A STIFFNESS MATRIX

FOR

TWIST BEND BUCKLING

OF

NARROW RECTANGULAR SECTIONS

BY

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ABSTRACT

The stiffness properties of a short narrow rectangular beam as modified by a primary bending moment and shear stress distribution in the major plane are presented. The beam is a segment taken from a longer member the "structure." A distribution of bending stress is assumed over the beam segment length and its effect on the stiffness properties in lateral bending and torsion obtained.

The stiffness matrix is used to obtain the critical value of load for a number of well known examples of narrow rectangular beams and the results are shown to be in good agreement.

The results of an energy solution, which produces a symmetrical matrix, are presented. Comparison with classical examples shows accurate results with the added benefit that the symmetrical matrix lends itself much more readily to more complicated problems.

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NOTATION

E	=	modulus of elasticity
G	=	shear modulus of elasticity
I	=	moment of inertia
J	=	torsion constant
L	=	length of "structure"
l	=	length of beam segment
a,b,c	=	centrodial, major and minor axis of beam element
x,y,z	=	fixed co-ordinate system
e	=	eccentricity of load applied to "structure"
g	=	eccentricity of restraint applied to "structure"
β	=	torsional deflection in z direction
μ	=	lateral deflection in x direction
Mx	=	moment about positive x axis
M_y	=	moment about positive y axis
M_z	=	moment about positive z axis
F_x	=	force along positive $x$axis
M_x	=	internal moment about axis b
M	=	internal moment about axis c
T	=	internal moment about axis a
V	=	internal shear along c
Q	=	internal shear along b
$\overline{M}$	=	average value of M_x in segment
P	=	vertical load on structure
M_F	=	fixed end moments

NOTATION Cont'd.

H	=	axial load in member
δ	=	deflection vector for beam segment
f	=	force vector for beam segment
$\bar{\delta}$	=	deflection vector for restrained beam segment
$\bar{f}$	=	force vector for restrained beam segment
$\bar{k}$	=	restrained 6 x 6 stiffness matrix
$\bar{\bar{k}}$	=	restrained 4 x 4 stiffness matrix
$K, K1$	=	6 x 6 stiffness matrices for beam segment
T_1, T	=	transformation matrices
σ_{cr}	=	fibre stress at critical load
γ	=	critical load coefficient
U	=	strain energy
q	=	lateral load
ϕ	=	stability function co-efficient
λ	=	stability function co-efficient

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CHAPTER I

INTRODUCTION

The effect of primary bending on the stability of narrow rectangular beams has been extensively investigated by, Timoshenko and Gere (1), Goodier (2), Bleich (3), Vlasov (4) and others. In Hartmann (5) and Bell (6) methods of determining critical loads for plane structures are presented.

A large amount of work has also been done on the effect of axial load on the stability of members and structures, in Gere and Weaver (7) and in McMinn (8) methods of modifying member stiffness matrices to show the effect of axial load on lateral stiffness are shown. This method readily adapts itself through the use of electronic computers to the prediction of critical load by evaluation of that load which produces a zero determinate for the structure stiffness matrix.

In this thesis a method of determining the effect of a primary bending stress distribution on the stiffness matrix of a narrow rectangular beam is presented. The well known differential equations governing the flexural-torsional behavior are determined and solved subject to the various boundary conditions as dictated by the definition of the stiffness matrix. The assumption that the beam segment is relatively short compared with the "structure" dimensions is made to simplify the solution of the equations.

Comparison of critical loads for structures determined by using the derived matrix agree very well with classical solutions available.

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* Numbers in brackets refer to the Bibliography

An alternate procedure using the energy method for determining a matrix is presented. The resulting symmetrical matrix which lends itself readily to more complex problems shows results which are consistant with available classical solutions.

CHAPTER II

BASIC EQUATIONS

2.1 General Remarks

In order to study the behavior of a beam segment under the influence of primary bending and associated shears it is necessary to derive the applicable differential equations linking bending and torsion.

For the purpose of this derivation an infinitesimal element of the beam segment of length dz as shown in Fig. 2.1 is considered. The element is shown in its general deformed position as defined by the deflections μ and β relative to the axes x,y,z which are fixed in space. Since the element is part of a beam of narrow rectangular cross section, deflections in the y-z plane are considered small and are neglected. Fig. 2.1 also shows the co-ordinate axes a,b,c of the element which are coincident with the centroidal, major and minor axes of the member and are in a translated rotated position to the axes x,y,z.

The internal forces acting on the element cross section are of two types:

(1) Primary forces M_x and V of large magnitude due to primary bending in the y-z plane.

(2) Secondary forces T,Q,M due to bending in the x-z plane and torsion, as these forces form resultant stresses parallel and perpendicular to the cross section, they are shown in the a,b,c axes system.

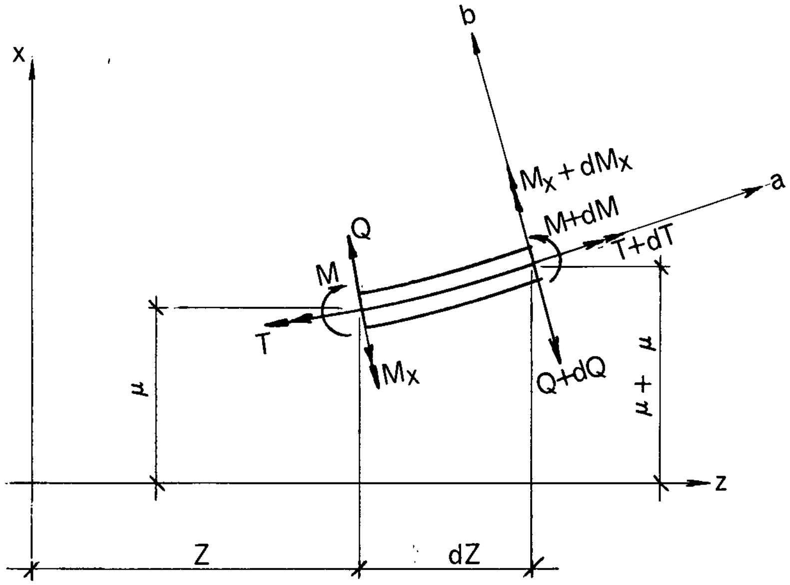
2.2 Derivation

The following four equations arise by demanding the element be in equilibrium in its deformed shape as shown in Fig. 2.1.a.

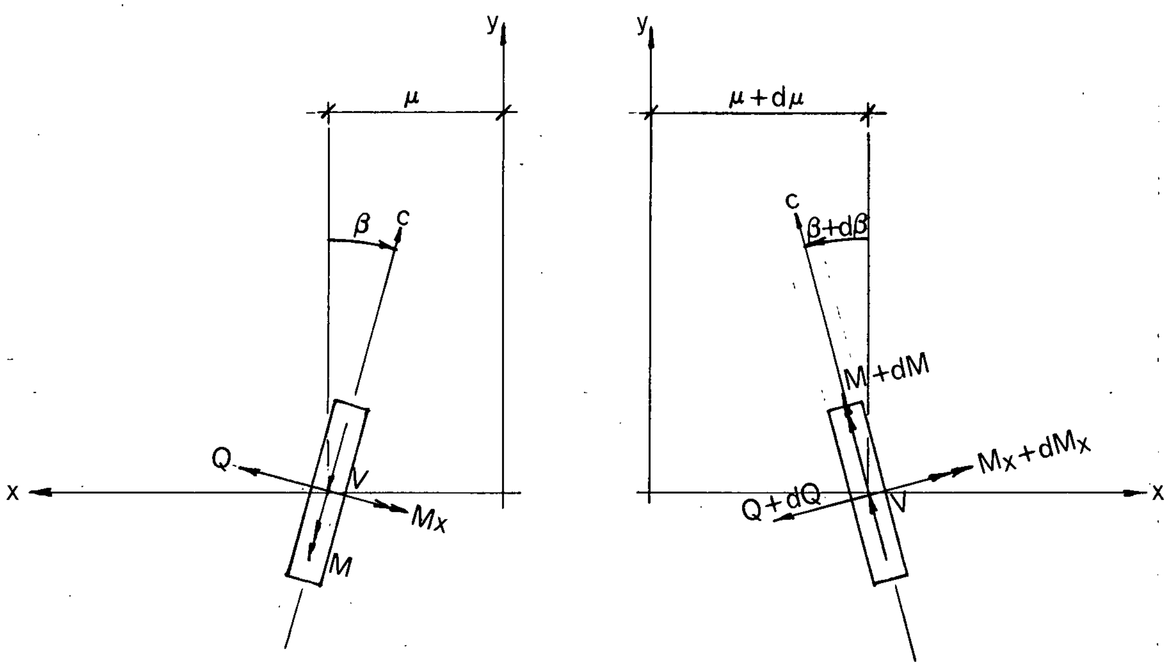
$$\Sigma\ Mx = 0 \quad dM_x - Vdz = 0$$

on rearrangement

$$M_x{}' - V = 0 \tag{1}$$



(a) PLAN VIEW



(b) ALONG +Z AXIS (c) ALONG −Z AXIS

Fig. 2.1 ELEMENTAL LENGTH OF BEAM SEGMENT

$$\Sigma F_x = 0 \quad - dQ - Vd\beta = 0$$

on rearrangement

$$Q' + V\beta' = 0 \tag{2}$$

$$\Sigma M_y = 0 \quad dM + M_x d\beta + dM_x d\beta - Qdz = 0$$

neglecting terms of higher order and

rearranging

$$M' + M_x\beta' - Q = 0 \tag{3}$$

$$\Sigma M_z = 0 \quad dT - M_x\mu'' dz - dM_x\mu'' dz = 0$$

neglecting terms of higher order and

rearranging $$T' - M_x\mu'' = 0 \tag{4}$$

From Hooke's Law of elastic behavior the following well known relationships are obtained.

$$EI\mu'' = M \tag{5}$$

$$GJ\beta' = T \tag{6}$$

Relationship (1) is a statical identity relating M_x to V. The remaining, (2) to (6) form 5 equations in the 5 unknowns M, Z, T, μ and β.

By differentiating equation (5) once and substituting in equation (3) the relationship

$$EI\mu''' + M_x\beta' - Q = 0 \tag{7}$$

is obtained.

By differentiating equation (7) once and substituting in equation (2) the relationship

$$EI\mu'''' + M_x\beta'' + 2V\beta' = 0 \tag{8}$$

results.

Similarly the relationship

$$GJ\beta'' - M_x\mu'' = 0 \tag{9}$$

is obtained.

Solution of (8) and (9) for μ and β in closed form is not possible in general since M_x, V and EI are functions of z. The beam could be considered split into many small parts wherein M_x, V and EI might be considered as constant. However simultaneous solution of the equations would be inconvenient and for this reason a stiffness procedure as outlined in Chapter III is preferred.

CHAPTER III

METHOD OF SOLUTION

3.1 General Remarks

The end of a beam has in general six degrees of freedom. In the previous derivation deflections in the y-z plane were considered small and neglected and the forces M_x and V were assumed to be defined. This then eliminates two degrees of freedom which when combined with the assumption that axial forces and deflections are small leaves three at each end of the beam. They are μ, μ^1 and β associated with the forces T, M, and Q.

A 6 x 6 stiffness matrix K will then govern the behavior of the beam segment Fig. 3.1 such that

$$K \; \delta = f \tag{10}$$

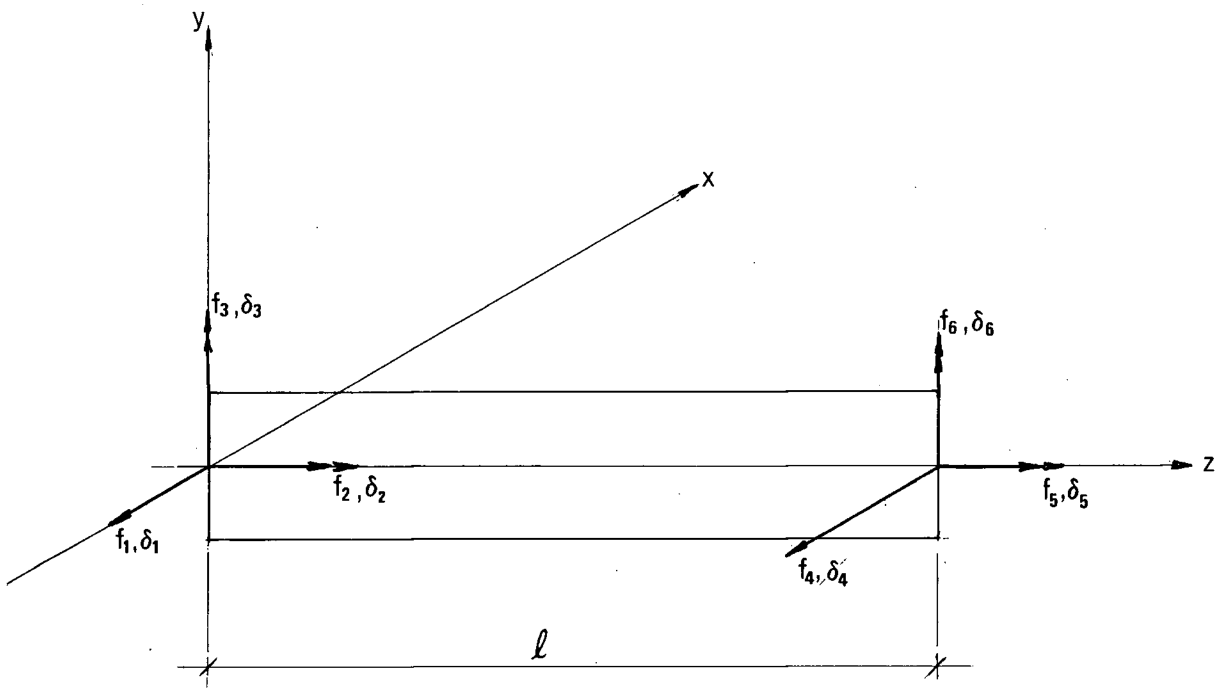


Fig. 3.1 DEFINED FORCES AND DEFLECTIONS

Where f and δ are 6 - component vectors which represent the forces and deflections of Fig. 3.1. The first column of K represents f when $\delta_1 = 1$ and all other displacements are zero. Similarly the second column represents the forces f when $\delta_2 = 1$ and all others are zero.

The deflections δ will be

$$\begin{aligned}
\delta_1 &= \mu && \text{at } Z = 0\\
\delta_2 &= \beta && \text{at } Z = 0\\
\delta_3 &= \mu' && \text{at } Z = 0\\
\delta_4 &= \mu && \text{at } Z = l\\
\delta_5 &= \beta && \text{at } Z = l\\
\delta_6 &= \mu' && \text{at } Z = l
\end{aligned}$$

The forces f will be

$$\begin{aligned}
f_1 &= -EI\mu''' - M_x\beta' - V\beta && \text{at } Z = 0\\
f_2 &= -GJ\beta' + M_x\mu' && \text{at } Z = 0\\
f_3 &= -EI\mu'' - M_x\beta && \text{at } Z = 0\\
f_4 &= EI\mu''' + M_x\beta' - V\beta && \text{at } Z = l\\
f_5 &= GJ\beta' - M_x\mu' && \text{at } Z = l\\
f_6 &= EI\mu'' + M_x\beta && \text{at } Z = l
\end{aligned}$$

These forces are merely the components of the stress resultants T, M, Q, M_x and V in the x,y,z directions. There is a force in f_1 and f_4 that is not obvious namely $M_x\beta'$ which arises because M_x acts on a twisted cross section.

In order to find any column of K it appears necessary to solve (8) and (9) subject to the boundary conditions on δ for that column. A method of successive approximations is chosen to satisfy (8) and (9). First they are rewritten as:

$$EI\mu'''' = -M_x\beta'' - 2V\beta' \tag{11}$$

and

$$GJ\beta'' = M_x\mu'' \tag{12}$$

If the beam segment is small enough such that the average M_x is much less than the critical moment for the length l, a suitable initial guess is the linear deflection curve for the column of K being considered. This guess is then substituted into the right hand side of (11) or (12) and the deflection curve μ or β is found. One such cycle is sufficient if the segment is short enough.

Before illustrating the method in detail it is necessary to define M_x as

$$M_x = (\overline{M} + \frac{Vl}{2}) - V(l - z) \tag{13}$$

where $\overline{M}$ represents the average moment over the length l, or the center line moment.

3.2 Derivation of Stiffness Matrix

3.2.1 Column 1

By the definition of K, $\delta_1 = 1$ and $\delta_2 \ldots \delta_6 = 0$ the boundary conditions on (11) and (12) are therefore:

$$\begin{aligned} \mu &= -1 && \text{at } Z = 0 \\ \beta &= 0 && \text{at } Z = 0 \\ \mu' &= 0 && \text{at } Z = 0 \\ \mu &= 0 && \text{at } Z = l \\ \beta &= 0 && \text{at } Z = l \\ \mu' &= 0 && \text{at } Z = l \end{aligned}$$

The solution by successive approximations will take the form:

$$\mu = \mu_1 + \mu_2 + \ldots + \mu_n$$

$$\beta = \beta_1 + \beta_2 + \ldots + \beta_n$$

Assume that μ_1 is the linear deflection shape for column 1 of K and is given by

$$\mu_1 = -1 + \frac{3z^2}{l^2} - \frac{2z^3}{l^3} \tag{14}$$

This assumption will satisfy equation (11) if the right hand side is zero and therefore if further accuracy is required μ_2 would be established to partly satisfy equation (11). Substituting μ_1'' into (12), where

$$\mu_1'' = \frac{6}{l^2} - \frac{12z}{l^3} \tag{15}$$

gives

$$GJ\beta_1'' = [(\bar{M} + \frac{Vl}{2}) - V(l - z)]\,[\frac{6}{l^2} - \frac{12z}{l^3}] \tag{16}$$

Integrating (16) twice and introducing the boundary conditions above gives

$$GJ\beta_1 = \bar{M}\,[\frac{-z}{l} + \frac{3z^2}{l^2} - \frac{2z^3}{l^3}] - \frac{Vl}{2}\,[\frac{3z^2}{l^2} - \frac{2z^3}{l^3}] + V\,[\frac{z}{2} + \frac{z^3}{l^2} - \frac{z^4}{l^3}] \tag{17}$$

Equations (14) and (17) for μ_1 and β_1 satisfy equation (12) exactly, however since μ_1 was assumed to be the linear deflection curve equation (11) is not fully satisfied. The normal procedure in this method of solution would be to substitute values of β_1' and β_1'' into the right hand side of (11) and solve again for μ, establishing μ_2. On the other hand the established values of μ_1 and β_1 may define the deflected shape of the beam segement to a sufficient degree of accuracy. To establish a measure of the accuracy of μ_1 and β_1 a comparison is made of the values of end moments produced; firstly by the requirements of the boundary conditions of the stiffness matrix acting on equation (11) with the right hand side zero, and secondly those produced by a lateral load of magnitude represented by the right hand side of (11) with $\beta = \beta_1$.

In the first case the value of end moments will be of the order represented by

$$M_F = \frac{6EI}{l^2} \tag{18}$$

In the second case the value of the lateral load must be first established. To simplify this, values of β_1', β_1'' and M_x will be assumed as constant and of value representing their order of magnitude only. These assumed values are then

$$M_x = \overline{M} + \frac{Vl}{2} \tag{19}$$

$$\beta_1' = \frac{\overline{M}}{GJl} + \frac{V}{GJ} \tag{20}$$

$$\beta_1'' = \frac{\overline{M}}{GJl2} + \frac{V}{GJl} \tag{21}$$

These values of M_x, β_1' and β_1'' will then define the lateral load on the beam segment as

$$q = - [\overline{M} + \frac{Vl}{2}] [\frac{\overline{M}}{GJl2} + \frac{V}{GJl}] - 2V [\frac{\overline{M}}{GJl} + \frac{V}{GJ}] \tag{22}$$

End moments corresponding to this load would be of the order

$$M_F = q \frac{l^2}{12} \tag{23}$$

Substituting q from equation (22) the end moments become

$$M_F = \frac{-\overline{M}^2}{12GJ} - \frac{7\overline{M}Vl}{24GJ} - \frac{5V^2l^2}{24GJ} \tag{24}$$

The values of M_F in equations (18) and (24) cannot be compared directly since equation (24) contains terms with $\overline{M}$ and V. These terms may be reduced to a more convenient form by considering the intended use of the matrix. It is intended to predict critical loads for structures comprised of several beam segments. The maximum values of $\overline{M}$ and V that will occur in a segment of the structure is then the moment and shear corresponding to the critical structure loading. For the purposes of this accuracy check, consider a simply supported beam with a point load at the center line. The well known value of critical load is

$$P_{cr} = \frac{16\sqrt{EIGJ}}{L^2} \tag{25}$$

Where L is the length of beam or "structure." By letting L take the value

$$L = a\,l \tag{26}$$

$$P_{cr} = \frac{16\ EIGJ}{a^2\ l^2} \tag{27}$$

The maximum values of moment and shear are then

$$\overline{M} = \frac{4\sqrt{EIGJ}}{a\ l} \tag{28}$$

$$V = \frac{8\sqrt{EIGJ}}{a^2\ l^2} \tag{29}$$

Substituting these values into equation (24) the end moment becomes:

$$M_F = -\frac{4\ EI}{3a^2 l^2} - \frac{28\ EI}{3a^3 l^2} - \frac{40\ EI}{3a^4 l^2} \tag{30}$$

Considering then equation (18) and (30) and eliminating the common factor EI/l^2 the requirement of further approximations to the deflection curve namely $\mu_2 \ldots \mu_n$ depends on the magnitude of equation (30) as compared to the value 6. However since several approximations were involved in obtaining equation (30) a more logical approach would be to consider the value of $1/a^2$ as compared to unity. Obviously a choice of 10 elements would lead to relative magnitudes of 1:.01. The determination of the ratio that will lead to satisfactory results is best determined by numerical trials since an increasing number of segments will likely lead to better accuracy. Numerical trials presented in Chapter V confirm that for certain situations 10 segments is very satisfactory.

With the assumption that further approximations are unnecessary and that equation (14) and (17) adequately define the deflected shape of the member the forces f_1 to f_6 are evaluated by substitution of M_x, β_1 and μ_1 into the equations of Part 3.1. The forces are as follows:

$$f_1 = \frac{12EI}{l^3} + \frac{\overline{M}^2}{GJl} - \frac{\overline{M}V}{GJ} + \frac{V^2 l}{4GJ}$$

In the above expression the last three terms as compared to the first term are small with increasing number of segments by the same reasons as presented previously. For this reason they will be neglected and the force f_1 taken as

$$f_1 = \frac{12EI}{l^3}$$

Similarly the remaining forces are:

$$f_2 = \frac{\overline{M}}{l} - \frac{V}{2}$$

$$f_3 = - \frac{6EI}{l^2}$$

$$f_4 = - \frac{12EI}{l^3}$$

$$f_5 = - \frac{\overline{M}}{l} - \frac{V}{2}$$

$$f_6 = - \frac{6EI}{l^2}$$

which constitute the first column of K Fig. 3.2.

3.2.2 Column 2

By the definition of K, $\delta_2 = 1$ with the remaining deflections zero. The boundary conditions on (11) and (12) are therefore:

$$\begin{array}{ll} \mu = 0 & \text{at } Z = 0 \\ \beta = 1 & \text{at } Z = 0 \\ \mu' = 0 & \text{at } Z = 0 \\ \mu = 0 & \text{at } Z = l \\ \beta = 0 & \text{at } Z = l \\ \mu' = 0 & \text{at } Z = l \end{array}$$

The solution by successive approximations will take the form $\beta = \beta_1 + \beta_2 + \ldots + \beta_n$ and $\mu = \mu_1 + \mu_2 + \ldots + \mu_n$

Assume that β_1 is the linear deflection curve for Column 2 of K, therefore:

$$\beta_1 = 1 - \frac{z}{l} \tag{30}$$

This assumption will satisfy equation (12) if the right hand side is zero. Proceeding as before by differentiating equation (30) and substituting into equation (11)

$$EI\mu_1'''' = \frac{2V}{l} \tag{31}$$

Integrating four times and introducing the boundary conditions above gives

$$EI\mu_1 = \frac{Vz^4}{12l} - \frac{Vz^3}{6} + \frac{Vlz^2}{12} \tag{32}$$

The accuracy of μ_1 and β_1 may be established as in Case 1 and therefore further approximations $\mu_2 \ldots \mu_n$ and $\beta_2 \ldots \beta_n$ are neglected.

With equations (30) and (32) the deflected shape of the member is adequately defined and the forces f_1 to f_6 may be evaluated. Proceeding as in Case 1 the forces are:

$$f_1 = \frac{\overline{M}}{l} - \frac{V}{2}$$

$$f_2 = \frac{GJ}{l}$$

$$f_3 = -\overline{M} + \frac{Vl}{3}$$

$$f_4 = -\frac{\overline{M}}{l} + \frac{V}{2}$$

$$f_5 = -\frac{GJ}{l}$$

$$f_6 = \frac{Vl}{6}$$

where additions to f_2 and f_5 are neglected. These values constitute the second column of K Fig. 3.2.

3.2.3 Column 3

By the definition of K, $\delta_3 = 1$ with the other deflections equal to zero. The boundary conditions on (11) and (12) are therefore:

$$\mu = 0 \qquad \text{at } Z = 0$$

$$\beta = 0 \qquad \text{at } Z = 0$$

$$\mu' = 0 \qquad \text{at } Z = 0$$

$$\mu = 0 \qquad \text{at } Z = l$$

$$\beta = 0 \qquad \text{at } Z = l$$

$$\mu' = 0 \qquad \text{at } Z = l$$

Taking the solution in the form of successive approximations as before the deflected shapes are:

$$\mu_1 = z + \frac{z^3}{l^2} - \frac{2z^2}{l} \tag{33}$$

and

$$GJ\beta_1 = \overline{M}\,[z + \frac{z^3}{l^2} - \frac{2z^2}{l}] - \frac{Vl}{2}\,[\frac{z^3}{l} - \frac{2z^2}{l} + \frac{2z}{3}] + V\,[\frac{z^4}{2l^2} - \frac{2z^3}{3l}] \tag{34}$$

On evaluation of the forces f_1 to f_6 the third column of K Fig. 3.2 is obtained.

Columns 4, 5 and 6 of K may be derived in a similar manner and the complete matrix is shown in Fig. 3.2. Before discussion of the use and accuracy of K an alternate method of derivation for K is presented in the following section.

$$K = \begin{bmatrix} \frac{12EI}{l^3} & \frac{\bar{M}}{l} - \frac{V}{2} & -\frac{6EI}{l^2} & -\frac{12EI}{l^3} & -\frac{\bar{M}}{l} - \frac{V}{2} & -\frac{6EI}{l^2} \\ \frac{\bar{M}}{l} - \frac{V}{2} & \frac{GJ}{l} & -\frac{Vl}{6} & -\frac{M}{l} + \frac{V}{2} & -\frac{GJ}{l} & +\frac{Vl}{6} \\ -\frac{6EI}{l^2} & -\bar{M} + \frac{Vl}{3} & \frac{4EI}{l} & \frac{6EI}{l^2} & \frac{Vl}{6} & \frac{2EI}{l} \\ -\frac{12EI}{l^3} & -\frac{\bar{M}}{l} + \frac{V}{2} & \frac{6EI}{l^2} & \frac{12EI}{l^3} & \frac{M}{l} + \frac{V}{2} & \frac{6EI}{l^2} \\ -\frac{\bar{M}}{l} - \frac{V}{2} & -\frac{GJ}{l} & \frac{Vl}{6} & \frac{\bar{M}}{l} + \frac{V}{2} & \frac{GJ}{l} & -\frac{Vl}{6} \\ -\frac{6EI}{l^2} & \frac{Vl}{6} & \frac{2EI}{l} & \frac{6EI}{l^2} & \bar{M} + \frac{Vl}{3} & \frac{4EI}{l} \end{bmatrix}$$

Fig. 3.2 STIFFNESS MATRIX K

CHAPTER IV

ALTERNATE METHOD OF SOLUTION

4.1 General Remarks

A more direct method of obtaining the stiffness matrix K is to consider a cantilever beam as shown in Fig. 4.1. The end $Z = l$ is considered free for deflections μ and β and deflections in the y - z plane are considered negligible as before.

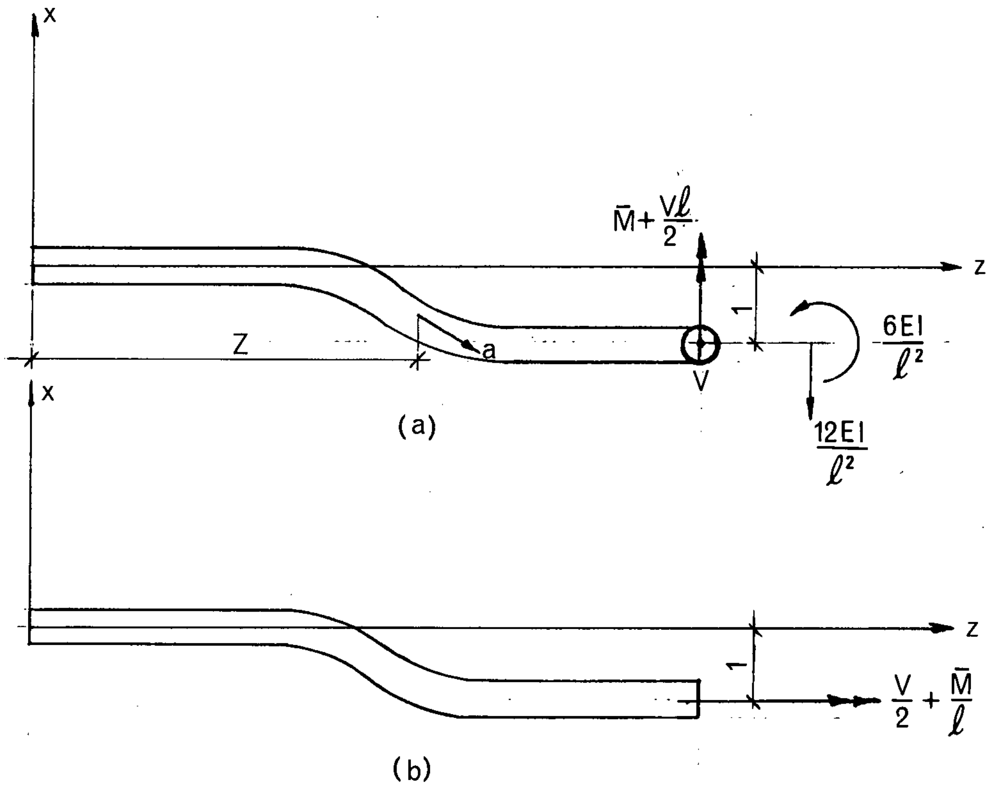


Fig. 4.1 FORCES FOR COLUMN 4 of K

The method of obtaining K is to place the free end $z = l$ into a deflected position as dictated by the boundary conditions for columns 4 to 6 of K. The deflected shape of the beam is assumed to be the linear deflection for the particular column of K being considered. This assumption corresponds to the one in Chapter III where no corrections to the first approximation (the linear deflection)were made. Equations for moments M and T at $Z = Z$ from a free body of the segment from $Z = Z$ to $Z = l$ in its deflected shape are then obtained. The moments T are required in columns 4 and 6 and M in column 5 since the basic shapes for these columns are respectively ones of lateral deflection and torsion. For this distribution of M or T the linear equations

$$EI\mu'' = M \tag{35}$$

and

$$GJ\beta' = T \tag{36}$$

are solved and give deflections μ or β at $Z = l$. End forces are then applied to provide equal and opposite end deflections to satisfy the deflection conditions for the column being considered. Forces 4 - 6 are then these correction forces plus the original linear forces along with those forces due to realignment of the primary forces at $Z = l$ due to deflections. The forces 1 - 3 are obtained from equilibrium of the element.

4.2 Derivation of Stiffness Matrix

4.2.1 Column 4

Assume a linear deflected shape as shown in Fig. 4.1 (a) such that

$$\mu = - \frac{3z^2}{l^2} + \frac{2z^3}{l^3} \tag{37}$$

and

$$\mu' = - \frac{6z}{l^2} + \frac{6z^2}{l^3} \tag{38}$$

The usual linear forces as shown are required to maintain the end $Z = l$ in this position.

Considering the free body from $Z = Z$ to $Z = l$ the equation for T at Z is

$$T = V\ (1 - \mu) + [(\overline{M} + \frac{Vl}{2}) - V\ (l - z)]\ \mu' \tag{39}$$

Substituting μ and μ' from equations (37) and (38) and introducing equation (36)

$$GJ\beta' = V[1 + \frac{3z}{l} - \frac{6z^2}{l^2} + \frac{4z^3}{l^3}] + \overline{M}[\frac{-6z}{l^2} + \frac{6z^2}{l^3}] \tag{40}$$

Integrating once and substituting the boundary condition, $\beta = 0$ at $Z = 0$ the deflection at $Z = l$ is:

$$\beta(l) = \frac{Vl}{2GJ} - \frac{\overline{M}}{GJ} \tag{41}$$

In order that the deflection $\beta(l)$ be zero the end torque

$$M_z = \frac{V}{2} + \frac{\overline{M}}{l} \tag{42}$$

as shown in Fig. 4.1.(b) is applied. The forces f_4 to f_6 are then the sum of the forces shown in Fig. 4(a) and (b) along the respective directions. The forces f_1 to f_3 are obtained from equilibrium of the beam. These forces then represent column 4 of K Fig. 3.2.

4.2.2 Column 5

Assume a linear deflected shape as shown in Fig. 4.2(a) such that

$$\beta = \frac{z}{l} \tag{43}$$

and

$$\beta' = \frac{1}{l} \tag{44}$$

The usual linear force as shown is required to maintain the end $Z = l$ in this position.

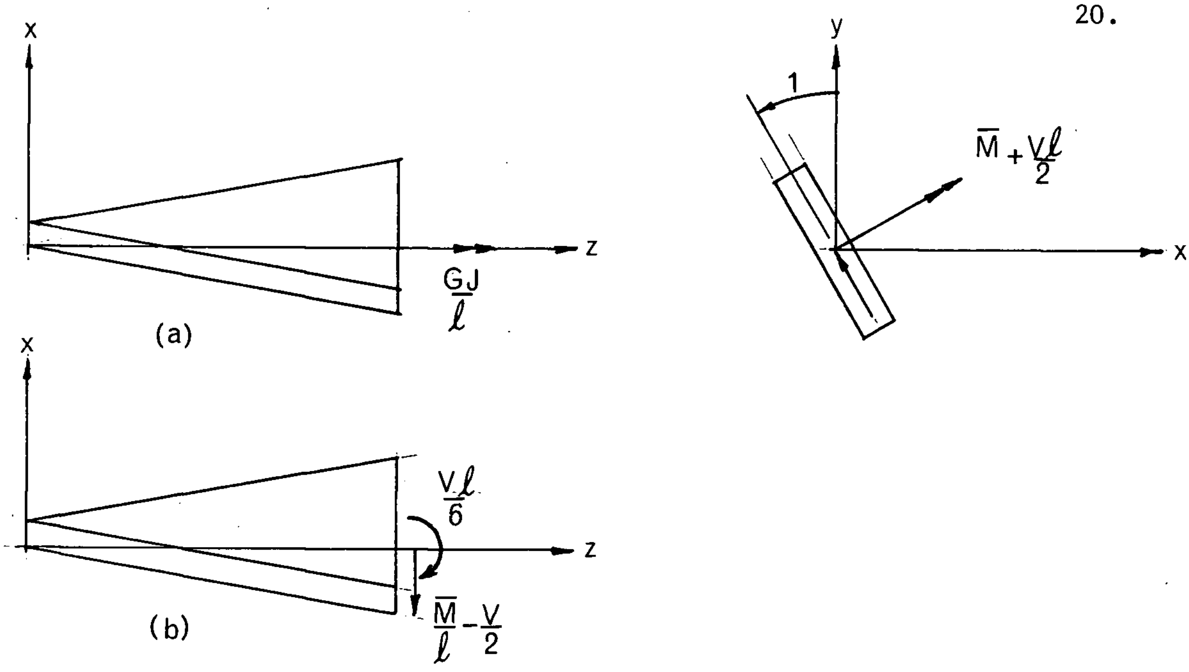


Fig. 4.2 FORCES FOR COLUMN 5 OF K

Considering the free body from $Z = Z$ to $Z = l$ and employing equation (43), (44) and (35)

$$EI\mu'' = \overline{M}\left(1 - \frac{z}{l}\right) \tag{45}$$

Integrating twice and introducing the coundary conditions, $\mu = \mu' = 0$ at $Z = 0$, the deflections at $Z = l$ are:

$$\mu'(l) = \frac{\overline{M}l}{2EI} - \frac{Vl^2}{12} \tag{46}$$

and

$$\mu(l) = \frac{\overline{M}l^2}{3EI} - \frac{Vl^3}{12} \tag{47}$$

In order that the deflections $\mu'(l)$ and $\mu(l)$ be zero the end forces

$$F_x = \frac{\overline{M}}{l} + \frac{V}{2} \tag{48}$$

and

$$M_y = \frac{Vl}{6} \tag{49}$$

as shown in Fig. 4.2.(b) are applied. The forces f_4 to f_6 are then the sum of the forces shown in Figs. 4.2.(a) and (b) along the respective directions noting the re-alignment of the primary bending moment and shear force due to the deflection β. Forces f_1 to f_3 are obtained from equilibrium. These forces then represent column 5 of K Fig. 3.2.

4.2.3 Column 6

Assume a linear deflected shape as shown in Fig. 4.3.(a) such that

$$\mu = -\frac{z^2}{l} + \frac{z^3}{l^2} \tag{50}$$

and

$$\mu' = -\frac{2z}{l} + \frac{3z^2}{l^2} \tag{51}$$

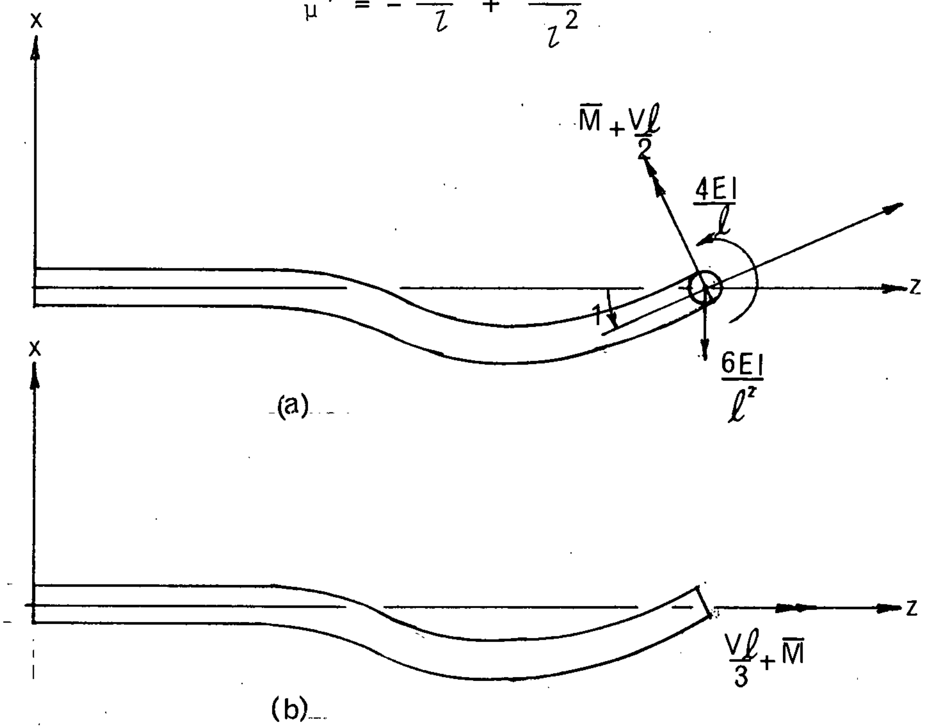


Fig. 4.3 FORCES FOR COLUMN 6 OF K

The usual linear forces as shown are required. Considering the free body $Z = Z$ to $Z = l$, introducing equations (50), (51) and (36), and integrating, the deflection $\beta(l)$ is

$$\beta(l) = -\frac{Vl^2}{3GJ} - \frac{\overline{M}l}{GJ} \tag{52}$$

The correction force required

$$M_z = \frac{Vl}{3} + \overline{M} \tag{53}$$

as shown in Fig. 4.3.(b). The forces f_4 to f_6 and f_1 to f_3 are evaluated in the previous manner. These forces are then those of column 6 as shown in Fig. 3.2.

The methods presented in Chapters III and IV give identical results; the latter however lends itself to a better physical understanding of the problem. In both methods better values of μ and β could be obtained, in the first by making further approximations and in the second by considering progressive modifications to assumed shapes. These are in fact the same procedure, however, as was shown in Chapter III the necessity of further approximations does not appear warranted. Numerical results presented in Chapter VI confirm this assumption.

4.3 Consideration of Eccentric Load

The stiffness matrix as derived in Chapter III and IV show the effect of a primary bending moment and shear force acting at the centroidal axes of the member. This primary force system is assumed to remain constant throughout the buckling deformation. This assumption is valid if the deformations are small and the load is applied at the centroid of the section.

However, if loads are applied above or below as shown in Fig. 4.5 it is evident that buckling deformations create torsional moments about the centroidal axes of the member. In order to determine the effect of placing a load in these positions the structure stiffness matrix must be modified accordingly.

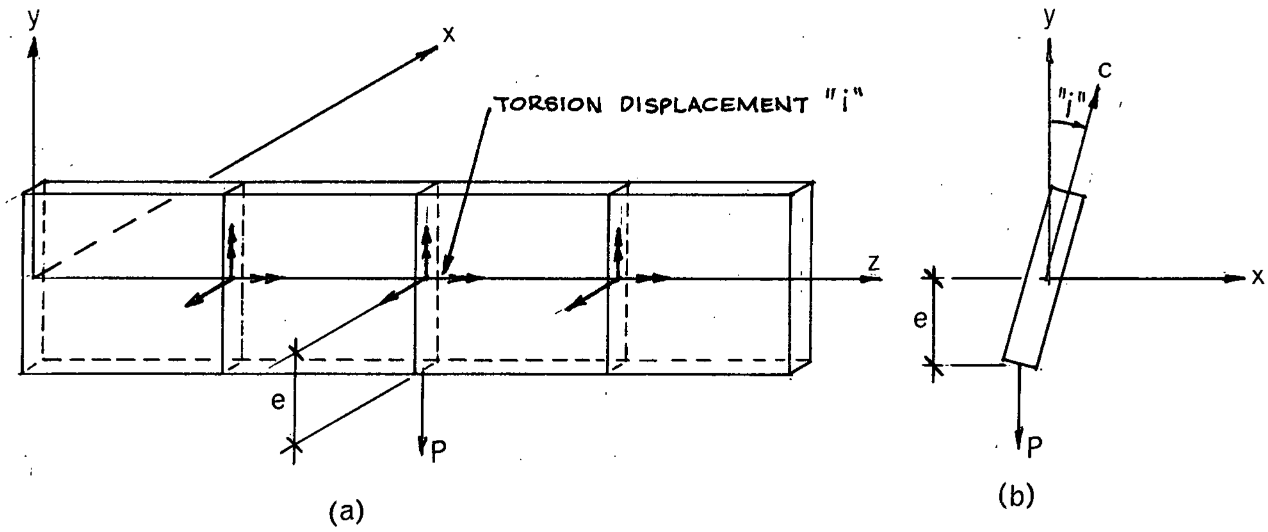


Fig. 4.5 STRUCTURE LOADED AT BOTTOM CHORD

Considering the structure in Fig. 4.5 the load on the bottom edge may be considered as a load at the centroid plus a torsional moment applied along the displacement. This moment is proportional to the magnitude of the load and the value of the displacement. Obviously it is identical in effect on the structure to the effect of a torsional spring of stiffness $P \times e$ and may be treated by introducing a torsional restraint of that value in the diagonal element of the structure stiffness matrix corresponding to the torsional displacement at that location. It is also evident that if the load were above the centroid of the section, introduction of a negative torsional restraint would have the correct effect on the structure.

Numerical trials as presented in Chapter VI establish the accuracy of this procedure.

4.4 Consideration of Restrained Centre of Rotation and Axial Loads

In the performance of tests on lateral buckling of beams the use of a restrained centre of rotation is convenient and naturally many building systems such as girders with decking display this type of restraint. Also the effect of axial loads, must be included in a buckling analysis.

Firstly the effect of axial load may be included by introducing the stability function S_i of Gere and Weaver (7). These functions, shown below, are shown in the stiffness matrix Fig. 4.6 where H is the axial load in the member.

$$S_1 = \lambda^3 \sin \lambda / 12\phi$$

$$S_2 = \lambda^2 (1-\cos \lambda)/6\phi$$

$$S_3 = \lambda \ (\sin \lambda - \lambda \cos \lambda)/4\phi$$

$$S_4 = \lambda \ (\lambda - \sin \lambda)/2\phi$$

where

$$\lambda^2 = HL^2/EI$$

$$\phi = 2 - 2 \cos \lambda - \lambda \sin \lambda$$

In addition, an extra term appears because of the effect of H on the torsional rigidity GJ and is represented by a modified torsional rigidity $\overline{GJ} = (GJ - \frac{Io}{A} H)$.

$$K' = \begin{bmatrix}
\frac{12EI}{l^3} \times S_1 & \frac{\overline{M}}{l} - \frac{V}{2} & -\frac{6EI}{l^2} \times S_2 & -\frac{12EI}{l^3} \times S_1 & -\frac{\overline{M}}{l} - \frac{V}{2} & -\frac{6EI}{l^2} \times S_2 \\
\frac{\overline{M}}{l} - \frac{V}{2} & \frac{\overline{GJ}}{l} & -\frac{Vl}{6} & -\frac{\overline{M}}{l} + \frac{V}{2} & -\frac{\overline{GJ}}{l} & \frac{Vl}{6} \\
-\frac{6EI}{l^2} \times S_2 & -\overline{M} + \frac{Vl}{3} & \frac{4EI}{l} \times S_3 & \frac{6EI}{l^2} \times S_2 & \frac{Vl}{6} & \frac{2EI}{l^2} \times S_4 \\
-\frac{12EI}{l^3} \times S_1 & -\frac{\overline{M}}{l} + \frac{V}{2} & \frac{6EI}{l^2} \times S_2 & \frac{12EI}{l^3} \times S_1 & \frac{\overline{M}}{l} + \frac{V}{2} & \frac{6EI}{l^2} \times S_2 \\
-\frac{\overline{M}}{l} - \frac{V}{2} & -\frac{GJ}{l} & \frac{Vl}{6} & \frac{\overline{M}}{l} + \frac{V}{2} & \frac{\overline{GJ}}{l} & -\frac{Vl}{6} \\
-\frac{6EI}{l^2} \times S_2 & \frac{Vl}{6} & \frac{2EI}{l} \times S_4 & \frac{6EI}{l^2} \times S_2 & \overline{M} + \frac{Vl}{3} & \frac{4EI}{l} \times S_3
\end{bmatrix}$$

Fig. 4.6 MATRIX WITH STABILITY FUNCTIONS

Consideration of a restrained centre of rotation at a distance g from the centroid as shown in Fig. 4.7 has the effect of reducing the number of degrees of freedom from 6 to 4.

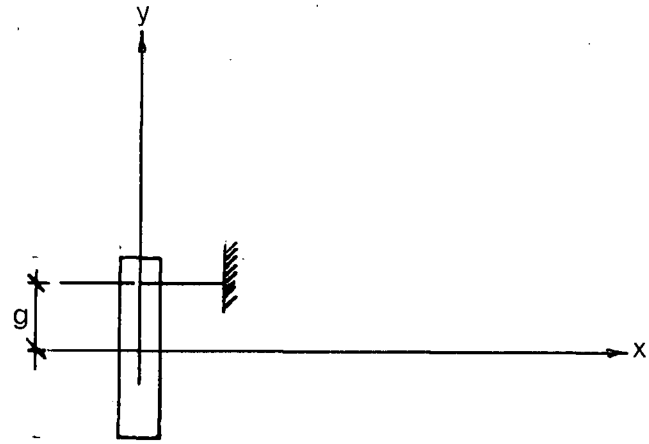
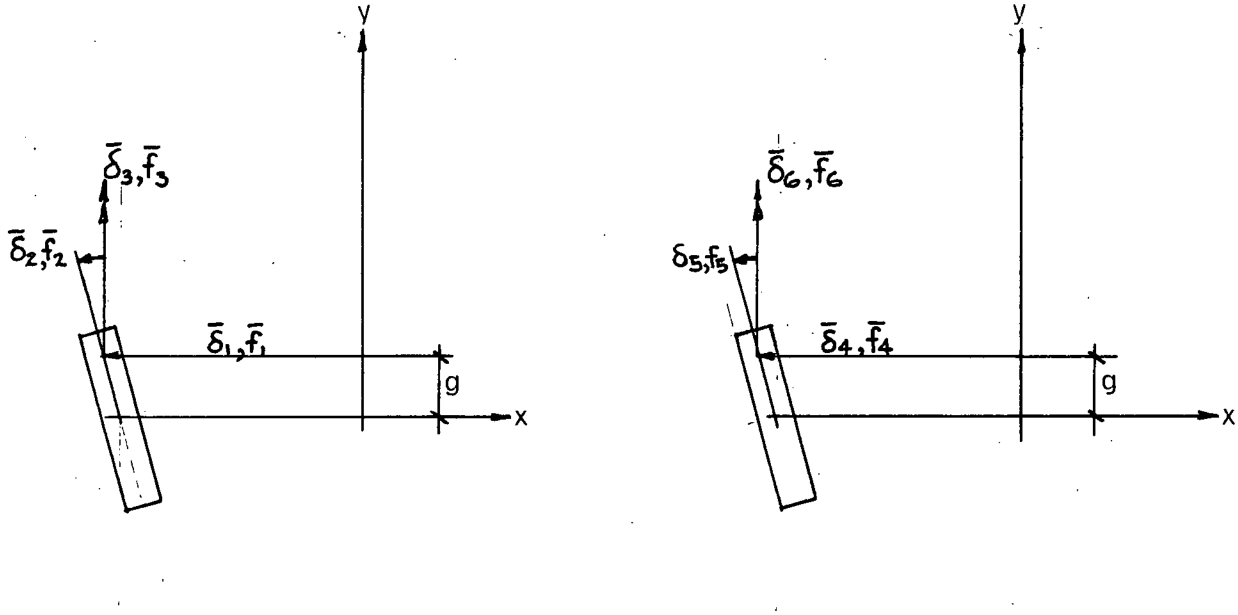


Fig. 4.7 RESTRAINED CENTRE OF ROTATION

With reference to the deformation and force system δ and f as shown in Fig. 4.8 and the system used in Chapters III and IV Fig. 3.1 the following equivalences are noted.

$$\bar{\delta}_1 = \delta_1 + g\,\delta_2 \qquad \bar{f}_1 = f_1$$
$$\bar{\delta}_2 = \delta_2 \qquad \bar{f}_2 = f_2 - f_1\,g$$
$$\bar{\delta}_3 = \delta_3 \qquad \bar{f}_3 = f_3$$
$$\bar{\delta}_4 = \delta_4 + g\,\delta_5 \qquad \bar{f}_4 = f_4$$
$$\bar{\delta}_5 = \delta_5 \qquad \bar{f}_5 = f_5 - f_4\,g$$
$$\bar{\delta}_6 = \delta_6 \qquad \bar{f}_6 = f_6$$



(a) LOWER END (b) UPPER END

Fig. 4.8 RESTRAINED FORCE AND DEFLECTION SYSTEM

These equivalences may be placed in the matrix form

$$\bar{\delta} = T\,\delta \tag{54}$$

and

$$\bar{f} = T_1 f \tag{55}$$

where T and T_1 are given by:

$$T = \begin{bmatrix} 1 & g & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & g & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix} \qquad T_1 = \begin{bmatrix} 1 & & & & & \\ -g & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & -g & 1 & \\ & & & & & 1 \end{bmatrix}$$

Fig. 4.9 TRANSFORMATION MATRICES

noting that $T^{-1} = T_1{*}$

$$(T_1 K T_1{*})\,\bar{\delta} = \bar{f} \tag{57}$$

from which

$$\bar{k} = T_1 K T_1{*} \tag{58}$$

A new 6 x 6 stiffness matrix $\bar{k}$ referenced to the deflections $\bar{\delta}$ and forces $\bar{f}$ is therefore available by transformation of K. However the deflections $\bar{\delta}_1$ and $\bar{\delta}_4$ are zero by virtue of the restraint. This enables the removal of rows and columns 1 and 4 of $\bar{k}$ and the development of a new 4 x 4 matrix $\bar{\bar{k}}$. The matrix $\bar{\bar{k}}$ is not shown since the multiplications in equation (58) are carried out numerically for each particular case. The reduction to $\bar{\bar{k}}$ enables the analysis of restrained centre of rotation problems.

CHAPTER V

DERIVATION USING THE ENERGY METHOD

5.1 General Remarks

Derivation of a stiffness matrix by differentiation of the strain energy function for the beam will by definition produce a symmetrical matrix. The results of such a derivation, as obtained in private consultation with Anderson (9), are presented in this section. For the purpose of this derivation the simplifying assumption that moment on the segment was constant, and hence the shear equal to zero, was made.

5.2 Brief Description of the Method

Bleich (8) presents an energy expression for the beam as follows

$$U = 1/2 \int_0^L (EI\mu''^2 + GJ\beta'^2 + 2M_x\mu''\beta)\, dz \tag{59}$$

which corresponds to the additional strain energy of a beam subjected to an initial moment M_x and a twist-bend deflection.

Assumptions as to the functions μ and β are the same as presented in Chapter III page 9 where

$$\mu = \mu_1 + \mu_2 + \ldots + \mu_n$$

and

$$\beta = \beta_1 + \beta_2 + \ldots + \beta_n$$

As before μ_1 and β_1 are taken as the linear values, μ_2 and β_2 calculated from the effect of the moment on the linear deflected shape and subsequent deflections neglected. The second derivitives of U with respect to the deflections $\delta_1 \ldots \delta_6$, which represent the components of the symmetrical matrix $K1$, are shown in Fig. 5.1

$$K1 = \begin{bmatrix} \frac{12EI}{l^3} & \frac{\overline{M}}{l} & -\frac{6EI}{l^2} & -\frac{12EI}{l^3} & -\frac{\overline{M}}{l} & -\frac{6EI}{l^2} \\ \frac{\overline{M}}{l} & \frac{GJ}{l} & -\overline{M} & -\frac{\overline{M}}{l} & -\frac{GJ}{l} & 0 \\ -\frac{6EI}{l^2} & -\overline{M} & \frac{4\,EI}{l} & \frac{6EI}{l^2} & 0 & \frac{2EI}{l} \\ -\frac{12EI}{l^3} & -\frac{\overline{M}}{l} & \frac{6EI}{l^2} & \frac{12EI}{l^3} & \frac{\overline{M}}{l} & \frac{6EI}{l^2} \\ -\frac{\overline{M}}{l} & -\frac{GJ}{l} & 0 & \frac{\overline{M}}{l} & \frac{GJ}{l} & \overline{M} \\ -\frac{6EI}{l^2} & 0 & \frac{2EI}{l} & \frac{6EI}{l^2} & \overline{M} & \frac{4EI}{l} \end{bmatrix}$$

Fig. 5.1 STIFFNESS MATRIX $K1$

CHAPTER VI

APPLICATION TO STABILITY PROBLEM

6.1 General Remarks

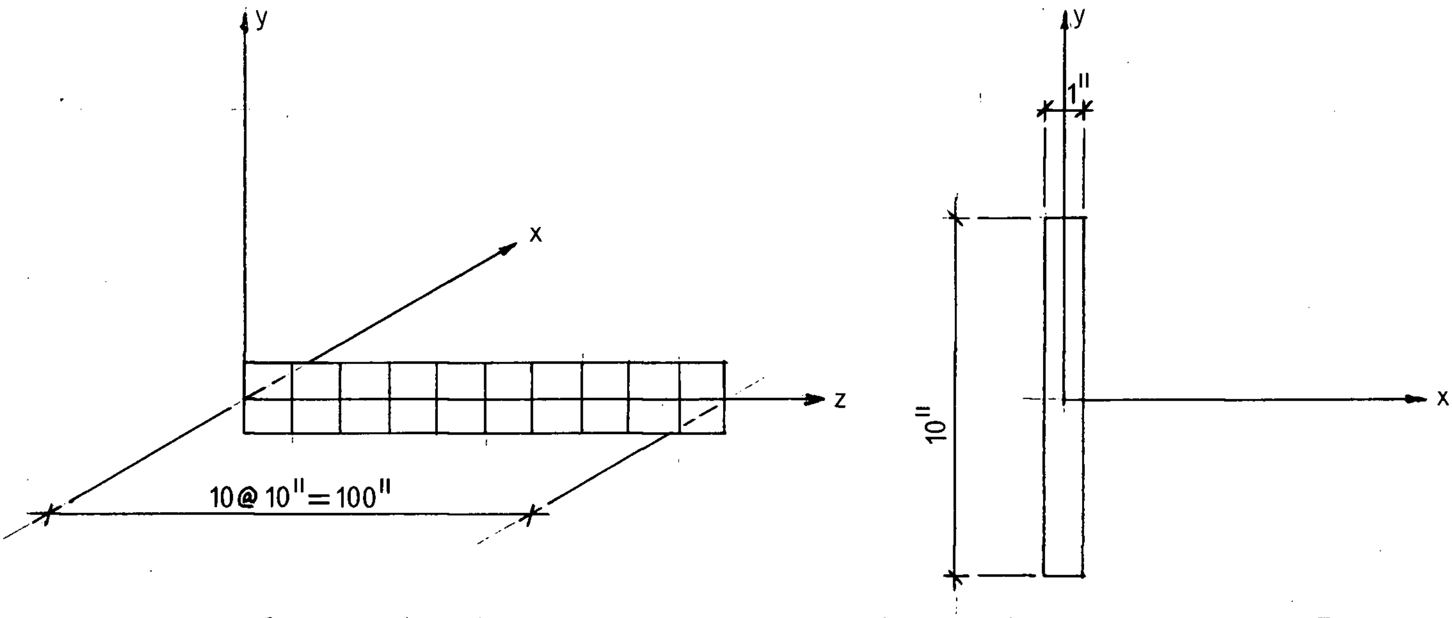
The stiffness matrix K as derived in Chapter III and IV contains the effect on the usual linear matrix of bending moments and shears in the plane of the beam element. This matrix may be employed to determine the critical value of load for narrow rectangular beams.

The procedure used is to formulate a "structure" composed of several small beam segments. With reference to the determination in Chapter III of the required number of segments, the ratio of segment to beam length of 1/10 was used. The member stiffness matrices are built up using values of $\overline{M}$ and V from a linear analysis of the structure. These member stiffness matrices are then entered into a structure stiffness matrix in the usual manner.

The actual determination of the critical load is obtained by first assuming a value for the external load, performing the linear analysis, building the structure stiffness matrix and evaluating the determinate. The external load is then incremented until the structure stiffness matrix is zero which by definition is the critical value of loading for the structure.

6.2 Numerical Examples

For the purpose of comparing the results of critical load evaluation by the above mentioned method with values as presented in the literature the beam shown in Fig. 6.1.a is used.



MEMBER DATA

$E = 30{,}000$ k.s.i. $\quad J = 3.333\ in^4 \quad A = 10\ in^2$

$G = 10{,}000$ k.s.i. $\quad I = 0.833\ in^4 \quad \ell = 10$ in

Fig. 6.1 EXAMPLE BEAM

The beam is of narrow rectangular cross section with a length depth width ratio of 100:10:1. The "structure" consists of 10 segments of length 10 and the structure stiffness matrix is of a size as determined by the boundary conditions as dictated by particular examples.

Six straight beams were investigated for critical load and the results in terms of γ are shown in Fig. 6.2.

Case #	End Cond.	Loading	γ *	γ_1**	γ_2***
1	S.S.	Equal End Couples	.1885	.1898	.1940
2	S.S.	Point at c.g. at	.2545	.2750	.2600
3	S.S.	Point at Bot. at	.2740	.2795	.2820
4	S.S.	Point at Top at	.2350	.2360	.2370
5	S.S.	Uniform	.2130	.2145	.2160
6	Cantilever	Point at c.g. at End	.2410	.2425	.2480

$\sigma_{cr} = \gamma\sqrt{EIGJ/l}$ *γ From available theory

γ_1 From test using matrix K *γ_2 From test using matrix K_1

Fig. 6.2 EXAMPLE RESULTS

Since the results agree with those available from the literature to within 3% it is assumed the method gives sufficiently accurate results for this simple type of structure.

CHAPTER VII

CONCLUSIONS

The author concludes that both matrices developed, namely K and $K1$, give satisfactory results when applied to critical load problems of simple structures. Investigation into the limits of the short length segment shows that a reduction in the number of elements from 10 to 3 decreases the accuracy of the result by approximately 5%. However due to the simple nature of the structures analysed, namely those with simple primary bending stress distributions it would be inadvisable to predict a definite number of segments for a specific degree of accuracy. A more logical procedure would be to plot critical load vs number of segments and determine convergence for each specific case.

The extension of the use of these matrices to more complex problems would be inadvisable until definite reasons are presented to explain why two entirely different matrices give consistent results to the same problems. Obviously the matrix $K1$ lends itself much more readily to numerical application since its symmetrical nature may be fitted into existing structure analysis programs.

When valid explanations for the above mentioned problems are obtained other effects such as the effect of axial load on torsional rigidity and previously derived effects of axial load on lateral stability may be added resulting in a matrix with which complex problems of flexural torsional stability under lateral and axial loads could be analysed.

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APPENDIX A

A.1. Description of Computer Program

In a brief manner the sequence of operations and the presentation of data required is as follows:

1. A card is read which bears the structure data.
   - (a) number of members, NM
   - (b) elastic and shear modulus, E.G
   - (c) moment of inertia and torsion constant, TT, T.
   - (d) length of each segment, AL
   - (e) number of degrees of freedom, NU
2. The next cards, one for each member, bear the average moment FM (I) and average shear VM (I) as determined by an independent elastic analysis under an assumed load.
3. Next, a series of cards, one for each member, with its code number, NCODE (II,J).
4. The above member data and moment and shear values are used to determine member stiffness matrices SM(I,J,K).
5. The above member stiffness matrices are then used in conjunction with the code numbers to build the structure stiffness matrix, SM (I,J).
6. A subroutine, "INVERT," is called from which the value of the determinate, "DETERM," is obtained.
7. The value of the determinate is printed out.
   If it is greater than zero the values of FMC(I) and VM(I) are incremented by a factor, FA, and steps 4 thru 7 repeated.
8. Termination of calculation is governed by obtaining a determinate value of zero or less.

A.2 FORTRAN SOURCE LIST

```
                                              FORTRAN SOURCE LIST
ISN         SOURCE STATEMENT

   0 ‡ $IBFTC  BAZ
   1 ‡         DIMENSION NCODE(25,6),SM(60,60),S(25,6,6),FM(25),VM(25)
     ‡ C THIS MATRIX IS THE COMPLETE SHEAR
   2 ‡ 99      CONTINUE
   3 ‡         READ(5,3333) NST
   5 ‡ 3333    FORMAT(I10)
   6 ‡         WRITE(6,3334) NST
   7 ‡ 3334    FORMAT(1X,22H THIS IS STRUCTURE NO.,I10,///)
  10 ‡         READ(5,100)NM,E,G,TT,T,AL,NU
  13 ‡ 151     FORMAT(1F10.2)
  14 ‡         WRITE(6,100)NM,E,G,TT,T,AL,NU
  15 ‡ 100     FORMAT(1I10,5F10.1,1I10)
  16 ‡         READ(5,101)(FM(I),I=1,24)
  23 ‡         WRITE(6,103)(FM(I),I=1,24)
  30 ‡         READ(5,101)(VM(I),I=1,24)
  35 ‡         WRITE(6,103)(VM(I),I=1,24)
  42 ‡         READ(5,151)TS
  43 ‡         WRITE(6,151)TS
  44 ‡ 103     FORMAT(1X,6F10.1)
  45 ‡ 101     FORMAT(6F10.1)
  46 ‡         DO 150 II=1,NM
  47 ‡         READ(5,102)(NCODE(II,J),J=1,6)
  54 ‡         WRITE(6,104)(NCODE(II,J),J=1,6)
  61 ‡ 104     FORMAT(1X,6I10)
  62 ‡ 102     FORMAT(6I10)
  63 ‡ 150     CONTINUE
  65 ‡         B=G*T
  66 ‡         A=E*TT
  67 ‡         C=A*B
  70 ‡         FA=1.0
  71 ‡         DO 800 KK=1,10
  72 ‡ 200     CONTINUE
  73 ‡         DO300 I=1,NM
  74 ‡         AA=(FM(I)**2)/C
  75 ‡         S(I,1,1)=(12.0*A/(AL**3))
  76 ‡         S(I,1,2)=(FM(I)/AL)+VM(I)/2.
  77 ‡         S(I,1,3)=-(6.0*A/(AL**2))
 100 ‡         S(I,1,4)=-S(I,1,1)
 101 ‡         S(I,1,5)=-FM(I)/AL+VM(I)/2.
 102 ‡         S(I,1,6)=S(I,1,3)
 103 ‡         S(I,2,1)=(FM(I)/AL)+(VM(I)/2.0)
 104 ‡         S(I,2,2)=B/AL
 105 ‡         S(I,2,3)=(VM(I)*AL/6.0)
 106 ‡         S(I,2,4)=-FM(I)/AL-VM(I)/2.
 107 ‡         S(I,2,5)=-S(I,2,2)
 110 ‡         S(I,2,6)=-(VM(I)*AL/6.0)
 111 ‡         S(I,3,1)=S(I,1,3)
 112 ‡         S(I,3,2)=-FM(I) -VM(I)*AL/3.
 113 ‡         S(I,3,3)=(4.0*A/AL)
 114 ‡         S(I,3,4)=-S(I,1,3)
 115 ‡         S(I,3,5)=-VM(I)*AL/6.
 116 ‡         S(I,3,6)=(2.0*A/AL)
 117 ‡         S(I,4,1)=-S(I,1,1)
 120 ‡         S(I,4,2)=-FM(I)/AL-VM(I)/2.
 121 ‡         S(I,4,3)=S(I,3,4)
```

```
                                   FORTRAN SOURCE LIST  BAZ
ISN       SOURCE STATEMENT

 122 +          S(I,4,4)=S(I,1,1)
 123 +          S(I,4,5)=FM(I)/AL-VM(I)/2.
 124 +          S(I,4,6)=S(I,4,3)
 125 +          S(I,5,1)=-(FM(I)/AL)+(VM(I)/2.0)
 126 +          S(I,5,2)=-S(I,2,2)
 127 +          S(I,5,3)=S(I,2,6)
 130 +          S(I,5,4)=(FM(I)/AL)-(VM(I)/2.0)
 131 +          S(I,5,5)=S(I,2,2)
 132 +          S(I,5,6)=S(I,2,3)
 133 +          S(I,6,1)=S(I,3,1)
 134 +          S(I,6,2)=-VM(I)*AL/6.
 135 +          S(I,6,3)=S(I,3,6)
 136 +          S(I,6,4)=S(I,4,6)
 137 +          S(I,6,5)=FM(I)-VM(I)*AL/3.
 140 +          S(I,6,6)=S(I,3,3)
 141 + 300      CONTINUE
 143 +          DO 400 J=1,NU
 144 +          DO 400 I=1,NU
 145 +          SM(J,I)=0.0
 146 + 400      CONTINUE
 151 +          DO 500 I=1,NM
 152 +          DO 500 K=1,6
 153 +          IF(NCODE(I,K))420,500,420
 154 + 420      M=NCODE(I,K)
 155 +          DO 502 J=1,6
 156 +          IF(NCODE(I,J))430,502,430
 157 + 430      N=NCODE(I,J)
 160 +          SM(M,N)=SM(M,N)+S(I,K,J)
 161 + 502      CONTINUE
 163 + 500      CONTINUE
 166 +          SM(15,15 )=SM(15,15)+TS*5.0
 167 + 501      FORMAT(6F20.6)
 170 +          DO 600  JJJ=1,NU
 171 +          DO 600  III=1,NU
 172 +          SM(JJJ,III)=SM(JJJ,III)/100.0
 173 + 600      CONTINUE
 176 +          CALL INVERT (SM,NU,60,DETERM,COND)
 177 +          WRITE(6,650)FA,DETERM
 200 + 650      FORMAT(13H LOAD FACTOR=,1F10.3,13H DETERMINATE=,1E20.6)
 201 +          WRITE(6,103)(FM(I),I=1,12)
 206 +          WRITE(6,103)(VM(I),I=1,12)
 213 +          WRITE(6,151)TS
 214 +          IF(DETERM)701,701,700
 215 + 700      FA=FA+0.05
 216 +          TS=TS*FA
 217 +          DO 680I=1,24
 220 +          FM(I)=FM(I)*FA
 221 +          VM(I)=VM(I)*FA
 222 + 680      CONTINUE
 224 + 800      CONTINUE
 226 + 701      CONTINUE
 227 +          GO TO 99
 230 + 702      CONTINUE
 231 +          STOP
 232 +          END
```