 0.682
	.625L	.210 .570	.224 .588	.258 .624	.329 .676	.332 .672	
	.75L	.198 .541	.206 .558	.224 .588	.266 .635	.355 .685	
	.875L	.192 .525	.198 .541	.210 .570	.238 .613	.400 .705	

STIFFNESS ANALYSIS — Load on Girder 2

GIRDER NO.	SECTION x	Point Load at .125L	.25L	.375L	.5L	UDL	HARMONIC ANALYSIS ρ
1	.125L	.178 .156	.279 .283	.288 .360	.260 .400	.242 .316	
	.25L	.279 .283	.224 .242	.271 .330	.266 .382	.252 .336	
	.375L	.288 .360	.271 .330	.225 .276	.260 .346	.253 .346	
	.5L	.260 .400	.266 .382	.260 .346	.223 .285	.254 .350	ρ_{12} .256 .338
	.625L	.230 .415	.242 .406	.258 .386	.260 .346	.253 .346	
	.75L	.209 .420	.222 .417	.242 .406	.266 .382	.252 .336	
	.875L	.198 .421	.209 .420	.230 .415	.260 .400	.242 .316	

STIFFNESS ANALYSIS — Load on Girder 3

GIRDER NO.	SECTION x	Point Load at .125L	.25L	.375L	.5L	UDL	HARMONIC ANALYSIS ρ
1	.125L	.066 .007	.133 .038	.180 .090	.202 .144	.148 .092	
	.25L	.133 .038	.114 .040	.161 .081	.190 .127	.162 .102	
	.375L	.180 .090	.161 .081	.133 .071	.168 .104	.169 .108	
	.5L	.202 .144	.190 .127	.168 .104	.137 .081	.171 .111	ρ_{13} .150 .104
	.625L	.206 .189	.202 .169	.191 .140	.168 .104	.169 .108	
	.75L	.204 .220	.205 .201	.202 .169	.190 .127	.162 .102	
	.875L	.200 .238	.204 .220	.206 .189	.202 .144	.148 .092	

Table 13. DISTRIBUTION FACTORS FOR LONGITUDINALS[a]

(a) The ρ_{ij} values, when multiplied by the simple moments, yield the longitudinal moments.
(b) The lower values are for zero torsion. (β = 0)

GIRDER NO.	SECTION x	STIFFNESS ANALYSIS — Load on Girder 1 — Point Load at ·125L	·25L	·375L	·5L	UDL	HARMONIC ANALYSIS ρ
2	·125L	·178 ·156	·279 ·283	·288 ·360	·260 ·400	·242 ·316	
	·25L	·279 ·283	·224 ·242	·271 ·330	·266 ·382	·252 ·336	
	·375L	·288 ·360	·271 ·330	·225 ·276	·260 ·346	·253 ·346	
	·5L	·260 ·400	·266 ·382	·260 ·346	·223 ·285	·254 ·350	ρ_{21} ·256 ·338
	·625L	·230 ·415	·242 ·406	·258 ·386	·260 ·346	·253 ·346	
	·75L	·209 ·420	·222 ·417	·242 ·406	·266 ·382	·252 ·336	
	·875L	·198 ·421	·209 ·420	·230 ·415	·260 ·400	·242 ·316	
3	·125L	·066 ·007	·133 ·038	·180 ·090	·202 ·144	·148 ·092	
	·25L	·133 ·038	·114 ·040	·161 ·081	·190 ·127	·162 ·102	
	·375L	·180 ·090	·161 ·081	·133 ·071	·168 ·104	·169 ·108	
	·5L	·202 ·144	·190 ·127	·168 ·104	·137 ·081	·171 ·111	ρ_{31} ·150 ·104
	·625L	·206 ·189	·202 ·169	·191 ·140	·168 ·104	·169 ·108	
	·75L	·204 ·220	·205 ·201	·202 ·169	·190 ·127	·162 ·102	
	·875L	·200 ·238	·204 ·220	·206 ·189	·202 ·144	·148 ·092	

GIRDER NO.	SECTION x	STIFFNESS ANALYSIS — Load on Girder 2 — Point Load at ·125L	·25L	·375L	·5L	UDL	HARMONIC ANALYSIS ρ
2	·125L	·567 ·665	·280 ·412	·180 ·278	·166 ·228	·294 ·375	
	·25L	·280 ·412	·405 ·505	·234 ·338	·178 ·256	·252 ·338	
	·375L	·180 ·278	·234 ·338	·358 ·445	·225 ·320	·234 ·318	
	·5L	·166 ·228	·178 ·256	·225 ·320	·347 ·434	·230 ·313	ρ_{22} ·263 ·335
	·625L	·174 ·218	·171 ·227	·179 ·254	·225 ·320	·234 ·318	
	·75L	·181 ·225	·176 ·222	·171 ·227	·178 ·256	·252 ·338	
	·875L	·183 ·232	·181 ·225	·174 ·218	·166 ·228	·294 ·375	
3	·125L	·148 ·171	·221 ·266	·220 ·276	·196 ·247	·193 ·231	
	·25L	·221 ·266	·178 ·215	·211 ·258	·204 ·252	·199 ·240	
	·375L	·220 ·276	·211 ·258	·178 ·214	·204 ·246	·198 ·242	
	·5L	·196 ·247	·204 ·252	·204 ·246	·177 ·212	·198 ·242	ρ_{32} ·201 ·237
	·625L	·178 ·212	·187 ·227	·200 ·244	·204 ·246	·198 ·242	
	·75L	·169 ·187	·175 ·202	·187 ·227	·204 ·252	·199 ·240	
	·875L	·166 ·174	·169 ·187	·178 ·212	·196 ·247	·193 ·231	

GIRDER NO.	SECTION x	STIFFNESS ANALYSIS — Load on Girder 3 — Point Load at ·125L	·25L	·375L	·5L	UDL	HARMONIC ANALYSIS ρ
2	·125L	·148 ·171	·221 ·266	·220 ·276	·196 ·247	·193 ·231	
	·25L	·221 ·266	·178 ·215	·211 ·258	·204 ·252	·199 ·240	
	·375L	·220 ·276	·211 ·258	·178 ·214	·204 ·246	·198 ·242	
	·5L	·196 ·247	·204 ·252	·204 ·246	·177 ·212	·198 ·242	ρ_{23} ·201 ·237
	·625L	·178 ·212	·187 ·227	·200 ·244	·204 ·246	·198 ·242	
	·75L	·169 ·187	·175 ·202	·187 ·227	·204 ·252	·199 ·240	
	·875L	·166 ·174	·169 ·187	·178 ·212	·196 ·247	·193 ·231	
3	·125L	·554 ·640	·252 ·384	·143 ·257	·126 ·208	·264 ·347	
	·25L	·252 ·384	·381 ·480	·202 ·314	·140 ·232	·220 ·311	
	·375L	·143 ·257	·202 ·314	·331 ·423	·192 ·292	·202 ·292	
	·5L	·126 ·208	·140 ·232	·192 ·292	·320 ·406	·196 ·287	ρ_{33} ·235 ·309
	·625L	·135 ·190	·132 ·200	·141 ·226	·192 ·292	·202 ·292	
	·75L	·144 ·179	·138 ·187	·132 ·200	·140 ·232	·220 ·311	
	·875L	·150 ·177	·144 ·179	·135 ·190	·126 ·208	·264 ·347	

Table 13. (cont'd) DISTRIBUTION FACTORS FOR LONGITUDINALS

STIFFNESS ANALYSIS — Load on Girder 1

GIRDER NO.	SECTION x	Point Load at ·125L	·25L	·375L	·5L	UDL	HARMONIC ANALYSIS ρ
4	·125L	·034 -·010	·073 -·019	·108 -·021	·136 -·013	·095 -·010	
	·25L	·073 -·019	·065 -·015	·098 -·017	·126 -·012	·105 -·011	
	·375L	·108 -·021	·098 -·017	·083 -·011	·110 -·011	·110 -·011	
	·5L	·136 -·013	·126 -·012	·110 -·011	·089 -·009	·113 -·011	ρ_{41} ·089 -·011
	·625L	·154 ·002	·146 -·003	·131 -·008	·110 -·011	·110 -·011	
	·75L	·165 ·012	·158 ·009	·146 -·003	·126 -·012	·105 -·011	
	·875L	·170 ·029	·165 ·012	·154 ·002	·136 -·013	·095 -·010	
5	·125L	·021 -·015	·046 -·032	·071 -·049	·096 -·065	·065 -·047	
	·25L	·046 -·032	·042 -·029	·065 -·045	·089 -·060	·072 -·051	
	·375L	·071 -·049	·065 -·045	·056 -·039	·077 -·052	·076 -·053	
	·5L	·096 -·065	·089 -·060	·077 -·052	·063 -·042	·080 -·054	ρ_{51} ·058 -·051
	·625L	·112 -·077	·104 -·072	·092 -·063	·077 -·052	·076 -·053	
	·75L	·124 -·085	·116 -·081	·104 -·072	·089 -·060	·072 -·051	
	·875L	·131 -·090	·124 -·085	·112 -·077	·096 -·065	·065 -·047	

STIFFNESS ANALYSIS — Load on Girder 2

GIRDER NO.	SECTION x	Point Load at ·125L	·25L	·375L	·5L	UDL	HARMONIC ANALYSIS ρ
4	·125L	·055 ·022	·111 ·068	·147 ·121	·165 ·160	·122 ·104	
	·25L	·111 ·068	·094 ·063	·132 ·105	·156 ·143	·133 ·115	
	·375L	·147 ·121	·132 ·105	·109 ·086	·138 ·118	·139 ·122	
	·5L	·165 ·160	·156 ·143	·138 ·118	·113 ·092	·141 ·125	ρ_{42} ·134 ·118
	·625L	·170 ·178	·167 ·167	·158 ·147	·138 ·118	·139 ·122	
	·75L	·170 ·182	·170 ·179	·167 ·167	·156 ·143	·133 ·115	
	·875L	·168 ·181	·170 ·182	·170 ·178	·165 ·160	·122 ·104	
5	·125L	·030 ·000	·064 ·003	·096 ·013	·123 ·031	·084 ·021	
	·25L	·064 ·003	·057 ·006	·087 ·014	·114 ·028	·093 ·023	
	·375L	·096 ·013	·087 ·014	·074 ·019	·099 ·023	·098 ·024	
	·5L	·123 ·031	·114 ·028	·099 ·023	·081 ·019	·102 ·025	ρ_{52} ·088 ·023
	·625L	·137 ·052	·129 ·044	·116 ·034	·099 ·023	·098 ·024	
	·75L	·146 ·070	·140 ·060	·129 ·044	·114 ·028	·093 ·023	
	·875L	·152 ·082	·146 ·070	·137 ·052	·123 ·031	·084 ·021	

STIFFNESS ANALYSIS — Load on Girder 3

GIRDER NO.	SECTION x	Point Load at ·125L	·25L	·375L	·5L	UDL	HARMONIC ANALYSIS ρ
4	·125L	·143 ·170	·210 ·264	·204 ·276	·178 ·254	·180 ·236	
	·25L	·210 ·264	·169 ·214	·197 ·260	·186 ·258	·184 ·244	
	·375L	·204 ·276	·197 ·260	·166 ·217	·189 ·252	·183 ·246	
	·5L	·178 ·254	·186 ·258	·189 ·252	·165 ·216	·182 ·248	ρ_{43} ·191 ·242
	·625L	·158 ·228	·168 ·240	·182 ·252	·189 ·252	·183 ·246	
	·75L	·149 ·210	·155 ·222	·168 ·240	·186 ·258	·184 ·244	
	·875L	·146 ·201	·149 ·210	·158 ·228	·178 ·254	·180 ·236	
5	·125L	·055 ·022	·111 ·068	·147 ·121	·165 ·160	·122 ·104	
	·25L	·111 ·068	·094 ·063	·132 ·105	·156 ·143	·133 ·115	
	·375L	·147 ·121	·132 ·105	·109 ·086	·138 ·118	·139 ·122	
	·5L	·165 ·160	·156 ·143	·138 ·118	·113 ·092	·141 ·125	ρ_{53} ·134 ·118
	·625L	·170 ·178	·167 ·167	·158 ·147	·138 ·118	·139 ·122	
	·75L	·170 ·182	·170 ·179	·167 ·167	·156 ·143	·133 ·115	
	·875L	·168 ·181	·170 ·182	·170 ·178	·165 ·160	·122 ·104	

Table 13. (cont'd) DISTRIBUTION FACTORS FOR LONGITUDINALS

GIRDER NO.	SECTION x	STIFFNESS ANALYSIS — Load on Girder 1 — Point Load at ·125L	·25L	·375L	·5L	UDL	HARMONIC ANALYSIS ρ
	·125L	·015 -·016	·030 -·035	·054 -·057	·070 -·078	·050 -·057	
	·25L	·030 -·035	·032 -·033	·050 -·052	·065 -·072	·056 -·061	
	·375L	·054 -·057	·050 -·052	·043 -·046	·056 -·063	·059 -·064	
6	·5L	·070 -·078	·065 -·072	·056 -·063	·046 -·051	·058 -·065	ρ_{61} ·042 -·062
	·625L	·088 -·098	·082 -·090	·072 -·078	·056 -·063	·059 -·064	
	·75L	·101 -·115	·094 -·105	·082 -·090	·065 -·072	·056 -·061	
	·875L	·109 -·125	·101 -·115	·088 -·098	·070 -·078	·050 -·057	

GIRDER NO.	SECTION x	STIFFNESS ANALYSIS — Load on Girder 2 — Point Load at ·125L	·25L	·375L	·5L	UDL	HARMONIC ANALYSIS ρ
	·125L	·021 -·015	·046 -·032	·071 -·049	·096 -·065	·065 -·047	
	·25L	·046 -·032	·042 -·029	·065 -·045	·089 -·060	·072 -·051	
	·375L	·071 -·049	·065 -·045	·056 -·039	·077 -·052	·076 -·053	
6	·5L	·096 -·065	·089 -·060	·077 -·052	·063 -·042	·080 -·054	ρ_{62} ·058 -·051
	·625L	·112 -·077	·104 -·072	·092 -·063	·077 -·052	·076 -·053	
	·75L	·124 -·085	·116 -·081	·104 -·072	·089 -·060	·072 -·051	
	·875L	·131 -·090	·124 -·085	·112 -·077	·096 -·065	·065 -·047	

GIRDER NO.	SECTION x	STIFFNESS ANALYSIS — Load on Girder 3 — Point Load at ·125L	·25L	·375L	·5L	UDL	HARMONIC ANALYSIS ρ
	·125L	·034 -·010	·073 -·019	·108 -·021	·136 -·013	·095 -·010	
	·25L	·073 -·019	·065 -·015	·098 -·017	·126 -·012	·105 -·011	
	·375L	·108 -·021	·098 -·017	·083 -·011	·110 -·011	·110 -·011	
6	·5L	·136 -·013	·126 -·012	·110 -·011	·089 -·009	·113 -·011	ρ_{63} ·089 -·011
	·625L	·154 ·002	·146 -·003	·131 -·008	·110 -·011	·110 -·011	
	·75L	·165 ·012	·158 ·009	·146 -·003	·126 -·012	·105 -·011	
	·875L	·170 ·029	·165 ·012	·154 ·002	·136 -·013	·095 -·010	

Table 13. (cont'd) DISTRIBUTION FACTORS FOR LONGITUDINALS

μ_{ij} - 6 GIRDERS

GIRDER NO.	SECTION x	STIFFNESS ANALYSIS, Load on Girder 1, Point Load at ·125L ($\mu \times 10^4$)	·25L	·375L	·5L	UDL	HARMONIC ANALYSIS
2	·25L	-45.0	-36.8	-41.4	-37.9	-37.9	-37.7
	·5L	-36.2	-37.9	-38.9	-34.8	-37.6	
	·75L	-26.0	-28.5	-32.8	-37.9	-37.9	
3	·25L	-25.8	-22.4	-33.2	-41.0	-34.4	-35.4
	·5L	-44.2	-41.0	-35.4	-28.4	-36.6	
	·75L	-46.5	-46.2	-45.0	-41.0	-34.4	
4	·25L	-11.3	-10.8	-17.3	-23.9	-20.1	-20.8
	·5L	-26.2	-23.9	-20.6	-16.8	-21.5	
	·75L	-37.5	-34.6	-30.0	-23.9	-20.1	
5	·25L	-3.7	-3.5	-5.9	-8.4	-7.2	-7.4
	·5L	-9.1	-8.4	-7.4	-6.0	-7.6	
	·75L	-14.9	-13.3	-11.0	-8.4	-7.2	

GIRDER NO.	SECTION x	STIFFNESS ANALYSIS, Load on Girder 2, Point Load at ·125L ($\mu \times 10^4$)	·25L	·375L	·5L	UDL	HARMONIC ANALYSIS
2	·25L	63.9	51.3	49.6	34.9	41.4	40.0
	·5L	28.2	34.9	42.3	41.2	38.6	
	·75L	5.5	10.8	20.6	34.9	41.4	
3	·25L	-13.1	-10.3	-2.8	7.6	-0.1	1.3
	·5L	-12.8	7.6	0.2	-3.9	2.7	
	·75L	-15.1	15.3	13.5	7.6	-0.1	
4	·25L	-12.2	-9.0	-11.7	-10.9	-9.2	-9.4
	·5L	-10.9	-11.0	-9.6	-7.0	-9.6	
	·75L	-1.0	-2.4	-7.0	-10.9	-9.2	
5	·25L	-3.9	-3.8	-5.6	-7.1	-5.8	-6.1
	·5L	-8.1	-7.1	-6.0	-4.8	-6.3	
	·75L	-7.8	-8.1	-8.0	-7.1	-5.8	

GIRDER NO.	SECTION x	STIFFNESS ANALYSIS, Load on Girder 3, Point Load at ·125L ($\mu \times 10^4$)	·25L	·375L	·5L	UDL	HARMONIC ANALYSIS
2	·25L	-4.5	-2.9	7.7	20.1	10.8	12.4
	·5L	26.0	20.1	11.3	5.2	14.1	
	·75L	29.3	29.3	27.2	20.1	10.8	
3	·25L	69.0	57.0	59.0	49.1	53.9	52.8
	·5L	43.9	49.1	55.0	52.0	51.8	
	·75L	35.0	36.6	40.9	49.1	53.9	
4	·25L	-6.6	-4.7	6.0	19.1	9.9	11.4
	·5L	24.8	19.1	10.5	4.4	13.1	
	·75L	32.9	31.4	27.4	19.1	9.9	
5	·25L	-7.0	-4.3	-4.5	-1.5	-1.3	-1.3
	·5L	-0.8	-1.5	-1.5	-0.8	-1.2	
	·75L	13.7	9.7	4.0	-1.5	-1.3	

Table 14. DISTRIBUTION FACTORS FOR TRANSVERSALS[a]

(a) The μ_{ij} values, when multiplied by the simple moments, yield the transverse moments per unit length of the transverse medium.
(b) The values given are for zero torsion only ($\beta = 0$).

CHAPTER 7. ANISOTROPIC PLATE THEORY

1. Brief Outline

Guyon's[8] method of planar grid analysis, based on anisotropic plate theory, was generalised by Massonet[4] to include the torsional resistance of the constituent members.

For the evaluation of longitudinal and transverse moments, M_x and M_y, due to the application of a concentrated load P at any point on the grid, Massonet derives the following equations in terms of Fourier series.

$$M_x\,(x,\,y) \;=\; \frac{P\,l}{b\,\pi^2} \sum_{m=1}^{\infty} \frac{1}{m^2}\, K_m\,(e,\,y)\,\sin\frac{m\,\pi\,c}{l}\,\sin\frac{m\,\pi\,x}{l} \qquad (88)$$

$$M_y\,(x,\,y) \;=\; \frac{2P\,b}{l} \sum_{m=1}^{\infty} \mu_m\,\sin\frac{m\,\pi\,c}{l}\,\sin\frac{m\,\pi\,x}{l} \qquad (89)$$

where, l = length of the grid,

b = half width of the grid,

K_m = transverse distribution coefficients of the load,

c = distance of the load from the origin of axes,

x = distance of the section at which the moment is calculated,

μ_m = characteristic coefficients for transverse bending moments.

The values of K_m and μ_m for a particular grid depend on the grid factor θ and its torsional parameter α, which are given by

$$\theta \;=\; \frac{b}{l}\,\sqrt[4]{\frac{\rho_P}{\rho_E}} \qquad (90)$$

$$\alpha = \frac{\gamma_P + \gamma_E}{2\sqrt{\rho_P\;\rho_E}} \qquad (91)$$

In the above expressions,

ρ_P = elastic rigidity per unit length in the longitudinal direction,

ρ_E = elastic rigidity per unit length in the transverse direction,

γ_P = torsional rigidity per unit length in the longitudinal direction,

γ_E = torsional rigidity per unit length in the transverse direction.

Evidently, for zero torsional rigidity $\alpha = 0$, and for full torsional rigidity $\alpha = 1$. For these boundary values, Massonet has prepared tables giving K and μ values for $\theta = 0$ to 3.162. For intermediate values of α, the interpolations recommended are

$$K_\alpha = K_o + (K_1 - K_o)\sqrt{\alpha} \tag{92}$$

$$\mu_\alpha = \mu_o + (\mu_1 - \mu_o)\sqrt{\alpha} \tag{93}$$

and, a linear interpolation is considered adequate for intermediate values of θ. Note, however, that for the Fourier series terms of Eqs. 88 and 89, $\theta_m = m\,\theta$. Therefore, it is only possible to consider the first 2 or 3 terms without falling outside the range of θ given by Massonet.

After the coefficients K and μ have been determined, the required longitudinal and transverse bending moments may be computed from Eqs. 88 and 89. However, as the Fourier series tends to converge somewhat slowly for the bending moments of loaded girders, it is advisable, in the author's opinion, to restrict the use of these equations to unloaded girders only. The bending moments of the loaded girders are obtained readily by subtracting the sum of the bending moments of the unloaded girders from the free bending moment, regarding the entire grid as a simply supported beam.

For the purpose of investigating the accuracy of the anisotropic plate theory coefficients given by Massonet, a five longitudinal grid with seven transverse beams was analyzed for several loading conditions, first neglecting and then including the torsional rigidities of the members. The comparative results of stiffness analysis and plate theory are summarized in Tables 17 and 18. In order to illustrate the procedure of Massonet's method, the typical calculations involved in the example are given below.

2. Five Girder Bridge Example

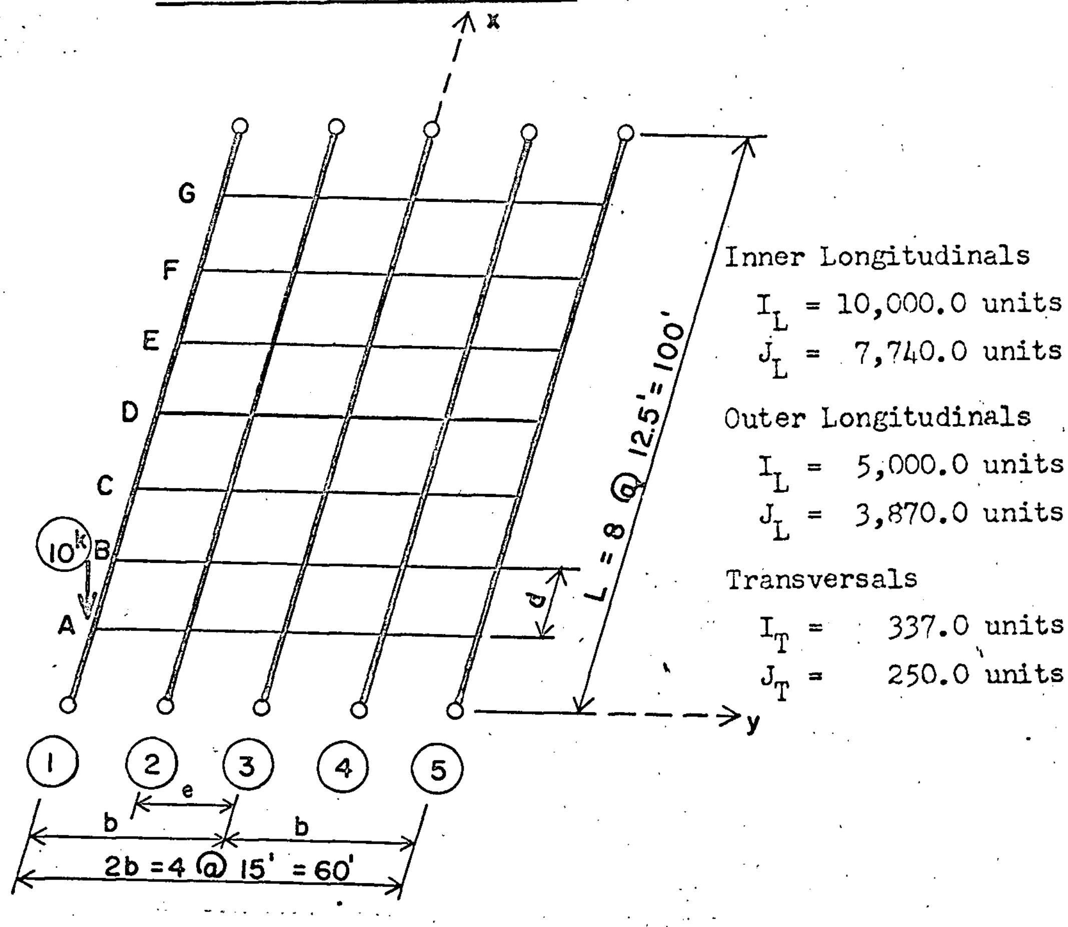


Fig. 32 Five girder bridge for the comparison of stiffness and anisotropic plate theory results

Step 1 The grid parameters, for $E = 1$ and $G = 0.5$, are

$$\rho_P = \frac{EI_L}{e} = \frac{10000}{15} = 666.67 \text{ units}$$

$$\rho_E = \frac{EI_T}{d} = \frac{337}{12.5} = 27.0 \text{ units}$$

$$\theta = \frac{b}{l}\sqrt[4]{\frac{\rho_P}{\rho_E}} = \frac{30}{100}\sqrt[4]{\frac{666.67}{27}} = 0.669$$

$$\alpha = \frac{\frac{GJ_L}{e} + \frac{GJ_t}{d}}{2\sqrt{\rho_P \rho_E}} = \frac{\frac{0.5 \times 7740}{15} + \frac{0.5 \times 250}{12.5}}{2\sqrt{666.67 \times 27}} = 1.0$$

Step 2 The longitudinal and transversal bending moment distribution factors K and μ, determined for various values of the grid factor θ by interpolation from Massonet's charts, are listed in Table 15.

Load on Girder	Grid Factor θ	Torsional Parameter α	K for Girders 1	2	3	4	5	$\mu \times 10^{-4}$ for transversals 1	2	3	4	5
1, $y = -b$	0.669	0	5.984	2.113	0.120	-0.517	-0.692	0	-2123	-1418	-435	0
		1	2.811	1.511	0.733	0.364	0.202	0	- 591	- 419	-213	0
	1.338	0	12.257	0.223	-0.562	-0.093	+0.113	0	- 873	- 256	- 73	0
		1	5.608	1.387	0.295	0.067	0.016	0	- 216	- 62	- 15	0
2, $y = -\frac{b}{2}$	0.669	0	2.113	1.919	1.055	0.195	-0.517	0	1306	- 103	-175	0
		1	1.511	1.477	0.995	0.580	0.364	0	30	- 322	-221	0
	1.338	0	0.237	3.115	0.736	-0.100	-0.093	0	892	- 150	- 60	0
		1	1.387	2.297	0.828	0.190	0.067	0	- 60	- 77	- 23	0
3, $y = 0$	0.669	0	0.120	1.055	1.618	1.055	0.120	0	266	1828	266	0
		1	0.733	0.995	1.248	0.995	0.733	0	- 86	1096	- 86	0
	1.338	0	-0.562	0.736	3.008	0.736	-0.562	0	- 62	962	- 62	0
		1	0.295	0.828	2.125	0.828	0.295	0	- 84	606	- 84	0
	2.007	0	-0.086	+0.120	4.460	+0.120	-0.086	0	- 80	596	- 80	0
		1	0.050	0.55	3.125	0.55	0.050	0	- 30	350	- 30	0

Table 15. Anisotropic Plate Theory Distribution Coefficients

Step 3 The longitudinal bending moments M_x, per unit width of the grid, due to a single concentrated load, $P = 10^K$, applied on girder 1 at $c = 1/8$, as shown in Fig. 32, in accordance with Eq. 88, are

$$M_x\,(x, y) = \frac{10 \times 100}{30\,\pi^2}\left(K_1 \sin\frac{\pi}{8}\sin\frac{\pi x}{L} + \frac{K_2}{4}\sin\frac{\pi}{4}\sin\frac{2\pi x}{L} + \frac{K_3}{9}\sin\frac{3\pi}{8}\sin\frac{3\pi x}{L} + \text{----}\right)$$

Substituting $x = L/8$, for the transverse section A,

$$M_x\,(L/8, y) = 3.38\left(K_1 \sin\frac{\pi}{8}\sin\frac{\pi}{8} + \frac{K_2}{4}\sin\frac{\pi}{4}\sin\frac{\pi}{4} + \frac{K_3}{9}\sin\frac{3\pi}{8}\sin\frac{3\pi}{8} + \text{---}\right)$$

Using the K values given in Table 15, the bending moments of the unloaded girders at section A, for $\alpha = 0$, are

$$M_{2A} = 3.38 \times 15(2.113 \times 0.382^2 + \frac{0.223}{4} \times 0.707^2) = 17.20 \text{ k.ft.}$$

$$M_{3A} = 3.38 \times 15(0.120 \times 0.382^2 - \frac{0.562}{4} \times 0.707^2) = -2.68 \text{ k.ft.}$$

$$M_{4A} = 3.38 \times 15(-0.517 \times 0.382^2 - \frac{0.093}{4} \times 0.707^2) = -4.44 \text{ k.ft.}$$

$$M_{5A} = 3.38 \times 7.5(-0.692 \times 0.382^2 + \frac{0.113}{4} \times 0.707^2) = \underline{-2.21 \text{ k.ft.}}$$

$$+7.87 \text{ k.ft.}$$

The bending moment of the loaded girder at section A is now obtained by subtracting the above sum of the loaded girder moments from the simple moment. Hence,

$$M_{1A} = \frac{10 \times 12.5 \times 87.5}{100} - 7.87 = 101.50 \text{ k.ft.}$$

<u>Step 4</u> The transverse bending moments M_y, per unit length of the grid, due to the same loading i.e. $P = 10^k$ at $c = L/8$ on girder 1, using Eq. 89, are

$$M_y\,(x, y) = \frac{2 \times 10 \times 30}{100}\left(\mu_1 \sin\frac{\pi}{8}\sin\frac{\pi x}{L} + \mu_2 \sin\frac{\pi}{4}\sin\frac{2\pi x}{L} + \text{----}\right)$$

For the transverse section A, at which the load is applied, substituting $x = L/8$,

$$M_y\,(L/8,\,y) = 6.0\,(\mu_1 \sin\frac{\pi}{8} \sin\frac{\pi}{8} + \mu_2 \sin\frac{\pi}{4} \sin\frac{\pi}{4} + \text{----})$$

Referring to Table 15 for the nontorsional μ values and taking the first two terms of the Fourier series of the above equation, the bending moments of the transversals are obtained as

$$M_{A1} = M_{A5} = 0$$

$$M_{A2} = 6.0 \times 12.5(-\,.2123 \times 0.382^2 - .0873 \times 0.707^2) = -\,7.56 \text{ k.ft.}$$

$$M_{A3} = 6.0 \times 12.5(-\,.1418 \times 0.382^2 - .0256 \times 0.707^2) = -\,2.52 \text{ k.ft.}$$

$$M_{A4} = 6.0 \times 12.5(-\,.0435 \times 0.382^2 - .0073 \times 0.707^2) = -\,0.75 \text{ k.ft.}$$

<u>Step 5</u> For the purpose of comparison, the transverse distribution factors, ρ at section A, are calculated for both the plate theory and stiffness analyses from the respective final longitudinal bending moments and shown in Table 16.

Girder	Longitudinal moments (k.ft)			ρ Distribution factors	
	M	Massonet	Stiffness	Massonet	Stiffness
1	M_{1A}	101.50	94.90	92.8%	86.7%
2	M_{2A}	17.20	21.69	15.7%	19.9%
3	M_{3A}	- 2.68	- 2.00	- 2.5%	- 1.8%
4	M_{4A}	- 4.44	- 3.18	- 4.0%	- 2.9%
5	M_{5A}	- 2.21	- 2.04	- 2.0%	- 1.9%
M_{free}		109.37	109.37	100. %	100. %

<u>Table 16.</u> Comparison of transverse distribution factors for section A.

Likewise, the comparative distribution factors of all the transverse sections were calculated for 12 different loading conditions, for both torsional and non torsional cases, and are shown in Table 17. In addition, the bending moments of the loaded transversals, as obtained by the plate theory and stiffness methods, are directly compared in Table 18.

SECTION	P=1 on Beam 1 at .125L DISTRIBUTION FACTORS FOR GIRDERS					P=1 on Beam 1 at .25L DISTRIBUTION FACTORS FOR GIRDERS					P=1 on Beam 1 at .375L DISTRIBUTION FACTORS FOR GIRDERS					P=1 on Beam 1 at .5L DISTRIBUTION FACTORS FOR GIRDERS				
	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5
α = 0																				
.125L	.866 .928	.198 .157	-.018 -.025	-.029 -.041	-.019 -.020	.739 .834	.377 .333	-.012 -.036	-.064 -.085	-.041 -.045	.631 .714	.508 .504	.024 -.018	-.099 -.127	-.065 -.075	.552 .589	.590 .634	.077 .045	-.128 -.153	-.090 -.108
.25L	.739 .834	.377 .333	-.012 -.036	-.064 -.085	-.041 -.045	.770 .838	.327 .303	-.000 -.022	-.059 -.076	-.038 -.043	.662 .727	.462 .463	.023 -.007	-.090 -.115	-.060 -.070	.581 .614	.556 .599	.064 .037	-.117 -.146	-.083 -.100
.375L	.631 .714	.508 .504	.024 -.018	-.099 -.127	-.065 -.075	.662 .727	.462 .463	.023 -.007	-.090 -.115	-.060 -.070	.714 .760	.388 .396	.026 .003	-.076 -.098	-.053 -.062	.630 .659	.495 .532	.049 .029	-.101 -.131	-.072 -.087
.5 L	.552 .589	.590 .634	.077 .045	-.128 -.153	-.090 -.108	.581 .614	.556 .599	.064 .037	-.117 -.146	-.083 -.100	.630 .659	.495 .532	.049 .029	-.101 -.131	-.072 -.087	.698 .721	.406 .434	.037 .022	-.082 -.108	-.059 -.070
.625L	.495 .428	.636 .776	.131 .122	-.148 -.188	-.114 -.140	.520 .480	.613 .724	.110 .096	-.138 -.175	-.104 -.128	.565 .561	.569 .640	.077 .063	-.123 -.155	-.090 -.109	.630 .659	.495 .532	.049 .029	-.101 -.131	-.072 -.087
.75L	.458 .291	.658 .864	.176 .214	-.159 -.202	-.133 -.168	.480 .368	.645 .814	.149 .163	-.152 -.192	-.122 -.152	.520 .480	.613 .724	.110 .096	-.138 -.175	-.104 -.128	.581 .614	.556 .599	.064 .037	-.117 -.146	-.083 -.100
.875L	.437 .202	.668 .906	.204 .287	-.165 -.209	-.146 -.187	.458 .291	.658 .864	.176 .214	-.159 -.202	-.133 -.168	.495 .428	.636 .776	.131 .122	-.148 -.188	-.114 -.140	.552 .589	.590 .634	.077 .045	-.128 -.153	-.090 -.108
α = 1																				
.125L	.713 .682	.212 .218	.051 .067	.020 .025	.004 .007	.467 .403	.366 .386	.113 .136	.045 .060	.009 .016	.316 .255	.426 .439	.172 .193	.071 .089	.016 .024	.233 .224	.430 .403	.220 .228	.097 .113	.021 .031
.25L	.467 .403	.366 .386	.113 .136	.045 .060	.009 .016	.546 .502	.302 .310	.102 .120	.041 .054	.009 .014	.372 .334	.391 .392	.156 .172	.066 .081	.014 .022	.269 .249	.419 .409	.203 .210	.090 .105	.020 .029
.375L	.316 .255	.426 .439	.172 .193	.071 .089	.016 .024	.372 .334	.391 .392	.156 .172	.066 .081	.014 .022	.474 .448	.323 .320	.132 .143	.057 .069	.013 .019	.340 .314	.390 .385	.176 .182	.078 .091	.017 .025
.5 L	.233 .224	.430 .403	.220 .228	.097 .113	.021 .631	.269 .249	.419 .409	.203 .210	.090 .105	.020 .029	.340 .314	.390 .385	.176 .182	.078 .091	.017 .025	.454 .430	.326 .325	.142 .150	.063 .074	.014 .021
.625L	.192 .234	.406 .357	.253 .241	.120 .131	.027 .038	.214 .223	.414 .388	.238 .231	.111 .123	.025 .035	.257 .240	.414 .408	.212 .238	.097 .123	.022 .035	.340 .314	.390 .385	.176 .182	.078 .091	.017 .025
.75L	.174 .202	.383 .364	.273 .240	.138 .141	.032 .042	.185 .218	.398 .369	.261 .240	.128 .134	.029 .039	.214 .223	.414 .388	.238 .231	.111 .123	.025 .035	.269 .249	.419 .409	.203 .210	.090 .105	.020 .029
.875L	.165 .150	.367 .407	.283 .228	.150 .170	.035 .046	.174 .202	.383 .364	.273 .240	.138 .141	.032 .042	.192 .234	.406 .357	.253 .241	.120 .131	.027 .038	.233 .224	.430 .403	.220 .228	.097 .113	.021 .031

Table 17. DISTRIBUTION FACTORS FOR LONGITUDINALS[(a)]

(a) These factors, when multiplied by the simple moments, yield the longitudinal moments.
(b) The lower values correspond to anisotropic plate theory, and the upper values correspond to stiffness analysis.

SECTION	P=1 on Beam 2 at .125L DISTRIBUTION FACTORS FOR GIRDERS					P=1 on Beam 2 at .25L DISTRIBUTION FACTORS FOR GIRDERS					P=1 on Beam 2 at .375L DISTRIBUTION FACTORS FOR GIRDERS					P=1 on Beam 2 at .5L DISTRIBUTION FACTORS FOR GIRDERS				
	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5
.125L	.099 .079	.774 .820	.141 .114	.000 .008	-.015 -.020	.189 .167	.578 .625	.238 .232	.018 .019	-.032 -.042	.254 .252	.443 .475	.303 .300	.042 .036	-.050 -.063	.295 .319	.368 .372	.316 .318	.086 .069	-.064 -.076
.25L	.189 .167	.578 .625	.238 .232	.018 .019	-.032 -.042	.164 .151	.636 .670	.208 .193	.021 .026	-.029 -.038	.231 .232	.493 .524	.277 .266	.045 .035	-.045 -.058	.278 .300	.402 .415	.304 .300	.075 .059	-.059 -.073
.375L	.254 .252	.443 .475	.303 .300	.042 .036	-.050 -.063	.231 .232	.493 .524	.277 .266	.045 .035	-.045 -.058	.194 .198	.575 .604	.223 .215	.040 .033	-.038 -.049	.248 .266	.465 .481	.278 .271	.061 .048	-.051 -.066
.5 L	.295 .319	.368 .372	.316 .318	.086 .069	-.064 -.076	.278 .300	.402 .415	.304 .300	.075 .059	-.059 -.073	.248 .266	.465 .481	.278 .271	.061 .048	-.051 -.066	.203 .217	.559 .583	.231 .216	.047 .037	-.041 -.054
.625L	.318 .388	.334 .331	.306 .300	.117 .076	-.074 -.094	.306 .362	.350 .343	.305 .303	.103 .079	-.069 -.087	.285 .320	.393 .400	.302 .292	.084 .065	-.061 -.078	.248 .266	.465 .481	.278 .271	.061 .048	-.051 -.066
.75L	.329 .432	.321 .320	.290 .233	.139 .116	-.080 -.101	.322 .407	.330 .310	.299 .280	.125 .100	-.076 -.096	.306 .362	.350 .343	.305 .303	.103 .079	-.069 -.087	.278 .300	.402 .415	.304 .300	.075 .059	-.059 -.073
.875L	.334 .453	.318 .316	.278 .202	.151 .133	-.082 -.105	.329 .432	.321 .320	.290 .233	.139 .116	-.080 -.101	.318 .388	.334 .331	.306 .300	.117 .076	-.074 -.094	.295 .319	.368 .372	.316 .318	.086 .069	-.064 -.076

α = 0

.125L	.106 .109	.705 .712	.136 .116	.042 .050	.010 .013	.183 .193	.468 .448	.237 .226	.090 .104	.022 .030	.213 .220	.334 .304	.281 .283	.137 .149	.036 .045	.215 .202	.274 .298	.288 .263	.175 .180	.049 .057
.25L	.183 .193	.468 .449	.237 .226	.090 .104	.022 .030	.151 .155	.550 .537	.197 .191	.082 .089	.021 .027	.195 .196	.390 .370	.258 .260	.125 .134	.033 .041	.210 .204	.304 .309	.280 .268	.162 .167	.045 .052
.375L	.213 .220	.334 .304	.281 .283	.137 .149	.036 .045	.195 .196	.390 .370	.258 .260	.125 .134	.033 .041	.162 .160	.489 .487	.213 .206	.106 .113	.029 .035	.195 .193	.367 .360	.258 .256	.141 .146	.039 .046
.5 L	.215 .202	.274 .298	.288 .263	.175 .180	.049 .057	.210 .204	.304 .309	.280 .268	.162 .167	.045 .052	.195 .193	.367 .360	.258 .256	.141 .146	.039 .046	.163 .163	.475 .469	.216 .214	.114 .119	.032 .037
.625L	.203 .179	.255 .291	.278 .269	.203 .196	.060 .066	.207 .194	.268 .285	.280 .273	.190 .187	.056 .061	.207 .204	.300 .302	.278 .270	.170 .170	.048 .055	.195 .193	.367 .360	.258 .256	.141 .146	.039 .046
.75L	.192 .182	.253 .345	.267 .262	.220 .201	.069 .071	.199 .184	.256 .309	.273 .236	.210 .204	.064 .067	.207 .194	.268 .285	.280 .273	.190 .187	.056 .061	.210 .204	.304 .309	.280 .268	.162 .167	.045 .052
.875L	.154 .204	.255 .374	.258 .140	.230 .198	.075 .085	.192 .182	.253 .345	.267 .202	.220 .201	.069 .071	.203 .179	.255 .291	.278 .269	.203 .196	.060 .066	.215 .202	.274 .298	.288 .263	.175 .180	.049 .057

α = 1

Table 17. (cont'd) DISTRIBUTION FACTORS FOR LONGITUDINALS

SECTION	P=1 on Beam 3 at .125L DISTRIBUTION FACTORS FOR GIRDERS					P=1 on Beam 3 at .25L DISTRIBUTION FACTORS FOR GIRDERS					P=1 on Beam 3 at .375L DISTRIBUTION FACTORS FOR GIRDERS					P=1 on Beam 3 at .5L DISTRIBUTION FACTORS FOR GIRDERS				
	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5
.125L	−.009 −.012	.141 .114	.738 .795	.141 .114	−.009 −.012	−.006 −.018	.238 .232	.515 .572	.238 .232	−.006 −.018	.012 −.009	.303 .300	.370 .420	.303 .300	.012 −.009	.038 .022	.316 .318	.292 .320	.316 .318	.038 .022
.25L	−.006 −.018	.238 .232	.515 .572	.238 .232	−.006 −.018	−.000 −.011	.208 .193	.584 .635	.208 .193	−.000 −.011	.012 −.003	.277 .266	.425 .475	.277 .266	.012 −.003	.032 .019	.304 .300	.328 .364	.304 .300	.032 .019
.375L	.012 −.009	.303 .300	.370 .420	.303 .300	.012 −.009	.012 −.003	.277 .266	.425 .475	.277 .266	.012 −.003	.013 .002	.223 .215	.518 .560	.223 .215	.013 .002	.024 .015	.278 .271	.396 .429	.278 .271	.024 .015
.5 L	.038 .022	.316 .318	.292 .320	.316 .318	.038 .022	.032 .019	.304 .300	.328 .364	.304 .300	.032 .019	.024 .015	.278 .271	.396 .429	.278 .271	.024 .015	.018 .011	.231 .216	.501 .546	.231 .216	.018 .011
.625L	.066 .061	.306 .300	.257 .281	.306 .300	.066 .061	.055 .048	.305 .303	.276 .298	.305 .303	.055 .048	.038 .031	.302 .292	.318 .354	.302 .292	.038 .031	.024 .015	.278 .271	.396 .429	.278 .271	.024 .015
.75L	.088 .107	.290 .233	.242 .320	.290 .233	.088 .107	.075 .081	.299 .280	.253 .278	.299 .280	.075 .081	.055 .048	.305 .303	.276 .298	.305 .303	.055 .048	.032 .019	.304 .300	.328 .364	.304 .300	.032 .019
.875L	.102 .144	.278 .202	.239 .308	.278 .202	.102 .144	.088 .107	.290 .233	.242 .320	.290 .233	.088 .107	.066 .061	.306 .300	.257 .281	.306 .300	.066 .061	.038 .022	.316 .318	.292 .320	.316 .318	.038 .022

$\alpha = 0$

.125L	.026 .034	.136 .116	.678 .703	.136 .116	.026 .034	.056 .068	.237 .226	.414 .413	.237 .226	.056 .068	.086 .096	.281 .283	.267 .242	.281 .283	.086 .096	.110 .114	.288 .263	.205 .246	.288 .263	.110 .114
.25L	.056 .068	.237 .226	.414 .413	.237 .226	.056 .068	.051 .060	.197 .191	.505 .499	.197 .191	.051 .060	.078 .086	.258 .260	.330 .308	.258 .260	.078 .086	.102 .105	.280 .268	.238 .254	.280 .268	.102 .105
.375L	.086 .096	.281 .283	.267 .242	.281 .283	.086 .096	.078 .086	.258 .260	.330 .308	.258 .260	.078 .086	.066 .072	.213 .206	.440 .441	.213 .206	.066 .072	.088 .091	.258 .256	.307 .306	.258 .256	.088 .091
.5 L	.110 .114	.288 .263	.205 .246	.288 .263	.110 .114	.102 .105	.280 .268	.238 .254	.280 .268	.102 .105	.088 .091	.258 .256	.307 .306	.258 .256	.088 .091	.071 .075	.216 .214	.425 .423	.216 .214	.071 .075
.625L	.127 .121	.278 .269	.191 .222	.278 .269	.127 .121	.119 .116	.280 .273	.202 .222	.280 .273	.119 .116	.106 .119	.278 .270	.233 .248	.278 .270	.106 .119	.088 .091	.258 .256	.307 .306	.258 .256	.088 .091
.75L	.137 .120	.267 .202	.194 .358	.267 .202	.137 .120	.131 .120	.273 .236	.193 .287	.273 .236	.131 .120	.119 .116	.280 .273	.202 .222	.280 .273	.119 .116	.102 .105	.280 .268	.238 .254	.280 .268	.102 .105
.875L	.142 .114	.258 .140	.200 .493	.258 .140	.142 .114	.137 .120	.267 .202	.194 .358	.267 .202	.137 .120	.127 .121	.278 .269	.191 .222	.278 .269	.127 .121	.110 .114	.288 .263	.205 .246	.288 .263	.110 .114

$\alpha = 1$

Table 17. (cont'd) DISTRIBUTION FACTORS FOR LONGITUDINALS

LOAD POSITION		TRANSVERSE BEAM	BENDING MOMENTS OF LOADED TRANSVERSALS Nontorsional Case ($\alpha = 0$)								BENDING MOMENTS OF LOADED TRANSVERSALS Torsional Case ($\alpha = 1$)							
			1	2		3		4		5	1	2		3		4		5
			Right	Left	Right	Left	Right	Left	Right	Left	Right	Left	Right	Left	Right	Left	Right	Left
P=10 kips on Beam 1 at section	.125L	A	0 0	-5.36 -7.56	-5.36 -7.56	-1.11 -2.52	-1.11 -2.52	-0.35 -0.75	-0.35 -0.75	0 0	7.61 0	-9.09 -1.77	-0.24 -1.77	-1.77 -0.69	-0.15 -0.69	-0.58 -0.29	-0.07 -0.29	-0.14 0
	.25L	B	0 0	-11.34 -15.90	-11.34 -15.90	-4.56 -7.24	-4.56 -7.24	-1.18 -2.33	-1.18 -2.33	0 0	11.91 0	-15.22 -4.02	0.31 -4.02	-5.26 -2.04	-0.38 -2.04	-1.97 -0.95	-0.23 -0.95	-0.48 0
	.375L	C	0 0	-14.21 -17.38	-14.21 -17.38	-7.71 -10.00	-7.71 -10.00	-2.23 -4.66	-2.23 -4.66	0 0	12.59 0	-16.75 -4.65	0.90 -4.65	-7.62 -2.92	-0.46 -2.92	-3.28 -1.43	-0.28 -1.43	-0.63 0
	.5 L	D	0 0	-14.97 -17.60	-14.97 -17.66	-8.39 -10.53	-8.39 -10.53	-1.88 -3.57	-1.88 -3.57	0 0	12.61 0	-16.99 -4.81	1.18 -4.81	-7.70 -3.14	-0.31 -3.14	-2.69 -1.60	-0.29 -1.60	-0.69 0
P=10 kips on Beam 2 at section	.125L	A	0 0	4.80 14.00	4.80 14.00	-2.20 -1.40	-2.20 -1.40	-0.21 -0.42	-0.21 -0.42	0 0	-2.93 0	4.66 -0.51	15.77 -0.51	-4.71 -0.65	0.09 -0.65	-1.06 -0.33	-0.06 -0.33	-0.24 0
	.25L	B	0 0	9.06 15.00	9.06 15.00	-3.10 -1.85	-3.10 -1.85	-1.10 -1.10	-1.10 -1.10	0 0	-3.41 0	7.43 -0.52	10.12 -0.52	-7.45 -1.80	0.80 -1.80	-3.08 -1.00	-0.15 -1.00	-0.78 0
	.375L	C	0 0	10.36 16.20	10.36 16.20	-2.65 -1.40	-2.65 -1.40	-1.63 -1.34	-1.63 -1.34	0 0	-2.66 0	7.88 -0.09	11.47 -0.09	-7.79 -2.34	1.40 -2.34	-4.37 -1.50	-0.15 -1.50	-1.27 0
	.5 L	D	0 0	10.53 16.60	10.53 16.60	-1.55 -1.44	-1.55 -1.44	-1.42 -1.31	-1.42 -1.31	0 0	-2.34 0	7.89 -0.15	11.75 -0.15	-7.76 -2.49	1.59 -2.49	-4.73 -1.65	-0.14 -1.65	-1.45 0

Table 18. COMPARATIVE RESULTS OF TRANSVERSAL BENDING MOMENTS (k.ft.) [(a)]

(a) The upper values correspond to Stiffness analysis, and the lower values correspond to Anisotropic plate theory.

CONCLUSIONS

The principles of the stiffness method have proved to be extremely efficient for the general analysis of grid frameworks. The technique employed is readily applicable to the analysis of both planar and spatial grids of any shape and complexity and, as no approximations are resorted to, the results obtained are exact. Furthermore, without any considerable change in the formulation of the problem, the effects of torsional, axial and shear deformations, variable moments of inertia and support settlements can be easily taken into account. It is also a point of interest that the same formulation remains valid for the analysis of lattice modelworks into which a plate or shell may be idealized.

In contrast, the numerous hand calculation methods developed so far are based on highly theoretical and complicated backgrounds, such as anisotropic plate theory and harmonic analysis. However, realizing the complexity of their own methods, several authors[3,4,6,13] provide numerical tables of moment distribution factors covering a limited range of problems. For a grid falling outside the range of these tables, extremely tedious calculations are involved and the possibility of a general solution becomes impractical.

Hendry & Jaeger's[3] and Massonet's[4] moment distribution factors are found to be reasonably accurate only in the immediate vicinity of the applied loads. In unloaded areas, the errors are substantial despite the fact that the example grids analyzed for comparison were especially arranged to have closely spaced transversals, in order to meet the authors' assumption of a uniform transverse medium.

In harmonic analysis the errors arise partly because the moment distribution factors are obtained without regard to the position of the load along the girders and, further, they are assumed to remain constant at every transverse section. This assumption is completely unjustified by stiffness analysis, because the exact distribution factors are different at every transverse section for every position of the load, as clearly indicated in Tables 5 to 14. For instance, in the torsional case of the five beam grid of Table 11, 35 different stiffness distribution factors ranging from .254 to .755 are tabulated against Hendry & Jaeger's single value of $\rho_{11} = 0.482$.

Although Massonet's anisotropic plate theory approach provides different distribution factors at every section for each load position, the degree of accuracy is still insufficient. Table 17 shows that, in loaded areas the errors normally lie within 10 - 20%, but, away from the loads discrepancies from 50 - 100% are not uncommon.

Besides being approximate, most of the hand calculation methods are limited to specific types of problems. Consequently, the same method may not be applicable to any two different grids. The stiffness method, however, is completely general and is not restricted to any particular type of grid. Moreover, the stiffness approach is so versatile that, once a general computer programme has been developed, the analysis of any type of grid is reduced merely to the clerical job of filling out the basic geometric and elastic data on a standard form.

In view of the facts stated above about the relative merits of stiffness analysis, the author is convinced that, upon the availability of a digital computer, hand calculation methods of grid analysis are rendered obsolete.

NOTATIONS

STIFFNESS ANALYSIS:

A	= Cross sectional area;
a,b,c	= End stiffness coefficients of a member with variable moment of inertia;
$\{D\}$	= Column vector of joint deformations of the sytem;
d	= Depth of a prismatic member;
EI	= Elastic rigidity of a member;
$\{F\}$	= Column vector of fixed end reactions of a member;
FEM,FER	= Fixed end moments and shears, respectively;
GJ	= Torsional rigidity of a member;
J	= Number of joints in a structure;
$[K]$	= Stiffness matrix of the sytem;
$[k]$	= Stiffness matrix of an individual member;
L	= Length of a member;
l,m,n	= Direction cosines of the member axes with respect to the common axes;
N	= Number of joint deformations;
$\{P\}$	= Column vector of joint loads of the structure;
$\{p\}$	= Column vector of final stress resultants of a member;
R	= Number of support restraints; or, radius of curvature;
$[T]$	= Transformation matrix;
xyz,x'y'z'	= Common and member axes;
α_t	= Coefficient of thermal expension;
$\{\delta\}$	= Column vector of final end deformations of a member;
Δ_t	= Change in temperature;
ε	= Shearing strain parameter;
λ	= Shearing stress numerical factor;

HARMONIC ANALYSIS:

EI, EI_T	= Elastic rigidities of the longitudinals and transversals, respectively;
F	= Shear force;
GJ_L	= Torsional rigidity of a longitudinal;

h	=	Spacing of the longitudinals;
L	=	Length of the grid;
M	=	Bending moment;
n	=	Number of cross girders;
w	=	Load;
y	=	Deflection;
α	=	Grid stiffness parameter;
β	=	Torsional parameter;
η	=	Ratio of EI_{outer}/EI_{inner};
θ	=	Slope ;
ρ	=	Transverse distribution factors;

ANISOTROPIC PLATE THEORY:

b	=	Half width of the grid;
c	=	Distance of load from the origin of coordinate axes;
d,e	=	Spacing of transversals and longitudinals, respectively;
K_m , m	=	Longitudinal and transversal bending moment coefficients, respectively;
l	=	Length of the grid;
x	=	Distance of the section at which the moment is calculated;
α	=	Torsional parameter;
γ_P , γ_E	=	Torsional rigidity per unit length in the longitudinal and transverse direction;
θ	=	Grid factor;
ρ_P , ρ_E	=	Elastic rigidity per unit length in the longitudinal and transversal direction;

APPENDIX B :

J	=	Torsional constant;

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APPENDIX

APPENDIX A

COMPUTER PROGRAMME FOR STIFFNESS ANALYSIS OF GRIDS

1. General Remarks

The maximum size of structures that can be handled is limited by the core memory capacity of the computer used. If no special technique is employed, the limiting number of unknowns, for example, is about 50 for the IBM 1620 and 160 for the IBM 7090. When the method of triple block band matrices described in Chapter 5 is used, the capacity can be increased to as high as 1200 unknowns with the IBM 1620 and well over 10,000 with the IBM 7090.

No accurate relation has been arrived at to estimate the execution time required for a particular grid problem. The time taken for the solution of linear simultaneous equations varies approximately with the cube of the number of unknowns, while for most of the remaining parts of the analysis the time required varies proportionately with the number of members. A relative idea can be formed from the fact that the execution time was 20 mins. for 22 unknowns and 60 mins. for 44 unknowns using an IBM 1620 computer with a 1405 Disk File and 1940 Printer. On the other hand, an execution time of only 68 seconds was required to analyze a grid with 68 unknowns using an IBM 7090.

The FORTRAN II programme presented in the following pages is capable of analyzing planar rectangular grids with a maximum of 47 unknowns and 43 members. For the general analysis of larger grids, diagrids and spatial grids, special large capacity programmes using triple block band matrices were developed. However, a discussion of these is beyond the scope of this thesis.

2. Input Data Cards

Card	Variables	Definition	Example Data for Problem of Fig. 9	FORMAT
1	ME, N	Number of members and deformations	4 3	24I3
2	(S(I), I = 1, ME)	Lengths of the members	10. 10. 10. 10.	13F6.0
3	(EI(I), I = 1, ME)	Elastic Rigidities of the members	100. 100. 100. 100.	13F6.0
4	(GJ(I), I = 1, ME)	Torsional rigidities of the members	30. 30. 30. 30.	13F6.0
5	(W(I), I = 1, ME)	Distributed loads on the members	0. 0. 0. 0.	13F6.0
6	(C(I), I = 1, ME)	Point loads on the members	10. 0. 0. 0.	13F6.0
7	(U(I), I = 1, ME)	Distances of point loads to the left hand support	5. 0. 0. 0.	13F6.0
8	((ICD(I,J), J=1, 6), I=1, ME)	Identification code numbers of each member (see Ch. 1)	0 3 0 2 0 1 3 0 2 0 1 0 0 1 0 2 0 3 1 0 2 0 3 0	24I3

3. Other Symbols Used in the Programme

Symbol	Definition
FIX(I,J)	Jth fixed end reaction of member I
SI(I,J)	Individual stiffness matrix of a member (6 by 6)
P(I)	Joint loads
SG(I,J)	Stiffness matrix of the system (N by N)
D(I)	Deformations of the system
DEL(I)	Deformations of a member
F(I)	Final stress resultants of a member

4. FORTRAN I PROGRAMME FOR PLANAR RECTANGULAR GRIDS FOR AN IBM 1620 WITH A 1405 DISK FILE

```
ID NUMBER - 1643
PRINTED FOR  R.KINRA                    ON  MAR. 31  AT  1 HR.  46.7 MIN.

FORTRAN 2 COMPILE.
      DIMENSION S(43),EI(43),ICD(43,6),W(43),C(43),U(43),FIX(6,43),
     1P(48),SI(6,6),GJ(43)
      COMMON ME,N,W,C,U,S,ICD,EI,GJ
      READ 2,ME,N
      PRINT 2,ME,N
      READ 202,(S(I),I=1,ME)
      READ 202,(EI(I),I=1,ME)
      READ 202,(GJ(I),I=1,ME)
      READ 202,( W(I),I=1,ME)
      READ 202,( C(I),I=1,ME)
      READ 202,( U(I),I=1,ME)
      READ 2,((ICD(I,J),J=1,6),I=1,ME)
      WRITE DISK1000,2,((ICD(I,J),J=1,6),I=1,ME)
      PRINT 215
      DO 808 I=1,ME
808   PRINT 214,I,S(I),EI(I),GJ(I),W(I),C(I),U(I),(ICD(I,J),J=1,6)
2     FORMAT(24I3)
202   FORMAT(13F6.0)
214   FORMAT(I2,F8.2,4F10.2,F9.2,3X,6I3)
215   FORMAT(/3HNO.,1X,6HLENGTH,6X,2HEI,8X,2HGJ,8X,3HUDL,6X,4HLOAD,7X,
     11HU,9X,11HCODE NUMBER)
      CALL FIXMOM
      CALL SITORS
      CALL LINK(1)
      END

TABLE OF MEMORY ALLOCATIONS.

  PROGRAM STARTS AT       15546
  PROGRAM ENDS AT         17746
  LOWER END OF COMMON AT  36119
```

```
FORTRAN 2 COMPILE.
      SUBROUTINE FIXMOM
      DIMENSION S(43),EI(43),ICD(43,6),W(43),C(43),U(43),FIX(6,43),
     1P(48),SI(6,6),GJ(43)
      COMMON ME,N,W,C,U,S,ICD,EI,GJ
      DO 209 I=1,ME
      B=S(I)-U(I)
      FIX(1,I)=-(W(I)*S(I)*S(I))/12.- C(I)*U(I)*B*B/(S(I)*S(I))
      FIX(2,I)= +W(I)*S(I)*S(I)/12.+ C(I)*U(I)*U(I)*B/(S(I)*S(I))
      FIX(3,I)=-0.5*W(I)*S(I)-C(I)*B/S(I) +  (FIX(1,I)+FIX(2,I))/S(I)
      FIX(4,I)=-W(I)*S(I)-C(I)-FIX(3,I)
      FIX(5,I)=0.
      FIX(6,I)=0.
209   CONTINUE
      LFX=2000
      WRITE DISK LFX,75,((FIX(J,I),J=1,6),I=1,ME)
      DO 45 I=1,N
45    P(I)=0.
      DO 325 I=1,ME
      DO 325 L=1,N
      DO 325 J=1,6
      IF (ICD(I,J)-L) 325,322,325
322   P(L)=P(L)-FIX(J,I)
325   CONTINUE
      LP=2000+ME
      WRITE DISK LP,75,(P(I),I=1,N)
75    FORMAT(6E14.8)
      RETURN
      END
```


TABLE OF MEMORY ALLOCATIONS.

```
  PROGRAM STARTS AT         15602
  PROGRAM ENDS AT           19688
  LOWER END OF COMMON AT    36119
```

```
FORTRAN 2 COMPILE.
          SUBROUTINE SITORS
          DIMENSION S(43),EI(43),ICD(43,6),W(43),C(43),U(43),FIX(6,43),
         1P(48),SI(6,6),GJ(43)
          COMMON ME,N,W,C,U,S,ICD,EI,GJ
          LSI=3000
          DO 26 I=1,ME
          PSI=2.*EI(I)/S(I)
          SI(1,1)=2.*PSI
          SI(2,1)=PSI
          SI(3,1)=(3./S(I))*PSI
          SI(4,1)=-SI(3,1)
          SI(2,2)=SI(1,1)
          SI(3,2)=SI(3,1)
          SI(4,2)=SI(4,1)
          SI(3,3)=6.*PSI/(S(I)*S(I))
          SI(4,3)=-SI(3,3)
          SI(4,4)=SI(3,3)
          DO 22 K=5,6
          DO 22 J=1,4
   22     SI(K,J)=0.
          SI(5,5)=GJ(I)/S(I)
          SI(6,5)      =-SI(5,5)
          SI(6,6)=SI(5,5)
          DO 23 L=1,5
          LP=L+1
          DO 23 J=LP,6
   23     SI(L,J)=SI(J,L)
          LSI=LSI+6
          WRITE DISK LSI,531,((SI(K,J),J=1,6),K=1,6)
  531     FORMAT(6E14.8)
   26     CONTINUE
          RETURN
          END
```

	PROGRAM STARTS AT	15612
	PROGRAM ENDS AT	19264
TABLE OF MEMORY ALLOCATIONS.	LOWER END OF COMMON AT	36119

```
FORTRAN 2 COMPILE.
      LINK(1)
      DIMENSION ICD(43,6),SI(6,6),SG(47,48)
      COMMON ME,N
      DO 2 I=1,N
      DO 2 J=1,N
2     SG(I,J)=0.
      READ DISK 1000,207,((ICD(I,J),J=1,6),I=1,ME)
      LSI=3000
      DO 841 I=1,ME
      LSI=LSI+6
      READ DISK LSI,234,((SI(K,J),J=1,6),K=1,6)
      DO 841 K=1,6
      IF(ICD(I,K)) 842,841,842
842   NN=ICD(I,K)
      DO 841ΔJ=1,6
      IF(ICD(I,J)) 843,841,843
843   MM=ICD(I,J)
      SG(NN,MM)=SG(NN,MM)+SI(K,J)
841   CONTINUE
      WRITE DISK 6000,234,((SG(I,J),J=1,N),I=1,N)
207   FORMAT(24I3)
234   FORMAT(6E14.8)
      CALL LINK(2)
      END
```

```
TABLE OF MEMORY ALLOCATIONS.

   PROGRAM STARTS AT        36366
   PROGRAM ENDS AT          38326
   LOWER END OF COMMON AT   39989

  SUBPROGRAMS CALLED     LINK

  LIBRARY FUNCTIONS CALLED.
```

```
FORTRAN 2 COMPILE.
          LINK(2)
C         SOLUTION OF LINEAR EQUATIONS
          DIMENSION SG(47,48),P(48),D(48)
          COMMON ME,N
          LP=2000+ME
          READ DISK LP,245,(P(I),I=1,N)
          READ  DISK6000,245,((SG(I,J),J=1,N),I=1,N)
          NP=N+1
          NM=N-1
          DO 2 I=1,N
2         SG(I,NP)=P(I)
          DO 3 I=1,NM
          IP=I+1
          DO 3 J=IP,N
          R=SG(J,I)/SG(I,I)
          DO 3 K=IP,NP
3         SG(J,K)=SG(J,K)-SG(I,K)*R
          D(N)=SG(N,NP)/SG(N,N)
          DO 4 I=2,N
          J=N-I+1
          JP=J+1
          D(J)=SG(J,NP)
          DO 5 K=JP,N
5         D(J)=D(J)-D(K)*SG(J,K)
4         D(J)=D(J)/SG(J,J)
          WRITE DISK 8500,245,(D(I),I=1,N)
          PRINT 832
          PRINT 833,(I,D(I),I=1,N)
833       FORMAT(5(I6,F10.4))
245       FORMAT(6E14.8)
832       FORMAT(/26HDEFORMATIONS OF THE SYSTEM)
          CALL LINK(3)
          END
```


TABLE OF MEMORY ALLOCATIONS.	PROGRAM STARTS AT	35696
	PROGRAM ENDS AT	38724
	LOWER END-OF COMMON AT	39989

```
FORTRAN 2 COMPILE.
      LINK(3)
C     FINAL END MOMENTS AND REACTIONS OF EACH MEMBER
      DIMENSION ICD(43,6),FIX(6,43),SI(6,6),D(48),DEL(6),F(6)
      COMMON ME,N
      READ DISK1000,224,((ICD(I,J),J=1,6),I=1,ME)
      READ DISK2000,447,((FIX(J,I),J=1,6),I=1,ME)
      READ DISK 8500,447,(D(I),I=1,N)
      LSI=3000
      PRINT 96
      DO 6 I=1,ME
      DO 3 L=1,6
      IF (ICD(I,L)) 31,31,32
31    DEL(L)=0.
      GO TO 3
32    M=ICD(I,L)
      DEL(L)=D(M)
3     CONTINUE
      LSI=LSI+6
      READ DISK LSI,447,((SI(K,J),J=1,6),K=1,6)
      DO 4 J=1,6
      F(J)=0.
      DO 5 K=1,6
5     F(J)=F(J)+SI(J,K)*DEL(K)
4     F(J)=F(J)+FIX(J,I)
      PRINT 2,I,(F(J),J=1,6)
6   Δ CONTINUE
2Δ    FORMAT(I2,6F9.3)
96    FORMAT(//31HFINAL END MOMENTS AND REACTIONS/8X,2HM1,6X,2HM2,7X,
     12HR1,7X,2HR2,8X,2HT1,7X,2HT2/)
224   FORMAT(24I3)
447   FORMAT(6E14.8)
      STOP
      END
```

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TABLE OF MEMORY ALLOCATIONS.

PROGRAM STARTS AT	17016
PROGRAM ENDS AT	19712
LOWER END OF COMMON AT	39989

APPENDIX B

TORSIONAL CONSTANTS

The torsional moment of inertia of rectangular sections is, in general, given by

$$J = \eta b^3 h$$

in which h is the depth of the section, b is the width and is the torsional factor given in Table A[(a)] for values of h/b between 1 and ∞.

$n = h/b$	1	1.2	1.5	2	2.5	3	4	5	10	∞
η	.141	.166	.196	.229	.249	.263	.281	.291	.312	.333

Table A. Torsional Constants for Rectangular Sections

Values of the torsional moment of inertia for various other types of sections are given in Tables B[(b)] and C[(c)].

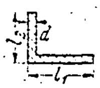
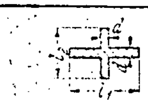
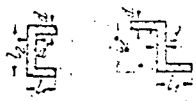
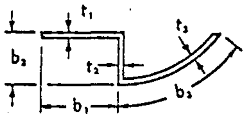
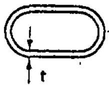
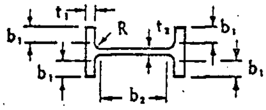
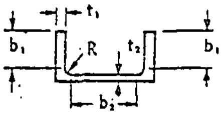
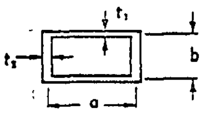
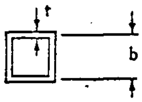
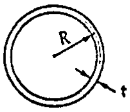
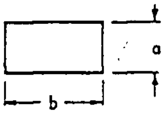
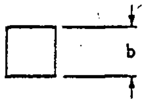
Steel Sections					
C. Webere Forsh.-Arb.Heft249	$J^* = \frac{d^3}{3}[l_1 + l_2 - 1{,}6d]$	$J^* = \frac{d^3}{3}[l_1 + l_2 - 0{,}9d]$	$J^* = \frac{d^3}{3}[l_1 + l_2 - 0{,}15d]$	$J^* = \frac{d^3}{3}[2l_1 + l_2 - 1{,}2d]$	$J^* = \frac{d^3}{3}[2l_1 + l_2 - 2{,}6d]$
L Schmiedene Z. angew. Math Mech Bd. 10 (1930)	$J^* = \frac{d^3}{3}\Sigma l_1 - 0{,}1444 d^4$	—	—	—	$J^* = \frac{d^3}{3}\Sigma l_1 - 0{,}0788 d^4$

Table B. Torsional Constants for Steel Sections

(a) Timoshenko, S., "Theory of Elasticity", McGraw Hill Inc., 1939, page 248.
(b) Poschl, Theodor, "Elemanter Mukavemet", Technical University of Istanbul, 1952.
(c) "The Strength of Aluminium", Aluminium Company of Canada, Ltd., page 76.

Section		Torsion Constant, J	Factor for Maximum Stress, C_s	Factor for Ultimate Torque, C_u
1. Thin Walled Open Section		$\frac{\Sigma bt^3}{3}$	$\frac{t}{J}$ $t =$ maximum thickness	$\frac{\Sigma bt^2}{2}$
2. Thin Walled Closed Section		$\frac{4A^2}{\int \frac{du}{t}}$	$\frac{1}{2At}$ $t =$ minimum thickness.	2At
3. Thin Walled Closed Section, Uniform Thickness		$\frac{4A^2t}{U}$	$\frac{1}{2At}$	2At
4. I Beam		$4\frac{b_1t_1^3}{3} + \frac{b_1t_1^3}{3} + 2\left[nt_1 - 0.35(t_1 - t_2)\right]^4$ See Fig. C for value of n.	$\left[1.5+\frac{1}{2}\left(\frac{R}{t_1}+\frac{t_1}{R}\right)\right]\frac{t_1}{J}$ $\frac{1}{4} < \frac{R}{t_1} < 2$	
5. Channel or Z Section		$2\frac{b_1t_1^3}{3} + \frac{b_2t_2^3}{3} + 2\left[nt_2 + 0.45(t_1 - t_2)\right]^4$ See Fig. C for value of n.	$\left[1+\frac{1}{3}\left(\frac{R}{t_1}+\frac{t_1}{R}\right)\right]\frac{t_1}{J}$ $\frac{1}{4} < \frac{R}{t_1} < 2$	
6. Rectangular Hollow		$\frac{2a^2b^2}{\left(\frac{a}{t_1}+\frac{b}{t_2}\right)}$	$\frac{1}{2abt}$ $t =$ minimum thickness.	2abt
7. Square Hollow		b^3t	$\frac{1}{2b^2t}$	$2b^2t$
8. Thin Walled Tube		$2\pi R^3t$	$\frac{1}{2\pi R^2t}$	$2\pi R^2t$
9. Solid Rectangle		$\frac{ba^3}{3}\left[1 - 0.63\frac{a}{b} + 0.052\left(\frac{a}{b}\right)^2\right]$ $a < b$	$\frac{3(b+0.6a)}{a^2b^2}$	$\frac{a^2}{6}(3b - a)$
10. Solid Square		$0.141\,b^4$	$\frac{1}{0.208b^3}$	$\frac{b^3}{3}$
11. Solid Round		$\frac{\pi R^4}{2}$	$\frac{2}{\pi R^3}$	$\frac{2\pi R^3}{3}$

Nomenclature: $A =$ area enclosed by the median line of the wall of a closed section,
$U =$ the perimeter of the median line of a closed section.

The highest shear stress in a shape is obtained by multiplying the applied torque by C_s.

The ultimate torque for a shape is given by the product of the ultimate shear stress and C_u.

TABLE C — TORSION CONSTANTS FOR STRUCTURAL SHAPES