SOME INVESTIGATIONS INTO THE FINITE ELEMENT METHOD WITH SPECIAL REFERENCE TO PLANE STRESS

by

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SOME INVESTIGATIONS INTO THE FINITE ELEMENT METHOD WITH SPECIAL REFERENCE TO PLANE STRESS

ABSTRACT

Plane stress stiffness matrices are derived explicitly for square isotropic elements under different assumptions on the stress distribution. An explicit (8 x 8) matrix is obtained under the assumption of uniform σ_x, σ_y, linear τ_{xy} and thus it is shown that the Gallagher matrix belongs to the class of parametric matrices. Two (10 x 10) matrices are obtained under the assumption of linear σ_x, σ_y, τ_{xy} using interior nodal translations and corner edge rotations respectively as additional generalized displacements. These two matrices do not appear suitable for general usage but will perform as well as the Turner matrix under the same nodal loads. A (12 x 12) matrix is derived under the assumption of hyperbolic σ_x, σ_y, and parabolic τ_{xy}, again exemplifying the use of corner edge rotations as additional generalized displacements. This matrix behaves unexpectedly with varying Poisson's ratio.

A method of evaluating stiffness matrices, which reduces the necessity of comparing finite element solutions with analytical ones, is formulated. In this method a comparison is made of the strain energy of deformation produced within a finite element by the different matrices under the same nodal loads. It is shown that such comparisons require the study of special matrices i.e. the stiffness difference matrix and the inverse difference matrix which are obtained from the matrices under comparison. It is proved that the results of the element matrix comparisons apply to the structure. It is shown that the strain energy of a finite element under normalised loads is bounded between the maximum and minimum eigenvalues of the inverse matrix.

The strain energy comparison criterion is used in the study of parametric matrices. An explicit parametric inverse is obtained. Explicit parametric eigenvalues are obtained for the inverse difference matrix and the stiffness difference matrix, and it is verified that they give identical results for the matrix comparisons. It is proved that the parametric matrices produce the exact strain energy under uniform nodal loads. It is shown that the stiffness matrix parameter and the inverse matrix parameter represent a measure of the strain energy under non-uniform nodal loads so that the strain energy can always be bounded by varying the parameter. It is proved that if strain energy curves are drawn with respect to structure sub-division then no two curves will intersect. It is proved that all

parametric strain energy curves will converge towards the true solution with progressive structure subdivision. A strain energy ordering is obtained for the parametric matrices and the following conclusions are drawn. The Pian matrix is the best displacement matrix. The Gallagher matrix is inferior to the Turner, Pian, and Argyris-Melosh matrices. Constant stress tri-nodal triangles are generally inferior to the use of square elements. Matrices satisfying microscopic equilibrium or capable of representing uniform stresses will not necessarily yield good results.

A method is proposed for obtaining upper bounds on the strain energy of a region under plane stress by replacing th continuum with a psuedo-truss system, the bar forces of which provide the equilibrium and self-straining solutions. Two examples of its application are presented, and an indication is obtained that upper bounding by varying the matrix parameter will give better results for the same structure subdivision.

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SOME INVESTIGATIONS INTO THE FINITE ELEMENT METHOD WITH SPECIAL REFERENCE TO PLANE STRESS

ABSTRACT

This study of the Finite Element Method is limited to static linear structural behavior under small displacements and involving strains small as compared to unity.

The first part of this study deals with the derivation of stiffness matrices for square isotropic elements under different assumptions on the stress distribution. An explicit (8 x 8) matrix is obtained under the assumption used by Gallagher of uniform σ_x, σ_y , linear τ_{xy}. Two (10 x 10) matrices are obtained under the assumption of linear σ_x, σ_y, τ_{xy} using interior nodal translations and corner edge rotations respectively as additional generalized displacements. These two matrices do not appear suitable for general use but will perform as well as the Turner matrix under the same nodal loads. A (12 x 12) matrix is derived under the assumption of hyperbolic σ_x, σ_y and parabolic τ_{xy}, again exemplifying the use of edge rotations at corners as additional generalized displacements. This matrix behaves unexpectedly with varying Poisson's ratio.

Since, in general, there may be a number of stiffness matrices available for different classes of finite elements (i.e., elements for plane stress, plate-bending, shells, etc.,) the second part proposes a method for choosing the best matrix from an available set. This "best" matrix is defined as the one which will yield the closest approximation to the true strain energy of deformation. In order to make this choice a comparison is made of the strain energy produced within a finite

element by the different matrices under the same nodal loads. It is shown that such comparisons require the study of special matrices, i.e., the stiffness difference matrix and the inverse difference matrix which are obtained from the matrices under comparison. It is proved that the results of the element matrix comparisons generally apply to the structure. It is shown that the strain energy of a finite element under normalized loads is bounded between the maximum and minimum eigenvalues of the inverse matrix.

Next, the proposed method for choosing stiffness matrices on the basis of strain energy comparisons is verified by a study of ten plane stress matrices for square isotropic elements, which conform to the parametric representation shown to apply to some plane stress matrices for square isotropic elements by Hooley and Hibbert. An explicit parametric inverse is obtained. Explicit parametric eigenvalues are obtained for the inverse difference matrix and the stiffness difference matrix, and it is verified that they give identical results for the matrix comparisons. It is proved that the parametric matrices produce the exact strain energy under uniform nodal loads. It is shown that the stiffness matrix parameter and the inverse matrix parameter represent a measure of the strain energy under non-uniform nodal loads. It is proved that if strain energy curves are drawn with respect to structure subdivision then no two curves will intersect. It is proved that all parametric strain energy curves will converge towards the true solution with progressive structure subdivision. In a specific problem, where the strain energy curves are observed to converge monotonically, it is shown that it is reasonable to expect bounding of the solution strain energy by varying the parameter according to the procedure suggested by Hooley and Hibbert. A strain energy ordering is obtained for the parametric matrices and the following conclusions are drawn. The Pian matrix is the best displacement matrix. The Gallagher matrix is inferior to the Turner, Pian, and Argyris-Melosh matrices. Constant stress tri-nodal triangles are generally inferior to the use of square elements. Matrices satisfying microscopic equilibrium or capable of representing uniform stresses will not necessarily yield good results.

Finally, a method is proposed for obtaining upper bounds on the strain energy of a region under plane stress by replacing the continuum with a psuedo-truss system, the bar forces of which provide the equilibrium and self-straining solutions. Two examples of its application are presented, and in a specific problem where bounding by variation of the Hooley-Hibbert parameter appears possible an indication is obtained that upper bounding by this latter method will give better results for the same structure subdivision.

TABLE OF CONTENTS

TABLE OF CONTENTS (Cont'd)

LIST OF TABLES

LIST OF FIGURES

LIST OF FIGURES (Cont'd) Page No.

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SOME INVESTIGATIONS INTO THE FINITE ELEMENT METHOD

WITH SPECIAL REFERENCE TO PLANE STRESS

CHAPTER I

INTRODUCTION

1.1 Description of the Finite Element Method

During the last decade, the discrete or finite element method has been established as an efficient and powerful tool for obtaining digital computer solutions to problems of the continuum.

This method requires the fictitious division of the continuum into contiguous finite elements, which are joined together at discrete points called Nodes.

Each element has a stiffness matrix which gives the relationship between the generalized forces and displacements of the element. The element stiffness matrices are summed to give the structure stiffness matrix.

Many stiffness matrices can be developed for a finite element depending upon the assumptions made on the variation of stresses or displacements over the element. Therefore the real problem lies in developing stiffness matrices with predictable solution behavior and in choosing the best stiffness matrix from an available set.

1.2 Some Problems of the Finite Element Method

1. Bases have been defined for obtaining stiffness matrices which will provide bounds on elastic behavior (Melosh, 1962, pp.14-17) but no theoretical criterion has been proposed for choosing the best matrix from an available set of bounding matrices.
2. Stiffness matrices have been generally evaluated by comparing finite element solutions to analytical ones, but difficulties arise in extrapolating the results to the infinity of problems that may be formulated.
3. A criterion has been developed for monotonic convergence (Melosh, 1962, pp. 20-23) but there is no guarantee that convergence will be to the solution. Another criterion has been proposed for convergence to the solution (Bazely et al, 1965, pp. 2-3) but its validity has not been rigorously established.
4. Success in obtaining bounding matrices has been limited. Also sometimes non-bounding matrices give better solutions than bounding matrices (Bazely et al, 1965, pp. 21-23; Hooley and Hibbert, 1966, pp. 46-47).

1.3 Scope of the Present Study

This study is limited to static structural behavior under small displacements and involving strains small as compared to unity.

A strain energy formulation is utilized to derive some new stiffness matrices for square isotropic elements under plane stress.

A strain energy criterion is developed to compare stiffness matrices. It is used in the study of plane stress matrices for square isotropic elements which conform to the parametric representation shown to apply to some plane stress matrices for square isotropic elements by Hooley and Hibbert.

A square element with special stress transmission properties is conceived to provide an upper bound on the strain energy.

1.4 Definition of terms

A stiffness matrix satisfying complete displacement compatibility within and across the boundaries of an element is termed a displacement stiffness matrix.

A stiffness matrix satisfying the differential equations of equilibrium within the element and complete stress continuity within and across the boundaries of an element is termed an equilibrium stiffness matrix.

A stiffness matrix which satisfies displacement compatibility and equilibrium only at the nodes but allows discontinuities in both stresses and displacements at the boundaries is termed a hybrid stiffness matrix.

A stiffness matrix from which rigid body modes have been eliminated is termed a natural stiffness matrix.

CHAPTER II

REVIEW OF PREVIOUS WORK

The Finite Element Method is a generalization of matrix structural analysis procedures, described comprehensively by Argyris (1954, 1955) so as to include the use of two- and three-dimensional elements.

Levy (1953, pp. 449-454) and Argyris (1955, pp. 125-126) developed stiffness matrices for specific two-dimensional structural components: an idealized quadilateral torsion box and a rectangular flanged panel under direct stress respectively.

Turner, Clough, Martin and Topp (1956, pp. 805-823) considered the use of more fundamental elements whose behavior would approach that of the continuous structure in the limit. They applied the idea to plane stress problems. Stiffness matrices for rectangular and triangular elements were obtained on the basis of assumed stress distributions. The number of modes in the chosen stress distribution was equal to the number of nodal displacements of the supported element (i.e. with rigid body motions prevented). Clough (1960, pp. 345-378) provided further applications to plane stress problems and noted an improvement in answers with increased subdivision.

Melosh (1962, pp. 14-17) showed that if stiffness matrices were developed so as to conform to the minimum potential energy and minimum complementary energy formulations, then the strain energy of the solution (as affected by the discretization errors) would be bounded. He indicated continuity requirements for displacement functions satisfying the minimum potential energy theorem, (1963, pp. 1632-1633). These requirements should be made more restrictive by stating that they must conform to those for admissible functions of the potential energy functional being considered. This statement would automatically include continuity of displacement as well as slope for the Kirchhoff plate element (Weinstock, 1962, p. 239), the lack of which produced a nullification of the lower bound character (Tocher and Kapur, 1965) of the rectangular plate matrix developed by Melosh (1963, p. 1634).

Melosh (1962, pp. 20-23) also developed a sufficient criterion for monotonic convergence but noted that convergence to the true solution can only be guaranteed if the displacement functions are complete. This requirement of completeness of displacement functions is extremely difficult to satisfy.

Using a suitable displacement function with modes equal in number to the nodal displacements Melosh (1962, pp. 31-32) developed a plane stress stiffness matrix for a rectangular element insuring monotonic convergence. This matrix is evaluated in Chapter VI.

Melosh (1962, pp. 68-72) also proposed an hypothesis, based on a study of prism stiffness matrices, for choosing the best matrix by comparing stiffness matrix invariants. A theoretical basis is provided for this hypothesis and its usefulness examined in Chapter IV.

De Veubeke (1965, pp. 145-197) has presented comprehensive theoretical procedures for the development of equilibrium and displacement matrices, and used them to develop such matrices for plane stress elements. He has also shown (1962, pp. 185-188) that the dual minimal analysis enables influence coefficients to be bounded.

In the case of equilibrium matrices, De Veubeke has shown (1965, pp. 183-188) that artificial kinematic modes may be introduced, so that special care is necessary in using them. Also the generalized displacements associated with equilibrium matrices are weighted averages taken over the element edges. Therefore, it seems to this author that the results obtained by their use would not easily give a clear picture of the physical behavior.

De Veubeke (1965, pp. 191-193) has shown how an upper bounding solution may be obtained in terms of equilibrium and self-straining stresses. In Chapter V, a special square element is proposed by means of which the equilibrium and self-straining force systems may be obtained as bar forces of a psuedo-truss system.

In Chapter VI, a method is discussed for obtaining upper bounds on the strain energy by varying a matrix parameter. This method can be expected to apply to those specific problems where the strain energy curves are observed experimentally to be convergent monotonically. In this method no artificial kinematic modes are created, and the generalized displacements refer to discrete points so that there is a better appreciation of physical behavior than with the De Veubeke equilibrium triangular element matrix.

Pian (1964, pp. 576-577) has shown that the number of modes of the assumed displacement function can be more than the number of nodal displacements of an element. Then the principle of minimum potential energy enables the derivation of the stiffness matrix. He has suggested that solutions obtained by taking more terms in the displacement function will represent an improvement in equilibrium conditions. Clough (1965, p. 91) has observed that this does not generally lead to an improvement in element stiffness matrices. In Chapter VI, an example will be shown for plane stress matrices where an improvement is noted.

Pian (1964, pp. 1333-1336) has also shown that the number of modes in the assumed stress distribution may exceed the number of nodal displacements of the supported element. Then the principle of minimum complementary energy enables the derivation of the stiffness matrix under prescribed boundary displacements for the element. It is suggested that this procedure will allow an improvement in displacement compatibility while ensuring stress equilibrium within the element. Plane stress stiffness matrices having either more displacement modes or more stress modes are presented by Pian for square isotropic elements. These are compared in Chapter VI.

Hooley and Hibbert (1966) have observed that, for square isotropic elements under plane stress, stiffness matrices may be generated by discrete values of a continuous stiffness matrix parameter. This allows a very simple representation of many plane stress stiffness matrices for square isotropic elements. It is shown in

Chapter VI that these parameters correspond to different strain energy levels.

Hooley and Hibbert have noted that amongst the plane stress matrices for square isotropic elements tested by them (Turner, Melosh, Hrennikoff, McCormick), the Turner matrix, which is hybrid (i.e., non-bounding), gives the best results. Bazely et al (1965, pp. 21-23) have also presented some results for triangular plate matrices, where similar behavior is observed. In Chapter VI, an explanation is provided for this behavior. Also additional hybrid plane stress matrices are developed in Chapter III.

Irons and Draper (1964) and Bazely et al (1965, pp. 2-3) have proposed that for convergence to the true solution, it should be possible to represent a constant state of stress within an element. Hrennikoff (1941, pp. A169-A170) originally used this basis to justify framework representation of continua. In Chapter VI, it is shown that parametric matrices, which are capable of representing constant stresses within the element, do provide convergence to the true solution with sufficient network refinement.

CHAPTER 3

DERIVATION OF PLANE STRESS STIFFNESS MATRICES

3.1 Common Basis for Derivation

For linearly elastic structures (under small displacements), the strain energy is given by

$$U = \frac{1}{2} \lfloor q \rfloor [K] \{q\}$$

where K is the stiffness matrix and q the vector of generalized displacements (Bisplinghoff et al (1955), p. 23).

Therefore the stiffness influence coefficients may be expressed as

$$K_{ij} = \frac{\partial^2 U}{\partial q_i \partial q_j} \qquad i,j = 1,n$$

This formulation has been used previously by Green, Strome and Weikel (1961, pp. 1-9) to obtain a stiffness matrix for a triangular plate element used in approximating arbitrary shell shapes.

The strain energy may be expressed as a homogeneous quadratic in the generalized displacements by either assuming a displacement function or by assuming stress distributions satisfying the differential equations of equilibrium as well as compatibility.

In the derivations made in this chapter, stress distributions are assumed. A matrix formulation for obtaining the stiffness matrix on the basis of these stress assumptions is set out as follows.

For this matrix formulation it is assumed that the stresses over the element are described by means of a local co-ordinate system for the element (i.e., in Fig. 3.2.1, if the axes are translated so that the origin coincides with corner 1 of the element, then a local co-ordinate system for the element is defined).

Let the assumed stress distribution over an element be given by

$$\{\sigma\} = [A] \quad \{a^*\}$$

where σ is the stress vector, A, the transformation matrix, and a*, a vector of constants.

Then, the strains are given by

$$\{\varepsilon\} = [D] \quad \{\sigma\}$$

where D is a matrix of elastic constants and ε is the strain vector.

On integration of strains, the nodal displacements may be expressed as

$$\{q\} = [B] \quad \{a_1\}$$

where q is the displacement vector, and a_1, the augmented vector of constants (i.e., constants a* plus the constants associated with rigid body motions). Note that the constants a_1 are equal in number to the element nodal displacements.

Now, the strain energy of the element may be expressed as

$$U = \frac{1}{2} \int_V \lfloor \sigma \rfloor \{\varepsilon\} \, dV$$

$$= \frac{1}{2} \int_V \lfloor \varepsilon \rfloor [D^{-1}] \{\varepsilon\} \, dV$$

Now $$\{a_1\} = [B^{-1}] \{q\}$$

Also the matrix A is augmented with null columns to accommodate coefficients a_1 and is redesignated A_1, so that

$$\{\sigma\} = [A_1] \{a_1\}$$

Therefore $$\{\varepsilon\} = [D] [A_1] \{a_1\}$$

$$= [D] [A_1] [B^{-1}] \{q\}$$

Substituting in the expression for strain energy

$$U = \frac{1}{2} \int_V \lfloor q \rfloor [B^{-1}]^T [A_1]^T [D] [A_1] [B^{-1}] \{q\} \, dV$$

Therefore

$$[K] = \frac{\partial^2 U}{\partial q_i \partial q_j} \qquad i,j = 1,n$$

$$= \frac{1}{2} \int_V [B^{-1}]^T [A_1]^T [D] [A_1] [B^{-1}] \, dV$$

This matrix formulation is best suited for evaluation by the computer for arbitrary shapes of the element and different assumptions on the stress distribution.

However, in the explicit derivations presented here, for square isotropic elements, it was found more convenient to use a simplified procedure wherein the stresses over the element were defined in terms of a global co-ordinate system. This artifice reduced the number of unknown coefficients which defined the stress distributions.

3.2 Stiffness Matrices and their assumptions

A constant-thickness, square isotropic element is considered. The element is assumed embedded in the region shown in Fig. 3.2.1 for which the stress distribution is assumed.

The positive directions of the displacements and nodal forces and their ordering are indicated in Fig. 3.2.2, for matrices developed in sections 3.2.1, 3.2.2, and 3.2.3.

3.2.1 Uniform σ_x, σ_y, τ_{xy}

These assumed stresses do not constitute enough independent modes for a square element with freedom of corner translations. Therefore the natural stiffness matrix obtained is singular.

However the derivation is completed because the stiffness matrix parameter obtained for this case is of interest in subsequent discussion (Chapter VI). Also the procedure for the other cases is illustrated.

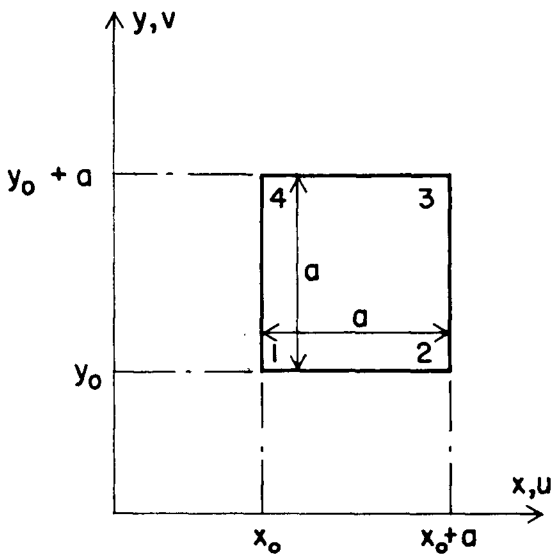


FIG. 3.2.1 ELEMENT IN A STRESS-FIELD

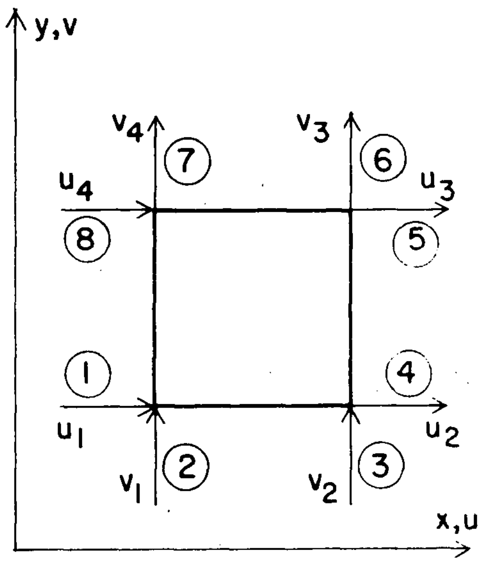


FIG. 3.2.2 ELEMENT-NODAL DISPLACEMENT NOMENCLATURE

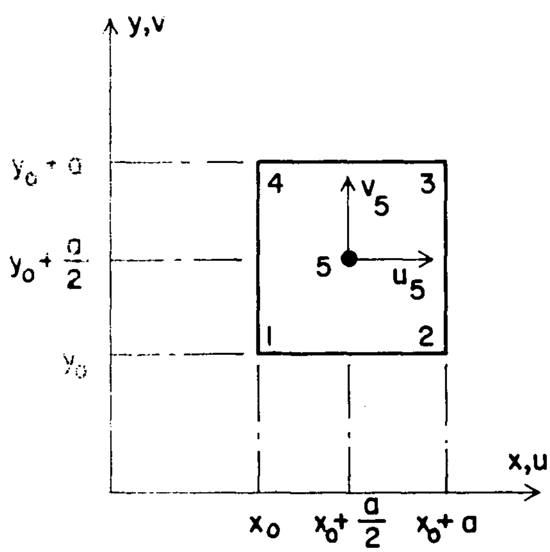


FIG. 3.24.1 ADDITIONAL GENERALIZED DISPLACEMENTS

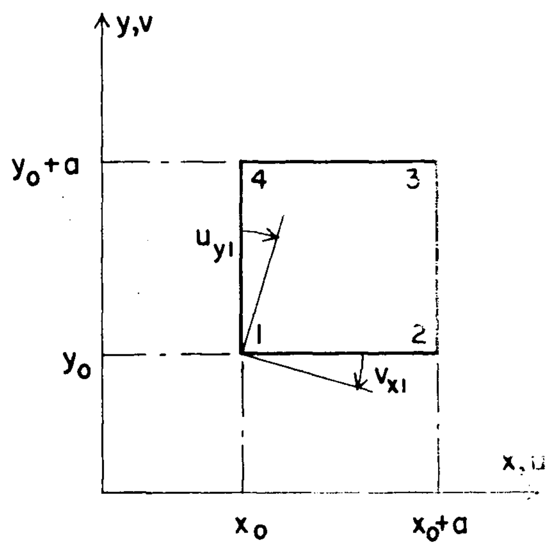


FIG. 3.24.2 CORNER EDGE ROTATIONS AS GENERALIZED DISPLACEMENTS

The stresses are obtained from an Airy's stress function satisfying the Biharmonic equation.

Thus

Airy's stress function $\Phi = \dfrac{a_1 y^2}{2} + \dfrac{a_2 x^2}{2} - a_3 xy$

Therefore

$$\sigma_x = \Phi_{yy} = a_1$$

$$\sigma_y = \Phi_{xx} = a_2$$

$$\tau_{xy} = -\Phi_{xy} = a_3$$

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \mu\sigma_y)$$

$$= \frac{1}{E}(a_1 - \mu a_2) = u_x$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \mu\sigma_x)$$

$$= \frac{1}{E}(a_2 - \mu a_1) = v_y$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$

$$= \frac{2(1+\mu)}{E} a_3 = u_y + v_x$$

whence on integrating

$$u = \frac{1}{E}(a_1 - \mu a_2)\, x + \frac{2(1+\mu)}{E}\, a_3 y + ky + c_1$$

$$v = \frac{1}{E}(a_2 - \mu a_1)\, y - kx + c_2$$

Hence, the nodal displacements may be expressed in terms of the arbitrary constants by inserting the nodal co-ordinates. Then

$$u_1 = (a_1 - \mu a_2)\frac{x_o}{E} + \frac{2(1+\mu)}{E}\, a_3 y_o + ky_o + c_1$$

$$v_1 = (a_2 - \mu a_1)\frac{y_o}{E} - kx_o + c_2$$

$$u_2 = (a_1 - \mu a_2)\frac{(x_o + a)}{E} + \frac{2(1+\mu)\, a_3 y_o}{E} + ky_o + c_1$$

$$v_2 = (a_2 - \mu a_1)\frac{y_o}{E} - k\,(x_o + a) + c_2$$

$$u_3 = (a_1 - \mu a_2)\frac{(x_o + a)}{E} + \frac{2(1+\mu)}{E}\, a_3\,(y_o + a) + k\,(y_o + a) + c_1$$

$$v_3 = (a_2 - \mu a_1)\frac{(y_o + a)}{E} - k\,(x_o + a) + c_2$$

$$u_4 = (a_1 - \mu a_2)\frac{x_o}{E} + \frac{2(1+\mu)}{E}\, a_3\,(y_o + a) + k\,(y_o + a) + c_1$$

$$v_4 = (a_2 - \mu a_1)\frac{(y_o + a)}{E} - kx_o + c_2$$

Now relations are derived between the arbitrary constants and the nodal displacements so that the strain energy may be expressed explicitly in terms of the nodal displacements.

$$\frac{a_1 - \mu a_2}{E} = \frac{u_2 - u_1}{a} = \frac{u_3 - u_4}{a} = \frac{u_2 - u_1 + u_3 - u_4}{2a} = \varepsilon_x$$

$$\frac{a_2 - \mu a_1}{E} = \frac{v_4 - v_1}{a} = \frac{v_3 - v_2}{a} = \frac{v_4 - v_1 + v_3 - v_2}{2a} = \varepsilon_y$$

$$\frac{2(1+\mu)}{E} a_3 = \frac{u_4 - u_1 - v_1 + v_2}{a} = \frac{u_3 - u_2 - v_4 + v_3}{a}$$

$$= \frac{u_4 - u_1 + u_3 - u_2 - v_1 + v_2 - v_4 + v_3}{2a} = \gamma_{xy}$$

Then the strain energy in the element is given by

$$U = \int_0^t \int_{y_o}^{y_o+a} \int_{x_o}^{x_o+a} \left\{ \frac{E}{2(1-\mu^2)} (\varepsilon_x^2 + \varepsilon_y^2 + 2\mu\varepsilon_x\varepsilon_y) + \frac{E}{2(1+\mu)} \frac{1}{2} \gamma_{xy}^2 \right\} dxdydz$$

$$= a^2 t \left\{ \frac{E}{2(1-\mu^2)} (\varepsilon_x^2 + \varepsilon_y^2 + 2\mu\varepsilon_x + \varepsilon_y) + \frac{E}{4(1+\mu)} \gamma_{xy}^2 \right\}$$

Substituting for ε_x, ε_y, γ_{xy} one obtains

$$U = \frac{Et}{8(1-\mu^2)} [(u_2 - u_1 + u_3 - u_4)^2 + (v_4 - v_1 + v_3 - v_2)^2 + 2\mu(u_2 - u_1 + u_3 - u_4)(v_4 - v_1 + v_3 - v_2)]$$

$$+ \frac{Et}{16(1+\mu)} [u_4 - u_1 + u_3 - u_2 - v_1 + v_2 - v_4 + v_3]^2$$

We know that the stiffness influence coefficients are given by

$$K_{ij} = \frac{\partial^2 U}{\partial q_i \partial q_j} \qquad i,j = 1,n$$

where q_i are the generalized displacements.

Here the generalized displacements are the nodal displacements. Hence we obtain

$$K_{11} = \frac{\partial^2 U}{\partial u_1^2} = \frac{\lambda}{24}(18-6\mu) \text{ where } \lambda = \frac{Et}{2(1-\mu^2)}$$

$$K_{21} = \frac{\partial^2 U}{\partial u_1 \partial v_1} = \frac{\lambda}{24}(6 + 6\mu)$$

$$K_{31} = \frac{\partial^2 U}{\partial u_1 \partial v_2} = \frac{\lambda}{24}(-6 + 18\mu)$$

$$K_{41} = \frac{\partial^2 U}{\partial u_1 \partial u_2} = \frac{\lambda}{24}(-6 - 6\mu)$$

$$K_{51} = \frac{\partial^2 U}{\partial u_1 \partial u_3} = \frac{\lambda}{24}(-18 + 6\mu)$$

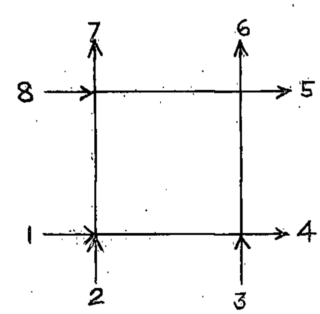
$$K_{61} = \frac{\partial^2 U}{\partial u_1 \partial v_3} = \frac{\lambda}{24}(-6 - 6\mu)$$

$$K_{71} = \frac{\partial^2 U}{\partial u_1 \partial v_4} = \frac{\lambda}{24} \; (6 - 18\mu)$$

$$K_{81} = \frac{\partial^2 U}{\partial u_1 \partial u_4} = \frac{\lambda}{24} \; (6 + 6\mu)$$

The remaining columns may be obtained by permutations of the elements of the first column, and the stiffness matrix is given in Table 3.2.1.1

<u>Table 3.2.1.1</u> - <u>Stiffness Coefficients for (8 x 8) Matrices</u>



$$\begin{array}{cc} u_1 & 1 \\ v_1 & 2 \\ v_2 & 3 \\ u_2 & 4 \\ u_3 & 5 \\ v_3 & 6 \\ v_4 & 7 \\ u_4 & 8 \end{array}
\begin{bmatrix}
K_{11} & & & & & & & \\
K_{21} & K_{11} & & & & & & \\
K_{31} & K_{81} & K_{11} & & & & & \\
K_{41} & K_{71} & -K_{21} & K_{11} & & \text{SYM.} & & \\
K_{51} & K_{61} & -K_{31} & K_{81} & K_{11} & & & \\
K_{61} & K_{51} & K_{41} & -K_{71} & K_{21} & K_{11} & & \\
K_{71} & K_{41} & K_{51} & -K_{61} & K_{31} & K_{81} & K_{11} & \\
K_{81} & K_{31} & -K_{61} & K_{51} & K_{41} & K_{71} & -K_{21} & K_{11}
\end{bmatrix}$$

If rigid body modes are eliminated from this matrix by removing the appropriate three rows and columns, the above matrix is still singular.

3.2.2 Linear σ_x, σ_y and Uniform τ_{xy}

Airy's stress function

$$\Phi = \frac{a_1 x^3}{6} + \frac{a_4 y^3}{6} - c_3 xy$$

$$\sigma_x = \phi_{yy} = a_4 y$$

$$\sigma_y = \phi_{xx} = a_1 x$$

$$\tau_{xy} = -\phi_{xy} = c_3$$

Proceeding, as before, obtaining and integrating the strains,

$$u = \frac{-\mu a_1 x^2}{2E} + \frac{a_4 xy}{E} - ky + \frac{2(1+\mu)}{E} c_3 y - \frac{a_1 y^2}{2E} + c_1$$

$$v = \frac{a_1 xy}{E} - \frac{\mu a_4 y^2}{2E} - \frac{a_4 x^2}{2E} + kx + c_2$$

Again, on expressing the nodal displacements in terms of the constants and x_o, y_o, and subsequently performing algebraic manipulations, one obtains

$$\frac{a_1}{E} = \frac{v_3 - v_4 - v_2 + v_1}{a^2}$$

$$\frac{a_4}{E} = \frac{v_3 - u_4 - u_2 + u_1}{a^2}$$

$$\frac{2(1+\mu)c_3}{E} = \frac{u_4 - u_1 + u_3 - u_2 + v_2 - v_1 + v_3 - v_4}{2a}$$

$$\frac{x_o}{a} = \frac{\mu^2}{2(1-\mu^2)} + \frac{\mu}{2(1-\mu^2)} \quad \frac{u_3-u_4+u_2-u_1}{v_3-v_4-v_2+v_1} + \frac{v_4 - v_1}{(1-\mu^2)(v_3-v_4-v_2+v_1)}$$

$$\frac{y_o}{a} = \frac{\mu^2}{2(1-\mu^2)} + \frac{\mu^2}{2(1-\mu^2)} \quad \frac{v_3 + v_4 - v_2 - v_1}{u_3 - u_4 - u_2 + u_1} + \frac{u_2 - u_1}{(1-\mu^2)(u_3-u_4-u_2+u_1)}$$

Now the strain energy may be expressed as

$$U = \frac{t}{2E} \int_{y_o}^{y_o+a} \int_{x_o}^{x_o+a} (\sigma_x^2 + \sigma_y^2 - 2\mu\sigma_x\sigma_y + 2(1+\mu)\tau_{xy}^2)\, dxdy$$

$$= \frac{a^4 t}{2E} \left\{ \left(\frac{a_1 x_o}{a}\right)^2 + a_1 \left(\frac{a_1 x_o}{a}\right) + \frac{a_1^{\,3}}{3} + \left(\frac{a_4 y_o}{a}\right)^2 + a_4 \left(\frac{a_4 y_o}{a}\right) + \frac{a_4^{\,2}}{3} \right.$$

$$\left. - 2\mu \left(\frac{a_1 x_o}{a} + \frac{a_1}{2}\right) \left(\frac{a_4 y_o}{a} + \frac{a_4}{2}\right) + \frac{2(1+\mu)}{a^2} \quad c_3^{\,2} \right\}$$

Substituting for the arbitrary constants and simplifying one obtains

$$U = \frac{Et}{8(1-\mu^2)} \left[(u_2-u_1+u_3-u_4)^2 + (v_4-v_1+v_3-v_2)^2 + 2\mu(u_2-u_1+u_3-u_4)(v_4-v_1+v_3-v_2) \right]$$

$$+ \frac{Et}{16(1+\mu)} [u_4 - u_1 + u_3 - u_2 + v_2 - v_1 + v_3 - v_4]^2$$

$$+ \frac{Et}{24} \left[(u_3 - u_4 - u_2 + u_1)^2 + (v_3 - v_2 - v_4 + v_1)^2 \right]$$

Again using the formulation

$$K_{ij} = \frac{\partial^2 U}{\partial q_i \partial q_j} \qquad i,j = 1,n$$

The stiffness matrix is of form given in Table 3.2.1.1 with

$$K_{11} = \frac{\lambda}{24}\,(22 - 6\mu - 4\mu^2) \text{ where } \lambda = \frac{Et}{2(1-\mu^2)}$$

$$K_{21} = \frac{\lambda}{24}\,(6 + 6\mu)$$

$$K_{31} = \frac{\lambda}{24}\,(-6 + 18\mu)$$

$$K_{41} = \frac{\lambda}{24}\,(-10 - 6\mu + 4\mu^2)$$

$$K_{51} = \frac{\lambda}{24}\,(-14 + 6\mu - 4\mu^2)$$

$$K_{61} = \frac{\lambda}{24}\,(-6 - 6\mu)$$

$$K_{71} = \frac{\lambda}{24}\,(6 - 18\mu)$$

$$K_{81} = \frac{\lambda}{24}\,(2 + 6\mu + 4\mu^2)$$

This matrix is identical to that derived by Turner et al (1956, p. 823) because the same assumptions have been made on the stress distribution.

Although the derivation given here has been found convenient for obtaining explicit results, if it is desired to conform to the matrix formulation of section 3.1, then the basic matrices and vectors to obtain the stiffness matrix will be given as follows.

In this case a local co-ordinate system is assumed for the element, in which corner 1 of the element (see Fig. 3.2.1) is taken to coincide with the origin.

Then the stress assumptions are written as

$$\sigma_x = a_3 + a_4 y$$

$$\sigma_y = a_2 + a_1 x$$

$$\tau_{xy} = c_3$$

The corresponding displacements over the element are

$$u = \frac{1}{E}(a_3 - \mu a_2)\,x - \frac{\mu a_1 x^2}{2E} + \frac{a_4 xy}{E} - \frac{ky}{E} + \frac{2(1+\mu)c_3 y}{E} - \frac{a_1 y^2}{2E} + \frac{c_1}{E}$$

$$v = \frac{1}{E}(a_2 - \mu a_3)\,y + \frac{a_1 xy}{E} - \frac{\mu a_4 y^2}{2E} - \frac{a_4 x^2}{2E} + \frac{kx}{E} + \frac{c_2}{E}$$

Then, in accordance with the previous formulation

$$\lfloor q \rfloor = \lfloor u_1, v_1, v_2, u_2, u_3, v_3, v_4, u_4 \rfloor$$

$$[A_1] = \begin{array}{c} \begin{matrix} a_1 & a_2 & a_3 & a_4 & c_3 & c_2 & c_1 & k \end{matrix} \\ \begin{bmatrix} 0 & 0 & 1 & y & 0 & 0 & 0 & 0 \\ x & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$[D] = \frac{1}{E} \begin{bmatrix} 1 & -\mu & 0 \\ -\mu & 1 & 0 \\ 0 & 0 & 2(1+\mu) \end{bmatrix}$$

$$[B] = \frac{1}{E} \begin{array}{c} \begin{matrix} a_1 & a_2 & a_3 & a_4 & c_3 & c_2 & c_1 & k \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{-a^2}{2} & 0 & 1 & 0 & a \\ \frac{-\mu a^2}{2} & -\mu a & a & 0 & 0 & 0 & 1 & 0 \\ \frac{-a^2(1+\mu)}{2} & -\mu a & a & a^2 & 2(1+\mu)a & 0 & 1 & -a \\ a^2 & a & -\mu a & \frac{-a^2(1+\mu)}{2} & 0 & 1 & 0 & a \\ 0 & a & -\mu a & \frac{-\mu a^2}{2} & 0 & 1 & 0 & 0 \\ \frac{-a^2}{2} & 0 & 0 & 0 & 2(1+\mu)a & 0 & 1 & -a \end{bmatrix} \end{array}$$

3.2.3 Uniform σ_x, σ_y, Linear τ_{xy}

Airy's stress function

$$\phi = \frac{a_2}{2} x^2 y + \frac{a_3}{2} xy^2$$

$$\sigma_x = \phi_{yy} = a_3 x$$

$$\sigma_y = \phi_{xx} = a_2 y$$

$$\tau_{xy} = -\phi_{xy} = -a_2 x \quad -a_3 y$$

Obtaining and integrating the strains one gets

$$u = \frac{a_3 x^2}{2E} - \frac{\mu a_2 xy}{E} - \frac{(2+\mu) a_3 y^2}{2E} - ky + c_1$$

$$v = \frac{a_2 y^2}{2E} - \frac{\mu a_3 xy}{E} - \frac{(2+\mu) a_2 x^2}{2E} + kx + c_2$$

Again, on expressing the nodal displacements in terms of the constants and x_o, y_o, and subsequently performing algebraic manipulations, one obtains

$$\frac{-\mu a_2}{E} = \frac{u_3 - u_4 - u_2 + u_1}{a^2}$$

$$\frac{-\mu a_3}{E} = \frac{v_3 - v_2 - v_4 + v_1}{a^2}$$

$$\frac{2ax_o + a^2}{E} = a_2 \frac{(X + \mu Y)}{K} \qquad \text{where } X = u_2 - u_1 + u_3 - u_4$$

$$Y = v_4 - v_1 + v_3 - v_2$$

$$\frac{2ay_o + a^2}{E} = a_3 \frac{(\mu X + Y)}{K} \qquad K = a_3 a_2 (1-\mu^2)$$

$$\frac{a_2}{E}(2ax_o+a^2) + \frac{a_3}{E}(2ay_o+a^2) = \frac{(u_4 - u_1 + u_3 - u_2 + v_2 - v_1 + v_3 - v_4)}{-2(1+\mu)}$$

Proceeding as before, the element strain energy may be expressed as

$$U = \frac{a^4 t}{24E}\left\{ (a_2{}^2+a_3{}^2)(3+2\mu)\right\} + \frac{Et(1+\mu)}{4}\left\{ a_2 \frac{(2ax_o+a^2)}{E} + a_3 \frac{(2ay_o+a^2)}{E}\right\}^2$$

$$+ \frac{Et}{8(1-\mu^2)} (X^2 + Y^2 + 2\mu XY)$$

Substituting for arbitrary constants and simplifying

$$U = \frac{Et}{8\ (1-\mu^2)} [\ (u_2 - u_1 + u_3 - u_4)^2 + (v_4 - v_1 + v_3 - v_2)^2 +$$

$$2\mu\ (u_2 - u_1 + u_3 - u_4)\ (v_4 - v_1 + v_3 - v_2)\ ]$$

$$+ \frac{Et}{16(1+\mu)} [\ u_4 - u_1 + u_3 - u_2 + v_2 - v_1 + v_3 - v_4]^2$$

$$+ \frac{Et}{24} \frac{(3+2\mu)}{\mu^2} [\ (u_3 - u_4 - u_2 + u_1)^2 + (v_3 - v_2 - v_4 + v_1)^2]$$

whence the stiffness matrix is of form given in Table 3.2.1.1 with

$$K_{11} = \frac{\lambda}{24}\left(18 - 6\mu + \frac{4\ (3 + 2\mu)(1 - \mu^2)}{\mu^2}\right) \text{ where } \lambda = \frac{Et}{2(1-\mu^2)}$$

$$K_{21} = \frac{\lambda}{24} (6 + 6\mu)$$

$$K_{31} = \frac{\lambda}{24} (-6 + 18\mu)$$

$$K_{41} = \frac{\lambda}{24}\left(-6 - 6\mu - \frac{4(3 + 2\mu)(1 - \mu^2)}{\mu^2}\right)$$

$$K_{51} = \frac{\lambda}{24}\left(-18 + 6\mu + \frac{4(3 + 2\mu)(1 - \mu^2)}{\mu^2}\right)$$

$$K_{61} = \frac{\lambda}{24}(-6 - 6\mu)$$

$$K_{71} = \frac{\lambda}{24}(6 - 18\mu)$$

$$K_{81} = \frac{\lambda}{24}\left(6 + 6\mu - \frac{4(3 + 2\mu)(1 - \mu^2)}{\mu^2}\right)$$

Gallagher et al (1962, pp. 27-29) have discussed the derivation of a stiffness matrix on the same basis. However, they left the results in the form of a matrix expression to be evaluated by the computer. Here the matrix is given explicitly.

3.2.4 Linear σ_x, σ_y, τ_{xy}

Airy's stress function

$$\Phi = \frac{a_1 x^3}{6} + \frac{a_2 x^2 y}{2} + \frac{a_3 x y^2}{2} + \frac{a_4 y^3}{6}$$

$$\sigma_x = \phi_{yy} = a_3 x + a_4 y$$

$$\sigma_y = \phi_{xx} = a_1 x + a_2 y$$

$$\tau_{xy} = -\phi_{xy} = -(a_2 x + a_3 y)$$

Obtaining and integrating strains,

$$u = (a_3 - \mu a_1)\frac{x^2}{2E} + (a_4 - \mu a_2)\frac{xy}{E} - \{a_1 + (2+\mu)\, a_3\}\frac{y^2}{2E} - ky + c_1$$

$$v = (a_2 - \mu a_4)\frac{y^2}{2E} + (a_1 - \mu a_3)\frac{xy}{E} - \{a_4 + (2+\mu)\, a_2\}\frac{x^2}{2E} + kx + c_2$$

from which

$$u_y = (a_4 - \mu a_2)\frac{x}{E} - \{a_1 + (2+\mu)\, a_3\}\frac{y}{E} - k$$

$$v_x = (a_1 - \mu a_3)\frac{y}{E} - \{a_4 + (2+\mu)\, a_2\}\frac{x}{E} + k$$

u_y and v_x represent rotations of lines parallel to the co-ordinate axes. They may be used as additional generalized displacements and will describe the rotations of the element edges.

Proceeding with the matrix derivation, the following results are obtained on expressing the translational nodal displacements in terms of the constants, and by performing algebraic manipulations.

$$\frac{a_4 - \mu a_2}{E} = \frac{u_3 - u_4 - u_2 + u_1}{a^2}$$

$$\frac{a_1 - \mu a_3}{E} = \frac{v_3 - v_2 - v_4 + v_1}{a^2}$$

$$\frac{2ax_o + a^2}{E} = \frac{X(a_2 - \mu a_4) - Y(a_4 - \mu a_2)}{K} \qquad \text{where}$$

$$X = u_2 - u_1 + u_3 - u_4$$

$$\frac{2ay_o + a^2}{E} = \frac{X(a_1 - \mu a_3) - Y(a_3 - \mu a_1)}{-K} \qquad Y = v_4 - v_1 + v_3 - v_2$$

$$K = (a_3 a_2 - a_1 a_4)(1-\mu^2)$$

$$a_2 \frac{(2ax_o + a^2)}{E} + a_3 \frac{(2ay_o + a^2)}{E} = \frac{(u_4 - u_1 + u_3 - u_2 + v_2 - v_1 + v_3 - v_4)}{-2(1 + \mu)}$$

Again the element strain energy may be expressed as

$$U = \frac{a^4 t}{24E} \{(a_1 - \mu a_3)^2 + (a_4 - \mu a_2)^2 + (a_2{}^2 + a_3{}^2)(3 + 2\mu - \mu^2)\}$$

$$+ \frac{Et(1+\mu)}{4} \left\{ a_2 \frac{(2ax_o + a^2)}{E} + a_3 \frac{(2ay_o + a^2)}{E} \right\}^2$$

$$+ \frac{Et}{8(1-\mu^2)} (X^2 + Y^2 + 2\mu XY)$$

Substituting for arbitrary constants and simplifying

$$U = \frac{Et}{8(1-\mu^2)} [(u_2 - u_1 + u_3 - u_4)^2 + (v_4 - v_1 + v_3 - v_2)^2 +$$

$$2\mu (u_2 - u_1 + u_3 - u_4)(v_4 - v_1 + v_3 - v_2)]$$

$$+ \frac{Et}{16(1+\mu)} [u_4 - u_1 + u_3 - u_2 + v_2 - v_1 + v_3 - v_4]^2$$

$$+ \frac{Et}{24} [(u_3 - u_4 - u_2 + u_1)^2 + (v_3 - v_2 - v_4 + v_1)^2]$$

$$+ \frac{a^4 t}{24E} (3 - \mu)(1 + \mu) [a_2{}^2 + a_3{}^2]$$

Note that it is not possible to evaluate the last term of the strain energy expressions in terms of the eight nodal translations.

Two additional generalized displacements are required. These may be supplied to the element in a variety of ways.

The translations of the point of intersection of the diagonals may be used, as in Fig. 3.2.4.1.

The values of u_5 and v_5 are obtained by inserting the co-ordinates of the nodal point 5 in the expressions for the displacements u and v, obtained by integrating the strains.

Then, by performing algebraic manipulations, the value of

$\frac{a^4 t}{24E} \; (3 - \mu)(1 + \mu) \; [a_2{}^2 + a_3{}^2]$ is obtained in terms of the ten nodal displacements, and the value of the element strain energy is given by

$$U = \frac{Et}{8(1-\mu^2)} \Big[(u_2 - u_1 + u_3 - u_4)^2 + (v_4 - v_1 + v_3 - v_2)^2$$

$$+ 2\mu \, (u_2 - u_1 + u_3 - u_4)(v_4 - v_1 + v_3 - v_2) \Big]$$

$$+ \frac{Et}{16(1+\mu)} [u_4 - u_1 + u_3 - u_2 + v_2 - v_1 + v_3 - v_4]^2$$

$$+ \frac{Et}{24} \Big[(u_3 - u_4 - u_2 + u_1)^2 + (v_3 - v_2 - v_4 + v_1)^2 \Big]$$

$$+ \frac{Et}{24} \frac{(3-\mu)}{(1+\mu)} \Big[\frac{\{8v_5 - 2(v_1+v_2+v_3+v_4)\}^2}{(1 + \mu)^2} - \frac{2\{8v_5 - 2(v_1+v_2+v_3+v_4)\}(u_3-u_2-u_4+u_1)}{(1 + \mu)}$$

$$+ \frac{\{8u_5 - 2(u_1+u_2+u_3+u_4)\}^2}{(1 + \mu)^2} - \frac{2\{8u_5 - 2(u_1+u_2+u_3+u_4)\}(v_3-v_2-v_4+v_1)}{(1 + \mu)}$$

$$+ (u_3 - u_2 - u_4 + u_1)^2 + (v_3 - v_2 - v_4 + v_1)^2 \Big]$$

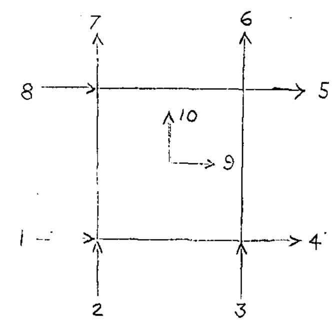
On using the formulae

$$K_{ij} = \frac{\partial^2 U}{\partial q_i \partial q_j} \qquad i,j = 1,n$$

a stiffness matrix of form given in Table 3.2.4.1 is obtained,

<u>Table 3.2.4.1</u> - <u>Stiffness Coefficients for (10 x 10) Matrix using Interior Nodal Translations</u>

$$\begin{array}{ll} u_1 & 1 \\ v_1 & 2 \\ v_2 & 3 \\ u_2 & 4 \\ u_3 & 5 \\ v_3 & 6 \\ v_4 & 7 \\ u_4 & 8 \\ u_5 & 9 \\ v_5 & 10 \end{array} \left[\begin{array}{cccccccccc} K_{11} & & & & & & & & & \\ K_{21} & K_{11} & & & & & & & & \\ K_{31} & K_{81} & K_{11} & & & & & & & \\ K_{41} & K_{71} & -K_{21} & K_{11} & & \text{SYM.} & & & & \\ K_{51} & K_{61} & -K_{31} & K_{81} & K_{11} & & & & & \\ K_{61} & K_{51} & K_{41} & -K_{71} & K_{21} & K_{11} & & & & \\ K_{71} & K_{41} & K_{51} & -K_{61} & K_{31} & K_{81} & K_{11} & & & \\ K_{81} & K_{31} & -K_{61} & K_{51} & K_{41} & K_{71} & -K_{21} & K_{11} & & \\ K_{91} & K_{10,1} & -K_{10,1} & K_{91} & K_{91} & K_{10,1} & -K_{10,1} & K_{91} & K_{99} & \\ K_{10,1} & K_{91} & K_{91} & -K_{10,1} & K_{10,1} & K_{91} & K_{91} & -K_{10,1} & K_{10,9} & K_{99} \end{array} \right]$$



with the coefficients given by

$$K_{11} = \frac{\lambda}{24}\left[34 - 22\mu + \frac{16(3 - 4\mu + \mu^2)}{(1+\mu)^2} \right] \quad \text{where} \quad \lambda = \frac{Et}{2(1-\mu^2)}$$

$$K_{21} = \frac{\lambda}{24}\left[6 + 6\mu + \frac{16\ (3 - 4\mu + \mu^2)}{(1+\mu)} \right]$$

$$K_{31} = \frac{\lambda}{24} [- 6 + 18\mu]$$

$$K_{41} = \frac{\lambda}{24} [- 22 + 10\mu + \frac{16 (3 - 4\mu + \mu^2)}{(1 + \mu)^2}]$$

$$K_{51} = \frac{\lambda}{24} [- 2 - 10\mu + \frac{16 (3 - 4\mu + \mu^2)}{(1 + \mu)^2}]$$

$$K_{61} = \frac{\lambda}{24} [- 6 - 6\mu + \frac{16 (3 - 4\mu + \mu^2)}{(1 + \mu)}]$$

$$K_{71} = \frac{\lambda}{24} [6 - 18 \mu]$$

$$K_{81} = \frac{\lambda}{24} [-10 + 22\mu + \frac{16 (3 - 4\mu + \mu^2)}{(1 + \mu)^2}]$$

$$K_{91} = \frac{\lambda}{24} [\frac{-64 (3 - 4\mu + \mu^2)}{(1 + \mu)^2}]$$

$$K_{10,1} = \frac{\lambda}{24} [\frac{-32 (3 - 4\mu + \mu^2)}{(1 + \mu)}]$$

$$K_{99} = \frac{\lambda}{24} [\frac{256 (3 - 4\mu + \mu^2)}{(1 + \mu)^2}]$$

$$K_{10,9} = 0$$

Another stiffness matrix may be obtained by choosing the additional generalized displacements as edge rotations at nodal point 1 as shown in Fig. 3.2.4.2.

The strain energy expression obtained, in this case, is given by

$$U = \frac{Et}{8(1-\mu^2)} \left[(u_2 - u_1 + u_3 - u_4)^2 + (v_4 - v_1 + v_3 - v_2)^2 + \right.$$

$$\left. 2\mu \, (u_2 - u_1 + u_3 - u_4)\,(v_4 - v_1 + v_3 - v_2) \right]$$

$$+ \frac{Et}{16(1+\mu)} \left[u_4 - u_1 + u_3 - u_2 + v_2 - v_1 + v_3 - v_4 \right]^2$$

$$+ \frac{Et}{24} \left[(u_3 - u_4 - u_2 + u_1)^2 + (v_3 - v_2 - v_4 + v_1)^2 \right]$$

$$+ \frac{Eta\,(3-\mu)}{24(1+\mu)} \left[\left\{ vx_1 - \frac{(v_2 - v_1)}{a} - \frac{(u_3 - u_4 - u_2 + u_1)}{2a} \right\}^2 + \right.$$

$$\left. \left\{ uy_1 - \frac{(u_4 - u_1)}{a} - \frac{(v_3 - v_2 - v_4 + v_1)}{2a} \right\}^2 \right]$$

On using the formulae

$$K_{ij} = \frac{\partial^2 U}{\partial q_i \partial q_j} \qquad i,j = 1,n$$

the stiffness matrix obtained is of form given in Table 3.2.4.2 with

$$K_{11} = \frac{\lambda}{24} \; [37 - 26\mu + \mu^2] \qquad \text{where} \qquad \lambda = \frac{Et}{2(1-\mu^2)}$$

$$K_{21} = \frac{\lambda}{24} \; [-6 + 22\mu - 4\mu^2]$$

$$K_{31} = \frac{\lambda}{24} \; [6 + 2\mu + 4\mu^2]$$

$$K_{41} = \frac{\lambda}{24} \; [-13 - 2\mu + 3\mu^2]$$

$$K_{51} = \frac{\lambda}{24} \; [-11 + 2\mu - 3\mu^2]$$

$$K_{61} = \frac{\lambda}{24} \left[-12 + 2\mu - 2\mu^2\right]$$

$$K_{71} = \frac{\lambda}{24} \left[12 - 26\mu + 2\mu^2\right]$$

$$K_{81} = \frac{\lambda}{24} \left[-13 + 26\mu - \mu^2\right]$$

$$K_{91} = \frac{\lambda a}{24} \left[12 - 16\mu + 4\mu^2\right]$$

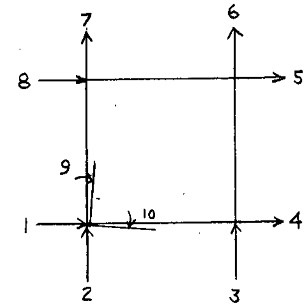
$$K_{10,1} = \frac{\lambda a}{24} \left[-6 + 8\mu - 2\mu^2\right]$$

$$K_{99} = \frac{\lambda a^2}{24} \left[12 - 16\mu + 4\mu^2\right]$$

$$K_{10,9} = 0$$

Table 3.2.4.2 - Stiffness Coefficients for (10 x 10) Matrix Using Corner Edge Rotations

$$
\begin{array}{ll}
u_1 & 1 \\
v_1 & 2 \\
v_2 & 3 \\
u_2 & 4 \\
u_3 & 5 \\
v_3 & 6 \\
v_4 & 7 \\
u_4 & 8 \\
u_{y1} & 9 \\
v_{x1} & 10
\end{array}
\left[
\begin{array}{cccccccccc}
K_{11} & & & & & & & & & \\
K_{21} & K_{11} & & & & & & & & \\
K_{31} & K_{81} & K_{11} & & & & & & & \\
K_{41} & K_{71} & K_{61} & K_{44} & & \text{SYM.} & & & & \\
K_{51} & K_{61} & K_{71} & K_{54} & K_{44} & & & & & \\
K_{61} & K_{51} & K_{41} & K_{64} & K_{74} & K_{44} & & & & \\
K_{71} & K_{41} & K_{51} & K_{74} & K_{64} & K_{54} & K_{44} & & & \\
K_{81} & K_{31} & K_{21} & K_{51} & K_{41} & K_{71} & K_{61} & K_{11} & & \\
K_{91} & K_{10,1} & -K_{10,1} & K_{94} & K_{94} & K_{10,1} & -K_{10,1} & -K_{91} & K_{99} & \\
-K_{10,1} & -K_{91} & K_{91} & K_{10,1} & -K_{10,1} & K_{94} & K_{94} & K_{10,1} & K_{10,9} & K_{99}
\end{array}
\right]
$$



$$K_{44} = \frac{\lambda}{24} \left[25 - 10\mu - 3\mu^2\right]$$

$$K_{54} = \frac{\lambda}{24} \left[-1 + 10\mu + 3\mu^2\right]$$

$$K_{64} = \frac{\lambda}{24} \left[-6 + 18\mu\right]$$

$$K_{74} = \frac{\lambda}{24} \left[6 + 6\mu\right] \qquad K_{94} = 0$$

3.2.5 Hyperbolic σ_x, σ_y, Parabolic τ_{xy}

Airy's stress function

$$\Phi = \frac{b_4 x^3 y}{6} + \frac{d_4 x y^3}{6}$$

$$\sigma_x = \phi_{yy} = d_4 xy$$

$$\sigma_y = \phi_{xx} = b_4 xy$$

$$\tau_{xy} = -\phi_{xy} = -\frac{b_4 x^2}{2} - \frac{d_4 y^2}{2}$$

Obtaining and integrating strains

$$u = (d_4 - \mu b_4)\frac{x^2 y}{2E} - \{b_4 + (2+\mu)\, d_4\}\frac{y^3}{6E} - ky + c_1$$

$$v = (b_4 - \mu d_4)\frac{x y^2}{2E} - \{d_4 + (2+\mu)\, b_4\}\frac{x^3}{6E} + kx + c_2$$

From which

$$u_y = (d_4 - \mu b_4)\frac{x^2}{2E} - \{b_4 + (2+\mu)\, d_4\}\frac{y^2}{2E} - k$$

$$v_x = (b_4 - \mu d_4)\frac{y^2}{2E} - \{d_4 + (2+\mu)\, b_4\}\frac{x^2}{2E} + k$$

Apart from the eight translational nodal displacements, four rotations u_{y1}, v_{x2}, u_{y3} and v_{x4} are chosen as the generalized displacements of the element. (Here also a choice is available).

As before, the displacements are expressed in terms of the arbitrary constants, and algebraic manipulations give the following relations.

$$(d_4 - \mu b_4)\quad (2ax_o + a^2)\ \frac{a}{2E} \quad = \quad u_3 - u_4 - u_2 + u_1$$

$$(b_4 - \mu d_4)\quad (2ay_o + a^2)\ \frac{a}{2E} \quad = \quad v_3 - v_2 - v_4 + v_1$$

$$(d_4 - \mu b_4)\quad \frac{(2ax_o + a^2)(2ay_o + a^2)}{2Ea} = u_3 - u_4 + u_2 - u_1$$

$$(b_4 - \mu d_4)\quad \frac{(2ax_o + a^2)(2ay_o + a^2)}{2Ea} = v_3 - v_2 + v_4 - v_1$$

$$\frac{2(1+\mu)}{E}\ \{b_4(ax_o{}^2 + a^2x_o + \frac{a^3)}{2} + d_4(ay_o{}^2 + a^2y_o + \frac{a^3)\}}{2} = -a(u_{y1}+u_{y3}+v_{x2}+v_{x4})$$

$$d_4 = \frac{6E}{a^2(3+\mu)(1+\mu)}\ [(2+\mu)\ \{\frac{u_4-u_1+u_3-u_2}{a} - (u_{y1}+u_{y3})\} - \{\frac{v_2-v_1+v_3-v_4}{a} - (v_{x2}+v_{x4})\}\ ]$$

$$(1+\mu)\ d_4\ \ \frac{(2ay_o + a^2)}{E} = u_{y1} - u_{y3} - \frac{(u_4 - u_1 - u_3 + u_2)}{a} - \frac{(v_3 - v_2 - v_4 + v_1)}{a}$$

$$(1+\mu)\ b_4\ \ \frac{(2ax_o + a^2)}{E} = -(V_{x2} - v_{x4}) - \frac{(u_3 - u_4 - u_2 + u_1)}{a} + \frac{(v_2 - v_1 - v_3 + v_4)}{a}$$

$$d_4(\frac{x_o}{a} + \frac{1}{2}) = \frac{E\mu}{2a^2(1+\mu)}\ [\ -(v_{x2} - v_{x4}) - \frac{(u_3-u_4-u_2+u_1)}{a} + \frac{(v_2-v_1-v_3+v_4)}{a}\ ]$$

$$+ \ \frac{E}{a^3}\ (u_3 - u_4 - u_2 + u_1)$$

$$b_4 = \frac{6E}{a^2(3+\mu)(1+\mu)}\ [(2+\mu)\ \{\frac{v_2-v_1+v_3-v_4}{a} - (v_{x2}+v_{x4}\} - \{\frac{u_4-u_1+u_3-u_2}{a} - (u_{y1}+u_{y3})\}\ ]$$

$$b_4 \left(\frac{y_o}{a} + \frac{1}{2}\right) = \frac{E\mu}{2a^2(1+\mu)} \left[u_{y1} - u_{y3} - \frac{(u_4 - u_1 - u_3 + u_2)}{a} - \frac{(v_3 - v_2 - v_4 + v_1)}{a}\right]$$

$$+ \frac{E}{a^3} (v_3 - v_2 - v_4 + v_1)$$

Now the strain energy in the element is given by

$$U = \frac{t}{2E} \int_{y_o}^{y_o+a} \int_{x_o}^{x_o+a} (\sigma_x^2 + \sigma_y^2 - 2\mu\sigma_x\sigma_y + 2(1+\mu)\,\tau_{xy}^2)\,dxdy$$

which on substitution and integration yields

$$U = \frac{ta^6}{2(1-\mu^2)E} \left[(b_4 - \mu d_4)^2 + (d_4 - \mu b_4)^2 + 2\mu\,(b_4 - \mu d_4)(d_4 - \mu b_4) \right] \times \left(\frac{x_o}{a} + \frac{1}{2}\right)^2 \left(\frac{y_o}{a} + \frac{1}{2}\right)^2$$

$$+ \frac{ta^6}{24E} (b_4^2 + d_4^2) \left\{ \left(\frac{x_o}{a} + \frac{1}{2}\right)^2 + \left(\frac{y_o}{a} + \frac{1}{2}\right)^2 \right\}$$

$$- \frac{ta^6}{12E} (1+2\mu)\, b_4 d_4 \left\{ \left(\frac{x_o}{a} + \frac{1}{2}\right)^2 + \left(\frac{y_o}{a} + \frac{1}{2}\right)^2 \right\}$$

$$+ \frac{ta^6}{4E} (1+\mu) \left[b_4 \left\{ \left(\frac{x_o}{a}\right)^2 + \left(\frac{x_o}{a}\right) + \frac{1}{2} \right\} + d_4 \left\{ \left(\frac{y_o}{a}\right)^2 + \left(\frac{y_o}{a}\right) + \frac{1}{2} \right\} \right]^2$$

$$+ \frac{ta^6}{2E} \frac{(-13-18\mu)}{720} (b_4^2 + d_4^2)$$

$$+ \frac{ta^6}{2E} \frac{(-4-5\mu)}{72} b_4 d_4$$

Substituting for the arbitrary constants and simplifying,

$$U = \frac{Et}{8(1-\mu^2)}\left[(u_3 - u_4 + u_2 - u_1)^2 + (v_3 - v_2 + v_4 - v_1)^2 + 2\mu\,(u_3 - u_4 + u_2 - u_1) \times (v_3 - v_2 + v_4 - v_1) \right]$$

$$+ \frac{Et}{24}\left[(u_3 - u_4 - u_2 + u_1)^2 + (v_3 - v_2 - v_4 + v_1)^2 \right]$$

$$+ \left\{ -(v_{x2} - v_{x4}) - \frac{(u_3 - u_4 - u_2 + u_1)}{a} + \frac{(v_2 - v_1 - v_3 + v_4)}{a} \right\}^2 \frac{Eta^2(1-3\mu)}{96\ (1+\mu)}$$

$$+ \left\{ (u_{y1} - u_{y3}) - \frac{(u_4 - u_1 - u_3 + u_2)}{a} - \frac{(v_3 - v_2 - v_4 + v_1)}{a} \right\}^2 \frac{Eta^2(1-3\mu)}{96\ (1+\mu)}$$

$$+ \left\{ -(v_{x2} - v_{x4}) - \frac{(u_3 - u_4 - u_2 + u_1)}{a} + \frac{(v_2 - v_1 - v_3 + v_4)}{a} \right\} (u_3 - u_4 - u_2 + u_1) \times \left[\frac{-Eta}{24} \right]$$

$$+ \left\{ (u_{y1} - u_{y3}) - \frac{(u_4 - u_1 - u_3 + u_2)}{a} - \frac{(v_3 - v_2 - v_4 + v_1)}{a} \right\} (v_3 - v_2 - v_4 + v_1) \times \left[\frac{-Eta}{24} \right]$$

$$+ \frac{Eta^2}{16(1+\mu)} (u_{y1} + u_{y3} + v_{x2} + v_{x4})^2$$

$$+ \left\{ \frac{(v_2 - v_1 + v_3 - v_4)}{a} - (v_{x2} + v_{x4}) \right\}^2 \left[\frac{Eta^2\ (15-2\mu-35\mu^2-18\mu^3)}{40(3+\mu)^2(1+\mu)^2} \right]$$

$$+ \left\{ \frac{(u_4 - u_1 + u_3 - u_2)}{a} - (u_{y1} + u_{y3}) \right\}^2 \left[\frac{Eta^2\ (15-2\mu-35\mu^2-18\mu^3)}{40(3+\mu)^2(1+\mu)^2} \right]$$

$$+ \left\{ \frac{(v_2 - v_1 + v_3 - v_4)}{a} - (v_{x2} + v_{x4}) \right\} \left\{ \frac{(u_4 - u_1 + u_3 - u_2)}{a} - (u_{y1} + u_{y3}) \right\} \times$$

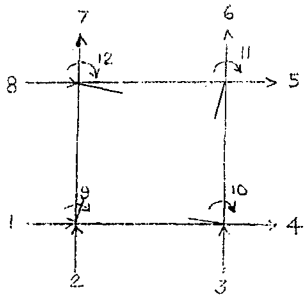
$$\frac{[-96-214\mu-168\mu^2-50\mu^3]}{40(3+\mu)^2(1+\mu)^2} \quad Eta^2$$

On applying the formulae

$$K_{ij} = \frac{\partial^2 U}{\partial q_i \partial q_j} \qquad i,j = 1,n$$

a stiffness matrix of form given in Table 3.2.5.1 is obtained.

Table 3.2.5.1 - Stiffness Coefficients for (12 x 12) Matrix



$$
\begin{array}{ll}
u_1 & 1 \\
v_1 & 2 \\
v_2 & 3 \\
u_2 & 4 \\
u_3 & 5 \\
v_3 & 6 \\
v_4 & 7 \\
u_4 & 8 \\
u_{y1} & 9 \\
-v_{x2} & 10 \\
u_{y3} & 11 \\
-v_{x4} & 12
\end{array}
\begin{bmatrix}
K_{11} & & & & & & & & & & & \\
K_{21} & K_{11} & & & & & & & & & & \\
K_{31} & K_{81} & K_{11} & & & & & & & & & \\
K_{41} & K_{71} & -K_{21} & K_{11} & & & & & & & & \\
K_{51} & K_{61} & -K_{31} & K_{81} & K_{11} & & & \text{SYM.} & & & & \\
K_{61} & K_{51} & K_{41} & -K_{71} & K_{21} & K_{11} & & & & & & \\
K_{71} & K_{41} & K_{51} & -K_{61} & K_{31} & K_{81} & K_{11} & & & & & \\
K_{81} & K_{31} & -K_{61} & K_{51} & K_{41} & K_{71} & -K_{21} & K_{11} & & & & \\
K_{91} & K_{12,1} & -K_{12,1} & K_{11,1} & -K_{11,1} & -K_{10,1} & K_{10,1} & -K_{91} & K_{99} & & & \\
-K_{10,1} & -K_{91} & K_{91} & -K_{12,1} & K_{12,1} & K_{11,1} & -K_{11,1} & K_{10,1} & -K_{10,9} & K_{99} & & \\
K_{11,1} & K_{10,1} & -K_{10,1} & K_{91} & -K_{91} & -K_{12,1} & K_{12,1} & -K_{11,1} & K_{11,9} & -K_{10,9} & K_{99} & \\
-K_{12,1} & -K_{11,1} & K_{11,1} & -K_{10,1} & K_{10,1} & K_{91} & -K_{91} & K_{12,1} & -K_{12,9} & K_{11,9} & -K_{10,9} & K_{99}
\end{bmatrix}
$$

with

$$K_{11} = \frac{\lambda}{24}\,[22-8\mu-2\mu^2+ A]$$

$$K_{21} = \frac{\lambda}{24}\,[12\mu- B]$$

$$K_{31} = \frac{\lambda}{24}\,[12\mu + B]$$

where $\lambda = \dfrac{Et}{2(1-\mu^2)}$

$$A = \frac{48(1-\mu)(15-2\mu-35\mu^2-18\mu^3)}{20(3+\mu)^2(1+\mu)}$$

$$K_{41} = \frac{\lambda}{24} \quad [-22+8\mu+2\mu^2+ A]$$

$$K_{51} = \frac{\lambda}{24} \quad [-2-8\mu-2\mu^2-A] \qquad B = \frac{48(1-\mu)(48+107\mu+84\mu^2+25\mu^3)}{20(3+\mu)^2(1+\mu)}$$

$$K_{61} = \frac{\lambda}{24} \quad [-12\mu+ B]$$

$$K_{71} = \frac{\lambda}{24} \quad [-12\mu- B]$$

$$K_{81} = \frac{\lambda}{24} \quad [2+8\mu+2\mu^2- A]$$

$$K_{91} = \frac{\lambda a}{24} \quad [1-4\mu+3\mu^2+ A] \qquad K_{9,9} = \frac{\lambda a^2}{24} \quad [7-10\mu+3\mu^2+ A]$$

$$K_{10,1} = \frac{\lambda a}{24} \quad [3-4\mu+\mu^2- B] \qquad K_{10,9} = \frac{\lambda a^2}{24} \quad [6-6\mu- B]$$

$$K_{11,1} = \frac{\lambda a}{24} \quad [-1+4\mu-3\mu^2+ A] \qquad K_{11,9} = \frac{\lambda a^2}{24} \quad [5-2\mu-3\mu^2+ A]$$

$$K_{12,1} = \frac{\lambda a}{24} \quad [-3+4\mu-\mu^2- B] \qquad K_{12,9} = \frac{\lambda a^2}{24} \quad [6-6\mu- B]$$

This matrix, along with the other matrices which have been developed, will be examined and evaluated in Chapter 6.

In addition a stiffness matrix based on linear edge displacements is derived in Appendix B, as this matrix is of interest in the discussions of Chapter VI.

CHAPTER IV

COMPARISON AND EVALUATION OF STIFFNESS MATRICES

4.1 Strain Energy Criterion for Element Matrix Comparisons

A theoretical basis for comparing stiffness matrices for all classes of finite elements (i.e., elements for plane stress, plate-bending, shells, etc.,) is the strain energy within a finite element under the same nodal loads. This basis is compatible with the recent development of the stiffness method for bounding elastic behavior wherein the minimal energy theorems have been applied to provide bounds on the strain energy. On this basis of strain energy, the "best" stiffness matrix from an available set is defined as the one which will yield the closest approximation to the strain energy of deformation. The results of such a choice will be of general validity.

Bases other than the strain energy of deformation, such as the maximum stress or the maximum displacement, may be more desirable from a practical viewpoint, but no results of general validity are available for bases other than strain energy.

The strain energy comparisons will be made by examination of one of two special difference matrices, as described below.

Inverse Difference Matrix

The strain energy of an element can be expressed as

$$U = \frac{1}{2} \lfloor q \rfloor \; [K] \; \{q\}$$

Now the equations of equilibrium can be written as

$$[\,K\,]\ \{\,q\,\} = \{\,p\,\}$$

where { p } is a nodal load vector. Therefore

$$\{\,q\,\} = [\,K^{-1}]\ \{\,p\,\}$$

and hence

$$U = \frac{1}{2} \lfloor p \rfloor\ [\,K^{-1}]\ \{\,p\,\}$$

Therefore, if two natural stiffness matrices K_1 and K_2 are available, then the difference in strain energy is

$$U_1 - U_2 = \frac{1}{2} \lfloor p \rfloor [K_1^{-1}]\ \{\,p\,\} - \frac{1}{2} \lfloor p \rfloor\ [K_2^{-1}]\ \{\,p\,\}$$

$$= \frac{1}{2} \lfloor p \rfloor [K_1^{-1} - K_2^{-1}]\ \{\,p\,\}$$

$$= \frac{1}{2} \lfloor p \rfloor\ [\,D\,]\ \{\,p\,\}$$

where $[\,D\,] = [K_1^{-1} - K_2^{-1}]$ may be called the Inverse Difference Matrix.

Therefore, on examining the quadratic form of D, it may be ascertained whether the stiffness matrix K_1 will provide greater, equal or lesser strain energy in the element than K_2, under arbitrary loading p. The properties of D may be obtained by finding out its eigenvalues. The results of such a study are indicated in Table 4.1.1

Table 4.1.1 - Examination of the Inverse Difference Matrix D

Quadratic form of D	Eigenvalues of D	Strain Energy Comparison
positive definite	all positive	$U_1 > U_2$
positive semi-definite	some positive, others zero	$U_1 \geqq U_2$
negative-definite	all negative	$U_1 < U_2$
negative semi-definite	some negative, others zero	$U_1 \leqq U_2$
indefinite	some positive, others negative	$U_1 \gtrless U_2$

If D is semi-definite, its vectors may be examined for linear dependence and hence the nodal loads causing equality of strain energy determined. An example of this is given in Chapter VI.

If D is indefinite, the loads making D positive or negative correspond to the eigenvectors related to these positive and negative eigenvalues. It may be possible in this case to choose a matrix on the basis of the rank and index of D from a probabilistic viewpoint. This aspect requires numerical experimentation to show the quantitative effects of the different sizes of positive and negative eigenvalues, and the loads defined by the corresponding eigenvectors.

Stiffness Difference Matrix

Now it will be shown with the help of a theorem which is enunciated and proved below, that the properties of the Inverse Difference Matrix, $D \equiv [K_1^{-1} - K_2^{-1}]$ may be obtained from a study of the properties of $[S] \equiv [K_1 - K_2]$ where S may be called the Stiffness Difference Matrix. This procedure affords a simplification by eliminating the extra computational work and numerical error in the inversion of matrices.

Theorem

If $[K_1]$ and $[K_2]$ are real symmetric positive definite matrices, and if $[K_1 - K_2]$ is positive definite then $[K_1^{-1} - K_2^{-1}]$ is negative definite.

Proof

Appeal is first made to a theorem of matrix algebra (Hohn (1958), p. 266) which states that

"If [A] and [B] are real matrices of order n, and if [A] is symmetric and [B] is positive definite, then there exists a real non-singular matrix [V] such that $[V^T]$ [A] [V] is diagonal and $[V^T]$ [B] [V] is the identity matrix".

Therefore, we know from the assumptions of the theorem enunciated above that there exists a non-singular matrix [V] such that

$$[V^T]\ [K_1]\ [V] = [d] \qquad (1)$$

where [d] = a diagonal matrix

and
$$[V^T]\ [K_2]\ [V] = [I] \qquad (2)$$

where [I] = the identity matrix

Now $[K_1 - K_2]$ is positive definite by assumption. Therefore $[V^T][K_1 - K_2][V]$ is also positive definite, because a congruent transformation maintains the form of $[K_1 - K_2]$.

That is

$$[V^T]\ [K_1]\ [V] \quad - [V^T]\ [K_2]\ [V] \quad \text{is pos. def.}$$

whence on substituting from (1) and (2)

$$[d] \quad - \quad [I] \quad \text{is pos. def.}$$

Therefore, $d_{ii} > 1$ for all i (3)

Also from (1)

$$[K_1] \;=\; [V^{T^{-1}}]\ [d]\ [V^{-1}]$$

hence $[K_1^{-1}] = [V]\ [d^{-1}]\ [V^T]$ (4)

Similarly from (2)

$$[K_2^{-1}] \quad = \quad [V]\ [I^{-1}]\ [V^T]$$

Hence

$$[K_1^{-1} - K_2^{-1}] \quad = \quad [V]\ [d^{-1}]\ [V^T] \quad - \quad [V]\ [I]\ [V^T]$$

$$= \quad [V]\ [d^{-1} - I] \quad [V^T]$$

Now $[d^{-1} - I]$ is negative definite since from (3) $\dfrac{1}{d_{ii}} < 1$

Therefore, $[V]\ [d^{-1} - I]\ [V^T]$ is negative definite since a congruent transformation maintains the form.

whence on substituting from (4) and (5)

$$[K_1^{-1} - K_2^{-1}] \text{ is negative definite}$$

QED.

Corollary 1

If $[K_1]$ and $[K_2]$ are real symmetric positive definite matrices, and if $[K_1 - K_2]$ is positive semi-definite, then $[K_1^{-1} - K_2^{-1}]$ is negative semi-definite.

Corollary 2

If $[K_1]$ and $[K_2]$ are real symmetric positive definite matrices, and if $[K_1 - K_2]$ is indefinite, then $[K_1^{-1} - K_2^{-1}]$ is indefinite.

Corollary 3

The converse of the Theorem and the corollaries is true.

The proof of the corollaries is analogous to the proofs of the theorem.

Therefore, analogous to the study of the Inverse Difference Matrix, the Stiffness Difference Matrix may be examined as shown in Table 4.1.2.

Table 4.1.2 - Examination of the Stiffness Difference Matrix S

Quadratic form of S	Eigenvalues of S	Strain Energy Comparison
negative-definite	all negative	$U_1 > U_2$
negative semi-definite	some negative, others zero	$U_1 \geqq U_2$
positive-definite	all positive	$U_1 < U_2$
positive semi-definite	some positive, others zero	$U_1 \leqq U_2$
indefinite	some positive, others negative	$U_1 \gtrless U_2$

In a set of displacement matrices, the "best" matrix will be the one providing the highest lower bound and hence the greatest strain energy. Now it is possible for the nodal loads consistent with a potential energy formulation, to be different for different displacement matrices. This can occur for distributed loading, body forces, and concentrated loads not at the nodes. However, it is felt that the differences in nodal loading will decrease with reduction in element size. Moreover, for loads applied directly to the nodes, the consistent nodal loading will be the same in each case. This also occurs for loads applied to the element boundaries, if the boundary displacements between nodes are the same in each case. Therefore, it is felt that the stiffness matrix selected as described earlier will be the "best".

In a set of equilibrium matrices, if available, the best matrix will provide the lowest upper bound and hence the least strain energy.

When hybrid matrices (i.e. those violating the requirements of both the potential and complementary energy formulations) are available, they can be evaluated with respect to a reference provided by a bounding matrix (displacement or equilibrium). An approximate reference may also be provided by a hybrid matrix for which some numerical comparisons with analytical solutions are available.

4.2 Application of Element Comparison Results to Structure

So far the strain energy comparisons have been made between single elements. It will now be shown that the results of the single element comparison can be applied to the structure stiffness matrix.

We know that the natural structure stiffness matrix is given by

$$[K_1] = [a^T]\,[k_1]\,[a]$$

where [a] = displacement transformation matrix

and $[k_1] = \begin{bmatrix} k_A & & & & \\ & k_B & & & \\ & & k_C & & \\ & & & \ddots & \\ & & & & k_N \end{bmatrix}$, a quasi-diagonal matrix of individual element natural stiffness matrices.

Using a different set of element stiffness matrices, the structure stiffness matrix would be given by

$$[K_2] = [a^T]\ [k_2]\ [a]$$

where $[k_2] = \begin{bmatrix} k_a & & & & \\ & k_b & & & \\ & & k_c & & \\ & & & \ddots & \\ & & & & k_n \end{bmatrix}$, a quasi-diagonal matrix with k_a, k_b, - - - k_n, another set of element natural stiffness matrices.

Therefore, the stiffness difference matrix for the structure is given by

$$[S] = [K_1 - K_2]$$

$$= [a^T]\ [k_1 - k_2]\ [a]$$

$$[S] = [a^T] \begin{bmatrix} k_A - k_a & & & & \\ & k_B - k_b & & & \\ & & k_C - k_c & & \\ & & & \ddots & \\ & & & & k_N - k_n \end{bmatrix} [a]$$

$$= [a^T] \begin{bmatrix} S_1 & & & & \\ & S_2 & & & \\ & & S_3 & & \\ & & & \ddots & \\ & & & & S_n \end{bmatrix} [a]$$

$$(m \times n) \qquad (n \times n) \qquad (n \times m)$$

where S_1, S_2, S_3, - - - - S_n are the element stiffness difference matrices.

It may be noted that a displacement transformation is applied to the quasi-diagonal matrix containing the element stiffness matrices. The order of the matrices are shown above. The rank of matrix [a] is m since the triple product $[a^T]$ $[K_1]$ [a] yields the non-singular (m x m) natural structure stiffness matrix. Also from physical consideration we know that $m < n$.

It will now be shown that the quadratic form of [S] will be governed by the quadratic form of $[k_1 - k_2] \equiv \begin{bmatrix} S_1 & & & \\ & S_2 & & \\ & & \ddots & \\ & & & S_n \end{bmatrix}$

Consider first the case when $[k_1 - k_2]$ is positive definite.

Then a non-singular matrix Q exists (Parker and Eaves, p.94) such that

$$[k_1 - k_2] = [Q^T]\ [Q]$$

Therefore, the stiffness difference matrix S may be written as

$$[S] = [a^T]\ [Q^T]\ [Q]\ [a]$$

$$= [P^T]\ [P]$$

where $$[P] = [Q]\ [a]$$

Note that [P] is of order (n x m) $n > m$ and the rank of [P] is m. Therefore $[P^T]$ [P] is a positive definite matrix. (Parker and Eaves, p.98) Hence [S] is positive definite if $[k_1 - k_2]$ is positive definite.

Consider next the case when $[k_1 - k_2]$ is positive semi-definite of rank r.

Then a rectangular matrix F of order (r x n) exists (Parker and Eaves, p.98) such that

$$[k_1 - k_2] = [F^T]\ [F]$$

Therefore S may be written as

$$[S] = [a^T]\ [F^T]\ [F]\ [a]$$

$$= [H^T]\ [H]$$

where $$[H] = [F]\ [a], \text{ a } (r \times m) \text{ matrix.}$$

Then if the rank of [H] is t, [S] is of positive semi-definite form of rank t. (Stoll, 1958, p. 124).

Therefore [S] is positive semi-definite if $[k_1 - k_2]$ is positive semi-definite.

The arguments used in these two cases apply analogously when $[k_1 - k_2]$ is negative definite and negative semi-definite respectively.

Finally, consider the case when $[k_1 - k_2]$ is indefinite. This may occur if different element matrices are used in different parts of the structure, or if individual element stiffness difference matrices are indefinite. Then, it will be shown that, in general, [S] will also be indefinite.

Let the matrices [a] and $[k_1 - k_2]$ be partitioned and some rows and columns interchanged so that the first (m x m) submatrix of [a] is non-singular. This is possible because the rank of the (n x m) matrix [a] is m.

Then S may be expressed as

$$[S] = [b^T \mid c^T] \left[\begin{array}{c|c} p & r \\ \hline r^T & q \end{array}\right] \left[\begin{array}{c} b \\ \hline c \end{array}\right]$$

where $[a] = \left[\begin{array}{c} b \\ \hline c \end{array}\right]$, [b] is non-singular of rank m and [c] is of order (n-m x m).

Similarly $[k_1 - k_2]$ is now partitioned into the (m x m) matrix [p], the (n-m x n-m) matrix q and the rectangular (m x n-m) matrix r.

On carrying out the multiplication of the partitioned matrices we get

$$[S] = \underbrace{[b^T]\ [p]\ [b]}_{\text{rank } m} + \underbrace{[c^T]\ [q]\ [c]}_{\text{rank} \leq n-m} + \underbrace{\{[c^T]\ [r^T]\ [b] + [b^T][r][c]\}}_{\text{rank} \leq n-m}$$

Note that the triple product $[b^T]$ $[p]$ $[b]$ has rank m, the remaining terms have rank $\leqslant$(n-m). In a practical problem m>> n-m. Therefore, the quadratic form of [S] will be governed mainly by the quadratic form of [p]. If [p] is indefinite, then in all probability so will be [S].

Note however the possibility that [p] is positive-definite, but [q] is indefinite, so that $[k_1 - k_2]$ is indefinite. This can happen if only a few element matrices are taken differently. Then it is conceivable that [S] may take on the form of [p].

In other words, the structure will portray the behavior of the vast majority of the element matrices. Note that if one element matrix is being used for all the elements of the structure, then the results of the element matrix comparisons will apply completely to the structure.

Thus it is seen that the results of the element matrix comparisons may, in general, be applied to the structure.

4.3 Comparison of Matrices of Different Orders

The order of the stiffness matrix of a finite element is equal to the number of nodal displacements allowed in the element, which generally varies with the shape of the element. Thus a plane stress rectangular element with corner nodes has an (8 x 8) matrix, and triangular element has a (6 x 6). If additional nodal displacements are specified, say by choosing extra nodes in the element or specifying edge rotations as shown in sections 3.2.4 and 3.2.5, then also the order of the matrices is increased.

Now the strain energy comparisons between matrices are made for equal volumes of the element (structure) and for the same set of nodal loads. This suggests the possibility of using the different shaped elements or elements with varying number of nodal displacements to fill the same volume, and then eliminating the extra nodes, if any, under the assumption of zero loads at those nodes.

In some cases this assumption of zero nodal loads may be unsatisfactory, if the consistent nodal loading requires loads at those nodes. However, the same nodal loads can always be one of the types of loading to which all matrices may be subjected, and it is felt that comparisons under these conditions will be useful in evaluating the matrices.

4.4 Maximum and Minimum Eigenvalues as Bounds on Strain Energy

Melosh (1963, pp. 222-223) has presented a hypothesis for choosing the best matrix from an available set. According to this hypothesis, the best stiffness matrix will have the smallest eigenvalues and the smallest trace.

This hypothesis will be examined on the basis of some theoretical results from matrix algebra and a strain energy connotation given to it.

The strain energy of an element (or structure) is given by the quadratic form

$$U = \frac{1}{2} \lfloor p \rfloor [K^{-1}] \{ p \}$$

where K is the natural stiffness matrix, and p the nodal load vector.

Now it is known from matrix algebra (Bodewig (1959), p.65) that for a normalised load vector

$$\text{i.e.} \qquad \lfloor p \rfloor \{p\} = 1$$

the maximum of the quadratic form $\lfloor p \rfloor [K^{-1}] \{p\}$ is given by λ_{max}, the largest eigenvalue of $[K^{-1}]$, and its minimum is given by λ_{min}, the smallest eigenvalue.

$$\text{i.e.} \qquad \lambda_{min} \leqq 2U \leqq \lambda_{max}$$

The load vectors giving these bounds are the eigenvectors corresponding to the maximum and minimum eigenvalues.

Similarly, the intermediate eigenvalues give the value of 2U, for load vectors defined by the corresponding eigenvectors.

It may be noted that the eigenvalues of $[K^{-1}]$ are reciprocals of the eigenvalues of [K]. Therefore, Melosh's hypothesis compares the maximum, minimum and other values of the strain energy for load vectors defined by the eigenvectors of each matrix, and selects the matrix giving the greatest strain energy.

Two characteristics of Melosh's hypothesis may be noted, on the strain energy basis.

1. The eigenvectors will be different for each matrix. Therefore, the strain energy comparisons are being made on the basis of different load vectors for each matrix.
2. The maximum strain energy criterion is being applied to all matrices without differentiating between displacement, equilibrium and hybrid matrices.

In Chapter 6, Melosh's hypothesis is compared with the theory developed in section 4.1 with respect to parametric plane stress matrices.

CHAPTER V

AN UPPER BOUND ON STRAIN ENERGY UNDER PLANE STRESS

5.1 Special Element for Constructing Equilibrium Field

By the theorem of Minimum Complementary Energy (Appendix A) it is known that any stress state satisfying the differential equations of equilibrium and the stress boundary conditions will provide an upper bound on the strain energy. Therefore, a finite element conforming to this theorem must provide a stress state in equilibrium within the element as well as continuity of shearing stresses and stresses normal to the boundary between adjacent elements.

De Veubeke (1962, pp. 170-171) has shown that a plane stress equilibrium field may be built up of triangular elements interconnected at the mid-point of the edges, and used in displacement analysis to obtain an upper bound. However in such an analysis the displacements are defined as weighted averages over the element edges, and additional kinematic modes may be introduced, so that it is not easy to get a clear picture of the displacement behavior.

De Veubeke (1965, pp. 191-193) has also noted that an equilibrium field may be obtained from equilibrium and self-straining stresses. In this chapter a square element is visualised by means of which the equilibrium and self-straining stresses may be obtained as bar forces of a psuedo-truss system, in which the bars correspond to the lines of stress transmission. It is not necessary to determine the elastic properties of the bars in order to find the equilibrium and self-straining solutions.

Consider a square element under the action of constant and equal normal and shear stresses along two of its adjacent edges as shown in Fig. 5.1.1.

Such a system of applied stresses is in equilibrium as indicated in Fig. 5.1.2.

Therefore an elasticity solution exists for the stresses and deformations within the element.

Note also that the applied stresses are such as to portray a transference of stresses from one edge to the other. Similar self-equilibrating stresses may be applied to other adjacent and opposite edges. Therefore, if such elements are interconnected at the mid-point of the edges, and equilibrium ensured at these nodes, then stress continuity will be established in the region.

Replacing the uniform stresses along the edges by their resultants acting at the nodes, it is seen that the transference of stresses across the element may be depicted by a psuedo-truss system which carries the resultant forces from one edge to the other. This transference is shown in Fig. 5.1.3., where the respective resultants are denoted by P_1, Q_1, R_1, S_1, T_1. All other constant self-equilibrating stresses may be represented by a superposition of these five resultants. Therefore, the transmission of stresses in a structure may be visualized as a transference through a psuedo-truss system superimposed on it.

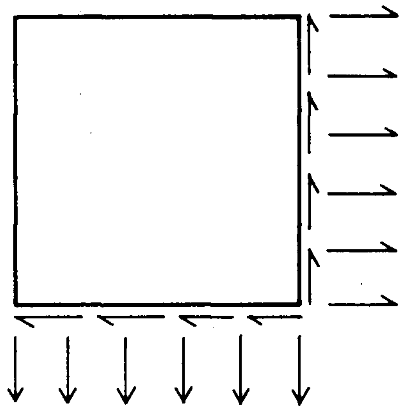


FIG. 5.1.1 BOUNDARY LOADING ON ELEMENT

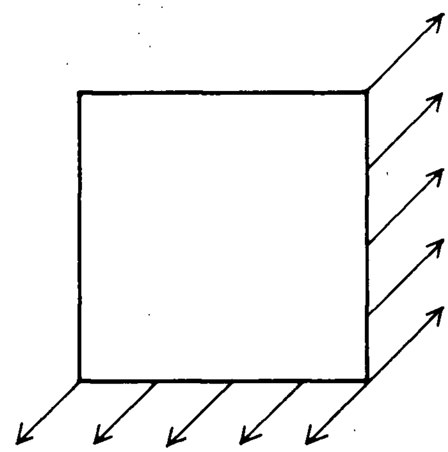


FIG. 5.1.2 EQUIVALENT BOUNDARY LOADING ON ELEMENT

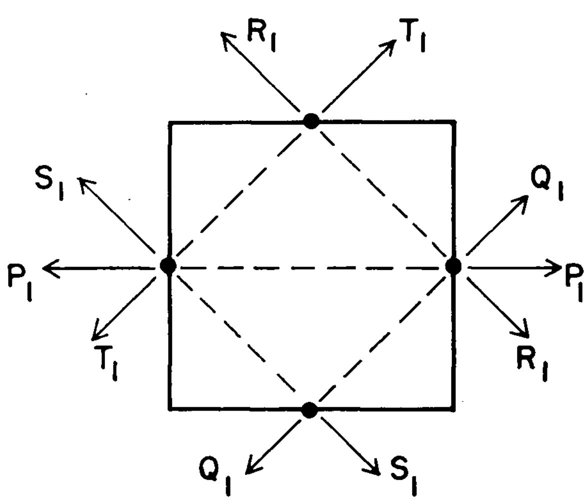


FIG. 5.1.3 TRANSMISSION OF GENERALIZED ELEMENT FORCES THROUGH PSUEDO- TRUSS SYSTEM

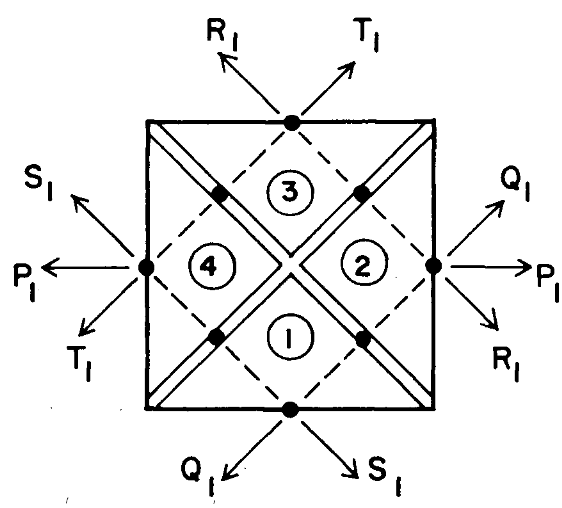


FIG. 5.2.1 SUBDIVISION OF SQUARE ELEMENT INTO FOUR EQUILIBRIUM TRIANGLES

5.2 Upper Bound on Strain Energy of Element

The deformation of the square element under the action of the equivalent nodal forces, and the strain energy of deformation could be ascertained exactly through an elasticity solution of the problem shown in Fig. 5.1.2. However on account of the singularities produced by the discontinuities in the applied shear stress distribution no closed form solution seems to be available. Therefore an upper bound on the strain energy of the element is obtained by considering the square as a combination of De Veubeke triangles, as shown in Fig. 5.2.1.

The transmission of constant stresses through the square produces four constant-stress triangles. The strain energy of each triangle is then evaluated by integrating the expression

$$U = \frac{t}{2E} \iint (\sigma_x^2 + \sigma_y^2 - 2\mu\sigma_x\sigma_y + 2(1+\mu)\tau_{xy}^2)\, dxdy$$

which gives the strain energy for an isotropic constant-thickness region.

On performing the evaluation for the four triangles and summing, an upper bound on the strain energy of the square element is given by

$$U = \frac{1}{2Et} [P_1^2+Q_1^2+R_1^2+S_1^2+T_1^2+\frac{(1-\mu)}{\sqrt{2}}P_1(Q_1+R_1+S_1+T_1)-\mu(Q_1R_1+R_1T_1+S_1T_1+Q_1S_1)]$$

in terms of the psuedo-truss forces.

This strain energy bound for the element can be utilized to obtain an upper bound for a structure composed of these elements. For this purpose the external nodal loads are transmitted to the supports through the psuedo-truss system. This gives an equilibrium solution.

Then self-straining solutions are obtained corresponding to the redundancies of the truss system in terms of arbitrary bar forces in the redundancies. The element strain energy is evaluated in terms of the psuedo-truss bar forces which are the sum of the bar forces of an equilibrium solution and the self-straining solutions. Then the element strain energies are summed to give the structure strain energy which represents an upper bound. The self-straining solutions are then evaluated so as to minimize the structure strain energy. Upper bounds obtained in this manner are shown in Chapter VI.

Note that the external loads are assumed to apply at the nodes. Now the stress-continuity will be ensured only if the external load possesses uniform distribution along an element edge. Otherwise an idealization error is introduced in the analysis. So in the case of non-uniform loading, the loading is approximated by a step-wise uniform distribution, the steps corresponding to the width of the elements.

This technique of obtaining upper bounds on the structure strain energy may be utilized to provide upper bounds on the flexibility influence coefficients as shown by De Veubeke (1962, pp. 185-188). Details of De Veubeke's development are given in Appendix C.

CHAPTER VI

APPLICATIONS TO PARAMETRIC PLANE STRESS STIFFNESS MATRICES

6.1 Comparison of Available Parametric Stiffness Matrices

Plane stress stiffness matrices of order (8 x 8), evaluated for a square isotropic constant-thickness element with corner nodes, are compared. The ordering and nomenclature of the nodal displacements is as shown in Fig. 3.2.2. A (6 x 6) triangular element stiffness matrix is included by forming a square element out of four triangular elements and reducing to (8 x 8) under assumption of zero loads at the extra node. The (10 x 10) and (12 x 12) matrices developed in section 3.2.4 and 3.2.5 are reduced likewise to (8 x 8).

Except for the reduced (12 x 12) matrix, the other matrices can be represented by discrete values of a continuous stiffness matrix parameter as observed by Hooley and Hibbert. In this representation the elements of the stiffness matrix marked K_{21}, K_{31}, K_{61}, K_{71} (Table 3.2.1.1) are invariants and K_{41}, K_{51}, K_{81} and K_{11} are linear functions of the parameter.

For those matrices conforming to the representation by the parameter taken as 'a', the elements of the matrices are given by

$$K_{11} = Et \{ a \}$$

$$K_{21} = Et \{ \frac{1}{8(1-\mu)} \}$$

$$K_{31} = Et \{ \frac{-1+3\mu}{8(1-\mu^2)} \}$$

$$K_{41} = Et \{- a + \frac{1}{4(1+\mu)} \}$$

$$K_{51} = Et \{a - \frac{3-\mu}{4(1-\mu^2)} \}$$

$$K_{61} = Et \{ - \frac{1}{8(1-\mu)} \}$$

$$K_{71} = Et \{ \frac{1-3\mu}{8(1-\mu^2)} \}$$

$$K_{81} = Et \{ - a + \frac{1}{2(1-\mu^2)} \}$$

The values taken by 'a' for the different matrices are shown in Table 6.1.1.

Table 6.1.1 - Parameters for Plane Stress Matrices under Comparison

No.	Matrix	Assumption	Classification	Parameter 'a'	'a' for Poisson's ratio = 1/3
1	Section 3.2.1	Uniform $\sigma_x, \sigma_y, \tau_{xy}$	-	$\frac{3-\mu}{8(1-\mu^2)}$	0.375000
2	Turner et al p. 823, Section 3.2.2	Linear σ_x, σ_y, uniform τ_{xy}	hybrid	$\frac{11-3\mu-2\mu^2}{24(1-\mu^2)}$	0.458333
3	Reduced (10 x 10) Section 3.2.4	Linear $\sigma_x, \sigma_y, \tau_{xy}$	hybrid	$\frac{11-3\mu-2\mu^2}{24(1-\mu^2)}$	0.458333
4	Pian (1) p.1335	quadratic $\sigma_x, \sigma_y, \tau_{xy}$	hybrid	-	0.468750
5	Pian (2) p.1336	Linear edge displacements, quartic in interior	displacement	-	0.473960
6	Argyris (1955, p.126) Melosh (1962, p.32), Appendix B	Linear edge displacements, quadratic in interior	displacement	$\frac{3-\mu}{6(1-\mu^2)}$	0.500000
7	amplified (6x6) Turner et al p. 816	composed of four constant stress triangles abutting along diagonals	displacement	-	0.515625
8	Hrennikoff as per Hooley and Hibbert	Lattice model	hybrid	$\frac{5-3\mu}{8(1-\mu^2)}$	0.562500

(Cont'd) Table 6.1.1 - Parameters for Plane Stress Matrices under Comparison

No.	Matrix	Assumption	Classification	Parameter 'a'	'a' for Poisson's ratio = 1/3
9	McCormick as per Hooley and Hibbert	Lattice model	hybrid	$\frac{11-9\mu}{16(1-\mu^2)}$	0.562500
10	Gallagher et al p.27-29, Section 3.2.3	Uniform σ_x, σ_y Linear τ_{xy}	hybrid	$\frac{3-\mu}{8(1-\mu^2)} + \frac{3+2\mu}{12\mu^2}$	3.125000
11	Reduced (12x12) Section 3.2.5	hyperbolic σ_x, σ_y parabolic τ_{xy}	hybrid	does not conform to representation by parameter	

While the comparisons will be made on the basis of reduced matrices, it should be noted that the (10 x 10) matrices developed in section 3.2.4 are capable of being used as such because the corresponding natural stiffness matrix is positive definite. However, the corresponding reduced matrix is identical to the (8 x 8) Turner matrix so that if the loads at the extra nodes of the (10 x 10) element are zero, the results obtained by using the (10 x 10) matrix will be identical to that obtained by using the Turner matrix.

The (12 x 12) matrix developed in section 3.2.5 behaves unexpectedly. For Poisson's ratio = 1/3, the natural stiffness matrix turns out to be singular on account of a singularity in the last (4 x 4) principal minor. The elements of this principal minor give the stiffness coefficients corresponding to edge rotations at the nodes. For other values of Poisson's ratio, the matrix is non-singular but indefinite. The reduced (8 x 8) matrix is also singular for Poisson's ratio = 1/3. But it is indefinite for Poisson's ratio less than 1/3, and positive definite for Poisson's ratio = 0.4. These reduced matrices do not conform to the parametric representation. The author can ascribe the unexpected behavior only to the assumption of hyperbolic normal stress distribution. Because of the indefinite quadratic form of this matrix, it is not considered any further.

Now the (8 x 8) stiffness matrices will be compared by forming the inverse difference matrices, and the stiffness difference matrices. It will be shown that comparison results are the same in either case as predicted by the theory developed in section 4.1.

For making the strain energy comparisons it is only necessary to compare the natural stiffness matrices since the rigid body modes do not produce any strain energy.

The natural stiffness matrix corresponding to the supported element can be obtained in two distinctly separate ways as shown in Fig. 6.1.1.

It will now be shown that either manner of obtaining the natural stiffness matrices yields the same comparison results.

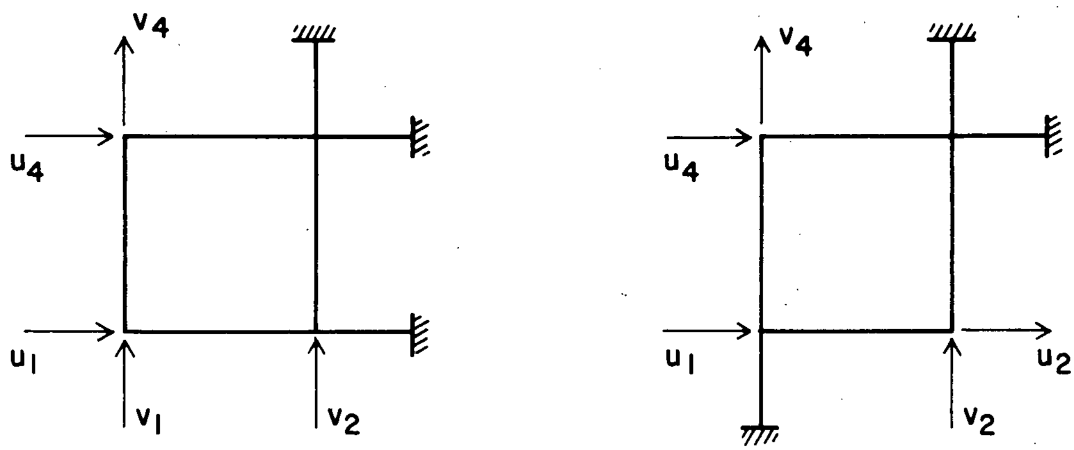
Support Case A

The natural stiffness matrix is given by

$$[K] \equiv \begin{array}{c} u_1 \\ v_1 \\ v_2 \\ v_4 \\ u_4 \end{array} Et \begin{bmatrix} a & & & & \\ \frac{1}{8(1-\mu)} & a & & \text{SYMMETRIC} & \\ \frac{-1+3\mu}{8(1-\mu^2)} & -a+\frac{1}{2(1-\mu^2)} & a & & \\ \frac{1-3\mu}{8(1-\mu^2)} & -a+\frac{1}{4(1+\mu)} & a-\frac{(3-\mu)}{4(1-\mu^2)} & a & \\ -a+\frac{1}{2(1-\mu^2)} & \frac{-1+3\mu}{8(1-\mu^2)} & \frac{1}{8(1-\mu)} & -\frac{1}{8(1-\mu)} & a \end{bmatrix}$$

The inverse of the natural stiffness matrix is given by

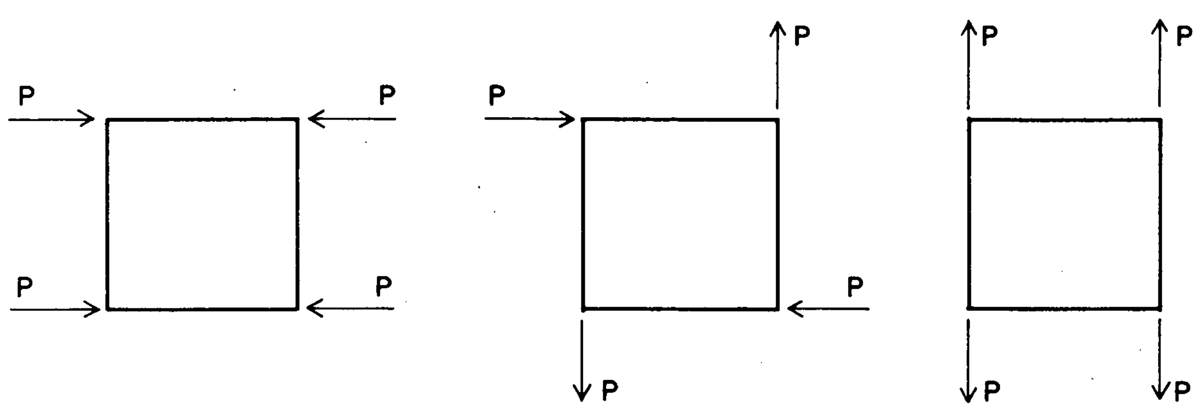
$$[K^{-1}] \equiv \frac{1}{Et} \begin{bmatrix} b & & & & \\ -b+(1-\mu) & b+2(1+\mu) & & \text{SYMMETRIC} & \\ -\mu & 1 & b & & \\ -b+1 & b+1+2\mu & b-1 & 2(b+\mu) & \\ -b+2 & b-(1+\mu) & -\mu & b-1 & b \end{bmatrix}$$



SUPPORT CASE A

SUPPORT CASE B

FIG. 6.1.1 SUPPORTED FINITE ELEMENT



LOADING 1

LOADING 2

LOADING 3

FIG. 6.1.2 INDEPENDENT LOADINGS PRODUCING EQUAL STRAIN ENERGY where $b = \left\{ \dfrac{1-\mu + 2\mu^2}{8(1-\mu^2)} - a \right\} \Big/ \left\{ \dfrac{3-\mu}{8(1-\mu^2)} - a \right\}$

may be considered the inverse matrix parameter.

The advantage of having the stiffness matrices and their inverses in parametric form is that the inverse difference matrix and the stiffness difference matrix may be formed in terms of these parameters, and results of general validity obtained for all plane stress matrices capable of being represented in this manner, some of which are shown in Table 6.1.1.

Inverse Difference Matrix

First the inverse difference matrix for these matrices is investigated.

Thus, two stiffness matrices with parameters a_1 and a_2 will have inverse matrix parameters of b_1 and b_2 where the 'b's and 'a's are related as shown above.

Therefore the inverse difference matrix is given by

$$[D] \equiv \frac{b_1-b_2}{Et} \begin{bmatrix} 1 & & & & \text{SYMMETRIC} \\ -1 & 1 & & & \\ 0 & 0 & 1 & & \\ -1 & 1 & 1 & 2 & \\ -1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

This is of form $c_2[B]$ where c_2 is a constant and [B] a matrix.

[B] has rank 2 and index 2, and the non zero eigenvalues of [D] are given by

$$\text{Eiv. no. 1} = \frac{b_1 - b_2}{Et} (3 + \sqrt{2})$$

$$\text{Eiv. no. 2} = \frac{b_1 - b_2}{Et} (3 - \sqrt{2})$$

Therefore, for $b_1 > b_2$, [D] is positive semi-definite. Consequently, the stiffness matrix with inverse parameter b_1 will have greater or equal strain energy than the matrix with inverse parameter b_2.

The equality of strain energy corresponds to the three zero eigenvalues which result from the singularities of [D] produced by linearly dependent vectors. Each column of [D] represents the difference in the displacements produced by using the two matrices under comparison, under unit nodal loads. Therefore, if a linear combination of some column vectors becomes the null vector, then the corresponding linear combinations of nodal loads will produce no difference in displacements when the two matrices are used. Whence by Clapeyron's Theorem, the strain energies will be equal.

The three independent load combinations producing equal strain energies in the element are given in Fig. 6.1.2.

Therefore, when $b_1 > b_2$ the arbitrary non-uniform loads will produce greater strain energy with the use of the corresponding matrices.

The relation between 'a' and 'b' is plotted in Fig. 6.1.3. Note that for $a = \frac{3-\mu}{8(1-\mu^2)} = a_{cr}$ the natural stiffness matrix is singular,

for $o \leq a < a_{cr}$ the natural stiffness is non-singular but indefinite, and for $a > a_{cr}$ the natural stiffness matrix is positive definite. For structural analysis only positive definite matrices are of interest, and hereafter in all references to parametric matrices it will be assumed that $a > a_{cr}$.

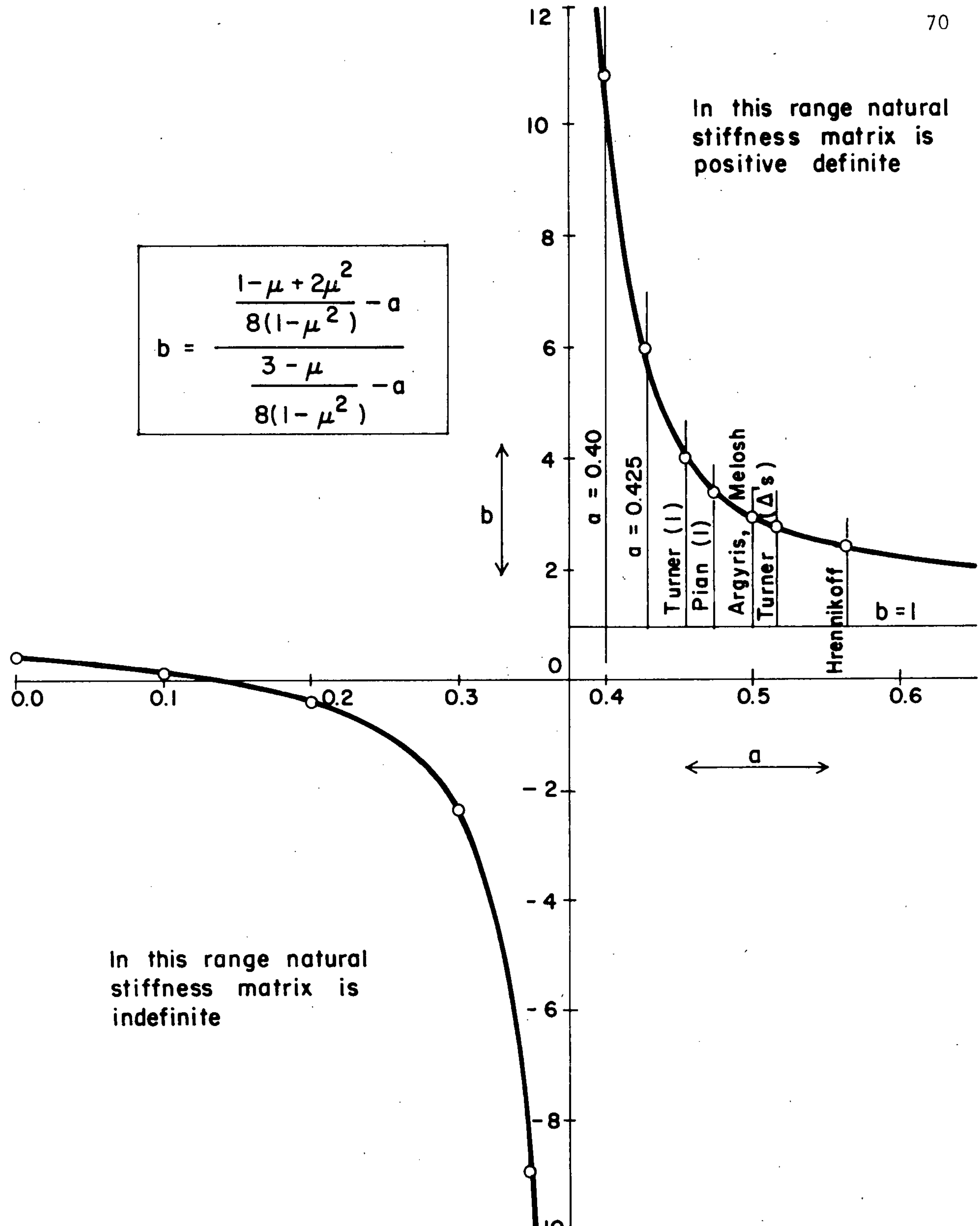


FIG. 6.1.3 RELATION BETWEEN STIFFNESS MATRIX PARAMETER 'a' AND INVERSE MATRIX PARAMETER 'b' PLOTTED FOR POISSON'S RATIO = 1/3

Note that $a = \frac{3-\mu}{8(1-\mu^2)}$ corresponds to Matrix No. 1 (Table 6.1.1).

Note also that for $\frac{3-\mu}{8(1-\mu^2)} \leqq a_1 < a_2$

the following holds

$$\infty \geqq b_1 > b_2$$

Thus it is seen that the stiffness matrix parameter and the inverse matrix parameter provide an index of the strain energy level under non-uniform loading.

Hence the stiffness matrices are ordered by strain energy in decreasing order of magnitude as shown in Table 6.1.1.

Stiffness Difference Matrix

Any two matrices under comparison differ only by the parameter 'a'. Therefore the stiffness difference matrix is given by

$$[S] = Et(a_1-a_2)\begin{bmatrix} 1 & & & & \\ 0 & 1 & & \text{SYMMETRIC} & \\ 0 & -1 & 1 & & \\ 0 & -1 & 1 & 1 & \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

This is of form $C_1[A]$ where C_1 is a constant and $[A]$, a matrix.

[A] has rank 2 and index 2 and the non-zero eigenvalues of [S] are given by

$$\text{Eiv. no. 1} = Et\,(a_1 - a_2) \quad (3)$$

$$\text{Eiv. no. 2} = Et\,(a_1 - a_2) \quad (2)$$

Therefore we see that for $a_1 < a_2$ the stiffness difference matrix is negative semi-definite. Consequently according to Table 4.1.2, the matrix with parameter a_1 will have greater or equal strain energy than the matrix with parameter a_2. By examining the linearly dependent vectors producing the singularities in [S], it is found that they occur under displacements of the element produced by uniform loading.* Consequently under non-uniform loading, the matrix with parameter a_1 will provide greater strain energy than the matrix with parameter a_2, if $a_1 < a_2$.

The same conclusions were drawn by examining the inverse difference matrix.

Support Case B

The natural stiffness matrix is given by

$$[K] \equiv Et\;\begin{array}{c} u_1 \\ v_2 \\ u_2 \\ v_4 \\ u_4 \end{array}\begin{bmatrix} a & & & & \\ \frac{-1+3\mu}{8(1-\mu^2)} & a & & \text{SYMMETRIC} & \\ -a+\frac{1}{4(1+\mu)} & \frac{-1}{8(1-\mu)} & a & & \\ \frac{1-3\mu}{8(1-\mu^2)} & a-\frac{(3-\mu)}{4(1-\mu^2)} & \frac{1}{8(1-\mu)} & a & \\ -a+\frac{1}{2(1-\mu^2)} & \frac{1}{8(1-\mu)} & a-\frac{(3-\mu)}{4(1-\mu^2)} & \frac{-1}{8(1-\mu)} & a \end{bmatrix}$$

---

* Uniform loads on the element implies that the element is stressed by constant normal and shear stresses.

The inverse of the natural stiffness matrix is given by

$$[K^{-1}] \equiv \frac{1}{Et} \begin{bmatrix} 4 & & & \text{SYMMETRIC} & \\ 1-\mu & b & & & \\ 3+\mu & 1 & b+2(1+\mu) & & \\ -1+\mu & b-2 & -1 & b & \\ 1-\mu & -\mu & b-(1+\mu) & \mu & b \end{bmatrix}$$

where, as in support Case A,

$$b = \left\{ \frac{1-\mu+2\mu^2}{8(1-\mu^2)} - a \right\} \Big/ \left\{ \frac{3-\mu}{8(1-\mu^2)} - a \right\}$$

<u>Inverse Difference Matrix</u>

Therefore the inverse difference matrix for any two matrices with inverse parameters b_1 and b_2 is given by

$$[D] \equiv \frac{b_1-b_2}{Et} \begin{bmatrix} 0 & & & & \\ 0 & 1 & & \text{SYMMETRIC} & \\ 0 & 0 & 1 & & \\ 0 & 1 & 0 & 1 & \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

This matrix has rank 2, and its non-zero eigenvalues are given by

$$\text{Eiv. no. 1} = \frac{b_1-b_2}{Et} \times (2)$$

$$\text{Eiv. no. 2} = \frac{b_1-b_2}{Et} \times (2)$$

Therefore, as for Support Case A, for $b_1 > b_2$ [D] is positive semi-definite. Again it may be shown that the equality of strain energy occurs under the uniform nodal loading shown in Fig. 6.1.2., so that under non-uniform loading the matrix with inverse parameter b_1 provides greater strain energy than the matrix with inverse parameter b_2 if $b_1 > b_2$.

Stiffness Difference Matrix

The stiffness difference matrix for Support Case B, with respect to any two stiffness matrices is given by

$$[S] \equiv Et(a_1-a_2)\begin{bmatrix} 1 & & & & \\ 0 & 1 & \cdots & & \text{SYMMETRIC} \\ -1 & 0 & 1 & & \\ 0 & 1 & 0 & 1 & \\ -1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

It has rank 2, and its non-zero eigenvalues are given by

$$\text{Eiv. no. 1} \quad = \quad Et\ (a_1 - a_2) \quad (3)$$

$$\text{Eiv. no. 2} \quad = \quad Et\ (a_1 - a_2) \quad (2)$$

Therefore, again, it is seen that for $a_1 < a_2$ the stiffness difference matrix is negative semi-definite. Again it can be shown that equality of strain energy occurs only under the uniform loadings shown in Fig. 6.1.2. Under non-uniform loading the matrix with parameter a_1 will provide greater strain energy than the matrix with parameter a_2 if $a_1 < a_2$.

Thus it is verified that the same strain energy comparison results are obtained by using the stiffness difference matrix and the inverse difference matrix. Also these results are independent of the manner in which the rigid body modes are eliminated to obtain the natural stiffness matrix.

Now the results obtained by using the special difference matrices will be compared with the results obtained by examining the element strain energy bounds provided by the stiffness inverse matrix eigenvalues.

First it will be shown that the results are independent of the manner of obtaining the natural stiffness matrices.

Table 6.1.2 gives the eigenvalues of the inverse matrices for support cases A and B.

Note that the ordering of the inverse matrix eigenvalues is independent of the support case, and hence independent of the manner of obtaining the natural stiffness matrix.

The maximum and minimum eigenvalues providing the strain energy bounds for support case A is plotted in Fig. 6.1.4. Note that the ordering of the matrices by strain energy obtained by comparing the bounds is the same as that obtained by using the special difference matrices. This correspondence will not always occur because the difference matrix comparisons compare strain energies under the same load vector, whereas the strain energy bounds are obtained for different load vectors (defined by the eigenvectors corresponding to the eigenvalues). Here the correspondence occurs fortuitously because the inverse matrix eigenvectors defining the loads are similar for all the matrices. This is shown in Table 6.1.3 for the Turner hybrid, Pian displacement, the Argyris-Melosh displacement matrix, Turner (Δ's) and the Hrennikoff matrix. The eigenvectors are normalized to have length of unity. A plot of the eigenvector coefficients is shown in Fig. 6.1.5.

Et λ

Table 6.1.2 -- Eigenvalues of the Inverse Matrices μ = 1/3

Support Case	Matrix a= / Eivs. in order of magnitude	Turner hybrid 0.45833	Pian displacement 0.47396	Argyris, Melosh disp. 0.50000	Turner disp. (from triangles) 0.51563	Hrennikoff, Lattice Model 0.56250	No. 10 hybrid 3.12500
A	Eiv. no. 1	17.26	15.26	13.09	12.19	10.44	6.28
	Eiv. no. 2	5.06	4.33	3.54	3.22	2.67	2.24
	Eiv. no. 3	2.26	2.22	2.16	2.13	1.97	1.06
	Eiv. no. 4	1.79	1.72	1.59	1.52	1.33	0.16
	Eiv. no. 5	0.966	0.962	0.953	0.946	0.923	0.119
B	Eiv. no. 1	10.28	9.68	9.11	8.89	8.50	7.70
	Eiv. no. 2	6.00	5.05	4.00	3.56	2.67	2.16
	Eiv. no. 3	3.84	3.54	3.16	2.99	2.67	0.87
	Eiv. no. 4	1.70	1.65	1.56	1.51	1.33	0.18
	Eiv. no. 5	0.849	0.848	0.845	0.844	0.837	0.119

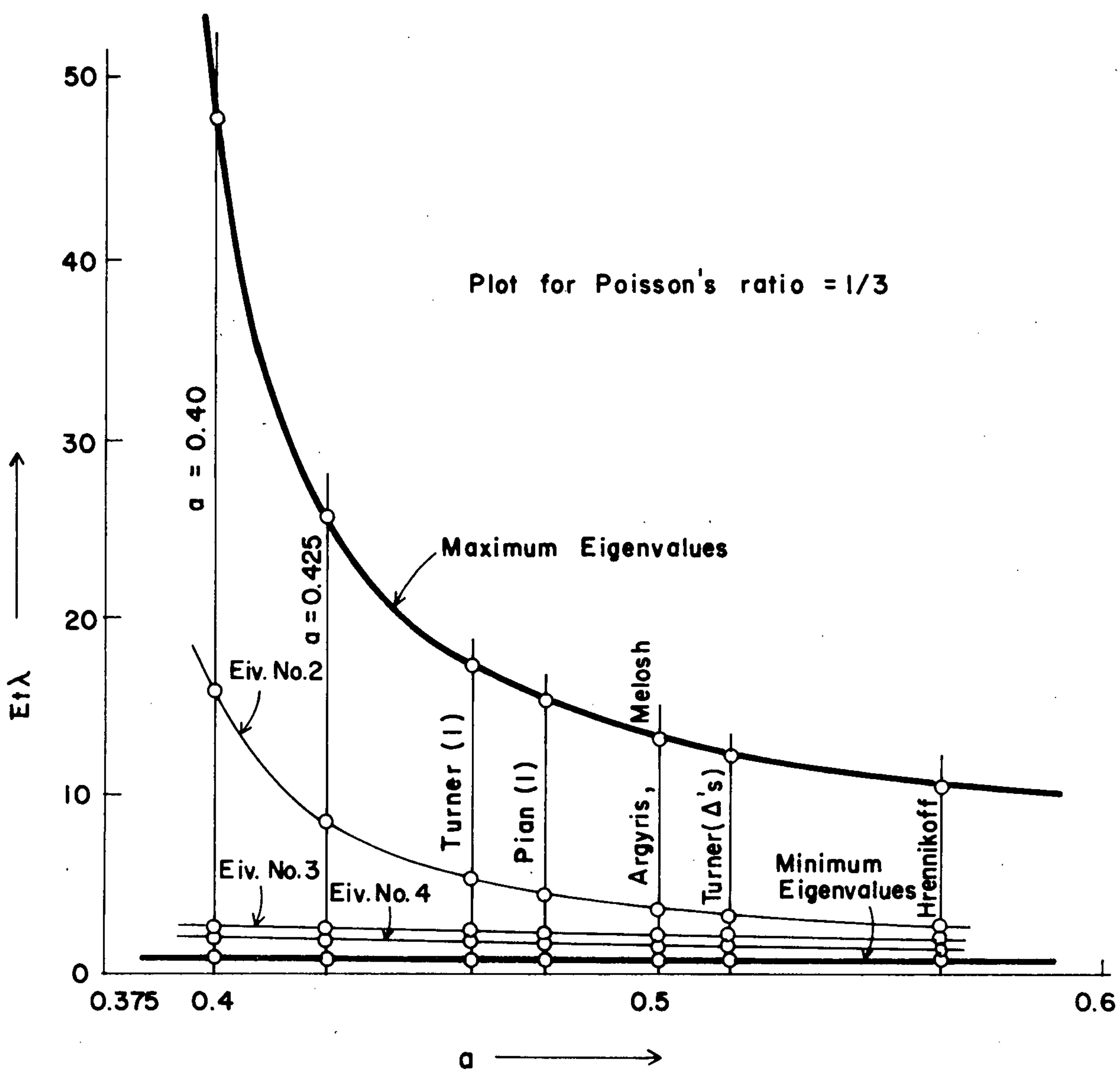


FIG. 6.1.4 STIFFNESS MATRIX INVERSE EIGENVALUES SHOWING RELATIVE BOUNDS ON ELEMENT STRAIN ENERGY

Table 6.1.3 - Eigenvectors of the Inverse Matrices

Matrix	Eiv. 1	Eiv. 2	Eiv. 3	Eiv. 4	Eiv. 5
Turner hybrid	0.3417	-0.3188	0.4825	-0.7092	0.2140
	-0.5581	0.1991	-0.4318	-0.5198	0.4385
	-0.1929	-0.7515	-0.0338	0.3747	0.5066
	-0.6627	-0.3181	0.2951	-0.1538	-0.5907
	-0.3089	0.4392	0.7018	0.2505	0.3954
Pian (2)	0.3262	-0.3072	0.5557	-0.6680	0.2103
	-0.5747	0.1744	-0.3838	-0.5388	0.4490
	-0.1933	-0.7544	-0.0229	0.3884	0.4920
	-0.6657	-0.2864	0.3244	-0.1116	-0.5975
	-0.2876	0.4733	0.6618	0.3165	0.3938
Argyris, Melosh	0.3027	-0.2659	0.6542	-0.6067	0.2042
	-0.5988	0.1284	-0.3247	-0.5473	0.4690
	-0.1935	-0.7572	0.0238	0.4170	0.4634
	-0.6688	-0.2217	0.3583	-0.0555	-0.6100
	-0.2551	0.5388	0.5811	0.3943	0.3893
Turner (Δ's)	0.2900	-0.2228	0.7037	-0.5751	0.2006
	-0.6113	0.0976	-0.2984	-0.5427	0.4827
	-0.1932	-0.7553	0.0710	0.4367	0.4430
	-0.6697	-0.1742	0.3715	-0.0311	-0.6181
	-0.2373	0.5829	0.5221	0.4276	0.3851
Hrennikoff	0.2577	0.0000	0.8042	-0.5000	0.1919
	-0.6415	0.0000	-0.2325	-0.5000	0.5332
	-0.1919	-0.7071	0.2858	0.5000	0.3625
	-0.6696	0.0000	0.3684	0.0000	-0.6448
	-0.1919	0.7071	0.2858	0.5000	0.3625

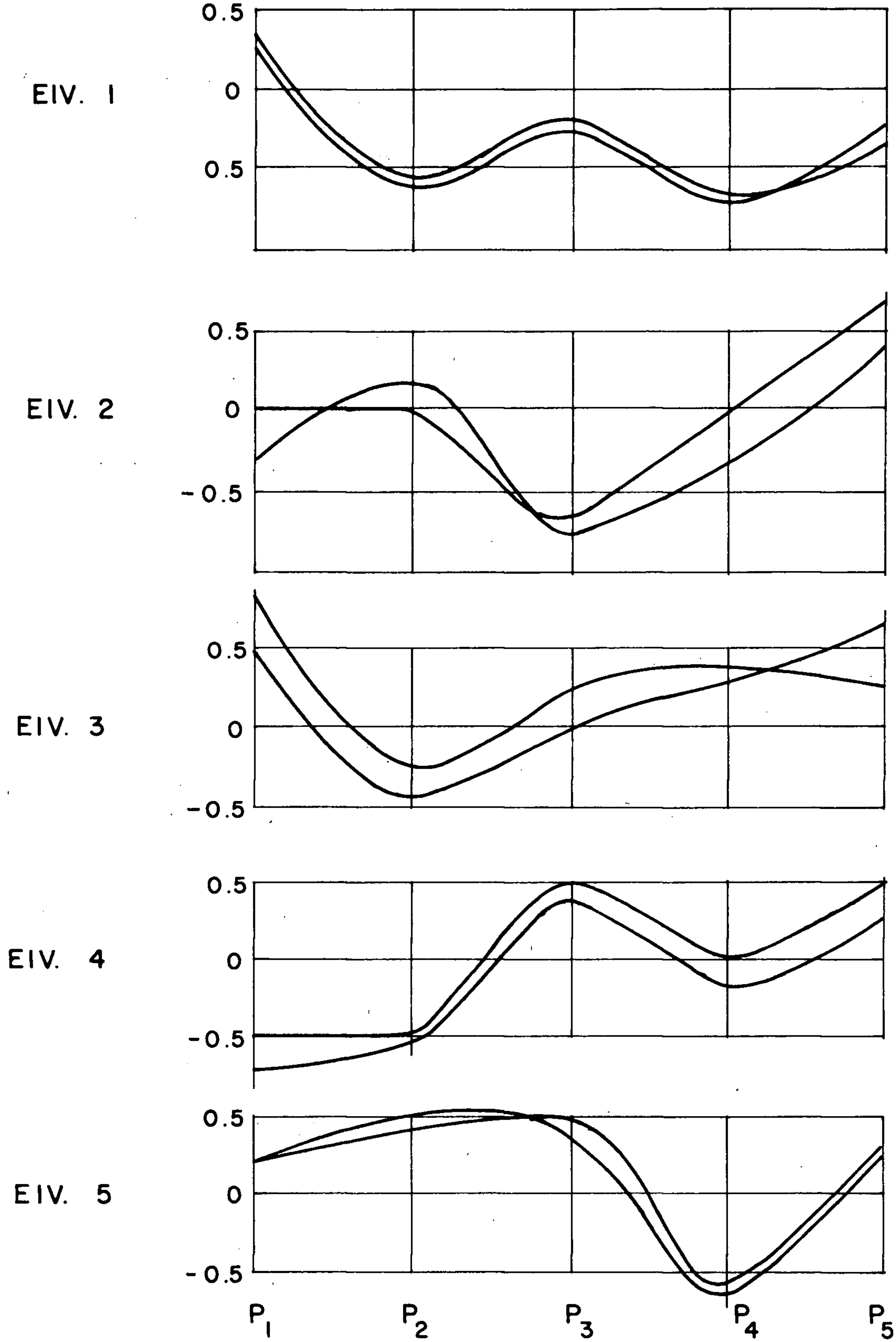


FIG. 6.1.5 LIMITS FOR EIGENVECTOR COEFFICIENTS

Now conclusions will be drawn on the basis of the strain energy ordering obtained by using the energy criterion for the stiffness matrices, for a square isotropic element, shown in Table 6.1.1.

Note that amongst the displacement matrices, the Pian matrix provides the greatest strain energy and hence is the best displacement matrix.

This matrix was obtained by Pian (1964, pp. 1336) by assuming more displacement modes than the number of element nodal displacements. This improvement in the displacement matrix is contrary to general experience (Clough, 1965, p. 91).and it is felt that such improvement will occur, in general, only if internal stress equilibrium is improved by doing so, as was conjectured by Pian for this case.

Also the assumption of more stress modes seems to result in less element strain energy under equal nodal loads. Thus as Pian (1964, pp. 1334-1336) induced more stress modes in the derivation of plane stress matrices the parameter 'a' increased from a = 0.45833 to a = 0.46875.

Also note that the use of triangular elements provides the least lower bound on the strain energy. Therefore it may be concluded that triangular elements would generally be inferior to the use of square elements.

Note also that the lattice model matrices (nos. 8 and 9) are inferior to the displacement matrices, and the Gallagher matrix (no.10) seems poor.

Note also that both the Turner Matrix (section 3.2.2) and Matrix No. 10 (section 3.2.3) satisfy microscopic equilibrium equations within the element. But Turner matrix is very good while Matrix No. 10 is poor. This shows that satisfaction of equilibrium within the element will not necessarily lead to a good stiffness matrix, as noted by Melosh (1962, p. 79) for elements for solids.

However, note that the hybrid Turner Matrix provides greater strain energy than the best displacement matrix. Therefore it will provide a higher lower bound on the strain energy than a displacement matrix or an upper bound. In the next section it will be shown that all these matrices will tend to converge towards the solution. In that case the Turner hybrid matrix may be considered the best parametric matrix. Note also that the (10 x 10) matrix (no. 3) on being reduced provides equal strain energy to the Turner hybrid matrix.

Finally, note that all parametric matrices are capable of representing constant stresses but not all of them yield good results.

6.2 Providing Bounds on Strain Energy by varying the Matrix Parameter

It has been shown that the stiffness matrix parameter 'a' provides an index of the strain energy level under non-uniform loading. Under uniform loading all the matrices provide equal strain energy.

An indication of this behavior can also be obtained by looking at the strain energy expressions for the (8 x 8) matrices derived in Chapter 3, and Appendix B. These are shown in Table 6.2.1.

Table 6.2.1 - Element Strain Energy Expressions for Different Matrices

Matrix	Element Strain Energy Expression in terms of nodal displacements
No. 1 Uniform $\sigma_x, \sigma_y, \tau_{xy}$ Section 3.2.1	$U_1 = \frac{Et}{8(1-\mu^2)} [(u_2 - u_1 + u_3 - u_4)^2 + (v_4 - v_1 + v_3 - v_2)^2 + 2\mu (u_2 - u_1 + u_3 - u_4)(v_4 - v_1 + v_3 - v_2)] + \frac{Et}{16(1+\mu)} [u_4 - u_1 + u_3 - u_2 - v_1 + v_2 - v_4 + v_3]^2$
Turner Linear σ_x, σ_y and Uniform τ_{xy} Section 3.2.2	$U_2 = U_1 + \frac{Et}{24} [(u_3 - u_4 - u_2 + u_1)^2 + (v_3 - v_2 - v_4 + v_1)^2]$
Argyris, Melosh Linear edge displacements, quadratic in interior Appendix B	$U_3 = U_1 + \frac{Et(3-\mu)}{48(1-\mu^2)} [(u_3 - u_4 - u_2 + u_1)^2 + (v_3 - v_2 - v_4 + v_1)^2]$
No. 10 Uniform σ_x, σ_y Linear τ_{xy} Section 3.2.3	$U_4 = U_1 + \frac{Et}{24} \frac{(3+2\mu)}{\mu^2} [(u_3 - u_4 - u_2 + u_1)^2 + (v_3 - v_2 - v_4 + v_1)^2]$

Note that the strain energy expressions for the Turner, Argyris-Melosh, and No. 10 Matrices contain the strain energy produced under uniform normal and shear stresses plus an additional strain energy term. This additional term

$$Et\ [\ (u_3 - u_4 - u_2 + u_1)^2 + (v_3 - v_2 - v_4 + v_1)^2]$$

is only altered by the value of its coefficient, say 'c', for each of these matrices. As 'c' increases from zero (for the uniform stress case), the element strain energy under arbitrary displacement vector increases, unless the displacement vector is such as to make the expression within brackets zero in which case no increase will take place. It can be shown that a displacement vector making the term zero corresponds to displacements under constant stresses.

Since the element strain energy increases, as 'c' increases, for non-uniform displacements, the element strain energy will decrease under non-uniform loading. This follows from the theory for the stiffness difference matrix developed in section 4.1.

Therefore, it is concluded that 'c' gives an indication of the element strain energy level.

Now it is found that the stiffness matrix parameter of Hooley and Hibbert is a linear function of 'c'.

Thus
$$a = \frac{3-\mu}{8(1-\mu^2)} + 2c$$

where 'a' = stiffness matrix parameter

and 'c' = coefficient of the strain energy term

$$Et\ [\ (u_3 - u_4 - u_2 + u_1)^2 + (v_3 - v_2 - v_4 + v_1)^2\ ]$$

Hence 'a' also gives an indication of the strain energy level.

This corroborates the same conclusion drawn in the last section.

Also note from Fig. 6.1.3 that for $\frac{3-\mu}{8(1-\mu^2)} \leq a \leq \infty$ we have $\infty \leq b \leq 1$

where 'b' is the inverse matrix parameter.

Therefore as 'a' varies from $\frac{3-\mu}{8(1-\mu^2)}$ to ∞, the strain energy under non-uniform load varies from infinity to a small magnitude represented by 'b' = 1, which is much smaller than b = 3 for the Argyris-Melosh displacement matrix $\left(a = \frac{3-\mu}{6(1-\mu^2)}\right)$ providing a lower bound.

Hence by varying the stiffness matrix parameter between $\frac{3-\mu}{8(1-\mu^2)} \leq a \leq \frac{3-\mu}{6(1-\mu^2)}$ it is possible to cover a range of element strain energy levels from ∞ to a lower bound given by a displacement matrix. Since the strain energy in an element is finite in magnitude, some value of the parameter exists which will provide an upper bound on the strain energy. However, while values of the parameter giving a lower bound are defined by displacement matrices, it has not yet been found possible to select a parameter providing an upper bound on the theoretical basis of equilibrium matrices.

Note that the ordering of element strain energy obtained by using different parameters is independent of element size. If the loading on the element is non-uniform, the strain energy will be ordered by parameter. If the loading is uniform the strain energy of the element will be independent of the parameter and equal to the exact strain energy.

Now it has been shown in section 4.2 that the results of the element matrix comparisons apply to a structure composed of them. This result is independent of the number of elements, their distribution or their size.

Therefore the ordering of structure strain energy depends upon the ordering of the element matrix parameters, but is independent of structure subdivision.

Hence if curves for the strain energy of a structure with respect to structure subdivision are drawn for different values of the matrix parameter, then no two curves will ever intersect. An example of this is shown in Fig. 6.3.1, where the deflection of a cantilever under the load (proportional to the strain energy) is plotted against structure subdivision.

It will now be shown that all such parametric curves must converge towards the solution.

Synge (1957, pp. 209-212) has proved that a given function and its first derivative can be approximated as closely as we like by a polyhedral function based on a suitable triangulation. This polyhedral function is defined within a triangle as a linear interpolation of the values at the vertices.

Using a linear displacement field, which corresponds to a representation of the interpolation function for a triangular element, Turner et al (1956) obtained the stiffness matrix for a constant-stress triangle. Therefore the stiffness matrix for such a triangular element will provide convergence to the solution with sufficient network refinement. This has been noted by Melosh (1962, pp. 81-82).

Note that this convergence to the solution is obtained for all load vectors.

Let us now divide a region of interest into triangles as follows. We first divide the region into squares, and subsequently subdivide each square into four triangles by joining the diagonals. One such progressive refinement of the region using such triangles is shown in Fig. 6.2.1.

Now we apply nodal loads to the subdivided region such that only the nodes corresponding to the corners of the squares are loaded, no loads being applied to the interior nodes. This represents a special load vector, and again convergence to the solution must take place.

But such a subdivision, and manner of loading corresponds to the use of the Turner-triangles matrix with parameter a = 0.515625 (Table 6.1.1). Therefore this specific parametric matrix must converge to the solution.

Now let us assume that the same nodal loads are applied to the Pian displacement and the Argyris-Melosh displacement matrices. Then these two parametric matrices must converge towards the solution defined by the actual strain energy U, to which the square element formed from the Turner-triangles has converged.

If not, let the solution from these two matrices converge to different values, U_1 and U_2 respectively.

Then, considering the hierarchy of strain energy levels (Table 6.1.1)

$$U_1 > U_2 > U$$

since parametric strain energy curves never intersect.

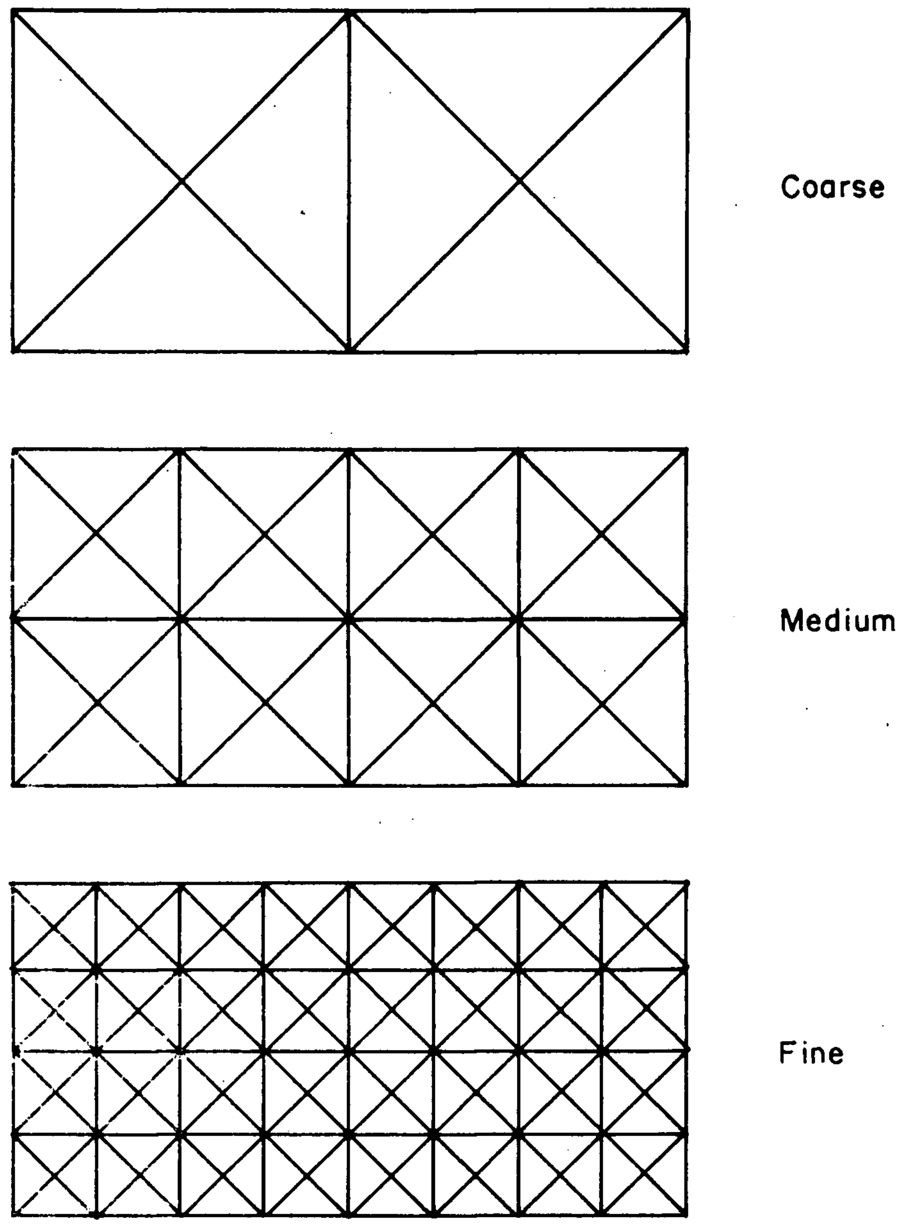


FIG. 6.2.1 PROGRESSIVE SUBDIVISION OF A REGION USING TRIANGULAR ELEMENTS

But U_1 and U_2 are lower bound solutions from displacement matrices. Therefore

$$U_1 \leqq U$$

and $$U_2 \leqq U$$

Therefore our assumption is false, and

$$U_1 \equiv U_2 \equiv U$$

Thus these three matrices must converge towards the solution under equal nodal loads.

Hence, the nodal displacements obtained in each case must be the same.

That is, for each infinitesimal element in the region, the vector of nodal displacements

$$\lfloor u_1, v_1, u_2, v_2, u_3, v_3, u_4, v_4 \rfloor$$

must be identical.

But the strain energy of the element in terms of these displacements is given for each of the matrices by

$$U_T = U_c + c_1 U_B \quad \text{(Turner-triangles)}$$

$$U_{A-M} = U_c + c_2 U_B \quad \text{(Argyris-Melosh)}$$

$$U_p = U_c + c_3 U_B \quad \text{(Pian displacement)}$$

where U_c and U_B are constant energy components and the c's are linear functions of the parameter. (As example see Table 6.2.1).

Now these strain energies must be equal since each represents the solution strain energy, say U.

This requires that $U_B = 0$ so that

$$U_T = U_{A-M} = U_p = U_c = U$$

We thus see that when the element size becomes infinitesimal U_B approaches zero.

This result is also verified by examining the strain energy expressions for the parametric matrices shown in Table 6.2.1.

When the element size becomes infinitesimal the nodal displacements at nodes 2, 3, 4 may be expanded by Taylor's series about node 1. Keeping only the first order terms of the expansion and substituting in the energy expression one gets the final result

$$U = U_c + c_1 \; [\; 0 \;]$$

where the coefficient of c_1 approaches zero and

$$U_c = Et \left[\frac{1}{2(1-\mu^2)} (\varepsilon_x{}^2 + \varepsilon_y{}^2 + 2\mu\, \varepsilon_x \varepsilon_y + \frac{1}{4(1+\mu)} \gamma_{xy}{}^2 \right] dx^2$$

which corresponds to strain energy in an infinitesimal element by linear elasticity theory.

But the strain energy of an element under all parametric matrices is given by

$$U_s = U_c + c_n \; U_B$$

see 91A

where c_n is a constant varying with the parameter.

As element size becomes infinitesimal $U_B \rightarrow 0$. Therefore

$$U_s = U_c$$

but $U_c = U$, the solution strain energy. Therefore $U_s = U$.

This proves that all parametric matrices must converge towards the solution with sufficient structure subdivision.

Therefore the parametric strain energy curves plotted against structure subdivision represent a one-parameter family of non-intersecting curves all of which approach the solution in the limit.

Melosh (1962, pp. 20-21) has shown that a sufficient condition for monotonic convergence of displacement matrices is that under progressive structure subdivision, the displacement field in a particular level of subdivision must be capable of representing the displacement field before that level of subdivision. He has noted (1962, pp. 31-32) that the Argyris-Melosh displacement matrix satisfies this criterion, and it is easily seen that the Turner-triangle matrix does the same.

Thus, it has been established that two strain energy curves corresponding to the parameters defined by the Argyris-Melosh and the Turner-triangle displacement matrices converge monotonically to the true solution. Strain energy curves conforming to other values of the parameter also converge to the solution but it has not been possible to prove that they do so monotonically.

In a specific problem, parametric strain energy curves may be obtained for different values of the parameter by the method suggested by Hooley and Hibbert. If these experimental curves appear to converge monotonically to some limit as the structure is progressively subdivided, it is a reasonable inference that the true strain energy will be between curves converging from above and below the supposed limit. The possibility must be admitted, however, that the apparent limit may not be the true solution. In this case the energy will not be bounded.

Note that the parametric matrices with parameter $a > a_{cr}$ are capable of representing constant stresses within the element, so that this capability has been found sufficient for providing convergence to the solution.

Note that the convergence criteria of Bazely et al (1965, pp. 2-3):

1. the displacement function must contain rigid body modes
2. the displacement function must be capable of expressing constant strain conditions

does not ensure that a matrix will provide good results with ordinary structure subdivision (not too coarse or too fine).

For example if plane stress matrices are chosen using $a = \frac{3-\frac{\mu}{2}}{8(1-\mu^2)}$ and $a = 10{,}000$, both satisfy the criteria but the first matrix will give a very high upper bound, and the second a very low lower bound, so that both results will be far away from the actual solution.

If, however, a set of matrices satisfying these criteria are derived then it is possible that one or more of them may be good matrices. The choice could be made by strain energy comparisons and comparisons with analytical solutions.

Thus amongst the parametric plane stress matrices, the Turner matrix is known to give the best results in specific cases (Hooley and Hibbert, pp. 46-47). Its convergence to the solution has been established. Therefore, it may be considered the best parametric plane stress matrix.

It should be pointed out that the best matrix on the basis of strain energy will not always give the best stresses. This is because the stresses are influenced by the structure subdivision and the manner of determining them, i.e.

1. within an element by linear transformation of element displacements

or 2. between nodes by displacement differentiation

or 3. at nodes by spreading nodal forces over tributary areas.

each of which may be most appropriate for a specific element matrix.

However a correspondence between the best matrix on the basis of strain energy and the best stresses is possible by judicious structure subdivision so as to pick up stress gradients and an appropriate manner of stress determination. Thus Hooley and Hibbert obtained the same ordering for extreme fibre stress of a cantilever, as is given for strain energy in Fig. 6.3.1, - the Turner matrix providing the best solution in both cases.

Note that the above arguments apply to the use of the same type of element matrix for the structure. If a structure contains element matrices with different strain energy levels, say a high strain energy matrix in one part (low value of parameter), and a low strain energy matrix in another (high value of parameter), then the combination may give a good overall strain energy result but poor local stresses.

6.3 Upper Bound on Strain Energy by using the Special Element

The use of the special square element developed in Chapter V for obtaining upper bounds, is illustrated by means of two examples, one of which has been previously analysed by Hooley and Hibbert.

Example 1 - Cantilever Beam as per Hooley and Hibbert (1966, pp. 46-47)

Hooley and Hibbert have compared the end deflection of a cantilever composed of a number of finite elements. The element matrices are varied by choosing different parameters. The results are reproduced with the nomenclature of this thesis in Fig. 6.3.1.

Note that the deflection under the load increases as the value

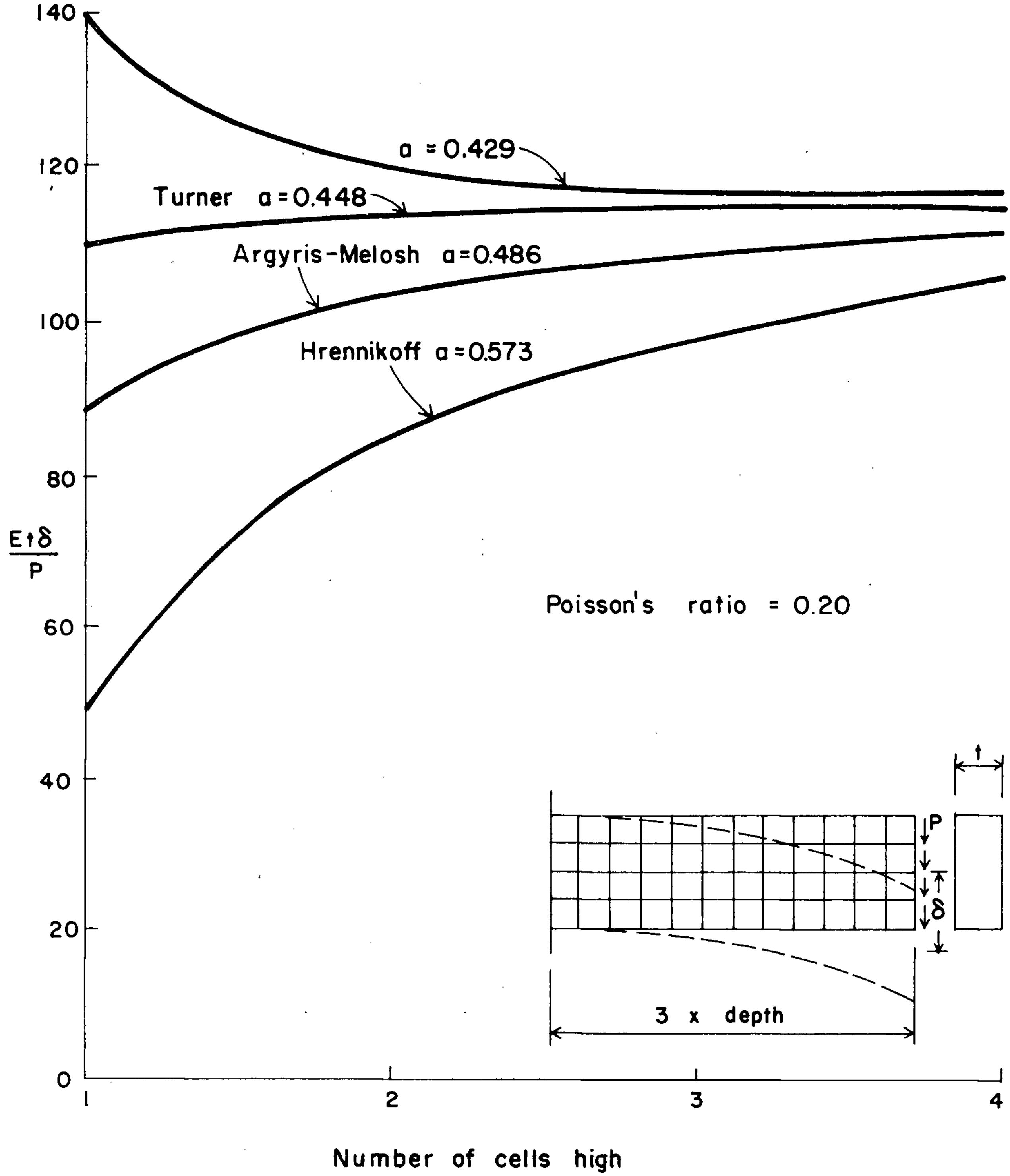


FIG. 6.3.1 BOUNDING THE CANTILEVER END DEFLECTION

of the parameter is reduced. Here the deflection is directly proportional to the strain energy, so that by varying the parameter for the element matrices the structure strain energy is varied. This behavior conforms to the theory developed in section 4.2.

Note also that all matrices converge towards the solution with increased subdivision as was anticipated by the discussions in section 6.2. Here the convergence in all cases is monotonic. However there is no guarantee for monotonic convergence in all plane stress problems.

Finally note than an upper bound subject to the limitations discussed on page 91 is obtained by a stiffness matrix with parameter $a = 0.429$.

Now upper bounds on the strain energy will be obtained by using the special element for two degrees of structure subdivision.

For the coarse subdivision, the psuedo-truss is statically determinate so that only the equilibrium solution is available for obtaining the upper bound.

For the fine subdivision, the psuedo-truss is statically indeterminate so that self-straining is possible, and an upper bound is obtained incorporating some self-straining.

The structure is divided into 12 square elements, forming the psuedo-truss system as shown in Fig. 6.3.2. This truss has

no. of joints, j = 32

no. of bars, n = 60

no. of constraints, R = 4 (for attachment to support)

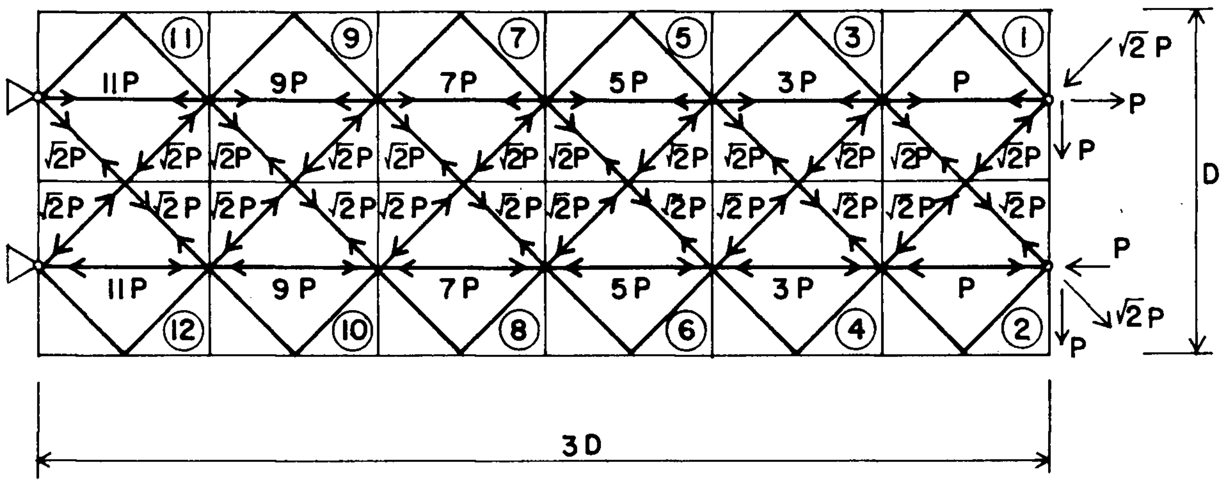


FIG. 6.3.2 PSUEDO-TRUSS FOR COARSE SUBDIVISION EQUILIBRIUM SOLUTION

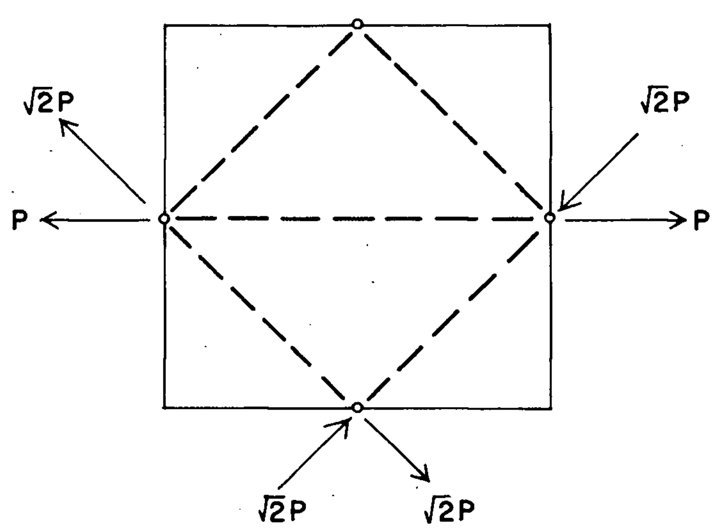


FIG. 6.3.3 BAR FORCES IN ELEMENT No. 1

Therefore no. of redundancies, $r = n + R - 2_j$

$$= 60 + 4 - 64$$

$$= 0$$

i.e. truss is statically determinate.

Uniform shear force is applied to the free end of the cantilever as two concentrated loads P at truss-joints at that end, and the bar forces determined.

Now the strain energy in each element is computed on the basis of the forces being transmitted through it.

Thus for element No. 1, the forces are transmitted as shown in Fig. 6.3.3.

From section 5.3, the strain energy on account of the transmission of these forces is given by

$$U_1^* = \frac{1}{2Et} \left[P_1^2 + Q_1^2 + R_1^2 + S_1^2 + T_1^2 \right.$$

$$\left. + \frac{1+\mu}{\sqrt{2}} P \ (Q_1 + R_1 + S_1 + T_1) - \mu \ (Q_1R_1 + R_1T_1 + S_1T_1 + Q_1S_1) \right]$$

where

$$P_1 = P$$

$$Q_1 = -\sqrt{2}P$$

$$S_1 = \sqrt{2}P$$

$$R_1 = 0$$

$$T_1 = 0$$

whence

$$U_1^* = \frac{1}{2Et} \ [5P^2 + 2\mu P^2]$$

$$= \frac{P^2}{2Et} \ [5 + 2\mu]$$

Obtaining the strain energy in the remaining elements in a similar way, the total strain energy in the structure is given by

$$U^{*} = \frac{2P^2}{Et}(155 + 6\mu)$$

Another equilibrium solution is obtained for psuedo-truss bars oriented as shown in Fig. 6.3.4. This psuedo-truss is also statically determinate, and the bar forces are as shown in Fig. 6.3.4.

The strain energy is calculated for each individual element on the basis of the forces being transmitted through it, and summed to give the structure strain energy, which is

$$U^{*} = \frac{2P^2}{Et}(155 + 6\mu)$$

This result is identical to the previous one. For $\mu = 0.2$

$$U^{*} = \frac{2P^2}{Et}(156.2)$$

For the fine subdivision, the structure is subdivided into 48 square elements, forming the psuedo-truss system shown in Fig. 6.3.6. This truss has

$$j = 112$$

$$n = 240$$

$$R = 8$$

Therefore

$$\begin{aligned} r &= n + R - 2j \\ &= 240 + 8 - 224 \\ &= 24 \end{aligned}$$

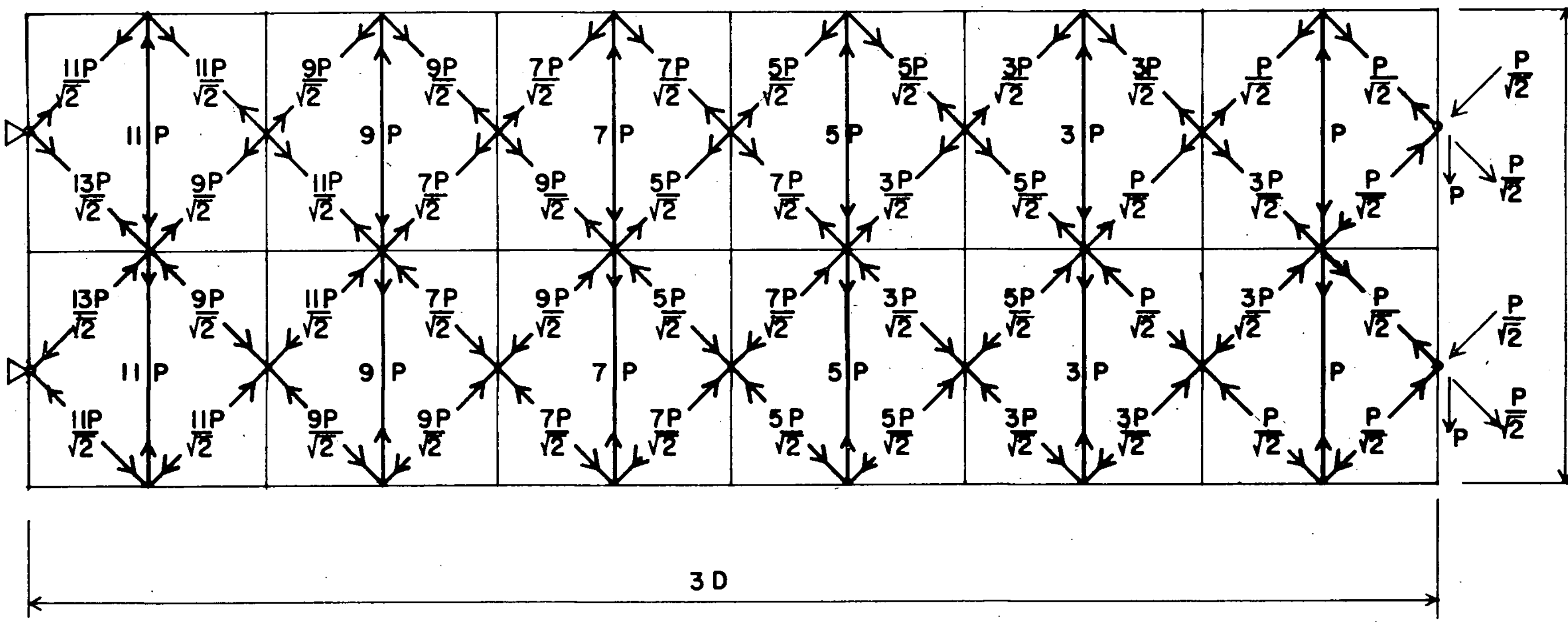


FIG. 6.3.4 PSUEDO-TRUSS FOR COARSE SUBDIVISION (DIFFERENT BAR ORIENTATION) EQUILIBRIUM SOLUTION

Therefore this truss is statically indeterminate and twenty-four self-straining solutions are possible.

However for the purpose of illustrating the procedure for incorporating the self-straining solutions with the equilibrium solution, three self-straining solutions are arbitrarily chosen as shown in Fig. 6.3.6. For obtaining the equilibrium solution, a step-wise distribution is assumed for the edge loading.

The element bar forces are now the sum of the equilibrium solution shown in Fig. 6.3.5 and the self-straining solution given in Fig. 6.3.6.

The element strain energies are summed to give the structure strain energy in terms of the applied load P, and the arbitrary self-straining forces Q, S and T.

The structure strain energy is given by

$$U^* = \frac{1}{9Et} \left[(2396+108\mu)P^2 + (69.5-3.5\mu)\,PQ + 180\,Q^2 - 124\,PS + 180\,S^2 - (130.5-10.5\mu) + 180\,T^2 \right]$$

Now Q, S, T are evaluated so as to minimize U^*, from the equations

$$\frac{\partial U^*}{\partial Q} = 0 \rightarrow (69.5-3.5\mu)\,P + 360\,Q = 0$$

$$\frac{\partial U^*}{\partial S} = 0 \rightarrow (-124)\,P + 360\,S = 0$$

$$\frac{\partial U^*}{\partial T} = 0 \rightarrow -(130.5-10.5\mu)\,P + 360\,T = 0$$

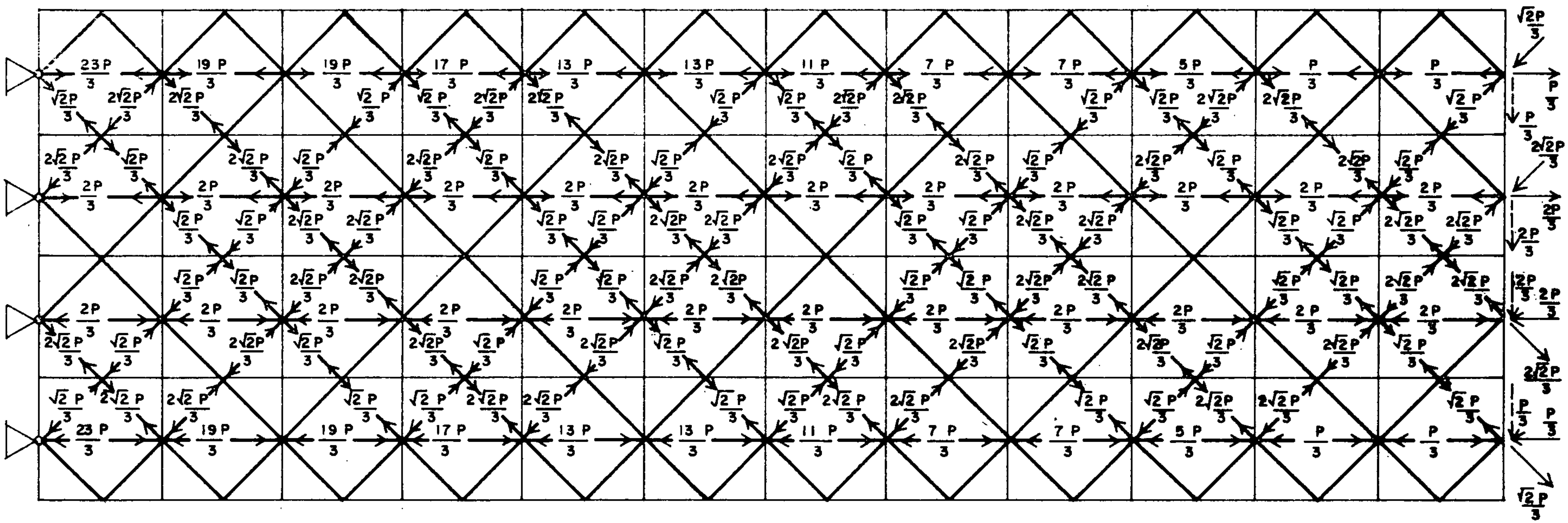


FIG. 6.3.5 CANTILEVER BEAM FINE SUBDIVISION — EQUILIBRIUM SOLUTION

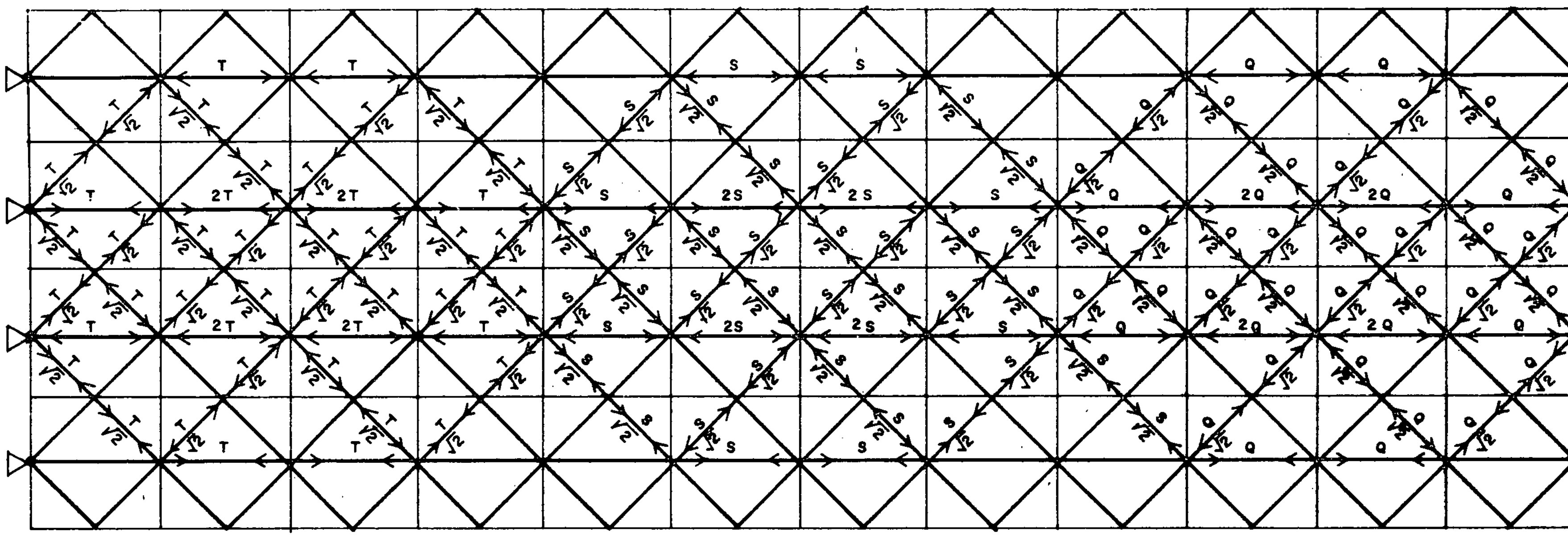


FIG. 6.3.6 CANTILEVER BEAM FINE SUBDIVISION - THREE SELF-STRAINING SOLUTIONS

From which

$$Q = \frac{-(69.5-3.5\mu)\ P}{360}$$

$$S = \frac{124}{360}\ P$$

$$T = \frac{(130.5-10.5\mu)\ P}{360}$$

Substituting

$$U^* = \frac{P^2}{9Et}\ [\ 2344.3 + 112.5\ \mu - .17\ \mu^2\ ]$$

For $\mu = 0.2$

$$U^* = \frac{2P^2}{Et}\ [\ 131.4]$$

The upper bound obtained by Hooley and Hibbert for the coarse subdivision is

$$U^* = \frac{2P^2}{Et}\ [\ 121\ ]$$

Table 6.3.1 shows a comparison of the results.

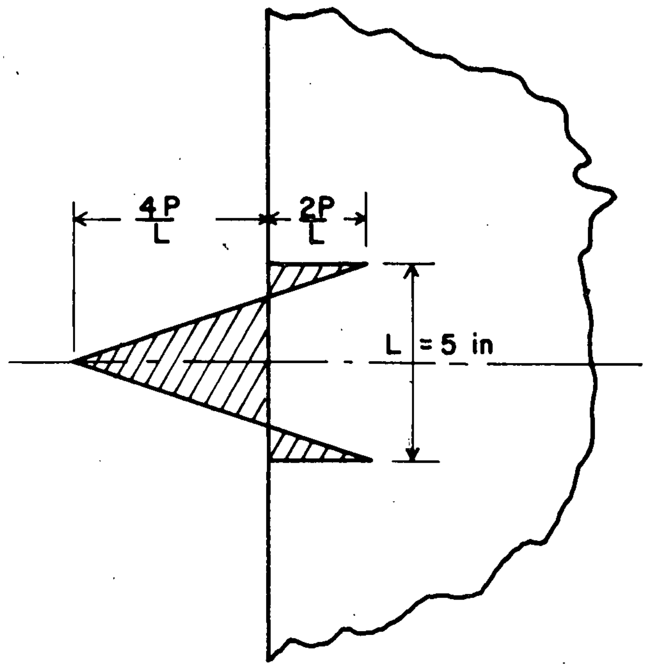
Table 6.3.1 - Upper Bound Comparisons for the Cantilever Beam

Method	Upper Bound on Strain Energy	% error w.r.t Hooley - Hibbert upper bound	Remarks
Special Element-Coarse sub-division	$\frac{2P^2}{Et}$ (156.2)	29.1%	
Special Element-Fine sub-division	$\frac{2P^2}{Et}$ (131.2)	8.4%	only 3 of 24 possible self-strainings used
Hooley-Hibbert variation of parameter	$\frac{2P^2}{Et}$ (121)	-	network corresponding to coarse subdivision

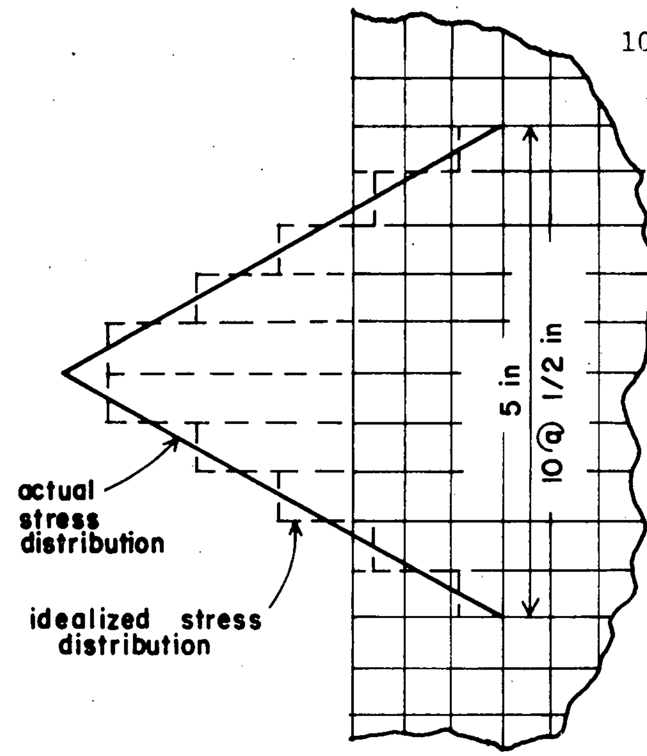
Note that the special element gives upper bounds which reduce with increased structure subdivision and inclusion of self-strainings. However it seems that for the same subdivision better results may be obtained by variation of the stiffness matrix parameter provided bounding by variation of parameter is valid.

Example 2 - Plate Under Action of Equal and Opposite End Loads

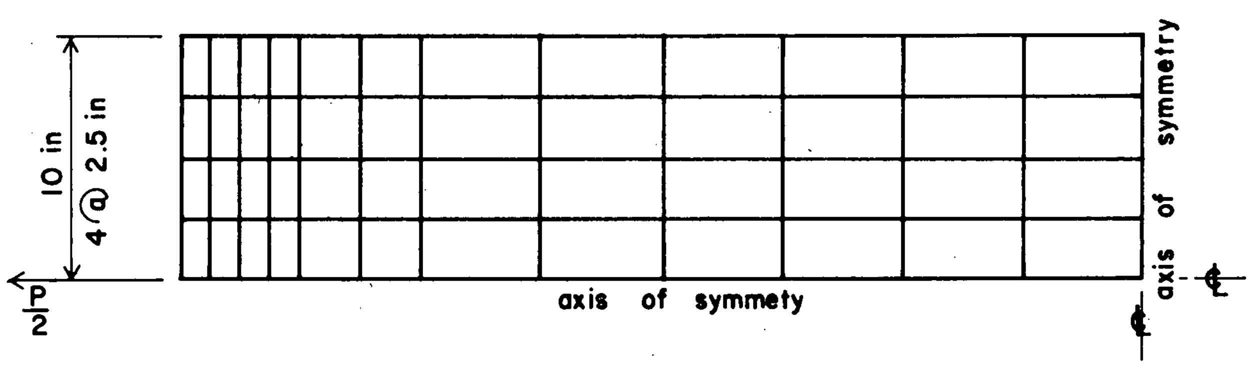
The example chosen is a simple adaptation of a problem analysed by Gallagher et al (1962, pp. 43-44) which consisted of a plate under plane stress acted upon by equal and opposite concentrated loads, as shown in Fig. 6.3.7. Using the Argyris-Melosh displacement stiffness matrix, the concentrated loading is equivalent to the linear stress distribution over the width of the subdivision, as shown. In order to use the special element for obtaining an upper bound, this linearly varying stress distribution would have to be approximated by a step-wise distribution of stress. One such step-wise distribution using 1/2 in. square elements is shown in Fig. 6.3.7. The corresponding psuedo-truss problem is large in magnitude as may be seen by the following data.



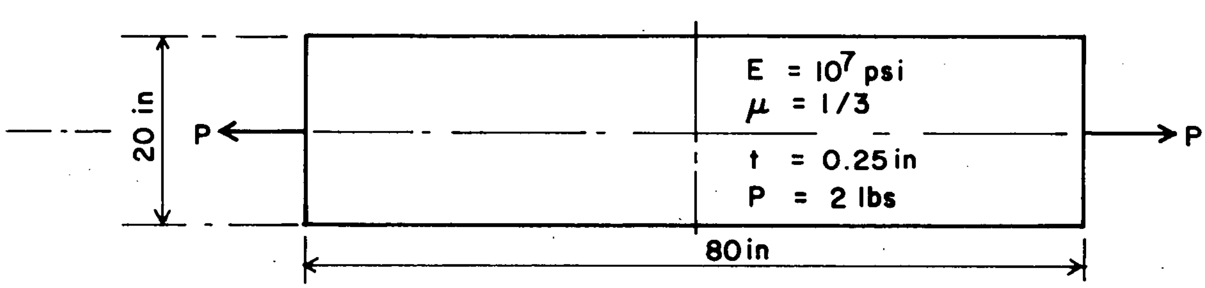
ACTUAL STRESS LOADING



POSSIBLE STRUCTURE SUBDIVISION AND LOAD IDEALIZATION FOR UPPER BOUNDING



ANALYTICAL MODEL OF ONE QUADRANT



GALLAGHER PROBLEM

FIG. 6.3.7 PLATE UNDER CONCENTRATED END LOADS

$$j = 13000$$

$$n = 32000$$

$$R = 4$$

$$r = 32000 + 4 - 26000$$

$$= 6004$$

Therefore for the purpose of illustration of the method a plate under distributed end loading as shown in Fig. 6.3.8 is solved using the Argyris displacement matrix to obtain a lower bound, and the special element to obtain an upper bound for the same network subdivision.

The displacements under the loads from the displacement analysis are respectively 0.0530 in. and 0.0364 in. so that a lower bound on the strain energy may be expressed as

$$\frac{(2P)^2}{Et} \quad (1.118)$$

The complete psuedo-truss has

$$j = 110$$

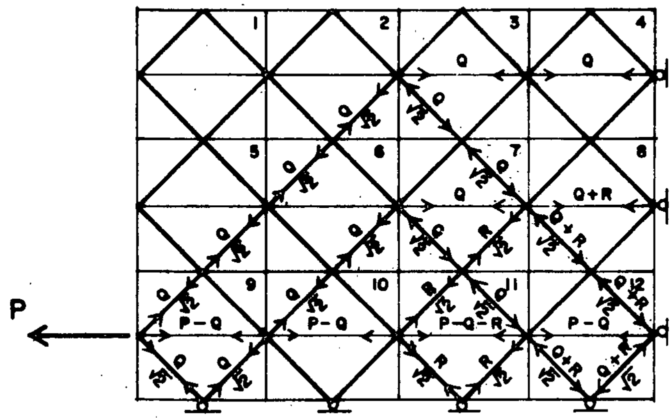
$$n = 240$$

$$R = 4$$

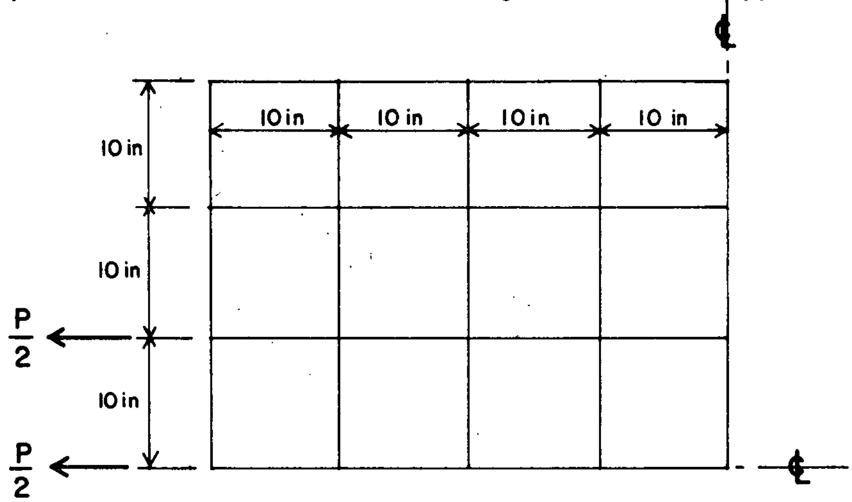
$$r = 240 + 4 - 220$$

$$= 24$$

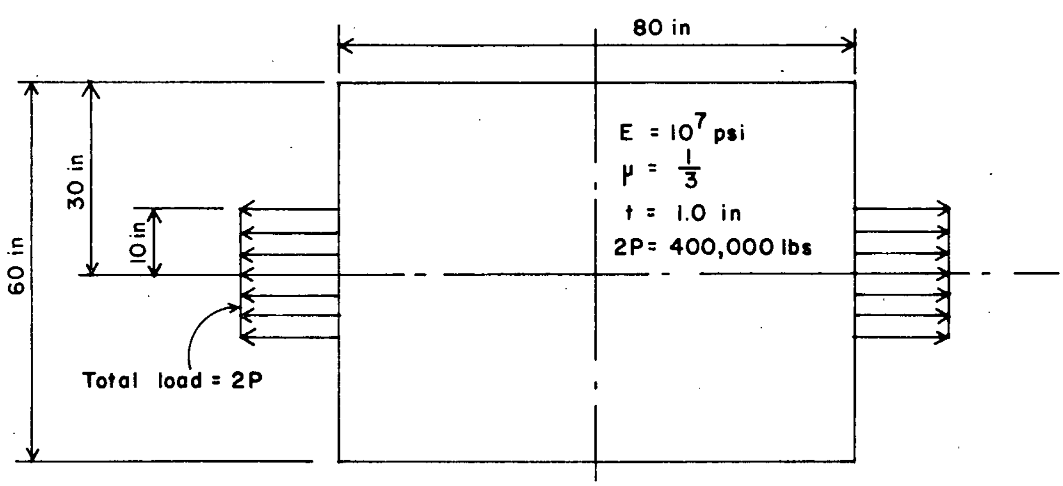
For obtaining the upper bound, two self-straining solutions are arbitrarily chosen. These are given by the Q's and R's bar forces, the equilibrium solution being given by P's.



Equilibrium and some self-straining solutions for upper bound



Analytical model of one quadrant for displacement analysis



Complete Problem

FIG. 6.3.8 PLATE UNDER DISTRIBUTED END LOADS

On summing the element strain energies, the structure strain energy is given by

$$U^* = \frac{1}{Et} \left[8P^2 - 16\,PQ + 28\,Q^2 - 4\,PR + 9\,R^2 + 16\,QR \right]$$

Again for minimizing U*

$$\frac{\partial U^*}{\partial Q} = 0 \rightarrow -16P + 56Q + 16\,R = 0$$

$$\frac{\partial U^*}{\partial R} = 0 \rightarrow -4P + 16Q + 18\,R = 0$$

whence

$$Q = 0.2978\,P$$

$$\text{and} \quad R = -0.04255\,P$$

So that the upper bound is

$$U^* = \frac{(2P)^2}{Et} \quad (1.425)$$

Therefore the strain energy of the solution is bracketed between

$$\frac{(2P)^2}{Et}\,(1.118) < U < \frac{(2P)^2}{Et}\,(1.425)$$

Note that the upper bound would come closer if more of the self-straining solutions were incorporated. Thus this method of obtaining upper bounds can be efficiently utilized only when the computations are performed with the help of a digital computer.

CHAPTER VII

SUMMARY AND RECOMMENDATIONS

The investigations made in this thesis are summarized as follows.

1. Plane stress stiffness matrices are derived explicitly for square isotropic elements under different assumptions on the stress distribution within the finite element.

   (i) An (8 x 8) matrix is obtained under the assumption originally used by Gallagher of uniform σ_x, σ_y, linear τ_{xy}.

   (ii) Two (10 x 10) matrices are obtained under the assumption of linear σ_x, σ_y and τ_{xy}, using interior nodal translations and corner edge rotations as additional generalized displacements. These matrices do not appear suitable for general usage but will perform as well as the Turner matrix under the same nodal loads.

   (iii) A (12 x 12) matrix is derived under the assumption of hyperbolic σ_x, σ_y and parabolic τ_{xy}, again, exemplifying the use of edge rotations at corners as additional generalized displacements. This matrix is found unsuitable for general usage as it behaves unexpectedly with varying Poisson's ratio.

2. A method is proposed for choosing stiffness matrices for all classes of finite elements (i.e., elements for plane stress, plate-bending, shells, etc.) on the basis of strain energy. The "best" stiffness matrix from an available set is defined as the one which yields the closest approximation to the true strain energy of deformation. In order to make this choice, a comparison is made of the strain energy of deformation produced within a finite element by the different matrices under the same nodal loads. It is shown that such comparisons require a study of the quadratic form of the inverse difference matrix i.e., $(K_1^{-1} - K_2^{-1})$.

   (i) It is proved that the quadratic form of the inverse difference matrix may be obtained by a study of the quadratic form of the stiffness difference matrix $(K_1 - K_2)$ with consequent simplification of the process of comparison.

   (ii) It is proved that the results of the element matrix comparisons, generally, apply to a structure composed of them.

   (iii) It is noted that comparisons under the same nodal loads do not always portray the behavior of matrices for which the consistent nodal loading may be different or for matrices of different orders. However, the same nodal loading can always be one of the types of loading to which all matrices may be subjected, and it is hypothesized that comparisons under this condition will be useful in evaluating the matrices.

   (iv) It is shown that the strain energy of a finite element under normalized loads is bounded between the maximum and minimum eigenvalues of the inverse matrix, and hence it is shown that Melosh's hypothesis for choosing matrices corresponds to a comparison of element strain energy.

3. The theoretical comparison procedures developed above are utilized in a study of parametric matrices for square isotropic elements.

   (i) It is shown that in addition to the lattice model and other matrices observed by Hooley and Hibbert to belong to the class of parametric matrices, other important matrices also belong to this class. These include the Pian displacement matrix, the displacement matrix formed by using the Turner triangles and the Gallagher matrix.

   (ii) An explicit parametric inverse is obtained for the parametric stiffness matrices.

   (iii) The quadratic form of the parametric matrices is studied and it is found that the natural stiffness matrix is indefinite for $0 \leq a < a_{cr}$, and positive definite for $a > a_{cr}$.

   (iv) Explicit parametric eigenvalues are obtained for the inverse difference matrix and the stiffness difference matrix, and it is verified that they give identical results for the matrix strain energy comparisons.

   (v) The explicit parametric inverse is used to prove that all parametric matrices give the exact strain energy under uniform nodal loads (i.e. the element deforms exactly under constant σ_x, σ_y, τ_{xy}). It is shown that the stiffness matrix parameter represents a measure of the strain energy under non-uniform nodal loads (i.e., under loads tending to bend the element). The critical value of the parameter corresponds to an unstable configuration in which the slightest non-uniform load produces infinite displacement. When the parameter approaches infinity, the element becomes so stiff in bending that no amount of non-uniform load can cause the element to bend.

(vi) It is proved that if strain energy curves are drawn with respect to structure subdivision, then no two curves will intersect. It is proved that all parametric strain energy curves will approach the true solution with progressive structure subdivision. This includes the Turner matrix, the Pian matrices, the Argyris-Melosh matrix and the Gallagher matrix. Monotonic convergence to the solution is not theoretically established for all parametric strain energy curves. If, however, in a specific problem, the strain energy curves are observed to converge monotonically, then it is a reasonable expectation (see page 91) that the strain energy of the solution may be bounded by varying the matrix parameter according to the procedure suggested by Hooley and Hibbert.

(vii) A strain energy ordering is obtained for the parametric matrices, and the following conclusions are drawn with respect to matrices for square isotropic elements. The Pian matrix is the best displacement matrix. The Gallagher matrix is inferior to the Turner, Pian and Argyris-Melosh matrices. Constant stress tri-nodal triangles are generally inferior to the use of square elements. Matrices satisfying microscopic equilibrium or capable of representing constant stresses will not necessarily yield good results.

(viii) The ordering of strain energy of the parametric matrices on the basis of the eigenvalues of the inverse, is examined, and it is found to correspond with that obtained by a study of the difference matrices. It is verified that the correspondence occurs because the eigenvectors of the inverse matrices are similar.

4. A method is proposed for obtaining upper bounds on the strain energy of a region under plane stress by replacing the continuum with a psuedo-truss system, the bar forces of which provide the equilibrium and self-straining solutions. Two examples of its application are presented and an indication is obtained in a specific case where bounding by variation of parameter seems valid that a better upper bound may be obtained by varying the matrix parameter for the same structure subdivision.

The following recommendations are made for further study.

1. The effect of using equal nodal loads instead of consistent nodal loading in making strain energy comparisons should be investigated so as to determine whether any qualifications are necessary in the comparison procedure.
2. When comparing matrices of different orders the effect of arbitrarily assuming some zero nodal loads should be investigated so as to improve upon the comparison procedure in such cases.

3. In the case where the special difference matrices are indefinite, the effect of the relative number and magnitude of the positive and negative eigenvalues and the load vectors defined by the corresponding eigenvectors should be studied numerically.
4. Sets of stiffness matrices for solids, plate and shell problems incorporating rigid body modes and capable of representing uniform strains should be developed so as to determine if parameters governing the element strain energy level may be found for obtaining bounds.
5. Available stiffness matrices for solids, plates and shells should be evaluated on the basis of the strain energy criterion developed in this thesis.
6. A digital computer program should be developed for the upper bounding procedures using the psuedo-truss system.
7. Studies are also desirable to investigate the relationship between the sensitivity of overall strain energy bounding and local bounds on stresses and displacements.

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APPENDIX A

TWO ENERGY THEOREMS

Theorem of Minimum Potential Energy

The theorem of minimum potential energy (Sokolnikoff, 1956, pp. 384-385) states that of all displacements satisfying the given displacement boundary conditions, those that satisfy the equilibrium configuration make the potential energy a minimum, where the potential energy is given by

$$V = U - \int_{\Gamma} T_i u_i d\Gamma - \int_{v} F_i u_i dv$$

where U = strain energy

Γ = the portion of surface where surface forces T_i are prescribed.

v = the portion of the body where the body forces F_i are prescribed.

Note that by Clapeyron's Theorem (Sokolnikoff, 1956, p.86), the strain energy of an equilibrium state is equal to one-half the work of the external forces on the displacements of the solution.

$$\text{i.e.} \quad 2U = -\int_{\Gamma} T_i u_i d\Gamma - \int_{v} F_i u_i dv$$

Therefore the potential energy

$$\begin{aligned} V &= U - 2U \\ &= -U \end{aligned}$$

Therefore a minimization of the potential energy implies a maximization of the strain energy.

Theorem of Minimum Complementary Energy

The theorem of minimum complementary energy (Sokolnikoff, 1956, p. 389) or least work states that of all the stress states satisfying the differential equations of equilibrium in the interior, and the stress boundary conditions, the actual state of stress (i.e. that satisfying compatability) makes the complementary energy a minimum where the complementary energy is given by

$$V^* = U - \int_\Gamma T_i u_i \, d\Gamma$$

where U = strain energy

and Γ = the portion of the surface where the displacements u_i are prescribed.

Note that under a stress boundary condition (i.e. no non-zero displacements are prescribed), the complementary energy

$$V^* = U$$

whence a minimization of the complementary energy implies a minimization of the strain energy.

APPENDIX B

STIFFNESS MATRIX UNDER ASSUMPTION OF DISPLACEMENT FUNCTION WITH LINEAR EDGE DISPLACEMENTS

Assumed displacement function

$$u = a_3xy$$

$$v = a_7xy$$

whence nodal displacements are (refer Fig. 3.2.2)

$$u_1 = a_3x_oy_o$$

$$u_2 = a_3(x_o+a)\ y_o$$

$$u_3 = a_3(x_o+a)\ (y_o+a)$$

$$u_4 = a_3x_o(y_o+a)$$

$$v_1 = a_7x_oy_o$$

$$v_2 = a_7(x_o+a)\ y_o$$

$$v_3 = a_7(x_o+a)\ (y_o+a)$$

$$v_4 = a_7x_o(y_o+a)$$

Strains from the displacement function are

$$\varepsilon_x = u_x = a_3 y$$

$$\varepsilon_y = u_y = a_7 x$$

$$\gamma_{xy} = u_y + v_x = a_3 x + a_7 y$$

Hence element strain energy is given by

$$U = \frac{Et}{2(1-\mu^2)} \iint [\varepsilon_x^2 + \varepsilon_y^2 + 2\mu\, \varepsilon_x \varepsilon_y + \frac{1-\mu}{2} \gamma^2_{xy}] dxdy$$

$$= \frac{Et}{2(1-\mu^2)} \int_{x_o}^{x_o+a} \int_{y_o}^{y_o+a} [(a_3 y)^2 + (a_7 x)^2 + 2\mu\, (a_3 y) \quad (a_7 x)$$

$$+ \frac{1-\mu}{2} (a_3 x + a_7 y)^2] \, dxdy$$

On integrating one obtains

$$U = \frac{Eta}{2(1-\mu^2)} [a_3^2 (y_o^2 a + y_o a^2 + \frac{a^3)}{3} + a_7^2 (\mathbf{x}_o^2 a + x_o a^2 + \frac{a^3)}{3}$$

$$+ 2\mu \{ \frac{a_3 a_7}{4a} (2ax_o + a^2)(2ay_o + a^2) \} + \frac{1-\mu}{2} \{ a_3^2 (x_o^2 a + x_o a^2 + \frac{a^3)}{3}$$

$$+ a_7^2 (y_o^2 a + y_o a^2 + \frac{a^3)}{3} + \frac{a_3 a_7}{2a} (2ax_o + a^2)(2ay_o + a^2) \}]$$

From nodal displacements the following relations are obtained

$$\left\{ \frac{a_3}{2} (2y_o + a) \right\} = \frac{u_2 - u_1 + u_3 - u_4}{2a}$$

$$\left\{ \frac{a_3}{2} (2x_o + a) \right\} = \frac{u_4 - u_1 + u_3 - u_2}{2a}$$

$$a_3 = \frac{u_3 - u_4 - u_2 + u_1}{a^2}$$

$$\left\{ \frac{a_7}{2} (2y_o + a) \right\} = \frac{v_2 - v_1 + v_3 - v_4}{2a}$$

$$\left\{ \frac{a_7}{2} (2x_o + a \ ) \right\} = \frac{v_4 - v_1 + v_3 - v_2}{2a}$$

$$a_7 = \frac{v_3 - v_4 - v_2 + v_1}{a^2}$$

On substituting these values for the arbitrary coefficients in the strain energy expression, one obtains

$$U = \frac{Et}{8(1-\mu^2)} \left[(u_2-u_1+u_3-u_4)^2 + (v_4-v_1+v_3-v_2)^2 + 2\mu(u_2-u_1+u_3-u_4)(v_4-v_1+v_3-v_2) \right]$$

$$+ \frac{Et}{16(1+\mu)} \left[u_4 - u_1 + u_3 - u_2 + v_2 - v_1 + v_3 - v_4 \right]^2$$

$$+ \frac{Et(3-\mu)}{48(1-\mu^2)} \left\{ (u_3 - u_4 - u_2 + u_1)^2 + (v_3 - v_2 - v_4 + v_1)^2 \right\}$$

Again using the formulation

$$K_{ij} = \frac{\partial^2 U}{\partial q_i \partial q_j} \qquad i,j = 1,n$$

The stiffness matrix is of form given in Table 3.2.1.1 with

$$K_{11} = \frac{\lambda}{24}(24 - 8\mu) \quad \text{where} \quad \lambda = \frac{Et}{2(1-\mu^2)}$$

$$K_{21} = \frac{\lambda}{24}(6 + 6\mu)$$

$$K_{31} = \frac{\lambda}{24}(-6 + 18\mu)$$

$$K_{41} = \frac{\lambda}{24}(-12 - 4\mu)$$

$$K_{51} = \frac{\lambda}{24}(-12 + 4\mu)$$

$$K_{61} = \frac{\lambda}{24}(-6 - 6\mu)$$

$$K_{71} = \frac{\lambda}{24}(6 - 18\mu)$$

$$K_{81} = \frac{\lambda}{24}(8\mu)$$

This stiffness matrix is identical to that obtained by Argyris (1955, pp. 125-126) and Melosh (1962, p. 31-32) as the same type of displacement function has been used in each case.

APPENDIX C

BOUNDING ELASTIC BEHAVIOR

Here the basis and procedure for obtaining bounds on the nodal displacements and the mean strain between nodes is given.

De Veubeke (1962, pp. 185-186) has shown that bounds on flexibility influence coefficients may be obtained from bounds on structure strain energy.

The bounds on the direct influence coefficients i.e. the diagonal elements of the flexibility matrix C_{ii} are obtained by applying single loads corresponding to the generalized displacements.

Thus under a single load P_i, with corresponding displacement δ_i, the influence coefficient C_{ii} is given by

$$\delta_i = C_{ii}P_i$$

and the strain energy can be written

$$U = \frac{1}{2} C_{ii}P_i^2 \qquad \text{(exact solution)}$$

From a compatible displacement analysis we obtain

$$\underline{U} = \frac{1}{2} \underline{C}_{ii}P_i^2 \qquad \text{(lower bound)}$$

From an equilibrium analysis we obtain

$$\overline{U} = \frac{1}{2} \overline{C}_{ii}P_i^2 \qquad \text{(upper bound)}$$

Therefore

$$\frac{1}{2} \quad \overline{C_{ii}} P_i^2 \quad \geq \quad \frac{1}{2} \; C_{ii} P_i^2 \quad \geq \quad \frac{1}{2} \; \underline{C_{ii}} P_i^2$$

and after dividing by $\frac{1}{2} P^2$

$$\overline{C}_{ii} \geq \quad C_{ii} \geq \quad \underline{C}_{ii}$$

The bounds on the cross-influence coefficients (off-diagonal elements of the flexibility matrix C_{ij} $i \neq j$) are obtained by applying two loads at a time, one corresponding to the displacement i, the other corresponding to the displacement j.

Let
$$\delta_i = C_{ii} P_i + C_{ij} P_j$$

$$\delta_j = C_{ij} P_i + C_{jj} P_j$$

be the exact displacements associated with the two loads P_i and P_j.

If $\underline{C}_{ii}$, $\underline{C}_{ij}$, $\underline{C}_{jj}$ are approximate influence coefficients obtained from a compatible approach, and $\overline{C}_{ii}$, $\overline{C}_{ij}$, $\overline{C}_{jj}$ those from an equilibrium approach, then from previous results we know

$$\overline{C}_{ii} \geq C_{ii} \geq \underline{C}_{ii} \qquad \text{and} \qquad \overline{C}_{jj} \geq C_{jj} \geq \underline{C}_{jj}$$

Let $P_j = \lambda P_i$

then the exact strain energy is given by

$$U = \frac{1}{2} \; P_i^2 \; (C_{ii} + 2\lambda \, C_{ij} + \lambda^2 \; C_{jj})$$

Obtaining the upper and lower bounds on the strain energy as before and dividing by $\frac{1}{2} P_i^{\,2}$ we can write

$$\bar{C}_{ii} + 2\lambda\,\bar{C}_{ij} + \lambda^2\bar{C}_{jj} \geqq C_{ii} + 2\lambda\,C_{ij} + \lambda^2 C_{jj} \geqq \underline{C}_{ii} + 2\lambda\,\underline{C}_{ij} + \lambda^2\underline{C}_{jj}$$

Taking the first inequality, and solving for C_{ij} assuming λ to be positive, we get

$$2C_{ij} \leqq \frac{1}{\lambda}\left(\bar{C}_{ii} - C_{ii}\right) + 2\bar{C}_{ij} + \lambda\left(\bar{C}_{jj} - C_{jj}\right)$$

$$\leqq \frac{1}{\lambda}\left(\bar{C}_{ii} - \underline{C}_{ii}\right) + 2\bar{C}_{ij} + \lambda\left(\bar{C}_{jj} - \underline{C}_{jj}\right)$$

The positive λ giving the smallest upper bound is found to be

$$\lambda = \sqrt{\left\{\left(\bar{C}_{ii} - \underline{C}_{ii}\right) \big/ \left(\bar{C}_{jj} - \underline{C}_{jj}\right)\right\}}$$

whence $$C_{ij} \leqq \bar{C}_{ij} + \sqrt{\left\{\left(\bar{C}_{ii} - \underline{C}_{ii}\right)\left(\bar{C}_{jj} - \underline{C}_{jj}\right)\right\}}$$

Similarly, taking λ to be negative we get from the first inequality

$$C_{ij} \geqq \bar{C}_{ij} - \sqrt{\left\{\left(\bar{C}_{ii} - \underline{C}_{ii}\right)\left(\bar{C}_{jj} - \underline{C}_{jj}\right)\right\}}$$

Treating the second inequality as for the first we get new bounds

$$C_{ij} \leqq \underline{C}_{ij} + \sqrt{\left\{\left(\bar{C}_{ii} - \underline{C}_{ii}\right)\left(\bar{C}_{jj} - \underline{C}_{jj}\right)\right\}}$$

$$C_{ij} \geqq \underline{C}_{ij} - \sqrt{\left\{\left(\bar{C}_{ii} - \underline{C}_{ii}\right)\left(\bar{C}_{jj} - \underline{C}_{jj}\right)\right\}}$$

Out of the two upper and lower bounds, the closest bounds are chosen.

Having obtained bounds on the flexibility influence coefficients, it is then possible to obtain bounds on the nodal displacements of the structure for any given nodal loads.

Thus the displacement δ_i would be given by

$$\delta_i = (\bar{C}_{i1}, \underline{C}_{i1})\ P_1 + (\bar{C}_{i2}, \underline{C}_{i2})\ P_2 + \ldots + (\bar{C}_{in}, \underline{C}_{in})\ P_n$$

The maximum and minimum value for each term on the right hand side can then be determined and appropriately summed to give bounds for δ_i.

Knowing the bounds on the displacements of adjacent nodes, bounds may be obtained on the mean strain between them.