RECTANGULAR BAR AND NO-BAR FINITE ELEMENTS

FOR

THREE DIMENSIONAL STRESS ANALYSIS

by

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ABSTRACT

A three dimensional bar cell in the form of a rectangular parallelepiped capable of imitating the action of elastic bodies of any value of Poisson's ratio is devised for three dimensional stress analysis by the finite element method and the stiffness matrix of the cell is derived. Furthermore a rectangular no-bar cell is also formed by extending the idea of linear displacement functions from two dimensional to three dimensional cases and the stiffness matrices of this cell are derived both by the method of virtual work and by statics.

Both these types of cells are used to solve three dimensional problems and the results in the form of displacements and stresses are compared with the exact elasticity solutions using different mesh sizes and different values of Poisson's ratio.

The stresses found by the Finite element method are calculated in two ways: by the Joint displacement method (using two different procedures) and by the Nodal force method and the quality of solution obtained by these methods is compared.

Finally the same examples are also analysed by using the framework models proposed by "Yettram and Robbins". (6)*

* Numbers signify the references, listed in Bibliography.
The results obtained by these several methods were good, with the ones corresponding to the proposed bar model being consistently the best of the three. The stresses found by both the Joint displacement and the nodal force methods were found comparable in quality.
DEFINITION OF SYMBOLS

\(a, b, c\) = Dimensions of the cell

\(A_x, A_y, A_z\) = Cross sectional areas of edge bars in \(x, y, z\) directions respectively

\(A_{xy}, A_{xz}, A_{yz}\) = Cross sectional areas of diagonal bars in \(xy, xz, yz\) planes respectively

\(A_{lx}, A_{lxz}, A_{lyz}\) = Cross sectional areas of corner bars in \(xy, xz, yz\) planes respectively

\(\sigma_x, \sigma_y, \sigma_z\) = Normal stresses

\(\tau_{xy}, \tau_{xz}, \tau_{yz}\) = Shear stresses

\(\varepsilon_x, \varepsilon_y, \varepsilon_z\) = Normal strains

\(\gamma_{xy}, \gamma_{xz}, \gamma_{yz}\) = Shearing strains

\(\sigma_r, \sigma_\theta, \tau_r\) = Radial, tangential and shear stresses in cylindrical coordinates respectively

\(v_r\) = Radial displacement

\(u, v, w\) = Displacements in \(x, y, z\) directions respectively

\(E\) = Modulus of elasticity of the solid and the members of the equivalent framework

\(\mu\) = Poisson's ratio for the material of the solid

\(P\) = Externally applied concentrated load

\([F]\) = Vector of direct joint forces

\([\Delta]\) = Vector of Joint deflections

\([K]\) = Element stiffness matrix

\(X_{ij}, Y_{ij}, Z_{ij}\) = Distribution factors as defined in the text

\([f]\) = Nodal force vector for each element
\[ \{ \delta \} \] = Deflection vector for each element
\[ x, y, z \] = Cartesian coordinates of a point
\[ x, r, \theta \] = Polar coordinates of a point
\[ x', y', z' \] = Non-dimensional coordinates of a point
\[ X, Y, Z \] = Body forces/unit volume
\[ D_1, D_2, D_3, L \] = Elastic constants as defined in the text
\[ \lambda_1 \] = Displacements perpendicular to a boundary surface of the cell
\[ \lambda_{11} \] = Displacements parallel to a boundary surface of the cell
\[ \{ S \} \] = Vector of known nodal displacements
\[ \{ D \} \] = Vector of unknown nodal displacements
\[ \{ P \}^* \] = Vector of fictitious nodal faces due to known displacements
\[ [B] \] = Strain matrix
\[ [M] \] = Stress matrix
<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Page No.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Introduction</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>Review of Previous Work</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>Derivation of Stiffness Matrices</td>
<td>7</td>
</tr>
<tr>
<td>3.1</td>
<td>Framework Cell (Type 1)</td>
<td>7</td>
</tr>
<tr>
<td>3.2</td>
<td>Requirements of the Framework cell</td>
<td>9</td>
</tr>
<tr>
<td>3.3</td>
<td>Deformability of an Elastic Body in Condition of Uniform Stress</td>
<td>10</td>
</tr>
<tr>
<td>3.4</td>
<td>Determination of Cell Characteristics</td>
<td>12</td>
</tr>
<tr>
<td>3.5</td>
<td>Stiffness Coefficients of the Cell</td>
<td>23</td>
</tr>
<tr>
<td>3.6</td>
<td>Framework Cell (Type 2)</td>
<td>42</td>
</tr>
<tr>
<td>3.7</td>
<td>No Bar Cells</td>
<td>42</td>
</tr>
<tr>
<td>3.8</td>
<td>Derivation of Statics Type Stiffness Matrix</td>
<td>50</td>
</tr>
<tr>
<td>3.9</td>
<td>Derivation of Energy Type Stiffness Matrix</td>
<td>66</td>
</tr>
<tr>
<td>4</td>
<td>Analysis of Model</td>
<td>81</td>
</tr>
<tr>
<td>4.1</td>
<td>Calculation of the Displacements at the Nodes</td>
<td>81</td>
</tr>
<tr>
<td>4.2</td>
<td>Calculation of Stresses at the Nodes</td>
<td>85</td>
</tr>
<tr>
<td>4.2a</td>
<td>Method 1 of Joint Displacements</td>
<td>85</td>
</tr>
<tr>
<td>4.2b</td>
<td>Method 2 of Joint Displacements</td>
<td>87</td>
</tr>
<tr>
<td>Section</td>
<td>Page No.</td>
<td></td>
</tr>
<tr>
<td>-------------------------------------------------------------------------</td>
<td>----------</td>
<td></td>
</tr>
<tr>
<td>4.2c Method of Nodal Forces</td>
<td>96</td>
<td></td>
</tr>
<tr>
<td>CHAPTER FIVE — Examples of Three Dimensional Solid Analysis</td>
<td>99</td>
<td></td>
</tr>
<tr>
<td>5.1 Example 1 - Boussinesq Problem</td>
<td>99</td>
<td></td>
</tr>
<tr>
<td>5.2 Example 2 - Infinite Body</td>
<td>103</td>
<td></td>
</tr>
<tr>
<td>5.3 Purposes of the Examples and Procedure Used</td>
<td>106</td>
<td></td>
</tr>
<tr>
<td>CHAPTER SIX — Results and Discussions</td>
<td>109</td>
<td></td>
</tr>
<tr>
<td>CONCLUSION</td>
<td>160</td>
<td></td>
</tr>
<tr>
<td>BIBLIOGRAPHY</td>
<td>162</td>
<td></td>
</tr>
</tbody>
</table>
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CHAPTER 1
Introduction

Rapid advance has been made during recent years in plane stress analysis of elastic plates by replacing the plate by a model consisting of finite elements or cells having the same external outline and boundary restraints and subjected to the same loads as the plate prototype. The success of this method in two dimensional cases makes it logical to expect similar results in extension of the method to the three dimensional stress conditions. Here again the continuous material of the elastic body is idealized into the assemblage of a number of structural elements or cells interconnected at the corners or nodal points. The cells used may be the framework cells composed of elastic bars or no-bar cells. The cells composing the model are usually made identical.

The cell model must imitate the structural action of the prototype as closely as possible. In the case of the framework model this may be accomplished by using cells whose deformations as described by their nodes, agree exactly with the deformations of the solid material in condition of an arbitrary uniform stress or strain. When solving the actual problem the state of stress in the prototype is not uniform and so the model solution will contain errors caused by the finite size of the cell. However as the size of the mesh decreases the results approach the true values in the limit. The fineness of mesh and with it the accuracy of the solution
is limited by the capacity of the computer. However even a coarser mesh model having proper elastic properties gives fairly accurate results which in fact are much better than by the finite difference approach using the same fineness of mesh(1).

The three dimensional cells forming the model are made up of a number of different elastic bars having the same modulus of elasticity as the prototype and endowed with appropriate tension-compression stiffnesses. The geometric pattern of the cells and the arrangement of the bars must be such that they are suitable for any value of Poisson's ratio and at the same time the cells behave as a rigid structure. The cell parameters such as the cross-sectional areas of the bars making the framework are determined from the condition that the framework deforms in the same manner as the elastic solid when placed in an arbitrary uniform stress field. If the number of independent parameters of the cell is greater than the number of equations corresponding to the uniform stress conditions the extra parameters may be assigned arbitrarily, but if it is less than the number of such equations the cell pattern must be modified to incorporate additional bars. Cell pattern short of one parameter may be suitable for one particular value of Poisson's ratio. Introduction of flexural rigidity into some bars may also provide additional parameters.

Once the stiffness properties of different bars composing the cells are known the analysis of model may be accomplished by finding the relationship between the corner
forces and corner displacements of each individual cell. When one of the corners of the cell is given unit displacement in the direction of one of the coordinate axes, while all other corners are held in their positions the forces required at the nodes to hold the cell in the deformed state are called distribution factors of the cell. The distribution factors corresponding to displacements of all the corners of the cell are arranged in the form of a matrix called the stiffness matrix of the cell; whose product with the column vector of the nodal-displacements gives the column vector of nodal forces of the cell. From cell matrices the computer forms the stiffness matrix of the whole model and sets up the equations expressing the equilibria of the nodes. These are solved for the nodal displacements. Once the displacements of various corners are known the stresses in the prototype may be evaluated either from the nodal displacements or from the nodal forces.

Cells not involving bars have also been suggested. In this case the elastic solid is assumed to consist of a number of rectangular blocks interconnected at the nodes. Unlike the framework cells the no-bar cells can not be constructed physically but like their framework counterpart they still allow to obtain a set of distribution factors which may be used in the solution of three dimensional problems. Cells of this type have also been developed in this thesis.

The application of the proposed methods is demonstrated on two examples whose rigorous elasticity solution are available: the Boussinesq problem-involving a concentrated
load applied normally on the surface of an elastic semi-space, and a concentrated load acting within an infinite solid.
Finite elements in the form of cells composed of bars were first introduced by Hrennikoff for the stress analysis of plates under plane stress and flexure in a paper published in 1941.\(^{(1)}\) Hrennikoff also extended the idea to three dimensional cases and devised a cubic cell whose six faces consisted of side bars, diagonal bars and auxiliary bars forming a square within each face. The latter were required to make the framework suitable for any value of Poisson's ratio. Later Clough, Turner, Martin and Topp\(^{(2)}\) introduced the idea of no-bar cells also suitable for the analysis of plane stress problems. Although the same general idea is clearly applicable to three dimensional cases only preliminary investigations of this type have been reported till now. In 1963 Melosh\(^{(3)}\) published a paper in which he developed stiffness matrices for a three dimensional solid from an assumed linear displacement function. In 1965 Clough and Rashid\(^{(4)}\) considered tetrahedral elements for the analysis of axisymmetric solids and Argyris used Tetrahedral elements in a general three dimensional case.\(^{(7)}\)

These applications of the method to the three dimensional cases have involved no bar cells. In 1967 Jobson\(^{(5)}\) has made a preliminary investigation to determine the characteristics of space frameworks. He avoided the use of Hrennikoff's

* Numbers signify the references, listed in Bibliography.
auxiliary members provided for Poisson's ratio values other than $\frac{1}{4}$ and attributed to the edge members instead of them flexural stiffnesses. He did not however derive the stiffness matrix and did not solve any examples.

Jobson's work was extended recently by Yettram and Robbins\(^{(6)}\) who proposed a cell substantially the same as Jobson and derived its stiffness matrix.
3.1 Framework Cell (Type 1)

Hrennikoff\(^{(1)}\) points out that for cells arranged in the form of a rectangular parallelepiped the deformability of the cell may be stated in terms of six stress conditions from which nine independent equations may be formulated. These will define nine independent characteristics or parameters such as cross-sectional areas of bars and angles between the bars. Framework cells having more parameters than nine can be permitted and in that case extra parameters may be assigned at will.

The type of cell used in the present work is presented in Fig. 1. The lengths of the cell edges parallel to the axis \(x, y\) and \(z\) are designated \(a, b, c\), respectively, and the cross-sectional area of corresponding edge bars \(A_x, A_y\) and \(A_z\). On each face of the cell there are diagonal bars and corner bars joining the mid-points of the adjacent edges. For reasons which will be explained later each side bar is split into three parts, one part going straight from corner to corner and the other two joined in the middle to the corner bars in the adjacent planes (Fig. 8). Thus the combined area \(A_x = A_{x'} + A_{x''} + A_{x'''}\), the three component areas being as follows:

\[
\begin{align*}
A_{x'} & = \text{the area of the bar connecting to the corner bars in the plane } xy. \\
A_{x''} & = \text{that of the bar connecting to the corner bars in the plane } xz. \\
A_{x'''} & = \text{the area of the unconnected bar.}
\end{align*}
\]
\[ A_x = A_{x'} + A_{x''} + A_{x'''} \]
\[ A_y = A_{y'} + A_{y''} + A_{y'''} \]
\[ A_z = A_{z'} + A_{z''} + A_{z'''} \]
Bars $A_y$ and $A_z$, situated along the $y$ and $z$ edges consist of the similar three parts. $A_y', A_y'', A_z', A_z''$ belong respectively to the faces $yz, yx, zx$ and $zy$.

The areas of the diagonal bars in the three faces $A_{xy}, A_{yz}, A_{zx}$ are identified by the subscripts belonging to the respective faces. The same symbols but provided with primes are used for the corner bars.

The total number of bar areas in the cell of Fig. 1 is then 15. Six of them will be assumed as explained further and nine determined from the deformability conditions.

3.2 Requirements of the Framework Cell

The state of stress at a point in a body is defined by six stress components $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$ and the state of strain by the components of strain $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$. In an elastic body the stress and strain components are inter-related through the elastic constants $E$ and $\mu$.

Any condition of the uniform stress or strain may be uniquely reproduced by the combination of six conditions, each of which is characterized by a single uniform strain component such as $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}...$ etc., the other five strain components being absent in each such condition.

The framework cells are so devised that they deform in the same manner as the solid elastic body when subjected to these simple strain conditions. Such cell will deform identically with the solid body in uniform strain or stress
condition of any kind.

3.3 Deformability of an Elastic Body in Condition of Uniform Stress (Fig. 2)

Condition 1 (Fig. 2a)

Consider an elastic body in the form of a cube 1\times1\times1 in size subjected to uniform stresses

\[
\begin{align*}
\sigma_x &= \frac{P}{E} \text{lb/in}^2 \\
\sigma_y &= \sigma_z = \frac{\mu P}{(1-\mu)} \text{lb/in}^2
\end{align*}
\]

while \(\tau_{xy} = \tau_{yz} = \tau_{zx} = 0\)

Then

\[
\epsilon_x = \left\{ \frac{\sigma_x}{E} - \frac{\mu}{E} (\sigma_y + \sigma_z) \right\} = \left\{ \frac{P}{E} - \frac{2\mu}{E} \left( \frac{\mu P}{(1-\mu)} \right) \right\} = \frac{(1+\mu)(1-2\mu)}{(1-\mu)} \frac{P}{E} \tag{2}
\]

\[
\epsilon_y = \left\{ \frac{\sigma_y}{E} - \frac{\mu}{E} (\sigma_x + \sigma_z) \right\} = \frac{1}{E} \left\{ \frac{\mu P}{1-\mu} - \mu \left( P + \frac{\mu P}{1-\mu} \right) \right\} = 0 \tag{3}
\]

Similarly \(\epsilon_z = 0\) \tag{4}
The condition 1 thus represents uniform strain in x direction with no strain in two other directions of coordinates.

Conditions 2 and 3 (Fig. 2b and 2c) similarly represent uniform strains in y and z directions respectively.

**Condition 4 (Fig. 2d)**

The same element is subjected to uniform shear stress \( \tau_{xy} = \tau_{yx} = p \text{ lbs/in}^2 \). The resultant unit shear deformation is

\[
\gamma_{xy} = \frac{2p(1+\mu)}{E}
\]

(5)

The normal strains in x, y and z planes and the shear strains \( \gamma_{yz} \) and \( \gamma_{yx} \) are zero.

The shear strains in two other directions obtained from Conditions 5 and 6 (Fig. 2e and 2f) are governed by similar relations.

### 3.4 Determination of Cell Characteristics

The framework cell is successively subjected to above six loading conditions from which bar areas are determined.

**Condition 1 (Fig. 3a and 3b)** \( \epsilon_x; \epsilon_y = \epsilon_z = \gamma_{xy} = \gamma_{xz} = \gamma_{yz} = 0 \)

The framework cell is subjected to \( p \text{ lb/in}^2 \) in x direction and \( \frac{\mu p}{1-\mu} \text{ lb/in}^2 \) in y and z directions.

The corner bars do not work, which follows from the symmetry of the loading; if the corner bar on the left...
in Fig. 3e considering the equilibrium in y direction

\[ S_{xy} \cos \alpha = \frac{H \rho a c}{4(1-\mu)} \quad (8) \]

Using equation (7) we get

\[ S_{xy} = \frac{H \rho a c}{4(1-\mu)} \frac{(a^2+b^2)^{1/2}}{b} \quad (9) \]

Similarly \( S_{xz} = \frac{H \rho a b}{4(1-\mu)} \frac{(a^2+c^2)^{1/2}}{c} \quad (10) \)

From \( \sum x = 0 \), \( S_x + S_{xy} \sin \alpha + S_{xz} \sin \beta = \frac{bbc}{4} \quad (11) \)

Substituting the value of \( S_{xy} \) and \( S_{xz} \) and simplifying

\[ S_x = \frac{b}{4(1-\mu)b c} \left\{ b^2 c^2 (1-\mu) - H a^2 (b^2+c^2) \right\} \quad (12) \]

The general relation between the stress and the elongation \( \delta \) of an elastic bar of the length \( l \) and the cross-sectional area \( A \) is

\[ S = \frac{A E \delta}{l} \quad (13) \]

The elongation of the bars \( A_x, A_{xy} \) and \( A_{xz} \) (Fig. 3f) are respectively
Fig. (3a and 3c) should be under tension so also should be
the corner bar on the right, but there is no member at the
mid length of the edge bars to resist the transverse components
of the stresses in the corner bars and so these stresses must
be zero.

Equating the elongation of the framework cell \( \delta \)
to the corresponding elongation of the elastic solid we get

\[
\delta = \frac{(1+\mu)(1-2\mu)}{(1-\mu)} \frac{PQ}{E} \tag{6}
\]

Since the strains in \( y \) and \( z \) directions are zero
the bars belonging to \( yz \) faces are unstressed i.e. \( S_y = 0 \),
\( S_z = 0 \) and \( S_{yz} = 0 \).

Denoting by \( \alpha \) and \( \beta \) the angles between the diagonals
and the \( y \) and \( z \) directions as shown in Fig. 3a and 3b:

\[
\begin{align*}
\tan \alpha &= \frac{a}{b} \\
\cos \alpha &= \frac{b}{(a^2+b^2)^{1/2}}
\end{align*}
\]

\[
\begin{align*}
\tan \beta &= \frac{a}{c} \\
\cos \beta &= \frac{c}{(a^2+c^2)^{1/2}}
\end{align*}
\tag{7}
\]

The free body diagram for the corner 1 is shown
\[ \delta_x = \delta \]
\[ \delta_{xy} = \delta \sin \alpha \]
\[ \delta_{xz} = \delta \cos \alpha \]

Substituting appropriate expression of \( S, \delta \) and \( \iota \) into Equation (13) the values of the three bar areas can be found as follows.

\[ A_x = \frac{b^2 c^2 (1-\mu) - \mu a^2 (b^2+c^2)}{4(1+\mu)(1-2\mu)bc} \]  \hspace{1cm} (15)
\[ A_{xy} = \frac{\mu c (a^2+b^2)}{4(1+\mu)(1-2\mu)ab} \]  \hspace{1cm} (16)
\[ A_{xz} = \frac{\mu b (a^2+c^2)}{4(1+\mu)(1-2\mu)ac} \]  \hspace{1cm} (17)

The cross sectional areas of other edge and diagonal bars are found following the same procedure in Condition 2 and 3 (Fig. 4 and 5). The resultant formulae may be obtained from Equations 15,16 and 17 by cyclic substitution.

**Condition 4**

The framework cell is subjected to uniform shear \( \tau_{xy} = \tau_{yx} = p \) lbs/in\(^2\).
Fig. 6a shows one xy face of the cell with the tributary corner forces applied to it. Of the bars located in the xy face the edge bars $A_x''$ and $A_y''$ unconnected to the corner bars develop no stresses because their lengths remain unchanged. Likewise unstressed are all bars located in the faces yz and zx. The diagonal bars $A_{xy}$ are evidently stressed and so are the corner bars $A_{1xy}$ and the subdivided side bars $A_x'$ and $A_y''$ as will be explained presently.

The free body diagram for forces acting in different bars in xy plane at the corner (5) is shown in Fig. 6b.
If $R$ is the resultant of the forces $\frac{pbc}{4}$ and $\frac{bac}{4}$ at the corner (5) then

$$R = \frac{pbc}{4} \left( \frac{a^2 + b^2}{2} \right)^{\frac{3}{2}}$$

(18)

and its direction is given by: $\tan \alpha = \frac{a}{b}$. Thus $R$ acts in the direction of the diagonal.

The horizontal displacement of the corner (5) in Fig. 6a is

$$\Delta = \frac{2pb(1+\mu)a}{E}$$

(19)

and the elongation of the diagonal bar is $\Delta \cos \alpha$

Then using the equation (13) and substituting the values of $A_{xy}$, $\Delta$ and $\cos \alpha$ the stress in the diagonal bar

$$S_{xy} = \frac{\mu cp(a^2 + b^2)^{\frac{3}{2}}}{2(1-2\mu)}$$

(20)

The difference of the corner load $R$ (Eq. 18) and the stress $S_{xy}$ i.e. the force $R_1 = R - S_{xy}$ must be carried by the subdivided bars situated along the $x$ and $y$ edges and connected to the corner bars in the face $xy$.

Thus

$$R_1 = \frac{(1-4\mu)c(a^2 + b^2)p}{4(1-2\mu)}$$

(21)

Note that $R_1$ acts in the direction of the diagonal.

Then

$$S_y'' = R_1 \cos \alpha = \frac{(1-4\mu)bcp}{4(1-2\mu)}$$

(22)
and

\[ S_x' = R_1 \sin \alpha \]

\[ = \frac{(1-4\mu)a\beta}{4(1-2\mu)} \]

(23)

It should be noted that at all four corners \( R_1 \), \( S_y'' \) and \( S_x' \) are of the same magnitude but different in direction or sign.

Now consider any intermediate joint such as \( B \).

Let \( S_{xy}' \) be the force in the corner bar. For equilibrium we have (Fig. 6c)

\[ 2S_y'' = 2S_{xy}' \cos \alpha \]

\[ \therefore S_{xy}' = \frac{(1-4\mu)c(a^2+b^2)^{\frac{1}{2}}}{4(1-2\mu)} \]

(24)

![Figure 6c](image)

Note that each corner bar carries a stress opposite in sign to the stresses in the adjacent edge bars.

The rectangle in Fig. 6d represents the xy face of the cell before the deformation, produced by the stresses in the corner and the edge bars. Deformation of each corner triangle such as \( ABC \), may be visualized separately from the other corner triangles. Let the sides \( AB \) and \( BC \) elongate under their respective tension stresses \( S_y'' \) and \( S_x' \) by the amounts \( AA' = db/2 \) and \( CC' = da/2 \). Since the corner bar \( AC \) is subjected to compression \( S_{xy} \) the points \( A' \) and \( C' \)
would move transversely through the distances $dx$ and $dy$ respectively into the positions $A^\prime\prime$ and $C^\prime\prime$ determined by the condition that the corner bar shortens by the amount $AC - A^\prime\prime C^\prime\prime = dl$.

The stresses in the members of the corner triangle $ADE$ are equal and opposite to the ones in the triangle $ABC$ and so its corners $A$ and $E$ would move through the same distances $da/2$, $db/2$, $dx$ and $dy$ but in the directions opposite to the ones in the triangle $ABC$, as is illustrated in the figure. The behaviour of the two lower triangles in Fig. 6d is similar to the upper ones.

Projecting the broken line $AA^\prime A^\prime\prime C^\prime C^\prime\prime$ on the line $AC$ we get

$$AC - A^\prime\prime C^\prime\prime = dl = dx \sin \alpha + dy \cos \alpha - \frac{da}{2} \sin \alpha - \frac{db}{2} \cos \alpha$$

From this

$$\left( a^2 + b^2 \right) \frac{1}{2} dl = a dx + b dy - \left( a \frac{da}{2} + b \frac{db}{2} \right)$$

The deformations $dl$, $da/2$ and $db/2$ may be expressed through the corresponding bar stresses, while the quantities $dx$ and $dy$ are still unknown.

It may be observed in Fig. 6d that the deformed positions of the horizontal and vertical sides of all corner triangles are respectively parallel to each other. By moving the triangle $E^\prime DA^\prime$ to the right a distance $2dy$, the triangle $G^\prime HC^\prime$ up a distance $2dx$ and the triangle $A^\prime\prime BC^\prime\prime - 2dx$ up and $2dy$ to the right the deformed face transforms into the parallelogram $DBHF$ of Fig. 7. The
deviations of the corner angles of this parallelogram from 90° represent the unit shear strain \( \gamma_{xy} = \frac{2b(1+\mu)}{E} \).

From fig. 7 thus

\[
\frac{2b(1+\mu)}{a} = \frac{2dy}{a} + \frac{2dx}{b}
\]  

(27)

Multiplying Equation (27) by ab and adding to Equation (26)

\[
(a^2 + b^2)^{1/2} dl + a da/2 + b db/2 = \frac{ab(1+\mu)}{E}
\]  

(28)

The length changes of the bars \( dl, da/2 \) and \( db/2 \) are expressed through their stresses \( S_{xy}', S_x', \) and \( S_y'' \)

known from Equations (24), (23), (22) and through the cross sectional areas \( A_{lxy}, A_x', \) and \( A_y'' \) still unknown. This gives

\[
\frac{PC(1-4\mu)(a^2 + b^2)^{3/2}}{4(1-2\mu)A_{lxy}} + \frac{Q^2C}{bA_x'} + \frac{Q^2C}{aA_y''} = \frac{8(1+\mu)(1-2\mu)}{(1-4\mu)}
\]  

(29)

This equation represents the basic relation following from the shear Condition 4 and serving for determination of the parameters of the cell. Two similar equations correspond to the shear conditions 5 and 6. They may be written directly from Equation (29) by cyclic substitution of terms.

These three equations are evidently insufficient for determination of the nine remaining bar areas, three of the corner bars and six of the edge bars joined to them. Since no additional equations are available the areas of edge bars will be assumed, and the assumption will be such as to make the expression for areas as simple as possible.
Assume

\[ A_x' = \frac{(1-4\mu) a^2 C}{2(1+\mu)(1-2\mu)b} \]  

(30)

\[ A_y'' = \frac{(1-4\mu) b^2 C}{2(1+\mu)(1-2\mu)a} \]  

(31)

Substituting these in Equation (29) we get

\[ A_{xy} = \frac{C(a^2+b^2)^{3/2}(1-4\mu)}{4(1+\mu)(1-2\mu)ab} \]  

(32)

The cross sectional areas of the corner bars in yz and xz planes may be found by cyclic substitution.

All bar areas of the cell are assembled in Table 1.

3.5 Stiffness Coefficients of the Cell

From the previous explanation it should be clear that the nth column of the stiffness matrix represents the forces arising at different corners of the cell when the latter is subjected to a single corner displacement \( A_n = 1 \), the unit displacement of the type corresponding to nth column. With the cross section areas of the bars being known the stiffness matrix \([K]\) may be generated column by column, using the conventional structural analysis for determination of the nodal forces corresponding to the individual unit displacements. These nodal forces are known as the distribution factors. The necessary structural analysis is much simplified if the condition of unit displacement of the corner in question is replaced by three component cases which are particularly easy to analyse. This is illustrated on the example of the y displacement of
**Expresssions for Parameters**

**OF CELL (TYPE 1)**

\[
A_x = A_x' + A_x'' + A_x'''
\]

\[
A_y = A_y' + A_y'' + A_y'''
\]

\[
A_z = A_z' + A_z'' + A_z'''
\]

\[
A_{xy} = \frac{A_{xy}}{4(1+H)(1-2H)\, bC}
\]

\[
A_{xz} = \frac{A_{xz}}{4(1+H)(1-2H)\, bC}
\]

\[
A_{yz} = \frac{A_{yz}}{4(1+H)(1-2H)\, bC}
\]

\[
A_{lxz} = \frac{A_{lxz}}{4(1+H)(1-2H)\, QC}
\]

\[
A_{lyz} = \frac{A_{lyz}}{(1-4H)\, a(2+C^2)\, 3/2}{4(1+H)(1-2H)\, bC}
\]

\[
A_{lxy} = \frac{A_{lxy}}{(1-4H)\, C(2+C^2)\, 3/2}{4(1+H)(1-2H)\, aC}
\]

\[
A_{lxx} = \frac{(1-4H)\, C^2a}{2(1+H)(1-2H)\, b}
\]

\[
A_{lxy} = \frac{(1-4H)\, b^2a}{2(1+H)(1-2H)\, C}
\]

\[
A_{lxx} = \frac{(1-4H)\, a^2b}{2(1+H)(1-2H)\, C}
\]

\[
A_{lxy} = \frac{(1-4H)\, b^3C}{2(1+H)(1-2H)\, aC}
\]

\[
A_{lxx} = \frac{(1-4H)\, C^2b}{2(1+H)(1-2H)\, bC}
\]
the corner 1 whose rectangular coordinates are $(+\frac{a}{2}, +\frac{b}{2}, +\frac{c}{2})$.

The displacement $\Delta_1 Y$ so produced is described as Action 2.

**Action 2**

As Fig. (10) shows the three component conditions of Action 2 are as follows.

Condition A - Uniform elongation of the face $+\frac{a}{2}$ with nodes 1 and 3 being moved in $y$ direction $\frac{\Delta_1 Y}{4}$ and the nodes 2 and 4, $-\frac{\Delta_1 Y}{4}$.

Condition B - The face $+\frac{a}{2}$ is being subjected to a flexure like deformation with the displacements in $y$ direction, the nodes 1 and 4 moving $\frac{\Delta_1 Y}{4}$ and the nodes 2 and 3, $-\frac{\Delta_1 Y}{4}$.

Condition C - A shear like deformations of the faces $+\frac{a}{2}$ and $+\frac{c}{2}$ with the corners 1 and 2 moving in $y$ direction through the distance $\frac{\Delta_1 Y}{2}$.

It is obvious that the combination of three component cases does result in the condition of a single displacement $\Delta_1 Y$. A little reflection shows that the deformations of the component cases A and B are both symmetrical about the mid plane $xz$ and thus these conditions make all corner bars of the cell in-operative. In Condition C however the corner
Action 2

Fig. 10d
bars situated in the faces $x = \pm \frac{a}{2}$ and $\pm \frac{c}{2}$ are stressed.

The method used in stress analysis of the component cases is to find the changes in lengths in all bars in terms of $\Delta_1^y$, then knowing the cross-sectional areas of the bars express their stresses through $\Delta_1^y$, break up these stresses into the components along the coordinate axes and finally sum up these components at all nodes.

The symbols for the corner forces in the Action 2, now under consideration, consist of a capital letter $X, Y$ or $Z$, depending on the direction in which the force acts, and two subscripts, the first indicating the number of the corner which the force is applied, and the second, the number 2 to designate the Action 2 or the letters $A, B$ or $C$ - to indicate the component case of this action.

Symbols "S" with appropriate subscripts of the bar in question are used for designating the bar stresses. The same "S" symbols are used in several stress conditions although they represent different stress values.

**Condition A - (Fig. 11)**

Face $x = \pm \frac{a}{2}$ (Fig. 11c)

$$S_y = \frac{A_y \Delta_1^y E}{2b}$$

$$= \frac{c^2 a^2 (1-\mu) - \mu b^2 (c^2 + a^2)}{8(1+\mu)(1-2\mu)abc} E_1^y$$

(33)
Bars which do not work in Condition "A" are not shown.

**Fig. 11a.**

**Bar Forces in Condition A**

**Fig. 11b.**

**Fig. 11c.**

**Face $z = \pm \frac{C}{2}$**

**Fig. 11d.**

**Fig. 11e.**

**Nodal Concentrations**

**Eq. 11b.**
\[ S_{yz} = \frac{A_{yz} A_f y \cos \beta E}{(b^2 + c^2)^{1/2}} \]
\[ = \frac{m a (b^2 + c^2)^{1/2} \Delta_{1} y}{8(1+\mu)(1-2\mu)c} \]  
\( (34) \)

Faces \( z = + \frac{c}{2} \) and \( z = - \frac{c}{2} \) (Figs. 11d and e)

\[ S_{xy} = \frac{A_{xy} A_f y \cos \alpha E}{4(a^2 + b^2)^{1/2}} \]
\[ = \frac{m c (a^2 + b^2)^{1/2} \Delta_{1} y}{16(1+\mu)(1-2\mu)a} \]  
\( (35) \)

The combined stresses in different bars are shown in Fig. 11a.

The \( x, y, z \) components of the equilibrants of several bar stresses meeting at the nodes are presented in Fig. 11b. They are the contributions of the condition 1 to the distribution factors of Action 2. Thus

\[ X_{1A} = X_{2A} = X_{3A} = X_{4A} = - X_{5A} = - X_{6A} = - X_{7A} = - X_{8A} \]

\[ = S_{xy} \sin \alpha \]

\[ = \frac{\mu c E \Delta_{1} y}{16(1+\mu)(1-2\mu)} \]  
\( (36) \)

\[ Y_{1A} = - Y_{2A} = Y_{3A} = - Y_{4A} = S_{y} + S_{yz} \cos \beta + S_{xy} \cos \alpha \]
\[ Y_5A = Y_6A = Y_7A = - Y_8A = S_{xy} \cos \alpha \]

\[ \frac{\mu bcE \Delta y}{16(1+\mu)(1-2\mu)a} \]  

\[ Z_{1A} = Z_{2A} = - Z_{3A} = - Z_{4A} = S_{xy} \cos \alpha \]

\[ \frac{\mu aE \Delta y}{8(1+\mu)(1-2\mu)a} \]  

\[ Z_{5A} = Z_{6A} = Z_{7A} = Z_{8A} = 0 \]  

**Condition B (Fig. 12)**

The antisymmetry of stress condition on the face \( x = + \frac{a}{2} \) leaves the diagonals of this face unchanged in length and hence unstressed.

**Face** \( x = + \frac{a}{2} \) (Fig. 12c)

\[ S_y = \frac{A_yE \Delta y}{2b} \]

\[ = \frac{[c^2a^2(1-\mu)-\mu b^2 c^2-\mu a^2b^2]E \Delta y}{8(1+\mu)(1-2\mu)a bc} \]  

**Faces** \( z = + \frac{c}{2} \) and \( z = - \frac{c}{2} \) (Figs. 12d and e)

\[ S_{xy} = \frac{A_{xy}E \Delta y \cos \alpha}{4(a^2+b^2)^{1/2}} \]
Bars which do not work in Condition B are not shown.

Bar Forces in Condition B

Fig. 12b

Face $x = \frac{a}{2}$

Fig. 12c

Face $z = \frac{c}{2}$

Fig. 12d

Nodal Concentrations

Fig. 12e
Then the condition B components of the distribution factors are (Fig. 12b)

\[ X_{1B} = X_{2B} = -X_{3B} = -X_{4B} = -X_{5B} = -X_{6B} = X_{7B} = X_{8B} \]

\[ = S_{xy} \sin \alpha \]

\[ = \frac{\mu \epsilon A_{1}}{16(1+\mu)(1-2\mu)} \]  

Then the condition C (Fig. 13)

Comparison of Condition C with the basic strain conditions used for determination of the bar areas shows that the faces \( z = + \frac{c}{2} \) and \( x = + \frac{a}{2} \) deform in exactly the same manner as under the uniform shear conditions. For this reason the corner forces in Condition C on faces \( z = + \frac{c}{2} \)

\[ Y_{1B} = -Y_{2B} = -Y_{3B} = Y_{4B} = S_y + S_{xy} \cos \alpha \]

\[ = \frac{(2c^2a^2(1-\mu)-2\mu b^2a^2-\mu b^2c^2)E_A}{16(1+\mu)(1-2\mu)abc} \]

\[ Y_{5B} = -Y_{6B} = -Y_{7B} = Y_{8B} = S_{xy}\cos \alpha \]

\[ = \frac{\mu \epsilon c e_A}{16(1+\mu)(1-2\mu)} \]

\[ Z_{1B} = Z_{2B} = Z_{3B} = Z_{4B} = Z_{5B} = Z_{6B} = Z_{7B} = Z_{8B} = 0 \]
Bars which do not work in Condition C are not shown.

BAR FORCES IN CONDITION C

Fig. 13a

\[ p = \frac{E \Delta_y}{4(1+\mu)C} \]

\[ \frac{bac}{4} = \frac{AE \Delta_y}{16(1+\mu)} \]

\[ \frac{bac}{4} = \frac{AE \Delta_y}{16(1+\mu)} \]

Fig. 13c

\[ \frac{bac}{4} = \frac{AE \Delta_y}{16(1+\mu)} \]

\[ \frac{bac}{4} = \frac{AE \Delta_y}{16(1+\mu)} \]

Fig. 13e

Nodal Concentrations

Fig. 13a

Face \( z = \frac{C}{2} \)

\[ \frac{bac}{4} = \frac{CE \Delta_y}{16(1+\mu)} \]

\[ p = \frac{AE \Delta_y}{4(1+\mu)C} \]
\( x = + \frac{a}{2} \) may be copied, with some changes in notation, from Fig. 6a used earlier for determination of the cross-section areas of the bars involved in the shear condition.

Face \( z = + \frac{c}{2} \) (Fig. 13e), (apart from the direction of shear) deforms in the manner presented in Fig. 6a with the exception of the corner displacement being \( \frac{\Delta y}{2} \) instead of \( \Delta \) in Fig. 6a.

Deformation of the face \( x = + \frac{a}{2} \) (Fig. 13d) is similar except that the edge "a" in Fig. 6a becomes the edge "c" now.

Then the contribution of the condition \( C \) to the distribution factors of Action 2 are

\[
\begin{align*}
X_{1c} &= - X_{2c} = X_{5c} = - X_{6c} = \frac{cEA_1 y}{16(1+\mu)} \\
X_{3c} &= X_{4c} = X_{7c} = X_{8c} = 0 \\
\end{align*}
\]

\[
\begin{align*}
Y_{1c} &= Y_{2c} = \frac{bcEA_1 y}{16(1+\mu)a} + \frac{baEA_1 y}{16(1+\mu)c} \\
&= \frac{b(c^2+a^2)EA_1 y}{16(1+\mu)ac} \\
Y_{3c} &= Y_{4c} = - \frac{baEA_1 y}{16(1+\mu)c}
\end{align*}
\]
\[ Y_{5c} = Y_{6c} = \frac{bcE\Delta_1 Y}{16(1+u)a} \]  \hspace{1cm} (51)

\[ Y_{7c} = Y_{8c} = 0 \]  \hspace{1cm} (52)

\[ Z_{1c} = -Z_{2c} = Z_{3c} = -Z_{4c} = \frac{aE\Delta_1 Y}{16(1+u)} \]  \hspace{1cm} (53)

\[ Z_{5c} = Z_{6c} = Z_{7c} = Z_{8c} = 0 \]  \hspace{1cm} (54)

As explained earlier the nodal forces of the Action 2 may be found by superposition of the corresponding values in Conditions A, B and C. Thus for example

\[ Y_{12} = \frac{\{4c^2a^2(1-\mu)+(1-4\mu)b^2(c^2+a^2)\}E\Delta_1 Y}{16(1+\mu)(1-2\mu)abc} \]  \hspace{1cm} (55)

The complete set of the expressions for the Action 2 distribution factors is given in Table 3. The distribution factors of the Actions 1 and 3 (node 1 moves in x and z directions respectively) may be found directly by following a procedure similar to the one just concluded, but it is more convenient to determine them by analogy with the Action 2 in a way to be described now.

In determining the distribution factors of the Action 1 (Fig. 14c), the cell subjected to the Action 2 in Fig. 14a, is rotated through 90° about the axis z and thus brought into the position of Fig. 14b. It is further reversed symmetrically about the yz plane as shown in Fig. 14c. The
<table>
<thead>
<tr>
<th>$F_1^x$</th>
<th>$F_1^y$</th>
<th>$F_1^z$</th>
<th>$F_2^x$</th>
<th>$F_2^y$</th>
<th>$F_2^z$</th>
<th>$F_3^x$</th>
<th>$F_3^y$</th>
<th>$F_3^z$</th>
<th>$F_4^x$</th>
<th>$F_4^y$</th>
<th>$F_4^z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_{11}$</td>
<td>$X_{12}$</td>
<td>$X_{13}$</td>
<td>$X_{21}$</td>
<td>$X_{22}$</td>
<td>$X_{23}$</td>
<td>$X_{31}$</td>
<td>$X_{32}$</td>
<td>$X_{33}$</td>
<td>$X_{41}$</td>
<td>$X_{42}$</td>
<td>$X_{43}$</td>
</tr>
<tr>
<td>$Y_{11}$</td>
<td>$Y_{12}$</td>
<td>$Y_{13}$</td>
<td>$Y_{21}$</td>
<td>$Y_{22}$</td>
<td>$Y_{23}$</td>
<td>$Y_{31}$</td>
<td>$Y_{32}$</td>
<td>$Y_{33}$</td>
<td>$Y_{41}$</td>
<td>$Y_{42}$</td>
<td>$Y_{43}$</td>
</tr>
<tr>
<td>$Z_{11}$</td>
<td>$Z_{12}$</td>
<td>$Z_{13}$</td>
<td>$Z_{21}$</td>
<td>$Z_{22}$</td>
<td>$Z_{23}$</td>
<td>$Z_{31}$</td>
<td>$Z_{32}$</td>
<td>$Z_{33}$</td>
<td>$Z_{41}$</td>
<td>$Z_{42}$</td>
<td>$Z_{43}$</td>
</tr>
</tbody>
</table>

**TABLE - 5**

$$X_{ij} = \sum_{k=1}^{3} \sum_{l=1}^{3} \sum_{m=1}^{3} F_{k}^{x} \cdot F_{l}^{y} \cdot F_{m}^{z}$$
**TABLE-2**

**DISTRIBUTION FACTORS FOR ACTION I, WHEN $\Delta_1^x = 1$**

**BAR MODEL (TYPE 1)**

$$L = \frac{E}{(1+\mu)(1-2\mu)}$$

<table>
<thead>
<tr>
<th>FORCES IN X-DIRECTION</th>
<th>FORCES IN Y-DIRECTION</th>
<th>FORCES IN Z-DIRECTION</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_{ii} = \left{4c^2b(1-\mu) + (1-4\mu)Q^2(c^2+b^2)\right} \frac{L}{16abc}$</td>
<td>$y_{ii} = \frac{C L}{16}$</td>
<td>$z_{ii} = \frac{b L}{16}$</td>
</tr>
<tr>
<td>$x_{2i} = - (1-4\mu) \frac{ACL}{16b}$</td>
<td>$y_{2i} = (1-4\mu) \frac{CL}{16}$</td>
<td>$z_{2i} = 0$</td>
</tr>
<tr>
<td>$x_{3i} = - (1-4\mu) \frac{Qbl}{16c}$</td>
<td>$y_{3i} = 0$</td>
<td>$z_{3i} = (1-4\mu) \frac{bL}{16}$</td>
</tr>
<tr>
<td>$x_{4i} = 0$</td>
<td>$y_{4i} = 0$</td>
<td>$z_{4i} = 0$</td>
</tr>
<tr>
<td>$x_{5i} = - \left{4c^2b(1-\mu) - a^2(c^2+b^2)\right} \frac{L}{16abc}$</td>
<td>$y_{5i} = -(1-4\mu) \frac{CL}{16}$</td>
<td>$z_{5i} = -(1-4\mu) \frac{bL}{16}$</td>
</tr>
<tr>
<td>$x_{6i} = - \frac{ACL}{16b}$</td>
<td>$y_{6i} = - \frac{CL}{16}$</td>
<td>$z_{6i} = 0$</td>
</tr>
<tr>
<td>$x_{7i} = - \frac{Qbl}{16c}$</td>
<td>$y_{7i} = 0$</td>
<td>$z_{7i} = - \frac{bL}{16}$</td>
</tr>
<tr>
<td>$x_{8i} = 0$</td>
<td>$y_{8i} = 0$</td>
<td>$z_{8i} = 0$</td>
</tr>
</tbody>
</table>
### TABLE 3

**DISTRIBUTION FACTORS FOR ACTION-Z, WHEN Δ^y_1 = 1**

**BAR MODEL (TYPE 1)**

\[
L = \frac{E}{(1+\mu)(1-2\mu)}
\]

<table>
<thead>
<tr>
<th>FORCES IN X-DIRECTION</th>
<th>FORCES IN Y-DIRECTION</th>
<th>FORCES IN Z-DIRECTION</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x_{12} = \frac{CL}{16})</td>
<td>(Y_{12} = \left[4C^2a^2(1-\mu) + (1-4\mu)b^2(c^2+a^2)\right] \frac{L}{16abc})</td>
<td>(Z_{12} = \frac{QL}{16})</td>
</tr>
<tr>
<td>(x_{22} = -(1-4\mu) \frac{CL}{16})</td>
<td>(Y_{22} = -\left[4C^2a^2(1-\mu) - b^2(c^2+a^2)\right] \frac{L}{16abc})</td>
<td>(Z_{22} = -(1-4\mu) \frac{QL}{16})</td>
</tr>
<tr>
<td>(x_{32} = 0)</td>
<td>(Y_{32} = -(1-4\mu) \frac{abL}{16c})</td>
<td>(Z_{32} = (1-4\mu) \frac{QL}{16})</td>
</tr>
<tr>
<td>(x_{42} = 0)</td>
<td>(Y_{42} = -\frac{abL}{16c})</td>
<td>(Z_{42} = -\frac{QL}{16})</td>
</tr>
<tr>
<td>(x_{52} = (1-4\mu) \frac{CL}{16})</td>
<td>(Y_{52} = -(1-4\mu) \frac{bCL}{16a})</td>
<td>(Z_{52} = 0)</td>
</tr>
<tr>
<td>(x_{62} = -\frac{CL}{16})</td>
<td>(Y_{62} = -\frac{bCL}{16a})</td>
<td>(Z_{62} = 0)</td>
</tr>
<tr>
<td>(x_{72} = 0)</td>
<td>(Y_{72} = 0)</td>
<td>(Z_{72} = 0)</td>
</tr>
<tr>
<td>(x_{82} = 0)</td>
<td>(Y_{82} = 0)</td>
<td>(Z_{82} = 0)</td>
</tr>
</tbody>
</table>
### TABLE-4

**DISTRIBUTION FACTORS FOR ACTION 3, WHEN Δ1=1**

**BAR MODEL (TYPE 1)**

\[ L = \frac{E}{(1+μ)(1-2μ)} \]

<table>
<thead>
<tr>
<th>FORCES IN X-DIRECTION</th>
<th>FORCES IN Y-DIRECTION</th>
<th>FORCES IN Z-DIRECTION</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X_{13} = \frac{bL}{16} )</td>
<td>( Y_{13} = \frac{QL}{16} )</td>
<td>( Z_{13} = \left{ 4b^2a^2(l-μ) + (l-2μ)C^2(3b^2+2a^2) \right} \frac{L}{16abc} )</td>
</tr>
<tr>
<td>( X_{23} = 0 )</td>
<td>( Y_{23} = (l-4μ) \frac{QL}{16} )</td>
<td>( Z_{23} = -(l-4μ) \frac{QCL}{16b} )</td>
</tr>
<tr>
<td>( X_{33} = -(l-4μ) \frac{bL}{16} )</td>
<td>( Y_{33} = -(l-4μ) \frac{QL}{16} )</td>
<td>( Z_{33} = \left{ 4a^2b^2(l-μ) - C^2(b^2+a^2) \right} \frac{L}{16abc} )</td>
</tr>
<tr>
<td>( X_{43} = 0 )</td>
<td>( Y_{43} = -\frac{QL}{16} )</td>
<td>( Z_{43} = -\frac{QCL}{16b} )</td>
</tr>
<tr>
<td>( X_{53} = (l-4μ) \frac{bL}{16} )</td>
<td>( Y_{53} = 0 )</td>
<td>( Z_{53} = -(l-4μ) \frac{bCL}{16a} )</td>
</tr>
<tr>
<td>( X_{63} = 0 )</td>
<td>( Y_{63} = 0 )</td>
<td>( Z_{63} = 0 )</td>
</tr>
<tr>
<td>( X_{73} = -\frac{bL}{16} )</td>
<td>( Y_{73} = 0 )</td>
<td>( Z_{73} = -\frac{bCL}{16a} )</td>
</tr>
<tr>
<td>( X_{83} = 0 )</td>
<td>( Y_{83} = 0 )</td>
<td>( Z_{83} = 0 )</td>
</tr>
</tbody>
</table>
stressed condition presented here is analogous to Action 1 depicted in Fig. 14d, and the expressions for the terms of Action 1 may be written down from Fig. 14c making allowance for the numbers of the corresponding corners and the positions of the corresponding edges. Thus

\[
X_{11} = Y_{12} \text{(modified)} = \frac{f^4 c^2 b^2 (1-u) + (1-4u) a^2 (c^2 + b^2) E}{16(1+u)(1-2u)abc} \tag{56}
\]

The distribution factors of the Action 3 are obtained in a similar manner.

The distribution factors of the Actions 1, 2 and 3 given in Tables 2, 3 and 4 respectively comprise the first three columns of \(24 \times 24\) stiffness matrix of the cell. The terms of the other columns of the matrix are equal or equal and opposite in sign to some terms in the first three columns and may be obtained from the first three columns by appropriate reversals about some coordinate planes. Thus in order to find the distribution factors corresponding to \(A^2 y = 1\), i.e. the column 5 of the matrix the cell undergoing the Action 2 (Fig. 14a) is reversed symmetrically about the plane xz as shown in Fig. 15a. By reversing the direction of the \(y\) displacement in Fig. 15a the required distribution factors are obtained as shown in Fig. 15b.

The complete \(24 \times 24\) stiffness matrix of the cell is given in Table 5. As mentioned earlier the terms of its first three columns repeat themselves, sometimes with an opposite sign, in the other 21 columns. Making use of the expressions
of the individual terms it may be found that the matrix is symmetrical about the principal diagonal. This is a general condition characteristic of all bar cells.

3.6 Framework Cell (Type 2)

In (6) a three dimensional framework cell in the form of a rectangular parallelepiped is assumed. It is made up of 12 bars situated along the edges of the cell and 12 diagonal bars placed in its six faces. These bars are endowed with the cross sectional area parameters as in the cell of this thesis. In addition to this the edge bars are provided also with the flexural stiffnesses, i.e. the moments of inertia $I$ for bending in the planes of the faces. All edge bars oriented in the direction of the same axis possess equal $I$ for bending in the planes of both faces, to which the edge belongs. This makes 9 independent parameters in the cell, six extensional and three flexural which are determined from the 9 independent equations as stated earlier.

Expressions for different parameters along with the stiffness matrix of the cell are given in (6). The expressions for the terms of this matrix are stated in Tables 10, 11 and 12. They are naturally different from the ones developed in the present thesis and compiled in the Tables 6, 7, 8; but the matrix formed by these terms and described by the symbols of the relevant corner forces is identical with the one given in Table 5.

3.7 No Bar Cells

An idealized elastic solid consists of a number of rectangular cells joined to each other at the nodes, where the
### Table 6

**Distribution Factors for Action \( A_1 \) When \( \Delta_1^x = 1 \)**

**Bar Model (Type 2)**

\[ L = \frac{E}{(1+\mu)(1-2\mu)} \]

<table>
<thead>
<tr>
<th>Forces in X-direction</th>
<th>Forces in Y-direction</th>
<th>Forces in Z-direction</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_{11} = \left{ 8b^2c^2(l-M) + 3(1-4\mu)a^2(b+c)^2 \right} \frac{E}{32abc} )</td>
<td>( y_{11} = \frac{CL}{16} )</td>
<td>( z_{11} = \frac{bL}{16} )</td>
</tr>
<tr>
<td>( x_{21} = -3(1-4\mu) \frac{ACL}{32b} )</td>
<td>( y_{21} = \left( 1 - 4\mu \right) \frac{CL}{16} )</td>
<td>( z_{21} = 0 )</td>
</tr>
<tr>
<td>( x_{31} = -3(1-4\mu) \frac{Abl}{32c} )</td>
<td>( y_{31} = 0 )</td>
<td>( z_{31} = \left( 1 - 4\mu \right) \frac{bL}{16} )</td>
</tr>
<tr>
<td>( x_{41} = 0 )</td>
<td>( y_{41} = 0 )</td>
<td>( z_{41} = 0 )</td>
</tr>
<tr>
<td>( x_{51} = -\left{ 8b^2c^2(l-M) - (1+4\mu)a^2(b+c)^2 \right} \frac{E}{32abc} )</td>
<td>( y_{51} = -\left( 1 - 4\mu \right) \frac{CL}{16} )</td>
<td>( z_{51} = -\left( 1 - 4\mu \right) \frac{bL}{16} )</td>
</tr>
<tr>
<td>( x_{61} = -\left( 1 + 4\mu \right) \frac{ACL}{32b} )</td>
<td>( y_{61} = -\frac{CL}{16} )</td>
<td>( z_{61} = 0 )</td>
</tr>
<tr>
<td>( x_{71} = -\left( 1 + 4\mu \right) \frac{Abl}{32c} )</td>
<td>( y_{71} = 0 )</td>
<td>( z_{71} = -\frac{bL}{16} )</td>
</tr>
<tr>
<td>( x_{81} = 0 )</td>
<td>( y_{81} = 0 )</td>
<td>( z_{81} = 0 )</td>
</tr>
</tbody>
</table>
### TABLE- 7

**DISTRIBUTION FACTORS FOR ACTION-2, WHEN Δ'_y = 1**

**BAR MODEL (TYPE 2)**

\[ L = \frac{E}{(1+\mu)(1-2\mu)} \]

### FORCES IN X-DIRECTION

- \( x_{12} = \frac{CL}{16} \)
- \( x_{22} = -(1 - 4\mu) \frac{CL}{16} \)
- \( x_{32} = 0 \)
- \( x_{42} = 0 \)
- \( x_{52} = (1 - 4\mu) \frac{CL}{16} \)
- \( x_{62} = -\frac{CL}{16} \)
- \( x_{72} = 0 \)
- \( x_{82} = 0 \)

### FORCES IN Y-DIRECTION

- \( y_{12} = \left\{ \frac{8c^2a^2(1-\mu)+3(1-4\mu)b^2(c^2+a^2)}{32abc} \right\} L \)
- \( y_{22} = -\left\{ \frac{8c^2a^2(1-\mu)-(1+4\mu)b^2(c^2+a^2)}{32abc} \right\} L \)
- \( y_{32} = -3(1-4\mu) \frac{abL}{32c} \)
- \( y_{42} = -(1+4\mu) \frac{abL}{32c} \)
- \( y_{52} = -3(1-4\mu) \frac{bCL}{32a} \)
- \( y_{62} = -(1+4\mu) \frac{bCL}{32a} \)
- \( y_{72} = 0 \)
- \( y_{82} = 0 \)

### FORCES IN Z-DIRECTION

- \( z_{12} = \frac{QL}{16} \)
- \( z_{22} = -(1 - 4\mu) \frac{QL}{16} \)
- \( z_{32} = (1 - 4\mu) \frac{QL}{16} \)
- \( z_{42} = -\frac{QL}{16} \)
- \( z_{52} = 0 \)
- \( z_{62} = 0 \)
- \( z_{72} = 0 \)
- \( z_{82} = 0 \)
### TABLE 8

**Distribution Factors for Action 3, When $\Delta^z=1$**

**Bar Model (Type 2)**

\[ L = \frac{E}{(1+\mu)(1-2\mu)} \]

\[ \Delta^z = 1 \]

<table>
<thead>
<tr>
<th>FORCES IN X-DIRECTION</th>
<th>FORCES IN Y-DIRECTION</th>
<th>FORCES IN Z-DIRECTION</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_{i3} = \frac{bL}{16}$</td>
<td>$y_{i3} = \frac{QL}{16}$</td>
<td>$z_{i3} = \left{ \frac{8b^2a^2(1-\mu)+3(1-4\mu)c(b^2+a^2)}{32ab} \right} \frac{L}{32abc} $</td>
</tr>
<tr>
<td>$x_{23} = 0$</td>
<td>$y_{23} = (1-4\mu) \frac{QL}{16}$</td>
<td>$z_{23} = -3(1-4\mu) \frac{aCL}{32b}$</td>
</tr>
<tr>
<td>$x_{33} = -(1-4\mu) \frac{bL}{16}$</td>
<td>$y_{33} = -(1-4\mu) \frac{QL}{16}$</td>
<td>$z_{33} = -\left{ \frac{8b^2a^2(1-\mu)-(1+4\mu)c(b^2+a^2)}{32abc} \right} \frac{L}{32abc}$</td>
</tr>
<tr>
<td>$x_{43} = 0$</td>
<td>$y_{43} = -\frac{QL}{16}$</td>
<td>$z_{43} = -(1+4\mu) \frac{aCL}{32b}$</td>
</tr>
<tr>
<td>$x_{53} = (1-4\mu) \frac{bL}{16}$</td>
<td>$y_{53} = 0$</td>
<td>$z_{53} = -3(1-4\mu) \frac{bCL}{32a}$</td>
</tr>
<tr>
<td>$x_{63} = 0$</td>
<td>$y_{63} = 0$</td>
<td>$z_{63} = 0$</td>
</tr>
<tr>
<td>$x_{73} = -\frac{bL}{16}$</td>
<td>$y_{73} = 0$</td>
<td>$z_{73} = -(1+4\mu) \frac{bCL}{32a}$</td>
</tr>
<tr>
<td>$x_{83} = 0$</td>
<td>$y_{83} = 0$</td>
<td>$z_{83} = 0$</td>
</tr>
</tbody>
</table>
adjoining cells maintain continuity with regard to their displacements in the direction of the $x$, $y$ and $z$ axes. The analysis of the model is based on the derivation of the stiffness matrix of the cell for which purpose it is necessary to find the nodal forces corresponding to the single displacements of the nodes. This may be accomplished by assuming what is known as a displacement function or functions defining the state of displacements within each cell. The displacement function is a polynomial in $x, y, z$ containing some constants determined from the conditions of continuity at the nodes.

Consider a rectangular cell of size $a \times b \times c$ (Fig. 16a) with the origin of coordinates at the center of the cell. For such a cell Zienkiewicz sets up a condition with regard to the displacement functions that the continuity of the displacements on the interfaces of the adjacent cells must be preserved. How he attains this aim will now be described.

Each corner or node "i" of the rectangular prism has three components of displacements $u_i, v_i, w_i$ in $x, y, z$ directions respectively. Since there are eight such nodes the total number of the displacement components is $2^4$. Zienkiewicz chooses to define each particular type of displacement within the cell (such as $u$) by a separate suitable polynomial in $x, y, z$ with coefficients containing solely the $u$ displacement components of the eight nodes of the cell. The polynomial for $v$ and $w$ must also contain only the $v$ and $w$ nodal displacements respectively.

Using non-dimensional coordinates $x' = \frac{2x}{a}$,
Fig. 12a  A Rectangular prism Element
\[ y' = \frac{2y}{b} \quad \text{and} \quad z' = \frac{2z}{c}. \] Zienkiewicz\(^{(7)}\) assumes the following expression for \( u \).

\[
\begin{align*}
u &= \frac{1}{8}[(1+x')(1+y')(1+z')u_1 + (1+x')(1-y')(1+z')u_2 \\
&+ (1+x')(1+y')(1-z')u_3 + (1+x')(1-y')(1-z')u_4 \\
&+ (1-x')(1+y')(1+z')u_5 + (1-x')(1-y')(1+z')u_6 \\
&+ (1-x')(1+y')(1-z')u_7 + (1-x')(1-y')(1-z')u_8]
\end{align*}
\tag{58}
\]

The expression contains eight terms, each one representing a product of one of the corner displacements \( u_i \) and three binomials of the form \((1+X'), (1+Y')\) and \((1+Z')\). The signs inside the binomials of each term, either plus or minus, agree with the coordinates of the node associated with this term. Thus the term corresponding to the displacement \( u_6 \) of the node 6 with the coordinates \( x' = -1, y' = -1 \) and \( z' = 1 \) is \((1-x')(1-y')(1+z')u_6\).

The assumed displacement polynomials for \( v \) and \( w \) are identical with the expression (58) except that they contain \( v_i \) and \( w_i \) nodal displacements, instead of \( u_i \). Thus

\[
\begin{align*}
v &= \frac{1}{8}[(1+x')(1+y')(1+z')v_1 + (1+x')(1-y')(1+z')v_2 \\
&+ (1+x')(1+y')(1-z')v_3 + (1+x')(1-y')(1-z')v_4 \\
&+ (1-x')(1+y')(1+z')v_5 + (1-x')(1-y')(1+z')v_6 \\
&+ (1-x')(1+y')(1-z')v_7 + (1-x')(1-y')(1-z')v_8]
\end{align*}
\tag{59}
\]
\[ w = \frac{1}{8} [(1+x')(1+y')(1+z')w_1 + (1+x')(1-y')(1+z')w_2 \\
+ (1+x')(1+y')(1-z')w_3 + (1+x')(1-y')(1-z')w_4 \\
+ (1-x')(1+y')(1+z')w_5 + (1-x')(1-y')(1+z')w_6 \\
+ (1-x')(1+y')(1-z')w_7 + (1-x')(1-y')(1-z')w_8 ] \] (60)

With the displacement functions satisfying equations (58), (59) and (60) the continuity of displacements on the interface of adjacent cells is satisfied. Thus the common points on the interface of the two cells (A) and (B) set on top of each other in Fig. 16b, have equal y and z coordinates, while their x-coordinate referred to the axes associated with their respective cells are equal and opposite in sign. Of the eight terms in the displacements \( u \) on the interface of the cells four vanish, and four others are equal since the corner displacements \( u_5^A, u_6^A, u_7^A \) and \( u_8^A \) in the lower cell agree exactly with the displacements \( u_1^B, u_2^B, u_3^B \) and \( u_4^B \) respectively in the upper cell. Thus

\[ u_A = u_B = \frac{1}{4} [(1+y')(1+z')u_{5A} + (1-y')(1+z')u_{6A} + (1+y')(1-z')u_{7A} \\
+ (1-y')(1-z')u_{8A}] = \frac{1}{4} [(1+y')(1+z')u_{1B} + (1-y')(1+z')u_{2B} + (1+y')(1-z')u_{3B} \\
+ (1-y')(1-z')u_{4B}] \] (61)

This proves the preservation of continuity on the
interface between the cells A and B and a similar reasoning demonstrate the continuity on the other interfaces.

Once suitable expressions for the displacement functions are found it is possible to obtain the stiffness matrix of the element: two methods are available for such purpose; the direct statics method and the virtual work method. The results of the two do not always coincide. Both these methods will be presented here.

3.8 Derivations of Statics Type Stiffness Matrix

The derivation of the distribution factors will be demonstrated on the example of Action 2, in which corner 1 of the cell is given a unit displacement in Y direction while all other corners are held in their original positions.

Substituting in Equations (58),(59),(60) zero values for all corner displacements other than \( v_1 = 1 \) we get,

\[
\begin{align*}
  u &= 0 \\
  v &= \frac{1}{6}[(1+x')(1+y')(1+z')] \\
  &\quad \times \left[ 1 + \frac{2x}{a} + \frac{2y}{b} + \frac{2z}{c} + \frac{4xy}{ab} + \frac{4xz}{ac} + \frac{4yz}{bc} + \frac{8xyz}{abc} \right] \\
  w &= 0
\end{align*}
\]

(62) (63) (64)

Unit strains are obtained by differentiation of these expressions. Thus

\[
\varepsilon_x = \frac{du}{dx} = 0
\]

(65)
The expressions for the stresses are obtained from the strains by using the usual elastic stress-strain relationships:

\[
\sigma_x = \frac{(1-\mu)E}{(1+\mu)(1-2\mu)} \varepsilon_x + \frac{\mu E}{(1+\mu)(1-2\mu)} \varepsilon_y + \frac{\mu E}{(1+\mu)(1-2\mu)} \varepsilon_z
\] (71)

\[
\sigma_y = \frac{(1-\mu)E}{(1+\mu)(1-2\mu)} \varepsilon_x + \frac{(1-\mu)E}{(1+\mu)(1-2\mu)} \varepsilon_y + \frac{\mu E}{(1+\mu)(1-2\mu)} \varepsilon_z
\] (72)

\[
\sigma_z = \frac{\mu E}{(1+\mu)(1-2\mu)} \varepsilon_x + \frac{\mu E}{(1+\mu)(1-2\mu)} \varepsilon_y + \frac{(1-\mu)E}{(1+\mu)(1-2\mu)} \varepsilon_z
\] (73)

\[
\tau_{xy} = \frac{E}{2(1+\mu)} \gamma_{xy}
\] (74)

\[
\tau_{xz} = \frac{E}{2(1+\mu)} \gamma_{xz}
\] (75)

\[
\tau_{yz} = \frac{E}{2(1+\mu)} \gamma_{yz}
\] (76)

The normal stresses in these expressions are assumed positive if they represent tension and the shearing
stresses are positive if they act in the positive direction of
the axis on the positive face of the element or in the negative
direction on its negative face. Positive values of strains
agree with the positive value of stresses.

Let

\[ D_1 = \frac{(1-\mu)E}{(1+\mu)(1-2\mu)} \]  

\[ D_2 = \frac{\mu E}{(1+\mu)(1-2\mu)} \]  

\[ D_3 = \frac{E}{2(1+\mu)} \]  

Substituting the expressions for strains (Equations 65 to 70) in Equations 71 to 76 we get,

\[ \sigma_x = \frac{D_2}{4} \left[ \frac{1}{b} + \frac{2x}{ab} + \frac{2z}{bc} + \frac{4xz}{abc} \right] \]  

\[ \sigma_y = \frac{D_1}{4} \left[ \frac{1}{b} + \frac{2x}{ab} + \frac{2z}{bc} + \frac{4xz}{abc} \right] \]  

\[ \sigma_z = \frac{D_2}{4} \left[ \frac{1}{b} + \frac{2x}{ab} + \frac{2z}{bc} + \frac{4xz}{abc} \right] \]  

\[ \tau_{xy} = \frac{D_2}{4} \left[ \frac{1}{a} + \frac{2y}{ab} + \frac{2z}{ac} + \frac{4yz}{abc} \right] \]  

\[ \tau_{xz} = 0 \]  

\[ \tau_{yz} = \frac{D_2}{4} \left[ \frac{1}{c} + \frac{2x}{ac} + \frac{2y}{bc} + \frac{4yx}{abc} \right] \]  

The material of the cell when acted upon by the
stresses of Equations (80) to (85) is not in equilibrium unless
it is subjected to the body forces with components along the coordinate axes \( x, y, z \) per unit volume of the cell. These may be found by substitution of the expressions for stresses into the equations of equilibrium as follows:

\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X = 0 \tag{86}
\]

\[
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + Y = 0 \tag{87}
\]

\[
\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Z = 0 \tag{88}
\]

From these the body forces are

\[
X = - \frac{(D_2 + D_3)}{2ab} \left( 1 + \frac{2z}{c} \right) \tag{89}
\]

\[
Y = 0 \tag{90}
\]

\[
Z = - \frac{(D_2 + D_3)}{2bc} \left( 1 + \frac{2x}{a} \right) \tag{91}
\]

Equations (80) to (85) give the expressions for the stresses within the cell brought about by the unit displacement \( v_1 = 1 \). Substituting into these expressions the coordinates of the six faces, the expressions for the boundary stresses on these faces are obtained. These are presented in Table 9 and they form the basis for the calculation of distribution factors corresponding to the displacement \( v_1 = 1 \). Conversion of the
### Table 9

Stresses on the faces of the element due to $v_1 = 1$

<table>
<thead>
<tr>
<th>Face (1234)</th>
<th>$x = \pm \frac{a}{2}$</th>
<th>$\sigma_x = \frac{D_2}{2} \left[ \frac{1}{b} + \frac{2z}{bc} \right]$</th>
<th>$\tau_{xy} = \frac{D_3}{4} \left[ \frac{1}{a} + \frac{2y}{ab} + \frac{2z}{ac} + \frac{4yz}{abc} \right]$</th>
<th>$\tau_{xz} = 0$</th>
</tr>
</thead>
</table>

| Face (5678) | $x = -\frac{a}{2}$ | $\sigma_x = 0$ | $\tau_{xy} = \frac{D_3}{4} \left[ \frac{1}{a} + \frac{2y}{ab} + \frac{2z}{ac} + \frac{4yz}{abc} \right]$ | $\tau_{xz} = 0$ |

| Face (1256) | $z = \pm \frac{c}{2}$ | $\sigma_z = \frac{D_2}{2} \left[ \frac{1}{b} + \frac{2x}{ab} \right]$ | $\tau_{zy} = \frac{D_3}{4} \left[ \frac{1}{c} + \frac{2x}{ac} + \frac{2y}{bc} + \frac{4xy}{abc} \right]$ | $\tau_{zx} = 0$ |

| Face (3478) | $z = -\frac{c}{2}$ | $\sigma_z = 0$ | $\tau_{zy} = \frac{D_3}{4} \left[ \frac{1}{c} + \frac{2x}{ac} + \frac{2y}{bc} + \frac{4xy}{abc} \right]$ | $\tau_{zx} = 0$ |

Continued...
Stresses on the faces of the element due to $v_1 = 1$

### Table 9 (Continued)

| Face (1357) | $y = + \frac{b}{2}$ | $\sigma_y = \frac{D_1}{4} \left[ \frac{1}{b} + \frac{2x}{ab} + \frac{2z}{bc} + \frac{4xz}{abc} \right]$ |  
|            |                    | $\tau_{yx} = \frac{D_3}{2} \left[ \frac{1}{a} + \frac{2z}{ac} \right]$ | (104)  
|            |                    | $\tau_{yz} = \frac{D_3}{2} \left[ \frac{1}{c} + \frac{2x}{ac} \right]$ | (105)  
| Face (2468) | $y = - \frac{b}{2}$ | $\sigma_y = \frac{D_1}{4} \left[ \frac{1}{b} + \frac{2x}{ab} + \frac{2z}{bc} + \frac{4xz}{abc} \right]$ | (106)  
|            |                    | $\tau_{yx} = 0$ | (107)  
|            |                    | $\tau_{yz} = 0$ | (108)  
|
face stresses into the corner forces at the adjacent corners is carried out in accordance with the laws of statics and consideration of symmetry, separately for every term of the stress polynomials. With some of these terms the conversion is quite definite. With others it is somewhat ambiguous and it requires some further assumptions and exercise of judgment.

The two terms composing the stress $\sigma_x$ (Equation 92) are disposed of in the only way consistent with the above enunciated principles (Fig. 17b and c). The constant component produces equal corner forces $F_1 = \frac{D_2}{2b} \times \frac{bc}{4} = \frac{D_2 c}{8}$. The corner contribution of the linear in $z$ component of $\sigma_x$ are found by the moment equation (Fig. 17c) $F_2 = \frac{1}{C} \left( \frac{D_2}{2b} \right) \left( \frac{b c^2}{2} \right) \left( \frac{c^2}{3} \right) = \frac{D_2 c}{24}$.

Corner concentrations resulting from similar normal stresses on the other five faces of the cell are found in the same way.

The situation is however different with regard to the normal stresses corresponding to the quadratic terms like the term with $xz$ in the stress $\sigma_y$, illustrated in Fig. 17g. These stresses numerically equal on both sides of the coordinate axes are positive in the first and third quadrants and negative in the two others. The total statical effect of these stresses is obviously equal to zero and so the equations of statics and the conditions of symmetry are insufficient to determine the equivalent corner forces in the distribution factors. To assume them all zero would be consistent with statics and so would be any other assumption as long as the forces at the
corners 1 and 7 are equal and opposite in sign to the ones at the corners 5 and 3. The second alternative seems more reasonable. The magnitude of these forces will be assumed such that its moment either about the ox or oz axis equals the moment of the stresses belonging to its quadrant. Then

\[ F_3 = \frac{2}{3} \int_0^{c/2} \int_0^{a/2} \frac{1}{abc} dxdz = \frac{Dc}{9b} \]

This \( F_3 \) comes out equal to \( \frac{2}{3} \) of the stress resultant in one quadrant. However assuming \( F_3 \) to be \( \frac{4}{9} \) of the resultant (i.e. \( F_3 = \frac{4}{9} \frac{Dc}{a} \frac{c}{4} \frac{a}{4} = \frac{Dc}{144b} \)), the stiffness matrix of the element coincides exactly with the one obtained by the method of virtual work, to be derived in the next section.

Turning now to the shear stresses on the face \( x = + \frac{a}{2} \), the uniform shearing stress \( \frac{D_bc}{4a} \) results in equal corner concentrations \( \frac{D_{bc}}{16a} \). It is also legitimate to treat the non-uniform shear components of \( \tau_{xy} \) varying with \( z \) (Fig. 18a) by the moment equation. Then each corner force coming from this source is

\[ F = \frac{1}{c(\frac{D_3}{4a} \times \frac{c}{4} \times \frac{b}{2} \times \frac{2c}{3}c)} = \frac{D_{bc}}{48a} \]

Strictly speaking the moment equation is inapplicable for the determination of the corner forces corresponding to the
components $\tau_{xy}$ shearing stresses $\frac{D_3}{2}(\frac{V}{ab} + \frac{2yz}{abc})$ (Figs. 18b and 18c) whose total statical effect on the whole face is zero. However in the absence of a rigorous alternative method the corner forces will still be determined by moments treating the shearing stresses in the manner of normal stresses discussed earlier. In other words each corner will be provided with a shearing force acting in the same direction as the stresses in that particular quadrant and having the same moment about oz axis as the stresses in that quadrant (Figs. 18b and 18c) visualising them as acting normally to the face $x = \frac{a}{2}$. Thus the contributions to the distribution factors of the shearing stresses considered are in Fig. 18b

$$F = \frac{1}{4a} \frac{D_3}{c} \cdot \frac{b}{4} \cdot \frac{c}{2} \cdot \frac{2}{3} = \frac{D_3bc}{48a}$$

in Fig. 18c

$$F = \frac{2}{b} \int_{0}^{c/2} \int_{0}^{b/2} \frac{D_3yz}{abc} ydydz = \frac{D_3bc}{96a}$$

The whole set of corner forces resulting from the stresses acting on the six faces of the cell corresponding to the corner displacements $v_1 = 1$ is summarized in Fig. 19. It may be observed that the corner forces acting on the positive and negative faces of the cell are opposite in sign, at the symmetrically opposite corners.
It must not be forgotten that body forces expressed by Equations (89) to (91) provide additional contributions to the distribution factors of Action 2. These consist of forces acting in x direction and varying only with z as shown in Fig. 18d, and the ones parallel to z direction and varying only with x (Fig. 18e). The equivalent corner forces are easily found by statics as follows.

The total body force in x direction is

\[-\frac{(D_2+D_3)}{ab} \cdot \frac{1}{2}c_{ab} = -\frac{(D_2+D_3)c}{2} \]

Two thirds of this force are carried equally by the nodes on the face \( z = + \frac{c}{2} \) and the one third on the opposite \( z \) face. Then the \( x \) corner forces are:

at each node of the face \( z = + \frac{c}{2}, \ X = \frac{-(D_2+D_3)c}{12} \) and the face \( z = - \frac{c}{2}, \ X = \frac{(D_2+D_3)c}{24} \). Similarly at the four corners of the face \( x = + \frac{a}{2} \) the \( z \) force at each corner is

\[-\frac{(D_2+D_3)a}{12} \]

and at the corners of the face \( x = - \frac{a}{2}, -\frac{(D_2+D_3)a}{24} \).

These additional components of the distribution factors are stated in Fig. 19 encircled.

The total expressions for the Action 2 distribution factors are given in Table 11. The distribution factors of the Actions 1 and 3 found by analogy with Action 2 distribution factors as explained in case of bar model are given in Tables 10 and 12.

Tables 10, 11 and 12 give the first three columns of \( 24 \times 24 \) stiffness matrix of the cell. The terms of the other
**TABLE-10**

**DISTRIBUTION FACTORS FOR ACTION WHEN $\Delta_i = 1$**

**NO-BAR MODEL (STATICS TYPE)**

\[
L = \frac{E}{(1+\mu)(1-2\mu)}
\]

**FORCES IN X-DIRECTION**

\[
X_{11} =\left\{22(1-\mu)b^3c^2+11(1-2\mu)a^2(b^3+c^3)\right\} \frac{L}{192abc}
\]

\[
Y_{11} = \frac{CL}{24}
\]

\[
Z_{11} = \frac{bl}{24}
\]

**FORCES IN Y-DIRECTION**

\[
X_{21} =\left\{10(1-\mu)b^3c^2-(1-2\mu)a^2(11b^2-5b^3)\right\} \frac{L}{192abc}
\]

\[
Y_{21} = (1-4\mu) \frac{CL}{24}
\]

\[
Z_{21} = \frac{bl}{48}
\]

**FORCES IN Z-DIRECTION**

\[
X_{31} =\left\{10(1-\mu)b^3c^2-(1-2\mu)a^2(11b^2-5c^3)\right\} \frac{L}{192abc}
\]

\[
Y_{31} = \frac{CL}{48}
\]

\[
Z_{31} = (1-4\mu) \frac{bl}{24}
\]

\[
X_{41} =\left\{6(1-\mu)b^3c^2-5(1-2\mu)a^2(b^3+c^3)\right\} \frac{L}{192abc}
\]

\[
Y_{41} = (1-4\mu) \frac{CL}{48}
\]

\[
Z_{41} = (1-4\mu) \frac{bl}{48}
\]

\[
X_{51} =-\left\{22(1-\mu)b^3c^2-5(1-2\mu)a^2(b^3+c^3)\right\} \frac{L}{192abc}
\]

\[
Y_{51} = -(1-4\mu) \frac{CL}{24}
\]

\[
Z_{51} = -(1-4\mu) \frac{bl}{24}
\]

\[
X_{61} =-\left\{10(1-\mu)b^3c^2+(1-2\mu)a^2(5c^3-3b^3)\right\} \frac{L}{192abc}
\]

\[
Y_{61} = -\frac{CL}{24}
\]

\[
Z_{61} = -(1-4\mu) \frac{bl}{48}
\]

\[
X_{71} =-\left\{10(1-\mu)b^3c^2+(1-2\mu)a^2(5b^3-3c^3)\right\} \frac{L}{192abc}
\]

\[
Y_{71} = -(1-4\mu) \frac{CL}{48}
\]

\[
Z_{71} = -\frac{bl}{24}
\]

\[
X_{81} =-\left\{6(1-\mu)b^3c^2+3(1-2\mu)a^2(b^3+c^3)\right\} \frac{L}{192abc}
\]

\[
Y_{81} = -\frac{CL}{48}
\]

\[
Z_{81} = -\frac{bl}{48}
\]
### Table II

**Distribution Factors for Action When \( \Delta_i^y = 1 \)**

**No-Bar Model (Statics Type)**

\[
L = \frac{E}{(1+\mu)(1-2\mu)}
\]

**Forces in X-Direction**

<table>
<thead>
<tr>
<th>Force Index</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>( \frac{CL}{24} )</td>
</tr>
<tr>
<td>22</td>
<td>(- (1-4\mu) \frac{CL}{24} )</td>
</tr>
<tr>
<td>32</td>
<td>( \frac{CL}{48} )</td>
</tr>
<tr>
<td>42</td>
<td>(- (1-4\mu) \frac{CL}{48} )</td>
</tr>
<tr>
<td>52</td>
<td>( (1-4\mu) \frac{CL}{24} )</td>
</tr>
<tr>
<td>62</td>
<td>(- \frac{CL}{24} )</td>
</tr>
<tr>
<td>72</td>
<td>( (1-4\mu) \frac{CL}{48} )</td>
</tr>
<tr>
<td>82</td>
<td>(- \frac{CL}{48} )</td>
</tr>
</tbody>
</table>

**Forces in Y-Direction**

<table>
<thead>
<tr>
<th>Force Index</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>( \left{ \frac{22(1-\mu)C^2a^2+11(1-2\mu)b^2(C^2+a^2)}{192abc} \right} \frac{L}{192abc} )</td>
</tr>
<tr>
<td>22</td>
<td>(- \left{ \frac{22(1-\mu)C^2a^2-5(1-2\mu)b^2(C^2+a^2)}{192abc} \right} \frac{L}{192abc} )</td>
</tr>
<tr>
<td>32</td>
<td>( \left{ \frac{10(1-\mu)C^2a^2-(1-2\mu)b^2(5a^2-3c^2)}{192abc} \right} \frac{L}{192abc} )</td>
</tr>
<tr>
<td>42</td>
<td>(- \left{ \frac{10(1-\mu)C^2a^2+(1-2\mu)b^2(5a^2-3c^2)}{192abc} \right} \frac{L}{192abc} )</td>
</tr>
<tr>
<td>52</td>
<td>( \left{ \frac{10(1-\mu)C^2a^2-(1-2\mu)b^2(11c^2-5a^2)}{192abc} \right} \frac{L}{192abc} )</td>
</tr>
<tr>
<td>62</td>
<td>(- \left{ \frac{10(1-\mu)C^2a^2+(1-2\mu)b^2(5c^2-3a^2)}{192abc} \right} \frac{L}{192abc} )</td>
</tr>
<tr>
<td>72</td>
<td>( \left{ \frac{6(1-\mu)C^2a^2-5(1-2\mu)b^2(C^2+a^2)}{192abc} \right} \frac{L}{192abc} )</td>
</tr>
<tr>
<td>82</td>
<td>(- \left{ \frac{6(1-\mu)C^2a^2+3(1-2\mu)b^2(C^2+a^2)}{192abc} \right} \frac{L}{192abc} )</td>
</tr>
</tbody>
</table>

**Forces in Z-Direction**

<table>
<thead>
<tr>
<th>Force Index</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>( \frac{QL}{24} )</td>
</tr>
<tr>
<td>22</td>
<td>(- (1-4\mu) \frac{QL}{24} )</td>
</tr>
<tr>
<td>32</td>
<td>( (1-4\mu) \frac{QL}{24} )</td>
</tr>
<tr>
<td>42</td>
<td>(- \frac{QL}{24} )</td>
</tr>
<tr>
<td>52</td>
<td>( \frac{QL}{48} )</td>
</tr>
<tr>
<td>62</td>
<td>(- (1-4\mu) \frac{QL}{48} )</td>
</tr>
<tr>
<td>72</td>
<td>( (1-4\mu) \frac{QL}{48} )</td>
</tr>
<tr>
<td>82</td>
<td>(- \frac{QL}{48} )</td>
</tr>
</tbody>
</table>
### Table 12

Distribution Factors for Action \( z \) When \( \Delta z = 1 \)

**Non-Bar Model (Statics Type)**

\[
L = \frac{E}{(1+\mu)(1-2\mu)}
\]

<table>
<thead>
<tr>
<th>Forces in X-Direction</th>
<th>Forces in Y-Direction</th>
<th>Forces in Z-Direction</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_{13} = \frac{bl}{24} )</td>
<td>( y_{13} = \frac{ql}{24} )</td>
<td>( z_{13} = \left{ \frac{22(1-\mu)b^3a^2 + 11(1-2\mu)c^2(b^2+q^2)}{192abc} \right} L )</td>
</tr>
<tr>
<td>( x_{23} = \frac{bl}{48} )</td>
<td>( y_{23} = (1-4\mu) \frac{ql}{24} )</td>
<td>( z_{23} = \left{ \frac{10(1-\mu)b^3a^2 - (1-2\mu)c^2(11a^2-5b^2)}{192abc} \right} L )</td>
</tr>
<tr>
<td>( x_{33} = -(1-4\mu) \frac{bl}{24} )</td>
<td>( y_{33} = -(1-4\mu) \frac{ql}{24} )</td>
<td>( z_{33} = \left{ \frac{22(1-\mu)b^3a^2 - 5(1-2\mu)c^2(b^2+q^2)}{192abc} \right} L )</td>
</tr>
<tr>
<td>( x_{43} = -(1-4\mu) \frac{bl}{48} )</td>
<td>( y_{43} = -\frac{ql}{24} )</td>
<td>( z_{43} = \left{ \frac{10(1-\mu)b^3a^2 + (1-2\mu)c^2(5a^2-3b^2)}{192abc} \right} L )</td>
</tr>
<tr>
<td>( x_{53} = (1-4\mu) \frac{bl}{24} )</td>
<td>( y_{53} = \frac{ql}{48} )</td>
<td>( z_{53} = \left{ \frac{10(1-\mu)b^3a^2 - (1-2\mu)c^2(11b^2-5a^2)}{192abc} \right} L )</td>
</tr>
<tr>
<td>( x_{63} = (1-4\mu) \frac{bl}{48} )</td>
<td>( y_{63} = (1-4\mu) \frac{ql}{48} )</td>
<td>( z_{63} = \left{ \frac{6(1-\mu)b^3a^2 - 5(1-2\mu)c^2(b^2+q^2)}{192abc} \right} L )</td>
</tr>
<tr>
<td>( x_{73} = -\frac{bl}{24} )</td>
<td>( y_{73} = -(1-4\mu) \frac{ql}{48} )</td>
<td>( z_{73} = \left{ \frac{10(1-\mu)b^3a^2 + (1-2\mu)c^2(5b^2-3a^2)}{192abc} \right} L )</td>
</tr>
<tr>
<td>( x_{93} = -\frac{bl}{48} )</td>
<td>( y_{83} = -\frac{ql}{48} )</td>
<td>( z_{83} = \left{ \frac{6(1-\mu)b^3a^2 + 3(1-2\mu)c^2(b^2+q^2)}{192abc} \right} L )</td>
</tr>
</tbody>
</table>
columns expressible through the ones in the first three and the whole matrix is given in Table 5.

3.9 Derivation of Energy Type Stiffness Matrix

Consider an elastic solid in the form of a rectangular cell subjected to a combination of stress conditions resulting in $2^4$ displacements $u_1, v_1, w_1, u_2, v_2, w_2, \ldots$ etc. described by the Equations (58), (59), (60). The body so deformed is held in equilibrium by the appropriate boundary stresses $\sigma$ perpendicular to the boundary and shear stresses $\tau$ parallel to the boundary, along with the body forces applied throughout the volume of the cell. For such a system let

$\sigma_A =$ boundary stresses normal to the surface

$\tau_{AB} =$ boundary stresses parallel to the surface in one direction

$\tau_{AC} =$ boundary stresses parallel to the surface in a perpendicular direction

$X, Y, Z =$ components of the body forces per unit volume at any point $(x, y, z)$ inside the cell

$\lambda_1, \lambda_{AB}, \lambda_{AC} =$ displacements on the boundary in the direction of the stresses $\sigma_A$, $\tau_{AB}$ and $\tau_{AC}$ respectively.

$\lambda_X, \lambda_Y, \lambda_Z =$ components of displacements at the point $(x, y, z)$ inside the cell

Let $F_{x1}, F_{y1}, F_{z1}, \ldots, F_{x8}, F_{y8}, F_{z8}$ be the corner forces statically equivalent to the above boundary stresses and body forces.

Consider the cell acted upon by the boundary stresses and the body forces corresponding to $2^4$ corner displacements
u_1, v_1, w_1 \ldots u_8, v_8, w_8 \) and separately by the corner forces statically equivalent to the boundary stresses and body forces. The cell is in equilibrium under both these sets of forces acting independently.

The corner forces may be determined by applying virtual displacements corresponding to different corner displacements one at a time. Let the cell be given a virtual displacement \( \text{du}_1 \) in \( x \) direction at corner 1. The virtual work of corner forces is done only by \( F_{1x}^x \) and it is equal to \( F_{1x}^x \text{du}_1 \).

Similarly the virtual work of deformation of the solid by the boundary stresses and body forces under a virtual displacement \( \text{du}_1 \) is given by

\[
\int_A \left( \frac{\partial \lambda_1^A}{\partial u_i} \text{du}_1 + \frac{\partial \lambda_{11}^{AB}}{\partial u_i} \text{du}_1 + \frac{\partial \lambda_{11}^{AC}}{\partial u_i} \text{du}_1 \right) \\text{d}A
+ \int_V \left( x \frac{\partial \lambda_{1z}}{\partial u_i} \text{du}_1 + y \frac{\partial \lambda_{1y}}{\partial u_i} \text{du}_1 + z \frac{\partial \lambda_{1z}}{\partial u_i} \text{du}_1 \right) \\text{d}V
\]

The symbols \( \lambda_1^A, \lambda_{11}^{AB}, \lambda_{11}^{AC}, (u_1), (u_1) \) and \( \lambda_1^A \) with superscripts \( A(u_1), AB(u_1), AC(u_1), u_1 \ldots \) etc. represent the displacements on the respective boundary and inside the cell caused by the single displacement \( u_1 \) of node 1.

Assuming that the energies of deformation of the cell produced by the boundary stresses together with the work of
the body forces on the one hand and corner forces on the other are equal we find after cancelling $\text{du}_1$

$$F_i^x = \int\int_A \left( \sigma_A \frac{\partial \lambda_A(u)}{\partial u} + \tau_{AB} \frac{\partial \lambda_{AB}(u)}{\partial u} + \tau_{AC} \frac{\partial \lambda_{AC}(u)}{\partial u} \right) dA$$

$$+ \int\int\int_V \left( X \frac{\partial \lambda_x(u)}{\partial u} + Y \frac{\partial \lambda_y(u)}{\partial u} + Z \frac{\partial \lambda_z(u)}{\partial u} \right) dV$$  \hspace{1cm} (113)

Since all displacements in the cell including the displacements along the boundary are proportional to the corner displacements Equation (113) reduces to

$$F_i^x = \int\int_A \left( \sigma_A \lambda_A(u=1) + \tau_{AB} \lambda_{AB}(u=1) + \tau_{AC} \lambda_{AC}(u=1) \right) dA$$

$$+ \int\int\int_V \left( X \lambda_x(u=1) + Y \lambda_y(u=1) + Z \lambda_z(u=1) \right) dV$$  \hspace{1cm} (114)

The stresses and body forces present under the integral signs in Equation (114) are produced by all $2^4$ independent corner displacements of the cell. If instead of all these displacements single unit displacements are used one at a time for calculation of stresses and body forces the equation will give expressions for the $2^4$ terms of the first row of the stiffness matrix of the cell. The symbols to be used for these distribution factors in row 1, as well as the ones in the other rows will consist of a capital letter $X,Y$ or $Z$ depending on the direction in which the force acts, with a subscript
indicating the number of the corner at which the force is applied, and the accompanying brackets, the letter and the number subscript inside which show the nature of the displacement (direction and node number) producing the distribution factors. Thus $x_n(v_m)$ is the symbol for the force in $x$ direction at the node $n$ due to the unit displacement of the corner "m" in $y$ direction. The simpler type symbols used earlier for the designation of terms in the statics stiffness matrix can not be employed here because they do not convey the information as to the number of the corner whose displacement produces the corner forces in question. In the statics type matrix the moved corner was always number one but in the energy matrix a possibility of the movement of any corner must be contemplated.

Close attention should be given to the signs of the stresses, strains and the body-forces while making use of Equation (114).

Although the basic Equation (114) is suitable for determination of all terms of the stiffness matrix, it is more convenient to apply it only for the first three columns and to find the rest of the factors by cyclic substitution and considerations of symmetry, as was already done in case of statics matrix.

The application of Equation (114) will be illustrated here on three examples. Table 13 gives boundary stresses and body forces due to $v_1 = 1$ obtained as explained before. Tables 14, 15 and 16 give boundary displacements and displace-
### Table 13

<table>
<thead>
<tr>
<th>Face $x = + \frac{a}{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_x = \frac{D_2}{2b}(1+\frac{2z}{c})$</td>
</tr>
<tr>
<td>$\tau_{xy} = \frac{D_2}{4a}(1+\frac{2y}{b})(1+\frac{2z}{c})$</td>
</tr>
<tr>
<td>$\tau_{xz} = 0$</td>
</tr>
<tr>
<td>$\lambda_1 = 0$</td>
</tr>
<tr>
<td>$\lambda_{11} = \frac{1}{4}(1+\frac{2y}{b})(1+\frac{2z}{c})$</td>
</tr>
<tr>
<td>$\lambda_{11} = 0$</td>
</tr>
</tbody>
</table>

### Table 14

<table>
<thead>
<tr>
<th>Face $x = + \frac{a}{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_x = \frac{D_2}{2b}(1+\frac{2z}{c})$</td>
</tr>
<tr>
<td>$\tau_{xy} = \frac{D_2}{4a}(1+\frac{2y}{b})(1+\frac{2z}{c})$</td>
</tr>
<tr>
<td>$\tau_{xz} = 0$</td>
</tr>
<tr>
<td>$\lambda_1 = 0$</td>
</tr>
<tr>
<td>$\lambda_{11} = \frac{1}{4}(1+\frac{2y}{b})(1+\frac{2z}{c})$</td>
</tr>
<tr>
<td>$\lambda_{11} = 0$</td>
</tr>
</tbody>
</table>

### Table 15

<table>
<thead>
<tr>
<th>Face $x = - \frac{a}{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_x = 0$</td>
</tr>
<tr>
<td>$\tau_{xy} = \frac{D_2}{4a}(1+\frac{2y}{b})(1+\frac{2z}{c})$</td>
</tr>
<tr>
<td>$\tau_{xz} = 0$</td>
</tr>
<tr>
<td>$\lambda_1 = 0$</td>
</tr>
<tr>
<td>$\lambda_{11} = \frac{1}{4}(1+\frac{2y}{b})(1+\frac{2z}{c})$</td>
</tr>
<tr>
<td>$\lambda_{11} = 0$</td>
</tr>
</tbody>
</table>

### Table 16

<table>
<thead>
<tr>
<th>Face $x = - \frac{a}{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_x = 0$</td>
</tr>
<tr>
<td>$\tau_{xy} = \frac{D_2}{4a}(1+\frac{2y}{b})(1+\frac{2z}{c})$</td>
</tr>
<tr>
<td>$\tau_{xz} = 0$</td>
</tr>
<tr>
<td>$\lambda_1 = 0$</td>
</tr>
<tr>
<td>$\lambda_{11} = \frac{1}{4}(1+\frac{2y}{b})(1+\frac{2z}{c})$</td>
</tr>
<tr>
<td>$\lambda_{11} = 0$</td>
</tr>
</tbody>
</table>

Continued.....
<table>
<thead>
<tr>
<th>Table 13</th>
<th>Table 14</th>
<th>Table 15</th>
<th>Table 16</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Boundary stresses &amp; body forces due to $V_1=1$</strong></td>
<td><strong>Boundary and inside displacements due to $V_1=1$</strong></td>
<td><strong>Boundary and inside displacements due to $U_5=1$</strong></td>
<td><strong>Boundary and inside displacements due to $W_8=1$</strong></td>
</tr>
<tr>
<td><strong>Face $y = + \frac{b}{2}$</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma_y = \frac{D_1}{4b}(1+\frac{2x}{a})(1+\frac{2z}{c})$</td>
<td>$\lambda_y^1 = \frac{1}{4}(1+\frac{2x}{a})(1+\frac{2z}{c})$</td>
<td>$\lambda_y^1 = 0$</td>
<td>$\lambda_y^1 = 0$</td>
</tr>
<tr>
<td>$\tau_{yx} = \frac{D_3}{2a}(1+\frac{2z}{c})$</td>
<td>$\lambda_{11}^y = 0$</td>
<td>$\lambda_{11}^y = \frac{1}{4}(1-\frac{2x}{a})(1+\frac{2z}{c})$</td>
<td>$\lambda_{11}^y = 0$</td>
</tr>
<tr>
<td>$\tau_{yz} = \frac{D_3}{2c}(1+\frac{2x}{a})$</td>
<td>$\lambda_{11}^y = 0$</td>
<td>$\lambda_{11}^y = 0$</td>
<td>$\lambda_{11}^y = 0$</td>
</tr>
<tr>
<td><strong>Face $y = - \frac{b}{2}$</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma_y = \frac{D_1}{4b}(1+\frac{2x}{a})(1+\frac{2z}{c})$</td>
<td>$\lambda_y^1 = 0$</td>
<td>$\lambda_y^1 = 0$</td>
<td>$\lambda_y^1 = 0$</td>
</tr>
<tr>
<td>$\tau_{yx} = 0$</td>
<td>$\lambda_{11}^y = 0$</td>
<td>$\lambda_{11}^y = 0$</td>
<td>$\lambda_{11}^y = 0$</td>
</tr>
<tr>
<td>$\tau_{yz} = 0$</td>
<td>$\lambda_{11}^y = 0$</td>
<td>$\lambda_{11}^y = 0$</td>
<td>$\lambda_{11}^y = \frac{1}{4}(1-\frac{2x}{a})(1-\frac{2z}{c})$</td>
</tr>
</tbody>
</table>

Continued.....
<table>
<thead>
<tr>
<th>Table 13</th>
<th>Table 14</th>
<th>Table 15</th>
<th>Table 16</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boundary stresses for body forces due to $V_1=1$</td>
<td>Boundary and inside displacements due to $V_1=1$</td>
<td>Boundary and inside displacements due to $U_5=1$</td>
<td>Boundary and inside displacements due to $W_8=1$</td>
</tr>
<tr>
<td>Face $z = + \frac{c}{2}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma_z = \frac{D_2}{2b}(1+\frac{2x}{a})$</td>
<td>$\lambda_1 = 0$</td>
<td>$\lambda_1 = 0$</td>
<td>$\lambda_1 = 0$</td>
</tr>
<tr>
<td>$\tau_{zy} = \frac{D_3}{4c}(1+\frac{2x}{a})(1+\frac{2y}{b})$</td>
<td>$\lambda_{11} = \frac{1}{4}(1+\frac{2x}{a})(1+\frac{2y}{b})$</td>
<td>$\lambda_{11} = 0$</td>
<td>$\lambda_{11} = 0$</td>
</tr>
<tr>
<td>$\tau_{zx} = 0$</td>
<td>$\lambda_{11} = 0$</td>
<td>$\lambda_{11} = 0$</td>
<td>$\lambda_{11} = 0$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Face $z = - \frac{c}{2}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma_z = 0$</td>
<td>$\lambda_1 = 0$</td>
<td>$\lambda_1 = 0$</td>
<td>$\lambda_1 = 0$</td>
</tr>
<tr>
<td>$\tau_{zy} = \frac{D_3}{4c}(1+\frac{2x}{a})(1+\frac{2y}{b})$</td>
<td>$\lambda_{11} = 0$</td>
<td>$\lambda_{11} = 0$</td>
<td>$\lambda_{11} = 0$</td>
</tr>
<tr>
<td>$\tau_{zx} = 0$</td>
<td>$\lambda_{11} = 0$</td>
<td>$\lambda_{11} = 0$</td>
<td>$\lambda_{11} = 0$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Continued.....</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Table 13</td>
<td>Table 14</td>
<td>Table 15</td>
<td>Table 16</td>
</tr>
<tr>
<td>----------</td>
<td>----------</td>
<td>----------</td>
<td>----------</td>
</tr>
<tr>
<td>Boundary stresses &amp; body forces due to $V_1=1$</td>
<td>Boundary and inside displacements due to $V_1=1$</td>
<td>Boundary and inside displacements due to $U_5=1$</td>
<td>Boundary and inside displacements due to $W_6=1$</td>
</tr>
<tr>
<td><strong>Body Forces and Inside Displacements</strong></td>
<td><strong>Body Forces and Inside Displacements</strong></td>
<td><strong>Body Forces and Inside Displacements</strong></td>
<td><strong>Body Forces and Inside Displacements</strong></td>
</tr>
<tr>
<td>$X = -\frac{(D_2+D_3)}{2ab}(1+\frac{2z}{c})$</td>
<td>$\lambda_x = 0$</td>
<td>$\lambda_x = \frac{1}{3}(1-\frac{2x}{a})(1+\frac{2y}{b})(1+\frac{2z}{c})$</td>
<td>$\lambda_x = 0$</td>
</tr>
<tr>
<td>$Y = 0$</td>
<td>$\lambda_y = \frac{1}{3}(1+\frac{2x}{a})(1+\frac{2y}{b})(1+\frac{2z}{c})$</td>
<td>$\lambda_y = 0$</td>
<td>$\lambda_y = 0$</td>
</tr>
<tr>
<td>$Z = -\frac{(D_2+D_3)}{2bc}(1+\frac{2x}{a})$</td>
<td>$\lambda_z = 0$</td>
<td>$\lambda_z = 0$</td>
<td>$\lambda_z = \frac{1}{3}(1-\frac{2x}{a})(1-\frac{2y}{b})(1-\frac{2z}{c})$</td>
</tr>
</tbody>
</table>
ments at any point \((x,y,z)\) inside the cell due to \(v_1 = 1\), \(u_5 = 1\) and \(w_8 = 1\) obtained by making use of Equations (58), (59), (60) respectively.

**Calculation of \(Y_1(v_1) = Y_{12}\)**

\[
Y_1(v_1) = \sum [ \int \int_{\text{area}} f(\sigma_A) (v_1=1) (\lambda_1 A) (v_1=1) + (\tau_{AB}) (v_1=1) (\lambda_{11} AB) (v_1=1) \\
+ (\tau_{AC}) (v_1=1) (\lambda_{11} AC) (v_1=1) ] dA \\
+ \int \int_{V} f(x) (v_1=1) (\lambda_x) (v_1=1) + (Y) (v_1=1) (\lambda_y) (v_1=1) \\
+ (Z) (v_1=1) (\lambda_z) (v_1=1) ] dv
\]

(115)

Making use of tables 13 and 14 and Equation (115)

\[
Y_1(v_1) = \int \int_{\frac{c}{2}}^{\frac{b}{2}} \int_{\frac{-b}{2}}^{\frac{-c}{2}} (\tau_{xy}) (v_1=1) (\lambda_{11} xy) (v_1=1) dydz + \\
+ \int \int_{\frac{b}{2}}^{\frac{-b}{2}} \int_{\frac{a}{2}}^{\frac{-a}{2}} (\tau_{zy}) (v_1=1) (\lambda_{11} zy) (v_1=1) dxdy + \\
+ \int \int_{\frac{-c}{2}}^{\frac{c}{2}} \int_{\frac{a}{2}}^{\frac{-a}{2}} (\tau_{yx}) (v_1=1) (\lambda_{11} yx) (v_1=1) dxdz
\]

(116)
\[
\frac{D^2}{16a} \int - \frac{c}{2} \int - \frac{b}{2} \frac{1}{(1+ \frac{2y}{b})(1+ \frac{2z}{c})} dydz + \\
\frac{D^2}{16c} \int - \frac{a}{2} \int - \frac{a}{2} \frac{(1+ \frac{2x}{a})^2 (1+ \frac{2y}{b})^2}{dx dy} \\
\frac{D^2}{16b} \int - \frac{c}{2} \int - \frac{a}{2} \frac{a}{2} \frac{(1+ \frac{2x}{a})^2 (1+ \frac{2z}{c})^2}{dx dz}
\]

(117)

\[
\frac{D^2_{bc}}{9a} + \frac{D^2_{ab}}{9c} + \frac{D^2_{ac}}{9b}
\]

\[
= \frac{2(1-\mu)c^2a^2 + (1-2\mu)b^2(c^2+a^2)}{18(1+\mu)(1-2\mu)abc}
\]

(118)

Calculation of \( X_5(v_1) = X_{52} \)

\[
X_5(v_1) = \sum \left[ \int \int_{A} \{ (\sigma_A)^{v_1=1} (\lambda_1^A)^{u_5=1} + (\tau_{AB})^{v_1=1} (\lambda_{11}^{AB})^{u_5=1} \\
+ (\tau_{AC})^{v_1=1} (\lambda_{11}^{AC})^{u_5=1} \} da \right] \\
+ \int \int_{V} \{ (I)^{v_1=1} (\lambda_x)^{u_5=1} + (Y)^{v_1=1} (\lambda_y)^{u_5=1} + (Z)^{v_1=1} (\lambda_z)^{u_5=1} \} dv
\]

(119)

Making use of tables 13 and 15 and Equation (119)
\[
X_5(v_1) = \int \int \left( \frac{c}{2} + \frac{a}{2} \right) (\tau_{yx}) v_1 = 1 \quad (\lambda_{11}) u_5 = 1 \\
\quad + \int \int \left( \frac{b}{2} + \frac{a}{2} \right) v_1 = 1 \quad u_5 = 1 \\
\quad + \int \int \left( \frac{a}{2} \right) (x) v_1 = 1 \quad (\lambda_x) u_5 = 1 \\
\quad \text{dxdz} + \\
\quad + \int \int \left( \frac{c}{2} + \frac{b}{2} + \frac{a}{2} \right) v_1 = 1 \quad u_5 = 1 \\
\quad \quad \text{dxdydz} \\
= \frac{D_3}{8a} \int \int \left( \frac{c}{2} \right) (1+\frac{2z}{c}) (1-\frac{2x}{a}) \text{dxdz} - \\
\quad \left( \frac{D_2 + D_3}{16ab} \right) \int \int \left( \frac{b}{2} + \frac{a}{2} \right) (1+\frac{2z}{c}) (1+\frac{2y}{b})(1-\frac{2x}{a}) \text{dxdydz} \\
= \frac{D_3c}{6} - \frac{(D_2 + D_3)c}{12} \\
= \frac{(1-4u)cE}{24(1+u)(1-2u)} \\
\text{(120)} \\
\text{(121)}
\]

Calculation of \(Z_8(v_1) = Z_{82}\)

\[
Z_8(v_1) = \sum [\int \int (c_A) v_1 = 1 \quad (\lambda_1 A) w_8 = 1 \\
\text{dA}] + (\tau_{AB}) v_1 = 1 \quad (\lambda_{11} AB) w_8 = 1 \\
\quad + (\tau_{AC}) v_1 = 1 \quad (\lambda_{11} AC) w_8 = 1 ]dA \\
+ \int \int \int (x) v_1 = 1 \quad (\lambda_x) w_8 = 1 \\
\quad + (y) v_1 = 1 \quad (\lambda_y) w_8 = 1 \\
\quad + (z) v_1 = 1 \quad (\lambda_z) w_8 = 1 ]dv \\
\text{(122)}
\]
Making use of Tables 13 and 16 and Equation 122

\[ Z_0(v_1) = \int \frac{c}{2} \int \frac{b}{2} \int \frac{a}{2} (Z_{v_1=1} \lambda_{z} w_{0=1}) \, dx \, dy \, dz \]

\[ = \frac{(D_0 + D_2)}{16bc} \left( \frac{c}{2} \int \frac{b}{2} \int \frac{a}{2} \left( \frac{4x^2}{a^2} \left( 1 - \frac{2y}{b} \right) \left( 1 - \frac{2z}{c} \right) \right) \, dx \, dy \, dz \right) \]

\[ = \frac{aE}{48(1+\mu)(1-2\mu)} \]  \hspace{1cm} (123)

The distribution factors for Action 1, 2 and 3, corresponding to the first three columns of the stiffness matrix are given in tables 17, 18 and 19. The stiffness matrix given in Table 5 is equally applicable with tables 17, 18 and 19 for its terms.

Melosh(3) has derived the energy stiffness matrix for a rectangular prismatic cell of an orthotropic elastic body. Once isotropic elastic constants are substituted into his stiffness matrix it becomes the same as the one developed here.
<table>
<thead>
<tr>
<th>FORCES IN X-DIRECTION</th>
<th>FORCES IN Y-DIRECTION</th>
<th>FORCES IN Z-DIRECTION</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_{1i} = \left[2(1-\mu)b^2c^2 + (1-2\mu)a^2(b^2+c^2)\right] \frac{L}{18abc}$</td>
<td>$Y_{1i} = \frac{CL}{24}$</td>
<td>$Z_{1i} = \frac{BL}{24}$</td>
</tr>
<tr>
<td>$X_{2i} = \left[2(1-\mu)b^2c^2 - (1-2\mu)a^2(2c^2-b^2)\right] \frac{L}{36abc}$</td>
<td>$Y_{2i} = (1-4\mu) \frac{CL}{24}$</td>
<td>$Z_{2i} = \frac{BL}{48}$</td>
</tr>
<tr>
<td>$X_{3i} = \left[2(1-\mu)b^2c^2 - (1-2\mu)a^2(2b^2-c^2)\right] \frac{L}{36abc}$</td>
<td>$Y_{3i} = \frac{CL}{48}$</td>
<td>$Z_{3i} = (1-4\mu) \frac{BL}{24}$</td>
</tr>
<tr>
<td>$X_{4i} = \left[(1-\mu)b^2c^2 - (1-2\mu)a^2(b^2+c^2)\right] \frac{L}{36abc}$</td>
<td>$Y_{4i} = (1-4\mu) \frac{CL}{48}$</td>
<td>$Z_{4i} = (1-4\mu) \frac{BL}{48}$</td>
</tr>
<tr>
<td>$X_{5i} = -\left[4(1-\mu)b^2c^2 - (1-2\mu)a^2(b^2+c^2)\right] \frac{L}{36abc}$</td>
<td>$Y_{5i} = -(1-4\mu) \frac{CL}{24}$</td>
<td>$Z_{5i} = -(1-4\mu) \frac{BL}{24}$</td>
</tr>
<tr>
<td>$X_{6i} = -\left[4(1-\mu)b^2c^2 + (1-2\mu)a^2(2c^2-b^2)\right] \frac{L}{72abc}$</td>
<td>$Y_{6i} = -\frac{CL}{24}$</td>
<td>$Z_{6i} = -(1-4\mu) \frac{BL}{48}$</td>
</tr>
<tr>
<td>$X_{7i} = -\left[4(1-\mu)b^2c^2 + (1-2\mu)a^2(2b^2-c^2)\right] \frac{L}{72abc}$</td>
<td>$Y_{7i} = -(1-4\mu) \frac{CL}{48}$</td>
<td>$Z_{7i} = -\frac{BL}{24}$</td>
</tr>
<tr>
<td>$X_{8i} = -\left[2(1-\mu)b^2c^2 + (1-2\mu)a^2(b^2+c^2)\right] \frac{L}{72abc}$</td>
<td>$Y_{8i} = -\frac{CL}{48}$</td>
<td>$Z_{8i} = -\frac{BL}{48}$</td>
</tr>
</tbody>
</table>
### TABLE - 18

**DISTRIBUTION FACTORS FOR ACTION \( z \), WHEN \( \Delta_z = 1 \)**

**NO-BAR MODEL (ENERGY TYPE)**

\[
 L = \frac{E}{(1+\mu)(1-2\mu)}
\]

**FORCES IN X-DIRECTION**

<table>
<thead>
<tr>
<th>( X_{12} ) = ( \frac{C_L}{24} )</th>
<th>( Y_{12} ) = ( \frac{2(1-\mu)C^2a^2+(1-2\mu)b^2(c^2+a^2)}{18abc} )</th>
<th>( Z_{12} ) = ( \frac{QL}{24} )</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>( X_{22} ) = (- (1-4\mu) \frac{C_L}{24} )</th>
<th>( Y_{22} ) = ( \frac{4(1-\mu)C^2a^2-(1-2\mu)b^2(c^2+a^2)}{36abc} )</th>
<th>( Z_{22} ) = (- (1-4\mu) \frac{QL}{24} )</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>( X_{32} ) = ( \frac{C_L}{48} )</th>
<th>( Y_{32} ) = ( \frac{2(1-\mu)C^2a^2-(1-2\mu)b^2(2a^2-c^2)}{36abc} )</th>
<th>( Z_{32} ) = ( (1-4\mu) \frac{QL}{24} )</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>( X_{42} ) = (- (1-4\mu) \frac{C_L}{48} )</th>
<th>( Y_{42} ) = ( \frac{4(1-\mu)C^2a^2+(1-2\mu)b^2(2a^2-c^2)}{72abc} )</th>
<th>( Z_{42} ) = (- \frac{QL}{24} )</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>( X_{52} ) = ( (1-4\mu) \frac{C_L}{24} )</th>
<th>( Y_{52} ) = ( \frac{2(1-\mu)C^2a^2-(1-2\mu)b^2(2c^2-a^2)}{36abc} )</th>
<th>( Z_{52} ) = ( \frac{QL}{48} )</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>( X_{62} ) = (- \frac{C_L}{24} )</th>
<th>( Y_{62} ) = ( \frac{4(1-\mu)C^2a^2+(1-2\mu)b^2(2c^2-a^2)}{72abc} )</th>
<th>( Z_{62} ) = (- (1-4\mu) \frac{QL}{48} )</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>( X_{72} ) = ( (1-4\mu) \frac{C_L}{48} )</th>
<th>( Y_{72} ) = ( \frac{(1-\mu)C^2a^2-(1-2\mu)b^2(c^2+a^2)}{36abc} )</th>
<th>( Z_{72} ) = ( (1-4\mu) \frac{QL}{48} )</th>
</tr>
</thead>
</table>

| \( X_{82} \) = \(- \frac{C_L}{48} \) | \( Y_{82} \) = \( \frac{2(1-\mu)C^2a^2+(1-2\mu)b^2(c^2+a^2)}{72abc} \) | \( Z_{82} \) = \(- \frac{QL}{48} \) |
**TABLE-19**

**DISTRIBUTION FACTORS FOR ACTION 3 WHEN $\Delta_1^2 = 1$**

**NO-BAR MODEL (ENERGY TYPE)**

$$L = \frac{E}{(1+\mu)(1-2\mu)}$$

<table>
<thead>
<tr>
<th>FORCES IN $X$-DIRECTION</th>
<th>FORCES IN $Y$-DIRECTION</th>
<th>FORCES IN $Z$-DIRECTION</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_{i3} = \frac{bl}{24}$</td>
<td>$y_{i3} = \frac{QL}{24}$</td>
<td>$z_{i3} = \left{2(1-\mu)b^2a^2 + (1-2\mu)c^2(b^2+a^2)\right} \frac{L}{18abc}$</td>
</tr>
<tr>
<td>$x_{23} = +\frac{bl}{48}$</td>
<td>$y_{23} = (1-4\mu) \frac{QL}{24}$</td>
<td>$z_{23} = \left{2(1-\mu)b^2a^2 - (1-2\mu)c^2(2a^2-b^3)\right} \frac{L}{36abc}$</td>
</tr>
<tr>
<td>$x_{93} = -(1-4\mu) \frac{bl}{24}$</td>
<td>$y_{93} = -(1-4\mu) \frac{QL}{24}$</td>
<td>$z_{93} = \left{4(1-\mu)b^2a^2 - (1-2\mu)c^2(b^2+a^2)\right} \frac{L}{36abc}$</td>
</tr>
<tr>
<td>$x_{43} = -(1-4\mu) \frac{bl}{48}$</td>
<td>$y_{43} = -\frac{QL}{24}$</td>
<td>$z_{43} = \left{-4(1-\mu)b^2a^2 + (1-2\mu)c^2(2a^2-b^3)\right} \frac{L}{72abc}$</td>
</tr>
<tr>
<td>$x_{53} = (1-4\mu) \frac{bl}{24}$</td>
<td>$y_{53} = \frac{QL}{48}$</td>
<td>$z_{53} = \left{2(1-\mu)b^2a^2 - (1-2\mu)c^2(2b^2-a^2)\right} \frac{L}{36abc}$</td>
</tr>
<tr>
<td>$x_{c3} = (1-4\mu) \frac{bl}{48}$</td>
<td>$y_{c3} = (1-4\mu) \frac{QL}{48}$</td>
<td>$z_{c3} = \left{(1-\mu)b^2a^2 - (1-2\mu)c^2(b^2+a^2)\right} \frac{L}{36abc}$</td>
</tr>
<tr>
<td>$x_{73} = -\frac{bl}{24}$</td>
<td>$y_{73} = -(1-4\mu) \frac{QL}{48}$</td>
<td>$z_{73} = \left{-4(1-\mu)b^2a^2 + (1-2\mu)c^2(2b^2-a^2)\right} \frac{L}{72abc}$</td>
</tr>
<tr>
<td>$x_{83} = -\frac{bl}{48}$</td>
<td>$y_{83} = -\frac{QL}{48}$</td>
<td>$z_{83} = \left{2(1-\mu)b^2a^2 + (1-2\mu)c^2(b^2+a^2)\right} \frac{L}{72abc}$</td>
</tr>
</tbody>
</table>
CHAPTER 4
Analysis of Model

Once the stiffness matrix of the element is obtained the analysis of the model is accomplished by standard operations performed by an electronic digital computer. It is desirable to present here some of the basic ideas related to the computer programming.

4.1 Calculation of the Displacements of the Nodes

When an interior joint like joint (1) (Fig. 21a) is displaced along one of the coordinate axes, nodal forces are set up along all three coordinate axes at the joint in question as well as the 26 immediately adjacent joints. This makes 27 joints affected by the movement of an interior joint. With three nodal forces per joint; 81 forces are thus brought in, although some of them have zero values. Displacements of a boundary joint affect less joints.

Thus equilibrium of each joint in x, y or z direction is determined by displacements of all neighbouring joints and the corresponding equation of equilibrium involves numerous unknown nodal displacements. In setting up these equations the computer forms the stiffness matrix of the whole model out of the stiffness matrices of the individual cells by adding up the appropriate terms from the adjacent cells. Each row of the stiffness matrix of the model is multiplied by the column matrix of the nodal displacements and is equated to the column
matrix of the external nodal forces, some of which are zeros. The numerous simultaneous equations so formed are solved for the nodal displacements. If some of the joints in the problem have zero displacements the corresponding equations are omitted from the solution.

There may be problems (the examples given in this thesis fall into their category) in which known displacements are specified at some nodes and known forces applied at other nodes. A substantial change in the procedure is required for problems of this type. One method is to solve the problem twice; firstly on the basis of the known displacements and secondly on the basis of the known loads and zero displacements, at the locations where they have been specified. The two solutions are then added up. The following adaptation of this method has been found convenient for the computer work.

Let \( \{S\} \) be the vector \( mx1 \) of the specified nodal displacements and \( \{D\} \) the vector \( nx1 \) of the unknown nodal displacements. The stiffness matrix \( [K]\) of the model is partitioned into four parts and the system of simultaneous equations used in the first step of the solution (known displacements and no applied loads) may be presented in this form\(^8\)

\[
\begin{bmatrix}
  [K_{11}]_{nxn} & [K_{12}]_{nxm} \\
  [K_{21}]_{mxn} & [K_{22}]_{mxm}
\end{bmatrix}
\begin{bmatrix}
  \{D\}_{nx1} \\
  \{S\}_{mx1}
\end{bmatrix} =
\begin{bmatrix}
  \{0\}_{nx1} \\
  \{Q\}_{mx1}
\end{bmatrix}
\] (124)
where \( \{Q\} \) represent the set of forces which causes the known deformations.

This equation gives

\[
[K_{11}]_{n \times n} \{D\}_{n \times 1} + [K_{12}]_{n \times m} \{S\}_{m \times 1} = \{0\}_{n \times 1}
\]  

(125)

The unknown displacements of the nodes involved in the vector \( \{D\} \) may be found from this relation, but more significant is the fact that the forces acting on these nodes and produced by their displacements are equal and opposite in sign to the forces acting on these nodes as result of the known displacements corresponding to the vector \( \{S\} \). This conclusion may be stated in this different form: fictitious nodal forces \( \{P\}^*_{n \times 1} = -[K_{12}]_{n \times m} \{S\}_{m \times 1} \) acting at all nodes with unknown displacements may be substituted for the known displacements involved in \( \{S\} \) while assuming the values of these displacements as zero. This result may be combined with the second step of the solution and the total values of the \( n \) unknown displacements may be obtained from the following equation solved by the computer:

\[
[K_{11}]_{n \times n} \{D\}_{n \times 1} = \{P\}_{n \times 1} + \{P\}^*_{n \times 1}
\]  

(126)

Here \( \{P\} \) and \( \{P\}^* \) are respectively the vectors of the given and the fictitious sets of loads applied at the nodes with the unknown displacements, while the displacements at the nodes of the vector \( \{S\} \) are implied to be zero.
4.2 Calculation of Stresses at the Nodes

After determining the displacements of the nodal points of the model, it is possible to calculate the stresses at the nodes in the prototype by one of the following methods.

4.2a Method 1 of Joint Displacements

This is the simpler of the two displacement methods available. The expressions for stresses in terms of strains in three dimensional solids are:

\[
\sigma_x = \frac{\mu E}{(1+\mu)(1-2\mu)} (\varepsilon_x + \varepsilon_y + \varepsilon_z) + \frac{E}{1+\mu} \varepsilon_x
\]

\[
\sigma_y = \frac{\mu E}{(1+\mu)(1-2\mu)} (\varepsilon_x + \varepsilon_y + \varepsilon_z) + \frac{E}{1+\mu} \varepsilon_y
\]

\[
\sigma_z = \frac{\mu E}{(1+\mu)(1-2\mu)} (\varepsilon_x + \varepsilon_y + \varepsilon_z) + \frac{E}{1+\mu} \varepsilon_z
\]

\[
\tau_{xy} = \frac{E}{2(1+\mu)} \gamma_{xy}
\]

\[
\tau_{yz} = \frac{E}{2(1+\mu)} \gamma_{yz}
\]

\[
\tau_{xz} = \frac{E}{2(1+\mu)} \gamma_{xz}
\]

The strains \(\varepsilon_x, \varepsilon_y, \varepsilon_z\), in the above expressions may be expressed in terms of the nodal displacements \(u\) and \(v\) and \(w\) of the adjacent cells approximately as follows (Fig. 21a).
\[ \varepsilon_x = \frac{1}{2} \left[ \frac{u_1-u_3}{a} + \frac{u_8-u_1}{a} \right] = \frac{(u_8-u_3)}{2a} \]

Similarly
\[ \varepsilon_y = \frac{(v_5-v_7)}{2b} \]
\[ \varepsilon_z = \frac{(w_9-w_{10})}{2c} \]

Substituting these expressions into the Equations (127), (128) and (129) we find

\[ \sigma_{1x} = \frac{\mu E}{(1+\mu)(1-2\mu)} \left[ \frac{u_8-u_3}{2a} + \frac{v_5-v_2}{2b} + \frac{w_9-w_{10}}{2c} \right] + \frac{E}{1+\mu} \left( \frac{u_8-u_3}{2a} \right) \]  (133)

\[ \sigma_{1y} = \frac{\mu E}{(1+\mu)(1-2\mu)} \left[ \frac{u_8-u_3}{2a} + \frac{v_5-v_2}{2b} + \frac{w_9-w_{10}}{2c} \right] + \frac{E}{1+\mu} \left( \frac{v_5-v_2}{2b} \right) \]  (134)

\[ \sigma_{1z} = \frac{\mu E}{(1+\mu)(1-2\mu)} \left[ \frac{u_8-u_3}{2a} + \frac{v_5-v_2}{2b} + \frac{w_9-w_{10}}{2c} \right] + \frac{E}{1+\mu} \left( \frac{w_9-w_{10}}{2c} \right) \]  (135)

The shear stresses at the node 1 are found from the Equations (130), (131) and (132) by using for \( \gamma \) the mean of the four values of the angle of distortion in the four cell faces adjacent to the node 1.

Thus for face A \[ \tau_{1Ax} = \frac{E}{2(1+\mu)} \left( \frac{u_1-u_2}{b} + \frac{v_1-v_3}{a} \right) \]

for face B \[ \tau_{1Bxy} = \frac{E}{2(1+\mu)} \left( \frac{u_5-u_1}{b} + \frac{v_1-v_3}{a} \right) \]

for face C \[ \tau_{1Cxy} = \frac{E}{2(1+\mu)} \left( \frac{u_5-u_1}{b} + \frac{v_8-v_1}{a} \right) \]
The mean of these gives

\[
\tau_{1xy} = \frac{E}{2(1+\mu)} \left\{ \frac{u_2-u_2}{2b} + \frac{v_8-v_3}{2a} \right\}
\]

(136)

Similarly

\[
\tau_{1xz} = \frac{E}{2(1+\mu)} \left\{ \frac{u_9-u_{10}}{2c} + \frac{w_8-w_3}{2a} \right\}
\]

(137)

and

\[
\tau_{1yz} = \frac{E}{2(1+\mu)} \left\{ \frac{w_2-w_2}{2b} + \frac{v_9-v_{10}}{2c} \right\}
\]

(138)

It should be noted that the expressions for \( \sigma \) and \( \tau \) stresses, Equations (133) to (138) inclusive are applicable only at the interior nodes. However similar formulae can be derived for the stresses at the boundary nodes taking into consideration the absence of the cell on one side of the node.

4.2b Method 2 of Joint Displacements

As explained in chapter 3 the displacement at any point \((x, y, z)\) within a cell may be expressed in terms of the displacements of the corners of the cell by the Equation (58), (59) and (60).

In matrix form this gives

\[
\{\varepsilon\}_{6 \times 1} = [B]_{6 \times 24} \{A\}_{24 \times 1}
\]

where

\[
\{\varepsilon\}_{6 \times 1} = \text{column matrix of strains}
\]
\[ \{\Delta\}_{24 \times 1} = \text{column matrix of nodal displacements} \]

\[ [B]_{6 \times 24} = \text{matrix relating } \{\epsilon\} \text{ and } \{\Delta\}. \]

For three dimensional bodies the six strain components at any point are expressed through the three displacement components at that point by:

the relation

\[
\begin{bmatrix}
\epsilon_x \\
\epsilon_y \\
\epsilon_z \\
\gamma_{xy} \\
\gamma_{yz} \\
\gamma_{zx}
\end{bmatrix} =
\begin{bmatrix}
\frac{\partial u}{\partial x} \\
\frac{\partial v}{\partial y} \\
\frac{\partial w}{\partial z}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \\
\frac{\partial v}{\partial x} + \frac{\partial w}{\partial z} \\
\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}
\end{bmatrix}
\]

(140)

The vector of strain components \( \{\epsilon\} \) may thus be found by differentiating the displacement components \( u,v,w \) given by Equations (58), (59) and (60) with respect to \( (x,y,z) \) which when written in the form of Equation (139) give matrix \( [B] \) explicitly.

In an isotropic elastic material the stresses are related to strains by two independent elastic constants \( E \) and \( \nu \). In matrix form:
\[[\sigma]_{6\times1} = [D]_{6\times6} [\epsilon]_{6\times1}\]  \hspace{1cm} (141)

where

\[[\sigma]_{6\times1} \text{ = column matrix of stresses} \]
\[[\epsilon]_{6\times1} \text{ = column matrix of strains} \]
\[[D]_{6\times6} \text{ = elasticity matrix containing appropriate elastic constants.}\]

Explicitely this relation is

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_z \\
\tau_{xy} \\
\tau_{yz} \\
\tau_{zx}
\end{bmatrix}
\begin{bmatrix}
\frac{E}{(1+\mu)(1-2\mu)} \\
1-\mu & \mu & 0 & 0 & 0 \\
\mu & 1-\mu & \mu & 0 & 0 \\
\mu & \mu & 1-\mu & 0 & 0 \\
0 & 0 & 0 & 1-\frac{2\mu}{2} & 0 \\
0 & 0 & 0 & 0 & \frac{1-2\mu}{2}
\end{bmatrix}
\begin{bmatrix}
\epsilon_x \\
\epsilon_y \\
\epsilon_z \\
\gamma_{xy} \\
\gamma_{yz} \\
\gamma_{zx}
\end{bmatrix}
\]

(142)

Substituting Equation (139) into Equation (141)

\[[\sigma]_{6\times1} = [D]_{6\times6} [B]_{6\times24} [\Delta]_{24\times1}\]

\[= [M]_{6\times24} [\Delta]_{24\times1}\]  \hspace{1cm} (143)

The matrix \([M]_{6\times24} = [D]_{6\times6}[B]_{6\times24}\) relating the column matrix of stresses at any point with the column
matrix of the nodal displacements is called the stress matrix. It is given in explicit form in table 20.

To calculate the stresses at the node (1) (Fig. 21a) Equation (141) is applied to all eight cells meeting at this node. In each cell these stresses are evaluated by substituting appropriate values for the coordinates of the node (1) in that particular cell. The eight values of each of the stresses at the node 1 in the eight cells containing this node are averaged up. It should be noted that stress calculation in individual cells and their averaging are performed automatically by the computer.

The same applies to the boundary nodes, where a smaller number of cells is involved in finding the mean values of stresses.

It may be mentioned that in case of bar models only method 1 of joint displacements is applicable because no points in the cells other than the nodes may be considered in stress analysis. On the other hand in the no-bar models both method 1 and method 2 are equally applicable. Furthermore a closer examination of these two methods would reveal that their results must coincide. To demonstrate this proposition follow the procedure of the method 2 in application to the model of Fig. 21a and evaluate the stresses at the node 1 in the several cells adjacent to it in the manner just explained by making use of Equations (58), (59), (60).

From basic stress strain relationship or Equation 142
**TABLE - 20**

**CALCULATION OF STRESSES FROM DISPLACEMENTS (STRESS MATRIX)**

\[
\begin{align*}
\{G\}_{ex1} &= \{P\}_{ex1} + \{Q\}_{ex1} + \{R\}_{ex1} + \{S\}_{ex1} \\
D_1 &= \frac{(1-\mu)E}{(1+\mu)(1-2\mu)} \\
D_2 &= \frac{\mu E}{(1+\mu)(1-2\mu)} \\
D_3 &= \frac{E}{2(1+\mu)}
\end{align*}
\]

$$
\begin{bmatrix}
P_1' \\
P_2' \\
P_3' \\
P_4' \\
P_5' \\
P_6'
\end{bmatrix}
\begin{bmatrix}
D_1 \\
D_2 \\
D_3 \\
D_4 \\
D_5 \\
D_6
\end{bmatrix}
= 
\begin{bmatrix}
P_{12} \\
P_{23} \\
P_{31} \\
P_{42} \\
P_{53} \\
P_{61}
\end{bmatrix}
$$

\[
\begin{array}{cccc}
\frac{D_1}{a} (1+y')(1+z') & \frac{D_2}{b} (1+x')(1+z') & \frac{D_3}{c} (1-x')(1+y') & \frac{D_4}{a} (1-y')(1+z') - \frac{D_2}{b} (1+x')(1+z') \\
\frac{D_1}{a} (1+y')(1+z') & \frac{D_2}{b} (1+x')(1+z') & \frac{D_3}{c} (1-x')(1+y') & \frac{D_4}{a} (1-y')(1+z') - \frac{D_2}{b} (1+x')(1+z') \\
\frac{D_1}{a} (1+y')(1+z') & \frac{D_2}{b} (1+x')(1+z') & \frac{D_3}{c} (1-x')(1+y') & \frac{D_4}{a} (1-y')(1+z') - \frac{D_2}{b} (1+x')(1+z') \\
\frac{D_1}{a} (1+y')(1+z') & \frac{D_2}{b} (1+x')(1+z') & \frac{D_3}{c} (1-x')(1+y') & \frac{D_4}{a} (1-y')(1+z') - \frac{D_2}{b} (1+x')(1+z') \\
\frac{D_1}{a} (1+y')(1+z') & \frac{D_2}{b} (1+x')(1+z') & \frac{D_3}{c} (1-x')(1+y') & \frac{D_4}{a} (1-y')(1+z') - \frac{D_2}{b} (1+x')(1+z') \\
\frac{D_1}{a} (1+y')(1+z') & \frac{D_2}{b} (1+x')(1+z') & \frac{D_3}{c} (1-x')(1+y') & \frac{D_4}{a} (1-y')(1+z') - \frac{D_2}{b} (1+x')(1+z')
\end{array}
\]

$$
\begin{bmatrix}
u_1 \\
u_2 \\
\omega_1 \\
u_3 \\
u_4 \\
\omega_2
\end{bmatrix}
$$
TABLE-20(Cont.)
CALCULATION OF STRESSES FROM DISPLACEMENTS (STRESS MATRIX)

\[
\left[\sigma\right]_{exi} = \left\{F\right\}_{exi} + \left\{Q\right\}_{exi} + \left\{R\right\}_{exi} + \left\{S\right\}_{exi}
\]

\[
D_1 = \frac{(1-\mu)E}{(1+\mu)(1-2\mu)}
\]

\[
D_2 = \frac{\mu E}{(1+\mu)(1-2\mu)}
\]

\[
D_3 = \frac{E}{2(1+\mu)}
\]

<table>
<thead>
<tr>
<th>(Q_1)</th>
<th>(Q_2)</th>
<th>(Q_3)</th>
<th>(Q_4)</th>
<th>(Q_5)</th>
<th>(Q_6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\frac{D_1}{A}(1+y')(1-z'))</td>
<td>(\frac{D_2}{B}(1+x')(1-z'))</td>
<td>(-\frac{D_3}{C}(1+x')(1+y'))</td>
<td>(\frac{D_1}{A}(1-y')(1-z'))</td>
<td>(-\frac{D_2}{B}(1+x')(1-z'))</td>
<td>(-\frac{D_3}{C}(1+x')(1-y'))</td>
</tr>
<tr>
<td>(\frac{D_2}{A}(1+y')(1-z'))</td>
<td>(\frac{D_1}{B}(1+x')(1-z'))</td>
<td>(\frac{D_2}{C}(1-x')(1+y'))</td>
<td>(\frac{D_3}{A}(1-y')(1-z'))</td>
<td>(-\frac{D_2}{B}(1+x')(1-z'))</td>
<td>(-\frac{D_3}{C}(1+x')(1-y'))</td>
</tr>
<tr>
<td>(\frac{D_2}{A}(1+y')(1-z'))</td>
<td>(\frac{D_3}{B}(1+x')(1-z'))</td>
<td>(\frac{D_1}{C}(1-x')(1+y'))</td>
<td>(\frac{D_2}{A}(1-y')(1-z'))</td>
<td>(-\frac{D_3}{B}(1+x')(1-z'))</td>
<td>(-\frac{D_1}{C}(1+x')(1-y'))</td>
</tr>
<tr>
<td>(0)</td>
<td>(\frac{D_2}{B}(1+x')(1-z'))</td>
<td>(\frac{D_1}{A}(1+y')(1-z'))</td>
<td>(0)</td>
<td>(-\frac{D_3}{B}(1+x')(1-z'))</td>
<td>(-\frac{D_2}{A}(1+y')(1-z'))</td>
</tr>
<tr>
<td>(0)</td>
<td>(-\frac{D_1}{C}(1-x')(1+y'))</td>
<td>(\frac{D_2}{B}(1+x')(1-z'))</td>
<td>(0)</td>
<td>(-\frac{D_3}{B}(1+x')(1-z'))</td>
<td>(-\frac{D_2}{A}(1+y')(1-z'))</td>
</tr>
<tr>
<td>(-\frac{D_3}{C}(1+x')(1+y'))</td>
<td>(0)</td>
<td>(\frac{D_2}{A}(1+y')(1-z'))</td>
<td>(\frac{D_1}{C}(1-x')(1+y'))</td>
<td>(0)</td>
<td>(\frac{D_3}{A}(1-y')(1-z'))</td>
</tr>
</tbody>
</table>
TABLE-20 (CONT.)

CALCULATION OF STRESSES FROM DISPLACEMENTS (STRESS MATRIX)

\[
\frac{\{G\}_{C_x l}}{D_1} = \{P\}_{C_x l} + \{Q\}_{C_x l} + \{R\}_{C_x l} + \{S\}_{C_x l}
\]

\[
D_1 = \frac{(1-\mu)E}{(1+\mu)(1-2\mu)} \quad D_2 = \frac{HE}{(1+\mu)(1-2\mu)} \quad D_3 = \frac{E}{2(1+\mu)}
\]

| \(R_1'\) | \(-\frac{D_a}{a}(1+y')(1+z')\) | \(\frac{D_b}{b}(1-x')(1+z')\) | \(\frac{D_c}{c}(1-x')(1+y')\) | \(-\frac{D_d}{d}(1-y')(1+z')\) | \(-\frac{D_e}{e}(1-x')(1+y')\) | \(\frac{D_f}{f}(1-x')(1-y')\) |
| \(R_2'\) | \(-\frac{D_a}{a}(1+y')(1+z')\) | \(\frac{D_b}{b}(1-x')(1+z')\) | \(\frac{D_c}{c}(1-x')(1+y')\) | \(-\frac{D_d}{d}(1-y')(1+z')\) | \(-\frac{D_e}{e}(1-x')(1+z')\) | \(\frac{D_f}{f}(1-x')(1-y')\) |
| \(R_3'\) | \(-\frac{D_a}{a}(1+y')(1+z')\) | \(\frac{D_b}{b}(1-x')(1+z')\) | \(\frac{D_c}{c}(1-x')(1+y')\) | \(-\frac{D_d}{d}(1+y')(1+z')\) | \(-\frac{D_e}{e}(1-x')(1+z')\) | \(\frac{D_f}{f}(1-x')(1-y')\) |
| \(R_4'\) | \(\frac{D_b}{b}(1-x')(1+z')\) | \(-\frac{D_a}{a}(1+y')(1+z')\) | \(0\) | \(-\frac{D_d}{d}(1-x')(1+z')\) | \(-\frac{D_e}{e}(1-y')(1+z')\) | \(0\) |
| \(R_5'\) | \(0\) | \(-\frac{D_a}{a}(1-x')(1+y')\) | \(\frac{D_b}{b}(1-x')(1+z')\) | \(0\) | \(-\frac{D_c}{c}(1-x')(1-y')\) | \(-\frac{D_d}{d}(1-x')(1+z')\) |
| \(R_6'\) | \(\frac{D_c}{c}(1-x')(1+y')\) | \(0\) | \(-\frac{D_a}{a}(1+y')(1+z')\) | \(\frac{D_b}{b}(1-x')(1-y')\) | \(0\) | \(-\frac{D_d}{d}(1-y')(1+z')\) |
### TABLE 20 ( CONT. )

**Calculation of Stresses from Displacements (Stress Matrix)**

\[
\{G\}_{ex1} = \{P\}_{ex1} + \{q\}_{ex1} + \{r\}_{ex1} + \{s\}_{ex1}
\]

\[
D_1 = \frac{(1-\mu)E}{(1+\mu)(1-2\mu)} \quad D_2 = \frac{\mu E}{(1+\mu)(1-2\mu)} \quad D_3 = \frac{E}{2(1+\mu)}
\]

| \( S_1 \) | \( -\frac{D_1}{a}(l+y')(l-z') \) | \( D_2^2(l-x')(l-z') \) | \( -\frac{D_2}{a}(l-y')(l-z') \) | \( -\frac{D_3}{b}(l-x')(l-z') \) | \( -\frac{D_3}{c}(l-x')(l-y') \) |
| \( S_2 \) | \( -\frac{D_2}{a}(l+y')(l-z') \) | \( D_2^2(l-x')(l-z') \) | \( -\frac{D_2}{a}(l-y')(l-z') \) | \( -\frac{D_2}{b}(l-x')(l-z') \) | \( D_2^2(l-x')(l-y') \) |
| \( S_3 \) | \( -\frac{D_2}{a}(l+y')(l-z') \) | \( D_2^2(l-x')(l-z') \) | \( -\frac{D_2}{a}(l-y')(l-z') \) | \( -\frac{D_2}{b}(l-x')(l-z') \) | \( D_2^2(l-x')(l-y') \) |
| \( S_4 \) | \( D_2^2(l-x')(l-z') \) | \( -\frac{D_3}{a}(l+y')(l-z') \) | \( 0 \) | \( -\frac{D_3}{b}(l-x')(l-z') \) | \( -\frac{D_3}{c}(l-y')(l-z') \) |
| \( S_5 \) | \( 0 \) | \( -\frac{D_3}{a}(l-x')(l+y') \) | \( D_2^2(l-x')(l-z') \) | \( 0 \) | \( -\frac{D_3}{b}(l-x')(l-z') \) |
| \( S_6 \) | \( -\frac{D_3}{c}(l-x')(l+y') \) | \( 0 \) | \( -\frac{D_3}{a}(l+y')(l-z') \) | \( D_2^2(l-x')(l-y') \) | \( 0 \) |
\[ \sigma_x = \frac{E}{(1+\mu)(1-2\mu)} \left[ (1-u)\varepsilon_x + u(\varepsilon_y + \varepsilon_z) \right] \]
\[ = \frac{E}{(1+\mu)(1-2\mu)} \left[ (1-u)\frac{\partial u}{\partial x} + \mu \left( \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right] \]
\[ \tau_{xy} = \frac{E}{2(1+\mu)} \gamma_{xy} = \frac{E}{2(1+\mu)} \left( \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} \right) \]

Substituting for \( u, v \) and \( w \) in these equations their values from Equations (58), (59) and (60) using the coordinates of the node 1 in the cell \( A_1 \) (i.e. \( x'=1, y'=1 \) and \( z'=1 \)) and making allowance for the fact that the numbering of the cell in Equations (58), (59) and (60) is different from those in Fig. 21a, we get

\[ \sigma_{lx}(A_1) = \frac{(1-u)E}{(1+\mu)(1-2\mu)} \left( \frac{u_1-u_2}{a} \right) + \frac{uE}{(1+\mu)(1-2\mu)} \left( \frac{v_1-v_2}{b} + \frac{w_1-w_{10}}{c} \right) \]
\[ \tau_{lxy}(A_1) = \frac{E}{2(1+\mu)} \left( \frac{u_1-u_2}{b} + \frac{v_1-v_3}{a} \right) \]

Close inspection of these formulae in conjunction with the observation of Fig. 21a and the way of numbering of its nodes reveals the manner in which the expressions for \( \sigma_x \) and \( \tau_{xy} \) at different nodes of a cell may be formed.

Later following the same operation for the node 1 in the adjacent cells and finally averaging up \( \sigma_x \) and \( \tau_{xy} \) stresses at the node 1 in all 8 cells adjacent to this node we get:
\[ \sigma_{1x} = \frac{(1-\mu)E}{(1+\mu)(1-2\mu)} \left( \frac{u_8-u_3}{2a} \right) + \frac{yE}{(1+\mu)(1-2\mu)} \left( \frac{v_5-v_2}{2b} + \frac{w_9-w_{10}}{2c} \right) \]

\[ \tau_{1xy} = \frac{E}{2(1+\mu)} \left( \frac{u_5-u_2}{2b} + \frac{v_8-v_3}{2c} \right) \]

These equations agree exactly with Equations (133) and (136) respectively obtained by making use of method 1 of calculation of stresses by the joint displacements. The same results may also be obtained with regard to the other stress components at the node.

Similarly it may be shown that the stresses obtained at the boundary nodes by both methods of joint displacements coincide if in the method 1 of joint displacement appropriate expressions for strains are substituted taking into consideration the absence of cells on one side of the node.

4.2c Method of Nodal Forces

Knowing the displacements of various nodes of the model the corner forces \( \{f\} \) of the individual cells may be obtained by making use of the relation:

\[ \{f\} = [K]\{\delta\} \quad (144) \]

where

\([k] = \text{stiffness matrix of the cell} \]

\(\{\delta\} = \text{displacement vector for each cell} \]

Fig. 21b shows the normal and tangential nodal concentrations acting on a plane perpendicular to the \( y \) axis.
These concentrations at each node such as node 1 are found by combining the corresponding nodal forces in the four cells containing the node 1 and located on one side of the given y plane. In the absence of external loads at the node 1 the nodal concentrations come out the same no matter whether the elements to the one or the other side of the plane y are considered.

The stresses in the prototype structure are found by dividing the nodal concentrations of the model by the tributary areas.

Thus

\[ \sigma_{1y} = \frac{N_{1y}}{ac}, \quad \tau_{1yx} = \frac{T_{1yx}}{ac}, \quad \tau_{1yz} = \frac{T_{1yz}}{ac} \]  \hspace{1cm} (145)

Smaller tributary areas are involved at the edge joints such as joint 2 and 3 for example where

\[ \sigma_{2y} = \frac{2N_{2y}}{ac} \]  \hspace{1cm} (146)

\[ \sigma_{3y} = \frac{4N_{3y}}{ac} \]

Some adjustments of these stresses may be required for compliance with statics. Thus conjugate shearing stresses on two perpendicular planes must be averaged. \[ \tau_{1yx} = \tau_{1xy} \]

The shear stress at the edge \( \tau_{2yx} \) must be zero unless an external shear load is applied to the prototype in the vicinity of the node 2. In the absence of such load the nodal concentrations \( T_{2yx} \) must be converted into stress at
a location away from the edge. In making these adjustments statics must be preserved. The error in stresses resulting from such adjustments will be purely local and minor by the St. Venant's principle.
CHAPTER 5

Examples of Three Dimensional Solid Analysis

The finite element method in its several versions has been applied to two problems for which the exact solutions by the theory of elasticity are available. The approximate values of stresses and displacements found by the finite element method have been compared with the exact ones for different values of Poisson's Ratio.

5.1 Example 1 - Boussinesq Problem (Elasticity Solution)

A concentrated load $P$ acts normally to the plane $H_0H_0$ representing the free boundary of an elastic half-space (Fig. 22a). The axisymmetric stress condition in the body is described by the stresses and displacements expressed in cylindrical coordinates $(x, \gamma, \theta)$ (Fig. 22b) as follows$^9$

$$\sigma_r = \frac{P}{2\pi} \left[ (1-2\mu) \left\{ \frac{1}{r^2} - \frac{x}{Rr^2} \right\} - \frac{3r^2x}{R^5} \right]$$  \hspace{1cm} (147)

$$\sigma_\theta = \frac{P}{2\pi} (1-2\mu) \left\{ - \frac{1}{r^2} + \frac{x}{Rr^2} + \frac{x}{R^3} \right\}$$  \hspace{1cm} (148)

$$\sigma_x = -\frac{3P}{2\pi} \frac{x^3}{R^5}$$  \hspace{1cm} (149)

$$\tau_{rx} = -\frac{3P}{2\pi} \frac{rx}{R^5}$$  \hspace{1cm} (150)

$$v_r = \frac{(1-2\mu)(1+\mu)P}{2\pi r^2} \left\{ \frac{x}{R} - 1 + \frac{1}{(1-2\mu)} \frac{r^2x}{R^3} \right\}$$  \hspace{1cm} (151)
The \( x=0 \) PLANE

\[ \sigma_x = \text{Vertical Stress} \]
\[ \sigma_\theta = \text{Tangential Stress} \]
\[ \sigma_r = \text{Radial Stress} \]
\[ \tau_{r\theta} = \text{Shear stress} \]
\[ \nu_r = \text{Radial displacement} \]
\[ u = \text{Vertical displacement} \]

Fig. 22a

Stresses in Cylindrical Co-ordinates

Fig. 22b

Stresses in Rectangular Co-ordinates

Fig. 22c
\[ u = \frac{P}{2\pi E} \left[ (1+\nu)\frac{x^2}{R^2} + \frac{2(1-\nu^2)}{R} \right] \]  

where in the above expressions

\[ R = \sqrt{x^2+y^2+z^2} \]  
\[ r = \sqrt{y^2+z^2} \]

For our purposes the cylindrical coordinates are replaced by the rectangular using the common axis x. If \( \theta \) is the angle between r and z axis so that \( \cos \theta = \frac{z}{r} \) and \( \sin \theta = \frac{y}{r} \) then the expressions for the stresses and displacements in rectangular coordinates can be obtained from above expressions by using the following relations:

\[ \sigma_z = \sigma_r \cos^2 \theta + \sigma_\theta \sin^2 \theta \]  
\[ \sigma_y = \sigma_r \sin^2 \theta + \sigma_\theta \cos^2 \theta \]  
\[ \tau_{zx} = \tau_{rx} \cos \theta \]  
\[ \tau_{yx} = \tau_{rx} \sin \theta \]  
\[ \tau_{zy} = \frac{1}{2}(\sigma_r - \sigma_\theta) \sin 2\theta \]  
\[ v = v_r \sin \theta \]  
\[ w = v_r \cos \theta \]
Making use of relations 147 to 161 the expressions for stresses and displacements in rectangular coordinates are obtained as

\[ \sigma_z = \frac{P}{2\pi} \left[ (1 - 2\mu) \left\{ \frac{(z^2 - y^2)}{R(R+x)r^2} + \frac{xy^2}{R^3 r^2} \right\} - \frac{3z^2}{R^5} \right] \]  \hspace{1cm} (162)

\[ \sigma_y = \frac{P}{2\pi} \left[ (1 - 2\mu) \left\{ \frac{(y^2 - z^2)}{R(R+x)r^2} + \frac{xz^2}{R^3 r^2} \right\} - \frac{3y^2}{R^5} \right] \]  \hspace{1cm} (163)

\[ \sigma_x = - \frac{3P}{2\pi} \frac{x^3}{R^5} \]  \hspace{1cm} (164)

\[ \tau_{zx} = - \frac{3P}{2\pi} \frac{x^2 z}{R^5} \]  \hspace{1cm} (165)

\[ \tau_{yx} = - \frac{3P}{2\pi} \frac{x^2 y}{R^5} \]  \hspace{1cm} (166)

\[ \tau_{zy} = \frac{P}{2\pi} \left[ (1 - 2\mu) \left\{ \frac{2x}{r^2} - \frac{2x}{Rr^2} - \frac{x^2}{R^3} \right\} - \frac{3x^2 y^2}{R^5 r^2} \right] \]  \hspace{1cm} (167)

\[ u = \frac{P}{2\pi E} \left[ (1+\mu) \frac{x^2}{R^3} + 2 \frac{1-\mu^2}{R} \right] \]  \hspace{1cm} (168)

\[ v = \frac{(1-2\mu)(1+\mu)}{2\pi Er} \frac{x}{R} - 1 + \frac{1}{1-2\mu} \frac{r^2 x y}{R^3} \]  \hspace{1cm} (169)

\[ w = \frac{(1-2\mu)(1+\mu)}{2\pi Er} \frac{x}{R} - 1 + \frac{1}{1-2\mu} \frac{r^2 x z}{R^3} \]  \hspace{1cm} (170)
5.2 Example 2 - Infinite Body (Elasticity Solution)

The second example represents an infinite elastic body with a point load applied inside it. Let \( P \) be the point load which acts at the origin in the positive direction of \( x \) axis.

For the above system the expressions for stresses and displacements described in cylindrical coordinates as in Figs. 22 a, b, c are given by (9)

\[
\sigma_r = \frac{P}{8\pi(1-\nu)}[(1-2\nu)\frac{x}{R^2} - \frac{3r^2\nu}{R^5}] \tag{171}
\]

\[
\sigma_\theta = \frac{P}{8\pi(1-\nu)}(1-2\nu)\frac{x^2}{R^3} \tag{172}
\]

\[
\sigma_x = -\frac{P}{8\pi(1-\nu)}[(1-2\nu)\frac{x^3}{R^3} + \frac{3x^3}{R^5}] \tag{173}
\]

\[
\tau_{rx} = -\frac{P}{8\pi(1-\nu)}[(1-2\nu)\frac{x^3}{R^3} + \frac{3rx^2}{R^5}] \tag{174}
\]

\[
v_r = \frac{P(1+\nu)}{8\pi E(1-\nu)} \frac{rx}{R^3} \tag{175}
\]

\[
u = \frac{P(1+\nu)}{8\pi E(1-\nu)} \left[\frac{4(1-\nu)}{R} - \frac{r^2}{R^3}\right] \tag{176}
\]

Making use of the relations (155) to (161) the expressions for stresses and displacements in rectangular coordinates are obtained as
\[ \sigma_x = -\frac{P}{8\pi(1-\mu)} \left[ (1-2\mu)\frac{x}{R^3} + \frac{3x^3}{R^5} \right] \quad (177) \]

\[ \sigma_y = \frac{P}{8\pi(1-\mu)} \left[ (1-2\mu)\frac{y}{R^3} - \frac{3xy^2}{R^5} \right] \quad (178) \]

\[ \sigma_z = \frac{P}{8\pi(1-\mu)} \left[ (1-2\mu)\frac{z}{R^3} - \frac{3xz^2}{R^5} \right] \quad (179) \]

\[ \tau_{xz} = -\frac{P}{8\pi(1-\mu)} \left[ (1-2\mu)\frac{z}{R^3} + \frac{3x^2z}{R^5} \right] \quad (180) \]

\[ \tau_{yx} = -\frac{P}{8\pi(1-\mu)} \left[ (1-2\mu)\frac{y}{R^3} + \frac{3x^2y}{R^5} \right] \quad (181) \]

\[ \tau_{yz} = -\frac{3P}{8\pi(1-\mu)} \frac{xyz}{R^5} \quad (182) \]

\[ u = \frac{P(1+\mu)}{8\pi E(1-\mu)} \left[ \frac{\mu(1+\mu)}{R} - \frac{r^2}{R^3} \right] \quad (183) \]

\[ v = \frac{P(1+\mu)}{8\pi E(1-\mu)} \frac{xy}{R^3} \quad (184) \]

\[ w = \frac{P(1+\mu)}{8\pi E(1-\mu)} \frac{xz}{R^3} \quad (185) \]

In view of the symmetry of the load conditions in both examples it is sufficient to analyse only a quarter of the half space in the first example and one eighth of the whole space in the second. This is because the stresses and displacements on the two sides of the \( y \) and \( z \) planes are symmetrical
Fig. 23.

ABEF - Horizontal boundary plane for semi-infinite body and plane of anti-symmetry for infinite body.

IDEALIZED ELASTIC SOLID.
and on the two sides of the horizontal plane $x$ in the second example - antisymmetrical.

5.3 Purpose of Examples and Procedure Used

Two objectives are pursued in solution of the examples outlined above.

1. To demonstrate the convergence of the finite element solutions to the elasticity values on reduction of the size of the mesh using two different stiffness matrices.

   The cell is assumed cubic and the Poisson's ratio has the value 0.2. The space quadrant (Example 1) or the octant (Example 2) of the size $3a$ (Fig. 24) is represented by the models $2 \times 2 \times 2$, $4 \times 4 \times 4$ and $6 \times 6 \times 6$ of successively smaller cubes.

2. To compare the precision of the finite element solutions at different locations obtained by the use of different stiffness matrices with noncubic and cubic parallelepiped cells and several values of Poisson's ratio. Comparison is also made of the two methods of determination of the prototype stresses: by the nodal concentrations and the nodal displacements. The quadrant or octant of the space is made into $8 \times 5 \times 5$ models of cells of the size $ax1.25axl.25a$, the dimension $a$ of the 8 layers of cells lying along the vertical axis $x$ (Fig. 23). In some cases cubic cells of the size $axaxa$ are also used as explained later.

   Proper boundary conditions must be provided on the six faces of the models taking into consideration the conditions
of symmetry. In Example 1 no displacement of the nodes lying in the planes of symmetry $y$ and $z$ are possible in the direction perpendicular to these planes i.e. the nodes on the plane ACGE (Fig. 23) have no $v$ displacements and the nodes in ACDB no $w$ displacements. The upper face nodes are free to move in any direction, while the nodes in the two remaining side faces and the bottom face must be given the displacements computed by the elasticity formulae Equation 168, 169 and 170.

In Example 2 similar boundary conditions are provided except on the planes of the antisymmetry $x$, where no $v$ and $w$ displacements of the nodes lying in this plane are possible.

Stresses as a rule were computed from the nodal displacements (Plates 1 to 21 and 38 to 46) but in plates 22 to 37 the nodal displacement and nodal concentration methods of stress determination were used both for their comparison.
Size of the Elements: 1.5a x 1.5a x 1.5a

Size of the Elements: .75a x .75a x .75a

Size of the Elements: .5a x .5a x .5a
CHAPTER 6

Results and Discussions

For convenience of discussion and comparison of solutions obtained with different types of cells, these and their results will be identified respectively by the following designations in brackets:
(Gant.) meaning Gantayat for the bar model Type 1.
(Yett.) i.e. Yettram for the bar model Type 2.
(Mel.) i.e. Melosh for the no-bar model using the energy matrix.

While pursuing objective 1 (demonstration of convergence), stresses and displacements are computed at two locations in the model - nodes 1 and 2 (Fig. 24). Table 22 and 23 present respectively the results of the Examples 1 and 2 determined using the Melosh's cell, and the Tables 24 and 25 - the Gantayat's. The figures stated are the coefficients before \( \frac{P}{Ea} \) and \( \frac{P}{a^2} \) for displacements and stresses respectively, corresponding to different numbers of cells in the model. Percentage deviation of the results from the elasticity solution, as well as the values in the latter, are also stated. The percentage deviations are computed by the following formula, making use of the absolute values of the appropriate quantities

\[
\text{% Error} = \frac{|\text{value by Elasticity}| - |\text{value by F.E.}|}{|\text{value by Elasticity}|} \times 100
\]  

(186)

If the values of the displacements and stresses determined by the Elasticity and the Finite Element happen
to be of opposite signs (this may occur when the functions are very small); plus sign should be used in the numerator of the expression for the °/o Error.

The same results of °/o error are presented graphically in Plates 1, 2 and 3. In all cases the °/o error decreases on increase of the number of cells, at different rates for different functions, usually faster than in inverse proportion to the number of cells in the model along each edge of the cube. In the most precise solutions corresponding to representation of the quadrant or octant models by the blocks of 6x6x6 cells, the errors in most functions are of the order of a few percent unless the function itself happens to be very small. In most but not in all functions Gantayat's model gives the more precise results.

The results of solutions carried out in pursuit of the objective 2 (comparison of different types of cells for different Poisson's ratios - Fig. 23), are presented graphically in Plates 4 to 21 inclusive. Each plate represents the cross section of the model along a horizontal or a vertical plane parallel to one of the coordinate planes, with all nodes marked in their proper locations. The plate is intended to describe the precision of values of one particular stress or displacement function such as the displacement \( u \) for one particular value of \( \mu \) on the plane chosen. At all nodes the percentage errors are stated of the values of the function determined by different methods, i.e. using (Gant.), (Yett.), and (Mel.) cells, thus providing data for the comparison of the results.
For greater clarity what may be called the contour lines of the function considered are drawn on the basis of the elasticity formulae. The figure thus presents not only the errors corresponding to different methods, but also the variation of the function itself through the field. Five different values of Poisson's ratio are used in solution 0, 0.1, 0.2, 0.25, 0.3. Stresses are determined by the method of joint displacements.

In most cases all three types of cells give reasonably good results with the error being of the order of a few percent sometimes even fraction of one percent. Nearly always Gantayat's cell is the best of the three followed by Melosh's with the Yettram's cell at the rear. However there are some exceptions, mostly in the range of higher values of $\mu$ (Plates 16 and 17) with some errors in stresses and displacements by the Yettram's model, becoming as high as $50\%$ and higher. This happens because for $\mu$ greater than 0.25, the flexural stiffnesses of the edge bars in the Yettram's cell become negative. Cells with negative areas or moments of inertia are of course physically impossible but this does not preclude their use in analysis. However the results obtained with frameworks made up of such cells are usually of a lower precision.

It may be pointed out that if the stress is small high percentage error has no practical significance.

For $\mu = 0.25$ the bars of the Yettram's cell lose their flexural stiffnesses and the cell become identical with the Gantayat's.
Plates 22 to 37 inclusive are devoted to comparison of the stresses both in the infinite and the semi-infinite bodies determined from the nodal displacements and nodal force concentrations. Only the Melosh's and Gantayat's cells are used in the comparison with a single value of $\mu = 0.2$.

As was explained earlier the two joint displacement methods (the more elaborate and the simple one) for determination of stresses coincide in case of the Melosh's cell. As to the Gantayat's cell only the simple displacement method is possible, as was stated earlier.

The results show that both the nodal displacement and nodal force methods are adequate for stress computation. Neither one seems to have an over-all advantage over the other. A higher precision in one location is offset by a lower one in the other. This applies both to the Melosh's and Gantayat's cell.

The precision of results obtained by using the finite element bar and no-bar models seems to depend on the Poisson's ratio. In the elasticity solution both displacements and stresses are functions of $\mu$. Similarly the distribution factors in the finite element method are also functions of $\mu$. Hence it is natural that the percentage error obtained by using the finite element method may vary depending on the value of the Poisson's ratio.

The effect of $\mu$ on the percentage error in stresses and displacements found by the finite element method is demonstrated in the Plates 38 to 46 inclusive. Four values of $\mu$,
0, 0.1, 0.2 and 0.3 are used for this purpose with Melosh's and Gantayat's model applied to both infinite and semi-infinite bodies. The scope of this investigation is rather narrow. It involves only 5 locations all situated in a horizontal plane $x = 4a$. Naturally the results must be viewed only as a partial indication of the trend in the effect of $\mu$ on the displacement and stress functions.

It appears that in most cases the percentage error in stresses and displacements varies with $\mu$ almost linearly. Sometimes rising and sometimes falling. The % error may be either positive or negative and in some cases it passes through zero at an intermediate value of $\mu$. No substantial difference is apparent in this respect in the results pertaining to the two examples, nor to the two types of cells, although the signs of the errors are often different in the Melosh's and Gantayat's methods.

Both examples have also been solved by using the statics matrix of the no-bar cell of this thesis. The results have been found comparable in precision to the ones obtained by the energy matrix.
**TABLE 22**

Effect of Mesh Size on Finite Element Solution

<table>
<thead>
<tr>
<th>JT. No.</th>
<th>Elasticity Solution</th>
<th>Finite Element Solution (No Bar Model-MEI)</th>
<th>Displacements $\frac{P}{Ea}$</th>
<th>Stresses $\frac{P}{a^2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>2x2x2 Mesh</td>
<td>4x4x4 Mesh</td>
<td>6x6x6 Mesh</td>
</tr>
<tr>
<td>1</td>
<td>u-displacement</td>
<td>0.1421</td>
<td>0.1357</td>
<td>4.5</td>
</tr>
<tr>
<td></td>
<td>v-displacement</td>
<td>0.0083</td>
<td>0.0021</td>
<td>74</td>
</tr>
<tr>
<td></td>
<td>w-displacement</td>
<td>0.0083</td>
<td>0.0021</td>
<td>74</td>
</tr>
<tr>
<td></td>
<td>$\sigma_x$-stress</td>
<td>-0.0136</td>
<td>-0.0095</td>
<td>29</td>
</tr>
<tr>
<td></td>
<td>$\sigma_y$-stress</td>
<td>-0.0095</td>
<td>-0.0027</td>
<td>71</td>
</tr>
<tr>
<td></td>
<td>$\sigma_z$-stress</td>
<td>-0.0095</td>
<td>-0.0027</td>
<td>71</td>
</tr>
<tr>
<td></td>
<td>$\tau_x y$-stress</td>
<td>-0.0087</td>
<td>-0.0098</td>
<td>-13</td>
</tr>
<tr>
<td></td>
<td>$\tau_x y$-stress</td>
<td>-0.0136</td>
<td>-0.0096</td>
<td>29</td>
</tr>
<tr>
<td></td>
<td>$\tau_x z$-stress</td>
<td>-0.0136</td>
<td>-0.0096</td>
<td>29</td>
</tr>
<tr>
<td>2</td>
<td>u-displacement</td>
<td>0.1440</td>
<td>0.1480</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>v-displacement</td>
<td>-0.0381</td>
<td>-0.0272</td>
<td>28</td>
</tr>
<tr>
<td></td>
<td>w-displacement</td>
<td>-0.0381</td>
<td>-0.0272</td>
<td>28</td>
</tr>
<tr>
<td></td>
<td>$\tau_y z$-stress</td>
<td>0.0212</td>
<td>0.0097</td>
<td>54</td>
</tr>
</tbody>
</table>

For joint 2, $\sigma_x$, $\sigma_y$, $\sigma_z$, $\tau_x y$, $\tau_x z$ are all equal to zero.
<table>
<thead>
<tr>
<th>JT. No.</th>
<th>Elasticity Solution</th>
<th>$2 \times 2 \times 2$ Mesh</th>
<th>$4 \times 4 \times 4$ Mesh</th>
<th>$6 \times 6 \times 6$ Mesh</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Absolute Value</td>
<td>% Error</td>
<td>Absolute Value</td>
</tr>
<tr>
<td>1</td>
<td>u-displacement</td>
<td>0.0582</td>
<td>-</td>
<td>0.0544</td>
</tr>
<tr>
<td></td>
<td>v-displacement</td>
<td>0.0076</td>
<td>0</td>
<td>0.0043</td>
</tr>
<tr>
<td></td>
<td>w-displacement</td>
<td>0.0076</td>
<td>0</td>
<td>0.0043</td>
</tr>
<tr>
<td></td>
<td>$C_x$ stress</td>
<td>-0.0068</td>
<td>41</td>
<td>-0.0039</td>
</tr>
<tr>
<td></td>
<td>$C_y$ stress</td>
<td>-0.0017</td>
<td>25</td>
<td>-0.0013</td>
</tr>
<tr>
<td></td>
<td>$C_z$ stress</td>
<td>-0.0017</td>
<td>25</td>
<td>-0.0013</td>
</tr>
<tr>
<td></td>
<td>$\tau_{xy}$ stress</td>
<td>-0.0042</td>
<td>-22</td>
<td>-0.0052</td>
</tr>
<tr>
<td></td>
<td>$\tau_{xy}$ stress</td>
<td>-0.0068</td>
<td>21</td>
<td>-0.0053</td>
</tr>
<tr>
<td></td>
<td>$\tau_{xz}$ stress</td>
<td>-0.0068</td>
<td>21</td>
<td>-0.0053</td>
</tr>
<tr>
<td>2</td>
<td>u-displacement</td>
<td>0.0619</td>
<td>3</td>
<td>0.0599</td>
</tr>
<tr>
<td></td>
<td>$\tau_{xy}$ stress</td>
<td>-0.0047</td>
<td>24</td>
<td>-0.0058</td>
</tr>
<tr>
<td></td>
<td>$\tau_{xz}$ stress</td>
<td>-0.0047</td>
<td>24</td>
<td>-0.0058</td>
</tr>
</tbody>
</table>

For Joint 2, v and stresses $C_x$, $C_y$, $C_z$, $\tau_{xy}$, $\tau_{xz}$, $\tau_{xz}$ stresses are all equal to 0.
# TABLE 24

**Effect of Mesh Size on Finite Element Solution**

**Semi-Infinite Body (μ = 0.2)**

<table>
<thead>
<tr>
<th>JT. No.</th>
<th>Elasticity Solution</th>
<th>Finite Element Solution (Bar Model - Govt.)</th>
<th>Common factor Displacements $\frac{P}{E_A}$</th>
<th>Stresses $\frac{P}{Q^2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>2x2x2 Mesh</td>
<td>4x4x4 Mesh</td>
<td>6x6x6 Mesh</td>
</tr>
<tr>
<td>1</td>
<td>u-displacement 0.1421</td>
<td>0.1345</td>
<td>0.1407</td>
<td>0.1413</td>
</tr>
<tr>
<td></td>
<td>v-displacement 0.0083</td>
<td>0.0052</td>
<td>0.0090</td>
<td>0.0086</td>
</tr>
<tr>
<td></td>
<td>w-displacement 0.0083</td>
<td>0.0052</td>
<td>0.0090</td>
<td>0.0086</td>
</tr>
<tr>
<td></td>
<td>$T_x$-stress -0.0136</td>
<td>-0.0230</td>
<td>-0.0169</td>
<td>-0.0151</td>
</tr>
<tr>
<td></td>
<td>$T_y$-stress -0.0095</td>
<td>-0.0094</td>
<td>-0.0099</td>
<td>-0.0099</td>
</tr>
<tr>
<td></td>
<td>$T_z$-stress -0.0095</td>
<td>-0.0094</td>
<td>-0.0099</td>
<td>-0.0099</td>
</tr>
<tr>
<td></td>
<td>$T_{yz}$-stress -0.0087</td>
<td>-0.0060</td>
<td>-0.0083</td>
<td>-0.0086</td>
</tr>
<tr>
<td></td>
<td>$T_{xy}$-stress -0.0136</td>
<td>-0.0151</td>
<td>-0.0143</td>
<td>-0.0143</td>
</tr>
<tr>
<td></td>
<td>$T_{xz}$-stress -0.0136</td>
<td>-0.0151</td>
<td>-0.0143</td>
<td>-0.0143</td>
</tr>
<tr>
<td>2</td>
<td>u-displacement 0.1440</td>
<td>0.1490</td>
<td>0.1456</td>
<td>0.1448</td>
</tr>
<tr>
<td></td>
<td>v-displacement -0.0381</td>
<td>-0.0306</td>
<td>-0.0386</td>
<td>-0.0385</td>
</tr>
<tr>
<td></td>
<td>w-displacement -0.0381</td>
<td>-0.0306</td>
<td>-0.0386</td>
<td>-0.0385</td>
</tr>
<tr>
<td></td>
<td>$T_{yz}$-stress 0.0212</td>
<td>0.0048</td>
<td>0.0131</td>
<td>0.0158</td>
</tr>
</tbody>
</table>

For Joint 2 $T_x, T_y, T_z, T_{xy}, T_{xz}$ are all equal to zero.
TABLE 25

Effect of Mesh size on Finite Element Solution

Infinite Body (H = 0.2)

<table>
<thead>
<tr>
<th>JT. No.</th>
<th>Elasticity Solution</th>
<th>Finite Element Solution (Bar Model - Gant.)</th>
<th>Common Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>2x2x2 Mesh</td>
<td>4x4x4 Mesh</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Absolute Value</td>
<td>% Error</td>
</tr>
<tr>
<td>1</td>
<td>u - displacement</td>
<td>0.0582</td>
<td>0.0615</td>
</tr>
<tr>
<td></td>
<td>v - displacement</td>
<td>0.0076</td>
<td>0.0055</td>
</tr>
<tr>
<td></td>
<td>w - displacement</td>
<td>0.0076</td>
<td>0.0055</td>
</tr>
<tr>
<td></td>
<td>Gx - stress</td>
<td>-0.0068</td>
<td>-0.0099</td>
</tr>
<tr>
<td></td>
<td>Gy - stress</td>
<td>-0.0017</td>
<td>-0.0019</td>
</tr>
<tr>
<td></td>
<td>Gz - stress</td>
<td>-0.0017</td>
<td>-0.0019</td>
</tr>
<tr>
<td></td>
<td>Gyz - stress</td>
<td>-0.0042</td>
<td>-0.0038</td>
</tr>
<tr>
<td></td>
<td>Gxy - stress</td>
<td>-0.0068</td>
<td>-0.0073</td>
</tr>
<tr>
<td></td>
<td>Gzx - stress</td>
<td>-0.0068</td>
<td>-0.0073</td>
</tr>
<tr>
<td>2</td>
<td>u - displacement</td>
<td>0.0619</td>
<td>0.0680</td>
</tr>
<tr>
<td></td>
<td>Gxy - stress</td>
<td>-0.0047</td>
<td>-0.0091</td>
</tr>
<tr>
<td></td>
<td>Gz - stress</td>
<td>-0.0047</td>
<td>-0.0091</td>
</tr>
</tbody>
</table>

For Joint 2, v & w displacements & Gx, Gy, Gz stresses are all equal to zero.
SEMII-INFINITE BODY WITH A CONC LOAD P (H = 0.2)

EFFECT OF MESH SIZE ON FINITE ELEMENT SOLUTION

VERTICAL DISPLACEMENTS X Eq.

VERTICAL DISPLACEMENTS IN TERM OF P

VERTICAL DISPLACEMENT FROM ELASTICITY

VERTICAL DISPLACEMENT FROM BAR MODEL

VERTICAL DISPLACEMENT FROM NO-BAR MODEL

NUMBER OF CELLS THICK
### Numerical Figures at Joints Represent Percentage Error

**Results for Bar Models and No Bar Model.**

<table>
<thead>
<tr>
<th>Axis of Symmetry</th>
<th>0.00</th>
<th>0.05</th>
<th>0.10</th>
<th>0.15</th>
<th>0.20</th>
<th>0.25</th>
<th>0.30</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Axis of Symmetry:**

- Represents lines of equal virtual displacement.

**Plan:**

- VERS.
- PLAN.
- VERT.
- DISPL.
- MOLD.

**Surface:**

- Y-DISP.
- X-DISP.
- Z-DISP.

**Error:**

- Y-ERROR
- X-ERROR
- Z-ERROR

---

**Plate 4**
INFINITE BODY UNDER CONC LOAD P(4.0,0)

**I**

STRESS ON PLANE \((x=40)\) MULTIPLES OF \(P\times 10^{-2}\)

- **ERROR** = \(\%\) of the cells \(-0.125\times 0.125\)

- **ERROR** = \(\%\) of the bars \(-0.125\times 0.03125\)

- **ERROR** = \(\%\) of the no-barc model \(-0.125\times 0.03125\)

Represents lines of equal \(I_{xy}\) stress.

**AXIS OF SYMMETRY**

**RESULTS FOR BAR AND NO-BAR MODEL**

**NUMERICAL FIGURES AT JOINTS REPRESENT PERCENTAGE ERROR**
**SEMI-INFINITE BODY UNDER CONC LOAD P (M=0.1)**

**VERTICAL DISPLACEMENT ON PLANE (x=40)**

<table>
<thead>
<tr>
<th>% ERROR B</th>
<th>% ERROR C</th>
<th>% ERROR D</th>
<th>% ERROR E</th>
<th>% ERROR F</th>
<th>% ERROR G</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.30</td>
<td>0.47</td>
<td>0.21</td>
<td>0.15</td>
<td>0.24</td>
<td>0.00</td>
</tr>
<tr>
<td>0.35</td>
<td>0.33</td>
<td>0.40</td>
<td>0.19</td>
<td>0.20</td>
<td>0.00</td>
</tr>
<tr>
<td>0.06</td>
<td>0.57</td>
<td>0.23</td>
<td>0.42</td>
<td>0.25</td>
<td>0.00</td>
</tr>
</tbody>
</table>

**RESULTS FOR BARRY MODEL**

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>z</th>
<th>% ERROR</th>
<th>% ERROR</th>
<th>% ERROR</th>
<th>% ERROR</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.21</td>
<td>-0.05</td>
<td>0.12</td>
<td>0.11</td>
<td>0.07</td>
<td>0.00</td>
<td></td>
</tr>
<tr>
<td>0.20</td>
<td>0.03</td>
<td>0.19</td>
<td>0.17</td>
<td>0.08</td>
<td>0.00</td>
<td></td>
</tr>
<tr>
<td>0.25</td>
<td>0.03</td>
<td>0.19</td>
<td>0.17</td>
<td>0.08</td>
<td>0.00</td>
<td></td>
</tr>
</tbody>
</table>

**PLATE**

**MULTIPLYING FACTOR FOR DISPLACEMENTS**

- **SIZE OF THE CELLS:** OX1, 25DX125D

- **EQUATION:**
  - % ERROR = |x-Dis|/(|x-Dis|) x 100%

- **% ERROR**:
  - x-disp (BAR MODEL - CANT) - TOP
  - x-disp (BAR MODEL - YETT) - MID
  - x-disp (NO-BARY MODEL - MEL) - BOL

**PERCENTAGE LINES OF EQUA VERTICAL DISPLACED**

**NUMERICAL FIGURES AT JOINTS REPRESENT PERCENTAGE ERROR**
NUMERICAL FIGURES AT JOINTS REPRESENT PERCENTAGE ERROR

RESULTS FOR BAR MODELS AND NO BAR MODEL

AXIS OF SYMMETRY

PLAN

ELUTION

SIZE OF THE CELLS = 20 x 20

AXIS OF SYMMETRY

PREPARED BY (REPLACE WITH NAME)

PRESENTED BY (REPLACE WITH NAME)

PLATEA
NUMERICAL FIGURES AT JOINTS REPRESENT PERCENTAGE ERROR

RESULTS FOR BAR MODELS AND NO BAR MODEL

AXIS OF SYMMETRY

SEMINECTIC BODY UNDER CAGING LOAD P(4,0)

PLATE II

STRESS ON PLANE (2x10) MULTIPLES OF EXACT STRESS

SIZE OF THE CRACK-GAP 1/3 1/2 1/4 1/8 1/16"
SEMINEFINITE BODY UNDER CONC LOAD P (M = 0.7)

VERTICAL DISPLACEMENTS ON PLANE (z = 45) AND % ERROR

MULTIPLYING FACTOR FOR DISPLACEMENTS, E \times 10^v

PLAN Y ELEVATION SIZE OF THE CELLS 0.01 \times 0.01

\[ \% \text{ERROR} = \frac{y_{\text{DIS}}(\text{ELAS}) - y_{\text{DIS}}(\text{NEW})}{y_{\text{DIS}}(\text{ELAS})} \times 100 \]

WHERE

\[ y_{\text{DIS}}(\text{BAR MODEL - GANT). TOP}) \]

\[ y_{\text{DIS}}(\text{NO BAR MODEL). BOTTOM}) \]

REPRESENTS LINES OF EQUAL VERTICAL DISPLACEMENT.

AXIS OF SYMMETRY

NUMERICAL FIGURES AT JOINTS REPRESENT PERCENT DISPLACEMENT.
SEMILINFITE BODY UNDER CONCLOAD P (μ = 0.2)

STRESS ON PLANE (α = 40) IN PATHS OF 6 BY 6 PERCENT ERROR.

RESULTS FOR BAR MODEL (GAUSS)

WHERE

J_x (EQU) = J_x (FIN.EL.)

J_x (FIN.EL.) = BY MODEL FORCES TOP

J_x (FIN.EL.) = BY DISPLACEMENT EQUATION

REPRESENTS LINES OF EQUAL J_x STRESS.
Semi-Infinite Body Under Conc. Load $P(\mu = 0.2)$

Stress on Plane ($\gamma = 220$) multiplies $\gamma = 10^4 \alpha$ x 10,000.

Size of the Cells: $0 \times 1.250 \times 1.250$

$\%$ Error = $|\gamma_{(El)} - \gamma_{(End)}| / \gamma_{(El)}$

where

$\gamma_{(El)}$: By Nodal Forces - Top

$\gamma_{(End)}$: By Displacements - Bottom

Results for SAP Model (Grant)

Numerical Figures at Joints Represent Percentage Error
SEMI-INFINITE BODY UNDER CONC. LOAD $P (H=0.2)$

$I_{xy}$ STRESS ON PLANE $(\alpha=40)$

\[ \frac{\Delta I_{xy}}{I_{xy}} \times 100 \text{%} \]

SIZE OF THE CELLS = O.2 x 0.2

BARE MODEL (GAIT)

\[ \frac{\Delta I_{xy}}{I_{xy}} = \left( \frac{I_{xy}(ECC) - I_{xy}(FIN)}{I_{xy}(ECC)} \right) \times 100 \text{%} \]

6.588

$I_{xy}$ (FIN.ECC) - BY NODAL FORCE METHOD

$I_{xy}$ (FIN.ECC) - BY DISPLACEMENT-THE

REPRESENTS LINES OF EQUAL $I_{xy}$ STRESS

AXIS OF SYMMETRY

RESULTS FOR BARE MODEL (GAIT)

NUMERICAL FIGURES AT JOINTS REPRESENT PERCENTAGE ERROR
SEMI-INFINITE BODY UNDER CONC. LOAD \( P(4=0.2) \)

**PLATE 25**

**STRESS ON PLANE \( (Z=250) \)**

- Multiples of \( \frac{P}{A} \times 10^{-3} \%

**ERROR:**

- Type (1) = Type (FIN.EL.)
- Type (2) = Type (FIN.EL. - BY NODAL FORCES)
- Type (3) = Type (FIN.EL. - BY DISPLACEMENT ERROR)

**SIZE OF CELLS:**

- \( X \times Y \)

**RESULTS FOR PARALLEL LINES:**

- \( X \)
- \( Y \)

**VALUES:**

- \( X \): \( 3.91 \, 3.47 \, 2.85 \, 2.21 \, -2.42 \)
- \( Y \): \( 3.27 \, 3.51 \, 4.15 \, 5.62 \, 7.25 \, 8.9)
INFINITE BODY UNDER CONCENTRATED LOAD \( P(x=0.2) \)

Stress on plane \( (y=40) \), multiples of \( 10^4 \), 2\% error

Size of the cells: \( 6 \times 125 = 1250 \)

Bar model (Gant)

\[ \% \text{ Error} = \left| \frac{T_a(EIGS)}{C} - \frac{T_b(FNL1)}{EIGS} \right| \]

\( T_a(FNL1) \) by nodal forces - Top
\( T_b(FNL1) \) by displacements, Bar model

Represents lines of equal \( T_y \) stress

Numerical figures at joints for bar model (Gant)

Axis of symmetry

Results for symmetry

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15

0 2.92 2.11 0.31 0.01 -0.43 -0.62 -0.92 -0.10 0 0.18 0.47 0.75 1.04 1.32 1.61

0 4.10 3.53 2.96 2.40 1.84 1.27 0.71 0.15 -0.49 -0.92 -1.35 -1.78 -2.21 -2.64 -3.07

0 4.51 3.93 3.35 2.78 2.20 1.63 1.05 0.48 0.12 -0.32 -0.69 -1.05 -1.42 -1.78 -2.15

0 4.39 3.91 3.43 2.95 2.47 1.99 1.51 1.03 0.55 0.10 -0.24 -0.66 -1.08 -1.50 -1.93

0 4.27 3.79 3.31 2.83 2.35 1.87 1.39 0.91 0.43 0.06 -0.06 -0.49 -0.92 -1.35 -1.78

0 4.15 3.67 3.19 2.71 2.23 1.75 1.27 0.79 0.31 0.04 -0.04 -0.47 -0.90 -1.32 -1.75

0 4.03 3.55 3.07 2.59 2.11 1.63 1.15 0.67 0.20 0.03 -0.03 -0.45 -0.88 -1.30 -1.73

0 3.91 3.43 2.95 2.47 1.99 1.51 1.03 0.55 0.08 -0.02 -0.44 -0.87 -1.30 -1.73 -2.15

0 3.79 3.31 2.83 2.35 1.87 1.39 0.91 0.43 0.06 -0.02 -0.42 -0.85 -1.28 -1.71 -2.14

0 3.67 3.19 2.71 2.23 1.75 1.27 0.79 0.31 0.04 -0.02 -0.40 -0.83 -1.26 -1.69 -2.12

0 3.55 3.07 2.59 2.11 1.63 1.15 0.67 0.20 0.02 -0.02 -0.38 -0.81 -1.24 -1.67 -2.10

0 3.43 2.95 2.47 1.99 1.51 1.03 0.55 0.08 -0.02 -0.36 -0.79 -1.22 -1.65 -2.08 -2.51

0 3.31 2.83 2.35 1.87 1.39 0.91 0.43 0.06 -0.02 -0.34 -0.77 -1.20 -1.63 -2.06 -2.49

0 3.19 2.71 2.23 1.75 1.27 0.79 0.31 0.04 -0.02 -0.32 -0.75 -1.18 -1.61 -2.04 -2.47

0 3.07 2.59 2.11 1.63 1.15 0.67 0.20 0.02 -0.02 -0.30 -0.73 -1.16 -1.58 -2.01 -2.44
ERRORS AT LOCATION 1, HORIZONTAL SECTION AB

EFFECT OF POISSON'S RATIO ON FINITE ELEMENT SOLUTION

\( \% \text{ERROR vs } \% \text{ERROR FOR STRESSES AND DISPLACEMENTS} \)

SEM-INFINITE BODY (JOINT NUMBER 1)

<table>
<thead>
<tr>
<th>PLAN-ART</th>
<th>ZATION RESULTS FOR BAR AND NO-BAR MODEL</th>
</tr>
</thead>
<tbody>
<tr>
<td>x1</td>
<td>0.0</td>
</tr>
<tr>
<td>x2</td>
<td>0.0</td>
</tr>
<tr>
<td>x3</td>
<td>0.0</td>
</tr>
</tbody>
</table>

\( \% \text{ERROR} \) = ELAB-SOL-FINED SOL

EAB-SOL

EAB MODEL (CPL)

NO-EAB MODEL (NPL)

u - DISPLACEMENT

v - DISPLACEMENT

w - DISPLACEMENT

\( \% \text{ERROR} \) vs \( \% \text{ERROR} \) for stresses and displacements.
EFFECT OF POISSON'S RATIO ON FINITE ELEMENT SOLUTION

%ERROR AT LOCATION 4, HORIZONTAL SECTION AB. (PLATE-40)

%ERROR VS. %ERROR FOR STRESSES AND DISPLACEMENTS.

SEM-INFINITE BODY (JOINT NUMBER 4)

RESULTS FOR BAP AND NO BAP MODEL

SIZE OF THE CELLS: OX0 X0

BAP MODEL (GANT)

NO BAP MODEL (MEL)

V-Displacement

-Y-Displacement

-Ty Stress

Tz Stress

Tyz Stress
EFFECT OF POISSON'S RATIO ON FINITE ELEMENT SOLUTION

% ERROR AT LOCATION 6, HORIZONTAL SECTION AB (PLATE-9)

% ERROR FOR STRESSES AND DISPLACEMENTS

SEMINEFINITE BODY (JOIN NUMBER a)

Y ELEVATION

SIZE OF THE CELLS (40X40)

ELEVATIONS

BAR MODEL (SANT)

NO BAR MODEL (BU)

ELAS. SOL. - FIN. EL. SOL.

U DISPLACEMENT

V & W DISPLACEMENTS

U & W STRESSES

VY & VZ STRESSES

0.0 0.1 0.2 0.3

G & G STRESSES

% ERROR
EFFECT OF POISSON'S RATIO ON FINITE ELEMENT SOLUTION

INFINITE POLY(JOINT NUMBER 1)

BAR MODEL (GANT)

WEAK MODEL (NO)

ERROR = [ERROR - LAB. SOL. - FIN. SOL.] / [ERROR - LAB. SOL.]

U-DISPLACEMENT

NEW - DISPLACEMENT

STRESS

XY STRESS

XI STRESS

X2 STRESS

YZ STRESS

XZ STRESS

XY STRESS

X2 STRESS

-% ERRORS AT LOCATION I, HORIZONTAL SECTION AB
EFFECT OF POISSON'S RATIO ON FINITE ELEMENT SOLUTION

INFINITE BODY (JOINT NUMBER 2)

RESULTS FOR BAR AND NO-BAR MODELS

BAR MODEL (CONT.)

NO-BAR MODEL (CONT.)
EFFECT OF POISSON'S RATIO ON FINITE ELEMENT SOLUTION

INFINITE BODY (JOINT NUMBER 3)

STRESS VS. V-ERROR FOR STRESSES AND DISPLACEMENTS

V-ERROR VS. V-ERROR FOR STRESSES AND DISPLACEMENTS

SPHERICAL BODY (JOINT NUMBER 3)

STRESS VS. V-ERROR FOR STRESSES AND DISPLACEMENTS

V-ERROR VS. V-ERROR FOR STRESSES AND DISPLACEMENTS
EFFECT OF POISSON'S RATIO ON FINITE ELEMENT SOLUTION

% ERROR AT LOCATION G, HORIZONTAL SECTION AB

INFINITE BODY (JOINT NUMBER 6)

RESULTS FOR BARE AND NO BAR MODEL

ELEVATION

SLICE OF THE MODEL: Gx0x0

FOR MODEL (GANT)

NO-BASE MODEL (med)

VERTICAL DISPLACEMENT

U DISPLACEMENT

STRESS

T_{xx} STRESS

T_{yy} STRESS

F_{xy} STRESS
The investigation described in this thesis warrant the following conclusions:

1. As judged by the examples of the infinite and semi-infinite elastic bodies acted upon by a concentrated force, the finite element method in its several versions making use of a model composed of cells in the shape of a rectangular parallelepiped, is a valid method of stress analysis, whose results in the form of displacements and stresses converge monotonically, to the true values. This applies both to the bar cells and the no-bar cells.

2. Of the two bar cells the Gantayat's cell has been found consistently superior in the precision of its results to the Yettram's cell.

3. On comparison with the Melosh's no-bar cell the Gantayat's bar cell has nearly always proved to be the better of the two. The reason for this may be explained as follows. The finite element idealization of a continuous medium makes it stiffer compared to the actual medium itself since the idealized structure has fewer degrees of freedom. In case of Melosh's no-bar model the continuity of displacement between adjacent elements is preserved and hence such an idealized structure is over stiff compared to the actual structure. However in the case of the bar model the discontinuities at the interfaces of the elements make the structure more flexible. This probably gives a solution
nearer to the true solution as compared to that obtained by using Melosh's no-bar model. It should be noted that lower strain energy is associated with a structure which is stiffer than the actual structure and hence probably higher strain energy is associated with the bar model compared to the energy associated with the no-bar model by Melosh.

4. After determination of the displacements the stresses may be found either by the nodal displacements or the nodal force concentrations. The results of the two solutions are comparable in precision.

5. The precision of results obtained seems to depend on the value of the Poisson's ratio of the body. The limited study of this subject attempted in the present investigation suggests that more often than not the % error decreases with increase in \( \mu \).

6. For value of \( \mu \) equal to 0.25 the edge bars in Yettram's cell lose their flexural stiffnesses and his cell becomes identical with Gantayat's. For \( \mu \) greater than 0.25 the flexural stiffnesses of the edge bars in the Yettram's cell become negative and the precision of the results obtained by the use of these cells is lowered. On the other hand the Gantayat's cell with \( \mu \) greater than 0.25 continues to give good results.
BIBLIOGRAPHY


10. A. Hrennikoff, Mimeographed Notes on Plane Stress and Plate bending, in Course CE551, University of British Columbia, Vancouver.