STeady State Vibrations of Framed Structures

by

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in the Department
of
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ABSTRACT

This thesis is concerned with the determination of Internal Stress Resultants produced in different structural elements of framed structures due to harmonic disturbances.

The analysis of frames under vibrating loads has so far been dealt with with physical lumping of the structural mass at the node points where the stiffness influence coefficients are defined. To improve the accuracy of the results, a consistent mass matrix approach is dealt with in few of the latest solutions giving higher degree of precision compared with the results of problems solved by lumped mass system.

Keeping in mind the criteria of efficiency for solving any structural dynamic response problem, a stiffness matrix is generated which depends upon the distributed mass of the member and the frequency of vibrations of the impressed force. The stiffness influence coefficients are derived for a plane frame member of uniformly distributed mass from the general differential equation of motion under longitudinal and lateral vibrations.

This concept is then extended to generate a stiffness matrix for a space frame member including torsional vibrations. The effects of rotary inertia and shear deformations being predominant for framed structures such as turbine foundations, are also included. The
generated stiffness matrix is called the "Frequency and Mass Dependent Stiffness Matrix" (F.M.) which is used for the dynamic analysis of some structural problems and the results are compared with those of lumped mass and consistent mass matrix approaches. The F. M. Stiffness Matrix approach is then applied to the dynamic analysis of full size turbine foundations in comparison with other methods of solution. In this thesis major emphasis is stressed upon the dynamic analysis of turbine foundations.
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### DEFINITION OF SYMBOLS

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<tr>
<td>{a}</td>
<td>Column vector of constants.</td>
</tr>
<tr>
<td>A</td>
<td>Area of cross-section.</td>
</tr>
<tr>
<td>G</td>
<td>Torsional Rigidity.</td>
</tr>
<tr>
<td>[d]</td>
<td>Diagonal matrix established out of eigenvalues of stiffness matrix.</td>
</tr>
<tr>
<td>[D]</td>
<td>Diagonal matrix established out of the eigenvalues of the structure.</td>
</tr>
<tr>
<td>E</td>
<td>Modulus of elasticity.</td>
</tr>
<tr>
<td>[F]</td>
<td>Transformation matrix.</td>
</tr>
<tr>
<td>G</td>
<td>Shear modulus.</td>
</tr>
<tr>
<td>I_{xx}, I_{zz}</td>
<td>Moment of inertia of the member about x-x and z-z axis of bending.</td>
</tr>
<tr>
<td>J</td>
<td>Polar moment of inertia.</td>
</tr>
<tr>
<td>k_{ij}</td>
<td>Stiffness Influence Coefficient.</td>
</tr>
<tr>
<td>K, K'</td>
<td>Coefficients defining the effects of rotary inertia and shear deformations on the transverse vibrations of a frame member.</td>
</tr>
<tr>
<td>[K]_{F,M.}</td>
<td>Frequency and mass dependent stiffness matrix of a frame member.</td>
</tr>
<tr>
<td>L</td>
<td>Length of the frame member.</td>
</tr>
<tr>
<td>[m]</td>
<td>Modal matrix established by the normalized eigenvectors of the stiffness matrix.</td>
</tr>
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<td>[M]</td>
<td>Diagonal mass matrix of a member.</td>
</tr>
<tr>
<td>[M]</td>
<td>Consistent mass matrix of a member.</td>
</tr>
<tr>
<td>Symbol</td>
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<td>-------------</td>
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<tr>
<td>$N$</td>
<td>Axial force due to longitudinal vibrations.</td>
</tr>
<tr>
<td>$P$</td>
<td>External forces existing on the structure.</td>
</tr>
<tr>
<td>$r^2$</td>
<td>Square of radius of gyration of the cross-section of a member.</td>
</tr>
<tr>
<td>${u}$</td>
<td>Generalized displacement vector.</td>
</tr>
<tr>
<td>$U$</td>
<td>Axial displacement of a member under longitudinal vibrations.</td>
</tr>
<tr>
<td>${U}$</td>
<td>Vector of displacements prescribed at the nodal points.</td>
</tr>
<tr>
<td>$V$</td>
<td>Shear force on the cross-section of a member.</td>
</tr>
<tr>
<td>$W_i$</td>
<td>Potential energy of the member.</td>
</tr>
<tr>
<td>${x}$</td>
<td>Displacement vector.</td>
</tr>
<tr>
<td>${X}$</td>
<td>Acceleration vector.</td>
</tr>
<tr>
<td>$X$</td>
<td>Body forces of the structural element.</td>
</tr>
<tr>
<td>$Y$</td>
<td>Lateral deflection of a beam under vibrating loads.</td>
</tr>
<tr>
<td>$z$</td>
<td>Lateral deflection of a beam, function of coordinate as well as of time.</td>
</tr>
<tr>
<td>$z_b$</td>
<td>Lateral deflection of a beam due to pure bending.</td>
</tr>
<tr>
<td>$z_s$</td>
<td>Lateral deflection of a beam due to shear forces only.</td>
</tr>
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</table>
(α, β, γ, δ, ε, θ) ... Coefficients defining the Stiffness Influence Coefficients of the Frequency and Mass dependent Stiffness Matrix of a fix-fix beam.

γ ... ... ... Unit shear strain.

ε ... ... ... Strain in a member.

θ ... ... ... Angle of twist of the member under torques.

μ ... ... ... Shape factor, depends upon the cross-section of the member.

ρ ... ... ... Mass per unit length of the member.

σ ... ... ... Normal stress at any section of the member.

τ ... ... ... Shear stress at any section of the member.

ω ... ... ... Frequency of vibrations of the impressed force in rad./sec.

ω_n ... ... ... Natural frequency of vibration of any structure.
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CHAPTER I

INTRODUCTION

The increasing popularity of high speed digital computers has made engineers understand that in structural engineering technology today there is greater need for efficient, accurate and economical methods of solving both statical and dynamical problems. The stiffness matrix formulation of general dynamic response problems has been handled by an influence coefficient approach using a stiffness matrix defining the elastic characteristics and a mass matrix defining the inertial characteristics of the structure. The stiffness matrix formulation for various kinds of structures has been thoroughly discussed and is well documented. The mass matrix construction, at the same time, is usually accomplished by the physical lumping of the structural mass at points where influence coefficients are defined. The computed natural frequencies and mode shapes using the lumped mass method often differ from the exact values. To improve the accuracy of the dynamical analysis as it is effected by the mass matrix, a consistent mass matrix construction was investigated by various authors such as Archer (1)*, Przemieniecki (2) and others. The consistent mass matrix accounts for the actual distribution of the mass throughout the structure.

*Numerals in parenthesis refer to References given at page
For best use of the computer and for better efficiency in the dynamic analysis of frame structures, a frequency and mass dependent stiffness matrix** approach is evolved in this thesis. In the technique employed to generate the frequency and mass dependent stiffness matrix, the differential equation of motion for a member of uniformly distributed mass is exploited. The general differential equation of motion is written for longitudinal, lateral and torsional vibrations. This equation is also extended for the effects of rotary inertia and shear deformations since they play an appreciable role in the dynamic analysis of turbine foundations. From the known sets of end conditions, the general differential equation of motion is solved to represent the response of the structure. The stiffness influence coefficients of the frequency and mass dependent stiffness matrix are derived conforming to the end conditions of the structural member. The generated frequency and mass dependent stiffness matrix is a function of the frequency of the impressed force. This frequency becomes natural frequency when the impressed force is of zero magnitude. In the dynamic analysis of the frame structures if the frequency of the impressed force is the same as the natural frequency of the structure, the determinant of the frequency and mass dependent stiffness matrix approaches zero. This is a case of resonance and needs special attention during the design

** The abbreviation F.M. = Frequency and Mass dependent.
of the structure. The frequency and mass dependent stiffness matrices generated in this thesis are for fix-fix, fix-pin, fix-free and pin-pin type of structural elements.

The turbine foundations are usually given the shape of frame structures. This is done for the reasons of easy and proper distribution of the vibrating loading produced by the self weight of the generator and the rotor which are accommodated by every turbine foundation. The rotor of the turbine usually has some tilt in its alignment due to some initial inclination in the axis of the rotor and due to thermal expansions. This tilt produces moments and extra loading. The other reason for which turbine foundations assume the shape of frame structure is to make the turbine foundation structure easily separated from adjoining structures. This technique of separating the structures is accomplished by providing a mat foundation resting on several layers of cork and bituminous material. The mat foundation holds the legs of the frame and absorbs all the vibrating shocks. In the analysis of turbine foundations using the frequency and mass dependent stiffness matrix method, the mat foundation is idealized by spring supports of equivalent spring constants. The legs of the turbine foundations are usually made out of steel and the remaining part of the frame is made out of reinforced concrete. Different materials are used for the different parts of the frame according to their strength requirements. The analysis of turbine foundations using the
frequency and mass dependent stiffness matrix is accomplished after
the whole structure has been idealized conforming to the prevailing
conditions in the structure.

The frame structure is mostly loaded along the length
of its members. The fixed end reactions produced under dynamic
loading cannot be calculated using the methods of statics. The
dynamic fixed end reactions are, therefore, evaluated using the slope
deflection equations written for the deflection z which is a function
of co-ordinate y as well as time t. In this thesis, dynamic fixed
end reactions are determined for point loading and uniformly
distributed loading for the types of end conditions such as fix-fix,
fix-pin and fix-free.
CHAPTER II

REVIEW OF PREVIOUS WORK

The first significant contribution to the dynamic analysis of frame structures was introduced by Rausch (3) through the lumped mass method. With the use of certain factors, the dynamic loading was approximated by static loading and the analysis of frames was accomplished using static methods. The lumped mass method could calculate only the fundamental natural frequency. Weber (4) improved this method to include the effects of ground motion.

Hohenemser and Prager (5) were the first to introduce the uniformly distributed mass concept for the dynamic analysis of frames. The method was employed for free as well as forced vibrations, but the effect of longitudinal vibration was not included. Kolousek (6) improved the differential equation for transverse vibrations by including the effect of longitudinal vibrations. The effect of rotary inertia and shear deformations was, however, not included. Though the uniformly distributed mass concept had been developed for structural elements, Lorenj (7,9) and Schrader (8) were the first to utilize the uniformly distributed mass concept for analysing plane frame structures. Lorenj (7,9) did not include the effect of longitudinal vibrations, but prepared nomograms for a fix-fix and a pin-pin beam. Schrader (8) took account of the effects of longi-
tudinal vibrations as well as ground motion for frame structures under forced vibrations using the distributed mass concept.

Klein (10) used the distributed mass system to analyse turbine foundations including the effect of longitudinal vibrations. A successful attempt was made to achieve a close approximation for the conditions prevailing in the turbine foundations under free and forced vibrations. Klein (10) compared the natural frequencies and the internal stress resultants calculated using the distributed mass method to those found by experiments on models. The effects of rotary inertia and shear deformations were, however, not included.

In this thesis, the uniformly distributed mass concept has been employed using the stiffness method for analysing dynamic response problems of frame structures. The effects of longitudinal vibrations, rotary inertia and shear deformations have also been included. In order to consider the relative merits of both the lumped mass method and the distributed mass method, comparisons were made in this thesis. The natural frequency and the internal stress resultants calculated by both the methods at the fundamental mode are quite comparable, but greater discrepancies are observed at higher modes. In summary, it may be stated that the results of internal stress resultants calculated for a turbine foundation using the lumped mass method may yield inaccurate results.
3.1 Reduction of a Dynamic Problem to a Static Problem — The framed structure subjected to a steady state vibration is idealized into an assembly of straight line elements of uniformly distributed mass. Each element will be assumed to vibrate in steady state with finite degrees of freedom specified only at the ends. If during such vibrations the end displacements of the elements are linearly related to the end forces, then the dynamic problem may be reduced to a static problem and subsequently the familiar stiffness approach becomes applicable. For the purpose of achieving such a reduction, the elements are considered to vibrate with unit amplitude in each of the specified degrees of freedom one at a time. If an instantaneous deflection of unit amplitude is provided in the jth direction, the amplitude of dynamic forces $k_{ij}$, produced in all the specified degrees of freedom of the member can be calculated using the differential equation of motion for longitudinal, lateral and torsional vibrations. It is assumed that for small deflections, the
longitudinal vibrations do not have any effect on the lateral or torsional vibrations and vice versa. Under this assumption, the combined forces produced in the various specified degrees of freedom of a member can be calculated by superimposing the forces produced by individual unit amplitudes. These forces are functions of the mass of the member and the frequency of the impressed force. This relationship between the deformations \{U\} and the forces \{P\} of the members can be represented in the matrix form by a static equation,

$$[K] \{U\} = \{P\} \quad (I)$$

in which, \([K]\) is a special static member stiffness matrix, which is a function of its mass and the frequency, \(\omega\), of the impressed force (*) . Under the small deflection theory, the individual frequency and mass dependent stiffness matrices can be combined to generate the master stiffness matrix of the structure as in the case of static analysis. Then the determination of the internal dynamic stress resultants follow the same approach of the ordinary stiffness method of analysis.

(*) The static matrix will be designated as Frequency and Mass dependent (F.M.)
3.1.1 **Plane Frame Member** - In the dynamic analysis of frame structures under forced vibrations using the frequency and mass dependent stiffness matrix, it is assumed that the material of the structural elements is homogenous, isotropic and obeys Hooke's law. A stiffness influence coefficient, $k_{ij}$, constituting the F.M. stiffness matrix for a frame member under steady state vibration is defined as the amplitude of the dynamic force required along direction $i$ to produce a unit amplitude along direction $j$, while all other specified deformations are prevented.

![FIG.1. - A PLANE FRAME MEMBER IN MEMBER AXES](image-url)
Fig. 1 illustrates a plane frame member in member axes specifying the degree of freedom at either end of the member. The y axis is defined as being along the axis of the member and its sense is determined as positive in the direction of the higher numbered end from lower numbered end. The x and z axes are determined by the right-hand screw rule. The x axis always takes the direction perpendicular to the plane of bending.

3.1.2. Longitudinal Vibrations - It is considered that during the longitudinal vibrations of a prismatic member the cross-section of the member remains plane and the particles in the cross-section undergo motion only in the axial direction of the member. The effect of flexural vibrations on the axial forces is neglected.

FIG. 2. - LONGITUDINAL VIBRATIONS OF A PRISMATIC MEMBER
Denoting $v = \text{the longitudinal displacement of any cross-section } \ell_m \text{ of the bar during vibrations}$, $\varepsilon = \frac{\partial v}{\partial y}$, the unit elongation, $\rho = \text{mass per unit length of the bar}$, 
the tensile forces at any cross-section $\ell_m$, and also at 
an adjacent cross-section $\ell'm'$ as shown in Fig.2, are 
stated as
\begin{align*}
\text{at } \ell_m & \quad N = AE \frac{\partial v}{\partial y} \quad \text{(1)} \\
\text{at } \ell'm' & \quad N + dN = AE \left( \frac{\partial v}{\partial y} + \frac{\partial^2 v}{\partial y^2} dy \right) \quad \text{(2)}
\end{align*}
The inertia force of the element $\ell \ell'm'm'$ from Newton's 
law should balance the net tensile force acting on it.
That is,
\begin{equation}
\rho dy \frac{\partial^2 v}{\partial t^2} = AE \frac{\partial^2 v}{\partial y^2} dy \quad \text{(3)}
\end{equation}
This simplifies to
\begin{equation}
\frac{\partial^2 v}{\partial t^2} = \frac{AE}{\rho} \frac{\partial^2 v}{\partial y^2} \quad \text{(4)}
\end{equation}
which indicates that the displacement $v$ is a function of the 
co-ordinate $y$ as well as time $t$. Thus the solution of the 
partial differential equation in Eq.4 can be taken as
\begin{equation}
v = U \left( A_\omega \cos \omega t + B_\omega \sin \omega t \right) \quad \text{(5)}
\end{equation}
in which
\begin{equation}
U = C_\omega \cos \frac{\lambda \omega}{L} y + D_\omega \sin \frac{\lambda \omega}{L} y \quad \text{(6)}
\end{equation}
and
\begin{equation}
\lambda \omega = L \left( \frac{\omega \rho}{EA} \right)^{1/2} \quad \text{(7)}
\end{equation}
The constants $A_\omega$ and $B_\omega$ are determined from the initial 
conditions and the constants $C_\omega$ and $D_\omega$ are determined.
from the end conditions. The initial conditions by definition of the stiffness at \( y \to 0 \),

\[
\begin{align*}
\nu &= 1 \quad @ \ t = 0 \\
\frac{\partial \nu}{\partial t} &= 0 \quad @ \ t = 0
\end{align*}
\]  

(a)

These solve the constants as

\[
\begin{align*}
A &= 1 \\
B &= 0
\end{align*}
\]  

(b)

which reduces the final form of the displacement, \( \nu \), to

\[ \nu = U \cos \omega t \]  

(8)

3.1.3. **F.M. Stiffness Influence Coefficients** - The Stiffness

---

**FIG. 3.** - UNIT DEFORMATION ALONG DIRECTION 1
influence coefficients as defined in Sec. 3.1. may be developed from Eq. 8 by providing a unit amplitude in direction 1 as shown in Fig. 3 and preventing all other specified displacements. Thus, the end conditions used in Eq. 8 to solve the constants $C_x$ and $D_x$ are

\[ \begin{align*}
  v(y = 0) &= 1 \\
  v(y = L) &= 0 
\end{align*} \]

which yields the constants as

\[ \begin{align*}
  C_x &= 1 \\
  D_x &= -\frac{1}{\tan\lambda_x} 
\end{align*} \]

This gives the final form to displacement, v and strain,

\[ \frac{\partial v}{\partial y} \]

as

\[ v = \left[ \cos \frac{\lambda_x}{L} y - \frac{1}{\tan\lambda_x} \sin \frac{\lambda_x}{L} y \right] \cos \omega t \]  \hspace{2cm} (9)

\[ \frac{\partial v}{\partial y} = -\frac{\lambda_x}{L} \left[ \sin \frac{\lambda_x}{L} y + \left( \frac{1}{\tan\lambda_x} \right) \cos \frac{\lambda_x}{L} y \right] \cos \omega t \]  \hspace{2cm} (10)

The axial Force, $N$, under longitudinal vibration is defined as

\[ N = AE \left( \frac{\partial v}{\partial y} \right) \]  \hspace{2cm} (11)

Therefore, the Stiffness Influence Coefficients occupying the 1st column of the F.M. stiffness matrix are derived as

\[ \begin{align*}
  k_{11} &= -N(y = 0) \\
  k_{11} &= -EA \left( \frac{\partial v}{\partial y} \right) y = 0 
\end{align*} \]  \hspace{2cm} (12)
From Eqs. 10 and 12, 
\[ k_{11} = \frac{EA\eta}{L} \]  
(13)
in which 
\[ \eta = \lambda \frac{1}{\tan \lambda \ell} \]  
(14)
and similarly for the other end of the member 
\[ k_{41} = N(y = L) \]  
(15)
\[ k_{41} = + EA \left( \frac{\partial y}{\partial y} \right) y = L \]  
Similarly, by providing a unit amplitude deflection in the direction 4, the Stiffness Influence Coefficients occupying the 4th column of F.M. stiffness matrix are developed as 
\[ k_{44} = + \frac{EA\eta}{L} \]  
(18)
\[ k_{14} = - \frac{EA\eta}{L} \]  
(19)

3.2. Lateral Vibrations - It is assumed that a flexure member under lateral vibrations, has a plane of symmetry and the vibrations occur in that plane only. The differential equation for a deflection curve is known as 
\[ EI \frac{d^2\omega}{dy^2} = + M \]  
(20)
in which $EI = \text{flexural rigidity}$ and $M = \text{the algebraic expression of bending moment at any cross-section.}$
Differentiating Eq. 20, the shear and load expressions are obtained as,

\[
\frac{d}{dy} \left( EI \frac{d^2 w}{dy^2} \right) = \frac{dM}{dy} = V \quad \text{(21)}
\]

\[
\frac{d^2}{dy^2} \left( EI \frac{d^2 w}{dy^2} \right) = \frac{dV}{dy} = -\varphi \quad \text{(21)}
\]

FIG. 4. - LATERAL VIBRATIONS OF A FLEXURE MEMBER

The inertia forces \( F \) produced in the vibrating bar of varying intensity along the length of the bar, are given by

\[
F = -\rho \ \frac{\partial^2 \omega}{\partial t^2} \quad \text{(22)}
\]

From Eqs. 21 and 22, the differential equation of motion for a bar under lateral vibrations can be written using D'Alembert's principle as

\[
\frac{\partial^2}{\partial y^2} \left( EI \frac{\partial^2 w}{\partial y^2} \right) = -\varphi - \rho \ \frac{\partial^2 \omega}{\partial t^2} \quad \text{(23)}
\]

in which \( \varphi \) = the intensity of loading.
For a prismatic bar under no external loading, Eq. 23 takes the form
\[
\frac{\partial^4 w}{\partial y^4} + \frac{p}{EI} \frac{\partial^2 w}{\partial t^2} = 0
\] (24)
The solution to this partial differential equation is
\[
W = Y (A_1 \cos \omega t + B_1 \sin \omega t)
\] (25)
in which
\[
Y = C_1 \cos \frac{\lambda}{L} y + C_2 \sin \frac{\lambda}{L} y + C_3 \cosh \frac{\lambda}{L} y
\]
\[+ C_4 \sinh \frac{\lambda}{L} y
\] (26)
and
\[
\lambda = L \sqrt{\frac{\omega^2 p}{EI}}
\] (27)
The constants \(A_1\) and \(B_1\) are obtained using the initial conditions and the constants \(C_1, C_2, C_3\) and \(C_4\) are obtained using the end conditions of the bar. The initial condition is
\[
\frac{\partial w}{\partial t} = 0 \quad \text{at} \quad t = 0
\] (e)
This yields the constant as
\[
B_1 = 0
\] (f)
which reduces the final form of deflection \(z\), as
\[
W = Y A_1 \cos \omega t
\] (28)
3.2.1. Derivation of the Stiffness Influence Coefficients for a Fix-Fix Beam - The stiffness influence coefficients forming the 2nd, 3rd, 5th and 6th column of F.M. stiffness matrix will be obtained in the following three steps.

Step 1. - By providing a unit amplitude in direction 2 as shown below in Fig. 5 and preventing all other specified displacements, the stiffness influence coefficients forming the 2nd column of the F.M. stiffness matrix will be developed.

FIG. 5. - UNIT DEFLECTION ALONG DIRECTION 2
The specified end conditions necessary to solve the constants in Eq.26, therefore, are

\[ \begin{align*}
W(y = 0) &= 1 \\
W(y = L) &= 0 \\
\left( \frac{\partial^2 w}{\partial y^2} \right)_{y = 0} &= 0 \\
\left( \frac{\partial^2 w}{\partial y^2} \right)_{y = L} &= 0
\end{align*} \tag{g} \]

which yields the constants as follows

\[ \begin{align*}
C_1 &= \frac{1 - \cos \lambda \cosh \lambda + \sin \lambda \sinh \lambda}{2(1 - \cos \lambda \cosh \lambda)} \\
C_2 &= -\frac{(\cos \lambda \sinh \lambda + \sin \lambda \cosh \lambda)}{2(1 - \cos \lambda \cosh \lambda)} \\
C_3 &= \frac{1 - \cos \lambda \cosh \lambda - \sin \lambda \sinh \lambda}{2(1 - \cos \lambda \cosh \lambda)} \\
C_4 &= \frac{\cos \lambda \sinh \lambda + \sin \lambda \cosh \lambda}{2(1 - \cos \lambda \cosh \lambda)} \tag{h}
\end{align*} \]

Then \( Y \) can be represented as

\[ Y = \left[ \begin{array}{c}
(1 - \text{C.CH} + \text{S.SH}) \cos \frac{\lambda}{L} y - (\text{C.SH} + \text{S.CH}) \sin \frac{\lambda}{L} y \\
+ (1 - \text{C.CH} - \text{S.SH}) \cosh \frac{\lambda}{L} y \\
+ (\text{C.SH} + \text{S.CH}) \sinh \frac{\lambda}{L} y
\end{array} \right] \frac{1}{2(1 - \text{C.CH})} \tag{29} \]
The 2nd and 3rd derivatives of $\omega$ with respect to $y$ are,

$$\frac{\partial^2 \omega}{\partial y^2} = \frac{\lambda^2}{L^2} \begin{bmatrix} -(1 - C.CH + S.SH) \cos \frac{\lambda}{L} y \\ +(C.SH + S.CH) \sin \frac{\lambda}{L} y \\ +(1 - C.CH - S.SH) \cosh \frac{\lambda}{L} y \\ +(C.SH + S.CH) \sinh \frac{\lambda}{L} y \end{bmatrix} + \frac{\text{Cos}\omega t}{2(1-C.CH)}$$

$$\frac{\partial^3 \omega}{\partial y^3} = \frac{\lambda^3}{L^3} \begin{bmatrix} (1 - C.CH + S.SH) \sin \frac{\lambda}{L} y \\ +(C.SH + S.CH) \cos \frac{\lambda}{L} y \\ +(1 - C.CH + S.SH) \sinh \frac{\lambda}{L} y \\ +(C.SH + S.CH) \cosh \frac{\lambda}{L} y \end{bmatrix} + \frac{\text{Cos}\omega t}{2(1-C.CH)}$$

in which $S = \sin \lambda$, $C = \cos \lambda$, $SH = \sinh \lambda$, and $CH = \cosh \lambda$

The Stiffness Influence Coefficients,

$$k_{22} = \frac{V(y = 0)}{\text{EI}} = \frac{\text{EI}}{L^3} \gamma$$

From Eqs. 31 and 33,

$$k_{22} = \frac{\text{EI}}{L^3} \gamma$$

in which

$$\gamma = \lambda^3 \left[ \frac{\cos \lambda \sinh \lambda + \sin \lambda \cosh \lambda}{1 - \cos \lambda} \right]$$
and

\[ k_{32} = -M(y = 0) \]
\[ = -EI \left( \frac{\partial^2 u}{\partial y^2} \right) y = 0 \]  

(36)

From Eqs. 30 and 36,

\[ k_{32} = \theta \frac{EI}{L^2} \]  

(37)

in which

\[ \theta = \lambda^2 \frac{\sin \lambda \sinh \lambda}{1 - \cosh \lambda \cos \lambda} \]  

(38)

also

\[ k_{52} = -V(y = L) \]
\[ = -EI \left( \frac{\partial^2 u}{\partial y^2} \right) y = L \]  

(39)

From Eqs. 31 and 39

\[ k_{52} = -\varepsilon \frac{EI}{L^2} \]  

(40)

where

\[ \varepsilon = \lambda^3 \left( \frac{\sin \lambda + \sinh \lambda}{1 - \cos \lambda \cosh \lambda} \right) \]  

(41)

also

\[ k_{62} = M(y = L) \]
\[ = EI \left( \frac{\partial^2 u}{\partial y^2} \right) y = L \]  

(42)

From Eqs. 30 and 42,

\[ k_{62} = \frac{EI}{L^2} \delta \]  

(43)

where

\[ \delta = \lambda^2 \left( \frac{\cosh \lambda - \cos \lambda}{1 - \cos \lambda \cosh \lambda} \right) \]  

(44)
Step 2 - The Stiffness Influence Coefficients forming the 3rd column of the F.M. Stiffness Matrix are developed by providing a unit amplitude in direction 3 and preventing all other specified displacements as shown in Fig. 6.

\[ k_{13} \]

\[ k_{23} \]

\[ k_{33} \]

\[ k_{43} \]

\[ k_{53} \]

\[ k_{63} \]

**FIG. 6. - UNIT ROTATION ALONG DIRECTION 3**

The specified end conditions, therefore, are,

\[ w(y = 0) = 0 \]

\[ w(y = L) = 0 \]

\[ \frac{\partial w}{\partial y}(y = 0) = 1 \]

\[ \frac{\partial w}{\partial y}(y = L) = 0 \]
The constants are then solved from Eq. 26 as given below,

\[
C_1 = \frac{L}{\lambda} \left( \frac{\sin \lambda \cosh \lambda - \cos \lambda \sinh \lambda}{2 (1 - \cos \lambda \cosh \lambda)} \right) \\
C_2 = \frac{L}{\lambda} \left( \frac{1 - \sin \lambda \sinh \lambda - \cos \lambda \cosh \lambda}{2 (1 - \cos \lambda \cosh \lambda)} \right) \\
C_3 = \frac{L}{\lambda} \left( \frac{\cos \lambda \sinh \lambda - \sin \lambda \cosh \lambda}{2 (1 - \cos \lambda \cosh \lambda)} \right) \\
C_4 = \frac{L}{\lambda} \left( \frac{1 - \cos \lambda \cosh \lambda + \sin \lambda \sinh \lambda}{2 (1 - \cos \lambda \cosh \lambda)} \right)
\]

which reduces the final form of Y as

\[
Y = \begin{pmatrix}
(S.CH - C.SH) \cos \frac{\lambda}{L} y \\
+ (1 - S.SH - C.CH) \sin \frac{\lambda}{L} y \\
+ (C.SH - S.CH) \cosh \frac{\lambda}{L} y \\
+ (1 - C.CH + S.SH) \sinh \frac{\lambda}{L} y
\end{pmatrix} \frac{L}{2\lambda (1 - C.CH)}
\]

The second and third derivatives of \( \omega \) with respect to y are,

\[
\frac{\partial^2 \omega}{\partial y^2} = \frac{\lambda}{L} \begin{pmatrix}
(SH.C - CH.S) \cos \frac{\lambda}{L} y \\
-(1 - S.SH - C.CH) \sin \frac{\lambda}{L} y \\
+(C.SH - S.CH) \cosh \frac{\lambda}{L} y \\
+(1 - C.CH + S.SH) \sinh \frac{\lambda}{L} y
\end{pmatrix} \frac{\cos \omega t}{2(1 - C.CH)}
\]

\[
\frac{\partial^3 \omega}{\partial y^3} = \frac{\lambda}{L^2} \begin{pmatrix}
(CH.S - SH.C) \sin \frac{\lambda}{L} y \\
-(1 - S.SH - C.CH) \cos \frac{\lambda}{L} y \\
+(C.SH - S.CH) \sinh \frac{\lambda}{L} y \\
+(1 - C.CH + S.SH) \cosh \frac{\lambda}{L} y
\end{pmatrix} \frac{\cos \omega t}{2(1 - C.CH)}
\]

in which S, C, SH and CH are as given by Eq. 32 and \( \lambda \) as given by Eq. 27.
The Stiffness Influence Coefficients, then, obtained for the moment at left-hand side, are,

\[ k_{33} = -M_{(y = 0)} = -EI \left( \frac{\partial^2 \omega}{\partial y^2} \right)_{y = 0} \]  

(48)

From Eqs. 46 and 48,

\[ k_{33} = \frac{EI}{L} \alpha \]  

(49)

in which

\[ \alpha = \lambda \left( \frac{\sin \lambda \cosh \lambda - \cos \lambda \sinh \lambda}{1 - \cos \lambda \cosh \lambda} \right) \]  

(50)

also

\[ k_{53} = -V_{(y = L)} = -EI \left( \frac{\partial^3 \omega}{\partial y^3} \right)_{y = L} \]  

(51)

From Eqs. 47 and 51

\[ k_{53} = \frac{EI}{L^2} \delta \]  

(52)

in which

\[ \delta = \lambda^2 \left( \frac{\cosh \lambda - \cos \lambda}{1 - \cos \lambda \cosh \lambda} \right) \]  

(53)

and

\[ k_{63} = M_{(y = L)} = EI \left( \frac{\partial^2 \omega}{\partial y^2} \right)_{y = L} \]  

(54)

From Eqs. 46 and 54,

\[ k_{63} = \frac{EI}{L} \beta \]  

(55)

in which

\[ \beta = \lambda \left( \frac{\sinh \lambda - \sin \lambda}{1 - \cos \lambda \cosh \lambda} \right) \]  

(56)

**Step 3.** - In order to develop the Stiffness Influence Coefficients for the 5th and 6th column of the F.M. stiffness matrix, deflection and rotation of unit amplitude are
provided in the direction 5 and 6, respectively as shown in Fig. 7. Since the end conditions at end \( j \) are the same as were when working at the end \( i \), therefore, the stiffness Influence Coefficients for the end \( j \) may be obtained from those of end \( i \) using the rules of symmetry.

FIG. 7. - UNIT DEFORMATION ALONG DIRECTION 5 and 6
As a summary, all the Stiffness Influence Coefficients both for the longitudinal as well as the lateral vibrations are combined and the F.M. stiffness matrix, $[k]_{F.M.}$, for a fix-fix beam becomes,

$$[k]_{F.M.} = \begin{bmatrix}
\gamma \frac{EI}{L^3} & \theta \frac{EI}{L^2} & 0 & -\varepsilon \frac{EI}{L^3} & \delta \frac{EI}{L^2} \\
0 & \theta \frac{EI}{L^2} & \alpha \frac{EI}{L} & 0 & -\delta \frac{EI}{L^2} & \beta \frac{EI}{L} \\
-\nu \frac{EA}{L} & 0 & 0 & +\nu \frac{EA}{L} & 0 & 0
\end{bmatrix}$$

where

$$\alpha = \lambda \left( \frac{\sin \lambda \cosh \lambda - \cos \lambda \sinh \lambda}{1 - \cos \lambda \cosh \lambda} \right)$$

$$\beta = \lambda \left( \frac{\sinh \lambda - \sin \lambda}{1 - \cos \lambda \cosh \lambda} \right)$$

$$\gamma = \lambda^3 \left( \frac{\cos \lambda \sinh \lambda + \sin \lambda \cosh \lambda}{1 - \cos \lambda \cosh \lambda} \right)$$

$$\delta = \lambda^2 \left( \frac{\cosh \lambda - \cos \lambda}{1 - \cos \lambda \cosh \lambda} \right)$$

$$\varepsilon = \lambda^3 \left( \frac{\sin \lambda + \sinh \lambda}{1 - \cos \lambda \cosh \lambda} \right)$$

$$\eta = \lambda_2 \left( \frac{1}{\tan \lambda_2} \right)$$

$$\theta = \lambda^2 \left( \frac{\sin \lambda \sinh \lambda}{1 - \cos \lambda \cosh \lambda} \right)$$

$$\nu = \lambda_2 \left( \frac{1}{\sin \lambda_2} \right)$$

Parameters $\lambda$ and $\lambda_2$ are as given in Eqs. 7 and 27.
3.2.3 The Frequency and Mass Dependent Stiffness Matrix for a Fix-Pinned Beam - The F.M. stiffness matrix for a fix-pinned beam can be obtained from that of fix-fix beam by taking advantage of the fact that moment at end \( j \) is zero.

The end force-displacements equation of the beam shown in
Fig. 8 are,

\[
\begin{bmatrix}
\frac{2\pi}{L} & 0 & 0 & -\frac{2\pi}{L} & 0 & 0 \\
0 & \frac{\theta}{L^2} & \frac{\beta}{L} & 0 & -\frac{\theta}{L} & \frac{\beta}{L} \\
-\frac{2\pi}{L} & 0 & 0 & \frac{2\pi}{L} & 0 & 0 \\
0 & -\frac{\epsilon}{L^3} & -\frac{\delta}{L^2} & 0 & \frac{\epsilon}{L^3} & -\frac{\delta}{L^2} \\
0 & \frac{\delta}{L^2} & \frac{\beta}{L} & 0 & -\frac{\delta}{L^2} & \frac{\beta}{L}
\end{bmatrix}
\begin{bmatrix}
U_1 \\
U_2 \\
U_3 \\
U_4 \\
U_5 \\
U_6
\end{bmatrix}
= 
\begin{bmatrix}
P_1 \\
P_2 \\
P_3 \\
P_4 \\
P_5 \\
P_6
\end{bmatrix}
\]

This equation can be represented as,

\[
\begin{bmatrix}
A & C \\
C^T & B
\end{bmatrix}
\begin{bmatrix}
U_j \\
U_k
\end{bmatrix}
= 
\begin{bmatrix}
P_j \\
0
\end{bmatrix}
\]

Solving Eq. 67 for \( U_2 \),

\[
U_2 = -B^{-1} C^T U_k
\]
in which \( U_k = U_6 \) and \( U_j = U_1, U_2, U_3, U_4, \) and \( U_5 \)

Or \[\begin{bmatrix}
A - C B^{-1} C^T
\end{bmatrix}
\begin{bmatrix}
U_j \\
U_k
\end{bmatrix}
= 
\begin{bmatrix}
P_j \\
0
\end{bmatrix}
\]

Or \[\begin{bmatrix}
K
\end{bmatrix}
\begin{bmatrix}
U_j
\end{bmatrix}
= 
\begin{bmatrix}
P_j
\end{bmatrix}
\]

in which

\[
\begin{bmatrix}
K
\end{bmatrix}
= 
\begin{bmatrix}
A - C B^{-1} C^T
\end{bmatrix}
\]

and \( \begin{bmatrix}
P_j
\end{bmatrix} \) is the force vector required in a fix-pinned beam to maintain a \( U_{ij} \) deflection along the degrees of
freedom at the two ends of the beam. Performing this operation on Eq.66 as,

\[
\begin{bmatrix}
\frac{L}{aEI} & 0 & \frac{\beta EI}{L} & 0 & -\frac{\theta EI}{L^2} \\
0 & 0 & 0 & 0 & 0 \\
\frac{\delta EI}{L^2} & \frac{\delta}{aL} & \frac{\beta EI}{L} & 0 & 0 \\
0 & 0 & \frac{\beta EI}{L} & 0 & 0 \\
-\frac{\theta EI}{L^2} & -\frac{\theta}{aL} & 0 & 0 & 0 \\
\end{bmatrix}
\]

The F.M. stiffness matrix, \( K \) F.M., for a fix-pinned beam, therefore, becomes,

\[
\begin{bmatrix}
+\frac{EA}{L} & 0 & 0 & -\nu \frac{EA}{L} & 0 & 0 \\
0 & \gamma' \frac{EI}{L^3} & \theta' \frac{EI}{L^2} & 0 & -\epsilon' \frac{EI}{L^3} & 0 \\
0 & \theta' \frac{EI}{L^2} & \alpha' \frac{EI}{L} & 0 & -\delta' \frac{EI}{L^2} & 0 \\
-\frac{\nu EI}{L} & 0 & 0 & +\eta \frac{EA}{L} & 0 & 0 \\
0 & -\epsilon' \frac{EI}{L^3} & -\delta' \frac{EI}{L^2} & 0 & +\chi' \frac{EI}{L^3} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

in which

\[
\alpha' = \alpha - \frac{\beta^2}{a} = \lambda \frac{(2 \sin \lambda \sinh \lambda)}{\sin \lambda \cosh \lambda - \cos \lambda \sinh \lambda}
\]

\[
\gamma' = \gamma - \frac{\delta^2}{a} = \lambda^3 \frac{(2 \cos \lambda \cosh \lambda)}{\sin \lambda \cosh \lambda - \cos \lambda \sinh \lambda}
\]

\[
\delta' = \delta - \frac{\beta \theta}{a} = \lambda^2 \frac{(\sin \lambda + \sinh \lambda)}{\sin \lambda \cosh \lambda - \cos \lambda \sinh \lambda}
\]

\[
\epsilon' = \epsilon - \frac{\theta \delta}{a} = \lambda^3 \frac{(\cos \lambda + \cosh \lambda)}{\sin \lambda \cosh \lambda - \cos \lambda \sinh \lambda}
\]
\[ n = \lambda \frac{1}{\tan \lambda} \]  
\[ \theta' = \theta - \frac{\delta}{a} = \lambda^2 \frac{(\cos \lambda \sinh \lambda + \sin \lambda \cosh \lambda)}{(\sin \lambda \cosh \lambda - \cos \lambda \sinh \lambda)} \]  
\[ \nu = \lambda \frac{1}{\sin \lambda} \]  
\[ \chi' = \gamma - \frac{\theta^2}{a} = \lambda^3 \frac{(1+\cos \lambda \cosh \lambda)}{(\sin \lambda \cosh \lambda - \cos \lambda \sinh \lambda)} \]  

Parameters \( \lambda \) and \( \lambda^2 \) are as given in Eqs. 7 and 27.

### 3.2.4. The Frequency and Mass Dependent Stiffness Matrix for a Cantilever Beam

The F.M. stiffness matrix for a cantilever beam will be obtained from the F.M. stiffness matrix of a fix-pinned beam by taking advantage of the fact that the shear at pinned end \( j \) is zero. Writing the stiffness

![Diagram of a cantilever beam](image-url)
equation of the fix-pinned beam in the partitioned form as,

\[
\begin{bmatrix}
+ \eta \frac{EA}{L} & 0 & 0 & -v \frac{EA}{L} & 0 & 0 \\
0 & +v \frac{EI}{L^3} & \delta' \frac{EI}{L^2} & 0 & -\epsilon \frac{EI}{L^3} & 0 \\
0 & \theta \frac{EI}{L^2} & \alpha \frac{EI}{L} & 0 & -\delta \frac{EI}{L^2} & 0 \\
-\nu \frac{EA}{L} & 0 & 0 & +\eta \frac{EA}{L} & 0 & 0 \\
0 & -\epsilon \frac{EI}{L^3} & -\delta' \frac{EI}{L^2} & 0 & \chi \frac{EI}{L^3} & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
U_1 \\
U_2 \\
U_3 \\
U_4 \\
U_5 \\
U_6
\end{bmatrix}
= 
\begin{bmatrix}
P_1 \\
P_2 \\
P_3 \\
P_4 \\
P_5 \\
P_6
\end{bmatrix}
\tag{80}
\]

Applying the same reduced stiffness matrix technique as followed in Sec. 3.2.3., the F.M. stiffness matrix

\[
[K]_{F.M.}
\]

for a cantilever beam is generated using Eq. 71 as follows,

\[
[C]^{-1} (C^T) =
\begin{bmatrix}
\frac{L^3}{\chi' \frac{EI}{L^3}} & 0 & -\epsilon \frac{EI}{L^3} & -\delta' \frac{EI}{L^2} & 0 \\
0 & 0 & 0 & 0 & 0 \\
-\epsilon \frac{EI}{L^3} & \frac{\epsilon}{\chi'} & 0 & \frac{\epsilon'^2 \frac{EI}{L^3}}{\chi'} & \frac{\delta' \epsilon' \frac{EI}{L^2}}{\chi'} \\
-\delta' \frac{EI}{L^2} & \frac{\delta' L}{\chi'} & 0 & \frac{\delta' \epsilon' \frac{EI}{L^2}}{\chi' L^2} & \frac{\delta'^2 \frac{EI}{L}}{\chi'} \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}_{4 \times 4}
\tag{81}
\]

Thus the F.M. stiffness matrix \([K]_{F.M.}\) for a cantilever beam becomes
\[
K = \begin{bmatrix}
\frac{\eta EA}{L} & 0 & 0 & -\nu \frac{EA}{L} & 0 & 0 \\
0 & \frac{\gamma'' EI}{L^3} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{\theta'' EI}{L^2} & 0 & 0 & 0 \\
-\nu \frac{EA}{L} & 0 & 0 & \frac{\eta EA}{L} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

in which
\[
\alpha'' = \alpha' - \frac{\delta^2}{\chi}, \quad \gamma'' = \gamma' - \frac{\epsilon''}{\chi}, \quad \eta = \lambda \frac{1}{\tan \lambda}, \quad \theta'' = \theta' - \frac{\delta \epsilon'}{\chi}, \quad \nu = \lambda \frac{1}{\sin \lambda}
\]

Parameter \( \lambda \) and \( \lambda_\ell \) are as given in Eq. 7 and 27.

3.2.5. The Frequency and Mass Dependent Stiffness Matrix for a Pin-Pin Beam - The F.M. stiffness matrix for a pin-pin beam.
beam will be obtained from a fix-pinned beam by considering the fact that the moment at the fixed end i is zero. The end force-displacement equation of the simply supported beam can be written as,

\[
\begin{bmatrix}
+ \eta \frac{EA}{L} & 0 & 0 & -\nu \frac{EA}{L} & 0 & 0 \\
0 & \gamma' \frac{EI}{L^3} & \delta' \frac{EI}{L^2} & 0 & -\epsilon' \frac{EI}{L^3} & 0 \\
0 & \theta' \frac{EI}{L^2} & \alpha' \frac{EI}{L} & 0 & -\delta' \frac{EI}{L^2} & 0 \\
-\nu \frac{EA}{L} & 0 & 0 & + \eta \frac{EA}{L} & 0 & 0 \\
0 & -\epsilon' \frac{EI}{L^3} & -\delta' \frac{EI}{L^2} & 0 & \chi' \frac{EI}{L^3} & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
U_1 \\
U_2 \\
U_3 \\
U_4 \\
U_5 \\
U_6
\end{bmatrix}
= \begin{bmatrix}
P_1 \\
P_2 \\
P_3 \\
P_4 \\
P_5 \\
P_6
\end{bmatrix}
\tag{86}
\]

Since the method of partitioning presents a little difficulty in solving Eq.86, the usual method of solving the simultaneous equations is employed. It solves Eq.86 as,

\[
\begin{bmatrix}
U_3 \\
U_4 \\
U_5 
\end{bmatrix}
= \begin{bmatrix}
-\frac{\theta'}{\alpha L} & \frac{\delta'}{\alpha L^3} \\
\frac{\theta'}{\alpha L} & \frac{\delta'}{\alpha L^3} \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
U_2 \\
U_5
\end{bmatrix}
\tag{87}
\]

Substituting in Eq.86 for U_3, P_2 and P_5 are calculated as

\[
\begin{bmatrix}
P_2 \\
P_5
\end{bmatrix}
= \begin{bmatrix}
\gamma''' \frac{EI}{L^3} & \epsilon''' \frac{EI}{L^3} \\
\epsilon''' \frac{EI}{L^3} & \gamma''' \frac{EI}{L^3}
\end{bmatrix}
\begin{bmatrix}
U_2 \\
U_5
\end{bmatrix}
\tag{88}
\]

in which

\[
\gamma''' = \gamma' - \frac{\theta' \delta'}{\alpha} = \frac{\lambda^3}{2} \left[ \frac{\sinh \lambda \cos \lambda - \cosh \lambda \sin \lambda}{\sin \lambda \sinh \lambda} \right]
\tag{89}
\]

\[
\epsilon''' = \frac{\theta' \delta'}{\alpha} - \epsilon' = \frac{\lambda^3}{2} \left[ \frac{\sinh \lambda - \sin \lambda}{\sin \lambda \sinh \lambda} \right]
\tag{90}
\]

The parameter \( \lambda \) is as given in Eq.27. Thus the F.M. stiffness matrix, \([K]_{F.M.}\), for a pin-pin beam takes the final
in which the coefficient $n$ and $\nu$ are as given by Eqs. 65 and 67, the coefficients $\gamma''$ and $\epsilon''$ are as given by Eqs. 87 and 88 and the parameters $\lambda$ and $\lambda_2$ are as given by Eqs. 7 and 27.

3.3. Effect of Rotary Inertia and Shear Deformations on the Lateral Vibrations
3.3.1. **Rotary Inertia** - The effect of rotary inertia on a beam under lateral vibrations may be taken into account by correcting the pure bending calculations. The horizontal displacement of a fibre at a distance $z$ from the axis of the beam is expressed as

$$v = z \frac{\partial w}{\partial y}$$

(92)

The deflection $w$ varies with time as well as distance along the axis of the beam. Thus it must follow that the horizontal displacement $v$ must also vary with time and distance. For every $dy$ length of the beam, the fibres of cross-sectional area $dA$ will have masses equal to $\frac{\rho}{A} dA dy$. The inertia force, $F$, due to rotary inertia thus exists having the magnitude,

$$F = m\ddot{v} = + \frac{\rho}{A} dA dy z \frac{\partial^3 w}{\partial y \partial t^2}$$

(93)

This inertia force, $F$, is a function of co-ordinate $z$ and its sign depends upon the positive dissection of axis $z$. For every such inertia force acting above the $y-y$ axis, there is an equal and opposite force below the axis, producing a time dependent couple, $M$, given by,

$$M = \int_A \frac{\rho}{A} dy z^2 \frac{\partial^3 w}{\partial y \partial t^2} dA$$

(94)

$$M = \int_A I_{xx} dy \frac{\partial^3 w}{\partial y \partial t^2}$$

(95)

where

$$I_{xx} = \int_A z^2 dA$$

(96)
For the given positive direction of the z axis, the couple, \( M \), is anticlockwise in direction. Fig. 11(C) shows the free body diagram of an element of length \( dy \) of a beam. The moment equilibrium for the element \( dy \) can be written as,

\[
\frac{\partial M}{\partial y} = V - \rho \frac{Ixx}{A} \frac{\partial^3 w}{\partial y \partial t^2} \tag{97}
\]

The equation of motion of length \( dy \) of the beam using D'Alembert's principle is written as,

\[
\frac{\partial v}{\partial y} = -q - \rho \frac{\partial^2 w}{\partial t^2} \tag{98}
\]

in which \( q \) = the uniformly distributed load, \( \rho \) = mass per unit length of the beam and \( V \) = shear force at a section.

From Eqs. 20, 97 and 98, the modified form of the general differential equation representing the lateral vibrations of the beam including rotary inertia effect is,

\[
\frac{\partial^2}{\partial y^2} (EI \frac{\partial w}{\partial y^2}) = -\rho \left[ \frac{\partial^2 w}{\partial t^2} + r^2 \frac{\partial^4 w}{\partial t^2 \partial y^2} \right] \tag{99}
\]

3.3.2. Shear Deformation - The effects of the shear deformations on the lateral vibrations represent the effects of the shear-forces on the deflections of a beam. Using Hooke's law the shearing stress (\( \tau \)) at any cross-section of the beam can be written as,

\[
\tau = G\gamma = \frac{V}{\mu A} \tag{100}
\]

in which slope of the axis of the beam (\( \gamma \)) due to shear is,

\[
\gamma = \frac{\partial u_s}{\partial y} \tag{101}
\]
or,

\[ V = AG \mu \frac{\partial \omega}{\partial y} \]  \hspace{1cm} (102)

in which \( \mu \) is the shape factor depends upon the shape of
the cross-section of the beam \( \omega_s = \) deflection contributed by
the shear deformations.

The lateral deflection, \( \omega \), produced at any section is
contributed by shear forces as \( \omega_s \) and bending moment
including the effect of rotary inertia as \( \omega_b \), or

\[ \omega = \omega_s + \omega_b \]  \hspace{1cm} (103)

The rotational and translational equilibrium of the beam
for no external loading is represented as,

\[ \frac{\partial M}{\partial y} = V - \rho \frac{I_{xx}}{A} \frac{\partial^2 \omega}{\partial y \partial t^2} \]  \hspace{1cm} (104)

\[ \frac{\partial V}{\partial y} = -\rho \frac{\partial^2 \omega}{\partial t^2} \]  \hspace{1cm} (105)

The differential equation of motion for a prismatic member
under lateral vibrations derived using above equations, is,

\[ \frac{\partial^4 \omega}{\partial y^4} + \frac{\rho}{EI} \frac{\partial^2 \omega}{\partial t^2} - \frac{1}{A} \left( \frac{\rho}{E} + \frac{\rho}{\mu G} \right) \frac{\partial^2 \omega}{\partial y^2 \partial t^2} \]

\[ + \frac{\rho}{A^2 \mu EG} \frac{\partial^4 \omega}{\partial t^4} = 0 \]  \hspace{1cm} (106)

The term \( \frac{\rho}{A^2 \mu EG} \frac{\partial^4 \omega}{\partial t^4} \) is neglected because it is very
small in magnitude. This reduces Eq.106 to,

\[ \frac{\partial^4 \omega}{\partial y^4} + \frac{\rho}{EI} \frac{\partial^2 \omega}{\partial t^2} - \frac{1}{A} \left( \frac{\rho}{E} + \frac{\rho}{\mu G} \right) \frac{\partial^2 \omega}{\partial y^2 \partial t^2} = 0 \]  \hspace{1cm} (107)
The solution to partial differential equation of Eq.107 including the effects of rotary inertia and shear deformations can be assumed as

$$w = Y \left( A_1 \cos \omega t + B_1 \sin \omega t \right)$$  (108)

in which.

$$Y = C_1 \cos K_y + C_2 \sin K_y + C_3 \cosh K'_y + C_4 \sinh K'_y$$  (109)

and $K, K'$ are defined by

$$K^4 - \left( \lambda^4_1 \right) r^2 + \frac{\rho \omega^2}{A \mu G} K^2 - \lambda^4_1 = 0$$  (110)

$$K'^4 + \left( \lambda^4_1 \right) r^2 + \frac{\rho \omega^2}{A \mu G} K'^2 - \lambda^4_1 = 0$$  (111)

$$r^2 = \frac{I_{xx}}{A}$$  (112)

$$\lambda_1 = \sqrt{\frac{\omega^2 \rho}{EI}}$$  (113)

The constants $A_1$ and $B_1$ of Eq.108 are determined using initial conditions, while the constants $C_1, C_2, C_3$ and $C_4$ of Eq.109 are determined using the end conditions of the beam.

The initial condition is

$$\frac{\partial w}{\partial t} = 0 \quad @ \quad t = 0$$  (1)

Utilizing these conditions in Eq.108 the constant is determined as,

$$B_1 = 0$$  (m)

and Eq.108 reduces to

$$w = YA_1 \cos \omega t$$  (114)
3.3.3. **Stiffness Influence Coefficients including the Effects of Rotary Inertia and Shear Deformations** - The Stiffness Influence Coefficients occupying the 2nd column of the F.M. stiffness matrix for a fix-fix beam including the effects of rotary inertia and shear deformations are developed by providing a unit amplitude in the direction 2 as shown in Fig.12, keeping all other deformations equal.
to zero. The end conditions to solve the constants in Eq.109 are,

\[
\begin{align*}
\omega(y = 0) &= 1 \\
\omega(y = L) &= 0 \\
\frac{\partial \omega}{\partial y} y = 0 &= 0 \\
\frac{\partial \omega}{\partial y} y = L &= 0
\end{align*}
\]

This solves the constants as,

\[
\begin{align*}
C_1 &= \frac{K'}{K} \left[ \frac{K' \sin K}{\sinh K'} - K \cos K \cosh K' + K \right] \\
C_2 &= -\frac{K'}{K} \left[ \frac{K' \cos K}{\sinh K} + K \sin K \cosh K' \right] \\
C_3 &= \frac{K'}{K} \left[ \frac{(1 - \cos K \cosh K') - K \sin K \sinh K'}{2K'(1 - \cos K \cosh K') + (K' - K^2) \sin K \sinh K'} \right] \\
C_4 &= \frac{K'}{K} \left[ \frac{K' \cos K}{\sinh K'} + K \sin K \cosh K' \right]
\end{align*}
\]

which reduces Eq.114 to,

\[
\omega = \left[ \left( \frac{K' \sin \sinh K' - K \cos \cosh K'}{D'} \right) \cos K y - \left( \frac{K' \sin \sinh K' + K \cos \cosh K'}{D'} \right) \sin K y \right] \cos \omega t
\]

\[
\frac{\partial^2 \omega}{\partial y^2} = \left[ -K^2 \left( \frac{K' \sin \sinh K'}{D'} \right) \cos K y + K^2 \left( \frac{K' \sin \sinh K' + K \cos \cosh K'}{D'} \right) \sin K y \right] \cos \omega t
\]
\[ \frac{\partial^3 w}{\partial y^3} = \left[ \frac{k^3 (K'S.H-K.C.CH+K)}{D'} \sin K y + \frac{k^3 (K'C.S+K.S.CH)}{D'} \cos K y \right] \cos \omega t \]

\[ + \frac{K^2}{D'} \left( \frac{K' (1-C.C.H)-K.S.H}{D'} \right) \sin K'y + \frac{K^2}{D'} \left( \frac{K'C.S+K.S.CH}{D'} \right) \cosh K'y \]

in which \( D' = \frac{1}{K'} \left[ \begin{array}{c} 2K' (1-Cos K \cosh K') \\ + (K'^2 - K^2) \sin K \sinh K' \end{array} \right] \)

and \( S = \sin K, C = \cos K, SH = \sinh K', CH = \cosh K' \)

The Stiffness Influence Coefficients are derived from Eq.115 as,

\[ k_{22} = V(y = 0) = EI \left( \frac{\partial^3 w}{\partial y^3} \right) y = 0 \]

From Eqs. 117 and 120,

\[ k_{22} = EI \gamma^* \]

where

\[ \gamma^* = (K^3 + KK') \left[ \frac{K' \cos K \sinh K' + K \sin K \cosh K'}{I/K'} \right] \frac{(2K'(1-\cos K \cosh K') + (K'^2 - K^2) \sin K \sinh K')}{K'} \]

also

\[ k_{32} = -M(y = 0) = -EI \left( \frac{\partial^2 w}{\partial y^2} \right) y = 0 \]

From Eqs. 116 and 123,

\[ k_{32} = EI \phi^* \]

where

\[ \phi^* = - \left[ \frac{K(K'^2 - K^2)(1-\cos K \cosh K') - 2K'K^2 \sin K \sinh K'}{I/K'} \right] \frac{2K'(1-\cos K \cosh K') + (K'^2 - K^2) \sin K \sinh K'}{K'} \]

\[ k_{52} = -V(y = L) = -EI \left( \frac{\partial^3 w}{\partial y^3} \right) y = L \]
From Eqs. 117 and 126,

\[ k_{52} = -EI \varepsilon^* \tag{127} \]

where

\[ \varepsilon^* = \left[ \frac{K(K^2 + K'^2)(K'SinhK' + K SinK)}{\frac{1}{K'}[2KK'(1-CosK CoshK')+(K'^2-K^2)SinK SinhK']} \right] \tag{128} \]

and

\[ k_{62} = M(y = L) = EI \left( \frac{\partial^2 \omega}{\partial y^2} \right)_{y = L} \tag{129} \]

From Eqs. 116 and 129,

\[ k_{62} = EI \delta^* \tag{130} \]

where

\[ \delta^* = \frac{K(K^2 + K'^2)(CoshK' - CosK)}{\frac{1}{K'}[2KK'(1-CosK CoshK')+(K'^2-K^2) SinK SinhK']} \tag{131} \]

Step 2 - The Stiffness Influence Coefficients occupying the 3rd column of the F.M. stiffness matrix for a fix-fix beam can be developed by providing a unit amplitude in direction 3 while keeping all other displacements equal to zero. The end conditions to solve the constants of Eq.109 are,

\[
\begin{align*}
\omega(y = 0) &= 0 \\
\omega(y = L) &= 0 \\
\frac{\partial \omega}{\partial y}(y = 0) &= 1 \\
\frac{\partial \omega}{\partial y}(y = L) &= 0
\end{align*}
\tag{q}
\]
The constants thus solved are given below.

\[ C_1 = \frac{K' \sin K \cosh K' - K \cos K \sinh K'}{2K(K' - \cosh K)} + (K'^2 - K') \sin K \sinh K' \]

\[ C_2 = \frac{K' - K \sin K \sinh K' - K' \cos K \cosh K'}{2K(K' - \cosh K)} + (K'^2 - K') \sin K \sinh K' \]

\[ C_3 = \frac{K \cos K \sinh K' - K' \sin K \cosh K'}{2K(K' - \cosh K)} + (K'^2 - K') \sin K \sinh K' \]

\[ C_4 = \frac{K(1 - \cos K \cosh K') + K' \sin K \sinh K'}{2K(K' - \cosh K)} + (K'^2 - K') \sin K \sinh K' \]

which gives the final form of displacement \( \omega \) as,

\[
\omega = \left[ \frac{K' \sinh K - K \sin K}{2K(K' - \cosh K)} \cosh K'y + \left( K' \sin K \cosh K' - K \sinh K' \right) \right] \cos \omega t
\]

\[
+ \left( K \sinh K' \cosh K'y + (1 - K \sin K \cosh K') \sin K \sinh K' \right) \sin \omega t
\]

(132)

The second and third derivatives of \( \omega \) with respect to \( y \) are given as,

\[
\frac{\partial^2 \omega}{\partial y^2} = \left[ -K^2 \frac{(K' \sinh K - K \sin K)}{2K(K' - \cosh K)} \cosh K'y - K^2 \frac{(1 - K \sin K \cosh K') \sin K \sinh K' \cosh K'y}{2K(K' - \cosh K)} \right] \cos \omega t
\]

\[
+ K^2 \frac{(K \sinh K' \cosh K'y + (1 - K \sin K \cosh K') \sin K \sinh K' \sinh K'y)}{2K(K' - \cosh K)} \sin \omega t
\]

(133)

\[
\frac{\partial^3 \omega}{\partial y^3} = \left[ -K^3 \frac{(K' \sinh K - K \sin K)}{2K(K' - \cosh K)} \sinh K'y - K^3 \frac{(1 - K \sin K \cosh K') \cosh K'y}{2K(K' - \cosh K)} \right] \cos \omega t
\]

\[
+ K^3 \frac{(K \sinh K' \cosh K'y + (1 - K \sin K \cosh K') \sin K \sinh K' \cosh K'y)}{2K(K' - \cosh K)} \sin \omega t
\]

(134)
In which S, C, SH, CH are as given by Eq.119, and,

\[ D'_1 = 2KK' (1-CosK CoshK')+(K'^2-K^2)SinK SinhK' \quad \text{(135)} \]

The Stiffness Influence Coefficients are derived using the Eq.132, as,

\[ k_{33} = -M(y=0) = -EI \left( \frac{\partial^2 w}{\partial y^2} \right) y=0 \quad \text{(136)} \]

From Eqs. 133 and 136,

\[ k_{33} = EI\alpha^* \quad \text{(137)} \]

where

\[ \alpha^* = \frac{(K'SinhK' - K CosK SinhK')(K^2+K'^2)}{2KK'(1-CosK CoshK')+(K'^2-K^2)SinK SinhK'} \quad \text{(138)} \]

\[ k_{53} = -V(y=L) = -EI \left( \frac{\partial^3 w}{\partial y^3} \right) y=L \quad \text{(139)} \]

From Eqs. 134 and 139,

\[ k_{53} = -EI\delta^* \quad \text{(140)} \]

where

\[ \delta^* = \frac{KK'(K^2+K'^2)(CoshK' - CosK)}{2KK'(1-CosK CoshK')+(K'^2-K^2)SinK SinhK'} \quad \text{(141)} \]

\[ k_{63} = M(y=L) = EI \left( \frac{\partial^2 w}{\partial y^2} \right) y=L \quad \text{(142)} \]

From Eqs. 133 and 142,

\[ k_{63} = EI\beta^* \quad \text{(143)} \]

where

\[ \beta^* = \frac{(K SinhK' - K' SinK)(K^2+K'^2)}{2KK'(1-CosK CoshK')+(K'^2-K^2)SinK SinhK'} \quad \text{(144)} \]

**Step 3** - Similarly the Stiffness Influence Coefficients occupying the 5th and 6th columns of the F.M. stiffness matrix for a fix-fix beam including the effects of rotary inertia and shear deformations are developed by
providing a unit amplitude in directions 5 and 6 respectively. Due to symmetry the end conditions while working at end \( j \) of the beam remain the same as were when working at end \( i \) of the beam. The Stiffness Influence Coefficients, therefore, remain the same as developed at the end \( i \). Combining all the Stiffness Influence Coefficients the F.M. stiffness matrix for a fix-fix beam is generated as,

\[
[K]_{\text{F.M.}} = \begin{bmatrix}
\eta^*EA & 0 & 0 & -\nu^*EA & 0 & 0 \\
0 & \gamma^*EI & \theta^*EI & 0 & -\epsilon^*EI & \delta^*EI \\
0 & \theta^*EI & \alpha^*EI & 0 & -\delta^*EI & \beta^*EI \\
-\nu^*EA & 0 & 0 & \eta^*EA & 0 & 0 \\
0 & -\epsilon^*EI & -\delta^*EI & 0 & \gamma^*EI & -\theta^*EI \\
0 & \delta^*EI & \beta^*EI & 0 & -\theta^*EI & \alpha^*EI
\end{bmatrix}
\]

in which,

\[
\alpha^* = \frac{(K^2+K'^2)(K' \sin K \cosh K' - K \sinh K' \cos K)}{2KK'(1-\cos K \cosh K')+(K'^2-K^2)\sin K \sinh K'}
\]

\[
\beta^* = \frac{(K^2+K'^2)(K \sin K' - K' \sin K)}{2KK'(1-\cos K \cosh K')+(K'^2-K^2)\sin K \sinh K'}
\]

\[
\gamma^* = \frac{KK'(K^2+K'^2)(K' \cos K \sinh K+K \sin K \cosh K')}{1 \left[ 2KK'(1-\cos K \cosh K')+(K'^2-K^2)\sin K \sinh K' \right]}
\]

\[
\delta^* = \frac{K(K^2+K'^2)(\cosh K' - \cos K)}{1 \left[ 2KK'(1-\cos K \cosh K')+(K'^2-K^2)\sin K \sinh K' \right]}
\]
The parameters $K, K', \lambda_\parallel$ are as given in Eqs. 110, 111 and 7 respectively.

\[
\varepsilon^* = \frac{K(K^2 + K'^2)(K' \sinh K' + K \sin K)}{1 - K' \left[ 2K' (1 - \cos K \cosh K') + (K'^2 - K^2) \sin K \sinh K' \right]} \quad (150)
\]

\[
\eta^* = \frac{\lambda_\parallel}{L} \left( \frac{1}{\tan \lambda_\parallel} \right) \quad (151)
\]

\[
\theta^* = \frac{2K' \sin K \sinh K' - K(K'^2 - K^2)(1 - \cos K \cosh K')}{2K' (1 - \cos K \cosh K') + (K'^2 - K^2) \sin K \sinh K'} \quad (152)
\]

\[
\gamma^* = \frac{\lambda_\parallel}{L} \left( \frac{1}{\sin \lambda_\parallel} \right) \quad (153)
\]
3.4. **Space Frame Member** - Assuming that the vibrations due to bending about one plane do not have any effect on the vibrations due to bending about the other plane, the stiffness influence coefficients for the Frequency and Mass dependent stiffness matrix for a space frame member can be developed using the stiffness matrix derived for a plane frame member. These coefficients occupy the corresponding locations in a $12 \times 12$ F.M. Stiffness Matrix. In order to complete the stiffness matrix the effects of torsional vibrations should be added in the fifth and eleventh columns of the matrix.
3.4.1. **Torsional Vibrations** - The free body diagram of an element dy of a space frame member is shown in Fig. 14. The element is subject to a time dependent torque, T. From the free body diagram, the equation of motion using Newton's law for the element dy can be written as,

\[(T + \frac{\partial T}{\partial y} \, dy) - T = (\frac{\rho J}{A} \, dy) \frac{\partial^2 \theta}{\partial t^2}\] (154)

in which

\[T = C \frac{\partial \theta}{\partial y}\] (155)

\[C = \text{the torsional rigidity of the beam}, \ \theta = \text{the angle of twist at any section}, \ J = \text{the polar moment of inertia of the cross-section of the beam}, \ \rho = \text{mass per unit length of the beam}.

From the Eqs. 154 and 155, the final form of equation of motion becomes,

\[\frac{\partial^2 \theta}{\partial y^2} = \frac{\rho J}{CA} \frac{\partial^2 \theta}{\partial t^2}\] (156)
The solution of the partial differential Eq. 156 can be taken as,

\[ \theta = Y_t \ (A_t \ \cos \omega t + B_t \ \sin \omega t) \]  

(157)

in which

\[ Y_t = C_t \ \cos \left( \frac{\lambda_t}{L} y \right) + D_t \ \sin \left( \frac{\lambda_t}{L} y \right) \] 

(158)

\[ \lambda_t = L \ \sqrt{\frac{2 \ \rho J}{CA}} \] 

(159)

The constants \( A_t \) and \( B_t \) are solved using the initial conditions and the constants \( C_t \) and \( D_t \) are solved using the end conditions of the beam. The initial condition

\[ \frac{\partial \theta}{\partial t} = 0 \quad \theta(t = 0) = 0 \]  

(s)

This yields

\[ B_t = 0 \]  

(t)

The Eq. 157 is then reduced to

\[ \theta = Y_t A_t \ \cos \omega t \] 

(160)

3.4.2. Stiffness Influence Coefficients - The stiffness influence coefficients occupying the fifth column of the F.M. stiffness matrix for a space frame member are developed by providing a unit amplitude in direction 5 of the member shown in Fig. 14 while keeping all other displacements as zero. The end conditions to solve the constants \( C_t \) and \( D_t \) in Eq. 158 are,

<table>
<thead>
<tr>
<th>( y = 0 )</th>
<th>( y = L )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta )</td>
<td>1</td>
</tr>
</tbody>
</table>

(u)
which after substitution in Eq. 158 yields the constant as,

\[ C_t = 1 \]
\[ D_t = -\frac{1}{\tan \lambda_t} \]

Then the Eq. 160 takes the final form as,

\[ \theta = \left\{ \cos \frac{\lambda_t}{L} \cdot y \cdot \frac{1}{\tan \lambda_t} \cdot \sin \frac{\lambda_t}{L} \cdot y \right\} \cos \omega t \] (161)

The first partial derivative of \( \theta \) with respect to \( y \) is given as,

\[ \frac{2\theta}{\partial y} = \frac{\lambda_t}{L} \left\{ -\sin \frac{\lambda_t}{L} \cdot y \cdot \frac{1}{\tan \lambda_t} \cdot \cos \frac{\lambda_t}{L} \cdot y \right\} \cos \omega t \] (162)

The stiffness influence coefficients as defined before are derived as,

\[ k_{55} = T (y = 0) = -C \left( \frac{2\theta}{\partial y} \right) y = 0 \] (163)

From Eqs. 162 and 163,

\[ k_{55} = \frac{C}{L} \eta_t \] (164)

in which

\[ \eta_t = \lambda_t \frac{1}{\tan \lambda_t} \] (165)

\[ k_{ii,5} = T (y = L) = C \left( \frac{2\theta}{\partial t} \right) y = L = -\frac{C}{L} \mu_t \] (166)

Similarly the stiffness influence coefficients occupying the eleventh column of the F.M. Stiffness Matrix are developed by providing a unit amplitude in direction 11 of the member shown in Fig. 14 while keeping all other displacements to zero. Since the end conditions remain the same when working at end i as are when working at end j of the member in Fig. 14, the stiffness influence coefficients remain the same.
The stiffness influence coefficients after occupying their respective positions generate the F.M. stiffness matrix for a space frame member as above.
in which

\[ \alpha_{x,z} = \lambda_{x,z} \left( CH.S - SH.C \right) D_1 \]  \hspace{1cm} (168)

\[ \beta_{x,z} = \lambda_{x,z} \left( SH.S \right) D_1 \]  \hspace{1cm} (169)

\[ \gamma_{x,z} = \lambda_{x,z} \left( C.SH + S.CH \right) D_1 \]  \hspace{1cm} (170)

\[ \delta_{x,z} = \lambda_{x,z} \left( CH - C \right) D_1 \]  \hspace{1cm} (171)

\[ \epsilon_{x,z} = \lambda_{x,z} \left( SH + S \right) D_1 \]  \hspace{1cm} (172)

\[ n = \lambda \frac{1}{\tan \lambda} \]  \hspace{1cm} (14)

\[ n_t = \lambda_t \tan \lambda \]  \hspace{1cm} (165)

\[ \theta_{x,z} = \lambda_{x,z} \left( S.SH \right) D_1 \]  \hspace{1cm} (173)

\[ \nu = \lambda \frac{1}{\sin \lambda} \]  \hspace{1cm} (18)

\[ D_1 = \frac{1}{1 - C.CH} \] \hspace{1cm} (174)

\[ S = \sin \lambda_{x,z} \]  \hspace{1cm} (175)

\[ C = \cos \lambda_{x,z} \]  \hspace{1cm} (175)

\[ SH = \sinh \lambda_{x,z} \]  \hspace{1cm} (175)

\[ CH = \cosh \lambda_{x,z} \]  \hspace{1cm} (175)

\[ \lambda_x = L \sqrt{\frac{\omega \rho}{E I_{xx}}} \] \hspace{1cm} (176)

\[ \lambda_z = L \sqrt{\frac{\omega \rho}{E I_{zz}}} \] \hspace{1cm} (177)

\[ \lambda_t = L \sqrt{\frac{\rho J \omega^2}{C A}} \] \hspace{1cm} (159)

\[ \lambda_{\lambda} = L \sqrt{\frac{\omega^2 \rho}{E A}} \] \hspace{1cm} (7)

\[ \psi_t = \lambda_t \frac{1}{\sin \lambda_t} \]  \hspace{1cm} (177A)
3.5 Dynamic Fixed End Reactions

3.5.1. A Fix-Fix Beam Under a Point Vibrating Load - The dynamic fixed end reactions for a bar can be calculated from the relation:

\[ M_1 = -EI (Y'')_{y=0} \quad (178) \]
\[ M_2 = EI (Y'')_{y=L} \quad (179) \]
\[ V_1 = EI (Y''')_{y=0} \quad (180) \]
\[ V_2 = -EI (Y''')_{y=L} \quad (181) \]

in which

\[ Y = C_1 \cos \lambda y + C_2 \sin \lambda y + C_3 \cosh \lambda y + C_4 \sinh \lambda y \quad (26) \]

Parameter \( \lambda \) is defined in Eq.27. The constants \( C_1, C_2, C_3, \) and \( C_4 \) are determined from the end conditions of the bar under consideration. The slope deflection equations thus obtained have to satisfy the conditions of equilibrium at point C, such as,

\[ M_{ci} + M_{cj} = 0 \quad (182) \]
\[ V_{ci} - V_{cj} = P \quad (183) \]
For a fix-fix beam substituting for $Y$ from Sec. 3.2.2. in Eqs. 178, 179, 180 and 181 for elements $ic$ and $cj$ of the beam, the fixed end reactions come out to be,

\[
M_1 = \frac{EI}{A} \left[ \delta(\lambda_A) \theta_C - \delta(\lambda_A) \frac{w_C}{A} \right]_{183A}
\]

\[
M_2 = -\frac{EI}{B} \left[ \beta(\lambda_B) \theta_C + \delta(\lambda_B) \frac{w_C}{B} \right]_{184A}
\]

\[
v_1 = \frac{EI}{A^2} \left[ \delta(\lambda_A) \theta_C - \epsilon(\lambda_A) \frac{w_C}{A} \right]_{185A}
\]

\[
v_2 = -\frac{EI}{B^2} \left[ \delta(\lambda_B) \theta_C + \epsilon(\lambda_B) \frac{w_C}{B} \right]_{186A}
\]

where

\[
\lambda_A = A \sqrt{\frac{\omega_P}{EI}} \quad (187)
\]

\[
\lambda_B = B \sqrt{\frac{\omega_P}{EI}} \quad (188)
\]

\[
\lambda = L \sqrt{\frac{\omega_P}{EI}} \quad (27)
\]

The coefficients

\[
\beta(\lambda) = \lambda \left( \frac{\sinh \lambda - \sin \lambda}{1 - \cos \lambda \cosh \lambda} \right) \quad (189)
\]

\[
\delta(\lambda) = \lambda^2 \left( \frac{\cosh \lambda - \cos \lambda}{1 - \cos \lambda \cosh \lambda} \right) \quad (190)
\]

\[
\epsilon(\lambda) = \lambda^3 \left( \frac{\sinh \lambda + \sin \lambda}{1 - \cos \lambda \cosh \lambda} \right) \quad (191)
\]

Where

\[
A = \text{Length of segment } ic,
\]

\[
B = \text{Length of segment } cj.
\]
To calculate \( \omega_c \) & \( \theta_c \), Eqs. 182 and 183 are used, which gives the final and modified form to the fixed end reactions as,

\[
M_1 = P_o L \left[ d(a + a') - b(t - t') \right] D_F \quad (192)
\]

\[
M_2 = -P_o L \left[ d'(a + a') + b'(t - t') \right] D_F \quad (193)
\]

\[
V_1 = P_o \left[ e(a + a') - d(t - t') \right] D_F \quad (194)
\]

\[
V_2 = P_o \left[ e'(a + a') + d'(t - t') \right] D_F \quad (195)
\]

where

\[
D_F = \frac{1}{(a + a')(g + g') - (t - t')^2} \quad (196)
\]

\[
a = \frac{\alpha(A_A)}{A/L} \quad (200)
\]

\[
b = \frac{\beta(\lambda_A)}{A/L} \quad (201)
\]

\[
t = \frac{\theta(\lambda_A)}{A/L} \quad (202)
\]

\[
d = \frac{\delta(\lambda_A)}{A/L} \quad (203)
\]

\[
g = \frac{\gamma(\lambda_A)}{A/L} \quad (204)
\]

\[
et = \frac{\epsilon(\lambda_A)}{A/L} \quad (205)
\]

\[
a' = \frac{\alpha(A_B)}{B/L} \quad (206)
\]

\[
b' = \frac{\beta(\lambda_B)}{B/L} \quad (207)
\]

\[
t' = \frac{\theta(\lambda_B)}{B/L} \quad (208)
\]

\[
d' = \frac{\delta(\lambda_B)}{B/L} \quad (209)
\]

\[
g' = \frac{\gamma(\lambda_B)}{B/L} \quad (210)
\]

\[
et' = \frac{\epsilon(\lambda_B)}{B/L} \quad (211)
\]

Where \( \lambda, \lambda_A \) and \( \lambda_B \) are defined in Eqs. 27, 187, and 188 and \( \beta(\lambda), \delta(\lambda) \) and \( \epsilon(\lambda) \) are defined in Eqs. 189, 190 and 191.

\[
\alpha(\lambda) = \lambda \left[ \frac{\cosh \lambda \sin \lambda - \sinh \lambda \cos \lambda}{1 - \cos \lambda \cosh \lambda} \right] \quad (212)
\]

\[
\theta(\lambda) = \lambda^2 \left[ \frac{\sinh \lambda \sin \lambda}{1 - \cos \lambda \cosh \lambda} \right] \quad (213)
\]

\[
\gamma(\lambda) = \lambda^3 \left[ \frac{\cosh \lambda \sin \lambda + \sinh \lambda \cos \lambda}{1 - \cosh \lambda \cos \lambda} \right] \quad (214)
\]

3.5.2. A Fix - Pinned Beam Under A Point Load - The dynamic fixed end reactions of the beam shown,
in Fig. 16 are represented by the same equations of Sec. 3.5.1, Eqs. 183, 184, 185 and 186, except that
\[ M_2 = 0 \]  
where
\[ a' = \frac{\alpha'(\lambda R)}{B/L} \]  
\[ \theta' = \frac{\theta'(\lambda R)}{B/L} \]  
\[ \gamma' = \frac{\gamma'(\lambda R)}{B/L} \]  
\[ \epsilon' = \frac{\epsilon'(\lambda R)}{B/L} \]

in which \( a' \), \( \theta' \), \( \delta' \), \( \gamma' \), and \( \epsilon' \) are as given by Eqs. 74, 75, 76, 77 and 78
\[ a'(\lambda) = a - \frac{\delta^2}{\alpha} = \lambda \left[ \frac{2 \sinh \lambda \sin \lambda}{\cosh \lambda \sin \lambda - \sinh \lambda \cos \lambda} \right] \]  
\[ \theta'(\lambda) = \theta - \frac{\beta \delta}{\alpha} = \lambda^2 \left[ \frac{\cosh \lambda \sin \lambda + \sinh \lambda \cos \lambda}{\cosh \lambda \sin \lambda - \sinh \lambda \cos \lambda} \right] \]
\[
\delta'(\lambda) = \delta - \frac{\delta \theta}{\alpha} = \lambda^2 \left[ \frac{\sinh \lambda + \sin \lambda}{\cosh \lambda \sin \lambda - \sinh \lambda \cos \lambda} \right] 
\]

\[
\gamma'(\lambda) = \gamma - \frac{\delta^2}{\alpha} = \lambda^3 \left[ \frac{2 \cosh \lambda \cos \lambda}{\cosh \lambda \sin \lambda - \sinh \lambda \cos \lambda} \right] 
\]

\[
\varepsilon'(\lambda) = \varepsilon - \frac{\theta \delta}{\alpha} = \lambda^3 \left[ \frac{\cosh \lambda + \cos \lambda}{\cosh \lambda \sin \lambda - \sinh \lambda \cos \lambda} \right] 
\]

3.5.3. A Cantilever Under a Point Vibrating Load - The equations for \( M_1 \) and \( V_1 \) defining

\[
\frac{M_1 \cos \omega t}{P = P_0 \cos \omega t} 
\]

\[
V_1 \cos \omega t 
\]

**FIG. 17 - A CANTILEVER UNDER A POINT VIBRATING LOAD**

The fixed end reactions of a cantilever beam under a point vibrating load are the same as of Sec. 3.5.1, Eqs. 183 and 184, except at second end,

\[
M_2 = 0 
\]

\[
V_2 = 0 
\]

\[
M_1 = P_0 L \left[ \frac{d(a + a') - b(t - t')}{(a + a')(g + g') - (t - t')^2} \right] 
\]

\[
V_1 = P_0 \left[ \frac{e(a + a') - d(t - t')}{(a + a')(g + g') - (t - t')^2} \right] 
\]

in which

\[
a' = \frac{a''(\lambda R)}{B/L} \quad (223); \quad t' = \frac{\theta''(\lambda R)}{B/L} \quad (224)
\]

\[
\varphi' = \frac{\gamma''(\lambda R)}{B/L} \quad (225)
\]
The coefficients $\alpha''$, $\theta''$ and $\gamma''$ are as given by Eqs. 83, 84 and 85.

\begin{equation}
\alpha''(\lambda) = \lambda \left[ \frac{\text{Sinh}\lambda \text{ Cos}\lambda - \text{Cosh}\lambda \text{ Sin}\lambda}{1 + \text{Cosh}\lambda \text{ Cos}\lambda} \right] \tag{83}
\end{equation}

\begin{equation}
\gamma''(\lambda) = -\lambda^3 \left[ \frac{\text{Sinh}\lambda \text{ Cos}\lambda + \text{Cosh}\lambda \text{ Sin}\lambda}{1 + \text{Cosh}\lambda \text{ Cos}\lambda} \right] \tag{84}
\end{equation}

\begin{equation}
\theta''(\lambda) = -\lambda^2 \left[ \frac{\text{Sinh}\lambda}{1 + \text{Cosh}\lambda} \text{ Cos}\lambda \right] \tag{85}
\end{equation}

3.5.4. Uniformly Distributed Vibrating Loads on a Fix-Fix Beam - The equations representing the fixed end reactions of a beam under uniformly distributed vibrating load can be obtained by replacing the point load with the distributed load and integrating the expressions.

\begin{equation}
M_1 = L^2 \int_0^1 \frac{d(a + a') - b(t - t')}{(a + a')(g + g') - (t - t')^2} \cdot \frac{d(\gamma)}{L} \tag{226}
\end{equation}

\begin{equation}
M_2 = -L^2 \int_0^1 \frac{d'(a + a') + b'(t - t')}{(a + a')(g + g') - (t - t')^2} \cdot \frac{d(\gamma)}{L} \tag{227}
\end{equation}

\begin{equation}
V_1 = L \int_0^1 \frac{e(a + a') - d(t - t')}{(a + a')(g + g') - (t - t')^2} \cdot \frac{d(\gamma)}{L} \tag{228}
\end{equation}

\begin{equation}
V_2 = L \int_0^1 \frac{e'(a + a') + d'(t - t')}{(a + a')(g + g') - (t - t')^2} \cdot \frac{d(\gamma)}{L} \tag{229}
\end{equation}

where $q$ is the uniformly distributed vibrating load acting on the member.

The expressions for fixed end reactions of beams with various end conditions and under uniformly distributed vibrating load can be obtained from the expressions developed for same end conditions for beams under point vibrating loads. The point
vibrating loads are replaced by uniformly distributed loads and the expressions are integrated between the two ends of the member.

3.6. **Determination of Natural Frequencies**

In order to determine the natural frequency of vibrations of any frame structure, the frequency and mass dependent stiffness matrix for every structural element is developed. The main F.M. stiffness matrix is then generated through a suitable superposition of the member stiffness matrices in accordance with the code number approach. The elements of F.M. stiffness matrix are the function of frequency \( (\omega) \) of vibrations of the impressed force. In the case of free vibrations, this frequency is the natural frequency of the structure which is determined by solving the equation,

\[
\text{Det} |K| = 0 \tag{230}
\]

in which \( \text{Det} |K| \) = the determinant of the main F.M. stiffness matrix.

The determination of frequencies for which \( |K| \) is equal to zero, can be accomplished either using the trial and error method or Newton-Raphson iteration method. In the trial and error method, a graph of determinant versus frequency of vibrations is obtained using the plotter routine available with the computer. The fundamental natural frequency occurs where
the value of the determinant first changes from positive to negative. The sign change in the determinant is achieved by calculating where the sign of the product of its values at two successive trial values, \( \omega_1 \) and \( \omega_2 \), becomes negative. The value of natural frequency \( (\omega_n) \) is then determined by means of an interpolation between \( \omega_1 \) and \( \omega_2 \). The problem is solved by giving first large increments of \( d \) to \( \omega_1 \), and then small increments to get a closer approximation to the exact value of the natural frequency. The degree of accuracy of \( \omega_n \) is observed by comparing the determinant values at the points used for the interpolation with the determinant value at the point obtained from the interpolation.

In the Newton-Raphson iteration method for determining the roots of an equation \( F(x) = 0 \), a point \( x_n \) is chosen as a close approximation to the first root. A tangent to the curve is drawn at \( x = x_n \) and the point \( x_n + 1 \) where it intersects the \( x \)-axis is determined. A tangent to the curve is drawn at the point where the perpendicular at \( x = x_n + 1 \) cuts the curve. The new point, \( x_n + 2 \), where the tangent to curve intersects the \( x \)-axis is determined. The process is continued until the two successive values of \( x \) are very close to each other. The whole process is illustrated in Fig. 18. The analytical representation of the Newton-Raphson method is

\[
x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)}
\] (231)
In order to determine the roots of Eq. 230, the first approximation can be taken as the frequency of vibrations of the impressed force. In the Eq. 231, the function $F(x)$ is same as $\text{Det.} |K|_{FM}$ and the values of natural frequencies of the structure can easily be determined from Eq. 230.

$$f(x) = 0$$

**FIG. 18. - A DIAGRAMIC REPRESENTATION OF NEWTON-RAPHSON METHOD**

3.7. **Determination of Internal Stress Resultants** - In the dynamic analysis of any frame structure, the F.M. stiffness matrix is developed for each of its members. Depending upon the size of the problem, the main stiffness matrix for the whole structure is generated through a proper superposition using the code number technique. The equation

$$ [K] \{U\} = \{P\} $$(231a)

in which $[K]$ = the main F.M. stiffness matrix for the whole
structure, \( \{ U \} \) = the vector of displacements for specified degrees of freedom, and \( \{ P \} \) = the vector of dynamic fixed end reactions, is solved for \( \{ U \} \). For the known displacements, \( \{ U \} \) of member 1, the internal stress resultants \( \{ f_1 \} \) are calculated for each member, using the equation

\[
[K_1]\{U_1\} + \{P_1\} = \{f_1\} \quad (231b)
\]

in which \( [K_1] \) = the F.M. stiffness matrix and \( \{ P_1 \} \) = vector of dynamic fixed end reactions of a member.

The presence of a concentrated mass (m) on a structural member is taken into account by providing a spring having \( (-mw^2) \) as its spring constant. The spring is considered at the location and in the directions of specified degrees of freedom of the concentrated mass.
CHAPTER IV

LUMPED MASS SYSTEM IDEALIZATION

To represent the inertial characteristics of any structural element, lumped mass system provides the simplest form of mathematical tool. A continuous media is idealized into concentrated masses stationed at the node points in the directions stating the degrees of freedom and referring to translational and rotational inertia of the element. The system assumes that the material within the nodal points acts as a rigid body and the point masses contribute to the motion. This amounts to say that the inertia force produced by a unit acceleration provided to the lumped mass at a nodal point in the direction of any degree of freedom is independent of the inertia force produced by a unit acceleration provided to the lumped mass at the same or different nodal point in any other direction of the degree of freedom of the element. The mass matrix so derived is dynamically uncoupled between the element displacements and is only a diagonal matrix.

The general equation of motion derived for any structural element of a frame structure using the Lumped Mass System takes the final form.

\[
[M] \{\ddot{x}\} + [K] \{x\} = \{P\} \tag{232}
\]

in which \([M]\) = the diagonal mass matrix, \([K]\) = the stiffness matrix of the element, \(\{x\}\) = the displacement vector, \(\{\ddot{x}\}\) = the acceleration vector, \(\{P\}\) = the load vector in which \(P = P_0 e^{\omega t}\).
The solution to the Eq. 232 can be assumed to be

\[ x = A e^{\omega t} \tag{233} \]

On substitution, the Eq. 232 reduces to the form

\[ \begin{bmatrix} [K] - \omega^2 [M] \end{bmatrix} \{A\} = \{F_o\} \tag{234} \]

For the case of free vibrations, Eq. 234 can be represented as

\[ \begin{bmatrix} [K] - \omega_n^2 [M] \end{bmatrix} \{A\} = 0 \tag{235} \]

The approach to stiffness matrices for various types of elements has been dealt with very widely and the literature is well publisized.

The accuracy of the results obtained using Eq. 232 depends mainly on the accuracy of the mass matrix. Though the approach in the formulation of mass matrix using the Lumped Mass System and the solution of the structural dynamic response problems is rather simple, but the results so obtained might differ appreciably from the exact one.

Since the accuracy of the results for any structural dynamic response problem depends mainly upon the accuracy of the mass matrix, the mass matrices (1, 2) have been improved to account for the actual distribution of the mass. In the subsequent chapter, the concepts of consistent mass matrix has been introduced and the mass matrices of several finite element have been presented.
CHAPTER V

CONSISTENT MASS MATRICES

In the dynamic analysis of structural problems, the accuracy of results is largely effected by the mass matrix. Archer (1) suggested a new technique of formulating a mass matrix and named it as "Consistent Mass Matrix" to obtain a higher degree of precision in the results. Przemieniecki (2) later suggested his technique of formulating the mass matrix "Equivalent Mass Matrix". Even though the mass matrix obtained by both the techniques is same but the approach in obtaining the matrix is somewhat different.

5.1. Derivation of Consistent Mass Matrices - A displacement function, \{u\}, for any structural element under plane stress or plate bending may be given in terms of the coordinates as

\[
\{u\} = [C] \{a\}
\]

(236)

in which \([C]\) = a matrix, function of coordinates only, \(\{a\}\) = a vector of unknown constants,

and \(\{u\} = \begin{bmatrix} u \\ v \end{bmatrix} \)

The constants vector, \(\{a\}\), is determined from Eq. 236 for a finite number of displacements defined at the nodes of the element in the directions of specified degrees of freedom.

This gives

\[
\{U\} = [A] \{a\}
\]

(237)
or \( \{ a \} = [A^{-1}] \{ U \} \) \hspace{1cm} (238)

in which \( \{ U \} \) = the displacements defined at nodes in the direction of the specified degrees of freedom.

Substitution of \( \{ a \} \) in Eq. 236 gives,

\[
\{ u \} = [F] \{ U \} \hspace{1cm} (239)
\]

in which \( [F] = [C] [A^{-1}] \) \hspace{1cm} (240)

A relationship of the displacements \( \{ U \} \) in terms of a finite number of displacements \( \{ U \} \) prescribed at the nodes in the direction of specified degrees of freedom is defined by Eq. 239. The matrix \( [F] \) is function of coordinates only.

Eq. 239 is valid only for small deflections.

Since the first differential of \( \{ u \} \) with respect to coordinate axes gives strain \( (e) \) along the corresponding axes, there exists a strain-displacement relationship as,

\[
\{ e \} = [b] \{ u \} \hspace{1cm} (241)
\]

in which \( b = b(x,y,z) \) is obtained by differentiating \( [F] \) in accordance with the definition of strain \( \{ e \} \).

Eq. 239 is valid for dynamic analysis only if the displacement, \( \{ U \} \) is determined from the equation of motion of the system.

For dynamic loading, the principle of virtual work is defined as

\[
\delta U_i = \delta W - \int_V \delta u^T \left[ \frac{\partial \sigma}{\partial A} \right] u^i \hspace{1cm} (242)
\]

in which \( W = \) Potential Energy and \( U_i = \int_V \sigma^T \delta u^T \) the strain energy of the element. The term \( \int_V \delta u^T \left[ \frac{\partial \sigma}{\partial A} \right] u^i \) is an increment in the kinetic energy due to the inertia force produced in the element.
under the dynamic loading. The virtual displacement, $\delta u$, and virtual strain $\delta \varepsilon$ can be obtained from Eqs. 239 and 241 respectively as,

$$\delta u = F \delta U \quad \text{(243)}$$

$$\delta \varepsilon = B \delta u \quad \text{(244)}$$

In Eq. 242, $\delta U_i$ and $\delta W$ are defined as

$$\delta U_i = \int \delta \varepsilon^T \sigma \, dv \quad \text{(245)}$$

$$\delta W = \int \delta u^T \phi \, ds + \int \delta u^T X \, dv + \delta U_T P \quad \text{(246)}$$

in which $\phi$ is the surface force, $X$ is the body force and $P$ is the external force on the structure.

After substituting from Eqs. 243, 244, 245 and 246 in Eq. 242 and using Hooke's law, $\varepsilon = E \sigma$, is defined as

$$\sigma = E \varepsilon \quad \text{(247)}$$

Eq. 242 is represented as

$$\int \delta u^T b^T E b u \, dv = \int \delta u^T T^T \phi \, ds + \int \delta u^T T^T X \, dv + \int \delta u^T T^T P \, dv \quad \text{(248)}$$

From Eq. 239,

$$\ddot{u} = F \dot{u} \quad \text{(249)}$$

Substituting in Eq. 248, the equation reduces to

$$M \ddot{u} + K u = P + \int F^T \phi \, ds + \int F^T X \, dv \quad \text{(250)}$$

in which $M = \int \frac{\partial}{\partial u} F^T F \, dv \quad \text{(251)}$

$$K = \int b^T E b \, dv \quad \text{(252)}$$

If the surface forces, $\phi$ and the body forces $X$ of the structure are zero, Eq. 250 changes to

$$M \ddot{u} + K u = P \quad \text{(253)}$$

in which $M = \text{Equivalent Mass Matrix}$ and $K = \text{Stiffness Matrix}$
of the finite element under consideration.

5.2. Mass Matrix For a Plane Frame Members - The displacement function best suitable for a

\[ u = 0 \]
\[ v = a_1 + a_2y - (a_4 + 2a_5y + 3a_6y^2)z \]
\[ w = a_3 + a_4y + a_5y^2 + a_6y^3 \]

FIG. 19. - A PLANE FRAME MEMBER IN MEMBER AXES

- plane frame member is

\[ u = 0 \]
\[ v = a_1 + a_2y - (a_4 + 2a_5y + 3a_6y^2)z \]
\[ w = a_3 + a_4y + a_5y^2 + a_6y^3 \]

The first two terms of displacement function of \( v \) represents constant axial force and \( a_4, a_5, \) and \( a_6 \) terms represent effect of lateral vibrations on axial forces.
The displacement function $w$ as given in Eq. 254 represents constant shear.

This can be written as $\{u\} = [C]\{a\}$

\[
\begin{bmatrix}
  v \\
  w
\end{bmatrix} =
\begin{bmatrix}
  1 & y & o & -z & -2yz & -3y^2z \\
  o & o & o & 1 & y^2 & y^3
\end{bmatrix}
\begin{bmatrix}
  a_1 \\
  a_2 \\
  a_3 \\
  a_4 \\
  a_5 \\
  a_6
\end{bmatrix}
\]  

(255)

If $U_i$ ($i = 1, \ldots, 6$) are the prescribed displacements at the nodal points of the member then
Eq. 256 can be written as
\[
\{\mathbf{u}\} = [\mathbf{A}] \{\mathbf{a}\}
\]

Solving for \(\mathbf{a}\) in Eq. 256

\[
\begin{bmatrix}
\mathbf{a}_1 \\
\mathbf{a}_2 \\
\mathbf{a}_3 \\
\mathbf{a}_4 \\
\mathbf{a}_5 \\
\mathbf{a}_6 \\
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
-3 & \frac{3}{L^2} & \frac{2}{L^2} & \frac{1}{L} & 0 & 0 \\
-2 & \frac{2}{L^2} & \frac{1}{L} & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
\mathbf{U}_1 \\
\mathbf{U}_2 \\
\mathbf{U}_3 \\
\mathbf{U}_4 \\
\mathbf{U}_5 \\
\mathbf{U}_6 \\
\end{bmatrix}
\]

\[
\{\mathbf{a}\} = \left[ \mathbf{A}^{-1} \right] \{\mathbf{U}\}
\]

Substituting for \(\mathbf{a}\) in Eq. 255,

\[
\{\mathbf{u}\} = \left[ \mathbf{F} \right] \{\mathbf{u}\}
\]

in which
\[
\left[ \mathbf{F} \right] = \left[ \mathbf{C} \right] \left[ \mathbf{A} \right]^{-1}
\]

then

\[
\left[ \mathbf{F} \right] = \left[ \begin{array}{c}
1 - \frac{y^2}{L} - \frac{6yz}{L^2} + \frac{6y^2z}{L^2} - \frac{6yz}{L^3} - \frac{6y^2z}{L^3} \\
0 \\
1 - \frac{3y^2}{L^2} - \frac{2y^3}{L^3} \\
0 \\
0 \\
0 \\
\end{array} \right]
\]

The mass matrix \(\mathbf{M}\) as defined in Eq. 251 is

\[
\mathbf{M} = \int_0^L \rho \mathbf{F}^T \mathbf{F} \, dy
\]
Substitution of values in Eq. 251 \( [M] \), the mass matrix for a plane frame member then becomes as,

\[
[M] = \frac{\rho L}{420} \begin{bmatrix} 140 & 70 & 0 & 0 & 0 & 0 \\ 70 & 140 & 0 & 0 & 0 & 0 \\ 0 & 0 & 156 & 54 & 22L & -13L \\ 0 & 0 & 54 & 156 & 13L & -22L \\ 0 & 0 & 22L & 13L & 4L^2 & -3L^2 \\ 0 & 0 & -13L & -22L & -3L^2 & 4L^2 \end{bmatrix}
\tag{259}
\]

The mass matrix \([M]\) for a plane frame member defined in Eq. 259 does not include the effects of rotary inertia and shear deformations.

5.3. Mass Matrix For a Space Frame Member - The displacement function, \([\mathbf{u}]\), of Eq. 255 for a space frame member is presented in the matrix form as follows,

\[ \mathbf{u} \]

\[ \begin{bmatrix} u \\ v \\ w \\ \theta_x \\ \theta_y \\ \theta_z \end{bmatrix} \]

FIG. 21. - SPACE FRAME MEMBER IN MEMBER AXES
The prescribed displacements, \( U_i \ (i = 1, \ldots, 12) \) in the directions shown in Fig. 21 at the two ends \( i \) and \( j \) of the member are obtained from Eq. 260 as follows.

\[
\begin{bmatrix}
1 & y & y^2 & y^3 & z & yz & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -x & -2yx & -3y^2x & 0 & 0 & 1 & y & 0 & -z & -2zy & -3zy^2 \\
0 & 0 & 0 & 0 & x & xy & 0 & 0 & 1 & y & y^2 & y^3
\end{bmatrix}
\]

\( \begin{array}{c}
a_1 \\
a_2 \\
a_3 \\
a_4 \\
a_5 \\
a_6 \\
a_7 \\
a_8 \\
a_9 \\
a_{10} \\
a_{11} \\
a_{12}
\end{array} \)
\[
\begin{bmatrix}
u_1 \\
u_2 \\
u_3 \\
u_4 \\
u_5 \\
u_6 \\
u_7 \\
u_8 \\
u_9 \\
u_{10} \\
u_{11} \\
u_{12}
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & L & L^2 & L^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & L & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & L & L^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2L \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -L & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & -2L & -3L^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
a_1 \\
a_2 \\
a_3 \\
a_4 \\
a_5 \\
a_6 \\
a_7 \\
a_8 \\
a_9 \\
a_{10} \\
a_{11} \\
a_{12}
\end{bmatrix}
\]

\[
\{\mathbf{u}\} = [A]\{a\}
\]

Solving for \{a\} from Eq.(261)
Substitution for \{a\} in Eq. 260 gives,

\[
\{u\} = [C][A^{-1}]\{u\} = [F]\{u\} \quad \text{(239)}
\]

in which
The mass matrix, \([M]\) as defined by Eq. 251 is generated for a space frame member as given below,

\[
[M] = \begin{bmatrix}
M_{11} & M_{12} \\
M_{21} & M_{22}
\end{bmatrix}
\]  

(264)

where \(M_{11}, M_{22}\) and \(M_{12} = M_{21}\) as given in matrix form as below,

\[
\begin{bmatrix}
M_{11} & M_{12} \\
M_{21} & M_{22}
\end{bmatrix} =
\begin{bmatrix}
1 - 3\xi^2 + 2\xi^3 & 6(\xi - \xi^2)\xi & 0 \\
3\xi^2 - 2\xi^3 & 6(-\xi + \xi^2)\xi & 0 \\
0 & (1 - \xi) & 0 \\
0 & \xi & 0 \\
0 & 6(\xi - \xi^2) & 1 - 3\xi^2 + 2\xi^3 \\
0 & 6(-\xi + \xi^2)\eta & 3\xi^2 - 2\xi^3 \\
0 & \text{Ln}(1 + 4\xi - 3\xi^2) & (\xi - 2\xi^2 + 3\xi^3)\text{Ln} \\
0 & (2\xi - 3\xi^2)\text{Ln} & (-\xi^2 + 3\xi^3)\text{Ln} \\
-(1-\xi)\text{Ln} & 0 & -(1-\xi)\xi \\
-L\xi\eta & 0 & -L\xi \xi \\
L(-\xi + 2\xi^2 - 3\xi^3) & (1 - 4\xi + 3\xi^2)\text{Ln} & 0 \\
L(\xi^2 - \xi^3) & (-2\xi + 3\xi^2)\text{Ln} & 0
\end{bmatrix}
\]  

(263)

where

\[
\begin{align*}
\xi &= \frac{x}{L} \\
\xi &= \frac{y}{L} \\
\eta &= \frac{z}{L}
\end{align*}
\]  

\[\text{--------(a)}\]
\[
[M_{11}] = \rho \\
\begin{bmatrix}
\frac{13}{35} + 6I_Z & \frac{9}{70} - 6I_Z & 0 & 0 & 0 & 0 \\
\frac{9}{70} - 6I_Z & \frac{13}{35} + 6I_Z & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{L}{3} & \frac{L}{6} & 0 & 0 \\
0 & 0 & \frac{L}{6} & \frac{L}{3} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{13}{35} + 6I_Z & \frac{9}{70} - 6I_Z \\
0 & 0 & 0 & 0 & \frac{9}{70} - 6I_Z & \frac{13}{35} + 6I_Z \\
\end{bmatrix}
\]

\[\text{Where}\ I_x = \int_A z^2 dA ;\ I_z = \int_A x^2 dA ;\]

\[J_y = \int_A (x^2 + z^2) dA\]
\[
\begin{bmatrix}
0 & 0 & 0 & 0 & \frac{11}{210}L^2 + \frac{I_x}{10A} & \frac{I_x}{10A} \\
0 & 0 & 0 & 0 & \frac{-13}{420}L^2 + \frac{I_x}{10A} & \frac{-11}{210}L^2 - \frac{I_x}{10A} \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
[M_{21}] = \int \begin{bmatrix}
\frac{-11}{210}L^2 - \frac{I_x}{10A} & \frac{13}{420}L^2 + \frac{I_x}{10A} \\
\frac{13}{420}L^2 - \frac{I_x}{10A} & \frac{11}{210}L^2 + \frac{I_x}{10A} \\
\end{bmatrix} \, dA
\]

Where \( I_x = \int_A z^2 \, dA \); \( I_z = \int_A x^2 \, dA \)

\( (264c) \)
Determination of Natural Frequencies - The problem of evaluating the natural frequencies of a frame structure using lumped mass method presents no difficulty. It is a straightforward eigen value problem. However when the mass matrix has off diagonal terms too, the determination of natural frequency requires an indirect approach as presented by Utku(11), given in the following steps.

The eigen value problem associated with a real positive definite symmetric matrix is

\[ \begin{bmatrix} A \end{bmatrix} \{ \mathbf{x} \} = \lambda \{ \mathbf{x} \} \]  \hspace{1cm} (265)

in which \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are the eigen values and \( \{X_1\}, \{X_2\}, \{X_3\}, \ldots, \{X_n\} \) are the associated eigen vectors.

Defining \( [D] \) and \( [M_a] \) as

\[
[D] = \begin{bmatrix}
\lambda_1 & 0 & 0 & \cdots & 0 \\
0 & \lambda_2 & 0 & & \\
0 & 0 & \ddots & \ddots & \\
& & \ddots & \ddots & \\
0 & & & \cdots & \lambda_n
\end{bmatrix}
\]  \hspace{1cm} (266)

\[
[M_a] = \begin{bmatrix}
\{X_1\} & \{X_2\} & \{X_3\} & \cdots & \{X_n\}
\end{bmatrix}
\]  \hspace{1cm} (267)

Eq. 265 can be restated as

\[ \begin{bmatrix} A \end{bmatrix} [M_a] = [M_a] [D] \]  \hspace{1cm} (268)

In order to solve the eigen value problem resulting from
the dynamic analysis of discrete structures, Eq. 235 must be reduced to standard form of Eq. 265. The procedure is as follows

Define

\[ \frac{1}{\omega^2} = \lambda_1 \]  

(269)

Then substituting from Eqs. 266, 267, and 269, Eq. 235 reduces to

\[ [K] [Ma] = \omega^2 [M] [Ma] \]  

(270)

in which M is a real positive definite symmetric mass matrix.

Post multiplying both sides of Eq. 270 by \([D]\), the equation takes the form

\[ [K] [Ma] [D] = \omega^2 [M] [Ma] [D] \]  

(271)

\[ = [M] [Ma] \]  

(272)

in which \(\omega^2 [D] = [I]\), Identity matrix.

Since \([K]\) is positive definite, it may be defined as

\[ [K] = [m][d][m]^T \]  

(273)

in which \([m]\)= Modal matrix established by the normalized eigenvectors of \([K]\), \([d]\)= Diagonal matrix established out of eigenvalues of \([K]\). From Eq. 273, \([K]\frac{1}{2}\) and \([K]^{-\frac{1}{2}}\) can be defined using matrix algebra as,

\[ [K]^\frac{1}{2} = [m][d]^\frac{1}{2} [m]^T \]  

(274)

\[ [K]^{-\frac{1}{2}} = [m][d]^{-\frac{1}{2}} [m]^T \]  

(275)

Eq. 272 can be rewritten as

\[ [K]^\frac{1}{2} [K]^\frac{1}{2} [Ma][D] = [M][K]^{-\frac{1}{2}} [K]^\frac{1}{2} [Ma] \]  

(276)
Pre multiplying both sides by $[K]^{-\frac{1}{2}}$, it reduces to

$$[K]^{-\frac{1}{2}}[Ma][D] = [K]^{-\frac{1}{2}}[M][K]^{-\frac{1}{2}}[K]^{-\frac{1}{2}}[Ma]$$ (277)

Then Eq. 235 reduces to the standard form of Eq. 268 as

$$[\bar{M}][\bar{Ma}] = [\bar{Ma}][D]$$ (278)

in which

$$[\bar{M}] = [K]^{-\frac{1}{2}}[M][K]^{-\frac{1}{2}}$$ (279)

and

$$[\bar{Ma}] = [K]^{-\frac{1}{2}}[Ma]$$ (280)

The whole theory of determining the eigenvalues of a discrete structure can be summarized into following steps.

1. Solve the eigenvalues $[d]$ and eigenvectors $[m]$ of the equation

$$[K]\{x\} = \lambda \{x\}$$ (281)

where $[d] = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & \ldots & \lambda_n \end{bmatrix}$

and $[m] = \{X_1 \} \{X_2 \} \ldots \{X_n \}$

2. After calculating $[d]$ and $[m]$, establish

$$[K]^{-\frac{1}{2}} = [m][d]^{-\frac{1}{2}}[m]^T$$

AND

$$[\bar{M}] = [K]^{-\frac{1}{2}}[M][K]^{-\frac{1}{2}}$$

3. Find eigenvalues of $[\bar{M}]$ as $[D]$ and eigenvectors as $[\bar{Ma}]$

4. (a) The required eigenvalues are $[D]$

(b) The required eigenvectors are determined from

$$[\bar{Ma}] = [K]^{-\frac{1}{2}}[Ma]$$
In order to demonstrate the effectiveness of the frequency and mass dependent stiffness matrix method, certain simple plane frame problems are solved in this chapter. The results obtained by this method are compared to those of lumped mass method and, consistent mass matrix method. It has been realized that if a frame structure element of uniformly distributed mass is replaced by five physically lumped masses, the results are very close to the exact values. The greater the number of lumped masses replacing the structural element of uniformly distributed mass, the higher the proximity to the exact values will be.

Example 1

The frame shown in Fig. 22 is analysed using the lumped mass, the consistent mass matrix and the F.M. stiffness matrix method to calculate the internal stress resultants. In analysing the frame with lumped mass method, all the members of the frame are replaced by 5 equal lumped masses. The results of different methods are presented in Table No. 2.
\[ 1.262T \cos \omega t \]

\[ \omega = 80.2 \text{ rad/ sec} \]

\[ E = 2.1 \times 10^7 \text{ T/m}^2 \]

**Fig. 22. - Plane Frame Structure.**

The structural properties of the different members of the frame in Fig. 22 are given in Table No. 1.

**Table No. 1. Structural Properties of Fig. 22**

<table>
<thead>
<tr>
<th>MEMBER</th>
<th>JOINT NO.</th>
<th>WEIGHT ((\frac{1}{m}))</th>
<th>I ((m^4))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.2</td>
<td>0.5</td>
<td>2 \times 10^{-4}</td>
</tr>
<tr>
<td>2</td>
<td>2.3</td>
<td>0.5</td>
<td>2 \times 10^{-4}</td>
</tr>
</tbody>
</table>
FIG. 23. - MEMBER AXIS

TABLE NO. 2. Internal Stress Resultants of Example I

<table>
<thead>
<tr>
<th>METHODS OF SOLUTION</th>
<th>MEMBER AB</th>
<th></th>
<th></th>
<th></th>
<th>MEMBER BC</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>JOINT A</td>
<td>JOINT B</td>
<td>JOINT C</td>
<td>JOINT D</td>
<td>JOINT A</td>
<td>JOINT B</td>
<td>JOINT C</td>
<td>JOINT D</td>
</tr>
<tr>
<td>F.M. STIFFNESS MATRIX</td>
<td>$Y_1$</td>
<td>$Z_1$</td>
<td>$M_1$ T-m</td>
<td>$Y_2$</td>
<td>$Z_2$</td>
<td>$M_2$ T-m</td>
<td>$Y_1$</td>
<td>$Z_1$</td>
</tr>
<tr>
<td>LUMPED MASS SYSTEM</td>
<td>0.0</td>
<td>0.914</td>
<td>1.186</td>
<td>0.0</td>
<td>0.637</td>
<td>-0.442</td>
<td>0.0</td>
<td>0.082</td>
</tr>
<tr>
<td>CONSISTENT MASS METHOD</td>
<td>0.0</td>
<td>0.911</td>
<td>1.184</td>
<td>0.0</td>
<td>0.633</td>
<td>-0.442</td>
<td>0.0</td>
<td>0.085</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>0.764</td>
<td>1.01</td>
<td>0.0</td>
<td>0.496</td>
<td>-0.342</td>
<td>0.0</td>
<td>0.134</td>
</tr>
</tbody>
</table>
Example 2.

The structure shown in Fig. 24 is solved by the F.M. stiffness matrix method to determine the natural frequencies of the structure. The results are compared with those of lumped mass and consistent mass matrix method. In the lumped mass method, two cases are adopted. For the first case, the distributed mass of the member is replaced by lumped masses at the ends only and for the second case, the distributed mass is replaced by five lumped masses of equal magnitude. The results of all the methods are given in Table No. 4.

\[ E = 21 \times 10^6 \text{ T/m}^2 \]

FIG. 24.

The structural properties of the members of the structure are given in Table No. 3.

**TABLE NO. 3. Structural Properties of Fig. 24**

<table>
<thead>
<tr>
<th>MEMBER</th>
<th>STD. AMERICAN SECTION</th>
<th>JOINT NO.</th>
<th>Ixx(ln^4)</th>
<th>A(ln^2)</th>
<th>LENGTH</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5 \times 3 @ #10.0</td>
<td>1,2</td>
<td>12.1</td>
<td>2.87</td>
<td>12'</td>
</tr>
<tr>
<td>2</td>
<td>7 \times 35 @ #20.0 \frac{8}{5}</td>
<td>2,3</td>
<td>41.9</td>
<td>5.83</td>
<td>8'</td>
</tr>
</tbody>
</table>
TABLE NO. 4. Comparison of Natural Frequencies for Example 2

<table>
<thead>
<tr>
<th>NO.</th>
<th>METHOD</th>
<th>$\omega_{n_1}$ (rad/sec)</th>
<th>$\omega_{n_2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>F.M. STIFFNESS MATRIX</td>
<td>179.3</td>
<td>456</td>
</tr>
<tr>
<td>2</td>
<td>LUMPED MASS METHOD (ONLY ONE LUMPED MASS)</td>
<td>226</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>LUMPED MASS METHOD (USING FIVE LUMPED MASSES)</td>
<td>182.5</td>
<td>491</td>
</tr>
<tr>
<td>4</td>
<td>CONSISTENT MASS MATRIX METHOD</td>
<td>198</td>
<td>646</td>
</tr>
</tbody>
</table>

FIG. 25 - A TWO BAY SINGLE STOREY PLANE FRAME

$\omega = 50.0 \text{ rad/sec}$  
$E = 21000000 \text{ T/m}^2$
Example 3.

A two bay single storey plane frame as shown in Fig. 25 is analysed using the lumped mass, the consistent mass matrix and the F.M. stiffness matrix method. In the lumped mass method, the structural elements of the frame are replaced by lumping the masses varying in number from zero to six. The internal stress resultants and the natural frequencies of the frame are determined using the three methods. The results obtained by the three methods are given in Table Nos. 6 and 7.

The structural properties of the different members of the frame in Fig. 25 are given in Table No. 5.

<table>
<thead>
<tr>
<th>Member</th>
<th>Joint No.</th>
<th>Weight (T/m)</th>
<th>I (m^4)</th>
<th>A(m^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1,2</td>
<td>0.125</td>
<td>.00244</td>
<td>0.156</td>
</tr>
<tr>
<td>2</td>
<td>2,3</td>
<td>1.0</td>
<td>.414</td>
<td>4.97</td>
</tr>
<tr>
<td>3</td>
<td>3,4</td>
<td>0.125</td>
<td>.00244</td>
<td>0.156</td>
</tr>
<tr>
<td>4</td>
<td>3,5</td>
<td>0.75</td>
<td>.01725</td>
<td>0.4655</td>
</tr>
<tr>
<td>5</td>
<td>3,6</td>
<td>0.125</td>
<td>.00244</td>
<td>0.156</td>
</tr>
</tbody>
</table>
FIG. 26 - MEMBER AXIS

TABLE NO. 6. Comparison of Internal Stress Resultants for Example 3

<table>
<thead>
<tr>
<th>Method of Solution</th>
<th>Member</th>
<th>Joint i</th>
<th>Joint j</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$Y_1 (T)$</td>
<td>$Z_1 (T)$</td>
</tr>
<tr>
<td>Lumped Mass Method</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Case I</td>
<td>7.165</td>
<td>-0.648</td>
<td>-1.692</td>
</tr>
<tr>
<td>Case II</td>
<td>7.121</td>
<td>-0.657</td>
<td>-1.711</td>
</tr>
<tr>
<td>Case III</td>
<td>7.076</td>
<td>-0.659</td>
<td>-1.717</td>
</tr>
<tr>
<td>Case IV</td>
<td>7.117</td>
<td>-0.660</td>
<td>-1.719</td>
</tr>
<tr>
<td>Case V</td>
<td>7.123</td>
<td>-0.660</td>
<td>-1.720</td>
</tr>
<tr>
<td>Case VI</td>
<td>7.124</td>
<td>-0.660</td>
<td>-1.720</td>
</tr>
<tr>
<td>Case VII</td>
<td>6.98</td>
<td>-0.661</td>
<td>-1.722</td>
</tr>
<tr>
<td>Method of Solution</td>
<td>Member</td>
<td>Joint i</td>
<td>Joint j</td>
</tr>
<tr>
<td>---------------------------</td>
<td>--------</td>
<td>---------------</td>
<td>---------------</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$Y_1 (T)$</td>
<td>$Z_1 (T)$</td>
</tr>
<tr>
<td>F.M. Stiffness Matrix</td>
<td></td>
<td>7.126</td>
<td>-0.663</td>
</tr>
<tr>
<td>Consistent Mass Matrix</td>
<td></td>
<td>7.314</td>
<td>-0.806</td>
</tr>
<tr>
<td>Lumped Mass</td>
<td></td>
<td>0.648</td>
<td>7.157</td>
</tr>
<tr>
<td>Case I</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Case II</td>
<td></td>
<td>0.648</td>
<td>7.117</td>
</tr>
<tr>
<td>Case III</td>
<td></td>
<td>0.648</td>
<td>7.073</td>
</tr>
<tr>
<td>Case IV</td>
<td></td>
<td>0.647</td>
<td>7.115</td>
</tr>
<tr>
<td>Case V</td>
<td></td>
<td>0.647</td>
<td>7.122</td>
</tr>
<tr>
<td>Case VI</td>
<td>2</td>
<td>0.647</td>
<td>7.123</td>
</tr>
<tr>
<td>Case VII</td>
<td></td>
<td>0.647</td>
<td>6.979</td>
</tr>
<tr>
<td>F.M. Stiffness Matrix</td>
<td></td>
<td>0.646</td>
<td>7.126</td>
</tr>
<tr>
<td>Consistent Mass Matrix</td>
<td></td>
<td>0.691</td>
<td>7.289</td>
</tr>
<tr>
<td>Lumped Mass</td>
<td></td>
<td>71.352</td>
<td>-0.679</td>
</tr>
<tr>
<td>Case I</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Case II</td>
<td></td>
<td>71.563</td>
<td>-0.679</td>
</tr>
<tr>
<td>Case III</td>
<td></td>
<td>71.573</td>
<td>-0.679</td>
</tr>
<tr>
<td>Case IV</td>
<td>3</td>
<td>71.554</td>
<td>-0.678</td>
</tr>
<tr>
<td>Method of Solution</td>
<td>Member</td>
<td>Joint i</td>
<td>Joint j</td>
</tr>
<tr>
<td>--------------------------</td>
<td>--------</td>
<td>---------</td>
<td>---------</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$Y_1 (T)$</td>
<td>$Z_1 (T)$</td>
</tr>
<tr>
<td>Case V</td>
<td></td>
<td>71.529</td>
<td>-0.678</td>
</tr>
<tr>
<td>Case VI</td>
<td></td>
<td>71.531</td>
<td>-0.677</td>
</tr>
<tr>
<td>Case VII</td>
<td></td>
<td>71.682</td>
<td>-0.678</td>
</tr>
<tr>
<td>F.M. Stiffness Matrix</td>
<td></td>
<td>71.527</td>
<td>-0.678</td>
</tr>
<tr>
<td>Consistent Mass Matrix</td>
<td></td>
<td>71.382</td>
<td>-0.837</td>
</tr>
<tr>
<td>Lumped Mass</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Case I</td>
<td></td>
<td>1.327</td>
<td>38.364</td>
</tr>
<tr>
<td>Case II</td>
<td></td>
<td>1.327</td>
<td>38.532</td>
</tr>
<tr>
<td>Case III</td>
<td></td>
<td>1.327</td>
<td>38.547</td>
</tr>
<tr>
<td>Case IV</td>
<td></td>
<td>1.325</td>
<td>38.579</td>
</tr>
<tr>
<td>Case V</td>
<td></td>
<td>1.324</td>
<td>38.562</td>
</tr>
<tr>
<td>Case VI</td>
<td></td>
<td>1.324</td>
<td>38.566</td>
</tr>
<tr>
<td>Case VII</td>
<td></td>
<td>1.326</td>
<td>38.574</td>
</tr>
<tr>
<td>F.M. Stiffness Matrix</td>
<td></td>
<td>1.324</td>
<td>38.580</td>
</tr>
<tr>
<td>Consistent Mass Matrix</td>
<td></td>
<td>1.301</td>
<td>38.498</td>
</tr>
<tr>
<td>Method of Solution</td>
<td>Member</td>
<td>Joint i</td>
<td>Joint j</td>
</tr>
<tr>
<td>-----------------------</td>
<td>--------</td>
<td>---------</td>
<td>---------</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$Y_1(T)$</td>
<td>$Z_1(T)$</td>
</tr>
<tr>
<td>Lumped Mass</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Case I</td>
<td>21.663</td>
<td>1.327</td>
<td>5.707</td>
</tr>
<tr>
<td>Case II</td>
<td>21.739</td>
<td>1.327</td>
<td>5.721</td>
</tr>
<tr>
<td>Case III</td>
<td>21.754</td>
<td>1.327</td>
<td>5.719</td>
</tr>
<tr>
<td>Case IV</td>
<td>21.770</td>
<td>1.325</td>
<td>5.717</td>
</tr>
<tr>
<td>Case V</td>
<td>21.758</td>
<td>1.324</td>
<td>5.713</td>
</tr>
<tr>
<td>Case VI</td>
<td>21.760</td>
<td>1.324</td>
<td>5.713</td>
</tr>
<tr>
<td>F.M. Stiffness Matrix</td>
<td>21.760</td>
<td>1.324</td>
<td>5.706</td>
</tr>
<tr>
<td>Consistent Mass Matrix</td>
<td>21.662</td>
<td>1.188</td>
<td>5.307</td>
</tr>
<tr>
<td>Method of Solution</td>
<td>$\omega_{n1}$</td>
<td>$\omega_{n2}$</td>
<td>$\omega_{n3}$</td>
</tr>
<tr>
<td>-------------------------</td>
<td>---------------</td>
<td>---------------</td>
<td>---------------</td>
</tr>
<tr>
<td>Lumped Mass Case I</td>
<td>91.622</td>
<td>1110</td>
<td>1436.27</td>
</tr>
<tr>
<td>Case II</td>
<td>93.001</td>
<td>748.91</td>
<td>1073.98</td>
</tr>
<tr>
<td>Case III</td>
<td>93.055</td>
<td>747.46</td>
<td>1191.22</td>
</tr>
<tr>
<td>Case IV</td>
<td>93.095</td>
<td>746.62</td>
<td>1204.94</td>
</tr>
<tr>
<td>Case V</td>
<td>93.110</td>
<td>745.91</td>
<td>1207.57</td>
</tr>
<tr>
<td>Case VI</td>
<td>93.115</td>
<td>745.49</td>
<td>1208.44</td>
</tr>
<tr>
<td>F.M. Stiffness Matrix</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Consistent Mass Matrix</td>
<td>93.11</td>
<td>865.44</td>
<td>1459.4</td>
</tr>
</tbody>
</table>
In Table Nos. 6 and 7,

Case I —— Lumped masses at the ends of the member only.
Case II —— One lumped mass on each member.
Case III —— Two lumped masses on equal magnitude on each member.
Case IV —— Three lumped masses of equal magnitude on each member.
Case V —— Four lumped masses of equal magnitude on each member.
Case VI —— Five lumped masses of equal magnitude on each member.
Case VII —— Six lumped masses of equal magnitude on each member.

The results of natural frequencies of the two bay single storey plane frame using the lumped mass, the consistent mass matrix and the F.M. stiffness matrix method are presented in the following graphs. The graphs represent the natural frequency variation for each mode shape under the three methods of calculations. The fundamental frequency calculated by any one of the methods is very close to that obtained by the other two. The graphs illustrate that the natural frequency converges to exact value using 5 lumped masses for each mode shape and F.M. stiffness matrix give exact value. The consistent mass matrix method (C.M.M.) gives upper bound to the values.
NATURAL FREQUENCY VS NUMBER OF LUMPED MASSES
FOR 6th MODE
NATURAL FREQUENCY VS NUMBER OF LUMPED MASSES
FOR 4\textsuperscript{th} MODE

F.M.
NATURAL FREQUENCY VS NUMBER OF LUMPED MASSES
FOR 3rd MODE
CHAPTER 7

APPLICATIONS

7.1. The numerical examples in Chapter 6 give an account of the validity of the F.M. stiffness matrix method in analysing any structural dynamic response problem accurately. The method can now be applied in analysing turbine foundations.

The example of an existing turbine foundation as given by Klein (10) is chosen to be solved by F.M. Stiffness matrix method. As shown in Fig. 27,(a), the turbine foundation assumes the shape of a space frame like a table with six legs. The loading of the structure is produced by the rotor and the generator resting on it. The base of the structure as shown in Fig. 27,(a), is made out of layers of cork and bituminous material. It is used to separate the frame from transferring any vibrating loads to the adjoining part of the turbine foundation which has only the static loading. This base of the foundation is idealized by assuming it to be made out of springs of equivalent spring constants. The dynamic analysis of the frames shown in Fig. 28, and Fig. 30, is achieved through the F.M. Stiffness Matrix approach and the
**Fig. 27. - Diagram Of A Turbine Foundation.**

**Structural Properties**

**Turbine Rotor**
- Weight = 3400 Kgs.
- Tilt = 0.05 mm.

**Generator**
- Weight = 10300 Kgs.

**Columns**
- Length = 6.4 m
- $I = 2.44 \times 10^{-3} \text{ m}^4$
- $A = 0.156 \text{ m}^2$
- Weight = 0.125 T/m

**Transverse Beams**
1. Length = 4.0 m
2. $I = 4.14 \times 10^{-1} \text{ m}^4$
   - $A = 4.97 \text{ m}^2$
   - Weight = 1.0 T/m

**Longitudinal Beams**
- Length = 5.26 m
- $I = 1.725 \times 10^{-2} \text{ m}^4$
- $A = 0.4655 \text{ m}^2$
- Weight = 0.75 T/m
results are compared to those of lumped mass system taking 5 lumped masses for each structural element. The results of internal stress resultants and natural frequencies of the frames in Figs. 28. and 30. are given by Table Nos. (9, 10) and (12, 13) respectively.

\[ \omega = 50 \text{ rad/sec} \]
\[ k_1 = 5265.0 \text{ T/m} \]
\[ k_2 = 26.55 \frac{\text{T-m}}{\text{rad}} \]

**Fig. 28.**
The magnitude of the lumped masses is:

\[ m_1 = 0.218 \left( \frac{T \cdot \text{sec}^2}{m} \right) \]
\[ m_2 = 0.218 \]
\[ m_3 = 0.218 \]

The structural properties of various members of the structure in Fig. 28 are given in Table No. 8.

**TABLE NO. 8. Structural Properties of Fig. 28**

<table>
<thead>
<tr>
<th>MEMBER</th>
<th>JOINT NO.</th>
<th>( W ) (T/m)</th>
<th>( I ) (m(^4))</th>
<th>( A ) (m(^2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1,2</td>
<td>0.125</td>
<td>2.44 ( \times 10^{-3} )</td>
<td>0.156</td>
</tr>
<tr>
<td>2</td>
<td>2,3</td>
<td>1.0</td>
<td>4.14 ( \times 10^{-1} )</td>
<td>4.97</td>
</tr>
<tr>
<td>3</td>
<td>3,4</td>
<td>0.125</td>
<td>2.44 ( \times 10^{-3} )</td>
<td>.156</td>
</tr>
<tr>
<td>4</td>
<td>3,5</td>
<td>0.75</td>
<td>1.725 ( \times 10^{-2} )</td>
<td>.4655</td>
</tr>
<tr>
<td>5</td>
<td>5,6</td>
<td>0.125</td>
<td>2.44 ( \times 10^{-3} )</td>
<td>.156</td>
</tr>
</tbody>
</table>
TABLE NO.9. Internal Stress Resultants for Fig. 28

<table>
<thead>
<tr>
<th>Method of Solution</th>
<th>Member</th>
<th>Joint i</th>
<th></th>
<th>Joint j</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Y₁ (T)</td>
<td>Z₁ (T)</td>
<td>M₁ (T-m)</td>
<td>Y₂ (T)</td>
</tr>
<tr>
<td>F.M. Stiffness Matrix</td>
<td>AB</td>
<td>37.357</td>
<td>-0.847</td>
<td>0.003</td>
<td>-37.357</td>
</tr>
<tr>
<td>Lumped Mass System</td>
<td>AB</td>
<td>37.293</td>
<td>-0.880</td>
<td>0.003</td>
<td>-37.293</td>
</tr>
<tr>
<td>F.M. Stiffness Matrix</td>
<td>BC</td>
<td>0.955</td>
<td>33.451</td>
<td>5.624</td>
<td>-0.955</td>
</tr>
<tr>
<td>Lumped Mass System</td>
<td>BC</td>
<td>0.953</td>
<td>32.929</td>
<td>5.817</td>
<td>-0.943</td>
</tr>
<tr>
<td>F.M. Stiffness Matrix</td>
<td>CD</td>
<td>47.284</td>
<td>-0.887</td>
<td>-5.172</td>
<td>-47.284</td>
</tr>
<tr>
<td>Lumped Mass System</td>
<td>CD</td>
<td>47.163</td>
<td>-0.889</td>
<td>-5.356</td>
<td>-47.16</td>
</tr>
<tr>
<td>F.M. Stiffness Matrix</td>
<td>CE</td>
<td>1.842</td>
<td>26.994</td>
<td>-28.49</td>
<td>-1.842</td>
</tr>
<tr>
<td>F.M. Stiffness Matrix</td>
<td>EF</td>
<td>47.581</td>
<td>1.842</td>
<td>12.695</td>
<td>-47.581</td>
</tr>
<tr>
<td>Lumped Mass System</td>
<td>EF</td>
<td>47.472</td>
<td>1.837</td>
<td>12.473</td>
<td>-47.471</td>
</tr>
</tbody>
</table>

FIG. 29 - MEMBER AXIS
TABLE NO.10. Natural Frequencies for Fig. 28

<table>
<thead>
<tr>
<th>Method of Solution</th>
<th>$\omega_{n_1}$</th>
<th>$\omega_{n_2}$</th>
<th>$\omega_{n_3}$</th>
<th>$\omega_{n_4}$</th>
<th>$\omega_{n_5}$</th>
<th>$\omega_{n_6}$</th>
<th>$\omega_{n_7}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>F.M. Stiffness Method</td>
<td>332.5</td>
<td>341.5</td>
<td>759</td>
<td>1130</td>
<td>1157</td>
<td>1518</td>
<td>2470</td>
</tr>
</tbody>
</table>

$\omega = 50$ rad/sec

$k_1 = 5260$ T/m

$k = 26.55 \frac{T - m}{\text{rad}}$

$m_2 = m_3 = 0.2176 \frac{T - \text{sec}^2}{m}$
The structural properties of various members of the structure in Fig. 30 given in Table No. 11.

**TABLE NO. 11. Structural Properties of Fig. 30**

<table>
<thead>
<tr>
<th>MEMBER</th>
<th>JOINT NO.</th>
<th>W ($\frac{T}{m}$)</th>
<th>I (m$^4$)</th>
<th>A (m$^2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1,2</td>
<td>0.125</td>
<td>$2.44 \times 10^{-3}$</td>
<td>0.156</td>
</tr>
<tr>
<td>2</td>
<td>2,3</td>
<td>1.0</td>
<td>$1.44 \times 10^{-1}$</td>
<td>4.97</td>
</tr>
<tr>
<td>3</td>
<td>3,4</td>
<td>0.125</td>
<td>$2.44 \times 10^{-3}$</td>
<td>0.156</td>
</tr>
</tbody>
</table>

**TABLE NO. 13. Natural Frequencies for Fig. 30**

<table>
<thead>
<tr>
<th>Method of Solution</th>
<th>$\omega_{n1}$</th>
<th>$\omega_{n2}$</th>
<th>$\omega_{n3}$</th>
<th>$\omega_{n4}$</th>
<th>$\omega_{n5}$</th>
<th>$\omega_{n6}$</th>
<th>$\omega_{n7}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>F.M. Stiffness Matrix</td>
<td>326.5</td>
<td>627</td>
<td>1058</td>
<td>1309</td>
<td>2200</td>
<td>3340</td>
<td>3450</td>
</tr>
</tbody>
</table>
TABLE NO. 12. Internal Stress Resultants for Fig. 30

<table>
<thead>
<tr>
<th>Method of Solution</th>
<th>Member</th>
<th>Joint i</th>
<th>Joint j</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$Y_1(T)$</td>
<td>$Z_1(T)$</td>
</tr>
<tr>
<td>F.M. Stiffness Matrix</td>
<td>AB</td>
<td>25.077</td>
<td>-0.019</td>
</tr>
<tr>
<td>Lumped Mass System</td>
<td>AB</td>
<td>25.218</td>
<td>-0.018</td>
</tr>
<tr>
<td>F.M. Stiffness Matrix</td>
<td>BC</td>
<td>0.018</td>
<td>22.455</td>
</tr>
<tr>
<td>Lumped Mass System</td>
<td>BC</td>
<td>0.018</td>
<td>22.115</td>
</tr>
<tr>
<td>F.M. Stiffness Matrix</td>
<td>CD</td>
<td>25.077</td>
<td>0.018</td>
</tr>
<tr>
<td>Lumped Mass System</td>
<td>CD</td>
<td>25.238</td>
<td>0.018</td>
</tr>
</tbody>
</table>

FIG. 31.- MEMBER AXES
CONCLUSIONS

The plane frame problems and the existing turbine foundations analysed in Chapters 6 and 7, using dynamic methods of analysis, stress the validity of following conclusions.

(1) The results of natural frequencies and internal stress resultants obtained using F.M. stiffness matrix method are quite close to the exact values for frame structures under vibrating loads.

(2) The two bay single storey plane frame solved in Chapter 6 proves that the natural frequencies for each mode converges to exact value for five lumped masses replacing each structural element. Same results are obtained using F.M. stiffness matrix method.

(3) F.M. stiffness matrix method is the most efficient method for calculating the internal stress resultants for any structural dynamic response problem when compared with the lumped mass and the consistent mass matrix method.

(4) The space frame structures can also be analysed using the F.M. stiffness matrix approach since the space frame member F.M. stiffness matrix has been developed in this thesis. Above conclusions will hold good for space frame structures too.
BIBLIOGRAPHY


APPENDIX

MASS MATRICES

ELEMENT

1. Pin Jointed Member

\[ M = -\mathbf{S} \mathbf{h} \]

2. Plane Frame Member

\[ \mathbf{u} = 0 \]
\[ v = a_1 + a_2 y \]
\[ w = 0 \]

DISPLACEMENT FUNCTIONS

\[ u = 0 \]
\[ v = a_1 + a_2 y \]
\[ w = 0 \]

FIG. 32

\[ [M] = \frac{\rho L}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \]

FIG. 33

For Mass Matrix of a plane frame member refer to page 71
3. SPACE FRAME MEMBER

\[ u = a_1 + a_2 y + a_3 y^2 + a_4 y^3 + a_5 z + a_6 yz \]
\[ v = a_7 + a_8 y - (a_2 + 2a_3 y + 3a_4 y^2)x \]
\[ w = a_9 + a_{10} y + a_{11} y^2 + a_{12} y^3 + a_5 x + a_6 yx \]

\[ [M] = \text{Mass Matrix}^* \]

* For Mass Matrix of a Space Frame Member refer to Pages 76 and 77.

4. TRIANGULAR PLATE WITH TRANSLATIONAL DISPLACEMENTS

\[ u = a_1 + a_2 + a_3 y \]
\[ [M] = \frac{9t}{12} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \]

in which \( t \) = thickness of the plate.
5. RECTANGLE PLATE WITH TRANSLATIONAL DISPLACEMENTS

\[
\mathbf{U} = a + a_x x + a_y y + a_{xy} xy
\]

\[
\mathbf{[M]} = \frac{\rho t}{36} \begin{bmatrix} 4 & 2 & 1 & 2 \\ 2 & 4 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 1 & 2 & 4 \end{bmatrix}
\]
in which \( \rho \) = mass per unit thickness (t) of the plate.

6. SOLID TETRAHEDRON

\[
\mathbf{U} = a_1 + a_2 x + a_3 y + a_4 z
\]

\[
\mathbf{[M]} = \frac{\rho t}{20} \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \end{bmatrix}
\]
in which \( \rho \) = mass per unit thickness (t) of the plate.
7. SOLID PARALLELOPIPED

\[ u = a_1 + a_2 x + a_3 y + a_4 z + a_5 xy + a_6 xz + a_7 yz + a_8 xyz \]

**DISPLACEMENT FUNCTION**

**MASS MATRIX**

\[
[M] = \frac{\mathbf{f} \mathbf{t}}{216}
\]

\[
\begin{bmatrix}
8 & 4 & 2 & 4 & 4 & 2 & 1 & 2 \\
4 & 8 & 4 & 2 & 2 & 4 & 2 & 1 \\
2 & 4 & 8 & 4 & 1 & 2 & 4 & 2 \\
4 & 2 & 4 & 8 & 2 & 1 & 2 & 4 \\
4 & 2 & 1 & 2 & 8 & 4 & 2 & 4 \\
2 & 4 & 2 & 1 & 4 & 8 & 4 & 2 \\
1 & 2 & 4 & 2 & 2 & 4 & 8 & 4 \\
2 & 1 & 2 & 4 & 4 & 2 & 4 & 8
\end{bmatrix}
\]