STRESS ANALYSIS OF WOOD STAVE PIPE

BY

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IN THE DEPARTMENT OF
CIVIL ENGINEERING

We accept this thesis as conforming
to the required standard

THE UNIVERSITY OF BRITISH COLUMBIA
1965
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Date July 1965.
ABSTRACT

At present the majority of wood-stave pipelines are supported on rigid cradles which bear on the lower 45% of the pipe circumference. The accepted method of analysis, developed by Regnell, completely ignores the stress concentrations induced in the staves just above the cradle tips. In this work, a full ring is proposed to distribute the support reaction to all staves and minimize deflections from a circular profile.

From a consideration of equilibrium and stress-displacement relations for a stave element two fourth-order partial differential equations in terms of the radial and tangential displacements of the element are developed. Trigonometric series are applied to their solution. The support ring displacements are similarly described in series form. A study of the compatibility of ring and stave deflections removes the indeterminacy and all stress resultants, as functions of the ring or stave deflections, are then available from back-substitution. The formulas established are sufficiently complex that access to an electronic computer is a great practical advantage.

In a numerical example, the effects of modifying the ring stiffness, hydraulic head, and the circumferential stiffness of the stave cylinder are investigated. The non-linear influence of ring and band tensions on the deformed shape of the structure is included. Design considerations are briefly discussed.
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For convenience an equation number defining the notation is indicated when applicable.

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The author wishes sincerely to thank his supervisor, Dr. R. F. Hooley, for the encouragement, guidance, and inspiration conveyed to him during the preparation of this work. Gratitude is also expressed to the National Research Council of Canada for financial support in the form of an assistantship, and to the U.B.C. Computing Centre for provision of an invaluable means of performing the formidable calculations.

Vancouver, B.C.
INTRODUCTION

Continuous wood stave pipelines have been used extensively throughout the world for many decades, principally as power development penstocks and industrial water supply lines. These structures have proved to be economical, relatively simple to fabricate, non-corrosive, and lasting. Most installations have an expected life of over 40 years, provided they are properly designed and maintained. The maximum static pressure head on the pipeline is about 200 feet. As far as this author is aware, the largest diameter presently in service is 18 feet [1].

The timber staves are two inches to six inches thick and three inches to eight inches wide, cut in random lengths. The cross-section of each stave is trapezoidal so that the completed pipe is circular. Post-tensioned steel bands carry the hoop tension.

Most wood stave pipes are supported every few feet by a timber, concrete, or steel cradle which encloses about 45% of the pipe circumference. Investigators have found that the portions of the pipe just above the tops of the cradles deteriorate more rapidly than other parts of the pipe. A tendency, particularly under low head, for the aging pipe to experience severe bending stresses in this region causes the compression fibres on the exterior to become spongy. With fluctuations of head, chiefly from water hammer, together

a. Numbers in parentheses refer to the bibliography.
with freezing and thawing action, excessive leakage develops in this region. Since the life of a wood stave pipeline is generally dependent on the amount of damage caused by decayed wood it is important to reduce stress concentrations as much as possible.

A full circular ring will reduce the stress riser effect of the rigid cradle support, distributing the support reaction over all staves, and keeping the pipe very close to circular. For this reason, this analysis is restricted to an investigation of a full ring support. To reduce the bending stresses that would occur if radial supports are used on the ring, it is logical to use two columns tangent to the ring. Lateral stability can be provided by a horizontal strut between the columns, pinned to the ring. See Fig. (1).

A further practical advantage of the full ring support is evident. Considerable difficulty would be encountered in mathematically describing the cradle support. A numerical technique could be used but the convenience of general expressions for deflection and stress is then sacrificed. With a full ring support, however, the boundary conditions at the ring are simple, and loadings on the ring that are a function of the angle from a vertical through the crown can be easily expanded in infinite series form.

Regnell [2] formulated the generally accepted Stave Pile method for spacing the cradle supports. However, the aforementioned stress concentrations at the cradles are neglected in application of this procedure. A more precise technique, using full ring supports,
Figure 1. Full Ring Support
was proposed by Hooley [3]. While employing similar assumptions and mathematical approach, this thesis is intended to further improve upon the work of Hooley, complicating the analysis by accounting for eccentricities in the ring loading, circumferential stiffness of the stave cylinder, and the non-linear effect of ring and band tension on the deformed shape of the pipe.

**OUTLINE OF INVESTIGATION TECHNIQUE**

In order to appreciate the significance of the mathematical steps which follow, a brief outline of the complete procedure is useful.

From a consideration of equilibrium and stress-displacement relations for a stave element two fourth-order partial differential equations in terms of the radial and tangential displacements of the element are developed. These two equations are reduced to one eighth-order partial differential equation in terms of the radial displacement. This equation is solved by describing the displacement in Fourier series form. Four constants of integration are eliminated immediately by demanding symmetry of deflections at midspan. Substitution of the radial deflection into one of the original fourth-order equations establishes the tangential displacement, again in Fourier series form.

A study of the equilibrium of the ring under the imposed stave shear loadings, coupled with stress-displacement relations for the ring, results in expressions involving both the stave and ring deflections. Since the stave cylinders and the supporting ring are
physically linked, stave and ring deflections at the ring must be compatible. Application of this concept in expressing the stave deflections in terms of the ring deflections removes the remaining four constants of integration. The analysis then becomes completely determinate. The ring displacements are found, and because all stress resultants are functions of the ring displacements, either directly, or related through the compatibility criterion, somewhat sophisticated back-substitutions establish all desired formulas.

THE STAVE CYLINDER

3.1 ASSUMPTIONS.

In accordance with customary procedures for shell analysis it is assumed that:

1. All points on a normal to the middle surface before deformation remain on the normal to the middle surface after deformation. (This is analogous to the hypothesis in beam theory that plane sections remain plane during bending.)

2. Deformations of the cylinder due to shear forces can be neglected.

3. The stave thickness is small compared with the cylinder radius. (Hence the bending stress has a linear distribution over a plane section. As is usually assumed for large arches, "curved beam" theory is not required.)

In addition, pertinent to the wood stave pipe, it is assumed that:
4. The pipe is flowing full of liquid.

5. The full ring supports are equally spaced.

6. The load on an individual stave is from internal liquid pressure (which acts only radially), the two adjacent staves, and the bands.

7. No longitudinal shear stress exists between staves.
   This infers that no longitudinal tension can exist in the staves because there is no way in which it can be developed. This assumption is very conservative when applied to a new pipe which behaves as an orthotropic cylindrical shell with longitudinal shear between staves developed by friction. With the passage of time however, loss of post-tensioning in the bands and organic growth and rot on the wood alter the shell behaviour. The friction-induced shear capacity becomes an unreliable source of strength.

8. The bands have been tightened sufficiently that separation of the staves does not occur under any conditions. The composite cross-section is thus capable of taking circumferential tension as relief of initial compression. The post-tensioning stresses must be added to the final stresses found from this analysis.

To justify the principle of superposition, in keeping all equations linear, it is generally assumed that all displacements are
sufficiently small that equations of equilibrium formulated from the initial and final (deformed) geometry are identical. However, it is an object of this investigation to discover the effect of the circumferential tension on the displacements (and hence stress resultants) so the "beam-column" factor is included, although the small displacement assumption is adhered to as it affects other kinematical relations.

Armed with the above assumptions the investigation proceeds.

3.2 DIFFERENTIAL EQUATIONS FOR STAVE DISPLACEMENTS.

A stave segment of thickness $t$ and width $w$ at distance $x$ from the ring support and angle $\phi$ from the vertical through the crown is illustrated in Fig. (2). The radial pressure $Z$ and stress resultants $V, V_1, V_\phi, N_\phi, M, M_1,$ and $M_\phi$ per unit length at the middle surface of the cylinder acting on this segment are defined in Fig. (3). The segment width $Rd\phi$ corresponds to the actual width $w$ of a stave. (In this respect this investigation differs from a conventional shell analysis.)

The pressure $Z$ is given by

$$Z = \gamma[h + R(1 - \cos\phi)]$$  \hspace{1cm} (1)

where $\gamma$ is the unit weight of the contained liquid and $h$ is the pressure head on the pipe measured above the crown.
Figure 2. Coordinates of a Stave Element
Figure 3. Stress Resultants and Load Acting on a Stave Element.
From an equilibrium study of the stave element five equations are developed.

\[ \Sigma(\text{Radial Forces}) = 0 \quad \frac{R\partial V}{\partial x} + N_\phi + \frac{\partial V}{\partial \phi} = ZR \]  
(2)

\[ \Sigma(\text{Tangential Forces}) = 0 \quad \frac{\partial N_\phi}{\partial \phi} + \frac{R\partial V}{\partial x} - V_\phi = 0 \]  
(3)

\[ \Sigma(\text{Moments})_{a-a} = 0 \quad V_\phi = \frac{1}{R} \left[ \frac{\partial M}{\partial \phi} - N_\phi \cdot \frac{\partial u}{\partial \phi} \right] \]  
(4)

\[ \Sigma(\text{Moments})_{b-b} = 0 \quad V = \frac{\partial M}{\partial x} \]  
(5)

\[ \Sigma(\text{Moments})_{c-c} = 0 \quad V_1 = \frac{\partial M_1}{\partial x} \]  
(6)

Since it is specified that no shear stress exists between staves an equilibrium check of forces in the longitudinal direction yields no information.

The non-linear term in Eqn. (4) contributed by the tension \( N_\phi \) arises from unequal radial displacement \( u \) of the stave edges, where \( u \) is positive in the direction of increasing radius \( R \). See Fig. (2).

Eqns. (2) to (6) include the eight unknowns \( V, V_1, V_\phi, M, M_1, M_\phi, N_\phi, \) and \( u \). It is necessary to introduce stress-deformation relationships to provide additional equations.

The tangential displacement \( y \), defined in Fig. (2), is positive in the direction of increasing \( \phi \). The tangential elongation
of the stave element due to \( u \), neglecting second order terms, is
\( u d\phi \), and due to \( y \) is \( \frac{\partial y}{\partial \phi} d\phi \). The tangential strain \( \varepsilon_\phi \) is then given by

\[
\varepsilon_\phi = \frac{1}{R} \left( \frac{\partial y}{\partial \phi} + u \right)
\]

During deformation a normal to the middle surface of the cylinder experiences a rotation \( \theta \). Fig. (4) indicates \( \theta \) is given by

\[
\theta = -\frac{du}{ds} + \frac{y}{R} = -\frac{1}{R} \left( \frac{\partial u}{\partial \phi} - y \right)
\]

Before deformation the angle between the normals at the stave element edges is \( d\phi \). After deformation this angle is

\[
d\phi - \frac{\partial \theta}{\partial \phi} d\phi = \left[ 1 + \frac{1}{R} \left( \frac{\partial^2 u}{\partial \phi^2} - \frac{\partial y}{\partial \phi} \right) \right] d\phi
\]

The curvature of the deformed cylinder is then

\[
\frac{d\phi}{ds} \text{ deformed} = \frac{\left[ 1 + \frac{1}{R} \left( \frac{\partial^2 u}{\partial \phi^2} - \frac{\partial y}{\partial \phi} \right) \right] d\phi}{R[1 + \varepsilon_\phi] d\phi}
\]

Since \( \varepsilon_\phi \) is very small in comparison with unity, as is usual in arch theory, the denominator becomes \( Rd\phi \). The curvature before deformation is \( 1/R \). The change in curvature in the tangential direction \( \chi_\phi \) is then
Figure 4. Radial and Tangential Displacements and Rotation of the Stave Element.
Alternatively, substitution of $\epsilon_\phi$ in the final curvature equation gives

$$\frac{\partial \phi}{\partial s_{\text{deformed}}} = \frac{1}{R} [1 + \frac{1}{R} (\frac{\partial^2 u}{\partial \phi^2} + u) - \epsilon_\phi]$$

Again, since $\epsilon_\phi$ is small in comparison with unity it can be neglected and the change of curvature becomes

$$\chi_\phi = \frac{1}{R} [1 + \frac{1}{R} (\frac{\partial^2 u}{\partial \phi^2} + u)] - \frac{1}{R} = \frac{1}{R^2} (\frac{\partial^2 u}{\partial \phi^2} + u) \quad (8b)$$

Some authors prefer the first equation for change of curvature, among them Timoshenko [4], Billington [5], and the A.S.C.E. Committee on thin shell design [6], while others prefer the second, among them Flügge [7], Chronowicz [8], and Borg [9]. Since the difference between the equations is of order $\epsilon_\phi/R$, as Novozhilov [10] points out, the equation which offers the greatest simplicity for computing should be used. In this case Eqn. (8b) is the most convenient to use.

Since the individual staves behave as beams in the longitudinal direction the changes of curvature $\chi$ and $\chi_1$ corresponding to $M$ and $M_1$ respectively, are, as usual

$$\chi = -\frac{\partial^2 y}{\partial x^2} \quad (9)$$
and, \( \chi_1 = \frac{\partial^2 u}{\partial x^2} \) \hspace{1cm} (10)

From Hooke's law relating stress and strain for a linearly elastic system (assumption 1):

\[
N = E_s A_t \epsilon_t \tag{11a}
\]
\[
M = E_s I_t x_\epsilon \tag{11b}
\]
\[
M = E_1 \frac{tw^2}{12} \tag{11c}
\]
\[
M_1 = E_1 \frac{t^3}{12} x_1 \tag{11d}
\]

where an analysis in the manner of an "uncracked" concrete section gives the equivalent transformed area \( A_t \) and moment of inertia \( I_t \).

From Fig. (5).

\[
A_t = \frac{A}{S} + \frac{E_2}{E_s} t \tag{12a}
\]

and \( I_t = \frac{A}{S} \left[ \frac{\varphi_b^2}{16} + \left( \frac{\varphi_b}{2} + t - x_{NA} \right)^2 \right] + \frac{E_2}{E_s} \left[ \frac{t^2}{12} + \left( \frac{t}{2} - x_{NA} \right)^2 \right] \tag{12b} \)

where

- \( A \) is the cross-sectional area of the band
- \( S \) is the band spacing
- \( \varphi_b \) is the band diameter
- \( E_s \) is the modulus of elasticity of the band material
Figure 5. Transformed Cross-Section

Actual Section

Transformed Section
(as band material)

Strain Distribution
\( E_1 \) is the modulus of elasticity of the stave material parallel to the grain (i.e., in the longitudinal direction \( x \)).

\( E_2 \) is the modulus of elasticity of the stave material perpendicular to the grain.

\( X_{NA} \) is the distance from the inside of the cylinder to the centroid of the composite section, given by

\[
X_{NA} = \left[ \frac{A}{S} \left( t + \frac{t_0}{2} \right) + \frac{t^2}{2} \cdot \frac{E_2}{E_s} \right] \cdot \frac{1}{A_t} \tag{12c}
\]

Substitution of the Eqn. (7) for the strain \( \varepsilon_\varphi \) and Eqns. (8b), (9), and (10) for the curvatures \( x_\varphi, x, \) and \( x_1 \) into Eqns. (11) furnishes the desired stress-displacement relations:

\[
N_\varphi = \frac{E A_t}{R} \left( \frac{\partial y}{\partial \varphi} + u \right) \tag{13}
\]

\[
M_\varphi = \frac{E S I_t}{R^2} \left( \frac{\partial^2 u}{\partial \varphi^2} + u \right) \tag{14}
\]

\[
M = \frac{E t w^2}{12 l^2} \frac{\partial^2 y}{\partial x^2} \tag{15}
\]

\[
M_1 = E_1 \frac{t}{12 l^2} \frac{\partial^2 y}{\partial x^2} \tag{16}
\]

The ten equations (1), (2), (3), (4), (5), (6), (13), (14), (15), and (16) contain ten unknowns. They are combined to give two partial differential equations in terms of \( u \) and \( y \):
\[ - \beta \frac{\partial^4 y}{\partial x^4} + \frac{\partial^2 y}{\partial \phi^2} - \mu \frac{\partial^3 u}{\partial \phi^3} + (1 - \mu + vN_\phi) \frac{\partial u}{\partial \phi} = 0 \]  

(17)

\[ \frac{\partial y}{\partial \phi} + u + \eta \frac{\partial^4 u}{\partial x^4} + \mu \frac{\partial^4 u}{\partial \phi^4} = (vN_\phi - \mu) \frac{\partial^2 u}{\partial \phi^2} = a + \rho \cdot (1 - \cos \phi) \]  

(18)

where the substitution parameters are:

\[ \nu = \frac{1}{A_t E_s} \]  

(19a)

\[ \beta = \frac{v E_s t w^2 R^2}{12} \]  

(19b)

\[ \eta = \beta \left( \frac{t}{w} \right)^2 \]  

(19c)

\[ \mu = \frac{v E_s I_t}{R^2} \]  

(19d)

\[ \alpha = v \gamma h R^2 \]  

(19e)

\[ \rho = \alpha \cdot \frac{R}{h} \]  

(19f)

Eqns. (17) and (18) incorporate the non-linear term in \( N_\phi \).

It is most convenient to look after this non-linearity by iteration, guessing a suitable value for \( N_\phi \) to initiate the process. \( N_\phi \) is then a known quantity. To avoid confusing this \( N_\phi \) with the stress resultant \( N_\phi \) to be found, the \( N_\phi \) in Eqns. (17) and (18) is re-labelled as \( T \).

When the iteration process is complete \( N_\phi \) should be equal to \( T \).

However, \( N_\phi \) is actually a function of both \( x \) and \( \phi \), so the value to be used in each ensuing iteration step should also be a variable.
function. This complicates the problem considerably. Although 
under low head \( N_\varphi \) varies markedly with \( \varphi \), its magnitude is relatively 
small, and the resultant non-linear effect is very small. With the 
large \( N_\varphi \) encountered under high head, however, the non-linear effect 
becomes important. Fortunately \( N_\varphi \) is then almost constant with \( \varphi \), so 
satisfactory results are obtained by using a weighted average value. 
Thus \( N_\varphi \) is taken to be constant (i.e. \( T \)) in writing the derivative 
of \( V_\varphi \) with respect to \( \varphi \) for substitution into Eqn. (2).

Rather than attempt to solve Eqns. (17) and (18) simultaneously, 
it is easier to reduce the system to one equation involving only 
one unknown, in the manner of treating ordinary algebraic equations. 
It is most convenient to find \( u \) first.

If Eqn. (17) is differentiated with respect to \( \varphi \), the required 
derivatives \( \frac{\partial^5 y}{\partial x^4 \partial \varphi} \) and \( \frac{\partial^3 y}{\partial \varphi^3} \) of Eqn. (18) may be substituted in, to 
give the eighth-order partial differential equation in \( u \):

\[
\beta_1 \frac{\partial^6 u}{\partial x^6} + \beta \frac{\partial^6 u}{\partial x^4 \partial \varphi^2} + \beta \frac{\partial^4 u}{\partial x^4} - (\beta \nu T - \beta \mu + \eta) \frac{\partial^6 u}{\partial x^4 \partial \varphi^2} - \mu \frac{\partial^6 u}{\partial \varphi^6} + (\nu T - 2\mu) \frac{\partial^4 u}{\partial \varphi^4} + (\nu T - \mu) \frac{\partial^2 u}{\partial \varphi^2} = -\rho \cos \varphi
\]

(20)

3.3. SOLUTION FOR THE RADIAL DISPLACEMENT

The radial displacement \( u \) is expressed as the product of 
a function of \( x \) alone and a function of \( \varphi \) alone (i.e. the variables 
are separable) in Fourier series form. Thus,
\[ u = \sum_{n=0}^{\infty} U_n \cos n\varphi \quad (21) \]

where \( U_n \) is a function of \( x \) only.

If Eqn. (21) is substituted into Eqn. (20), and if for each value of \( n \) the coefficients of \( \cos n\varphi \) of the expanded expression are equated, then a set of eighth-order linear ordinary differential equations result from which the values of \( U_n \) are found.

For \( n = 0 \)
\[ a \frac{d^8 U}{dx^8} + b \frac{d^4 U}{dx^4} = 0 \quad (22a) \]

For \( n = 1 \)
\[ a \frac{d^8 U}{dx^8} + b \frac{d^4 U}{dx^4} = -1 \quad (22b) \]

For \( n > 1 \)
\[ a \frac{d^8 U}{dx^8} + b \frac{d^4 U}{dx^4} + c U_n = 0 \quad (22c) \]

when for \( n = 0, 1 \) or \( n > 1 \)
\[ a = \frac{\beta \eta}{\rho} \quad (23a) \]
\[ b_n = \frac{\beta}{\rho} \left( 1 + \mu \left[ n^4 - n^2 \right] + \nu T n^2 \right) + \frac{\eta n^2}{\rho} \quad (23b) \]
\[ c_n = \frac{n^2 (n^2 - 1)(\mu [n^2 - 1] + \nu T)}{\rho} \quad (23c) \]

Eight constants of integration are found from each of Eqns. (22) but because of the prescribed symmetry of deflection about midspan (assumption 5) half of these are eliminated immediately. The complete
solution for $u$ is then

$$
u = C_0 + D_0 x^2 + \theta_{10}(x)$$

$$+ (C_1 + D_1 x^2 + \theta_{11}(x) - \frac{x^4}{24b_1}) \cos \varphi$$

$$+ \sum_{n=2}^{\infty} \left[ \theta_{1n}(x) + \theta_{2n}(x) \right] \cos n\varphi \quad (24)$$

where

for $0 \leq n \leq \infty$ 

$$\theta_{1n}(x) = A_n \sin \beta_{1n} x \sinh \beta_{1n} x$$

$$+ B_n \cos \beta_{1n} x \cosh \beta_{1n} x \quad (25a)$$

for $2 \leq n \leq \infty$ 

$$\theta_{2n}(x) = C_n \sin \beta_{2n} x \sinh \beta_{2n} x$$

$$+ D_n \cos \beta_{2n} x \cosh \beta_{2n} x \quad (25b)$$

and

for $n = 0$ 

$$\beta_{10}^4 = \frac{b_0}{4a} = \frac{1}{4\bar{b}} \quad (26a)$$

for $n = 1$ 

$$\beta_{11}^4 = \frac{b_n}{4a} \quad (26b)$$

for $n > 1$ 

$$\beta_{1n}^4 = \frac{b_n}{8a} \left( 1 + \frac{\sqrt{1 - \frac{4ac_n}{b_n^2}}}{b_n} \right) \quad (26c)$$

$$\beta_{2n}^4 = \frac{b_n}{8a} \left( 1 - \frac{\sqrt{1 - \frac{4ac_n}{b_n^2}}}{b_n} \right) \quad (26d)$$
and $A_0, B_0, C_0, D_0, A_1, B_1, C_1, D_1, A_n, B_n, C_n, D_n$ are constants to be determined.

3.4 SOLUTION FOR THE TANGENTIAL DISPLACEMENT.

With $u$ known, the tangential displacement $y$ is found by integration of Eqn. (18). Then

$$y = \left[ a + \rho - C_0 - D_0 x^2 + (4 \eta \beta_{10}^4 - 1) \theta_{10}(x) \right] \varphi + f(x)$$

$$+ \left[ \frac{1}{b_1} - \rho - (C_1 + D_1 x^2 - \frac{x^4}{12}) (1 + \nu \tau) \right] \sin \varphi$$

$$- \sum_{n=1}^{\infty} \frac{1}{n} \left[ 1 + \mu (n^4 - n^2) + \nu \tau n^2 - 4 \eta \beta_{1n}^4 \right] \theta_{1n}(x) \sin n \varphi$$

$$- \sum_{n=2}^{\infty} \frac{1}{n} \left[ 1 + \mu (n^4 - n^2) + \nu \tau n^2 - 4 \eta \beta_{2n}^4 \right] \theta_{2n}(x) \sin n \varphi$$

Application of Eqn. (26a) reduces the first term to

$$[a + \rho - C_0 - D_0 x^2] \varphi.$$ This term represents a twist of the entire cylinder, a case that would prevail if the ring support legs settled differentially. Under the prescribed loading and support conditions a twist does not occur, so the term is set equal to zero. Since this must be true for any $x$, $D_0$ must be zero, leaving $C_0 = a + \rho$.

The expression for $y$ is abbreviated by letting

$$\tau_{1n} = \frac{1}{n} \left[ 1 + \mu (n^4 - n^2) + \nu \tau n^2 - 4 \eta \beta_{1n}^4 \right]$$

$$\tau_{2n} = \frac{1}{n} \left[ 1 + \mu (n^4 - n^2) + \nu \tau n^2 - 4 \eta \beta_{2n}^4 \right]$$
Now
\[ y = f(x) + \left[ \frac{\pi}{b_1} - \rho - \left( C_1 + D_1 x^2 - \frac{x^4}{24 b_1} \right) (1 + vT) \right] \sin \phi \]
\[ \sum_{n=1}^{\infty} \tau_{1n} \theta_{1n}(x) \sin n\phi \]
\[ \sum_{n=2}^{\infty} \tau_{2n} \theta_{2n}(x) \sin n\phi \tag{28} \]

To determine the integration \( f(x) \) it is necessary to substitute the solutions for \( u \) and \( y \), Eqns. (24) and (28), into Eqn. (17). Since this equation holds for any value of \( n \), equating coefficients of \( \sin n\phi \) for \( n = 0 \) results in \( \frac{d^4 f(x)}{dx^4} = 0 \). However, \( y \) does not possess a term in \( n = 0 \). Thus \( f(x) = 0 \). Finally, the solution for \( y \) is given by
\[ y = \left[ \frac{\pi}{b_1} - \rho - \left( C_1 + D_1 x^2 - \frac{x^4}{24 b_1} \right) (1 + vT) \right] \sin \phi \]
\[ \sum_{n=1}^{\infty} \tau_{1n} \theta_{1n}(x) \sin n\phi \]
\[ \sum_{n=2}^{\infty} \tau_{2n} \theta_{2n}(x) \sin n\phi \tag{29} \]

For each value of \( n \), four constants of integration that influence the shape of the stave cylinder remain to be determined. The supporting ring evidently has a "pinching" tendency on the pipe, and in proportion to its flexural stiffness attempts to maintain a circular cross-section. Therefore, it is necessary to inspect the behaviour of the ring.

**THE FLEXIBLE RING SUPPORT**

4.1 DIFFERENTIAL EQUATIONS FOR THE RING

The ring element illustrated in Fig. (6) is under the influence
of a radial load per unit length $p$, a tangential load per unit length $q$, and a moment $m$ created by any eccentricity of the load $q$. The desired ring stress resultants are the tension $T_R$, shear $V_R$, and moment $M_R$.

The three equations of equilibrium are satisfied:

$$\Sigma(\text{Radial Forces}) = 0 \quad T_R = -\frac{\partial V_R}{\partial \phi} - pR^R \quad (30)$$

$$\Sigma(\text{Tangential Forces}) = 0 \quad V_R = qR^R + \frac{dT_R}{d\phi} \quad (31)$$

$$\Sigma(\text{Moments}) = 0 \quad V_R = \frac{1}{R^R} \left( \frac{\partial M_R}{\partial \phi} - T_R \frac{du_R}{d\phi} \right) + m \quad (32)$$

where $R^R$ is the radius to the ring centroid and $u_R$ is the radial displacement of the ring element.

The term $T_R/R^R \cdot \partial u_R/\partial \phi$ in Eqn. (32) accounts for the moment-relieving tendency of the ring tension. Just as in the parallel case of the stave cylinder, $T_R$, an average value of $T_R$, is assumed to be known at each stage of an iteration procedure.

These three equations involve seven unknowns: $M_R, T_R, V_R, p, q, m$, and $u_R$. From Hooke's law applied to strain-displacement relations, two equations, but also one unknown, are added:

$$M_R = \frac{E R^T R}{R^R} \left( \frac{\partial^2 u_R}{\partial \phi^2} + u_R \right) \quad (33)$$

$$T_R = \frac{E A R}{R^R} \left( \frac{\partial v_R}{\partial \phi} + u_R \right) \quad (34)$$
Figure 6. Stress Resultants and Load Acting on a Ring Element
where \( y_R \) is the tangential displacement of the ring element. For consistency the same curvature relation is used for the ring as for the cylinder.

To provide the required three independent equations the ring loads \( p, q, \) and \( m \) must be expressed in terms of the ring deflections \( u_R, y_R \).

### 4.2 THE RING LOADING.

The radial load on the ring, \( p \), expanded as a Fourier series, is the sum of the end shears \( (V_1) \) from the adjacent cylinders:

\[
p = \sum_{n=0}^{\infty} p_n \cos n\varphi = 2 \frac{R}{R_R} V_1 \quad x = 0
\]

where the ratio \( R/R_R \) accounts for the shear force \( V_1 \) being distributed over a greater arc length on the ring than on the cylinder. This requires that the ring and cylinder be assumed to behave as one unit. (At the common surface there is no relative displacement.)

From Eqns. (6) and (16)

\[
V_1 = \frac{E_1}{12} \frac{t^3}{3} \frac{\partial^3 u}{\partial x^3}
\]

Hence

\[
p = \frac{R}{R_R} \frac{E_1}{6} \frac{t^3}{3} \frac{\partial^3 u}{\partial x^3} \quad x = 0
\]

The tangential load on the ring, \( q \), is due to the combined effects of the cylinder shears \( (V) \) and the column reaction at \( \varphi = \varphi_0 \).
Since from Eqns. (5) and (15) \( V \) is given by

\[
V = -E_1 \frac{tw^2}{12} \frac{\partial^3 y}{\partial x^3}
\]  

(38)

the shear component of \( q \) is

\[
\bar{q} = \sum_{n=1}^{\infty} \bar{q}_n \sin n\theta = 2 \frac{R.V}{R} \left[ x=0 \right] = -\frac{R}{R} \frac{E_1 tw^2}{6} \frac{\partial^3 y}{\partial x^3} \]  

(39)

Expansion of the column reaction as a Fourier series gives

\[
\bar{q} = \sum_{n=1}^{\infty} \bar{q}_n \sin n\theta = \sum_{n=1}^{\infty} \frac{R^2L_\theta}{R} \frac{\sin n\theta}{\sin \theta} \sin n\theta
\]

(40)

Then \( q = \bar{q} - \bar{q} \)

(41)

The moment \( m \) from eccentric loading of the ring by \( \bar{q} \) and \( \bar{q} \) is

\[
m = (R - R) \bar{q} + e\bar{q}
\]

(42)

where \( e \) is the distance from the centroid of the ring to the line of action of the support reaction.

Eqns. (37) and (39) include the stave deflections \( u \) and \( y \) rather than the ring deflection \( u_R \) and \( y_R \) required to determine the state of stress in the ring. A coupling of the ring and stave deflections, effected by the remaining integration constants, is expected of course, and it is most convenient to introduce the final
boundary conditions at this stage.

4.3 COUPLING OF THE STAVE CYLINDER AND RING SUPPORT.

The deflections of the stave cylinder and the ring support must be compatible. In addition, since the stave deflections must be symmetrical about the ring, curves of \( u \) and \( y \) plotted against \( x \) have zero slope at the ring. The boundary conditions for the case of a flexible ring support are then:

\[
\begin{align*}
\frac{\partial u}{\partial x} &= 0 \\
\frac{\partial y}{\partial x} &= 0
\end{align*}
\]

where \( u_R \) and \( y_R \) are expressed as

\[
\begin{align*}
u_R &= \sum_{n=0}^{\infty} u_{nR} \cos n\phi \\
y_R &= \sum_{n=1}^{\infty} y_{nR} \sin n\phi
\end{align*}
\]

The infinitely stiff ring or fixed end case can be considered as the special instance of a "flexible" ring with an infinitely large
area and moment of inertia.

Application of the boundary conditions given by Eqns. (43) to Eqns. (24) and (29) gives

\[ u = a + \rho + \left( K_{21} + \frac{a}{b_1 \eta} \left[ \frac{\eta}{b_1} - \rho \right] - \frac{x^2(L - x^2)}{24b_1} \right) \cos \phi \]

\[ + \sum_{n=0}^{\infty} K_{1n} \delta_{1n} (x) \cos \phi + \sum_{n=2}^{\infty} K_{2n} \delta_{2n} (x) \cos \phi \]

\[ y = \left[ \frac{\eta}{b_1} - \rho - (K_{21} + \frac{a}{b_1 \eta} \left[ \frac{\eta}{b_1} - \rho \right] - \frac{x^2[L - x^2]}{24b_1})(1 + vT) \right] \sin \phi \]

\[ - \sum_{n=1}^{\infty} K_{1n} \tau_{1n} \delta_{1n} (x) \sin \phi - \sum_{n=2}^{\infty} K_{2n} \tau_{2n} \delta_{2n} (x) \sin \phi \]

where

for \( n \geq 1 \)

\[ K_{2n} = \frac{n}{\Delta_n} \left( \frac{R}{R} \cdot Y_{nR} + \tau_{1n} u_{nR} \right) \]

\[ \Delta_n = 4 \eta (\beta_{2n}^4 - \beta_{1n}^4) \]

and

for \( n = 0 \)

\[ K_{10} = -(a + \rho - u_{0R}) \]

for \( n = 1 \)

\[ K_{11} = -K_{21} + u_{1R} - \frac{a}{b_1 \eta} \left( \frac{\eta}{b_1} - \rho \right) \]

for \( n > 1 \)

\[ K_{1n} = u_{nR} - K_{2n} \]
The terms in $x$ are:

$$
\delta_{1n}(x) = E(\beta_{1n} \ln) \left[ [1 - C (\beta_{1n} \ln)] [D(\beta_{1n} x) + D (\beta_{1n} \ln - x)] \right] \\
+ [1 - A(\beta_{1n} \ln)] [B(\beta_{1n} x) + B(\beta_{1n} \ln - x)] \right] \tag{51a}
$$

$$
\delta_{2n}(x) = E(\beta_{2n} \ln) \left[ [1 - C(\beta_{2n} \ln)] [D(\beta_{2n} x) + D(\beta_{2n} \ln - x)] \right] \\
+ [1 - A(\beta_{2n} \ln)] [B(\beta_{2n} x) + B(\beta_{2n} \ln - x)] \right] \tag{51b}
$$

where the functions of $x$ are defined by the following general relations:

$$
A(\varepsilon) = e^{-\varepsilon} \left( \cos \varepsilon + \sin \varepsilon \right) \tag{52a}
$$

$$
B(\varepsilon) = e^{-\varepsilon} \sin \varepsilon \tag{52b}
$$

$$
C(\varepsilon) = e^{-\varepsilon} \left( \cos \varepsilon - \sin \varepsilon \right) \tag{52c}
$$

$$
D(\varepsilon) = e^{-\varepsilon} \cos \varepsilon \tag{52d}
$$

$$
E(\varepsilon) = \frac{1}{2} \left( \frac{e^\varepsilon}{\sin \varepsilon + \sinh \varepsilon} \right) \tag{52e}
$$

The ring loads $p$, $q$, and $m$ are next expressed in terms of the above parameters for $u_{\text{p}}$ and $y_{\text{n}}$. Defining $J_{1n}$ and $J_{2n}$ by

$$
J_{1n} = \left. \frac{d^3 \delta_{1n}(x)}{dx^3} \right|_{x=0} = 4 \beta_{1n}^3 E(\beta_{1n} \ln) \left[ 1 - 2D(\beta_{1n} \ln) + e^{-2\beta_{1n} \ln} \right] \tag{53a}
$$
and

\[ J_{2n} = \frac{d^3 b_2}{dx^3} \bigg|_{x=0} = 4b_2 E(\beta_2 L) \left[ 1 - 2D (\beta_2 L) + e^{-2\beta_2 L} \right] \] (53b)

the radial load \( p \), from Eqns. (37) and (46) becomes

\[
p = \sum_{n=0}^{\infty} p_n \cos n\phi
\]

\[
= \frac{R}{6} \cdot \frac{E_t}{t^3} \left[ \frac{L}{2b_1} \cos \phi + \sum_{n=0}^{\infty} K_{1n} J_{1n} \cos n\phi \right.
\]

\[
+ \sum_{n=2}^{\infty} K_{2n} J_{2n} \cos n\phi \right]\] (54)

Using Eqns. (41) and (47) the tangential load \( q \) becomes

\[
q = \sum_{n=1}^{\infty} q_n \sin n\phi = \sum_{n=1}^{\infty} \tilde{q}_n \sin n\phi = \sum_{n=1}^{\infty} \bar{q}_n \sin n\phi
\]

\[
= \frac{R}{6} \cdot \frac{E_t w^2}{t} \left[ ([1 + vT] \frac{L}{2b_1} + \tau_{11} K_{11} J_{11}) \sin \phi \right.
\]

\[
+ \sum_{n=2}^{\infty} (\tau_{1n} K_{1n} J_{1n} + \tau_{2n} K_{2n} J_{2n}) \sin n\phi \left. \right] + \sum_{n=2}^{\infty} \sin n\phi + \sum_{n=1}^{\infty} \sin n\phi \] (55)

Eqns. (30) to (34), (42), (54) and (55) involve the eight unknowns \( T_R, V_R, M_R, p, q, m, u_R, \) and \( y_R \). Since the number of independent equations available is sufficient, the problem is completely determinate at this stage. It is advantageous to first find \( u_R \) and \( y_R \).
4.4 SOLUTION FOR THE RING DISPLACEMENTS.

Eqns. (30) to (34) are satisfied by taking

\[
M_R = \sum_{n=0}^{\infty} M_{nR} \cos n\phi
\]

(56)

\[
T_R = \sum_{n=0}^{\infty} T_{nR} \cos n\phi
\]

(67)

\[
V_R = \sum_{n=1}^{\infty} V_{nR} \sin n\phi
\]

(58)

If Eqn. (31) is absorbed in Eqns. (30) and (32), then a comparison of the coefficients of \(\sin n\phi\) and \(\cos n\phi\) in Eqns. (30), (32), (33) and (34) results in the system:

For \(n = 0\)

\[
T_{oR} + p_{oR} = 0
\]

(59a)

\[
M_{oR} = \frac{E_{IR}}{R_R} \frac{2}{R_R} u_{oR}
\]

(59b)

\[
T_{oR} = \frac{E_{A}}{R_R} \frac{2}{R_R} u_{oR}
\]

(59c)

For \(n \geq 1\)

\[
(1 - n^2)T_{nR} + R_R(p_n + nq_n) = 0
\]

(60a)

\[
-\frac{n}{R_R} M_{nR} + nT_{nR} = R_R q_n - \frac{T_R}{R_R} n u_{nR} - (R_R - R) \bar{q}_n - e \bar{q}_n
\]

(60b)

\[
M_{nR} = \frac{E_{IR}}{2} \frac{2}{R_R} (1 - n^2) u_{nR}
\]

(60c)
\[ T_{nR} = \frac{E A}{R} \cdot (u_{nR} + n y_{nR}) \]  

Eqns. (42), (54), and (55) are substituted in the above, and after some manipulation the Fourier coefficients \( u_{nR} \) and \( y_{nR} \) are found.

For \( n = 0 \)

\[ u_{0R} = \frac{a + \rho}{1 + \frac{E A R}{E_t^R}} \]

\[ y_{0R} \] cannot exist.

For \( n = 1 \)

\[ \left[ A R + \frac{(1 + \nu T) J_{11}}{2\nu R^4} \right] u_{1R} + \left[ A R + \frac{J_{11}}{2\nu R^4} \right] y_{1R} \]

\[ = - \frac{1}{b_1^2} \left[ L \eta + \frac{\rho(1 + \nu T) J_{11}}{2\nu R^4} \right] - R L V [R - R + e] \]

Another independent equation is required to separate out \( u_{1R} \) and \( y_{1R} \). If the supporting column is assumed rigid axially, then at \( \varphi = \varphi_0 \) the tangential deflection \( y_R \) must be zero.

Hence

\[ y_{1R} = \frac{-\sum_{n=2}^{\infty} y_{nR} \sin n\varphi_0}{\sin \varphi_0} \]

This infers that all terms for \( n > 1 \) must be summed until convergence is satisfactory before \( y_{1R} \) is calculated. Then \( u_{1R} \) is found from
Eqn. (62).

For \( n > 1 \)

The coefficients of \( u_{nR} \) and \( y_{nR} \) are very complicated so the writing of a general equation for each cannot be justified. Instead, it is more convenient to solve the equations

\[
\omega_{11} u_{nR} + \omega_{12} y_{nR} = nR^2 \frac{L \gamma}{\sin \varphi_o} \sin \varphi_o
\]

(64a)

\[
\omega_{21} u_{nR} + \omega_{22} y_{nR} = -R^2 \frac{L \gamma}{R_R + e} \sin \varphi_o
\]

(64b)

where

\[
\omega_{11} = (1-n^2) \frac{A_R \frac{E_R}{R}}{R_R} + \frac{RE \frac{t^3}{6}}{R_R} \left[ J_{1n} + \frac{n \tau_{1n}}{\Delta_n} (J_{2n} - J_{1n}) \right]
\]

(65a)

\[
\omega_{12} = n(1-n^2) \frac{A_R \frac{E_R}{R}}{R_R} + \frac{RE \frac{t^3}{6}}{R_R} \cdot \frac{nR}{nR} \cdot \left[ J_{2n} - J_{1n} \right]
\]

(65b)

\[
\omega_{21} = n(n^2-1) \frac{E_R \frac{t}{R}}{R^2} + nA_R \frac{E_R}{R} + nT_R
\]

(65c)

\[
\omega_{22} = n \frac{2A_R \frac{E_R}{R}}{R^2} - \frac{R^2 \frac{E_R \frac{t^2}{R}}{6}}{nR} \cdot \frac{nR}{nR} \left[ J_{2n} - J_{1n} \right]
\]

(65d)
Solution of Eqns. (64) requires a considerable number of computations as the system must be solved for every value of n from 2 until convergence is adequate.

With the $u_{nR}$ and $y_{nR}$ completely determined for all values of $n$, the analysis problem is reduced to the bookkeeping chore of back-substituting $u_{nR}$ and $y_{nR}$ into the appropriate formulas for displacement and stress.

**STRESS RESULTANTS AND DISPLACEMENTS**

5.1 RING AND STAVE DISPLACEMENTS.

For any angle $\phi$ the ring displacements $u_R$ and $y_R$ are found from the expansion of Eqns. (44) and (45).

The stave deflections, $u$ and $y$, are evaluated for any $x$ and $\phi$ from Eqns. (46) and (47).

If the displacements are desired in the horizontal (positive outward) and vertical (positive downward) directions Fig. (6) shows that:

\[
\text{Horizontal Displacement} = u \sin \phi + y \cos \phi \quad (66a)
\]

\[
\text{Vertical Displacement} = - u \cos \phi + y \sin \phi \quad (66b)
\]

5.2. STRESS RESULTANTS FOR THE RING.

From Eqns. (56), (59b) and 60c)

\[
M_R = \frac{EI_R}{R^2} \sum_{n=0}^{\infty} (1-n^2) u_{nR} \cos n\phi \quad (67)
\]
from Eqns. (57), (59c) and (60d)

\[
T_R = \frac{A R^2 E}{R} \cdot \sum_{n=0}^{\infty} (u_{nR} + ny_{nR}) \cos \varphi,
\]

and from Eqns. (32), (42), (44), (58) and (67)

\[
V_R = \frac{1}{R} \sum_{n=1}^{\infty} \left[ \frac{E R^2}{R^2} (n^2 - 1) + T'_R \right] n u_{nR} \sin \varphi
\]
\[+ \left( R_{R} - R \right) \sum_{n=1}^{\infty} \frac{\bar{q}_n}{\bar{q}_n} \sin \varphi + \frac{e R^2 L V}{R} \sum_{n=1}^{\infty} \frac{\sin \varphi}{\sin \varphi} \sin \varphi \]

where \( \bar{q}_n \) is defined by Eqn. (55).

5.3 STRESS RESULTANTS FOR THE STAVES.

The stave moments, shears, and tangential tension are as follows:

From Eqn. (15)

\[
M = - \frac{E t w^2}{12} \left[ \left( L^2 - 6Lx + 6x^2 \right) \frac{(1 + \nu T)}{12b_1} \sin \varphi
\]
\[- \sum_{n=1}^{\infty} \tau_{1n} K_{1n} a_{1n}(x) \sin \varphi
\]
\[- \sum_{n=2}^{\infty} \tau_{2n} K_{2n} a_{2n}(x) \sin \varphi \]

From Eqn. (16)

\[
M_1 = - \frac{E t^3 y}{12} \left[ \left( L^2 - 6Lx + 6x^2 \right) \frac{1}{12b_1} \cos \varphi \right]
\]
\[
\sum_{n=0}^{\infty} K_{1n} a_{1n}(x) \cos n\phi
\]

\[
\sum_{n=2}^{\infty} K_{2n} a_{2n}(x) \cos n\phi
\]

From Eqn. (14)

\[
M_\phi = \frac{EI}{R^2} \left[ \alpha + \rho - \sum_{n=0}^{\infty} (n^2-1)K_{1n} \delta_{1n}(x) \cos n\phi \right] - \sum_{n=2}^{\infty} (n^2-1)K_{2n}\delta_{2n}(x) \cos n\phi
\]

From Eqn. (5)

\[
V = -\frac{E_I t^2}{12} \left[ (L-2x)(1+v) \frac{\sin \phi}{2b_1} \right] - \sum_{n=1}^{\infty} \tau_{1n} K_{1n} \phi_{1n}(x) \sin n\phi
\]

\[
- \sum_{n=2}^{\infty} \tau_{2n} K_{2n} \phi_{2n}(x) \sin n\phi
\]

From Eqn. (6)

\[
V_1 = \frac{E_I t^3}{12} \left[ (L-2x) \frac{1}{2b_1} \cos \phi \right] + \sum_{n=0}^{\infty} K_{1n} \phi_{1n}(x) \cos n\phi
\]

\[
+ \sum_{n=2}^{\infty} K_{2n} \phi_{2n}(x) \cos n\phi
\]
From Eqn. (4)

\[
\phi = \frac{E I t}{R^3} \left[ \sum_{n=2}^{\infty} \frac{n(n^2-1)[K_{1n} \delta_{1n}(x) + K_{2n} \delta_{2n}(x)] \sin n\varphi}{\ln^2} \right] \\
+ \frac{T}{R} \left[ K_{21} + \frac{a}{b_1} \frac{\ln^2}{b_1} - \rho \right] - x^2 \frac{[L-x]^2}{24b_1} \sin \varphi \\
+ \sum_{n=1}^{\infty} nK_{1n} \delta_{1n}(x) \sin n\varphi + \sum_{n=2}^{\infty} nK_{2n} \delta_{2n}(x) \sin n\varphi \right] \tag{75}
\]

From Eqn. (13)

\[
\phi = \frac{A_t E}{R} \left[ \left( \frac{\eta}{b_1} - \rho \right) - [K_{21} + \frac{a}{b_1} \frac{\ln^2}{b_1} - \rho \right] - \frac{x^2}{24b_1} \sin^2 T \right] \cos \varphi \\
+ \alpha + \rho + \sum_{n=0}^{\infty} (1-n\tau_{1n}) K_{1n} \delta_{1n}(x) \cos n\varphi \\
+ \sum_{n=2}^{\infty} (1-n\tau_{2n}) K_{2n} \delta_{2n}(x) \cos n\varphi \right] \tag{76}
\]

where

\[
a_{1n}(x) = \frac{d^2 \delta_{1n}(x)}{dx^2} \\
= 2\beta_{1n}^2 E(\beta_{1n} L) \left( \left[ 1 - A(\beta_{1n} L) \right] \left[ -D(\beta_{1n} L) \right] -D(\beta_{1n} [L-x]) \right] \\
+ \left[ 1 - C(\beta_{1n} L) \right] B(\beta_{1n} x) + B(\beta_{1n} [L-x]) \right) \right) \tag{77}
\]

and
\[ \psi_{1n}(x) = \frac{d^3 \delta_{1n}(x)}{dx^3} \]

\[ = 2 \beta_{1n}^3 \mathbb{E}(\beta_{1n}L) \left( [1-A(\beta_{1n}L)][A(\beta_{1n}x) - A(\beta_{1n}[L-x])] \right) \]

\[ + \left[ 1 - C(\beta_{1n}L)[C(\beta_{1n}x) - C(\beta_{1n}[L-x])] \right) \]  

(78)

The expressions for \( a_{2n}(x) \) and \( \psi_{2n}(x) \) are similar, with \( \beta_{2n} \) replacing \( \beta_{1n} \) in the above equations. \( \delta_{1n}(x) \) and \( \delta_{2n}(x) \) are defined by Eqns. (51).

At the ring Eqns. (70) to (76) are reduced in complexity.

\[ M \left|_{x=0} = \frac{-E_{1} t w^2}{12} \left[ \frac{L^2 (1+\nu T)}{12b_1} \right] \sin \varphi \right. \]

\[ - \sum_{n=1}^{\infty} K_{1n} \tau_{1n} G_{1n} \sin n \varphi = \sum_{n=2}^{\infty} K_{2n} \tau_{2n} G_{2n} \sin n \varphi \]  

(79)

\[ M \left|_{x=0} = \frac{-E_{1} t^3}{12} \left[ \frac{L^2}{12b_1} \right] \cos \varphi - \sum_{n=0}^{\infty} K_{1n} \tau_{1n} G_{1n} \cos n \varphi = \sum_{n=2}^{\infty} K_{2n} \tau_{2n} G_{2n} \cos n \varphi \]  

(80)

\[ M \left|_{x=0} = \frac{E \mathcal{I} t}{2} \left[ a + \rho + \sum_{n=0}^{\infty} (1-n^2)K_{1n} \ln \ln \sin \varphi \right. \]

\[ + \sum_{n=2}^{\infty} (1-n^2)K_{2n} \tau_{2n} G_{2n} \cos n \varphi \]  

(81)

\[ V \left|_{x=0} = \frac{E_{1} t w^2}{12} \left[ \frac{L(1+\nu T)}{2b_1} \right] \sin \varphi + \sum_{n=1}^{\infty} K_{1n} \tau_{1n} J_{1n} \sin \varphi \right. \]

\[ - \sum_{n=2}^{\infty} K_{2n} \tau_{2n} J_{2n} \sin \varphi \]  

(82)
\[
V_I \bigg|_{x=0} = \frac{E_I t^3}{12} \left[ \frac{L}{2b_1} \cos\phi + \sum_{n=0}^{\infty} K_1 J_1 \cos \phi \cdot \sum_{n=2}^{\infty} K_2 J_2 \cos \phi \right] \tag{83}
\]

\[
V_\varphi \bigg|_{x=0} = \frac{E I t}{R^3} \left[ \sum_{n=2}^{\infty} (n^2 - 1)(K_1 P_{1n} + K_2 P_{2n}) \sin \varphi \right]
+ \frac{T}{R} \left[ (K_2 + \frac{a_1}{b_1} \eta - \rho) \sin \phi \right] + \sum_{n=1}^{\infty} n K_1 P_{1n} \sin \phi
+ \sum_{n=2}^{\infty} n K_2 P_{2n} \sin \phi \tag{84}
\]

\[
N_\varphi \bigg|_{x=0} = \frac{A E s}{R} \left[ (-K_2 \nu T + \frac{\eta}{b_1} (1 - \frac{a_1}{b_1} \nu T)) \cos \phi + a + \rho \right]
+ \sum_{n=0}^{\infty} (1 - n \tau_{1n}) K_1 P_{1n} \cos \phi \cdot \sum_{n=2}^{\infty} (1 - n \tau_{2n}) K_2 P_{2n} \cos \phi \tag{85}
\]

where

\[
P_{1n} = 5_{1n}(x) \bigg|_{x=0} = E(\beta_{1n} L) [1 + 2B(\beta_{1n} L) - e^{-2\beta_{1n} L}] \tag{86}
\]

and

\[
G_{1n} = 6_{1n}(x) \bigg|_{x=0} = 2\beta_{1n}^2 E(\beta_{1n} L) \left[-1 + 2B(\beta_{1n} L) + e^{-2\beta_{1n} L} \right] \tag{87}
\]

The expression for \( P_{2n} \) and \( G_{2n} \) are similar.

5.4 SIMPLIFICATIONS TO FORMULAS.

In the numerical evaluation of the stress resultants certain terms in the expressions can be simplified by using very good approximations. For most practical cases \( \beta_{1n} L \) (but not usually \( \beta_{2n} L \) however) is sufficiently large that \( e^{-\beta_{1n} L} \) is a negligibly small number compared
to unity. For example: if $\ln L = 10$, then $e^{\ln L} = 22026$ and $e^{-\ln L} = 0.0000454$. The following simplifications are then quite acceptable:

$$\sinh \ln L = \frac{e^{\ln L} - e^{-\ln L}}{2} \approx \frac{e^{\ln L}}{2}$$

Since $\sin \ln L$ is very small compared to $\sinh \ln L$ Eqn. (52e) becomes

$$B(\ln L) = \frac{1}{2} \cdot \frac{e^{\ln L}}{\sin \ln L + \frac{e^{\ln L}}{2}} \approx 1$$

By Eqs. (52a) to (52d) $A(\ln L)$, $B(\ln L)$, $C(\ln L)$, and $D(\ln L)$ are negligible compared to unity.

Then

from Eqn. (86) $P_{ln} \approx 1$ (88a)

from Eqn. (87) $G_{ln} \approx -2\beta_{ln}^2$ (88b)

from Eqn. (53) $J_{ln} \approx 4\beta_{ln}^3$ (88c)

from Eqn. (51) $\delta_{ln}(x) \approx [e^{-\ln x} + \sin \ln L e^{-\ln (L-x)}][\sin \ln x + \cos \ln x] + \cos \ln L e^{-\ln (L-x)}][\cos \ln x - \sin \ln x]$ (89a)

from Eqn. (77) $a_{ln}(x) \approx 2\beta_{ln}^2 [e^{-\ln x} - \sin \ln L e^{-\ln (L-x)}][\sin \ln x - \cos \ln x] - 2\beta_{ln}^2 \cos \ln L e^{-\ln (L-x)}][\sin \ln x + \cos \ln x]$ (89b)
from Eqn. (78) \( f_{1n}(x) \approx 4\beta_{1n}^3 \left[ e^{-\beta_{1n}^x} - \cos\beta_{1n} \ln \beta_{1n}^{(L-x)} \right] \cos\beta_{1n}^x \)

\[ - 4\beta_{1n}^3 \sin\beta_{1n} \ln \beta_{1n}^{(L-x)} \sin\beta_{1n}^x \]  

(89c)

If \( \beta_{2n} L \) is found to be satisfactorily large then similar approximate expressions can be written to reduce the number of computations required for terms in \( \beta_{2n}^L \).

**NUMERICAL EXAMPLE**

6.1 **PENSTOCK DESIGN VARIABLES.**

There are so many variables involved in this analysis that only a few of the possible combinations of these variables can be presented. This discussion is thus limited to an investigation of the effects of modifying the following:

1. The flexural stiffness \( (I_R) \) of the supporting ring.
2. The circumferential stiffness \( (E_z) \) of the stave material.
3. The pressure head \( (h) \) on the pipe
4. The "non-linear" terms \( (T, T_R') \) at high head.
5. The eccentricity \( (e) \) of the column support.
6. The number of terms \( (N) \) in the series summations.

At low head the average magnitudes of \( N, \phi \) and \( T_R \) are so low that the non-linear effect is relatively insignificant.

As an illustration, a penstock with an inside diameter of 8' 0" is considered. The following alternatives are selected:
1. A rolled steel 5110 section ("stiff ring") vs A fictitious section ("flexible ring") with the same depth and area as the 5110 but with only 10% of the moment of inertia of 5110.

2. $E_2 = 0$ ksi vs $E_2 = 20$ ksi

3. $h = 0.5$ ft. (low head) vs $h = 150$ ft. (high head).

4. $T, T_R = 0$ ("non-linear" effect neglected) vs $T, T_R = \text{average values of } N$ and $T_R$ respectively, from previous iteration.

5. $e = 2.65$ in. vs $e = 0$ in.

6. $N = 300$ vs $N = 9$

The alternatives for $E_2$ and $N$ require some explanation. Since the stave material becomes spongy with the passage of time, the value of $E_2$ is certain to be low. In the extreme case, neglecting entirely the circumferential stiffness of the staves, $E_2 = 0$ ksi. The second alternative, one quarter of the usual 1/20 of $E_1$, is probably more realistic.

The stave cylinder is not an orthodox shell structure. It is
composed of discontinuous elements of a definite width w, but this width is translated as Rdφ in order that the conventional techniques of calculus may be used. The Fourier series representation can not at once express the state of stress within individual staves as well as bridge the discontinuity between the staves, yet this latter assignment is essential to the analysis of the complete cylinder. The series should, then, technically not be summed to an impossible infinite number of terms, or even until convergence is deemed satisfactory, but only to an N where, for instance, at least three staves are enveloped by each half-wave of the trigonometric terms.

The penstock under consideration has 56 staves around the circumference. If three staves are permitted in each half-wave then only nine terms should be used in the series. However, in order to detect the stress concentrations a large number of terms must be used. The shear V, in particular, converges very slowly, yet the tangential loading q on the ring depends on this shear. For this reason, N is chosen as 300 for the majority of the tests.

Fifteen combinations of the above alternatives are considered. These are listed in Fig. (7).

The quantities required for the numerical work are:

For the Ring

\[ R_R = 54.125 \text{ in} \]
\[ A_R = 2.870 \text{ in}^2 \]
\[ I_R = 12.10 \text{ or } 1.21 \text{ in}^4 \]
\[ E_R = 29600 \text{ ksi} \]
\[ e = 2.65 \text{ in. or } 0 \text{ in.} \]
\[ \phi_0 = 95 \text{ deg.} \]
For the Staves
\[ R = 49.812 \text{ in.} \]
\[ t = 3.625 \text{ in.} \]
\[ w = 5.625 \text{ in.} \]
\[ E_1 = 1600. \text{ ksi} \]
\[ E_2 = 0. \text{ or } 20. \text{ ksi} \]

For the Bands
\[ \varphi_b = 0.725 \text{ in.} \]
\[ S = 3.0 \text{ in.} \]
\[ E_s = 29600. \text{ ksi} \]

Span
\[ L = 10. \text{ ft.} \]

Water
\[ \gamma = 62.4 \text{ lb/cu.ft.} \]
\[ h = 5 \text{ or } 150 \text{ ft.} \]

"Non-Linear" Factors
\[ T'_R = 0 \text{ or } T = 2.8 \text{ k/in,} \]
\[ T_R = 32.0 \text{ k} \]

Only one iteration is used in this example since the application of \( T \) and \( T'_R \) does not significantly change the average values of \( N_\varphi \) and \( T_R \). \( T'_R \) is taken as the average value of \( T_R \) in the "linear" case, while \( T_R \) is estimated from an analysis of a cylinder acted upon by a uniform internal pressure (i.e. from \( 2N_\varphi = 2R\gamma h_{\text{avg}} \) where \( h_{\text{avg}} \) is the pressure head to the centreline of the penstock), modified to account
### Table: NUMERICAL EXAMPLE CASES

<table>
<thead>
<tr>
<th>Case</th>
<th>Ring</th>
<th>$E_2$ ksi</th>
<th>h ft</th>
<th>T k/in</th>
<th>$T^\prime$ K</th>
<th>e in</th>
<th>N</th>
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Fig. 7 NUMERICAL EXAMPLE CASES
for the pinching effect of the ring (which decays quite rapidly with x). An average value of \( N_\varphi \) at the ring would be much too low because at this point the ring is assisting the bands.

Figs. (8) to (13) show the ring displacements and stress resultants \( u_R \), \( y_R \), \( M_R \), \( T_R \), and \( V_R \) while Figs. (14) and (15) indicate the stave bending moments \( M \) and \( M_1 \) evaluated at \( x = 0 \) (i.e., at the ring). Cases 13, 14, and 15 are plotted in Fig. (13) only.

6.3 DISCUSSION OF THE RESULTS.

Since the penstock is loaded symmetrically about a vertical axis through its centre line, it is only necessary to consider what occurs between \( \varphi = 0^\circ \) and \( \varphi = 180^\circ \). As expected, the curves for \( u_R \), \( M_R \), \( T_R \), and \( M \) have zero slope at \( \varphi = 0^\circ \) and \( \varphi = 180^\circ \), while the curves for \( y \), \( V_R \), and \( M \) are zero at these points.

In general, the graphs confirm anticipated variations in the results due to modification of the design parameters. The majority of the following remarks, however, are concerned exclusively with the stiff ring cases because the flexible ring results are not as readily understood and believed. The Fourier coefficients \( u_{nR} \) and \( y_{nR} \) are very sensitive to changes in the design parameters. Indeed, for practical examples, very careful attention must be devoted to their calculation from Eqns. (64). The determinant of the coefficients \( \omega_{11}' \), \( \omega_{12}' \), \( \omega_{21} \) and \( \omega_{22} \) is numerically the very small difference of two large numbers. Therefore a small adjustment in any of these coefficients will dramatically alter \( u_{nR} \) and \( y_{nR} \). Yet all stress resultants...
Figure 8. $u_R$ Ring Radial Displacements
Figure 10. $M_R$ Flexible Ring Moments
Figure II. $M_R$ Stiff Ring Moments
Figure 13. $V_R$ Ring Shears
are evaluated in terms of \( u_{nR} \) and \( y_{nR} \) so when the \( u_{nR} \) curve fluctuates considerably then all other curves relating to the same case can be predicted to do likewise. This is lucidly illustrated by the flexible ring curves.

An increase in the flexural stiffness of the supporting ring corresponds to a reduction in the combined flexure stress in the staves. Similarly, the provision of circumferential stiffness to the staves, while not significantly altering \( M \) and \( M_1 \), notably reduces \( M_R \), \( T_R \), and \( V_R \) in the ring. (The stresses produced by \( M_\varphi \) are meanwhile low).

The liquid pressure distribution consists of a uniform part, directly proportional to \( h + R \), and a triangular part, independent of \( h \). The uniform part appears in the terms for \( n = 0 \), while the triangular part appears in the terms for \( n = 1 \). In the linear situation, therefore, the curves for \( u_R \), \( M_R \), \( T_R \), \( M_1 \), \( M_\varphi \), \( V_1 \), and \( N_\varphi \) are varied by a quantity independent of \( \varphi \) as \( h \) is varied. This is not true for \( y_R \), \( V_R \), \( M \), \( V \), and \( V_\varphi \) which do not contain terms in \( n = 0 \).

The non-linear effect, in straightening the curves, notably reduces the magnitude of the displacements and stress resultants. Naturally the effect is most pronounced when the ring is very flexible. Case 13 on Fig. (11) shows that \( T \) has more influence on \( M_R \) than does \( T' \). It is logical that the most flexible system, the stave cylinder, will be most affected by consideration of the non-linear factor, and this tendency to keep the pipe circular is directly reflected in the ring moments.

The discontinuity in the \( T_R \) curves is clearly attributed
to the support reaction, while the discontinuity in the $M_R$ curves is created by both the eccentricity $e$ of the support reaction and the eccentricity $R_R - R$ of the stave shear $V$. In Case 14 on Fig. (11) the ring is assumed to be supported such that $e = 0$. The still considerable moment jump is generated by a sharp concentration of the shear $V$ at that point. Using $N = 9$, as in Case 15 on the same graph, the discontinuity in $M_R$ is not developed at all, although in the region some distance from the ring the agreement with Case 12 is very good.

Since the shear $V$ indirectly depends on the ring stiffness $I_R$, the ring moment $M_R$ must also be a function of $I_R$. (In the unrealistic situation where $R = R_R$ there is no moment contribution from $V$, so $M_R$ is not a function of $I_R$.)

No significant change in $M_R$ is noted when Case 12 is subject to either of the modifications: $s = 6$ in, or $t = 2^{5/8}$ in., all other variables remaining unaltered. These two cases are not plotted here but are mentioned for completeness.

**CONCLUSIONS**

Stress concentrations in a wood stave pipeline are substantially reduced if the support reaction is distributed to all staves by a full ring that minimizes departure from a circular profile.

Unequivocal rules for design cannot be promulgated until many more combinations of the design variables have been tested to establish patterns of behaviour. The numerical example included here
serves primarily to illustrate the character of information to be determined. For each practical problem the analytical results must be tempered by engineering judgment and experience. However, attention to the following conclusions will assist in adjustment of the design parameters:

1. The maximum combined flexural stress in the staves is reduced by a coincidental increase in the flexural stiffness of the supporting ring.

2. The maximum stresses in the ring are reduced if circumferential bending stiffness can be attributed to the stave cylinder.

3. With the assumption of linear behaviour the ring moment, tension, and radial deflection vary with the liquid pressure head as a quantity independent of the angle \( \phi \).

4. The inclusion at high head of the non-linear effect of ring and band tension on the deformed system produces a notable reduction in the ring moment and tension. The resultant saving in material may be large.

5. The eccentricity of the ring support column can be adjusted in order that the moment created by the eccentric reaction has a favourable influence on the maximum ring stress.

It is of paramount importance that a sufficient number of significant figures be carried in the calculations. Slide rule accuracy is not adequate. A desk calculator, or preferably an electronic computer, must be used.
The stress concentrations are not detected unless enough terms are used to encourage convergence of the series expressions. If these stresses are sufficiently high that they cannot be developed by the materials anyway, then the concentrations will not exist to a degree as well defined as the example graphs would infer. A redistribution of stress from the theoretical results must occur. Since this is the case, for design purposes it is adequate to sum only the first few terms. A difficulty arises regarding ring flexure stresses, however. The discontinuity in the ring moment distribution from the eccentric column reaction is always present, and therefore must be considered. A suggested technique to overcome this design problem is as follows:

1. Sum the series terms, including the eccentric column effect, only until no less than three staves are enveloped within the half-wave of the final trigonometric term. Plot the results as in Fig. (11).

2. Find the magnitude of the discontinuity as the product of the reaction and its eccentricity.

3. Plot this discontinuity, at the support column position, on the graph half above and half below the first stage ordinate, and sketch in the design moment curve.

There is a mathematical interpretation for the above dilemma. The equations derived here are exact for a certain set of assumptions and boundary conditions -- but this mathematical model contains some flaws. However, the physical properties of timber cannot be universally defined to begin with, so the most rigorous stress solution is at best
an adequate approximation.

A more comprehensive solution involving more variables, (e.g. time, temperature), is certainly available, yet the complex equations proposed here are indeed not conveniently evaluated. A computer program incorporating all of the required equations is invaluable to the designer. The work is performed once, for all time.
BIBLIOGRAPHY


APPENDIX I - COMPUTER PROGRAM

The formidable assignment of determining the distribution with \( x \) and \( \phi \) of the deflections and stress resultants for any particular wood stave pipe design is most conveniently completed by delegating the numerical calculations to the electronic computer. For this investigation a program evaluating the deflections and stresses at the ring (i.e. \( x = 0 \)) is written in Fortran IV language for the IBM 7040 computer. A copy of this program is contained in Appendix IV, while a list of the symbols used in this program is included in Appendix III.

The programming scheme is relatively routine although the program appears to be lengthy and complicated. Responsible for the program length are the lengthy deflection and stress resultant equations themselves, the elaborate output "formats", the explanatory comments, and the inclusion of the calculations for combined stresses in addition to those for moments, shears and axial forces. Many substitution factors are introduced to avoid repeated multiplications of the same terms.

The formula simplifications are not used in the program although for the numerical examples selected in the text the minimum value of \( \beta_{1nL} \) is greater than 10, and the use of the simpler equations would be justified.

To provide greater accuracy in the evaluation of \( u_{nR} \) and \( y_{nR} \) from Eqns. (64) for \( n \) greater than one, a "double precision" feature (16 significant figures) is employed. For most practical
designs this precaution is unwarranted — eight significant figures is adequate.

As an aid to understanding the programming scheme a block diagram (or flow sheet) is furnished in Appendix II. The equations employed during each stage are indicated by the numbers in parentheses.

In order to comply with Eqn. (63) (calculation of \( y_{1R} \)) the terms of \( n \) greater than one are summed first. During each cycle of a programming "loop", deflections and stress contributions for each desired angle \( \varphi \) are added to their respective series expansions. The loop terminates when the equations for the upper limit \( N \) are exhausted. The expressions for deflections, moments, shears, and tensions are completed by superimposing the terms evaluated for \( n = 0 \) and \( n = 1 \), and are then routinely converted to stresses by using the conventional relations in the following manner:

a) For the Ring:

Normal Stresses \( f_R \), outside fibres \( = \frac{1}{A_R} \left( T_R - \frac{M_{R}^{2}R}{2K_R} \right) \)

Normal Stresses \( f_R \), inside fibres \( = \frac{1}{A_R} \left( T_R + \frac{M_{R}^{2}R}{2K_R} \right) \)

Shear Stress \( v_{web} = \frac{V_R}{d_R t_w} \)

The program is set up to handle an "I" shaped supporting ring but by careful manipulation of the input data any other cross-section could be used. In the above equations the ring
properties are defined by

\[ k = \text{radius of gyration in the plane of bending} \]
\[ d = \text{depth} \]
\[ t = \text{web thickness}. \]

b) For the Staves:

On the "end" of the stave element:

**Flexure Stress**

\[ f_{\text{maximum}} = \frac{6}{t} \left( \frac{|M_1|}{t} + \frac{|M|}{d} \right) \]

**Shear Stress**

\[ v_{\text{stave}} = \frac{3}{2t} \sqrt{v^2 + v_1^2} \]

On the "edge" of the stave element:

**Normal Stress**

\[ f_{s, \text{outside fibres}} = \frac{E_2}{E_s} \left( \frac{N}{A_t} - \frac{M}{I_t} [t - X_{NA}] \right) \]
\[ f_{s, \text{inside fibres}} = \frac{E_2}{E_s} \left( \frac{N}{A_t} + \frac{M}{I_t} \cdot X_{NA} \right) \]

At the ring, perpendicular to the grain:

**Compressive Stress**

\[ C = \frac{2V_1}{w_f} \cdot \left( \frac{R + t}{2} \right) \]

where \( w_f = \text{ring flange width}. \)
c) For the Bands:

\[ \text{Normal Stress } f_{\text{band}} = \frac{N_{\theta}}{A} + \frac{M_{\theta}}{I_t} \left( t + \frac{d_b}{2} - x_{NA} \right) \]

\[ \text{Shear Stress (average) } v_{\text{band}} = \frac{V_{\theta} \cdot S}{A} \]

assuming \( V_{\theta} \) is taken entirely by the bands.

The program receives punched card data in the following form:

Card #1 \( N, M \)

#2 \( R_R, A_R, k_R, d_R, w_f, t_w, e \)

#3 \( \theta_0, E_R, L \)

#4 \( \theta_b, S, E_s \)

#5 \( R, t, w, E_1, E_2 \)

#6 \( \gamma, h \)

#7 \( T, T_R \)

#8, #9 Angles \( \varphi \) (M required)

where \( M \) is the number of angles \( \varphi \) for which the deflections and stresses are desired.

\( L \) and \( h \) are in units of feet, \( \gamma \) in pounds per cubic foot, \( \varphi \) and \( \varphi_0 \) in degrees. All other data is in kip, inch units.

The provision of extensive "formats" obviates the necessity of explaining in detail the output form of the results. The completely labelled output appears in the following order:
Page 1  Input information
Page 2  Ring deflections
Page 3  Ring stress resultants
Page 4  Stave cylinder stress resultants
Page 5  Stave and band stresses.
APPENDIX II - BLOCK DIAGRAM

Read and Print Data →

Transformed Section Properties (12) →

Parameters $\beta, \eta, \mu, \nu, \rho, \sigma$ (19) →

Substitution Factors Independent of $n$

$n = 0$ →

$b_0 (23)$
$\beta_{10} (26)$
$P_{10} (86)$
$G_{10} (87)$
$J_{10} (53)$
$u_{oR} (61)$
$K_{10} (50)$

$n = 1$

$b_1, (23), \beta_{11} (26)$
$P_{11} (86), G_{11} (87)$
$J_{11} (53), T_{11} (27)$
$[\gamma_{1R} (63)]$
$u_{1R} (62)$
$K_{21} (48), K_{11} (50)$
$\overline{q}_1 (40), \overline{q}_1 (39)$

$n > 1$

$b_n, \hat{c}_n (23), \beta_{1n}, \beta_{2n} (26)$
$\tau_{1n}, \tau_{2n} (27), \Delta_n (49)$
$P_{1n}, P_{2n} (86), G_{1n}, G_{2n} (87)$
$J_{1n}, J_{2n} (53), \omega_{11}, \omega_{12}, \omega_{21}, \omega_{22} (65)$

$u_{nR} (64)$
$K_{2n} (48), K_{1n} (50)$
$\overline{q}_n (40), \overline{q}_n (39)$

Substitution Factors Dependent on $n$ but Independent of $\phi$

Sum

Yes $n = N' →$ no

$M_R (67), T_R (68), V_R (69), u_R (44), y_R (45)$

at ring: $M (79), M_1 (80), M_\phi (81), V (82), V_1 (83), V_\phi (84), N_\phi (85)$

Ring Horizontal, Vertical Deflections (66) →

Print $u_R', y_R'$ Horiz., Vert.

Ring Normal, Shear Stresses →

Print $M_R$, $T_R$, $V_R$, stresses

Stave, Band Normal and Shear Stresses →

Print $M, M_1, M_\phi, V, V_1, V_\phi, N_\phi$

Print Stave, Band Stresses →

Return to Start for Another Structure
APPENDIX III - PROGRAM SYMBOL TABLE

The following list relates the programming symbol to the text notation:

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The subscript I defines the i-th term in the series expansion. 
The maximum value of I is N.

The subscript J identifies the angle \( \phi \) used in evaluating 
the stress resultants. There are M possible values for \( \phi \).

In addition to those in the above table, the following symbols 
are introduced to facilitate the calculation of normal and shear stresses 
(see Appendix I) in the ring, staves, and bands:

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</tr>
<tr>
<td>( v_{web} )</td>
<td>VRING</td>
</tr>
<tr>
<td>( w_{f} )</td>
<td>FLGE</td>
</tr>
</tbody>
</table>

The horizontal and vertical deflections of the ring, at an 
angle \( \phi \), are symbolized as DEPH\((J)\) and DEPV\((J)\) respectively.

The letter Q followed by an integer between 0 and 100 (e.g. 
Q63) is utilized to reduce the number of repeated multiplications.
WOOD STAVE PIPE STRESS ANALYSIS

0D0UBLE PRECISION BN(300), CN(300), FN(300), B1N4(300), B2N4(300), 1B1N(300), B2N(300), TAU1N(300), TAU2N(300), DELTA(300), 2CNPHI(30), SNPHI(30), ANGLE(30), DEFH(30), DEFV(30), UR(30), YR(30), 3P1N(300), P2N(300), G1N(300), G2N(300), AJ1N(300), 4AJ2N(300), UNR(300), YNR(300), AKN(300), AK4N(300), TNBB(300), 5BMR(30), TR(30), VR(30), BM(30), 6BM(30), BMFIB(30), VB(30), V1B(30), QFIB(30), TFIB(30) DOUBLE PRECISION DSIN, DCO5, DEXP, DSQ RT, SINH2, EB2N, BB2N, 1DB2NL, RHI5, RHIS, DEN, W11, W12, W21, W22, Q1, Q2, Q3, Q4, Q5, 2Q8, Q9, Q10, Q11, Q12, Q17, Q18, Q19, Q25, Q29, Q30, Q31, Q32, 3Q34, Q35, Q37, Q39, Q40, Q41, Q42, Q44, Q45, Q97, Q98, Q99, CONST, 4Q20, Q21, Q22, Q23, Q24, Q26, Q28, Q54, ANU, AMU, ETA, BETA, 5RHO, ALPHAl, AN, FN1, FN2, YPH10, Q47, Q48, Q49, Q50, SINH1, EB1NL, 6BB1NL, DB1NL

333 CONTINUE
C INPUT DATA AND PRINT OUT. SPAN, H, IN FT, GAM IN LB/CU FT,
C PHIO IN DEGREES, ALL ELSE IN KIP, INCH UNITS
C READ NUMBER OF TERMS TO BE USED IN FOURIER SERIES (MAX 300),
C NUMBER OF ANGLES TO BE USED
READ 4, N, M
C READ RING RADIUS, AREA, RADIUS OF GYRATION, DEPTH, FLANGE WIDTH,
C WEB THICKNESS, DISTANCE FROM CENTRE OF RING TO SUPPORT
C POINT, ANGLE PHI AT WHICH RING IS SUPPORTED, MODULUS OF
C ELASTICITY, SPAN BETWEEN RINGS
READ 1, RR, AR, GYR, DEPR, FLGE, TWEB, EB
READ 1, PHIO, ER, SPAN
C READ BAND DIAMETER, SPACING, MODULUS OF ELASTICITY
READ 1, DIAM, S, ES
C READ STAVE RADIUS, THICKNESS, WIDTH, MODULUS OF ELASTICITY
C PARALLEL AND PERPENDICULAR TO GRAIN
READ 1, R, THICK, WIDTH, E1, E2
C READ LIQUID UNIT WEIGHT, PRESSURE HEAD ABOVE TOP OF PIPE
READ 1, GAM, H
C READ ASSUMED BAND TENSION, RING TENSION
READ 1, TNL, TRNL
C  READ ANGLES IN DEGREES (MAX 30)
READ 84, (ANGLE(J), J = 1, M)
1  FORMAT (7F10.3)
4  FORMAT (7I10)
84  FORMAT (16F5.0)
CALL SKIP TO (1)

PRINT 57

57  FORMAT (17X, 15HRING PROPERTIES//)
CR = 0.5 * DEPR
A = 3.1415927*0.25*DIA*DIAM
PRINT 58, RR, ER, AR, GYR, PHI0, EB, FLGE, DEPR, TWEB

58  FORMAT (12H MEAN RADIUS, 18X, FB.3, 3H IN//22H MODULUS OF ELASTICI
1TY, 9X, F7.0, 4H KSI/5H AREA, 23X,F10.3, 6H SQ IN//19H RADIUS OF
2GYRATION, 9X, F10.3, 3H IN//26H RING SUPPORTED AT PHI0 OF, 5X,
3F7.0, 4H DEG//25H DIST FROM CENTRE OF RING/ 7X, 16HTO SUPPORT POIN
4T, 8X, F7.3, 3H IN//713H FLANGE WIDTH, 15X,F10.3, 3H IN//6H DEPT,
522X,F10.3, 3H IN//14H WEB THICKNESS, 14X,F10.3, 3H IN//)
PRINT 59

59  FORMAT (12X, 26HWOOD STAVE PIPE PROPERTIES//)
PRINT 60, R, E1, E2, ES, A, S, WIDTH, THICK, SPAN
60  FORMAT (12H MEAN RADIUS, 18X, FB.3, 3H IN//30H MODULUS OF ELASTICI
1TY OF WOOD, F8.0, 23H KSI PARALLEL TO GRAIN//30X, F8.0, 28H KSI
2PERPENDICULAR TO GRAIN//31H MODULUS OF ELASTICITY OF BANDS, F7.0,
34H KSI//13H AREA OF BAND, 18X, F7.3, 6H SQ IN//13H BAND SPACING,
418X, F7.3, 3H IN//15H WIDTH OF STAVE, 16X, F7.3, 3H IN//19H THICKN
5ESS OF STAVE, 12X, F7.3, 3H IN//19H SPAN BETWEEN RINGS, 12X, F7.3,
63H FT////)
PRINT 61, H, GAM, TNL, TRNL
61  FORMAT (14H PRESSURE HEAD, 17X, F7.3, 3H FT//22H UNIT WEIGHT OF LIQUID
1.9X,F7.2, 9H LB/CU FT:///21H ASSUMED BAND TENSION, 10X, F7.3, 4H K
2IP//21H ASSUMED RING TENSION, 10X, F7.3, 4H KIP)
CALL SKIP TO (1)
2  FORMAT (1X, 7F10.3/)
3  EBB = RR - R
7  C CONVERT TO KIP, INCH, RADIAN UNITS
5  SPAN = 12. *SPAN

12
11
10
9
8
7
6
5
4
3
2
1
H = 12. * H
GAM = GAM/1728000.
CONST = 3.14159265356/180.
PHIO = CONST*PHIO
DO 85 J = 1, M
   85 ANGLE(J) = CONST*ANGLE(J)
C FIND TRANSFORMED STAVE SECTION PROPERTIES
   RATE = E2/ES
   STAVE = RATE*S*THICK
   AM = A*(1.0 + STAVE/A)
   XNA = (A*(THICK + 0.5*DIAM) + STAVE*0.5*THICK)/AM
   OBNERT = (A*(DIAM*DIAM/16. + (0.5*DIAM + THICK - XNA)**2) / 1 + STAVE*(THICK*THICK/12. + (0.5*THICK-XNA)**2))/S
C FIND GREEK CONSTANTS
   ANU = S/(AM*ES)
   AMU = S*BNERT*E2/(AM*R*R*E2)
   ETA = ANU*THICK**3*R*R*R1/12.
   BETA = ETA * (WIDTH/THICK) * (WIDTH/THICK)
   RHO = ANU * GAM* R**3
   ALPHA = RHO * H/R
C FIND SUBSTITUTION FACTORS INDEPENDENT OF N
   AN = BETA * ETA/RHO
   FN1 = 4. * AN
   FN2 = 2. * FN1
   Q23 = AR*ER
   Q26 = R/RR
   Q54 = R*R*SPAN*GAM
   Q17 = Q54/DSIN(PHIO)
   Q18 = -Q17*(RR + EB)
   Q20 = Q23/RR
   Q21 = R*ER*(THICK/SPAN)**3/6.
   Q22 = R*Q21*(WIDTH/THICK)**2
   Q24 = Q23*(GYR/RR)**2
   Q27 = Q227(R*RR)
   Q33 = AMU/(ANU*R)
   Q36 = TNL/R
   Q69 = ALPHA + RHO
Q56 = E1*THICK/(12.*SPAN*SPAN)
Q57 = Q56*WIDTH*WIDTH
Q58 = Q56*THICK*THICK
Q59 = Q57/SPAN
Q60 = Q58/SPAN
Q61 = E2*(THICK/R)**3/12.
Q62 = Q61*K
Q63 = SPAN**4
Q81 = l./(R*ANU)
Q82 = ANU*TNL
Q71 = 1.0 + Q82
Q6 = E2*BNERT*ES/(R*R*E2)
Q7 = Q6/R

C SET DEFLECTION AND FORCE VARIABLES EQUAL TO ZERO INITIALLY
DO 99 J = 1, M
BMR(J) = 0.0
TR(J) = 0.0
VR(J) = 0.0
BMB(J) = 0.0
BM1B(J) = 0.0
BMFIB(J) = 0.0
VB(J) = 0.0
V1B(J) = 0.0
QFIB(J) = 0.0
TFIB(J) = 0.0
UR(J) = 0.0
YR(J) = 0.0

99 CONTINUE
YPHIO = 0.

C CALCULATE TERMS FOR N GREATER THAN 1. SUM CONTRIBUTIONS FROM EACH
C VALUE OF N.
DO 100 I = 2, N
EN = I
C FIND SUBSTITUTION FACTORS DEPENDENT ON N
Q38 = EN*EN
Q28 = Q38 - 1.0
Q1 = (AMU*Q28 + ANU*TNL)*Q38
\[ Q19 = \text{DSIN}(\text{EN*PHIO}) \]
\[ BN(I) = (\text{BETA}*(1.0 + Q1) + \text{ETA} \cdot Q38)/\text{RHO} \]
\[ CN(I) = Q1 \cdot Q28/\text{RHO} \]
\[ FN(I) = FN1 \cdot CN(I)/(\text{BN}(I) \cdot \text{BN}(I)) \]
\[ Q2 = \text{BN}(I)/\text{FN2} \]
\[ Q3 = \text{DSQRT}(1.0 - \text{FN}(I)) \]
\[ B1N4(I) = Q2 \cdot (1.0 + Q3) \]
\[ B2N4(I) = Q2 \cdot (1.0 - Q3) \]

C CHECK FOR SQUARE ROOT OF NEGATIVE NUMBER IN EQUATIONS (26)

IF (B2N4(I) \cdot \text{LT} \cdot 0.0 \text{OR} \cdot B1N4(I) \cdot \text{LT} \cdot 0.0) \text{GO TO 496}

\[ \text{TAU1N}(I) = (1.0 + Q1 - 4.0 \cdot \text{ETA} \cdot B1N4(I))/\text{EN} \]
\[ \text{TAU2N}(I) = (1.0 + Q1 - 4.0 \cdot \text{ETA} \cdot B2N4(I))/\text{EN} \]
\[ \text{DELTA}(I) = 4.0 \cdot \text{ETA} \cdot (B2N4(I) - B1N4(I)) \]
\[ B1N(I) = (B1N4(I))^{.25} \]
\[ B2N(I) = (B2N4(I))^{.25} \]

IF (E2 \cdot \text{EQ} \cdot 0.0) B2N(I) = 0.0

\[ Q4 = B1N(I) \cdot \text{SPAN} \]
\[ Q5 = B2N(I) \cdot \text{SPAN} \]

IF (Q4 \cdot \text{GE} \cdot 20.0) \text{GO TO 435}
\[ Q47 = \text{DEXP}(-Q4) \]
\[ Q48 = 2.0 \cdot Q4 \]
\[ Q49 = \text{DEXP}(-Q48) \]
\[ Q50 = \text{DEXP}(Q4) \]
\[ \text{SINH}1 = (Q50 - Q47) \cdot 0.5 \]
\[ \text{EB1NL = 0.5*Q50/(DSIN(Q4) + SINH1)} \]
\[ \text{BB1NL = Q47*DSIN(Q4)} \]
\[ \text{DB1NL = Q47*DC0S(Q4)} \]
\[ P1N(I) = \text{EB1NL*(1.0 + 2.0*BB1NL - Q49)} \]
\[ G1N(I) = 2.0*Q4*Q4*EB1NL*(-1.0 + 2.0*BB1NL + Q49) \]
\[ AJ1N(I) = 4.0*Q4*3*EB1NL*(1.0 - 2.0*DB1NL + Q49) \]

GO TO 436

\[ P1N(I) = 1.0 \]
\[ G1N(I) = -2.0*Q4*Q4 \]
\[ AJ1N(I) = 4.0*Q4*3 \]

CONTINUE

C CHECK MAGNITUDE OF (BETA 2N)*L TERMS

IF (Q5 \cdot \text{EQ} \cdot 0.0) \text{GO TO 856}
IF (Q5 .GE. 20.) GO TO 858
Q8 = DEXP(-Q5)
Q9 = 2.*Q5
Q10 = DEXP(-Q9)
Q11 = DEXP(Q5)
SINH2 = (Q11 - Q8)*0.5
EB2NL = 0.5*Q11/(D31N(Q5) + SINH2)
BB2NL = Q8*DSIN(Q5)
DB2NL = Q8*DCOS(Q5)
P2N(I) = EB2NL*(1.0 + 2.*BB2NL - Q10)
G2N(I) = 2.*Q5*Q5*EB2NL*(-1.0 + 2.*BB2NL + Q10)
AJ2N(I) = 4.*Q5**3*EB2NL*(1.0 - 2.*DB2NL + Q10)
GO TO 857
858 P2N(I) = 1.0
G2N(I) = -2.*Q5*Q5
AJ2N(I) = 4.*Q5**3
GO TO 857
856 P2N(I) = 0.0
G2N(I) = 0.0
AJ2N(I) = 0.0
CONTINUE
857 RHS1 = Q17*EN*Q19
RHS2 = Q18*Q19
Q99 = EN/DELTA(I)
Q25 = Q26*Q99
Q97 = AJ2N(I) - AJ1N(I)
Q98 = TAU2N(I)*AJ2N(I) - TAU1N(I)*AJ1N(I)
W11 = -Q28*Q20 + Q21*(AJ1N(I) + Q97*TAU1N(I)*Q99) + (AJ1N(I) +
1Q98*Q99)*TAU1N(I)*EN*Q22/R
W12 = -EN*Q28*Q20 + Q21*Q25*Q97 + EN*Q25*Q98*Q22/R
W21 = EN*(Q28*Q24 + Q23 + TRNL) - Q22*(Q98*EN+DELTA(I)*AJ1N(I))*
1TAU1N(I)/DELTA(I)
W22 = Q23*Q38 - Q22*Q98*Q25
DEN = W11*W22 - W21*W12
UNR(I) = (RHS1*W22 - RHS2*W12)/DEN
YNR(I) = (RHS2*W11 - RHS1*W21)/DEN
AK4N(I) = EN*(Q26*YNR(I) + TAU1N(I)*UNR(I))/DELTA(I)
$\text{AKN}(I) = \text{UNR}(I) - \text{AK4N}(I)$

$Q30 = \text{AKN}(I)G1N(I) + \text{AK4N}(I)G2N(I)$

$Q31 = \text{AKN}(I)\text{AJ1N}(I) + \text{AK4N}(I)\text{AJ2N}(I)$

$Q32 = \text{AKN}(I)P1N(I) + \text{AK4N}(I)P2N(I)$

$\text{TNBB}(I) = Q27*\text{TAU1N}(I)*\text{AKN}(I)*\text{AJ1N}(I) + \text{TAU2N}(I)*\text{AK4N}(I)*\text{AJ2N}(I))$

$\text{YPH10} = \text{YPH10} + \text{YNR}(I)*Q19$

**C** FIND SUBSTITUTION FACTORS DEPENDENT ON N BUT INDEPENDENT OF PHI

$Q34 = Q28*Q32$

$Q35 = Q7*Q34$

$Q37 = Q32*Q32$

$Q39 = Q17*Q19$

$Q39 = \text{EN}*(Q35 + Q37)$

$Q40 = Q28*\text{UNR}(I)$

$Q41 = \text{UNR}(I) + \text{EN}*\text{YNR}(I)$

$Q42 = (Q24*Q28 + \text{TRNL})*\text{EN}*(\text{UNR}(I) + \text{RR}*(\text{RR-R})*\text{TNBB}(I)) + \text{EB}Q29$

$Q44 = \text{AKN}(I)*\text{TAU1N}(I)*G1N(I) + \text{AK4N}(I)*\text{TAU2N}(I)*G2N(I)$

$Q45 = \text{AKN}(I)*\text{TAU1N}(I)*\text{AJ1N}(I) + \text{AK4N}(I)*\text{TAU2N}(I)*\text{AJ2N}(I)$

$Q12 = (1.0 - \text{EN})*\text{TAU1N}(I)*\text{P1N}(I) + (1.0 - \text{EN})*\text{TAU2N}(I)$

$\text{AK4N}(I)*P2N(I)$

**C** FIND FORCES AND DEFLECTIONS FOR ALL VALUES OF PHI

DO 101 J = 1, M

**C** FIND SIN(N*PHI) AND COS(N*PHI)

$\text{ENPHI} = \text{EN} * \text{ANGLE}(J)$

$\text{SNPHI}(J) = \text{DSIN}(...)$

$\text{CNPHI}(J) = \text{DCOS}(...)$

**C** FOR RING

$\text{BMR}(J) = \text{BMR}(J) - Q40*\text{CNPHI}(J)$

$\text{TR}(J) = \text{TR}(J) + Q41*\text{CNPHI}(J)$

$\text{VR}(J) = \text{VR}(J) + Q42*\text{SNPHI}(J)$

$\text{UR}(J) = \text{UR}(J) + \text{UNR}(I)*\text{CNPHI}(J)$

$\text{YR}(J) = \text{YR}(J) + \text{YNR}(I)*\text{SNPHI}(J)$

**C** FOR STAVES

$\text{BMB}(J) = \text{BMB}(J) + Q44*\text{SNPHI}(J)$

$\text{BMB}(J) = \text{BMB}(J) + Q44*\text{CNPHI}(J)$

$\text{BMFIB}(J) = \text{BMFIB}(J) - Q34*\text{CNPHI}(J)$

$\text{VB}(J) = \text{VB}(J) + Q45*\text{SNPHI}(J)$

$\text{V1B}(J) = \text{V1B}(J) + Q31*\text{CNPHI}(J)$
QL(B) = QI(B) + Q3*SNPHI(B)

TFIB(B) = TFIB(B) + Q12*CNPHI(B)

CONTINUE

CALCULATE TERMS FOR N = 0

BN = BETA/RHO

B104 = 0.25/ETA
B10 = B104**0.25
Q78 = B10*SPAN
P10 = 1.0
G10 = -2.0*Q78*Q78
AJ10 = 4.0*Q78**3
Q16 = 6.0*Q23*(SPAN/THICK)**3/(E1*RR*R*AJ10)
UR = Q69/(1.0 + Q16)
AK0 = UR - Q69
Q73 = AK0*G10
Q79 = AK0*P10
Q83 = AK0*AJ10
Q74 = Q79 + Q69

CALCULATE TERMS FOR N = 1

BN(1) = (BETA*(1.0 + ANU*TNL) + ETA)/RHO

B1N4(1) = BN(1)/FN1
B1N(1) = B1N4(1)**0.25
Q46 = B1N(1)*SPAN
P1N(1) = 1.0
G1N(1) = -2.0*Q46*Q46
AJ1N(1) = 4.0*Q46**3

FIND U1R AND Y1R

Y1R = -YPHIO/SIN(PIHQ)
Q85 = AJ1N(1)/(2.0*B1N4(1)*Q63)
Q90 = Q54*(EB*EB) + (ETA + BETA*Q85*Q71)*SPAN/(BN(1)*ANU)
Q84 = Q23 + SPAN*Q85/ANU + TRNL + TNL*SPAN*Q85
Q86 = Q23 + Q85*Q26*SPAN/ANU
U1R = -(Q90 + Q86*Y1R)/Q84
Q96 = U1R + Y1R

FIND TERMS DEPENDENT ON U1R AND Y1R

Q77 = AN/(BN(1)*ETA)
Q76 = ETA/BN(1) - RHO
AK4N(1) = -Q77*(Q26*Y1R + Q71*U1R) + U1R
AKN(1) = -AK4N(1) + U1R - Q77*Q76
TAUIN(1) = Q71 - BN(1)*RHO/BETA
Q64 = AKN(1)*G1N(1)
Q65 = AKN(1)*P1N(1)
Q66 = AKN(1)*A1IN(1)
Q55 = Q64*TAUIN(1) - Q71*Q63/(12.*BN(1))
Q68 = Q63/(2.*BN(1))
Q67 = Q64 - Q68/6.
Q70 = Q66*TAUIN(1) - Q68*Q71
Q72 = Q66 + Q68
Q75 = (AK4N(1) + Q77*Q76 + Q65)*TNL/R
TNBB(1) = Q27*(Q68*Q71 + TAUIN(1)*Q66)
Q52 = TRNL*U1R + (RR-R)*RR*TNBB(1) + EB*Q54
Q53 = RR*RR*TNBB(1) - Q54 - Q23*(U1R + Y1R)
Q80 = -AKN(1)*Q82 + Q76*(1. - Q77*Q82) + (1.0 - TAU1N(1))*Q65
PRINT 450

OFORMAT (30X,16HRING DEFLECT IONS//5H PHI, 17X, 6HRADIAL, 22X,
110HTANGENTIAL, 20X, 10HORIZONTAL, 22X, 8HORIZONTAL, 19X, 2HORIZONTAL, 28X, 2HORIZONTAL//)

C ADD TERMS FOR N = 0, N = 1, TO GET FINAL RESULTS
DO 102 J = 1, M
CNPHI(J) = COS(ANGLE(J))
SNPHI(J) = SIN(ANGLE(J))

C FOR RING
BMR(J) = Q24*(BMR(J) + U0R)
TR(J) = (TR(J) + UOR + (U1R + Y1R)*CNPHI(J))*Q20
VR(J) = (VR(J) + Q52*SNPHI(J))/RR
UR(J) = UOR + U1R*CNPHI(J) + UR(J)
YR(J) = Y1R*SNPHI(J) + YR(J)
DEFH(J) = UR(J)*SNPHI(J) + YR(J)*CNPHI(J)
DEFV(J) = -UR(J)*CNPHI(J) + YR(J)*SNPHI(J)
ANGLE(J) = ANGLE(J)/CONST

C PRINT RING DEFLECTIONS
PRINT 451, ANGLE(J), UR(J), YR(J), DEFH(J), DEFV(J)
PRINT 452, ANGLE(J), UR(J), YR(J), DEFH(J), DEFV(J)
FORMAT (1X, F6.1, 13X, 1PE10.3, 3720X, 1PE10.3/)
C FOR STAVES

\[ BMB(J) = (BMB(J) + Q55*SNPHI(J))*Q57 \]
\[ BM1B(J) = (BM1B(J) + Q73 + Q67*CNPHI(J))*Q58 \]
\[ BMFIB(J) = (BMFIB(J) + Q74)*Q6 \]
\[ VB(J) = (VB(J) + Q70*SNPHI(J))*Q59 \]
\[ V1B(J) = (V1B(J) + Q72*CNPHI(J) + Q83)*Q60 \]
\[ QhlB(J) = Uf-IB(J) + Q75*SNPHI(J) \]
\[ TFIB(J) = (TFIB(J) + 069 + Q79 + Q80*CNPHI(J))*Q81 \]

CONTINUE

CALL SKIP TO (1)
PRINT 453

453 OFORMAT (30X,24HRING FORCES AND STRESSES//74X, 13HNORMAL STRESS/
15H PHI,7X,10HMOMENT MK, 8X, 10HTENSION TR, 9X, 8HSHEAR VR, 11X, 27HOUTSIDE, 11X, 6HINSIDE, 9X, 12HSHEAR STRESS/5H DEG, 10X, 6HKIP 3IN, 13X, 4HKIPS, 14X, 4HKIPS> 15X, 3HKSI, 14X, 3HKSI, 15X, 3HKSI/)  

C FIND RING STRESSES

\[ \text{CONST}1 = CR/(GYR*GYR) \]
\[ \text{DO 454 J = 1, M} \]
\[ \text{AXRO} = (TR(J) - BMR(J)*CONST1)/AR \]
\[ \text{AXRI} = (TR(J) + BMR(J)*CONST1)/AR \]
\[ VRING = VR(J)/(DEPR*TWEB) \]

C PRINT RING FORCES AND STRESSES

\[ \text{PRINT 455, ANGLE(J), BMR(J), TR(J), VR(J), AXRO, AXRI, VRING} \]
\[ \text{455 FORMAT (1X,F6.1, 5X, 1PE12.5, 5(6X, 1PE16.5)/)} \]

CALL SKIP TO (1)
PRINT 456

456 OFORMAT(30X,20HSTAVE FORCES AT RING//5H PHI,6X,8HMOMENT M, 7X, 12HSTAVE M1, 5X, 12HMOMENT MPH, 6X, 8HSHEAR V, 8X,8HSHEAR VI, 27X, 10HSHEAR VPHI, 5X,12HTENSION NPHI/5H DEG,7X, 6HKIP IN, 10X, 36HKIP IN, 11X, 3HKIP, 13X, 3HKIP, 13X, 3HKIP, 13X, 3HKIP,13X, 43HKIP//)

\[ \text{DO 457 J = 1, M} \]
\[ \text{C PRINT STAVE AND BAND FORCES AT THE RING} \]
\[ \text{PRINT 458, ANGLE(J), BMB(J), BM1B(J), BMFIB(J), VB(J), V1B(J), QFIBC(J), TFIB(J)} \]
\[ \text{458 FORMAT (1X,F6.1, 1PE14.5, 6(1PE16.5)/)} \]

CALL SKIP TO (1)
PRINT 459

459 0FORMAT (30X, 43HMAXIMUM STAVE AND BAND STRESSES AT THE RING//
127X, 9HSTAVE END, 30X, 11HSTAVE EDGE, 25X, 4HBAND/
25H PHI, 7X, 7HFLEXURE, 10X, 5HSHEAR, 8X, 11HCOMPRESSION, 12X,
313HNORMAL STRESS, 15X, 6HNORMAL, 10X, 5HSHEAR/
442X, 11HPERP, GRAIN, 7X, 7HOUTSIDE, 9X, 6HINSIDE/
55H DEG, 9X, 3HKSI, 13X, 3HKSI, 13X, 3HKSI, 13X, 3HKSI, 13X, 3HKSI,
613X, 3HKSI, 13X, 3HKSI/) C
C FIND STAVE AND BAND STRESSES AT THE RING

C       CONST2 = 6.0/THICK
C       CONST3 = 1.5/THICK
C       CONST4 = 2.0*(R + 0.5*THICK)/(R*FLGE)
C       CONST5 = RATE/BNERT
C       CONST6 = CONST5*XNA
C       CONST7 = CONST5*(THICK - XNA)
C       CONST8 = (THICK + 0.5*DIAM - XNA)/BNERT
C       CONST9 = S/AM
C       CNST10 = CONST9*RATE

DO 460 J = 1, M

FLXMM1 = (ABS(BM1B(J)/THICK) + ABS(BMB(J)/WIDTH))*CONST2
SHRJV1 = CONST3*SQRT(VB(J)*VB(J) + VI*B(J))
CPERP = CONST4*VIB(J)

AXSO = TFIB(J)*CNST10 - BMFIB(J)*CONST7
AXSI = TFIB(J)*CNST10 + BMFIB(J)*CONST6

AXB = TFIB(J)*CONST9 - BFTIB(J)*CONST8

SHRB = S*QFIB(J)/A

C PRINT STAVE AND BAND STRESSES AT RING

460 PRINT 458, ANGLE(J), FLXMM1, SHRJV1, CPERP, AXSO, AXSI, AXB, SHRB

C FIND MINIMUM VALUE OF (BETA 2N)*L. Q78 OR (BETA 10)*L IS MINIMUM
C VALUE OF (BETA 1N)*L.

FNM = FN(2)
MF = 2
DO 336 I = 3, N

IF (FNM - FN(I)) 337, 336, 336

337 FNM = FN(I)
MF = I

336 CONTINUE
B2NMIN = B2N(2)
MIN = 2
DO 338 I = 3, N
 339 B2NMIN = B2N(I)
  MIN = I
338  CONTINUE
B2MINL = B2NMIN*SPAN
PRINT 334, Q78, B2MINL, FNM, MF
334  FORMAT (29H MIN. VALUES OF (BETA*L) ARE , F7.2, 5H AND , F7.2,
110X, 19H MAX. VALUE OF F IS , F7.4, 8H AT N = , I3//)
PRINT 335, N
335  FORMAT (IX, I4, 21H TERMS USED IN SERIES)
497  GO TO 333
496  PRINT 495
495  FORMAT (31H SQUARE ROOT OF NEGATIVE NUMBER)
GO TO 333
523  STOP
END