A SIMPLE SHEAR MACHINE FOR SOIL

by

D.J. PICKERING
B.Sc. (Tech.), Manchester University, 1943

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We accept this thesis as conforming to the
required standard

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April, 1969
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Department of Civil Engineering

The University of British Columbia
Vancouver 8, Canada

Date April 30, 1969
ABSTRACT

The new shear machine enforces more uniform deformations than the conventional triaxial test. A low compliance pore pressure measuring device is an integral part of the machine.

The machine described is capable of applying cyclic normal stress up to 1000 lb. per sq. in. and alternating cyclic shear stress up to ± 500 lb. per sq. in. Static loads can be taken 50 percent higher.

The test specimen can be cut from an ordinary undisturbed drill hole sample, being 2 ins. square and 1 1/8 in. high. Height variation of ± 1/8 in. is permitted during testing, but there is no "dead" zone; the entire sample is subjected to the applied shear.

An analytic solution is presented, for the boundary value problem of an anisotropic elastic sample in the tests to be described. This solution shows the variation of the stress field and deformations throughout the sample. For the benefit of any future simple shear machine designs, the theoretical relationship was also examined between the ratio of sample length to height and the uniformity of stresses and displacements within the sample.

In comparing test results from the new machine with conventional triaxial tests, it was found that the measured strengths are different. Some of the results suggest that the triaxial test could over-estimate the strength of undrained sand. The difference between simple shear and triaxial conditions is, therefore, of more than theoretical interest.

Liquefaction of undrained sand was readily induced by alternating shear in the new machine. It was found that liquefaction alters the structure of a sand sample, rendering it more susceptible to re-liquefaction.
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CHAPTER I
INTRODUCTION

Objectives. A few years ago, the mathematics of most soil engineering problems could not be solved without broad simplifying assumptions. For many problems, the necessary assumptions were probably the greatest source of error in an engineering solution. Recent advances in computer techniques have reduced the necessity of simplification, so much that inaccuracies of the mathematical treatment are no longer the prime source of error in soils engineering.

Improved laboratory results are now needed, to take full advantage of the modern computer techniques. The objectives of the work described herein were:

a) To build a new design of simple shear machine, to assist in obtaining improved laboratory results.

b) To conduct a series of preliminary tests with the machine.

Other Types of Apparatus. Laboratory shear testing of soils is most commonly done by triaxial apparatus, with direct shear tests sometimes employed for simplicity and speed. Various designs have also appeared for testing equipment using basically triaxial apparatus (i.e. load plunger and confining chamber pressure), but modified to give plane strain conditions; the apparatus developed at Imperial College, London\(^1\) is an example. Other types of shear testing apparatus have also been developed. For example, a machine capable of applying three separately controlled principal stresses was described by Kjellman\(^2\) and a more

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recent version by Ko and Scott\(^3\). In dynamic testing, velocity of shear and longitudinal compression waves of small amplitude are measured (e.g. Hardin\(^4\)) to obtain a shear modulus and Poisson’s ratio but test procedures do not yield a complete stress-strain relationship.

**History of Simple Shear Testing.** An apparatus for simple shear testing has been described by K.H. Roscoe\(^5\). The apparatus now to be described is intended to do basically the same type of testing, but it allows increased flexibility of procedure and seeks to eliminate one or two disadvantages in the earlier design.

Various design advances have been made by Roscoe since his original model. Details have not been published of all these later machines, but some information on the latest (Mark 6) model has been given by Roscoe, Bassett & Cole\(^6\).

Machines for simple shear have also been developed in Scandinavia,\(^7,^8\) but these suffer from certain disadvantages inherent in a test specimen of circular section. That is, conditions are neither plane stress nor plane strain and, as the vertical boundaries are flexible, the shapes of deformations around them are functions of the material as

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well as the applied load. The result is that the actual stress field is very hard to analyse. Also deformations can become very far from uniform.

A simple shear machine has also been developed by the University of California at Berkeley. The principles of operation are basically the same as Roscoe's third machine, but the actual means employed to put them into effect are different.

The machine at Berkeley has been used largely to examine the tendency of sand to liquefy under cyclic shear stress. The various machines at Cambridge have been used for more conventional types of shear test, mostly on dry granular materials, but meticulous endeavours were made to determine variations of conditions from point to point in the sample. For example, at Cambridge, several load cells around the boundaries were used to measure the distribution of normal force, eccentricity and friction force. Experiments were also done with lead shot distributed in sand samples and X-ray photographs were taken at successive small strain increments; this permitted reconstruction of the complete pattern of deformations within the sample.

Disadvantages of Common Soil Testing Procedures. In considering the use of an uncommon type of test, it appears desirable to explain what advantages are hoped for, over standard procedures. Engineering laboratory testing consists of observing either the behaviour of a model or the behaviour of a specimen of material. A model is essentially a test in which conditions, and possibly materials, vary from point to point in space. If scaling has been done correctly a model gives a complete answer, but to one problem only. The other approach to an engineering problem is to

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calculate stresses at certain points, relate these to strains at the same points, and hence determine displacements. A laboratory investigation is required for this second approach to supply the relationship between stresses and strains at a point and to ensure that all the variables which affect the behaviour of the material are taken into account. When conditions within a specimen are uniform at every point, it is reasonable to assume that the behaviour of the specimen represents the behaviour at a point within a mass of the same material. Uniformity of conditions is thus the goal in testing a specimen, but in practice it is probably unattainable.

If conditions are not uniform within the specimen, the measured quantities are averages. This would be the case when an overconsolidated clay or other material with peak strength fails in a triaxial test. In such a test, as stresses rise towards failure, some slight variation of material will lead to one zone exceeding its peak strength and shedding load onto neighbouring zones while undergoing increases of strain. The neighbouring zones will then be overstressed and, in the standard triaxial test, will also shed load, and contribute further large strains in narrow zones. The failure progresses in the manner described until a thin zone, right across the sample, has passed its peak strength and the sample "fails".

Both of the common laboratory shear tests for soil involve non-uniform conditions within the samples. In the direct shear box the non-uniform deformations are imposed by the apparatus and the shearing effect is propagated through the sample in an unknown way. In the triaxial test, non-uniform conditions develop as the test proceeds. They stem from the frictional restraint at the ends of the sample and from the non-uniform deformations which can start in a weak zone and spread outwards. The frictional end restraint
in the triaxial test has been largely eliminated by Rowe and Barden\textsuperscript{10} by the use of specially designed end platens, but although this treatment improves conditions they may still be non-uniform, due to the progressive spread of large deformations already described.

Non-uniformity of deformations leads to the following specific difficulties in triaxial and direct shear tests.

\textbf{a)} Measurement of volumetric strain. As soon as deformations are non-uniform, an unknown proportion of the total sample is contributing to the volume change. This means that the denominator of the expression for strain is not known. In the direct shear box the error is compounded by difficulty in measuring the volume change accurately.

\textbf{b)} Measurement of shear strain. The relative movement of the two halves of the direct shear apparatus can be measured, but how this compares with "average" shear strain is not known. In the triaxial test, the axial deformation gives a value for average axial strain, which can be directly related to shear strain. However, the high degree of uniformity of conditions within the sample at the start of a triaxial test, is progressively destroyed. Cone-shaped zones at the ends of a triaxial sample, are restrained from shearing. The points of these cones point towards each other on the central axis of the specimen. The result is that the axial length which is deforming is least between the points of the cones and increases toward the cylindrical surface of the sample. The

total axial movement, however, is generally uniform, hence, due to the variation in its denominator, the axial strain is greatest at the centre and diminishes toward the surface. An additional source of non-uniformity of shear strains in a triaxial sample is the progressive development of zones of large deformation already described. It is probable that the true shear strain in a zone approaching failure is far greater than any measured strain.

c) Stresses within the sample. In the triaxial test, the average stress on any plane is known at all times, but non-uniformity of strain leads to non-uniformity of pore pressure and effective stresses may vary considerably from the measured values. In the direct shear test, average normal and shear stresses on a horizontal plane are known, but again, non-uniformity of conditions may lead to non-uniform effective stresses.

In addition to the results of non-uniform deformations, listed above, there are various other disadvantages inherent in the usual soil laboratory shear tests. For example, the step-shaped discontinuity produced at the ends of a direct shear box, cannot be readily enclosed by a membrane. In the direct shear testing, it thus becomes impractical to run undrained tests, use back pressure on the pore fluid or take useful pore pressure measurements. Also, pre-selection of the failure plane in a direct shear test means that the material may contain layers which are weaker than the measured strength. In a triaxial test, the necessity of sealing the sample and the complications of the necessary drainage lines through a sealed pressure chamber lead to possible trouble from leakages and trapped air.

A more serious disadvantage of the triaxial test is that even if conditions were perfectly uniform within the
sample, they would usually be different from those in the ground because, in actual engineering problems, conditions are usually closer to plane strain than to triaxial test stresses. In fact, if the test specimen is supposed to represent a point in the ground, the only points that can be properly represented by a triaxial test, are points vertically below the centre of a circular load in level ground. Another way in which the triaxial test fails to duplicate most real problems is that in a triaxial sample the planes of maximum shear stress are always in the same directions, at 45 degrees to the vertical. In the ground it is very likely that shear strain during ground deposition will be followed by developing shear stresses on different planes as conditions change. The result is that there is likely to be a rotation of the principal planes as the stress field develops.

Intended Characteristics of the Machine to be Described. It is unlikely that any testing equipment can be developed which will overcome all the disadvantages listed above, but some of them are minimized in the apparatus to be described. It is designed for the application of normal load to the sample, with no lateral strain, and for deformation of the type shown in Fig. 1. that is, simple shear strain. The sequence of

![Fig. 1. Type of Deformation](image)

normal load with no lateral strain (maximum shear planes at 45 degrees) followed by horizontal shear (maximum shear planes horizontal and vertical) involves a rotation of the principal planes. This corresponds to conditions which often occur in practical engineering. The correspondence is
increased because the shearing takes place in conditions of plane strain. It will be shown in a later section that during shear the stress field within the sample is complicated, with stresses close to the desired values through the interior but varying considerably near the boundaries. However, the boundaries are approximately rigid, forcing the deformations to be almost uniform. The variation of stress represents a departure from the desired conditions, but its importance is believed to be minimized by the following considerations:

a) The uniformity of deformation prevents excessive strains in narrow zones and actual strains, through the interior of the sample, will be close to the measured strain. Thus measured deformation will represent strain behaviour reasonably closely.

b) The stresses through the interior of the sample do not vary much from the average. Thus behaviour of the sample is largely governed by stresses which are close to the average, that is, close to the applied stresses.

c) Since the measured strain closely represents sample behaviour and the behaviour is largely governed by stresses close to the applied stresses, it follows that a good approximation of the true stress-strain relationship should be obtained.

Some details of the machine to be described, are intended to maximize its application to practical engineering problems. For example, dimensions of the specimen are such that it can be cut from ordinary undisturbed drill hole samples. The specimen is placed in the machine and consolidated with the same stress orientation, relative to any layering, as would be expected in nature. A membrane is used to enclose the sample, permitting drained or undrained tests, application of back pressure and measurement of pore pressure.
The machine is designed to work with loads up to 1000 lb. per sq. in. cyclic normal pressure and 500 lb. per sq. in. cyclic shear stress. Steady loads 50 percent higher can be applied.
CHAPTER II
DESCRIPTION OF THE APPARATUS

Size. The apparatus is designed to apply normal loads up to 1000 lb. per sq. in. and shear loads up to 500 lb. per sq. in., under cyclic dynamic conditions, to a sample 2 in. square and initially 1-1/8 in. high. As will be described in Chapter III, it would be theoretically preferable to have a sample of flatter proportions. The actual dimensions, however, were chosen because the height is considered to be about the minimum to give meaningful, representative results from unconsolidated alluvial deposits, while a 2 in. square horizontal area is a reasonable maximum size (allowing for trimming) to cut from a 3 in. piston or Shelby sample or from a sample obtained by driving casing.

The apparatus has three main assemblies and various accessories and associated equipment.

Horizontal Motion. The base of the sample is supported on a carriage with horizontal linear roller bearing motion. The carriage sides extend upward, to form the lower halves of the metal shear box sides. Pressed into the carriage are bearing spindles, which form hinge points for the lower edges of the shear box ends. The shear load is applied to the sample by a horizontal force, acting on the end of the carriage, (Fig. 2a.).

Vertical Motion. A guide frame is constrained between roller bearing cam guides, so that it can move freely in a vertical line, with no rotation and no lateral movement. The bearings are well separated, to give a good resisting moment to tilting. This frame carries spindles, forming hinges for the upper edges of the shear box ends. It also carries the removable loading head, by which the normal load is made to bear on the top of the sample. By this means, the whole of the upper loading head and hinges for the upper edges of the box ends are carried as a single floating assembly. Dilation
A = Linear roller motion.
B = Hinge spindles, lower edges of box ends.
C = Sample
D = Applied shear load.

FIG. 2a. HORIZONTAL MOTION.

A = Removable frame holding upper half of box sides (shaded).
B = Roller bearing guides for vertical motion.
   (Six more inside underneath).
C = Bed for horizontal motion.

FIG. 2c. BODY FRAME. CUT AWAY VIEW.

A = Roller bearing guides.
B = Hinge spindles, upper edges of box ends.
C = Removable loading head (shaded).
D = Sample.
E = Normal load.

FIG. 2b. VERTICAL MOTION.

A = Slotted and circular holes for hinge bearings.
B = Sample.

FIG. 2d. END PIECE. 

Fig. 2. Machine Sub-Assemblies
or contraction of the sample can therefore be allowed and the "dead" zone, of sample subjected to compression but not shear, is eliminated (Fig. 2b.).

The Body Frame. (Fig. 2c.) The various assemblies are united in a heavy steel mounting, to which are attached the upper halves of the box sides, the guides for the horizontal and vertical motions and other stationary parts. Fig. 3 shows the carriage for horizontal motion, mounted on its roller assemblies, in the body frame.

Fig. 3. Horizontal Carriage, on its Rollers, in Body Frame

Shear Box Ends. (Fig. 2d.) In addition to the three main assemblies, there are the end pieces of the box. These are connected between the vertical and horizontal motions in such a way as to enclose the sample ends. The distance between the corners at one end of the box varies when shear occurs at constant volume (i.e. constant perpendicular height). The distance also varies for any dilation or contraction of the sample, or as a combination of volume change and shear. Each end piece has one pair of bearings providing rotation
only and another pair, riding in slotted holes, allowing both rotation and variation of distance. By this means, allowance is made for the change in dimension of the ends during shear. The end pieces are interchangeable and are arranged so that one end has its slotted holes at the top and the other at the bottom. By this means, any friction, between the sample and the box end, is made to act in opposite directions at the two ends, retaining cross-symmetry conditions. Also, this arrangement puts the pinned edge of one end at the top and of the other end at the bottom and, since they are always parallel, if one end is swinging upward, the other must be swinging on a downward arc. The weights of the two end pieces therefore counterbalance and the machine's only resistance to horizontal motion is the friction of the oiled bearings. In dynamic testing, of course, the inertia of the horizontal carriage is an added complication. The vertical motion, with shear box ends attached, is shown being installed in the body frame in Fig. 4.

Fig. 4. Installing Vertical Motion
A = Normal load, to loading head.
B = Shear load, to carriage on rollers. Shown at limit of travel.
C = Vertical motion, between roller guides.
D = End plates.
E = Membrane.

Fig. 5. Schematic Diagram of Complete Machine. Section along Plane of Centre Line.
A schematic illustration of the apparatus, as described so far, is given in Fig. 5.

**Pore Pressure Measurement.** In order to minimize compliance, pore water pressure is measured by a cell, mounted inside the loading head (Fig. 6). It consists of a heat treated beryllium copper diaphragm 0.22 in. diameter and 0.015 in. thick. Fixed edge conditions are obtained by forming the diaphragm integrally with a supporting ring of 0.566 in. diameter and 0.20 in. thickness.

![Diagram of Pore Pressure Measurement](image)

A diaphragm type foil strain gauge, providing a half bridge, is cemented to the side of the diaphragm remote from the pore water. An additional, waterproofed gauge, on the other side of the diaphragm, would provide a full bridge but, so far, all tests have been short duration, temperature has not had time to rise noticeably and a half bridge has been adequate.
The cell is held in place by a stainless steel screw ring and is sealed against leaks by an O-ring in its seating. The groove for the O-ring is designed to be completely filled when the cell is screwed fully in. Metal to metal contact is obtained, hence the compliance of the O-ring is considered to be negligible under change of water pressure. After mounting a sample, water for saturation is supplied through a tube with a non-displacement shut-off valve, in the body of the load head. A similar tube and valve diametrically opposite the first, permits flushing water right through the head, over the pressure cell, to remove air. Removal of air is facilitated by having interrupted threads for the stainless steel ring, connected to the water passages by wide slots over the upper face of the ring and by 1/32 in. diameter holes, just below the diaphragm, in the beryllium copper support ring, so that air cannot be trapped in this hollow. The arrangement of the non-displacement valves is also shown in Fig. 6.

Figs. 7a and 7b show the loading head assembly. The handles for the two valves are prominent in Fig. 7a, while Fig. 7b clearly shows the stainless steel ring retaining the diaphragm and the entrances to the water passages on either side. Expansion of the measuring system is practically eliminated because the non-displacement valves are built into the loading head, negligible compliance of the O-ring seal and all water passages between them and the sample and the pressure cell are machined out of the solid stainless steel head. The only appreciable expansion is that due to the deflection of the diaphragm. The diaphragm is calculated to have a maximum deflection of $39 \times 10^{-6}$ in. under 100 p.s.i. pressure, involving a volume of $0.49 \times 10^{-6}$ cu. in. In other words, the change in volume of the measuring system, including leads, etc., is approximately $4.9 \times 10^{-9}$ cu. in. per lb.
Fig. 7a. Loading Head

Fig. 7b. Underside of Loading Head
per sq. in. The volume of the passageways below the cell and between the shut-off valves is approximately .066 cu. in. so, taking the compressibility of water as $3.4 \times 10^{-6}$ sq. in. per lb., a change of pressure of 1 lb. per sq. in. causes water to flow out of the sample, into the head, as follows

\[
\begin{align*}
\text{vol. change} & = 0.0049 \times 10^{-6} \\
\text{compression of water} & = 0.224 \times 10^{-6} \\
\text{Total} & = 0.229 \times 10^{-6}
\end{align*}
\]

Compliance of modern laboratory pore pressure apparatus usually depends on the length of a piece of flexible metal tube, connecting the sample to the transducer. However, assuming that this is minimized, a typical value would be approximately $0.5 \times 10^{-6}$ cu. in. per lb. per sq. in.

Hvorslev's equation for the response time of a piezometer\(^{11}\) can be adapted to investigate the time lag in pressure recording, due to compliance. First consider a sample of infinite volume and permeability $k$ in. per sec. If there is a change of pressure $\Delta p$ at time $t = 0$, the equation is

\[
\frac{p_1 - P}{p_0 - p_1} = \frac{p_1 - P}{\Delta p} = 1 - \exp \left[ -\frac{Fkt}{V\gamma_w} \right]
\]

where $p_0$ is the pressure which existed before time $t = 0$, $p_1$ is the new pressure which occurred at $t = 0$ and $p$ is the pressure recorded at a later time $t$, $F$ is the intake factor, allowing for the porous disc through which the pressure must pass, $V$ is the compliance of the cell and $\gamma_w$ is the density of water. Taking $F = 2.75d$, for a flat disc filter $\frac{Fkt}{V\gamma_w} = 0.167 kt \times 10^9$. A value of $\frac{Fkt}{V\gamma_w} = 4.605$ is enough to

make \( \frac{p_1 - p}{p_0 - p_1} = 99 \) percent. So, putting \( 4.605 = 0.167kt \times 10^9 \), the recorded value is correct within 1 percent of the amount of the change when

\[ k \cdot t = 27.6 \times 10^{-9} \text{ in.} \]

So, for the range of typical soils, the time theoretically required for \( p \) to equal \( p \pm 0.01 \Delta p \) is approximately as follows:

<table>
<thead>
<tr>
<th>Permeability (in. per sec.)</th>
<th>Clay</th>
<th>Silty Sand</th>
<th>Clean Sand</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0.4 \times 10^{-9} )</td>
<td>( 0.4 \times 10^{-6} )</td>
<td>( 0.4 \times 10^{-3} )</td>
<td>( 0.4 )</td>
</tr>
<tr>
<td>Time (sec.)</td>
<td>69</td>
<td>( 69 \times 10^{-3} )</td>
<td>( 69 \times 10^{-6} )</td>
</tr>
</tbody>
</table>

This result is adequate for tests on sand. If greater stiffness is required, the volume of water can be cut down considerably. Ample diameter passages were allowed for waterways, since they are simpler to make from a machinist's point of view. However, they could be reduced in volume considerably by driving in liner tubes or by taking the time to drill them finer. Furthermore, the little retaining ring could be made with a smaller diameter hole and with a projection to take up most of the space in the hollow of the diaphragm ring. By these means the volume of enclosed water could be reduced to about .01 cu. in. This would reduce the total compliance to \( 0.039 \times 10^{-6} \) cu. in. per lb. per sq. in., as compared to \( 0.23 \times 10^{-6} \) for the existing arrangement. This would result in recorded pressures being within 1 percent of \( \Delta p \) when \( kt \) is only \( 4.7 \times 10^{-9} \) in. So, for samples with permeabilities of \( 0.4 \times 10^{-9} \), \( 0.4 \times 10^{-6} \), \( 0.4 \times 10^{-3} \) and \( 0.4 \) in. per sec., the times required would become 12, \( 12 \times 10^{-3} \), \( 12 \times 10^{-6} \) and \( 12 \times 10^{-9} \) sec. respectively.

The effect of a non-infinite sample is that withdrawal of the volume required to record the pressure, reduces the pressure being recorded. (Hence the time to record the
pressure is affected, but this is a second order small quan-
tity and tends to reduce the delay.) The amount by which
the pressure is affected depends on the relative compres-
sibility of the water and the soil. The ratio of the pressure
actually occurring, to the pressure which would occur if the
compliance of the apparatus were zero, is given by

\[
\frac{1}{1 + K \cdot B}
\]

Where \( K \) is the compliance of the apparatus and \( B \) is given by

\[
B = \frac{1}{V} \cdot \frac{1}{C_w \cdot n + C_s}
\]

In which

- \( V \) = Volume of sample
- \( C_w \) = Compressibility of water
- \( C_s \) = Compressibility of soil skeleton
- \( n \) = Porosity

(See Appendix A for proof of this statement.)

The volume of the sample is approximately

\[2 \times 2 \times 1.125 = 4.5 \text{ cu. in.}\]

\( C_w \) is approximately \( 3.4 \times 10^{-6} \) sq. in. per lb.

It is hard to imagine a porosity of less than about .25 or a soil skeleton stiffer than \( 10^{-6} \) sq. in. per lb. and these figures make \( B = 0.12 \times 10^6 \). With \( K \) equal to

\[0.229 \times 10^{-6}\]

the pressure ratio \( = \frac{1}{1 + 0.12 \times 0.229} \) or 97 percent. This is with an exceedingly stiff soil. Also,
if \( K \) were modified as described, the resulting pressure ratio
would become \( \frac{1}{1 + 0.12 \times 0.039} \) or 99.5 percent.

Therefore, the effect of the finite size of the sample on the pressure being recorded, is less than 3 percent even with extremely stiff, dense soil and would be reduced to
less than 0.5 percent with the water passages modified, as
described.
Details. Some details remain to be described. In order to reduce friction between the sample and the sides and ends of the box, a lubricated rubber membrane is employed. It takes the form of a box-shaped latex envelope, 0.012 to 0.014 in. thick and its linear dimensions are 5 percent less than the sample it encloses. The small dimensions are needed to provide enough stretch to eliminate buckling on the shorter diagonal of the deformed side during a test. The top of the membrane has a half-inch diameter hole in it, to communicate with the recess in the loading head, where pore pressure measurements are made. Hardened steel plates are inserted in the top and bottom of the membrane, to spread it to full size, prevent tension at the corners from deforming the sample and anchor the top and bottom of the membrane to their respective mountings. The four vertical corners of the membrane tend to stretch away from the corners of the box. This tendency can be overcome by back pressure in the pore fluid,

![Fig. 8. End Plates. A Groove, for retaining a membrane corner shows through slotted hole of plate to right.](image-url)
but during placement of a sample in the box, back pressure cannot be applied. For this reason, during placement of samples formed in situ, the vertical corners of the membrane are retained by 1/16 in. diameter rubber moulding, 7/16 in. long, placed inside the membrane and then inserted in grooves, tucked around the sides of the box ends (see Fig. 8.).

These grooves have to be polished and have rounded corners where the membrane fits. Otherwise the stretching and working of the membrane over these parts during shear deformation can lead to rupture. The membranes described are thick enough to act as a measure of padding for sand grains, reducing any tendency to press through the membrane, and rub the box sides. Such padding action probably saves recurrent trouble from torn membranes, also, without it, the frictional drag of a sand sample through the membrane, onto the box sides would probably be excessive. For silt or clay samples, a thinner membrane might prove adequate. The elasticity of the membrane must be high, as the samples are inserted through the 1/2 in. diameter hole in the top and, for the purpose, this hole has to be stretched to approximately 3 in. square, as will be described in the section dealing with preparation of samples.

Associated Equipment. The method of applying loads is, of course, more or less optional. In the case now being described, loads are applied by means of pistons. A range of four sizes, on interchangeable mounts, allows selection of suitable ratios of normal to shear loads with a maximum piston pressure of about 120 lb. per sq. in. The piston pressures are controlled, leading to load controlled testing. The apparatus was designed for any type of shear testing programme, but dynamic testing was in mind and, in order to allow flexibility of loading cycles, the loading pistons are double acting. The two ends of the shear load piston are controlled by two different valves, see Fig. 9. The valves revolve at equal speeds, but can be advanced and retarded
relative to each other. They consist of stems with ports, rotating at constant speed in a valve body. Different loading cycles and different forms of load application can be introduced by varying the shapes of the valve stem ports and by varying the relative setting of the two valves. The pistons were designed to run filled with water, to stiffen them, but air has given sufficiently rapid response (about .02 sec. rise time at 2 cycles per sec.) and is simpler to use. The house supply pressure is passed, via adjustable regulators, into a bank of small reservoirs. These, in turn, are connected to the various valve inlets, as required. The vertical loading piston is mounted on a gantry, which can be swung aside to allow access to the machine and installation of samples.
Fig. 10a. General Arrangement with Vertical Load Piston in Place.

Fig. 10b. General Arrangement. Vertical Loading Frame Swung Aside.
The shear machine itself, is practically symmetrical and a load could, if desired, be applied at the opposite end of the horizontal motion. Space has been left for this purpose, so that a strain controlled loading system can be added later, without having to dismantle the piston.

Loads are measured with circular beryllium copper diaphragm load cells, having four strain gauges set around the periphery to give a full resistance bridge. Strains are calculated from the displacement of a linear differential transformer, which reads the displacement of the horizontal motion. Outputs of shear load, pore pressure and displacement are carried to a multi-channel recording oscillograph. This is capable of linear response up to 1000 cycles per sec. The normal load could be recorded on the same instrument but, in the tests run so far, it is convenient to record it on another machine.

Apparatus for supplying de-aired water at adjustable pressure is required. It is not an integral part of the machine, but a description of it is desirable, in order to understand the method of installing samples. This apparatus is, therefore, described in Appendix B.

The general arrangement of the machine and some of the associated equipment are shown in Figs. 10a and 10b.
CHAPTER III
THEORETICAL CONDITIONS OF TEST

Definition and Explanation. Simple shear in an \( x - y \) plane is defined as pure shear in which either \( \frac{\partial u}{\partial y} \) or \( \frac{\partial v}{\partial x} = 0 \), where \( u \) and \( v \) are displacements in the \( x \) and \( y \) directions respectively. In Chapter I it was stated that the apparatus under discussion is intended to apply normal load and simple shear deformation to a sample of soil. Uniformity of sample deformation was a stated objective. This chapter considers the boundary conditions around the sample and describes tests of deformation under various boundary conditions. A theoretical elastic solution for the stress distribution under shear load is described and the resulting theoretical deformations are compared with those actually obtained. The effect of designing a machine for samples of different proportions is considered. The application of normal load and the combined effect of normal load and shear load are examined.

Boundary Conditions. Fig. 1 illustrated the sample deformation. Fig. 11 shows this deformation transferred to a system of co-ordinates. A sample of length \( l \) and height equal to \( 2h \) has been strained through a small shear angle, \( \omega \). The \( x \)-and \( y \)-axes are horizontal and vertical respectively. Taking

\[ x = \text{length} \]
\[ y = \text{height} \]

\[ \omega = \text{shear angle} \]

\[ h = \text{height of sample} \]

\[ 2h = \text{total height of sample} \]

\[ \frac{\partial u}{\partial y} \] or \[ \frac{\partial v}{\partial x} \] = 0

Fig. 11. Co-ordinate System.
the case of an undrained test (no volume change), the top and bottom boundaries cannot deform vertically. Each point on the end boundaries must move by an amount \( \omega \cdot y \) in the \( x \)-direction. It follows that the top corners both move \( \omega \cdot h \) and the bottom corners \( -\omega \cdot h \), therefore the total lengths of top and bottom boundaries do not change. If the friction at the top and bottom boundaries is adequate to prevent slipping, every point on these boundaries moves the same distance and there is no extensional strain. If we could stretch the ends of the box sufficiently to allow for their changing dimensions as \( \omega \) increases, the extensional strains would not vary over the ends. This would make all boundary deformations exactly the same as simple shear and hence conditions would be uniform, simple shear throughout the sample. Unfortunately it does not appear that the metal ends can be made to extend uniformly in this way. It becomes necessary to provide for the increasing length by having the metal ends long enough for maximum strain and having a slip contact at their top or bottom. In the arrangement described, for one of the two directions of shearing, friction would act in a direction opposite to that required for conjugate shear stress on the sample ends. So, since the shear box is designed to deform in either direction, the best condition to aim for, is zero friction. Thus, out of four boundary deformation conditions required for simple shear, three can be provided and one cannot.

<table>
<thead>
<tr>
<th>Desired</th>
<th>Provided</th>
</tr>
</thead>
<tbody>
<tr>
<td>Top and Bottom</td>
<td>( u = k \cdot h ) and (-k \cdot h) ( v = 0 )</td>
</tr>
<tr>
<td>Ends</td>
<td>( u = k \cdot y ) ( v = 0 )</td>
</tr>
</tbody>
</table>

Since one of the boundary conditions is not compatible with simple shear, conditions through the sample cannot be uniform simple shear and it is necessary to consider
how actual conditions differ from those desired. In order to test the uniformity of deformations, dummy samples made of two colours of plasticine were strained in the machine.

Such a sample, before testing, is shown in Fig. 12a. A plasticine sample, after undergoing three full cycles of reversing strain is shown in Fig. 12b. For the trial in Fig. 12b, sand-blasted metal boundary plates at the top and bottom of the sample were placed in direct contact with the plasticine and allowed to develop their full "friction". After the test, these plates adhered to the plasticine and had to be cut away with fine piano wire; it was assumed that "friction" had been enough to prevent slipping. Looking at Fig. 12b, it will be noted that strains consist of practically uniform shear through most of the sample, of a magnitude which approximately corresponds to the boundary deformation. Around the boundaries there are areas of reduced shear deformation, but even after three cycles, of strain (30 degrees each way), there has been no development of any failure plane or of zones with shear strains markedly higher than the average.

While working with dummy samples, it was decided to see what would happen if the top and bottom friction were inadequate to prevent slipping at these boundaries. Some tests were performed with several narrow bands of oiled paper introduced at the sample top and bottom. These strips of paper effectively broke the frictional restraint on the plasticine. The result of such a test is shown in Fig. 12c. It will be noted that shear strain varies markedly through the sample and only a small zone near the centre approaches a shear strain corresponding to the boundary deformations.

It may be argued that because plasticine is incompressible compared to a soil skeleton, the deformations are not comparable. The plasticine tests described so far, were performed under plane strain and the practically incompressible plasticine could only accommodate itself to the uniform boundary deformations. As a possible method of allowing the
Fig. 12a. Undeformed

Fig. 12b. Plane Strain, Friction, 3 Cycles

Fig. 12c. Plane Strain, No Friction, 1/2 Cycle

Fig. 12d. Plane Stress, Friction, 1/2 Cycle

Fig. 12e. Plane Stress, No Friction, 1/2 Cycle

Fig. 12f. Plane Stress, No Friction, 1 Cycle
plasticine to behave, two dimensionally, as though it were compressible, three more plasticine samples were tested, but they were cut narrower than the width of the shear box. The boundary surfaces were in contact with the box in the \( x-y \) plane, but there was room on each side of the sample for lateral expansion. Thus material which would have compressed if it were soil, could expand sideways instead. The result is, of course, not a real proof of what happens to a different material (soil) under different conditions (no total volume change). However, the narrower sample represents plane stress instead of plane strain, and since the two conditions should give the same displacements for given boundary conditions, the results should be somewhat analogous. It must be remembered, though, that although squeezing a narrow sample may allow plasticine to represent compression, boundary tensile stresses are not available to make it represent expansion. It is emphasized that, in the above context, compression and expansion are meant to indicate local changes in the density of a soil skeleton. In an undrained test on soil, total volume changes are very small and are due to compliance under changes of total stress; the average density of the whole sample remains effectively constant. Looking at Fig. 12d it will be seen that the deformations are almost the same as in Fig. 12b and hence the satisfactory results with proper boundary friction are not due simply to the incompressible nature of plasticine. Fig. 12d differs from 12b chiefly in the slight slipping at the joints in the plasticine. This is because, with compressive stresses relieved, the layers of different colours do not adhere to each other so well. Fig. 12d, however, still shows the same favourable distribution of shear as Fig. 12b and most of the sample can still be seen to have undergone shear strain corresponding to the boundary deformation. Fig. 12e shows that frictional restraint of the top and bottom boundaries would assume extra importance with a compressible soil skeleton. Fig. 12d and 12e are directly comparable and whereas 12d
shows conditions close to ideal, Fig. 12e shows them very far from being so. Fig. 12f shows what would have happened to the sample in Fig. 12e, after one reversal of shearing action. In Fig. 12f much of the original shear strain has been removed but the plasticine cannot offer any further shearing resistance. After the stage shown in Fig. 12f, the plasticine can slide back and forth under cyclic shear, undergoing no further deformation. Fig. 12f does not exactly represent the behaviour of a sand sample when the top and bottom boundaries slide. As mentioned above, once the corners of the plasticine sample "compress", they will not re-expand. On the other hand, a non-cohesive soil skeleton would be expected to re-occupy empty corners even if it only fell into them. However, the question does arise whether a sand sample with inadequate top and bottom boundary friction would exhibit reduced shear resistance after undergoing one cycle of large strain. Such a sand sample could be experiencing alternate densification and loosening of the corners; the central portion of the sample could be sliding bodily to and fro, with little deformation, as in Fig. 12f. As a result of the tests on plasticine samples, it was decided that the machine would probably achieve the intended aims, provided that top and bottom boundary friction were adequate to prevent slipping at the sample boundary.

A Theoretical Elastic Solution. The results of trials with plasticine dummies appear satisfactory. However, an empirical investigation can never be as helpful as an analytic solution in examining the effects of changes in variables, for example, sample dimensions. A truly analytic solution yields the displacements and stresses and strains through the sample in general terms. The obvious way to obtain such a solution is to solve the appropriate boundary value problem in terms of linear elasticity. Soil is not linearly elastic, and it may be objected that such a solution cannot apply to soil. However, plasticine is not linearly elastic either, but it will be shown that the elastic analytic solution leads to
displacements very close to those of the plasticine samples. If the linear elastic solution can predict the displacements of plasticine, it implies that very great departures from linear elasticity do not invalidate the elastic solution to this particular problem. It therefore seems reasonable to assume that the elastic solution will yield an approximate resemblance to conditions in a soil sample, and such a solution is therefore worth pursuing. The elastic solution for isotropic material was given by Roscoe\textsuperscript{5}, with constants for a certain size of sample and a Poisson's Ratio of 0.5. The solution for the isotropic case is given in more general terms in Appendix C, together with the solution for anisotropic material having radial symmetry about any vertical axis (viz. equal elastic constants in all horizontal directions).

In developing and solving the anisotropic equations, there was no thought of actual evaluation, for real soils, of the five anisotropic elastic constants; the intention was only to study the stress field and deformation pattern in the simple shear machine. In order that the study could be extended to include the behaviour of materials representing as wide a range of soils as reasonably possible, anisotropy was included in the development.

Results of the elastic solution, plotted in dimensionless form by computer, are given in Figs. 13, 14, 15, 16 and 17. Each of these figures gives a comprehensive summary of stresses and displacements in a sample after a small unit angle of shear deformation. The different figures represent varying dimensions of sample and varying material. Because these figures contain a great deal of condensed material, the first, Fig. 13 will now be explained in detail. Some information has been added to Fig. 13 to assist in following the explanation.

Fig. 13 is divided into six panels; the bottom right hand one gives details of the example. In this bottom
panel the first four figures give the values of the elastic constants which adequately define the material for the example illustrated. For anisotropic material of the type under discussion, there are five independent elastic constants. They can be taken as:

Young's modulus in horizontal direction,
\[ E_x \] (EX on the computer solution)

Young's modulus in vertical direction,
\[ E_y \] (EY on the computer solution)

Poisson's ratio in horizontal plane,
\[ \mu_{xx} \] (MUXX on the computer solution)

Poisson's ratio in vertical plane,
\[ \mu_{xy} \] (MUXY on the computer solution)

Shear modulus in vertical plane,
\[ G_{xy} \] (GXY on the computer solution)

The shear modulus in a horizontal plane is not independent, it is given by:

\[ G_{xx} = \frac{E_x}{2(1 + \mu_{xx})} \]

Any material of the type under discussion is isotropic if

\[ E_x = E_y \]

and

\[ G_{xx} = G_{xy} \]

and \[ \mu_{xx} = \mu_{xy} \]

In the example in Fig. 13 the bottom right hand panel shows that:

\[ \frac{E_x}{E_y} = 1 \] therefore \[ E_x = E_y \]

\[ \mu_{xy} = 0.5 \text{ and } \mu_{xx} = 0.5 \] therefore \[ \mu_{xx} = \mu_{xy} \]

\[ \frac{E_x}{G_{xy}} = 3.00 = 2(1 + \mu_{xx}) \] therefore \[ G_{xy} = G_{xx} \]

and the material is isotropic with \[ \mu = 0.5 \].

Also in the bottom right hand panel of Fig. 13 are values for \( K_1, K_2, K_3 \) and \( A \). These are functions of the
elastic constants, printed by the computer for checking purposes, but of no interest to the present discussion. The value $L/HT$ is the ratio of the length to height of the sample. It should be noted that $HT$ is the computer abbreviation for height; it is not equal to $h$. In Fig. 13, since $L/HT$ is unity, the sample is square. The value of $\lambda = .776$ is a coefficient of stress uniformity, which will be defined presently.

The remaining five panels of Fig. 13 are arranged in a row of three along the top, each of which plots a different stress, and two in the remaining spaces along the bottom, where two displacements are plotted. The three stresses are $\sigma_x$ (the normal stress in the $x$-direction), $\sigma_y$ (the normal stress in the $y$-direction) and $\tau_{xy}$ (the shear stress on an $x$-$y$ plane) and they have all been reduced to dimensionless form by dividing them by $\tau_{av}$ where $\tau_{av}$ is the applied shear force divided by the area of a horizontal section of the sample. In this dimensionless form, the stresses have been labelled SIG X, SIG Y and TAU XY respectively.

Starting with the left hand top panel, this shows the variation of dimensionless normal stress in the $x$-direction, SIG X. The panel has an upper portion and a smaller, lower portion. The upper portion shows SIG X as a function of distance from the end of the sample. Referring back to Fig. 11, $x = 0$ at the left end of the sample, and, as $x$ increases, $x/L$ (labelled X/L in Fig. 13) varies from 0.0 to 1.0. These values are written as abscissae. The two vertical axes of SIG X at $X/L = 0.0$ and $X/L = 1.0$, represent the two ends of the sample. It will be seen that in the upper part of the panel there are five curves showing SIG X vs. X/L. Each of these curves represents the variation of $\sigma_x$ at a different sample height ($y = 0.2h$, $y = 0.4h$, $y = 0.6h$, $y = 0.8h$, and $y = 1.0h$). At the upper boundary of the sample, $y = h$ and on the panel, at the top left hand corner, the symbols $Y = H$ appear. These symbols opposite the value of
SIG X = 1.8, indicate the start of the curve for SIG X at the upper boundary, as a function of X/L. In the corner, SIG X actually goes to infinity, but to maintain a reasonable scale in plotting, a cut-off was introduced at 1.8 and this is the reason for the flat top of the plot of SIG X on the top boundary. All stresses and strains are skew-symmetric so negative values can be cut off closer to zero without loss of information. Negative values are cut off at -0.6 and in the top left hand panel, two curves become cut off in this way. On further inspection of the left hand side of the upper portion of the left hand upper panel, the legend "Y = 0" will be seen. This represents the start of a plot of SIG X at the mid-height of the sample and it actually represents a sixth line, with SIG X = 0 for all values of X/L. The remaining four curves on this graph represent SIG X at the intermediate heights between \( y = 0 \) and \( y = h \). It will be seen that at any depth in the sample, SIG X reverses sign, passing through zero at mid-length. SIG X is zero at mid-height throughout the length of the sample, but above mid-height it increases gradually near the ends. It also increases with \( y \). The contours described are enough to give the approximate value of SIG X at any point in the sample, by interpolation. However, to make the plotted picture as comprehensive as possible, the lower portion of the upper left hand panel shows SIG X re-plotted. This time it is plotted as a function of height variation from \( Y/H = 0 \), at the mid-height of the sample, to \( Y/H = 1 \) and \( Y/H = -1 \), at the top and bottom boundaries. Since SIG X is plotted as a function of height, the vertical axis is used for \( Y/H \) and the axes of SIG X are set out horizontally. The half length of the sample was taken in ten equal increments of length and the eleven contours of SIG X represent its values at the boundaries of these segments from \( X = 0 \) at the end through \( .05L, .10L, .15L, .20L, .25L \) etc. to \( X = 0.5L \) at mid-length. The actual values on the contours repeat, of course, the pattern given in the upper portion of the panel, with SIG X = 0 at mid-height for
all lengths, increasing toward the corners and with maximum values just inside the boundaries. Values of SIG X have again been cut off at 1.8 and -0.6 to keep the plot within reasonable bounds.

The second panel of Fig. 13 shows the dimensionless vertical normal stress, SIG Y. This has been plotted in the same way as SIG X, first as a function of X/L for different values of Y and, below that, as a function of Y/H for different values of X. The values of SIG Y near the corners are greater than SIG X and this leads to more curves being cut off at the upper left hand corner of the panel. These curves are cut off at different heights from each other, to assist in identifying them. The top boundary curve is cut off at SIG Y = 1.8, but the curve representing SIG Y at .8h (one-tenth HT below the top boundary) is cut off just below 1.8 and the curve at .6h is cut off lower still. It will be seen that the general distribution of SIG Y is the same as SIG X, except that it is greater and the maximum points on the curves tend to occur at the end boundaries (instead of inside them) and further from the top boundaries than SIG X maxima.

The right hand top panel of Fig. 13 shows the dimensionless shear stress, TAU XY, in the same way as the two previously described stresses. In Fig. 13 the different contours of TAU XY are fairly well separated at mid-sample length (X/L = .500) and the contours for Y = H and Y = 0 could be labelled (although not at the left hand end, like SIG X and SIG Y). However, in most cases (see the following figures) the contours of TAU XY are too close together to be legibly labelled by the computer. The labels were therefore omitted and to distinguish the contours, it is necessary to use both portions of the upper right hand panel. For example, it is easy to see that the highest curve with a maximum value of approximately TAU XY = 1.1 has the same value in both plots, so the maximum curve in the upper plot must represent
SIG X as a function of X/L for Y = 0, Y = 2H, Y = 4H, Y = H

SIG Y as a function of X/L for Y = 0, Y = 2H, Y = 4H, Y = H

TAU XY as a function of X/L for Y = 0, Y = 2H, Y = 4H, Y = H

SIG X as a function of Y/H for X = 0, X = 0.05L, X = 0.10L, X = 0.5L

SIG Y as a function of Y/H

TAU XY as a function of Y/H

U/H as a function of X/L

V/H as a function of X/L

U/H as a function of Y/H

V/H as a function of Y/H

DIMENSIONLESS STRESS

U/H as a function of X/L

V/H as a function of X/L

EX/EY = 1.00
L/HT = 1.000
MUXT = .50
MUXX = .50
EX/GXY = 3.00
λ = .776
K1 = .750
K2 = .750
K3 = .750
A = .000

CONSTANTS FOR CASE IN THIS FIGURE

DIMENSIONLESS DISPLACEMENTS

Fig. 13. Isotropic Material. Sample Length Equal to Height.
Fig. 14. Isotropic Material. Sample Length Equal to Twice Height.
Fig. 15. Isotropic Material. Sample Length Equal to Three Times Height.
Fig. 16. Isotropic Material. Sample Length Equal to Six Times Height.
Fig. 17. Anisotropic Material. Sample Length Equal to Twice Height.
the mid-height (Y = 0) contour since this value occurs at Y = 0 on the lower plot. Similarly, since the maximum of this contour on the upper plot occurs at X/L = .500 (half the length), the contour containing this maximum in the lower plot must be the one for mid-length. Having identified the Y = 0 contour in the upper plot, it will be seen that it decreases from the centre to the end, with no other maximum. Therefore, at Y = 0 in the lower plot, the contours must be in a steady order of decreasing magnitude. Therefore the successive contours in the lower plot represent sections closer and closer to the end of the sample.

In all the stress plots, the upper plot shows six contours, one of which may lie along the axis, and the lower plot shows eleven contours, one of which may lie along the axis. The stress plots show the stress distribution in a linearly elastic sample, when subjected to shear forces with the boundary conditions already described. If the boundary conditions included zero vertical displacement instead of zero friction at the ends, stresses throughout the sample would be ideal, as follows:

a) SIG X = 0 for all values of X and Y. Therefore the upper plot of SIG X would contain six lines, all lying along the axis. The lower plot would contain eleven vertical lines, all coinciding with the axis.

b) SIG Y = 0 for all values of X and Y. The plots of SIG Y would therefore be as described for SIG X.

c) TAU XY = 1.0 for all values of X and Y. The upper plot of TAU XY would contain six horizontal lines, all superimposed at TAU XY = 1.0. The lower plot of TAU XY would contain eleven vertical lines, all at TAU XY = 1.0.

The plots of stresses are thus an indication of how far the real conditions are from being ideal.
In Fig. 13, two panels still remain to be described. They are the plots showing displacements. In the bottom left hand panel are two plots of horizontal displacements, \( u \), which have been divided by height, \( h \), to make them dimensionless. The displacement axes have been labelled \( U/H \) by the computer, and the dimensionless horizontal displacement has been plotted in the same way as the stresses. Contours of displacement as functions of \( X/L \) have been plotted for six different heights in the upper plot and as functions of \( Y/H \) for eleven different values of \( x \) in the lower plot.

The other displacement plots show the dimensionless vertical displacement \( V/H \) as functions of \( X/L \) and \( Y/H \).

If boundary conditions could be made ideal, the values of \( U/H \) would increase steadily from \( y = 0 \) to \( y = h \), but would not vary with \( x \). This would lead to the upper plot of \( U/H \) showing six equally spaced horizontal lines and the lower plot showing eleven superimposed straight lines from \( U/H = 1 \) (since unit angular shear displacement was assumed) through \( U/H = 0 \) at mid-height. The value of \( V \) would be zero everywhere, leading to six superimposed straight lines along the axis of the upper plot of \( V/H \) and eleven superimposed vertical lines on the axis of the lower plot.

If a network of horizontal and vertical lines were drawn on the side of a sample before deforming it, the vertical lines would be displaced horizontally during deformation and their final shape should show \( u \) as a function of \( y \). In other words the lower plot of \( U/H \) should be a set of drawings (to some scale) of the final shapes of lines which started vertical. Similarly any vertical displacement of horizontal lines should show \( v \) as a function of \( x \), so the upper plot of \( V/H \) should be a set of drawings of the final shapes of lines which started horizontal. A picture of a deformed sample from the machine, as calculated can therefore be drawn, by replotting the relevant \( U/H \) and \( V/H \) lines at a scale which
will make unit deformation equal to $h/\sqrt{3}$ (30 degrees shear). The lines must, of course, be separated and each plotted at its proper position along the length and height of the "sample". This was not done by computer, but computer print out of the values of $U/H$ and $V/H$ was used to plot such a picture by hand. It is shown in Fig. 18 for isotropic material and for a sample with $L/HT = 1.778$ (the same as the size in the machine described). Thus Fig. 18 is for comparison with the deformed plasticine samples. Fig. 18 forms a diagram of an imaginary elastic sample that started with horizontal lines at $y = 0$ (centre line), $y = -0.5h$, $y = 0.5h$, $y = -1.0h$, $y = 1.0h$ (top boundary) and with vertical lines at $x = 0$ (left hand end), $x = 0.143\ell$, $x = 0.286\ell$, $x = 0.429\ell$, $x = 0.571\ell$, ..................... $x = 1.0\ell$ (right hand end). It will be seen that the linear elastic solution yields displacements which are closely resembled by the results of the plasticine trials.

Figs. 13 to 16 are all examples of conditions in the same isotropic material. They vary in that each figure represents different sample proportions from a square (Fig. 13), to a length equal to six times the height (Fig. 16). Comparing these figures shows that, as hoped, the displacements are almost uniform in all cases and that, this having been obtained with non-ideal boundary conditions,
the stresses are not ideal except near the centre of the specimen. However, as the length of the specimen increases compared to its height, the stress distribution soon becomes more favourable and displacements become even more uniform.

Fig. 17 represents conditions in an anisotropic sample. Examination of the lower right hand part of the figure shows that the horizontal Young's modulus is 50 percent greater than the vertical, the Poisson's ratios are equal and, since $E_x/G_{xx} = 2(1 + \mu_{xx}) = 2.4$ and $E_x/G_{xy} = 6.00$, the shear modulus in a vertical plane is 0.4 times the shear modulus in a horizontal plane.

It may be questioned whether material of this type described can exist. In order to ensure that the anisotropic results should represent possible material, the bounding values of the elastic constants were examined. This examination is set out in Appendix D. In passing, it is interesting to note that Appendix D shows that material with the type of anisotropy described, can show a net volume increase under uniaxial vertical compression. Thus if heavily consolidated clays are regarded as possessing this type of anisotropy, dilation during triaxial shear may represent elastic behaviour; it is not necessary to assume release of energy. It is also interesting to note that Poisson's ratios $\mu_{xx}$ and $\mu_{xy}$ may both be greater than 0.5, although only for compressible material. Checking in Appendix D, it will be found that the material represented in Fig. 17 can exist and is mildly anisotropic. The length to height ratio in Fig. 17 will be seen to be 2.000 and it is therefore to be compared with Fig. 14, for isotropic material. It was noted by Zienkiewicz, Cheung and Stagg, that a considerable degree of anisotropy was unimportant.

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because it had little effect on the stresses around a tunnel, where the boundary conditions consisted of known stresses. It is interesting to note here that with boundary conditions consisting mostly of known displacements, anisotropy had some effect on internal displacements, but a much larger effect on stresses. The implication appears to be that where boundary conditions consist of known stresses, even though anisotropy has little effect on stress distribution, it may have a considerable effect on displacements. In support of this suggestion, it may be noted that near the sample ends, where one of the boundary conditions is a known stress, the variation of the type of material has its greatest effect on displacements and least effect on stresses.

As sample length increases, from Fig. 13 to Fig. 16, the region of zero normal stresses, near the sample centre, increases in size and increases its percentage proportion of the whole sample. It will be seen that as the region of zero normal stresses increases, the same region is a region where the desired unit shear stress acts and, for example in Fig. 16, the desired ideal conditions are achieved throughout a large proportion of the sample. Because the region of ideal conditions for one stress coincides with a region of ideal conditions for the others, and the displacements vary little from one example to another, it follows that the distribution of any one stress can be taken as indicative of a complete case. The various examples are therefore compared most easily by looking at the top right hand panels, which indicate the dimensionless shear stress. This seems an appropriate medium of comparison, since the test being considered is a shear test. For comparison of different cases it appears appropriate to define a coefficient of stress uniformity. This coefficient will be referred to as $\lambda$ and it will be made to depend on the distribution of shear stress. The point closest to ideal conditions is at the centre of the sample, and writing $\tau_c$ equal to the shear stress at this point, true uniformity would give $\tau_{xy} = \tau_c$
throughout the sample. To indicate the degree of non-uniformity, we can add together all departures from $\tau_c$ and average them through the sample, adding the result to $\tau_c$. Since the variation above $\tau_c$ is as serious as one below $\tau_c$, the absolute values of these departures must be added and the result can be expressed as

$$\tau_c + \frac{1}{2hl} \int_0^L \left[ \left| \tau_{xy} - \tau_c \right| + h \right] dy \, dx$$

In order to be able to compare different cases, it seemed desirable to divide by $\tau_c$, leaving

$$\lambda = 1 + \frac{1}{2hl} \int_0^L \left[ \left| \frac{\tau_{xy}}{\tau_c} - 1 \right| + h \right] dy \, dx$$

The ideal condition (complete uniformity of shear stress) is then represented by $\lambda = 1$, any actual condition will exceed this value, but the closer to unity, the more uniform the shear stress through the sample. The values of $\lambda$ shown in Figs. 13 to 17 should have unity added to them; they would then be as defined here. Figs. 13 to 17 show stresses and displacements through the sample in considerable detail. However, having summarized the trend of variation with the series of computer plots, it seems desirable to present results more concisely, at the same time obtaining a quantitative comparison. This is achieved by plotting graphs of $\lambda$. The value of $\lambda$ depends on both sample proportions and elastic constants. It is difficult to make a meaningful plot of $\lambda$ as a function of elastic constants. It is better to plot $\lambda$ as a function of sample proportions, for various materials. This has been done in Fig. 19. In looking at Fig. 19, it should be remembered that all isotropic materials would not yield the same results. This is because of the mixed boundary conditions; changes of $\nu$, implying variations of compressibility, lead to somewhat different stress patterns.
It will be noted in Fig. 19 that the curvatures reverse and in one case \( \lambda \) actually has a maximum. In case such a curve seems unlikely, consider an extreme case with a sample several times as high as it is long. In such a case, the boundary conditions would demand no friction, on the ends, which would be tall, but only on the short top and bottom. All deformation would therefore be confined to zones near the extremities and most of the sample would be at a uniform shear stress of zero. This would not be the desired condition, but it would be very nearly uniform. Thus \( \lambda \) would tend towards unity as the length to height ratio tends towards zero, and so the curves must have maxima.

![Fig. 19. Variations of Stress Coefficient with Proportions of Sample.](image-url)
Effect of Proportion of Length to Height. In considering the proportions of a possible apparatus, it appears desirable to have a length to height ratio large enough for $\lambda$ to have passed its maximum for any possible material. In fact, the larger the length to height ratio, the more theoretical conditions approach the ideal. However, the rate of change of $\lambda$ can be seen to get less and there seems little point in making the length to height ratio greater than about three or four. There is also need for caution in making the sample too long. It will be seen that for a very long low sample, although conditions are very close to uniform, most of the applied stress field depends on friction at the top and bottom surfaces; the rigid ends have proportionately less effect. If there is any slipping of the sample against the top and bottom plates, the entire distribution changes and becomes non-uniform, most of the sample ceasing to resist shear. The proportions of the machine described, having $L/HT = 1.778$, were chosen for practical reasons, already described. However, it is believed that Figs. 12 through 19 vindicate this choice, as producing a compromise of uniformity of stress with reduced liability to non-uniform displacements.

The Complete Stress Field. The foregoing theoretical development dealt with the effect of an applied shear load. Most soil tests, however, will involve a vertical normal load, and the complete stress field includes its effect. The vertical load, usually applied before commencement of shear, is accompanied by vertical strain but no lateral strain. It follows that the sample undergoes shear and that the directions of principal stress are vertical and horizontal. When horizontal shear is applied, the $x$-direction can no longer be a direction of principal stress and the directions of maximum shear must rotate.

It follows from the above that the $x - y$ shear need not be the maximum shear and that at the beginning of the test, it is not.
Relationship Between Experimental Measurements and Theory.

During the development of simple shear machines at Cambridge, England, theoretical boundary stresses and centre line displacements were calculated for a sample whose length was three times its height. From these, it was concluded that uniform conditions should occur throughout the middle third of the sample length. The more complete results presented in Fig. 15 confirm the earlier theory. Based on the theoretical results, Cole\(^{13}\) and earlier workers at Cambridge, set out to measure the actual displacements and boundary stresses. The Cambridge results showed that for the middle third there were uniform displacements under uniform stress, and hence that the stress-strain behaviour of the middle third was the relationship sought, for the soil being tested. So, if we define the idealized vertical normal and horizontal shear stresses as \(\sigma_y\) and \(\tau_{xy}\), these are the stresses in the middle third. Cole's work included extensive testing of dry sand; he did series of "drained" and "undrained" (constant volume) tests on dense, medium and loose samples. His machine was so made that he could measure separately the normal force and its eccentricity and the surface shear force (friction) on the two ends of the sample and, also separately, on the middle third and each end of both top and bottom sample boundaries. From the detailed information obtained, Cole was able to use statics to determine the maximum and minimum stresses and their directions for the (uniform) middle third of the sample length. It follows that he could also determine the vertical normal stress, \(\sigma_y\), and the horizontal shear, \(\tau_{xy}\), for the middle portion.

In the apparatus now being described, horizontal normal stresses cannot be measured. However, it will now be argued that the average measured stresses are close to the

stresses actually operating in the zone of approximate uniformity, that is, close to $\sigma_y$ and $\tau_{xy}$.

Cole showed that in all his tests, the vertical normal stress at the middle was close to the average vertical normal stress. Results were especially close for undrained tests. That is to say, $\sigma_y$ can be taken as equal to the measured average normal stress. In this context, a measured average stress is the measured applied load, divided by the sample area. Cole also showed that the middle third shear stress was higher than the average shear stress on the top and bottom boundaries. However, in the Cambridge apparatus, the average top friction was found to be different from the average bottom friction. The difference was ascribed largely to side friction; each side was in one piece and fixed. The result was that the two sides exerted a net frictional force in one direction, thus relieving force on the stationary sample boundary and increasing it on the moving one. In the apparatus being described, the sides were split longitudinally at mid-height. The result is assumed to give negligible net friction on the sides and, hence, equal top and bottom friction. Theory showed that the average friction on the top and bottom boundaries, for a sample with length to height ratio of three (Cambridge apparatus), should be about 10 percent less than for the middle third. Cole showed that after allowing for side friction, this result was approximately achieved, and, hence, that the stresses forecast by theory, were being achieved in practice. The fact that the middle third shear stress, $\tau_{xy}$, is different from the average boundary shear stress causes no difficulty because, since actual conditions are in accordance with theory, we can go straight from the applied shearing force to the middle third shear stress, $\tau_{xy}$. Fig. 15 shows that in the central zone of uniformity, the dimensionless $x - y$ shear stress is close to unity. That is to say, $\tau_{xy}$ is approximately equal to the applied shear force divided by the sample area. For shear strains it was shown by both theory and plasticine models
that, in the zone of uniform stress, displacements closely approximate uniform shear strain equal to the boundary displacement of the sample.

Summarizing the conclusions reached above, the stress-strain relationship in $x - y$ co-ordinates, for the zone of approximately uniform conditions can be obtained directly, using the average measured deformation for strain and using the applied loads, divided by the horizontal sample area, for stresses.

The machine now being described, was not capable of the detailed stress measurements accomplished by Cole. There was no method of measuring horizontal normal stresses and results are necessarily given in terms of $\sigma_y$, $\tau_{xy}$ and shear strain.

**Experimental Comparison of Maximum and $x - y$ Shear.** Maximum shear stress could not be ascertained in the apparatus now being described because, as stated earlier, horizontal normal stress cannot be measured and so there are not enough data to fix the Mohr stress circle. However, it may be of interest to consider how the measured shear stress ($\tau_{xy}$) compares with the maximum. It has been shown by Cole\textsuperscript{13} that for drained tests on sand, shear stresses on planes at 45 degrees to the horizontal (initial maximum) becomes zero at strains of 1 or 2 degrees and subsequently reverse. The reversal for dense samples takes place at smaller strains than for loose, and proceeds to a greater magnitude. The reason for reversal of initial shear in drained tests is that as the sample dilates, it can expand vertically but not laterally. The horizontal reaction to the dilatant tendency will be a passive pressure, greater than the vertical, hence causing a shear stress opposite to the initial shear. No drained tests are reported in the work now being described, and having noted the results presented by Cole, the matter will not be pursued further. However, for undrained tests a more complete resumé of Cole's results is presented in Fig. 20. This shows a series of tests
in which both $\tau_{\text{max}}$ and $\tau_{xy}$ are plotted as functions of shear strain. Also in Fig. 20 the ratio $\tau_{xy}/\tau_{\text{max}}$ is plotted, as this makes for easier comparison after the first stages of a test. Fig. 20 is a plot of results taken from figures in Cole's dissertation, it is not a copy of one of his figures, as his results were not plotted in the form desired for the present discussion. It will be seen that for undrained

![Diagram](image)

Fig. 20. After Cole (1967). Comparison of Maximum Shear and Horizontal Shear.
tests, the initial shear is soon largely destroyed, (at a
strain of 0.1 degrees or less) and after that it remains very
small. The result is that, after a small strain in an
undrained test on sand, the applied shear is approximately
the maximum shear.

Manner of Presentation of Experimental Results. There are at
least two possible methods of presenting the results of simple
shear tests. Shear strain could be given as a function of
maximum shear stress (or principal stresses) or it could be
given as a function of shear stress on a horizontal plane.

In the results to be presented, the second method
is selected, for the following reasons:

a) It is considered that where shear behaviour is of
interest in practical engineering problems, the
situation usually involves a normal load on a
given plane and variation of shear load on the
same plane.

b) It appears as logical to trace the development of
shear stress on a given plane, which rapidly
becomes a plane of maximum shear, as to trace
changes of maximum shear (on a succession of
planes). In fact, the first method is, perhaps,
to be preferred on logical grounds; the measured
shear strains are strains in the direction of the
horizontal-vertical shear, with which they are
associated. On the other hand, some of the strains
associated with the maximum shear stress, occur
during application of normal load and are usually
ignored.

c) The apparatus is greatly simplified when measure-
ment of horizontal normal stress is not required.

In summary, the vertical load and horizontal shear
load are measured; hence the average values of $\sigma_y$ and $\tau_{xy}$ are
known. The height and horizontal deflection of the samples
are known and so the average shear strain is obtained.
The apparatus described is designed to allow a shear strain of ± 30 degrees. It also allows ± 11 percent change in volume. If the axial compressive strain of a triaxial specimen is \( \varepsilon_A \), the shear strain can be shown to be:

\[
\frac{\pi}{2} - 2 \tan^{-1}(1-\varepsilon_A)^{3/2}
\]

For four values of axial strain, the resulting shear strain is:

<table>
<thead>
<tr>
<th>( \bar{\varepsilon} )</th>
<th>5%</th>
<th>10%</th>
<th>20%</th>
<th>30%</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \omega )</td>
<td>4°12'</td>
<td>9°1&quot;</td>
<td>18°50'</td>
<td>29°17'</td>
</tr>
</tbody>
</table>

It is, therefore, believed that ± 30° strain in the simple shear apparatus covers the range included in most triaxial tests.
CHAPTER IV

EXPERIMENTAL PROCEDURE

Installing Samples. A considerable amount of time was spent developing a technique for sample installation, especially sand samples. In order that the experience gained may be available to others, the procedure as developed to date, is set out in detail in Appendix E. A brief review of the concepts involved is given here. There are basically two types of sample; they are (i) samples formed outside the box and then placed in position with a minimum of strain or further disturbance and (ii) samples formed in situ in the box. It is evident that any undisturbed sample must be of type (i). Disturbed samples could be of either type.

Pre-formed samples must have a dimension approximately .005" less than the width of the box (after allowing for membrane thickness), in order to get them into place without undue disturbance. Subsequent expansion to fill the space represents a linear strain of about 0.25 percent. The length of such samples can be very close to the actual box length. It thus seems preferable to form samples in situ when possible and this method has been adopted for sand samples.

In situ Samples (Sand). The membrane has all necessary holes (for screws) punched and is then mounted on its base block, with the bottom friction plate inside it (Fig. 21). This little assembly is then placed in position in the machine and the drainage hole in the top of the membrane is stretched wide open, the full size of the sample (Fig. 22).

The space for the sample is filled with de-aired water and then de-aired saturated sand is placed under water by means of a tube. With the top of the sand, level and under water, at the correct height, the top friction plate is located and the membrane allowed to close over it. The cap
and loading head are then assembled over the sample and the de-airing, loading, etc., can commence.
In the above procedure, the chief points of difficulty were as follows:

a) The stretching of the membrane in the sample space tends to pull the vertical corners away from their position. The vertical corners are retained in position by short lengths of neoprene moulding, inserted inside the membrane and then placed in special grooves, described in Chapter II. Locating the lengths of neoprene moulding in their grooves adds to the complications of preparing in situ samples.

b) A considerable degree of care is required in actually placing the sand. It has to be as uniform as possible, the correct amount of material must be used (to get the desired density) and the process must leave a fairly level surface. After pouring, surface adjustment can be carried out, but this must be a minimum, to avoid sample disturbance.

c) After the sample has been placed, it is a delicate matter to place the upper plate on the sand surface and close the membrane over it. Everything must be done gently to avoid disturbance and also to avoid letting any portion of membrane snap into place. Any sudden movement, such as snapping by the membrane, will not only disturb the sample but is extremely likely to wash sand grains onto the top of the steel plate. Sand grains on top of the steel plate prevent proper seating of the loading head above it and hence lead to leaking, and probable puncturing of the membrane. It is essential to exercise proper control over the membrane when closing it, in order to ensure that the moulded corners of the membrane actually seat on the corners of the upper plate. If they do not, the membrane will be likely to form a fold and, also,
the plate corners are more likely to cause a puncture.

Undisturbed Samples. It would probably be very difficult to form even remoulded clay samples in situ and the procedure described now covers all samples formed outside the machine. After preparing the membrane, and mounting it on its block, in the same way as for in situ samples (Fig. 21) the sub-assembly is mounted in a square box with porous sides and open top. Vacuum can be applied to the sides of the membrane to stretch it to about 2-1/2 in. square. The sample is cut to the correct size, using a jig and can easily be lowered into place inside the membrane. The vacuum is then released and the top plate and porous stone put in place. The membrane is then closed over the top plate, lifting over one corner at a time and ensuring that the moulded corners of the membrane fit to the plate corners. Although it was essential to hold the vertical corners of the membrane for in situ samples, this is not necessary for samples of the type described unless they are very soft. This is not so much because the sample corners are strong, as because volume inside the membrane is now held constant and yielding at the corners would involve bulging and yielding at the mid-sides. Clays so far tested have resisted this type of deformation. The sample, on its base block, is now lubricated and installed in the machine, taking care not to scrape the material of the sample sides as it is lowered into the box. The upper part of the box sides are placed in position while ensuring that the membrane is not pinched. The rest of the machine can then be put together ready to commence testing procedure.

Loading. Application of normal load, flushing air out of the loading head and application of back pressure are three operations which must be combined in a logical order. With sand samples, the height locating pin may be left in place while air is flushed away and back pressure applied. Then,
if normal load is added until the back pressure is just balanced, the pin may be removed and the intended effective normal load may be added with the water supply valve open. It is important that the loading head and upper plate should not be resting on some odd grain of sand, above the general level.

Chapter III discussed the necessity of developing sufficient friction at the top and bottom sample boundaries, to prevent the soil slipping on the boundary plates. Depending on the soil being tested, the necessary friction at the top and bottom surfaces may be provided by gluing emery paper or sand grains to the plates or sandblasting. In the machine described, one set of plates was sandblasted but did not develop adequate friction for sands. This became evident, because during tests on sand, the surface of the hardened steel became scratched and after several tests in which full strain was allowed to develop under cyclic load, the sandblasted surface was polished off. The pattern of the polishing was most intense over approximately the mid-half of the length of the plates, dying away from there to the ends.

This pattern coincides with the intensity of boundary friction, calculated in Chapter III.

The same pair of plates was re-sandblasted and used for testing clay. With this material, absence of polishing, the strength of adhesion of samples to plates after testing and the satisfactory appearance of deformed samples on removal, all combined to suggest that adequate friction was developed with clay. For testing sands, something very positive was sought; a second pair of hardened steel plates was made with transverse ribs ground on their faces. The ribs were .05 mm. high, with sides at 30 degrees from the vertical. The ridges were at 2 mm. centres, with fairly sharp edges. These plates appeared to develop adequate friction because there was no sign of polishing of their surface, after many cycles of strain. It was found that with
light normal loads, the ribbed type of upper plate might be supported on its ribs, without bringing its main surface into contact with the sand. For this reason a seating load is required. This seating load depends on the density and grain characteristics of the sand and on the size and shape of the plate ribs. In the experiments described, seating loads of the order of 2 to 4 kg. per sq. cm. were used.

With clay, standard size samples could exhaust all the available vertical travel while consolidating. Therefore, samples are likely to require cutting about 0.1 in. higher than the nominal size. Thus the vertical height locating pin cannot be used and application of back pressure would drive the loading head upward, and allow the clay to start swelling. It is better to flush out the air rapidly with water at a minimal pressure and then apply both back pressure and normal load to balance each other, until the desired back pressure is reached. After this, the drainage valve can be closed and full normal load applied. This causes slight lateral strain to occur before consolidation starts and so permits a better check on events. When the lateral strains are complete, the vertical dial gauge will indicate no further change in height and the drainage valve may be opened to back pressure, allowing consolidation to start.

Shear load could be strain controlled but this method has not, as yet, been applied. The device adopted has been a double acting piston, allowing shear loads to be applied in either direction. The desired direction may be selected, pulling the valves round by hand, if a static test is to be run. A motor and gear box are provided for cyclic load tests, capable of continuous speed variation from about one-twentieth of a cycle to about ten cycles per second.

The desired loads are produced by setting the required pressures on three supply tanks. One tank is common to both ends of the piston and the other two tanks supply
one to each end. By putting a given pressure in the common tank and nil in the other two, it is possible to guarantee equal alternating loads, while back pressure in one makes unequal loads. With three tanks, the total load on the piston can be either the pressure in any one tank or the difference between any two tanks. The rotating valves, interposed between the piston and tanks, allow almost any cyclic pattern of loading.
The Test Programme. It seems reasonable that the initial programme of tests for a novel machine, should establish whether the results obtained are different from those of usual test procedures. In other words, the initial test programme should discover whether the differences in test conditions lead to appreciable differences in soil behaviour. Simple shear has already been compared with triaxial testing ($\sigma_2 = \sigma_3$) for cyclic loading on saturated sands by Peacock and Seed. For the present work, it was decided to compare simple shear with triaxial testing under static loading of undrained saturated sand.

Because the machine described herein was intended to be capable of dynamic testing, the initial test programme was designed to include tests of liquefaction under cyclic loading.

During testing, attention was given to the boundary conditions assumed and discussed in Chapter III. These conditions included:

a) No slipping of the sample on the top and bottom boundary plates.
b) Zero friction on the sample ends.
c) Rigid boundaries.

The effect of friction on the sample ends was not investigated in the initial test programme, as the volume of testing was already considerable without this extra variable.

The rigidity of metal sides and needle bearings is far higher than that of water or a soil skeleton. The finished machine had negligible "backlash". Thus rigid boundaries were achieved within the limits of practical purposes and this boundary condition was not examined further.
With the above points in mind, a programme was planned to:

a) Establish test procedures for the simple shear machine.

b) Perform undrained tests to
   (i) Compare the results from two types of top and bottom plates in the simple shear machine (Ribbed plates and plain plates).
   (ii) Compare the results from undrained triaxial tests with simple shear using ribbed plates.

c) Run cyclic simple shear tests to obtain a general picture of conditions which would lead to liquefaction and give a comparison of results using plain and ribbed plates.

The material for most of the tests was Ottawa sand, as described in A.S.T.M. Standard C109. This material was selected as, being well known, it would require a minimum of tests of a descriptive nature. Furthermore, being widely available, the same material can readily be used by others for comparison with the results presented here.

The maximum void ratio was found by letting dry sand fall in air from a low height and the result checked by allowing de-aired sand to fall through water from a low height. Both methods gave a maximum void ratio of 0.78 (Dry density = 93 lb. per cu. ft.). A dry sample was vibrated under no normal load, until a steady volume was reached. The resulting void ratio was 0.54 (Dry density = 107.5 lb. per cu. ft.) and this was taken as the minimum void ratio, to be comparable with normal practice. In passing, however, it should be remarked that much greater densities can be achieved by applying a large normal load and adequate cyclic shear under drained conditions. In fact, in the course of other testing, a void ratio of about 0.47 was reached, without specifically attempting to achieve high density. All tests were performed with controlled load rather than controlled strain.
In addition to the tests on Ottawa sand, a few tests were also performed on a local clay. These were done as a trial of methods of test preparation and procedure, and they are described in Appendix F.

All the tests to be described were run with back pressure on the pore fluid. The back pressure for any given test was chosen to optimize use of the transducer over the anticipated range of pressures. Providing the back pressure is high enough to prevent cavitation, its actual value is of little importance. So, to allow easy comparison of tests under different back pressures, the back pressure has been subtracted from all reported pore pressure readings. In all cases, the given values of pore pressure represent change from initial conditions (drainage against back pressure).

In presenting test results, the friction angle used is not the angle of maximum stress obliquity, \( \phi \), usually adopted with reference to triaxial testing. The basic reason for this is that, as explained in Chapter III, the apparatus used could only measure stresses on a horizontal plane. To obtain the stresses on other planes, and hence the angle of maximum stress obliquity, involves making an assumption. Such an assumption will be made to compare simple shear and triaxial tests, but it is considered that the simple shear tests should be presented in terms of the measured quantities, without any assumed relationship. The friction angle for the simple shear tests will therefore be defined as the stress obliquity on a horizontal plane. To emphasize that it is different from the usual \( \phi \), the angle will be denoted by a different symbol, \( \theta \). The notations of primes and subscripts usually applied to \( \phi \) will be applied to \( \theta \). In other words the ratio of applied simple shear stress to applied normal stress is \( \tan \theta \) and \( \tan \theta' = \frac{\tau_{xy}}{(\sigma_y - u)} \).

To compare simple shear and triaxial test results, the assumption will be made that after a small strain (less than that accompanying \( \theta_{max}' \)) in simple shear \( \tau_{xy} = \tau_{max} \).
This assumption is based on testing by Cole\textsuperscript{13}, described in Chapter III. If $\tau_{xy} = \tau_{\text{max}}$, the complete Mohr stress circle is known and $\phi$ could be determined, since, in this case $\sin \phi = \tan \theta$. However, this would not be true at the beginning of a test and hence results are given in terms of $\theta$. The simple shear test results, therefore, follow the stresses and stress obliquity on one plane, that is the plane of ultimate maximum shear stress. To make them comparable with simple shear, triaxial tests will be reported in terms of stresses and stress obliquity on planes of maximum shear stress (i.e. planes sloping at 45 degrees). This representation is shown diagrammatically in Fig. 23.

Fig. 23. Average Stresses Used In Presenting Experimental Results.
Undrained Static Tests on Ottawa Sand. To compare the results with and without proper top and bottom friction, stress-controlled simple shear tests were done on Ottawa sand in the middle range of relative density. Ribbed plates, described in Chapter IV were used to develop proper friction and plain plates, which had been found liable to slip, were used for the comparison tests. The normal pressures chosen were 2.0 and 4.5 kg. per sq. cm., as these are in the middle and upper ranges of common laboratory testing. The results are presented in terms of shear stress and developed friction angle and as functions of shear strain (Fig. 24a. and Fig. 24b.).

It will be noted in Figs. 24a. and 24b. that $\theta'_{max}$ is one or two degrees lower when plain plates are used and that this difference remains more or less constant as strains increase beyond the value at which $\theta'_{max}$ is reached. Below strains about 2 degrees, the effect of the type of plate appears to decrease as the normal stress increases.

On examining the curves of shear stress, it will be noted that at strains of 1 or 2 degrees a large deformation occurs at constant stress, going as far as 8 or 10 degrees. There is a considerable gap in the points plotted, extending over this range. The cause of the gap is an unstable condition which may develop as follows. Soon after the beginning of a test, shear strain resulting from the applied shear stress, causes pore pressure increase. The resulting reduction in effective normal stress reduces the shear resistance mobilized by a given magnitude of strain increment. A point is reached where the pore pressure is still increasing with strain, but shear resistance is not. A small increment of shear stress at this stage means that the shear stress exceeds the strength and, under the resulting shear strains, pore pressure continues to rise. Thus the strength reduces further, leaving an out of balance shear force. As a result, deformation accelerates until, at a considerable strain, the pore pressure begins to drop again;
Fig. 24a. Static Undrained Simple Shear. Ottawa Sand.
Normal load 2.0 kg. per sq. cm.
Comparison of Two Types of Top and Bottom Plates.
Fig. 24b. Static Undrained simple Shear. Ottawa Sand.
Normal Load 4.5 kg. per sq. cm.
Comparison of Two Types of Top and Bottom Plates.
shear resistance then increases until stability is re-established. The whole of this large strain increment occurs within a fraction of a second. After the large deformations, \( \tan \theta' \) remained approximately constant, while the applied shear stress increased and pore pressure decreased. The region of large deformation just described will be referred to again later and will be termed "yield".

In comparing the values of stress resisted by plain and ribbed plates, it will be seen that "yield" occurs at slightly lower stress with plain plates. In the test at 2.0 kg. per sq. cm. initial effective normal stress, the strength is greater with ribbed plates throughout the test. In the other test, plain plates show greater strength immediately after yield, but apparently this is because yield deformation was smaller and the sample began to re-mobilize strength earlier. However, the rate of increase of strength with strain was less with plain plates and it will be noted that after 19 degrees strain the ribbed plates again show higher shearing resistance, as well as higher \( \tan \theta' \).

It is considered that these tests show that in general, the plain plates did not mobilize adequate friction and the recorded results failed to indicate the true strength of the sand. With high normal stress at small strains, the results were approximately the same for the two types of plate; it appears that in this particular case, the plain plates were adequate. In all cases, the friction developed by ribbed plates appears to be adequate, judging by the absence of abrasion of the plates after a series of tests. However, it is not claimed that ribbed plates have been proved ideal; the ribs may cause undue disturbance of the sample surface when the top plate is lowered into place, and ultimately some other type of surface may prove preferable.

Turning now to the comparison between static undrained triaxial and simple shear tests, the same simple shear tests with ribbed plates as have already been quoted,
were used. In order to render tests comparable, triaxial tests were done with the same void ratios and with initial confining pressures equal to the initial normal stresses of the simple shear tests. As described earlier, the friction angle $\theta$ has been used instead of $\phi$ to present results, but assuming $\tau_{max}$ equal to $\tau_{xy}$ at $\theta'_{max}$, values of $\phi$ can be given and they are shown in the test results.

In discussing the experimental results, strength in terms of the ratio of the relevant effective stresses will be referred to as $\theta'$ or $\tan \theta'$ and strength in terms of applied shear stress will be referred to as shear strength.

For the simple shear tests, electronic transducers were used to measure strain, normal stress and pore pressure and the outputs were fed to chart recorders. The shear stress increments, although recorded on the oscillograph, were checked, when possible, by a mercury manometer, reading the pressure in the load piston. The results, in terms of $\tan \theta'$, shear stress and pore pressure as functions of shear strain, are shown for initial pressures of 2.0 and 4.5 kg. per sq. cm. in Figs. 25a and 25b respectively.

It can be seen that $\theta'$ is higher for simple shear than for triaxial tests, especially after the simple shear tests "yielded". However, the simple shear test seems far more sensitive to variation in relative density than the triaxial. The triaxial tests show $\theta'$ of 26 1/2° and 25 1/4° for relative densities of 52 and 46 percent. On the other hand, the relative densities of the simple shear specimens vary less but $\theta'$ varies more. In simple shear, $\theta'$ varies from a range of 33 degrees for 46 percent to 39 degrees for 50 percent relative density. The difference in $\theta'$ for the simple shear does not appear to depend on the difference in initial stress conditions for, as will be seen presently, the effective normal stresses overlap within the stated ranges of $\theta'$ values.
Fig. 25a. Static Undrained Shear. Ottawa Sand.
Initial Pressure 2.0 kg. per sq. cm.
Comparison of Simple Shear and Triaxial Results.
Fig. 25b. Static Undrained Shear. Ottawa Sand.
Initial Pressure 4.5 kg. per sq. cm.
Comparison of Simple Shear and Triaxial Results.
Considering the applied stress resisted, at shear strains higher than 1 or 2 degrees, the strengths of the triaxial specimens were considerably higher than the simple shear specimens. The results show that higher strength and lower $\theta'$ are accompanied by lower pore pressure for the triaxial test. At very small strains, pore pressures were higher and shear stress lower in the triaxial specimen, leading to values of $\theta'$ nearly as high as simple shear. However, as "yield" is reached, simple shear develops far higher pore pressures which so reduce the effective normal stress that little shear strength can be mobilized, strain becomes far higher and although $\theta'$ is greater, the actual strength is less than in the triaxial test.

It will be noted that within the range of strains roughly corresponding to "yield" in simple shear, the triaxial shear stress shows a minimum slope. For ease of discussion, this minimum slope will also be referred to as "yield".

In simple shear tests, a continuous record of pore pressure was obtained, showing quite clearly that its peak occurs during this yield stage. In the triaxial tests, each reading of pore pressure had to be balanced, thus values during yield could not be obtained and it could not be proved that the peak pressure occurred during yield. From the results obtained, it was concluded that a strain controlled test would show a definite reduction in shear force at yield in the simple shear tests. The yield phenomenon involved a much smaller range of strain in the triaxial test (1 or 2 degrees compared to 7 to 10 degrees in simple shear). Also, no peak shear stress was measured at yield in the triaxial test; such a peak may not occur and if it does it is likely to be small. The gap in the line of reliable points is smaller in the triaxial test, the yield strain is less; if the simple shear tests had not been run as a comparison, the possibility of a "yield" peak in triaxial shear stress might have been overlooked.
The importance of "yield" in practice involves the sudden occurrence of large strains. If the simple shear test gives a true picture of this phenomenon, it could imply sudden failures of which the triaxial test would have given little or no warning. The yield strains in the triaxial test are too small to imply failure in a typical engineering application, and hence a possible type of failure could be overlooked.

A comparison of the simple shear and triaxial testing is shown in Fig. 26. The same tests as were illustrated in Figs. 25a and 25b are shown again but effective stress paths have been plotted.

Fig. 26. Effective Stress Paths for Static Tests.

As before, maximum shear and normal stress on the plane of maximum shear have been plotted for the triaxial and
are compared with shear and normal stress on the horizontal plane for simple shear.

It appears that in each test the ultimate value of $\theta'$ is a function of relative density and of whether conditions are plane strain or triaxial. Thus, when the effective normal stress is 3.0 kg. per sq. cm. two plane strain tests show increase of $\theta'$ with density, but the densest sample of all is triaxial with a lower $\theta'$ than either. Comparing the two triaxial tests the denser one has a slightly higher $\theta'$. The general indication is that, within the range of densities used, the high value of $\theta_{\text{ult}}$ in simple shear is due to plane strain, but this difference might soon be exceeded if densities varied more widely. It will also be noted that there is a tendency for the initial directions of the stress paths to be different in simple shear tests from triaxial tests. This difference in initial directions appears likely to be due to the difference between principal stress loading and simple shear.

The greater initial stiffness of the simple shear specimens (see Figs. 25) is possibly due, in part, to the shear pre-stress and pre-strain, set up during application of the normal load. The rapid development of higher pore pressures in the simple shear could then be due to the rotation of the plane of maximum shear stress, releasing the pre-stress during the early part of the test. It is not considered that there is, as yet, sufficient evidence to prove this, it is offered only as a possible explanation of the difference in the early portions of the effective stress paths.

Summarizing the results of the undrained static shear tests on sand, it was concluded that:

a) Friction at the top and bottom boundaries is important. Insufficient tests have been done to say what would be "adequate friction", as this will depend on the material being tested. However, it has been shown that special attention must be
given to the plate surfaces. Otherwise, significant slipping and loss of strength may occur.

b) In terms of effective stress ratio the simple shear machine shows greater shearing resistance than the triaxial test. It develops $\phi'$ values of 30 to 37 degrees ($\phi' = 37$ to 53 degrees), compared to triaxial values for $\theta'$ of 26 degrees ($\phi' = 30$ degrees). Some of the increase is presumably due to the effect of plane strain. However, results are not really comparable with other types of plane strain tests, because the machine imposes extra restraints, and they would be expected to increase the stiffness of the specimen.

c) In terms of total resistance to shear force, the simple shear machine shows greater initial stiffness but lower strength. This is because the pore pressures in simple shear rapidly increase to greater values than in triaxial testing. It is interesting to note that in undrained cyclic tests Peacock and Seed$^9$ found lower strength in simple shear than in triaxial samples of Monterey sand.

d) The strain during "yield" is much greater in simple shear. Both this and the higher pore pressure may be the results of instability, introduced as the rotation of the principal axes of stress unlocks the pre-stress set up during application of normal load.

On the other hand it is possible that yield represents an inherently unstable condition (due to increasing pore pressures at certain densities). In this case it would seem to be a considerable vindication of the simple shear machine compared to triaxial testing as, presumably when instability is reached the triaxial allows large strains on several narrow zones, but the average strain remains only small. This would be in accordance
with the anticipated advantages, claimed for the new machine in Chapter I.

In considering these points, it should be remembered that looser or denser sands may show different results.

**Cyclic Shear Tests on Ottawa Sand.** The intention of performing cyclic shear tests was to investigate the phenomenon of liquefaction. As cyclic shear load is applied to an undrained sand sample, the pore water pressure may rise. If the pore water pressure rises to equal the confining pressure, the sand will exhibit zero shear strength and behave as a liquid. If the water pressure rises but not sufficiently to equal the effective confining pressure the shear strength of the sand will be reduced, but not to zero. If the pore pressure rises to a value where the reduced shear strength of the sand is less than the applied cyclic shear stress, large alternating shear strains will occur. Such a condition, where failure (i.e. sudden large shear strains) is brought on by rising pore pressure, will be referred to herein as "liquefaction", even though, unlike a liquid, the sand may still retain shear strength. In this definition, it does not matter whether the pore pressure rises to equal the confining pressure or to some lower value, provided only that a sudden increase in strain is attributable to rising pore pressure.

The cyclic shear tests were all run at about two cycles per second and the oscillograph paper was required to run at a higher speed than for the static tests, in which stress increments were typically at two minute intervals. The mechanics of the oscillograph used, are such that the light beams are bright at higher paper speeds and the oscillograph records of cyclic tests can be reproduced. Such a record of a typical cyclic test, leading to liquefaction, is shown in Fig. 27. This figure shows a cycling shear load (2 cycles per sec.) of equal intensity in alternate directions. It also shows the build-up of pore pressure, until there is a sudden onset of large strains when the pore pressure
reaches a certain value, less than the normal load. After liquefaction there are wide fluctuations of pore pressure for each cycle but the fluctuation of both pore pressure and strain soon become regular. In looking at Fig. 27 and other oscillograph recordings it will be noted that strain is shown in inches so that the scale will coincide with the grid lines on the oscillograph paper; it is necessary to divide strains by the sample height (typically 1.07 in.) to obtain the tangent of the strain angle. In view of the very rapid fluctuations of pore pressure after liquefaction it seems reasonable to question whether the response of the oscillograph is rapid enough to follow these changes without distortion. The specified response of the oscillograph galvanometers is linear up to 1000 cycles per second, and it will presently be seen (Fig. 29) that the fastest fluctuations recorded are of the order of 25 cycles per second, except for some higher harmonics. These harmonics probably include frequencies higher than
1000 cycles per second, and so there is probably some overshoot of the galvanometers, but the amplitude of the higher frequencies is small so the overshoot should not be of major importance. The galvanometers are therefore capable of dealing with the frequencies experienced. The linearity of the amplified output was investigated (Fig. 28) by allowing a liquefied sample in which the cycle of fluctuations has become steady, to perform several cycles with different amplifications of the electric signal. The proportionality between the amplification and the magnitude on the record indicates linearity of response. Therefore, the sharp peaks and valleys can be taken as a fair representation of true changes of pore pressure at the transducer diaphragm.

If the shear load is compared in Figs. 27 and 28, it will be noted that the wave takes a different form. The

**Fig. 28. Linearity of Pore Pressure Recording.**
form in Fig. 27 will be referred to as Load Form A and that in Fig. 28, as Load Form B. The Load Form A was intended to be a square wave but the load rise time was approximately 0.02 sec. Load Form B will be described hereafter.

In order to show events after liquefaction in more detail, a high speed recording is shown in Fig. 29. This shows shear load, relative movement (strain) and pore pressure for a few cycles of Load Form A on a sample which had just liquefied under the recorded loads. The marked troughs in the pore pressure will be seen to occupy a time of approximately 0.02 sec., representing a cycle time of 0.04 sec. or 25 cycles per sec., as already stated. The high frequency ripple on the record of shear load and the smaller ripple on the record of pore pressure were due to electrical noise. In this figure, the rise of the shear load is clearly seen, as also are the relative portions of the cycle for the reversal

![Fig. 29. High Speed Record After Liquefaction. Shear Load Form A.](image-url)
of shear, pore pressure variation and relative movement. It will be seen from Fig. 29 that, taking the start of a new cycle as occurring when the shear stress begins to act in a new direction, the pore pressure begins to rise when the rate of change of shear is close to its maximum. The pore pressure reaches its maximum at approximately 0.01 sec. after the shear stress. When the pore pressure reaches a certain value, the strain motion begins and continues either a) until the dilatent tendencies of the shearing material cause a drop in pore pressure, arresting the strain or b) until the machine hits the travel stop. In Fig. 29 strain was arrested through dilatancy of the material. In very few of the tests leading to liquefaction were the travel stops actually hit. The very marked trough in pore pressure at the end of each movement is probably due to inertia effects, considerable velocity having been achieved by fairly heavy machine parts, strains would be expected to go beyond the value at which the static resistance would have equalled a steady force of the same magnitude. It is also evident from the recording that the strain in fact ceases very suddenly. It will be noted that the strain occurred only during that part of the cycle when pore pressure was at a peak. This aspect of the situation is somewhat analogous to the condition described by Seed\textsuperscript{14} as partial liquefaction, but in the work described herein all cyclic failures had this characteristic and this is the reason for adopting the definition of liquefaction already given.

Fig. 29 also shows that the time from the beginning of each stress reversal until the completion of the associated movement and pore pressure changes was about 0.09 sec. out of a half cycle time of about 0.28 sec. It was therefore considered permissible to alter the shear load to Form B, making quarter cycles in alternate directions separated by quarter

cycles under zero shear stress. This wave form was designed in order to allow any elastic rebound to occur after unloading shear in one direction and before applying it in another. By this means, it was hoped to separate plastic and elastic components of strain. By increasing the amplification of the strain record, movements of about 0.0002 in. can easily be read and the rebound on each unloading can be seen. The record of a few cycles of such a test is shown in Fig. 30.

Fig. 30. Medium High Speed Record of Liquefaction. 
Shear Load Form B.

During the course of a liquefaction test, the only quantity showing significant response to the cyclic shear load is pore pressure, strains only becoming large as the test comes to its completion. For this reason, it was considered that pore pressure was the only directly measured quantity giving a real indication of progress and the events leading up to liquefaction. The strains, although recordable when suitably amplified, were of approximately constant
magnitude until just before the onset of liquefaction. It was therefore decided to analyse the progress of any particular test in terms which would reflect the rise of pore pressure as a function of the number of cycles. In order to compare events after a few cycles with those after several thousand cycles it was decided to use the log of the number of cycles in plotting results. Accordingly, the rise of pore pressure was converted into dimensionless form, dividing it by the applied normal pressure, to yield the parameter $U$, expressed as a percentage. Some results are presented in terms of $U$ and some in terms of $\tan \theta'$.

Fig. 31a shows the plots of some typical tests in terms of $\tan \theta'$. The same tests are shown again in Fig. 31b, but $U$ is plotted instead of $\tan \theta'$. Figs. 31 show the course of three tests. Curve No. 1 shows a test in which the void ratio was low and although the applied $\tan \theta$ was over 0.14, as opposed to 0.12 and 0.10 for the other two curves, the pore pressure showed only a small rise and then levelled off after a hundred cycles. No recordable increase of pore pressure was observed from about 113 to 213 cycles. At this stage, it was assumed that a steady state had been reached and the test was switched off. Curve No. 2 shows a test in which a somewhat lower alternating stress was applied to a fairly loose sample, which proceeded steadily to liquefaction. Curve No. 3 shows a sample which had a lower void ratio and lower cyclic shear than No. 2, it also went steadily to liquefaction but after many more cycles than No. 2. It will be noted that the general form of the tests is the same in Fig. 31a as in Fig. 31b.

Commenting on the results in general, no test ever failed to liquefy if $U$ reached a value of 30 percent and once 50 percent was reached, liquefaction ensued within a comparatively few extra cycles.

To plot the full course of each test on one combined graph, leads to a meaningless jumble of data and it was
Fig. 31a. Cyclic Load Tests in Simple Shear. 
Tan θ' vs. Log Cycles.

Fig. 31b. Same tests as above. 
U vs. Log Cycles.
considered necessary to adopt some means of summarizing many
test results in one plot. Thus it was desirable to reduce
the results of a complete test to one plottable point and the
logical figures to use appeared to be void ratio, applied
shear and normal stresses and the number of cycles to liquef
action. However, because of instability of the speed control
equipment after running for some time, tests of more than
about half an hour's duration, gave questionable results. So,
instead of using the number of cycles to liquefaction, use
was made of the number of cycles to make \( U \) rise to 20 percent.
The pore pressure rise leading to this value of \( U \), represents
a very definite loss of strength which might be as important
as liquefaction. Furthermore, in general, it is an indica
tion of the relative tendency to liquefaction in the various
tests.

Of the total number of tests completed, many
results were discarded for such reasons as

a) Erratic tendencies in the rate of pore pressure
rise, implying slight intermittent leakage.

b) Improvements in technique, leading to rejection
of earlier results as not being comparable with
later results. Examples of such changes are,
proper retention of vertical corners, methods of
installing the top boundary plate, etc.

Figs. 32a and 32b summarize the tests which are
considered valid. Fig. 32a shows results of tests carried
out with a normal load of 2.0 kg. per sq. cm. and Fig. 32b
with a normal load of 4.5 kg. per sq. cm. Both show contours
of the applied shear stress to cause a 20 percent rise of
pore pressure, as a function of void ratio and the number of
cycles required to cause the given rise. The contours rep
resent the results of tests done with ribbed plates, giving
proper frictional restraint of the top and bottom boundaries.
The tests from which these contours are drawn are represented
by triangles and squares. It was desired to compare results
Fig. 32a. Cyclic simple shear with $\sigma = 2.0$ kg/cm$^2$.
Shear to make $u$ rise 0.4 kg/cm$^2$ (i.e. $U = 20\%$)

Fig. 32b. Cyclic simple shear with $\sigma = 4.5$ kg/cm$^2$.
Shear to make $u$ rise 0.9 kg/cm$^2$ (i.e. $U = 20\%$)
obtained with ribbed plates and plain plates. Accordingly, representative tests with plain plates are superimposed on the contours of Figs. 32. The tests with plain plates are indicated by circles with the applied shear written beside them.

If the points representing plain plate tests are compared with the contoured surface representing ribbed plate tests, it will be found that in Fig. 32a the resistance to liquefaction with ribbed plates is greater than the resistance with plain plates in the ratios 2.2, 2.3, 2.3, 2.0, 3.2, and 6.2. In Fig. 32b, there are only three results with plain plates and two of these are in a region where the contour surface is so steep that its value is hard to compare. However, none of the plain plate tests in Fig. 32b show more than a third of the resistance represented by the contours for ribbed plate tests.

It is not to be expected that the results reported herein would agree numerically with those obtained by Peacock and Seed, as the tests were performed on a different material. Much less would the results be expected to agree numerically with the triaxial cyclic shear tests reported by Seed and Lee. However, it will be seen that the results all agree over the following main relationships.

a) For a given void ratio, resistance to liquefaction increases with initial effective normal stress.

b) For a given initial normal stress, resistance to liquefaction increases with decreasing void ratio.

c) For a given void ratio and normal load, the number of cycles to liquefaction decreases if the cyclic shear load is increased.

In considering the above relationship, Seed and Lee suggested that relationship No. 1 rendered the concept of a critical void ratio invalid. It is suggested, however, that the results obtained should lead to questioning the definition rather than the existence of a critical void ratio.
The critical void ratio is now generally regarded as the terminal void ratio for large strains in drained shear under a given normal load. This void ratio decreases as normal load increases. However, the original concept of a critical void ratio was to express the state in which a sand might liquefy under shock load. It took time and experimental evidence to show that such susceptibility to liquefaction would depend on more variables than density; Taylor\(^\text{15}\) (for example) insisted that for a critical void ratio to be valid, it must be determined in completely specified conditions. If this view is accepted, it is not valid to compare liquefaction tendencies under cyclic shear load with a critical void ratio determined under other types of test. In the examples quoted by Seed and Lee, the sample on the "safe" side of the critical void ratio line having liquefied, the critical void ratio must be wrongly defined for its original intention. A possible explanation of some of the inconsistencies is that the various definitions of critical void ratio involve equality of pore pressure, or volume, at two different stages of the shear test; this ignores the energy barrier which may exist between these stages. In the cyclic shear tests so far performed, liquefaction has occurred at very small net shear strains of the order of 0.05 to 0.1 degrees. This suggests that to be a valid criterion of liquefaction under cyclic shear stress, the definition of critical void ratio should involve no dilatent tendency from the strain of zero to a strain of 0.5 degrees. In other words, the critical void ratio for a given confining pressure should perhaps be the void ratio for which the volume change (or pore pressure for an undrained test) has zero rate of change at the start of the test. As yet there are insufficient test results to say whether this would prove to be a true criterion and in the meantime, it will be necessary to specify the loading

completely if any consideration of critical void is required.

Having expressed the results of the cyclic tests in the form of a set of shear contours for each normal load, it seems natural to attempt to combine them by using the applied tan $\theta$. If this is tried with Figs. 32a and 32b, dividing all shear levels by the relevant total normal stresses, it will be found that the results expressed in terms of the applied tan $\theta$ do not, in general, coincide and contours can no longer be drawn. In other words, for a given rise of pore pressure, $\tau$ is not usually directly proportional to $\sigma_n$. This non-linearity shows in Figs. 33 to be described presently.

It is therefore evident that the number of variables to be handled cannot be reduced in the way described. Instead, Figs. 32a and 32b were combined by plotting contours of shear stress to cause a 20 percent rise of pore pressure at 1000 cycles, as a function of void ratio and normal stress (Figs. 33). It will be seen from Figs. 32 that after a 100 cycles the actual number of cycles makes little difference, while at a 1000 cycles the shear stress level is almost asymptotic, thus the shear stress values plotted are close to the values below which liquefaction would never occur, and the surface represented by these contours is close to a critical state for liquefaction. The result of this plot is shown in Fig. 33a. It will be readily seen that if a smaller number of cycles is taken as the criterion, the contour levels are higher. Fig. 33b shows a sketch of a similar surface but for $U = 20\%$ in 10 cycles. Any given number of cycles would thus yield a sloping curved surface whose contours could be plotted in a manner similar to Fig. 33. This set of surfaces would vary from the lowest or critical one, below which liquefaction would not occur, up to the highest, which would presumably be close to the static undrained "yield" strength.

In Figs. 33, the applied friction angle, $\theta_a$, for a given shear stress, normal stress and void ratio, is the slope of a line to the $\epsilon$-axis from the point representing the given
Alternating Shear Stress to cause a Rise in $U$ of 20% in 1000 Cycles vs. Void Ratio and Initial Effective Normal Stress. For a large region, slope is constant for a constant value of $e$. Hence slope can be plotted against $e$.

Fig. 33b. Alternating Stress to cause a Rise in $U$ of 20% in 10 Cycles vs. Void Ratio and Initial Effective Normal Stress. Applied Shear Angle is no longer independent of initial normal stress in any region and so cannot be plotted against $e$. 
conditions. In Fig. 33a there is a considerable region of the plotted surface where such a line would lie along the surface. Therefore, in this region $\tau$ and $\sigma$ could be replaced by the single variable $\theta_a$. However, as $\sigma$ increases, this $\theta_a$ is no longer sufficient to define the surface and when the number of cycles is reduced (Fig. 33b) no part of the surface can be defined by $\theta_a$. This is the reason why Figs. 32 could not be combined into one plot.

It is, of course, accepted that Fig. 33 is based on the few plotted points available from Fig. 32. The resulting contours are not claimed to predict the behaviour of Ottawa sand; they are rather intended to show the general form of relationship between the variables. Further to the preceding discussion of critical void ratio, it will be seen from Fig. 33 that for any given cyclic shear stress the

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**MONTEREY SAND**

**CYCLIC SIMPLE SHEAR LIQUEFACTION IN 10 CYCLES**

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Fig. 34. After Peacock and Seed (1968). Results in Reference 9 for Comparison with Fig. 33.
critical void ratio increases with normal pressure. It should be noted though, that this conclusion may not apply in all possible combinations of void ratio and normal pressure. In order to emphasize the similarity of results obtained by Peacock and Seed for simple shear under cyclic load, their results have been re-plotted in Fig. 34 in a similar way to the plot of Fig. 33. Fig. 34 shows contours of the cyclic shear load required to cause initial liquefaction in 10 cycles as a function of void ratio and normal stress for Monterey sand. It will be seen that the form of the results is similar to that obtained in Fig. 33 but the range of normal stresses is greater, giving a picture of somewhat wider scope.

In order to discover the type of results which might be obtained, one drained cyclic shear test was performed on the Ottawa sand. The results of this test are shown in Fig. 35. It was found that for a given normal load and cycling shear load, the drained test reached a terminal void ratio of nearly 0.668. The void ratio was then reduced to .666, by increasing the normal load for a minute or two, after which the normal load was returned to its original value and the cyclic shear load was re-imposed. The void ratio did not then show a tendency to increase to the value that it had reached before, but remained at .666. From this it is suggested that under the same normal load and cyclic shear load, in an undrained test, any samples starting at a higher void ratio than .666 would ultimately have liquefied and any sample starting at a lower void ratio would never have liquefied. It this is true, a series of drained cyclic shear tests under varying normal loads and varying cyclic shear loads, with terminal void ratios recorded, could be used to make a plot of stress contours as a function of void ratio and normal pressure, similar to Fig. 33, which would define the critical surface already indicated.

The question naturally arises as to whether the cyclic shear stresses accomplish the recorded results simply because they vary, or whether the results are due to such causes
as inertia of sand grains and passage of stress waves, hereafter referred to as dynamic effects.

The process of liquefaction involves transfer of normal stress from the soil skeleton to pore water with consequent reduction in shear strength. This transference of normal stress is, in effect, a relaxation of the soil skeleton at almost constant volume. The drained cyclic shear tests, on the other hand, could be compared to a process of creep. The cyclic shear stress is thus able to cause relaxation or
creep of the soil skeleton which would not be accomplished by static shear stresses of the same magnitude. From the stress paths in the static tests (Fig. 26) it does not appear that the relaxation effect would be produced by gradually applied increments and decrements of stress. In general, for the void ratios tested, cyclic shear stresses of about one-tenth of the initial normal stress have been found sufficient to cause liquefaction. Stresses of this order in Fig. 26 generally tend to cause increases in the effective normal stress rather than decreases. If, on the other hand, the effect of relaxation depends on dynamic effects, the stress paths of static tests would not be expected to be comparable to the cyclic tests and the relaxation or creep effect are probably due to slight readjustment of particle contacts, which may be jarred apart during stress reversals. Dynamic effects such as those just mentioned might be proportional to the rate of change of stress, in which case liquefaction may be expected to be more rapid for more rapid stress cycling, or they might simply require some threshold level of the governing dynamic parameter. Published results on the effect of using different speeds are somewhat inconclusive as the experimental scatter is of the same order of magnitude as the effect of the change of frequency. However, in one of the tests now being reported, the already mentioned instability of the speed control led to a sudden increase of speed for about a dozen cycles in the middle of a long test. Thus, with zero change in void ratio, packing of grains or normal load, all other variables except speed remained unchanged. The result is therefore considered particularly significant.

Fig. 36 shows a reconstruction of the relevant portion of oscillograph record. A reconstruction was necessary because, for a long term test of this nature, the oscillograph paper speed was .01 in. per sec. and the individual cycles do not show. Instead the cycle counter was used to mark the paper every 10 cycles. In the actual record, the regularly spaced marks, .05 in. apart, suddenly show two
narrow gaps representing 20 cycles where extra cycles had occurred within an abnormal (short) time interval. Knowing the paper speed, the time within which the 20 cycles occurred, could be ascertained as approximately 6 sec. From the actual record, magnitude of the shear strains can be read quite accurately and the total rise of pore pressure during these twenty cycles can easily be read. After rising 0.28 kg. per sq. cm. in 6630 cycles (average rise $0.042 \times 10^{-3}$ per cycle) the pore pressure reached 0.37 kg. per sq. cm. at 6650 cycles (average rise $4.5 \times 10^{-3}$ per cycle). These events have been reconstructed for Fig. 36. The actual maximum rate of cycling and the number of cycles affected is not known; it has

\[
\text{OTTAWA SAND} \quad \text{Normal load} \quad 4.5 \text{kg/cm}^2
\]

\[
\begin{array}{l}
\text{SHEAR} \quad \text{DEFLECTION} \quad \text{inches} \\
\text{SHEAR} \quad \text{LOAD} \quad \text{kg/cm}^2 \\
\text{PORE} \quad \text{PRESSURE} \quad \text{kg/cm}^2
\end{array}
\]

\[
\begin{array}{l}
\text{TIME} \\
1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad 1 \quad 2 \quad \text{SEC}
\end{array}
\]

Fig. 36. Cyclic Simple Shear. Simulated Oscillograph Record Showing Pore Pressure Increase During Temporary Speed Increase.

been estimated by the information, from the record, that 20 cycles occurred within 6 secs. and from the rhythm of the exhaust noises during the test which changed for an estimated
time of 2 or 3 secs. before returning to normal. After the sudden sharp increase the rate of rise of pore pressure returned to what it has been.

In view of the foregoing remarks, about effective stress paths and the jump in pore pressure just described, it is considered that liquefaction is likely to be a phenomenon of dynamic, as opposed to just cyclic shear. However, it is realized that more testing is desirable before this can be stated definitely.

The following points summarize the cyclic test results:

a) The effect of inadequate top and bottom friction involves severe lowering of resistance to liquefaction.

b) Confirmation was obtained of the general relationships discovered by Peacock and Seed of the factors governing liquefaction of sand.

c) Drained cyclic shear under given stresses leads to a terminal void ratio, below which the sand is stable and will never liquefy under the same stresses.

d) Liquefaction is probably not a result of merely cyclic stress. It is likely that it is at least partly due to inertia effects during passage of strain waves in the sand mass.

Re-Liquefaction. Early during the testing reported herein, it was found that if a sample was liquefied and then redrained and subjected to the original cycling load, it would re-liquefy almost immediately. An example is given in Fig. 37a. This figure shows a first test, starting with a void ratio of 0.68 and an effective normal stress of 1.0 kg. per sq. cm., liquefying after 36 1/2 cycles. After liquefaction, the shear load was returned to zero. The pore pressure valve was then opened, allowing the pore pressure to return to the original back pressure. As excess pore water escaped, the
original normal stress became effective again and the void ratio decreased to 0.595. The pore pressure valve was then reclosed and the cyclic shear load was re-applied for the second test. In the second test $U$ went to 18 percent in half a cycle and 90 percent with liquefaction, in one cycle. This residual weakness after liquefaction was first noticed while plain upper and lower plates were in use. As has been previously mentioned, the plates were observed to be first scratched and then polished in a longitudinal direction. The area polished on either plate was in a band across its width, distributed symmetrically about the transverse centre line and occupying about half the length of the plate. The ends of the plate were unmarked. At first it was thought that the central region of the sample was becoming densified into a mass somewhat as in the plasticine example in Fig. 12f where densification was simulated by lateral expansion. If this were so, under the large cyclic deformations after liquefaction, the central region of sand was capable of sliding bodily to and fro without significant distortion and the loosened corners, alternately densifying and dilating, were not offering appreciable resistance. If the centre of the sample were sliding bodily to and fro in the manner suspected, redraining would not be likely to disperse the central core and almost immediate re-liquefaction would be expected. However, when ribbed top and bottom plates were used, abrasion of the plate surface did not occur, therefore the central mass of the sample cannot have been moving bodily, but almost immediate re-liquefaction still occurred (see Fig. 37b). It is emphasized that Fig. 37a shows one sample exhibiting a tremendous decrease in resistance to liquefaction and an accompanying increase in density. Ribbed top and bottom plates still showed a similar tendency to immediate re-liquefaction, but it could now be ascertained that the top and bottom boundaries were behaving as intended and it was therefore considered impossible that the sample could be forming into zones of zero deformation. If the sample was behaving more or less
Fig. 37a. Re-liquefaction of Ottawa sand with plain plates. Two tests on sample, drained between.

Fig. 37b. Re-liquefaction of Ottawa sand with ribbed plates. Two tests on same sample drained between.
uniformly, the rapid re-liquefaction could not be due simply to machine characteristics; something must be affecting the interparticle contacts, to make them all change behaviour in the same way, and, presumably, the same change would occur in the field in a layer subjected to the same set of events. It was considered that the change in interparticle contacts would consist of readjustment in either of two ways:

a) Re-orientation in such a way as to make all the contacts unstable and therefore more liable to liquefaction.

b) Re-arrangement so that high points on one particle, which were hindering the motion of a neighbour, were crushed or pushed aside.

If the first type of readjustment were occurring, it appears to imply some factor at liquefaction making particles seek positions with unstable contacts. Otherwise, it is hard to imagine that out of all the contacts through the mass, such a high percentage would reach an unstable position as to render resistance to re-liquefaction practically nil. It is still harder to imagine that at different void ratios and under different stress fields, large percentages of contacts would reach unstable positions unless some factor favours such a type of contact. In considering these types of readjustment, it appears reasonable that mechanical instability would involve transferrence of contacts to "peaks" and high points on the grains. Such a transfer would imply increase in void ratio, which did not happen. The second type of readjustment would not imply increase of void ratio. On re-liquefaction, the amount of shear deformation was always approximately the same as in the first liquefaction. This would be rather a coincidence if readjustment of contacts was of the first type, but would be expected for readjustment of the second type in which the large cyclic strains of the first liquefaction had cleared a low resistance path. Inductive reasoning, therefore, seems to favour readjustment of the second type. To investigate the matter experimentally,
it was decided to run some tests with a limit on horizontal travel, to keep deformation very small and comparable with typical pre-liquefaction deformations. A mechanical stop was accordingly devised, allowing deformations of approximately ± .002 in. (approximately ± 0.01 degree). This stop did not interfere with liquefaction until the last few cycles of a test.

As the sample approached liquefaction and the stop came into play, the carriage could be heard, tapping the stop, at every stroke. During a limited travel test, of the type described, the pore pressure would rise and then level out at approximately the value which would have been expected as the maximum pressure in an ordinary test. On switching off a limited travel test, draining and re-testing at the resulting higher density, the sample showed an increase in resistance to liquefaction. This was contrary to the earlier tests quoted. The results of such a series of tests are shown in Fig. 38. In the limited travel tests the full cyclic shear
load was applied to the sample at each reversal. After liquefaction, the shear load acted on the sample for only a very small fraction of a second before being transferred to the stop. The horizontal shear deformation started and stopped with a jerk at every stroke. It was hoped that, as far as possible, the conditions of an ordinary liquefaction test were repeated but with the sole omission of the considerable alternating strains. It is realized, of course, that the exact forces acting on a sand grain are not the same in the two types of test, but if some factor makes the grains seek an unstable contact during the jarring of liquefaction strains, it was considered that it should still make them form the same kind of contact under jarring by the mechanical travel stop. However, if the sensitivity to re-liquefaction is due to the particle contacts arranging themselves with a "cleared path" of the necessary length, the limited travel tests would prevent such arrangement. The experimental evidence is not conclusive, but it appears to add weight to the belief that liquefaction makes a sand more sensitive to re-liquefaction, by arranging the grains in a position which is stable but in which shear resistance has been minimized over some limited, but definite degree of strain. Such an arrangement would be met by having inter-particle contacts arranged so that the tangent plane to any contact is normal to a line from the contact to the centres of rotation of the two grains forming the contact. Such centres of rotation need be effective for only small displacements involved with small strains.

In natural conditions liquefaction may involve small shear strains, because once a layer has liquefied, it is hard to see how it can propagate the shear stresses needed to cause large strains. If the shear strains resulting from liquefaction of natural deposits are as small as those obtained in limited travel tests, one liquefaction might make a sand deposit safer for future occasions. If the shear
strains are a little larger, as in tests without a travel stop, one liquefaction of a natural deposit may make it far more susceptible to future liquefactions. It is considered that the phenomenon of re-liquefaction would repay further investigation.
CHAPTER VI
SUMMARY AND CONCLUSIONS

The difference between plane strain simple shear and ordinary triaxial testing for soil is of considerable interest. The typical triaxial test involves compression of a sample under equal all round stress, which sets up no shear, followed by a steady change of one or more of the principal stresses. The shear, therefore, is applied to a sample which has experienced zero shear stress and strain since removal from the ground. The shear is then developed, on a set of planes with fixed orientation. In the simple shear test, the sample is set up with zero lateral strain. Hence, compression is not equal in all directions, but is similar to conditions to be expected in the ground. Shear is then applied on horizontal planes, with the result that the principal axes rotate, as they would usually do in the ground, if the shear field changed.

It might be supposed that the considerations above, are of merely theoretical interest. However, test results, to be summarized presently, show considerable practical differences between results of triaxial and simple shear tests.

The simple shear machine was designed not only to improve on the sequence of stresses provided by the triaxial test, but also to constrain the sample to deform in a uniform manner. The deformation patterns actually achieved in the test were checked by dummy plasticine samples. These showed a high degree of uniformity through most of the sample, variations occurring only in the regions near the boundaries. The pattern of deformation shown by plasticine samples was confirmed by means of elastic analysis. The validity of elastic analysis in the present case, was demonstrated by the similarity of displacements predicted by analysis and those exhibited by plasticine. The elastic analysis also showed the degree to which conditions are theoretically improved by
increasing the ratio of sample length to height. However, caution is required in choosing the length to height ratio for a machine, since although conditions improve for a long sample if the assumed boundary conditions are realized, they become harder to realize. In the machine described, the sample dimensions were chosen primarily to maximize application to practical engineering, and greater length might be preferable for considerations of stress distribution. However, even with the dimensions chosen, realization of proper boundary conditions needed considerable care in the case of sand samples.

Using results obtained from experiments at Cambridge, England\textsuperscript{13}, indicating that the central portion of a sample would behave in a manner very close to the desired stress-strain relationship, it was deduced that the stresses and strains at the central portion of the sample would be very close to the applied stresses and measured strains. Hence the desired stress-strain relationship for a sample of soil is obtained directly from the measured average stresses and deformations.

The analytic elastic investigation of the stress distribution was extended to cover anisotropic material of the type which would be associated with horizontally layered soil. From this investigation it was concluded that where boundary conditions consist of known displacements, mild anisotropy would have an appreciable effect on the distribution of stresses, but not of displacements. Similarly where boundary conditions consist of known stresses, the same degree of anisotropy would have an appreciable effect on displacements but not on stresses.

In dealing theoretically with anisotropic material, it was considered necessary to investigate the bounding values of the elastic constants, in order to ensure that the materials selected could theoretically exist. The bounding values were
calculated by energy considerations and during the course of this investigation it was noted that

a) Conditions for a zero bulk modulus are narrow. If the average normal stress is not zero, the material changes in volume unless

$$\mu_{xx} + \mu_{xy} = 1$$

and

$$\frac{E_x}{E_y} = 2\mu_{xy}$$

b) If $\sigma_2$ and $\sigma_3$ are small, anisotropic material may increase in volume under a compressive $\sigma_1$, (and vice versa) without violating energy principles. In other words, an anisotropic soil such as pre-compressed or flocculated clay may dilate under triaxial compression without violating any of the rules of elastic behaviour. This conclusion might have significance in finite element analyses by computer, since it shows that dilatent material can be analysed by elastic methods.

Techniques were developed for setting up samples of cohesive soil and disturbed samples of sand.

Sand samples are more difficult to install than clay and considerable care is needed with sand. Failure to observe the proper precautions will probably lead to leakage, which could be slight or intermittent. Results from experiments where leakage is slight may appear to follow a normal pattern and it is desirable to avoid questionable results, by avoiding procedural errors which could lead to them.

Another condition is essential to distinguish valid tests from apparently valid tests. This essential condition involves the development of adequate friction, so that no part of the sample slips against the top and bottom boundaries. The only direct test so far devised for adequate friction is the condition of the boundary plates after several similar tests. If friction is inadequate there is an area of abrasion and polishing on the plates, distributed symmetrically about a
transverse centre line. To achieve adequate friction, a pair of plates was made with sharp transverse ribs projecting from their surface. Unfortunately, there are some difficulties with ribbed plates. In the first place there are probably slight inhomogeneities at the bottom surface of the sample, due to ribs interfering with the positions of the sand grains as they fall into place. During sample placement, the ribs may cause abnormal horizontal motion of grains bouncing off their sides. A more serious trouble is likely to occur at the top surface as the ribbed boundary plate is lowered onto the prepared sample. As the top plate is lowered, its ribs must dig into the sand before the plate is bedded down properly. It is believed that research is necessary to establish a test for proper seating, and to discover whether a temporary seating load affects the strength of the sample.

Stress-controlled, undrained experiments were performed on Ottawa sand. In static tests it was found that

a) The angle of friction in simple shear tests requires a somewhat different concept than the usual $\phi$, associated with the Mohr-Coulomb failure criterion. The angle of friction developed in simple shear tests was higher than the comparable angle in triaxial tests on the same material. However, the pore pressure developed was much higher in undrained simple shear than in undrained triaxial tests. The result of the high pore pressure was that, in spite of the higher friction angle, simple shear tests showed a lower strength than corresponding triaxial tests.

b) It was notable that a "yield" occurred which was more marked in simple shear than in a corresponding triaxial test. In simple shear, the yield consisted of high strain at constant load, well below the maximum shear load, but approximately at the stage where the developed friction angle reached its maximum. In the triaxial tests a
slight "yield" was noted at approximately the same strain as in simple shear, but it involved only a slight "flattening" of the stress-strain curve, not a large deflection at constant stress.

c) Use of sand blasted top and bottom plates did not provide adequate boundary friction. For this reason they usually indicated a low apparent strength, less than that obtained by using ribbed plates. Under high normal loads the sand blasted plates showed almost as high a strength (for a given density) as ribbed plates, especially at small strains, but under lower normal loads the sand blasted plates gave apparent strengths as much as 13 1/2% lower.

The main results of cyclic shear tests on sand are summarized below:

a) Confirmation was obtained of the effect on liquefaction of most of the parameters described by Peacock and Seed. The results showed that the number of cycles to liquefaction increases with
   (i) Density of the sand and
   (ii) Initial normal pressure on the sand.

The number of cycles to liquefaction is reduced if the cyclic shear load is increased.

b) Contrary to Peacock and Seed's results, it was found that there was a tendency for increased frequencies of stress alternation to reduce the number of cycles to liquefaction.

c) If, after liquefying, a sample was drained to its initial normal effective load (thereby densifying it) and then subjected to the original cyclic load without drainage, it was found that the sample was far more sensitive to a second liquefaction than to the first. It was concluded that the structure
of the sand was altered by liquefaction.

d) Liquefaction with sand blasted top and bottom plates occurred at a cyclic shear often as low as 1/3 of that required for liquefaction with ribbed plates.

Several questions have been left unanswered. It is, perhaps, to be expected that work of the type which has been described, would raise more queries than it can solve immediately. In closure, it seems desirable to comment on some of these questions as lines for future research.

a) The apparatus described is not equipped to measure horizontal normal stresses. However, any work done to amplify knowledge of the complete stress field and behaviour of the sample at large strains would be beneficial.

b) The tests described were almost entirely on sand and it was considered that compliance of the pore water measurement was obviously negligible. Before testing clays, it would be advisable to measure compliance and, if necessary, modify the apparatus.

c) Sample interference by ribs on the top and bottom plates may be appreciable. It would be desirable to investigate other methods of developing friction and better tests for slipping at the boundary.

d) The vertical seating load, applied to ensure contact between the upper plate and the sample, has undesirable features. It would be valuable to know whether the resulting precompression affects the measured stress-strain relationship.

e) The cyclic shear tests leave many questions for investigation. In particular, the phenomenon of re-liquefaction appears to need more testing. Also, there are many parameters of cyclic loading which have not been investigated at all. Such parameters include the shape of different wave forms and the
question of whether liquefaction depends simply on shear stress amplitude or whether it depends more on rate of change of stress. These last questions have considerable importance, because recommended methods of seismic design suggest simplifying the design earthquake into its fundamental frequency, amplitude and number of cycles. The simplification might lead to a very different range of frequencies and numbers of cycles, if the critical factor were rate of change of stress rather than amplitude of stress.
APPENDIX A

THE EFFECT OF COMPLIANCE ON THE PORE PRESSURE RECORDED

It was mentioned in the text that if the pore pressure rises, a certain volume of water is withdrawn from the sample to record the rise, thereby reducing the recorded pressure.

Consider a sample of volume $V$, subjected to a change in total pressure $\Delta \sigma$. Take subscript $A$ to denote conditions if the pressure were not affected by compliance and $B$ to indicate actual conditions. Also put

\[ \Delta u = \text{change in pore pressure due to } \Delta \sigma. \]
\[ C_w = \text{Compressibility of water.} \]
\[ C_s = \text{Compressibility of the soil skeleton.} \]
\[ V_1 = \text{Volume of pressure measuring device.} \]
\[ S = \text{Change in } V_1 \text{ due to unit pressure change.} \]
\[ K = \text{Compliance of the apparatus, as worked out in the text.} \]
\[ = S + C_w \cdot V_1 \]
\[ n = \text{porosity of the sample.} \]

Assuming that the compressibility of the solid mineral grains is small compared with that of water, we have, for case $A$, the decrease in volume of the sample, and also the decrease in volume of the system is equal to the decrease in volume of the water or
\[ \Delta V_A = C_S \Delta \bar{\sigma}_A \cdot V \]
\[ = C_w \Delta u_A \cdot V \cdot n \]

But \[ \Delta \bar{\sigma}_A = \Delta \sigma - \Delta u_A \]

So \[ C_w \Delta u_A \cdot V \cdot n = C_S (\Delta \sigma - \Delta u_A) \cdot V \]

\[ \Delta u_A = \Delta \sigma \left( \frac{C_w \cdot n}{C_S} + 1 \right)^{-1} \]

In case B, the decrease in volume of the system is equal to the decrease in volume of the pore water. But it is also equal to the decrease in volume of the soil skeleton minus the increase in volume of the pressure device.

So
\[ C_w \cdot \Delta u_B \cdot (n \cdot V + V_1) = C_S \cdot \Delta \bar{\sigma}_B \cdot V - S \cdot \Delta u_B \]

or
\[ C_w \cdot \Delta u_B \cdot n \cdot V + \Delta u_B (C_w \cdot V_1 + S) = C_S \cdot \Delta \bar{\sigma}_B \cdot V \]

or
\[ \Delta u_B \cdot (C_w \cdot n \cdot V + K) = C_S \cdot \Delta \bar{\sigma}_B \cdot V \]

And \[ \Delta \bar{\sigma}_B = \Delta \sigma - \Delta u_B \]

So
\[ \Delta u_B = \Delta \sigma \cdot \frac{C_S \cdot V}{C_S \cdot V + K + C_w \cdot n \cdot V} \]

\[ = \Delta \sigma \cdot \frac{1}{K \left( \frac{C_S}{C_S \cdot V} + \frac{C_w \cdot n}{C_S} \right)} \]
and the ratio of the recorded pressure to the ideal pressure is

\[ \frac{\Delta u_B}{\Delta u_A} = \frac{1}{1 + \frac{K}{c_s \cdot V} \left[ \frac{c_s}{c_s} + 1 \right]} \cdot \frac{c_s \cdot n}{c_s} + 1 \]

\[ = \frac{1}{1 + \frac{K}{c_s \cdot V} \left[ \frac{c_s}{c_s} + 1 \right]} \]

\[ = \frac{1}{1 + K \cdot \frac{1}{V} \left[ \frac{1}{c_s \cdot n + c_s} \right]} \]
APPENDIX B

DESCRIPTION OF THE WATER SUPPLY

In order to use de-aired water, as an aid to saturation of samples, the water supply must contain a small supply reservoir. To prevent this supply dissolving atmospheric air, free surfaces must be avoided. To flush water back and forth, through the loading head and the space above the sample, it must be possible to put pressure behind the water supply and lead it to either side of the loading head. To allow the passage of water, it must be possible to release the pressure on either side of the loading head, at will. For checking the assembly for leaks and various other purposes, it is useful to be able to connect at least one side of the loading head to a vacuum. However, it is not advisable to use vacuum for de-airing loose samples, as the inward pressure on the membrane would be liable to disturb them.

In order to supply back pressure and do drained tests under back pressure, test for leaks and test for saturation, it is necessary to be able to apply adjustable pressure to the de-aired water supply and allow one side of the loading head to admit or expel water, under the set pressure, with the other side shut.

These objectives are achieved by the arrangement shown in Fig. 39, at the end of Appendix B.

The supply pressure is adjusted to the desired level by regulator A. Pressure or vacuum can then be applied to either side of the apparatus by appropriate settings of the assembly of valves B and C. Atmospheric pressure can be admitted to either side by opening one of the valves D. The apparatus is set up so that usually flow is from the right side of the diagram, towards the left, being reversed on comparatively rare occasions. For this reason, the sealed water reservoir E is on the right side. The surface of this water is protected from air contamination by a flexible diaphragm
and valve D2 is normally kept closed except when sucking de-aired water into E, by vacuum. The auxiliary tank F is interposed between the supply and E, so that inadvertent admission of an air bubble does not require the tedious procedure of dismantling and refilling E. Instead, the unwanted air rises to the top of F, where it is collected by the conical top and can be released through the small needle valve, G. As a rule, the tube below D2 is kept below water level, ready to suck more water up when required, while D1 is usually employed as a drain. When calibrating the electrical apparatus, valves B1, C1, C2, D1, D2 are closed, and valves B2, J and the loading head valves are open. After attaching a gauge tester, valve H may be opened and the system is closed under an adjustable pressure, which will be monitored by the gauge tester.

When a sample is being de-aired, pressure at two or three atmospheres is admitted to the right side, with valve K2 open. B1, C1 and H are closed, with D1 open. Then K1 is opened slightly for short periods. By not opening K1 fully, partial pressure is maintained and any dissolved air, in the sample, does not start forming bubbles. Intermittent opening of K1 allows water with dissolved air to flush away. Reclosing K1 allows time for new water to dissolve its quota of air. If the supply in E becomes exhausted during de-airing, valves B2, J and K1 may be closed for a moment, while a fresh supply is sucked through D2.

When de-airing is complete, E is left half full, so as to receive or supply water as necessary. K1 is closed and the desired back pressure is set, leaving B2, J and K2 open.
Fig. 39. Water Supply and Back Pressure System.
APPENDIX C

ANALYTIC ELASTIC SOLUTION FOR THE

BEHAVIOUR OF A SAMPLE

1. The Material

It is assumed that the material is homogeneous and that for any given direction, the elastic parameters are constant, not varying with stress, strain or time.

It is further assumed that the material is transversely isotropic. That is, taking the \( y \)-direction as vertical and \( x \)- and \( z \)-directions as horizontal, the elastic parameters are the same for all directions in the \( x-z \) plane. Such a type of anisotropy would usually represent a homogeneous or evenly bedded soil.

It has been shown\(^{16}\) that such a material has five independent elastic constants, which can be taken as

- \( E_x \): Young's modulus in any horizontal direction.
- \( E_y \): Young's modulus in a vertical direction.
- \( \mu_{xy} \): Poisson's ratio for strain in the vertical direction due to a horizontal stress.
- \( \mu_{xx} \): Poisson's ratio for strain in any horizontal direction due to a horizontal stress at right angles.
- \( G_{xy} \): Modulus of shear deformation in a vertical plane.

It can be proved that

\[
\frac{\mu_{xy}}{E_x} = \frac{\mu_{yx}}{E_y} \quad \text{and} \quad G_{xx} = \frac{E_x}{2(1 + \mu_{xx})}
\]

Five elastic parameters, which will be of use in

---

\(^{16}\) See, for example, A.E.H. Love "A Treatise on the Mathematical Theory of Elasticity", 1892.
later discussion, are introduced here;

\[ K_1 = \frac{E_x}{E_y} - \mu_{xy}^2 \]

\[ K_2 = \frac{E_x}{2G_{xy}} - \mu_{xy} (1 + \mu_{xx}) \]

\[ K_3 = 1 - \mu_{xx}^2 \]

\[ \alpha_1 = \frac{K_2 + \sqrt{K_2^2 - K_1 K_3}}{K_1} \]

\[ \alpha_2 = \frac{K_2 - \sqrt{K_2^2 - K_1 K_3}}{K_1} \]

The question naturally arises whether \( \alpha_1 \) and \( \alpha_2 \) are real or complex. For any isotropic material

\[ E_x = E_y = E \]

\[ \mu_{xx} = \mu_{xy} = \mu \]

\[ G_{xx} = G_{xy} = G \]

Substituting these values into the expressions for \( K \) yields

\[ K_1 = K_2 = K_3 = 1 - \mu^2 \]

therefore

\[ \alpha_1 = \alpha_2 = 1 \]

for all isotropic materials.

Examination of the possible values of the elastic moduli (See Appendix D) shows that \( G_{xy} \) must always be positive, but it is completely independent of all the other elastic parameters; it also shows that \( E_x \) is always positive. It follows that a material may exist which is isotropic.
except that $G_{xy}$ is slightly larger than $G_{xx}$. In this case, $K_2$ will be smaller than $1 - \mu^2$ without affecting $K_1$ and $K_3$. The square root of a negative quantity will then be included in the expressions for $\alpha_1$ and $\alpha_2$. If, on the other hand $G_{xy}$ is slightly less than $G_{xx}$, $K_2$ will be larger than $K_1$ and $K_3$. Thus $\alpha_1$ and $\alpha_2$ may be real or complex.

2. The Governing Differential Equation

The conditions for equilibrium and compatibility of strain are independent of the nature of the material, which affects only the relationships of Hooke's Law. The biharmonic equation for the material can therefore be developed in the usual way, as follows, for the case of plane strain.

Using the five parameters already listed, and writing Hooke's Law in matrix form, if we multiply through by $E_x$, for simplicity later, we get

$$
\begin{vmatrix}
1 & -\mu_{xy} & -\mu_{xx} \\
-\mu_{xy} & \frac{E_x}{E_y} & -\mu_{xy} \\
-\mu_{xx} & -\mu_{xy} & 1
\end{vmatrix}
\begin{vmatrix}
\sigma_x \\
\sigma_y \\
\sigma_z \\
\tau_{xy} \\
\tau_{yz} \\
\tau_{zx}
\end{vmatrix}
= 
\begin{vmatrix}
\varepsilon_{xx}E_x \\
\varepsilon_{yy}E_x \\
\varepsilon_{zz}E_x \\
\gamma_{xy}E_x \\
\gamma_{yz}E_x \\
\gamma_{zx}E_x
\end{vmatrix}
$$

where $\mu_{xx} = 2(1 + \mu)$. 

$$
\frac{E_x}{G_{xy}}
\begin{vmatrix}
\sigma_x \\
\sigma_y \\
\sigma_z \\
\tau_{xy} \\
\tau_{yz} \\
\tau_{zx}
\end{vmatrix}
= 
\begin{vmatrix}
\varepsilon_{xx}E_x \\
\varepsilon_{yy}E_x \\
\varepsilon_{zz}E_x \\
\gamma_{xy}E_x \\
\gamma_{yz}E_x \\
\gamma_{zx}E_x
\end{vmatrix}
$$

Thus $\alpha_1$ and $\alpha_2$ may be real or complex.
For plane strain,
\[ \varepsilon_z = 0 \]
\[ \tau_{yz} = \tau_{zx} = 0 \]
\[ \gamma_{yz} = \gamma_{zx} = 0 \] everywhere.

Since \( \varepsilon_z = 0 \)
\[ -\mu_{xx}\sigma_x - \mu_{xy}\sigma_y + \sigma_z = \varepsilon_z E_x \]
\[ = 0 \]

therefore
\[ \sigma_z = \mu_{xx}\sigma_x + \mu_{xy}\sigma_y \quad \text{and} \]
\[ E_x \varepsilon_x = \sigma_x - \mu_{xy}\sigma_y - \mu_{xx}\sigma_z \]
\[ = (1 - \mu_{xx})\sigma_x - \mu_{xy}(1 + \mu_{xx})\sigma_y \]
\[ E_x \varepsilon_y = -\mu_{xy}\sigma_x + \frac{E_x}{E_y} \cdot \sigma_y - \mu_{xy}\sigma_z \]
\[ = -\mu_{xy}(1 + \mu_{xx})\sigma_x + \frac{E_x}{E_y} - \mu_{xy}^2 \sigma_y \]

Similarly,
\[ E_x \varepsilon_y = -\mu_{xy}\sigma_x + \frac{E_x}{E_y} \cdot \sigma_y - \mu_{xy}\sigma_z \]

Differentiating,
\[ E_x \frac{\partial^2 \varepsilon_x}{\partial y^2} = (1 - \mu_{xx}) \frac{\partial^2 \sigma_x}{\partial y^2} - \mu_{xy}(1 + \mu_{xx}) \frac{\partial^2 \sigma_y}{\partial y^2} \]
\[ E_x \frac{\partial^2 \varepsilon_y}{\partial x^2} = -\mu_{xy}(1 + \mu_{xx}) \frac{\partial^2 \sigma_x}{\partial x^2} + \left( \frac{E_x}{E_y} - \mu_{xy}^2 \right) \frac{\partial^2 \sigma_y}{\partial x^2} \]
\[ E_x \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{E_x}{G \gamma_{xy}} \cdot \frac{\partial^2 \tau_{xy}}{\partial x \partial y} \]

For equilibrium, neglecting body forces,
\[ \frac{\partial^2 \tau_{xy}}{\partial x \partial y} = -\frac{1}{2} \frac{\partial^2 \sigma_x}{\partial x^2} - \frac{1}{2} \frac{\partial^2 \sigma_y}{\partial y^2} \]

So
\[ E_x \cdot \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = -\frac{E_x}{2 G \gamma_{xy}} \cdot \frac{\partial^2 \sigma_x}{\partial x^2} - \frac{E_x}{2 G \gamma_{xy}} \cdot \frac{\partial^2 \sigma_y}{\partial y^2} \]
For compatibility of strain

\[ \frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = 0 \]

If we multiply through by \( E_x \), substituting and regrouping terms, yields

\[ \frac{\partial^2}{\partial x^2} [K_2 \sigma_x + K_1 \sigma_y] + \frac{\partial^2}{\partial y^2} [K_3 \sigma_x + K_2 \sigma_y] = 0 \]

where \( K_1, K_2 \) and \( K_3 \) have the values already assigned.

Introducing an Airy stress function such that

\[ \sigma_x = \frac{\partial^2 \phi}{\partial y^2} \]
\[ \sigma_y = \frac{\partial^2 \phi}{\partial x^2} \]
\[ \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} \]

the partial differential equation reduces to

\[ k_1 \frac{\partial^4 \phi}{\partial x^4} + 2k_2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + k_3 \frac{\partial^4 \phi}{\partial y^4} = 0 \quad (1) \]

3. Solution of the equation

Referring back to the co-ordinate system of Fig. 8 and the boundary conditions discussed in the text, it will be noted that for any point along the upper half of the boundary, conditions are the exact opposite of those on the lower half boundary at the same \( x \)-value. In other words, the boundary conditions could be represented by the product of some function of \( x \) and an odd function of \( y \). This fact suggests that

\[ \phi(x, y) = f_1(x) \cdot f_2(y) \]

Trying a solution of this form we note that for zero friction on the end boundaries, \( \frac{\partial^2 \phi}{\partial x \partial y} \) must be zero for \( x = 0 \) and
for \( x = I \) independently of \( y \). This suggests

\[ f_1(x) = \cos \frac{n\pi x}{l} \]

giving

\[ \frac{\partial^4 \phi}{\partial x^4} = \frac{n^4 \pi^4}{l^4} \cos \frac{n\pi x}{l} \cdot f_2(y) \]

\[
\frac{\partial^4 \phi}{\partial x^2 \partial y^2} = -\frac{n^2 \pi^2}{l^2} \cos \frac{n\pi x}{l} \cdot f_2(y)
\]

\[ \frac{\partial^4 \phi}{\partial y^4} = \cos \frac{n\pi x}{l} \cdot f_2(y) \]

Substituting into Equation (1) and dividing by \( \cos \frac{n\pi x}{l} \) yields the ordinary differential equation

\[ K_1 \frac{n^4 \pi^4}{l^4} f_2(y) - 2K_2 \frac{n^2 \pi^2}{l^2} f_2(y) + K_3 f_2(y) = 0 \]

a linear equation with constant coefficients, with solutions of the type \( f_2(y) = A e^{\alpha y} \) leading to the auxiliary equation

\[ K_1 \frac{n^4 \pi^4}{l^4} - 2K_2 \frac{n^2 \pi^2}{l^2} m^2 + K_3 m^4 = 0 \]

Solving this equation for \( m^2 \) gives

\[ m^2 = \frac{n^2 \pi^2}{l^2} \cdot \frac{K_2 \pm \sqrt{K_2^2 - 4K_3K_1}}{2K_3} \]

\[ = \frac{n^2 \pi^2}{l^2} \cdot \frac{K_1 K_2 \pm \sqrt{K_2^2 - 4K_3K_1}}{2K_1 K_3} \]

\[ = \frac{n^2 \pi^2}{l^2} \frac{1}{\alpha_1} \text{ or } \frac{n^2 \pi^2}{l^2} \frac{1}{\alpha_2} \]

where \( \alpha_1 \) and \( \alpha_2 \) have the values already defined. Thus the four required values of \( m \) are

\[ \pm \frac{n\pi}{l} \frac{1}{\sqrt{\alpha_1}} \text{ and } \pm \frac{n\pi}{l} \frac{1}{\sqrt{\alpha_2}} \]

and \( f_2(y) = A \exp \frac{n\pi y}{l\sqrt{\alpha_1}} + B \exp \frac{-n\pi y}{l\sqrt{\alpha_1}} + C \exp \frac{n\pi y}{l\sqrt{\alpha_2}} + D \exp \frac{-n\pi y}{l\sqrt{\alpha_2}} \)
For vertical antisymmetry, it is evident that $A$ must equal $-B$ and $C$ must equal $-D$. Also, since every integer value of $n$ represents a different solution, we finally have

$$
\phi = \sum_{1}^{\infty} \cos \frac{n \pi x}{l} \left( A_{1n} \sinh \frac{n \pi y}{l \sqrt{\alpha}} + A_{2n} \sinh \frac{n \pi y}{l \sqrt{\alpha}} \right)
$$

(2)

where the subscript $n$ indicates a function of $n$.

If $K^2 = K_1 K_3$ then $a_1 = a_2 = a$ and in the above expression, $f_2(y)$ reduces to a single term. In this case, by the usual methods, we get

$$
f^2_2(y) = A \exp \frac{n \pi y}{l \sqrt{\alpha}} + B \exp \frac{-n \pi y}{l \sqrt{\alpha}} + C \exp \frac{n \pi y}{l \sqrt{\alpha}} + D \exp \frac{-n \pi y}{l \sqrt{\alpha}}
$$

the antisymmetry of conditions still applies, leading to $A = -B$ and, since the factor $y$ renders the last two terms cross-symmetric, $C = D$. Hence, we get, for $K^2 = K_1 K_3$

$$
\phi = \sum_{1}^{\infty} \cos \frac{n \pi x}{l} \left( A_{1n} \sinh \frac{n \pi y}{l \sqrt{\alpha}} + A_{2n} y \cosh \frac{n \pi y}{l \sqrt{\alpha}} \right)
$$

(3)

4. Evaluation of Arbitrary Constants

Taking first the case where $a_1 \neq a_2$, differentiating $\phi$ yields

$$
t_{xy} = \frac{\partial^2 \phi}{\partial x \partial y} = - \sum \frac{n^2 \pi^2}{l^2} \sin \frac{n \pi x}{l} \sum_{i=1}^{2} \alpha_i \frac{-1}{2} A_i \sinh \frac{n \pi y}{l \sqrt{\alpha}_i}
$$

(4)

$$
\sigma_x = \frac{\partial^2 \phi}{\partial y^2} = \sum \frac{n^2 \pi^2}{l^2} \cos \frac{n \pi x}{l} \sum_{i=1}^{2} \alpha_i \frac{-1}{2} A_i \sinh \frac{n \pi y}{l \sqrt{\alpha}_i}
$$

(5)

$$
\sigma_y = \frac{\partial^2 \phi}{\partial x^2} = - \sum \frac{n^2 \pi^2}{l^2} \cos \frac{n \pi x}{l} \sum_{i=1}^{2} \alpha_i \sinh \frac{n \pi y}{l \sqrt{\alpha}_i}
$$

(6)
It has already been shown, during development of the differential equation, that

\[ \gamma_{xy} \frac{E_x}{G_{xy}} = \frac{E_x}{G_{xy}} \tau_{xy} \]

\[ \epsilon_{x}E_{x} = (1 - \mu_{xx}^2)\sigma_{x} - \mu_{xy}(1 + \mu_{xx})\sigma_{y} \]

\[ \epsilon_{y}E_{x} = -\mu_{xy}(1 + \mu_{xx})\sigma_{x} + \left(\frac{E_{x}}{E_{y}} - \mu_{xy}^2\right)\sigma_{y} \]

But

\[ \epsilon_{x}E_{x} = E_{x} \frac{\partial u}{\partial x} \quad \text{and} \quad \epsilon_{y}E_{x} = E_{x} \frac{\partial v}{\partial y} \]

so, substituting \( \sigma_{x}, \sigma_{y} \) and \( \tau_{xy} \) from equations (4), (5) and (6) into the above equations and integrating the resulting derivatives of \( u \) and \( v \), we get

\[ E_{x}u = (1 - \mu_{xx}^2) \sum_{i=1}^{\infty} \frac{n\pi}{L} \sin \frac{n\pi x}{L} \frac{2}{\alpha_i} A_{in} \sinh y_i \]

\[ + \mu_{xy}(1 + \mu_{xx}) \sum_{i=1}^{\infty} \frac{n\pi}{L} \sin \frac{n\pi x}{L} \frac{2}{\alpha_i} A_{in} \sinh y_i \]

\[ + F_1(y) \]  

(7)

where

\[ y_1 = \frac{n\pi y_1}{L\alpha_1} \quad \text{and} \quad y_2 = \frac{n\pi y_2}{L\alpha_2} \]

Similarly

\[ E_{x}v = -\mu_{xy} \sum_{i=1}^{\infty} \frac{n\pi}{L} \cos \frac{n\pi x}{L} \frac{2}{\alpha_i} A_{in} \cosh y_i \]

\[ - \left(\frac{E_{x}}{E_{y}} - \mu_{xy}^2\right) \sum_{i=1}^{\infty} \frac{n\pi}{L} \cos \frac{n\pi x}{L} \frac{2}{\alpha_i} A_{in} \cosh y_i \]

\[ + F_2(x) \]

(8)

But

\[ E_{x} \gamma_{xy} = E_{x} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \]

so, differentiating (7) & (8)

yields

\[ E_{x} \gamma_{xy} = \sum_{i=1}^{\infty} \frac{n^2\pi^2}{L^2} \sin \frac{n\pi x}{L} \sum_{i=1}^{\infty} \frac{2}{\alpha_i} \left\{ (1 - \mu_{xx}^2)\alpha_{i}^{3} + \mu_{xy}(1 + \mu_{xx})\alpha_{i}^{2} \right\} 

+ \mu_{xy}(1 + \mu_{xx})\alpha_{i}^{-1} + \left(\frac{E_{x}}{E_{y}} - \mu_{xy}^2\right)\alpha_{i}^{2} A_{in} \cosh y_i \]

\[ + F_1'(y) + F_2'(x) \]

(9)
Equation (4) already gives an expression for $\tau_{xy}$ and Equation (9) must be made to agree, through the relationship between $\tau_{xy}$ and $\gamma_{xy}$. By factoring $\alpha_i^{-\frac{3}{2}}$ out of Equation (9), and writing the remaining elastic coefficients in terms of $K_1$, $K_2$, and $K_3$, it will be found that the required identity holds if $-F'_2(x) = F'_1(y)$. Hence these derivatives must both be a constant, to render them equal for all $x$ and all $y$. The remaining boundary conditions require

\[ u = \omega y \quad \text{on all boundaries} \]

and

\[ v = 0 \quad \text{for } y = \pm h \]

The equation for $E_xu$, Equation (7), contains $\sin \frac{n\pi x}{l}$ in all terms except $F_1(y)$, and, for the end boundaries, where $x = 0$ or $x = l$, it therefore reduces to

\[ E_xu \rvert_{x=0} = F_1(y) = E_x\omega y \]

Therefore

\[ F'_1(y) = E_x\omega \]

and so

\[ F'_2(x) = -E_x\omega \]

On the top and bottom boundaries, where $u = \omega y$, since $u$ and $\omega y$ are both odd functions of $y$, constants which fit the conditions for $y = h$ will also fit for $y = -h$. Putting $y = h$

\[
\sum_{i=1}^{2} \alpha_i^{-1}(1 - \mu_{xx}^2) + \mu_{xy}(1 + \mu_{xx})x A_{i, n} \sinh h_i + E_x\omega h
\]

where

\[ h_i = \frac{n\pi h}{l\alpha_i} \]

hence, on the top boundary $u = \omega h$, as required, if

\[
\sum_{i=1}^{2} \left\{ \alpha_i^{-1}(1 - \mu_{xx}^2) + \mu_{xy}(1 + \mu_{xx}) \right\} A_{i, n} \sinh h_i = 0
\]

which can be re-written

\[
\left\{ \frac{E_x}{2G_{xy}} - \sqrt{K_2^2 - K_1 K_3} \right\} A_{1, n} \sinh h_1
\]

\[
+ \left\{ \frac{E_x}{2G_{xy}} + \sqrt{K_2^2 - K_1 K_3} \right\} A_{2, n} \sinh h_2 = 0 \quad (10)
\]
To obtain \( v = 0 \) for \( y = \pm h \), re-write Equation (8) as

\[
E_x v \bigg|_{y=h} = -\mu_{xy}(1 + \mu_{xx}) \sum \frac{n \pi}{l} \cos \frac{n \pi x}{l} \sum_{i=1}^{2} \frac{1}{2} A_i \sinh h_i \\
- \left( \frac{E_x}{E_y} - \mu_{xy}^2 \right) \sum \frac{n \pi}{l} \cos \frac{n \pi x}{l} \sum_{i=1}^{2} \frac{1}{2} A_i \cosh h_i \\
+ F_2(x) \\
= 0
\]

But

\[ F'_2(x) = -E_x \omega \]

so

\[ F_2(x) = -E_x \omega x + \kappa \]

giving

\[
\sum_{1}^{2} \sum_{1}^{2} \left\{ \mu_{xy}(1 + \mu_{xx}) \alpha_i^{-3} + \left( \frac{E_x}{E_y} - \mu_{xy}^2 \right) \alpha_i^{-3} \right\} A_i \sinh h_i \sum \frac{n \pi}{l} \cos \frac{n \pi x}{l} \\
- \kappa + E_x \omega x \\
= 0
\]

Thus the double summation must converge to \( \kappa = E_x \omega x \). If \( P \) is a constant, the series

\[
\sum_{1}^{2} \sum_{1}^{2} P n^{-2} \pi^{-2} \cos \frac{n \pi x}{l} \\
_{n=1,3,5}
\]

converges to \( P \left( 1 - \frac{2x}{l} \right) \)

The required boundary value of \( v \) is therefore obtained if the even values of \( n \) yield zero coefficients \( (A_{i\,n} = 0 \text{ for } n \text{ even}) \) and

\[
\sum_{i=1}^{2} \left\{ \mu_{xy}(1 + \mu_{xx}) \alpha_i^{-3} + \left( \frac{E_x}{E_y} - \mu_{xy}^2 \right) \alpha_i^{-3} \right\} A_{i\,n} \frac{n \pi}{l} \sum \frac{1}{2} \sinh h_i \\
= P \text{ const} \\
\]

for any odd value of \( n \).

If it is established that \( P \) is a constant, the equation for \( v \) becomes

\[
E_x v \bigg|_{y=h} = \sum_{1}^{2} P n^{-2} \pi^{-2} \cos \frac{n \pi x}{l} - \kappa + E_x \omega x \\
_{1,3,5}
= \frac{P}{8} \left( 1 - \frac{2x}{l} \right) - \kappa + E_x \omega x \\
= 0
\]
This results in
\[ \frac{P_x}{4L} = E_x \omega x \]
so \( P = 4LE_x \omega \)
and \( k = \frac{\omega L}{2E_x} \).

In equation (11), \( \mu_{xy} (1 + \mu_{xx}) = \frac{E_x}{2G} \frac{x}{xy} - K_2 \)
and
\[ \frac{E_x}{E_y} - \mu_{xy}^2 = K_1 \]
so, re-writing Equation (11), we now get, for odd values of \( n \)
\[
\sum_{i=1}^{2n} \left\{ \left( \frac{E_x}{2G} \frac{x}{xy} - K_2 \right) \alpha_i^{-\frac{1}{2}} + K_1 \alpha_i^{\frac{1}{2}} \right\} A_i \cosh h_i = 4L^2 \omega \]
or
\[
\sum_{i=1}^{2n} \alpha_i^{\frac{1}{2}} \left( \frac{E_x}{2G} \frac{x}{xy} - K_2 + K_1 \alpha_i \right) A_i \cosh h_i = \frac{4L^2 E_x \omega}{n^3 \pi^3} \]
and, simplifying terms,
\[
\alpha_1^{-\frac{1}{2}} \left\{ \frac{E_x}{2G} \frac{x}{xy} + \sqrt{K_2^2 - K_1 K_3} \right\} A_1 \cosh h_1
\]
\[ + \alpha_2^{-\frac{1}{2}} \left\{ \frac{E_x}{2G} \frac{x}{xy} - \sqrt{K_2^2 - K_1 K_3} \right\} A_2 \cosh h_2 = \frac{4L^2 E_x \omega}{n^3 \pi^3} \]  \hspace{1cm} (12)

Combining Equations (10) and (12) and writing
\[ B_1 = \frac{E_x}{2G} \frac{x}{xy} + \sqrt{K_2^2 - K_1 K_3} \]
\[ B_2 = \frac{E_x}{2G} \frac{x}{xy} - \sqrt{K_2^2 - K_1 K_3} \]

We get
\[ A_{1n} = \frac{4L^2 E_x \omega}{n^3 \pi^3} \cdot \frac{\sqrt{\alpha_1 \alpha_2} B_1 \sinh h_1}{2 \cosh h_1 \sinh h_2 - \sqrt{\alpha_1} B_2 \sinh h_1 \cosh h_2} \]
\[ A_{2n} = \frac{4L^2 E_x \omega}{n^3 \pi^3} \cdot \frac{\sqrt{\alpha_1 \alpha_2} B_2 \sinh h_2}{2 \cosh h_1 \sinh h_2 - \sqrt{\alpha_1} B_2 \sinh h_1 \cosh h_2} \]
Checking that $P$ is a constant, we now have, for any value of $n$

$$P = \frac{\pi^2}{l} \sum_{i=1}^{2} \left\{ \left( \frac{E}{2G} \right) \alpha_i - K \alpha_i \right\} A_i \, \cosh h_i$$

$$= \frac{\pi^2}{l} \left\{ \alpha_1 \left( \frac{E}{2G} \right) - \sqrt{K^2 - K_1} \right\} A_1 \, \cosh h_1$$

$$+ \alpha_2 \left( \frac{E}{2G} \right) - \sqrt{K^2 - K_1} \right\} A_2 \, \cosh h_2 \right\}$$

$$= \frac{\pi^2}{l} \left\{ \alpha_1 B \left[ \cosh h_1 + \alpha_2 B \cosh h_2 \right] \right\}$$

$$= \frac{4\pi l^2 \omega x}{\pi^2} \left\{ \alpha_1 B \cosh h_1 + \alpha_2 B \cosh h_2 \right\}$$

$$= 4\pi l^2 \omega \quad \text{as required.}$$

Summarizing the results obtained so far, we have, for $K_2 > K_1$

$$\phi = \sum_{n=1,3,5}^{\infty} \cos \frac{n\pi x}{l} \{ A_1^1 \, \sinh y_1 + A_2^1 \, \sinh y_2 \}$$

$$\frac{1}{E \omega} \sigma_x = \sum_{n=1,3,5}^{\infty} \frac{n^2 \pi^2}{l^2} \cos \frac{n\pi x}{l} \{ \alpha_1^{-1} A_1^1 \, \sinh y_1 + \alpha_2^{-1} A_2^1 \, \sinh y_2 \}$$

$$\frac{1}{E \omega} \sigma_y = -\sum_{n=1,3,5}^{\infty} \frac{n^2 \pi^2}{l^2} \cos \frac{n\pi x}{l} \{ A_1^1 \, \sinh y_1 + A_2^1 \, \sinh y_2 \}$$

$$\frac{1}{E \omega} \tau_{xy} = \sum_{n=1,3,5}^{\infty} \frac{n^2 \pi^2}{l^2} \sin \frac{n\pi x}{l} \{ \alpha_1^{-1} A_1^1 \, \cosh y_1 + \alpha_2^{-1} A_2^1 \, \cosh y_2 \}$$

$$\frac{1}{\omega} \mu = (1 - \mu^2 \frac{x}{x}) \sum_{n=1,3,5}^{\infty} \frac{n\pi}{l} \sin \frac{n\pi x}{l} \sum_{i=1}^{2} \alpha_i^{-1} A_i^1 \sinh y_i$$

$$+ \mu \frac{x}{x} (1 + \mu^2 \frac{x}{x}) \sum_{n=1,3,5}^{\infty} \frac{n\pi}{l} \sin \frac{n\pi x}{l} \sum_{i=1}^{2} A_i^1 \sinh y_i$$

$$+ y$$
\[
\frac{1}{\omega^2} = -\mu xy (1 + \mu \mu) \sum_{1,3,5} \frac{n \pi}{l} \cos \frac{n \pi x}{l} 2 \alpha_{l}^{-1} A_{l} \cosh y_l
\]

\[-(\frac{E}{E_y} - \mu \mu) \sum_{1,3,5} \frac{n \pi}{l} \cos \frac{n \pi x}{l} 2 \alpha_{l}^{-1} A_{l} \cosh y_l
\]

\[+ \frac{1}{2} l - x\]

In the above equations

\[A'_{1n} = \frac{1}{E \omega} A_{1n}\]

\[4 l^2 \frac{\sqrt{\alpha} \alpha B \sinh h_1}{n^3 \pi^3 \sqrt{\alpha} B^2 \cosh h_1 \sinh h_2 - \sqrt{\alpha} B^2 \sinh h_1 \cosh h_2}\]

\[A'_{2n} = \frac{1}{E \omega} A_{2n}\]

\[-4 l^2 \frac{\sqrt{\alpha} \alpha B \sinh h_1}{n^3 \pi^3 \sqrt{\alpha} B^2 \cosh h_1 \sinh h_2 - \sqrt{\alpha} B^2 \sinh h_1 \cosh h_2}\]

Consider now the case where

\[K^2 = \frac{K^2}{1,3,5}\]

This case includes all isotropic materials and some others. As pointed out previously, we find that

\[a_1 = a_2 = a\]

Using the solution already discussed, and writing

\[y_n = \frac{n \pi y}{l \sqrt{\alpha}} \text{ and } h_n = \frac{n \pi h}{l \sqrt{\alpha}}\]

the boundary constants are obtained by the same methods as those in the case just completed. The results are

\[\frac{1}{E \omega} \sigma_x \]

\[= \sum_{1,3,5} \frac{n^2 \pi^2}{l^2} \cos \frac{n \pi x}{l} \left\{ (A_{1n} + 2A_{2n}) \sinh y_n + A_{2n} y_n \cosh y_n \right\} \frac{1}{\alpha}\]
\[
\frac{1}{E_x \omega} \sinh y \\
= - \sum_{1,3,5} \frac{n^2 \pi^2}{l^2} \cos \frac{n \pi x}{l} \left\{ A_{1n} \sinh y_n + A_{2n} y_n \cosh y_n \right\}
\]

\[
\frac{1}{E_x \omega} \sin y \\
= \sum_{1,3,5} \frac{n^2 \pi^2}{l^2} \sin \frac{n \pi x}{l} \left\{ (A_{1n} + A_{2n}) \cosh y_n + A_{2n} y_n \sinh y_n \right\} \frac{1}{\sqrt{\alpha}}
\]

\[
\frac{1}{\omega \nu} \\
= (1 - \mu_x^2) \sum_{1,3,5} \frac{n \pi}{l} \sin \frac{n \pi x}{l} \left\{ (A_{1n} + 2A_{2n}) \sinh y_n + A_{2n} y_n \cosh y_n \right\} \frac{1}{\sqrt{\alpha}}
\]

\[
\mu_{xy} (1 + \mu_x^2) \sum_{1,3,5} \frac{n \pi}{l} \sin \frac{n \pi x}{l} \left\{ A_{1n} \sinh y_n + A_{2n} y_n \cosh y_n \right\} 
\]

\[
\frac{1}{\omega \nu} \\
= - \mu_{xy} (1 + \mu_x^2) \sum_{1,3,5} \frac{n \pi}{l} \cos \frac{n \pi x}{l} \left\{ (A_{1n} + A_{2n}) \cosh y_n + A_{2n} y_n \sinh y_n \right\} \frac{1}{\sqrt{\alpha}}
\]

\[
- \left( \frac{E_x}{E_y} - \mu_{xy} \right) \sum_{1,3,5} \frac{n \pi}{l} \cos \frac{n \pi x}{l} \left\{ (A_{1n} + A_{2n}) \cosh y_n + A_{2n} y_n \sinh y_n \right\} \frac{1}{\sqrt{\alpha}}
\]

\[
- x + \frac{1}{2} l
\]

In which, if \( C = \frac{3E_x}{2G_{xy}} - 4 \mu_{xy} (1 + \mu_x^2) \frac{1}{\sqrt{\alpha}} \)

\[
A_{1n} = \frac{4l^2}{n^3 \pi^3} \frac{4 G_{xy} K_2}{E_x} \sinh h_n + h_n \cosh h_n
\]

\[
C \sinh h_n \cosh h_n + \frac{E_x h_n}{2 G_{xy} \sqrt{\alpha}}
\]

\[
A_{2n} = - \frac{4l^2}{n^3 \pi^3} \frac{\sinh h_n}{C \sinh h_n \cosh h_n + \frac{E_x h_n}{2 G_{xy} \sqrt{\alpha}}}
\]
If $K^2 < K_{13}$

The same solution holds as for $K^2 > K_{13}$ but the various expressions contain many complex quantities. The actual stresses and displacements are, of course, real and hence the imaginary portions all cancel out. It is therefore possible to write the equations for this case in terms of real variables, which use less computer storage than complex.

$$\phi = \sum_{1,3,5} \cos \frac{n \pi \alpha_1}{2} \left\{ \frac{Z_1 \cosh y_1 \sin y_1 - Z_2 \sinh y_1 \cos y_1}{\eta \sinh h_1 \cosh h_1 - \xi \sin h_1 \cos h_1} \right\}$$

in which expression, the undefined variables are obtained as follows

$$\alpha_1' = \frac{2K_1}{\sqrt{K_1 K_3} + K_2}, \quad \alpha_2' = \frac{2K_3}{\sqrt{K_1 K_3} - K_2}$$

$$y_1' = \frac{n \pi y_1}{2 \sqrt{\alpha_1}}, \quad y_2' = \frac{n \pi y_2}{2 \sqrt{\alpha_2}}, \quad h_1' = \frac{n \pi h_1}{2 \sqrt{\alpha_1}}, \quad h_2' = \frac{n \pi h_2}{2 \sqrt{\alpha_2}}$$

$$Z_1n = \frac{4l^3}{n^3 \pi^3} \left\{ \frac{E_x}{2G_{xy}} \sinh h_1 \cos h_2 - \frac{2K_3}{\sqrt{\alpha_1 \alpha_2}} \cosh h_1 \sin h_2 \right\}$$

$$Z_2n = \frac{4l^3}{n^3 \pi^3} \left\{ \frac{2K_3}{\sqrt{\alpha_1 \alpha_2}} \sinh h_1 \cos h_2 + \frac{E_x}{2G_{xy}} \cosh h_1 \sin h_2 \right\}$$

$$\xi = \frac{1}{\sqrt{\alpha_1}} \left\{ \left( \frac{E_x}{2G_{xy}} \right)^2 - \frac{4K^2}{\alpha_1 \alpha_2} \right\} + \frac{2}{\sqrt{\alpha_2}} \frac{E_x}{G} \frac{K_3}{\sqrt{\alpha_1}}$$

$$\eta = \frac{1}{\sqrt{\alpha_2}} \left\{ \left( \frac{E_x}{2G_{xy}} \right)^2 - \frac{4K^2}{\alpha_1 \alpha_2} \right\} - \frac{2}{\sqrt{\alpha_1}} \frac{E_x}{G} \frac{K_3}{\sqrt{\alpha_1}}$$

now, putting $D = \eta \sinh h_1 \cosh h_1 - \xi \sin h_2 \cos h_2$

$$\chi_1n = \frac{Z_1n}{\sqrt{\alpha_1}}, \quad \chi_2n = \frac{Z_2n}{\sqrt{\alpha_2}}$$

we obtain the following relationships
\[
\begin{align*}
\frac{1}{E_x} \sigma \propto &= \sum_{1,3,5} \frac{n^2 \pi^2}{l^2} \cos \frac{n \pi x}{l} \left\{ \left( \frac{X_{1n}}{\sqrt{\alpha_1}} - \frac{X_{2n}}{\sqrt{\alpha_2}} \right) \cosh y_1 \sin y_2 \\
&+ \left( \frac{X_{1n}}{\sqrt{\alpha_1}} + \frac{X_{2n}}{\sqrt{\alpha_2}} \right) \sinh y_1 \cos y_2 \right\} \frac{1}{D} \\
\frac{1}{E_x} \sigma \propto &= -\sum_{1,3,5} \frac{n^2 \pi^2}{l^2} \cos \frac{n \pi x}{l} \left\{ Z_{1n} \cosh y_1 \sin y_2 \\
&- Z_{2n} \sinh y_1 \cos y_2 \right\} \frac{1}{D} \\
\frac{1}{E_x} \tau_{xy} &= \sum_{1,3,5} \frac{n^2 \pi^2}{l^2} \sin \frac{n \pi x}{l} \left\{ \chi_{1n} \sinh y_1 \sin y_2 \\
&+ \chi_{2n} \cosh y_1 \cos y_2 \right\} \frac{1}{D} \\
\frac{1}{\omega} \mu &= \sum_{1,3,5} \frac{n \pi}{l} \frac{1}{D} \sin \frac{n \pi x}{l} \left\{ \left[ \left( \frac{X_{1n}}{\sqrt{\alpha_1}} - \frac{X_{2n}}{\sqrt{\alpha_2}} \right) \cosh y_1 \sin y_2 \\
&+ \left( \frac{X_{1n}}{\sqrt{\alpha_1}} + \frac{X_{2n}}{\sqrt{\alpha_2}} \right) \sinh y_1 \cos y_2 \right] \left[ 1 - \mu^2_{xx} \right] \\
&+ \left[ Z_{1n} \cosh y_1 \sin y_2 \\
&- Z_{2n} \sinh y_1 \cos y_2 \right] \mu_{xy} \left[ 1 + \mu^2_{xx} \right] \right\} + y \\
\frac{1}{\omega} \nu &= -\sum_{1,3,5} \frac{n \pi}{l} \frac{1}{D} \cos \frac{n \pi x}{l} \left\{ \left[ \chi_{1n} \sinh y_1 \sin y_2 \\
&+ \chi_{2n} \cosh y_1 \cos y_2 \right] \mu_{xy} \left[ 1 + \mu^2_{xx} \right] \\
&+ \left[ \left( Z_{1n} \sqrt{\alpha_1} - Z_{2n} \sqrt{\alpha_1} \right) \sinh y_1 \sin y_2 \\
&- \left( Z_{1n} \sqrt{\alpha_1} + Z_{2n} \sqrt{\alpha_2} \right) \cosh y_1 \cos y_2 \right] \sqrt{\frac{\alpha_1 \alpha_2}{\alpha_1 + \alpha_2}} \left[ -\frac{E_x}{E_y} - \mu^2_{xy} \right] \right\} + \frac{\nu}{2} - x
\end{align*}
\]
APPENDIX D

LIMITING VALUES OF ELASTIC CONSTANTS

Considering a material characterized by the type of anisotropy already described, the number and relationship of the various elastic constants has been mentioned earlier. Investigating the behaviour of such material, with numerical values for the constants, would not be valid if the constants exceed their limiting values. The usual criterion for limiting values is that the material be stable or, to ensure this, that the strain energy should always be positive (Love\textsuperscript{16}).

Taking the elastic constants in the form:

\[
\begin{pmatrix}
\frac{1}{E_x} & -\frac{\mu_{xy}}{E_x} & -\frac{\mu_{xx}}{E_x} \\
-\frac{\mu_{xy}}{E_x} & \frac{1}{E_y} & -\frac{\mu_{xy}}{E_y} \\
-\frac{\mu_{xx}}{E_x} & -\frac{\mu_{xy}}{E_y} & \frac{1}{E_x}
\end{pmatrix} = C
\]

\[\frac{1}{G_{xy}}, \quad \frac{1}{G_{xy}}, \quad \frac{2(1 + \mu_{xx})}{E_x}\]

The strain energy caused by a set of stresses $\sigma$ is $\frac{1}{2} \sigma^T C \sigma$ per unit volume. If this quadratic form is positive definite, the strain energy will then be positive as required. The problem has been stated by Lempriere\textsuperscript{17}, who then deduced the bounding values of $\mu$ by considering various possible stress fields and the conditions necessary to ensure that the

\textsuperscript{17} Lempriere, B.M., "Poisson's Ratio in Orthotropic Materials". Journal, American Institute of Aeronautics and Astronautics. Vol. 6, No. 11, November 1968.
resulting strain energy is positive. A more rigorous form of analysis, which considers all the elastic parameters, is given below. The necessary and sufficient conditions that the quadratic form should be positive definite are, that all the principal minors of $C$ should be positive. So, putting

\[
\begin{vmatrix}
\frac{1}{E_x} & -\frac{\mu_{xy}}{E_x} \\
\frac{1}{E_x} & \frac{1}{E_y} \\
-\frac{\mu_{xy}}{E_x} & \frac{1}{E_y}
\end{vmatrix} = D_1
\]

\[
\begin{vmatrix}
1 & -\frac{\mu_{xy}}{E_x} & -\frac{\mu_{xx}}{E_x} \\
\frac{1}{E_x} & \frac{1}{E_x} & -\frac{\mu_{xy}}{E_x} \\
-\frac{\mu_{xy}}{E_x} & \frac{1}{E_x} & \frac{1}{E_x}
\end{vmatrix} = D_2
\]

\[
\begin{vmatrix}
D_3 & 0 & 0 \\
0 & \frac{1}{G_{xy}} & 0 \\
0 & 0 & \frac{1}{G_{xy}}
\end{vmatrix} = D_4
\]

All the $D$ values should be positive.

\[
\begin{vmatrix}
D_3 & 0 & 0 \\
0 & \frac{1}{G_{xy}} & 0 \\
0 & 0 & \frac{1}{G_{xy}}
\end{vmatrix} = D_5
\]

\[
\begin{vmatrix}
D_3 & 0 & 0 & 0 \\
0 & \frac{1}{G_{xy}} & 0 & 0 \\
0 & 0 & \frac{1}{G_{xy}} & 0 \\
0 & 0 & 0 & \frac{2 + 2\mu_{xx}}{E_x}
\end{vmatrix} = D_6
\]
Since $D = D \frac{1}{G_{xy}}$ it is evident that if $D$ is positive, $D$ will be positive if $G_{xy} > 0$, and therefore $D$, which is equal to $D \frac{1}{G_{xy}}$ will be positive. In whatever way the rows and columns of $C$ are interchanged, $G_{xy}$ always appears by itself, on the main diagonal, and thus its only requirement is that it be positive. The requirement that $D$ be positive is satisfied if $E_x$ is positive and hence, multiplying the elements of the other minors by $E_x$, $E_x^2$ or $E_x^3$ will not change their sign so, the criteria for positive strain energy reduce to

$$D_1 = \frac{1}{E_x} \text{ therefore } E_x \text{ must be positive}$$

$$D_2 \cdot E_x^2 = E_x/E_y - \mu_{xy}^2 \text{ must be positive}$$

$$D_3 \cdot E_x^3 = [E_x/E_y (1 - \mu_{xx}) - 2 \mu_{xy}^2] [1 + \mu_{xx}] \text{ must be positive}$$

$$G_{xy} \text{ must be positive}$$

$$2(1 + \mu_{xx}) \text{ must be positive}$$

Since $-\mu_{xy}^2$ is evidently always negative, $D_2$ can be positive only if $E_x/E_y$ is positive and greater than $\mu_{xy}^2$. Since $E_x$ is positive, this means that $E_y$ must be positive.

The last condition requires that $1 + \mu_{xx}$ be positive and this factor can then be omitted from $D_3$, since, if one factor is always positive, the other must be too.

In $D_3$ the term $E_x/E_y (1 - \mu_{xx}) - 2\mu_{xy}^2$ remains to be discussed. It is evident that $-2\mu_{xy}^2$ is negative and $E_x/E_y$ is always positive so, to make the term positive, $1 - \mu_{xx}$ must be positive. Since $1 - \mu_{xx}$ and $1 + \mu_{xx}$ must both be positive, $-1 < \mu_{xx} < 1$ and the conditions we are
finally left with are:

a) \( E_x, E_y \) and \( G_{xy} \) each greater than zero

b) Absolute value of \( \mu_{xx} \) less than one.

c) \( E_x/E_y \left( 1 - \mu_{xx} \right) - 2 \mu_{xy}^2 > 0 \)

By equating expression c) to zero, we can obtain the bounding surface between \( \mu_{xx} = 1 \) and \( \mu_{xx} = -1 \) in \( E_x/E_y, \mu_{xx}, \mu_{xy} \) space. The surface includes all the parameters except \( G_{xy} \), which is completely independent of the others. The shape of the energy boundaries is shown in Fig. 40a. The shape of the boundary of expression c) is such that \( \mu_{xx} \) could never exceed unity and this limit is part of the continuous bounding surface. However, there is a discontinuity at \( \mu_{xx} = -1 \), where the solid containing possible values of the
constants, is cut off by a vertical wall. In addition to $G_{xy}$, it must be remembered that $E_x$ and $E_y$ are also independent (so long as all three are positive) and really, it is only the Poisson's Ratios which are bounded by the ratio of $E_x/E_y$. The actual shape of the bounding surface is paraboloid, any vertical section with constant $\mu_{xx}$ is a parabola and any horizontal section with constant $E_x/E_y$ is a parabola. It is interesting to note that the surface could be generated by straight lines passing through the point $E_x/E_y = 0$, $\mu_{xx} = 1$, $\mu_{xy} = 0$.

Some further points of interest emerge. The independence of $G_{xy}$ is emphasized by the line AB (Fig. 40b). This line represents all materials for which $E_x = E_y$ and $\mu_{xx} = \mu_{xy}$. Hence isotropic materials must lie on this line. It will be seen that, as expected, it is cut off at $\mu = 0.5$ and $\mu = -1$. However, material represented by a point on this line is isotropic if and only if $G_{xy} = \frac{E}{2(1 + \mu)}$. It is thus possible to have a material behaving isotropically under equal normal stresses and anisotropically under shear stress. The strain energy of a material subjected to equal all round normal stress $\sigma$ is given by

$$\frac{1}{2E_x} \left\{ 2 + E_x/E_y - 4 \mu_{xy} - 2 \mu_{xx} \right\} \sigma^2$$

If this is zero, the material has zero bulk modulus and $E_x/E_y + 2 = 4 \mu_{xy} + 2 \mu_{xx}$. In Figure 40, such material would be represented by a plane surface passing through the
This surface is tangential to the energy boundary already obtained, touching it along a line from the point \( E_x/E_y = 0, \mu_{xy} = 0, \mu_{xx} = 1 \) to the point \( E_x/E_y = 4, \mu_{xy} = 2, \mu_{xx} = -1 \). All incompressible materials must lie on this line.

If a uniaxial stress \( \sigma_y \) is applied to the material, the volumetric strain is

\[
\frac{1}{2E_x} \left\{ \frac{E_x}{E_y} - 2 \mu_{xy} \right\} \sigma_y
\]

Thus, there is no volumetric strain if \( E_x/E_y = 2 \mu_{xy} \).

This condition is represented by a plane, sloping upward in the direction of \( \mu_{xy} \) and containing the axis of \( \mu_{xx} \). If a material has \( \mu_{xy} \) large enough to put it outside this plane, volumetric strain is negative and the material swells under uniaxial compression in the \( y \) direction. Fig. 40b shows the portion of the energy boundary which lies outside this plane and, hence shows the possible materials which actually swell when subjected to compressive \( \sigma_y \). It follows, of course, that with a large \( \sigma_y \) and small \( \sigma_x \) and \( \sigma_z \), a material may swell when compressed. This point may be of interest in considering the behaviour of precompressed clays under triaxial test. Parallelism and subparallelism of clay particles would seem to indicate anisotropy of the kind under discussion, and it can be seen that it may be possible for such soil to swell without violating the usual energy criterion.

The same considerations may apply to sands, but caution is needed, for anisotropy in natural sands may be due to a certain amount of layering (greater or smaller) but, especially in uniform sands, it might also be due to a
statistical orientation of interparticle contacts. Such anisotropy as the second, depending on arrangement of grains would be likely to take a more complicated form than that discussed herein.
Installing Disturbed Sand Samples. The specially manufactured membrane must first be prepared. The membranes are delivered with a short tubular neck attached to the half inch diameter hole in the upper face. This tube must be cut off without any remainder (which might fold in and prevent proper seating) and without unduly enlarging the hole. Four small holes in the top and bottom must then be made, matching the screw holes in the steel top and bottom spreader plates. The bottom plate is then inserted, convex face downward, in the membrane and the membrane is stretched across it so that the bottom corners of the membrane fit properly to the plate corners. The corners must fit, as otherwise the plate corners are sharp enough to hold the membrane out of place and it is then liable to puncture. Also, folds of badly fitting membrane will interfere with seating or with the sample. The fit is then secured by screws through the plate and membrane, into the mounting block. Fig. 21 shows this stage. An aluminum block is then required, about .005 in. less than sample size, and having a hole in the top into which a finger can be inserted for later removal of the block. The membrane is then stretched to allow the block to be pushed through the top hole and into the sample space, and the corners of the membrane are adjusted to fit to the corners of the block. Pieces of rubber or neoprene moulding, 1/16 in. diameter and 3/16 to 7/32 in. long are then placed in the corners of the membrane, between the membrane and the aluminum block. The purpose of the block is to hold these pieces of moulding in place while they are inserted in place. The mouldings are adjusted to a vertical position, in the corner, equidistant from top and bottom. Once located, the stretch of the membrane retains these mouldings in position, against the block. The exterior sides and ends of the membrane are then lubricated with silicone oil. The membrane is now ready.
The next step is to ensure that the vertical and horizontal travel stops are in place in the shear machine, holding the moving parts in mid-position, and the upper box sides and upper extension of the projecting end plate (see Fig. 41) are removed. The mounting block and membrane, with the aluminum block inside, are then lowered carefully into the machine, ensuring that the rubber mouldings enter their housings in the sides of the end plates. Once they are correctly entered, the lower halves of the box sides will retain them in place and the mounting block must be secured in place, using the taper pins in its base. Using a piece of shim material, it is then fairly easy to press the upper halves of the rubber mouldings into their grooves, doing both mouldings on one side at the same time, and holding them in place while the box upper sides are placed in position. They must then be pressed inward until they are secured, by screwing their locating dowels down into the lower sides, if this is
not done, the rubber mouldings spring out of their grooves. Once this has been done, the membrane corners will be retained in place.

It is now necessary to open the mouth of the membrane, ready to form the sample. Four hooks of a special design are inserted through the centre top opening of the membrane, right into its upper corners, and pulled outwards until the holes in their handles can be retained on the screws, specially provided, at the four corners of the top of the machine. The aluminum block may now be removed from inside the membrane, and the stage illustrated in Fig. 22 has been reached. The sample space is then filled with de-aired water, at ambient temperature and any air bubbles removed from corners, etc. A special hopper, made of transparent plastic is then installed and filled nearly to the top with water (Fig. 42). This hopper has plain, vertical interior faces and its interior horizontal cross-section is the same as the sample, so that the sand can fall straight down into place. Around the base of the hopper there is an

Fig. 42. Loading Hopper in Place.
O-ring seal, so that when it is secured in place, the water does not leak away. A previously prepared supply of sand is then required.

To prepare a sand supply, a slight excess of dry sand is weighed and placed in a flask. A 500 ml. flask is suitable. The amount of sand required depends on the desired void ratio of the specimen, but about 115 or 120 grams is usually adequate. It is desirable to have only a slight excess of sand, because the sand segregates somewhat during placing and the finest grains form the surplus, leaving a non-representative sample. With a well graded or silty sand, it would be desirable to separate the material into two or three different ranges of grain size and make up one supply for each range. This would require, of course, that the amounts of supply have to be properly proportioned, the exact weight is required of any size present as a small percentage, the slight surplus being apportioned to the prevalent range of grain sizes. A half flask of water is added to the weighed material and boiled for long enough to remove the air. Vacuum is applied during cooling. Before using the sample supply, the flask must be filled to the brim with de-aired water. A stopper is then installed, having a glass tube about 1/4 in. diameter, pushed just through, and a slightly conical hollow at the inner face. (This hollow reduces any tendency to trap sand.) The glass tube is then filled with water to its very tip. The flask is now in a condition such, that if it is inverted, atmospheric pressure will prevent its contents from coming out, but the de-aired sand will fall into its neck and fill the glass tube. The instant that the tip of the inverted tube is placed below the surface of free water in the hopper, sand will begin to pour out, under water, and its volume will be replaced by water, drawn up into the flask. Good control can be kept of the situation, because all flow stops the moment the tip of the tube is lifted from the water surface. The sand grains pour out more gently in water than in air, and by not keeping the tube in one place long enough to
establish any density current, it is possible to obtain very loose sand. The deposition of the sand must be carefully controlled. It is probably impossible to get it truly uniform, but by pouring in a regular pattern, it is possible to ensure that the variations form a regular pattern of inhomogeneities, which are small compared to the sample size. Where silt is present, or the sizes are sufficiently different to require two or more supply flasks, uniformity can be approached by using the various flasks in turn and allowing only thin layers of one size at a time. The mixing due to grains rolling sideways tends towards a uniform sample. On the other hand, if it is desired to imitate a natural deposit, this can be attempted by deliberately pouring definite layers of different sizes. It is not yet known whether such a sample has a structure that truly imitates a natural layered deposit, or what difference this makes to the strength of the sand.

Techniques for completing the sample depend to some extent on the relative density desired. A very dense sample must be jarred and vibrated to get it as dense as possible before closing the membrane and placing the loading head. If this is not done, the entire vertical travel may well be used in densifying the sand, making it impossible to apply the desired normal load. Very loose samples are difficult to place, and the height is critical. They should start at their full nominal height, but with no excess. Excess height would involve carrying the weight of the vertical motion during later stages of preparation, and this would unduly compress a loose sample. The aim is to arrive at a placed sample as close as possible to the desired density with a level, even sand surface at the same height as the top of the box sides.

The corners cause unexpected interference with the pattern of falling sand and care and frequent pauses are necessary to bring the sand up in an approximately uniform manner. The upper edges of the hopper are milled level and
by laying a T-shaped spatula over them with its length exactly right, the blade just reaches to the correct height and it can be used to smooth the upper surface and draw any excess into a small pile in the middle. It is, of course, desirable to minimize this disturbance of the upper surface, by pouring so that it is as even as possible in the first place. In finishing the sample, it is desirable to ensure that the correct amount of sand has been placed to reach the intended density. Of course, the excess sand will be dried and weighed when the sample is complete but this will not help in achieving the correct weight of sample. A rough guide, enough to indicate within one percent of the void ratio, can be obtained by fixing a paper scale to the neck of the flask. With the flask inverted and the sand pouring out, the scale is marked with the weight of sand remaining in the bottle, for various levels.

Having deposited the correct amount of sand in the sample space and levelled its surface, the excess water is siphoned out of the hopper, together with any excess sand which was scraped into a pile at the middle. It is necessary to take care that the intake of the siphon is not close enough to the sand surface for the flow to pick up part of the sample. As the excess water is removed and its level approaches the sand, the flow must be very gentle. It is also necessary to ensure that the siphon starts properly, at the first attempt, and is not subsequently broken by accident. Otherwise the reverse uncontrolled flow will cause a jet of water into the hopper and probably upset the placed sample. Once the water level has been brought within about a 1/4 in. of the sand, the hopper may be removed, gently, and the surplus water, will be found to remain within the up-turned edges of the membrane.

In closing the membrane, very delicate handling is required. Most of the precautions described are intended to help in achieving very gentle, gradual movements until the
membrane is sealed. This is because any sudden movement is sure to wash the particles of sand into undesirable places.

Three rods are needed with one end threaded to fit the screw holes in the upper spreader plate. Each rod has a neck of smaller diameter. Also required are two support strips, 1/8 in. thick, long enough to reach across the end frames of the machine and having, one each side of their mid-point, two slots 1 in. apart and wide enough to accommodate the necks in the rods. These rods appear, in use, in Fig. 43.

The upper spreader plate is placed under water in a basin and the porous plug (de-aired) is pushed gently into place. This allows the de-aired plug to be kept under water. The special rods are screwed into three of the four holes in the plate and the support strips are engaged in the necks of the rods. The plate, rods and strips are then lifted, and the plate is slipped under the water above the sample with a slight tilt, to prevent trapping air, but taking care not to let a corner dig into the sample. The supporting strips should rest on the end members of the machine. If the rods were screwed the correct distance into the plate, the plate should now be located with its surface just under water and above the sand surface. The support strips must then be adjusted by eye, so that they are supporting the spreader plate square and true, with its edges lined up above the sides of the sample. By partially unscrewing the rods, the plate is now lowered about half way to the sand surface. This must be done carefully as, if it slips, the resulting disturbance is sure to wash sand grains onto the top of the plate, where they cannot be tolerated. Removing any such grains is tedious and if any are overlooked, their later removal will not be effected except by means which will also densify the sample.

Having partially lowered the plate, the membrane must be allowed to close, over it. The forces in the stretched membrane are easily sufficient to move the plate from its
correct position, so it is best to do two opposite corners at once, lifting the hooks clear of the machine and then letting them come forward gently and evenly, one with each hand. This is a critical operation, the forces in the membrane are enough to snap it hard if it is allowed to slip and the result is a disturbance in the water and sand washed onto the top of the spreader plate. As the membrane closes over the plate, its capacity to hold free water is reduced and some of the surplus water will be spilled. If all machine parts are rust-proof, this is not serious but for the capacity of load intended in the design, needle bearings were required and stainless steel ones were not available in the necessary sizes. It is desirable to make frequent pauses, using a plastic tube to suck away surplus water. In doing so, it is necessary to keep the water surface level above the porous disc, so that the disc cannot begin to dry and entrap air. It is also desirable to suck the excess water through the disc, otherwise it must flow over the edges of the plate and is liable to wash sand over. After all four hooks have been removed there will be one corner with the membrane lying smoothly over it and three with the membrane held back by supporting rods. Before proceeding further it is advisable to check whether there appear to be any grains trapped between the membrane and the top of the plate. A visual inspection is not very satisfactory, as small grains may become almost invisible surrounded by water and viewed through the membrane. A finger tip will detect even very fine grains and, although the area is somewhat encumbered, it is possible to make a reasonable check. It is, however, essential that the touch is very light and does not drag the membrane with it.

Although the membrane snaps down hard when released by mistake, in fact it does not cling very closely to the edges of the plate, except at the corners. Thus, any sudden movement will force a small quantity of water out, round the edge of the plate, onto its upper surface. This would also
be likely to wash sand over the top and it is still necessary to use great care. If any grains of sand are discovered on top of the plate at this stage, it is often possible to ease them along, with the end of a small screwdriver and either work them out of the hole in the top of the membrane or back under the edge of the plate. Such an operation must be done with great care, not to drag other grains over the top, by pulling the membrane too far. It is, of course, only suitable for removing odd single grains; any considerable amounts of sand indicate that there must have been enough disturbance to alter the sample appreciably. It is not, usually, of any avail to try to undo anything at this stage. Reinsertion of the hooks while keeping the level of the water above the sand surface is difficult and it would have to be accomplished without dragging at the membrane sides unevenly or in any manner which would shift the sand.

The upper plate can now be lowered, very gently, onto the sand surface, by unscrewing the rods. The rods must be unscrewed, a little at a time in turn, to lower the plate.
material would need to have samples built into the machine, which would have to be modified for the purpose.

**Remoulded Clay and Silt Samples.** To accomplish tests on remoulded clays and silts, it is necessary to remould them, cut an appropriate block of remoulded material, and set it up in the same way as an undisturbed sample.

**Checking the Sample.** The principal matters for concern are that there should be no leaks and that the water passages in the loading head should be truly full of water without air bubbles. Before opening any valves, it is desirable to set a dial gauge at mid-travel on the vertical motion. Once back pressure is admitted, it will start to carry the weight of the loading head, lifting the vertical motion one or two thousandths of an inch against the slack of the locating dowel. Thus, there will be no later chance to set the dial gauge, as once pressure has been admitted, complete release appears to cause slight membrane movements, which disturb the sample at least, and may start leaking or ruptures. Having set the dial gauge, water can be admitted into the head at about five pounds pressure and then allowed to flow right through by a slight opening of the outlet valve. Most air will be flushed through during the first few seconds, and the outlet valve may then be closed while the pressure is raised, in steps of ten pounds per square inch, to above the highest anticipated pressure during the test. As each pressure increment is added, the apparatus must be checked for leaks. The outlet valve can then be opened slightly to flush out water with dissolved air and the pressure reduced in steps to approximately the mid-range expected during the test. Water is flushed through at each pressure reduction and the decrement is allowed to stand, under pressure for a minute or so. By this means, most bubbles and dissolved air are flushed out. Having reached mid-range, it is necessary to check the success of the de-airing process. The pore pressure read-out is switched on and both inlet and outlet
valves are closed in the head. If the pressure drops off with the valves closed, it is an indication of air being dissolved, or a leak. The pressure drop may be imperceptibly slow, but it becomes readily apparent by the jump when the inlet valve is re-opened. Any entrapped air is usually flushed out quite readily by opening the outlet valve for a few seconds every minute or so until the air has been removed.

Application of Normal Load. When all air has been dissolved or flushed out, the outlet valve must be closed and will not be opened again until the test is finished, but the inlet valve is left open. The pressure in the water system is then reduced to the back pressure that is desired for the test, a minimum of one and a half kilograms per square centimetre is recommended, to ensure that the membrane is always kept fully expanded against the walls of the shear box.

The gantry carrying the normal loading piston is then swung into place and secured. The piston is usually maintained with an upward pressure which just balances the weight of the vertical motion of the machine. In this condition, the piston must be pushed down by hand, to screw its connection into the loading head. When this connection has been made, the forces on the vertically moving parts are as follows:

a) Weight acting down, balanced by the pre-set upward pressure in the piston.

b) The back pressure in the pore water, acting up on the loading head and balanced by the vertical motion locating pin, acting in shear.

The piston is, therefore, exerting a zero net downward force on the sample. After connecting the piston to the loading head, the lead to the normal load read-out can be plugged into its diaphragm, and this read-out is switched on and set to a suitable zero and scale factor, depending on the back pressure and the intended normal load.
The top of the machine, at this stage, is illustrated in Fig. 44. Pressure is then slowly admitted to force the piston downward, gradually transferring the resistance to back pressure from the locating pin to the piston. When the piston load just balances the back pressure, the vertical dial gauge will move, quite suddenly back to its original setting (about .002 in.). This is the most sensitive method, so far found, of knowing when the back pressure is just balanced. It will be found that the locating pin now moves freely in its hole and it may be withdrawn. The normal load read-out is now giving zero effective load and the value must be noted or a new zero set on the scale.

Fig. 44. Top of Machine, Assembled Over Sample.

After withdrawing the vertical locating pin, the normal load is increased to whatever value is desired for the test and the sample is then allowed to consolidate for whatever time is wished.
APPENDIX F

CYCLIC SHEAR TESTS ON HANEY CLAY

Haney clay is a local British Columbia clay. Its natural moisture content is about 42 percent, with a plastic limit of 26, a liquid limit of 44 percent, and a sensitivity of 5 or 6. The maximum past pressure is 2.7 kg. per sq. cm. Few tests were done, as the only intention was to establish experimental procedure.

It was found that the simple shear strength was about 1.6 to 1.7 kg. per sq. cm. when consolidated for twenty hours at 5.27 kg. per sq. cm. (75 lb. per sq. in.) and then loaded with shear stress increments of 0.1 kg. per sq. cm. per minute. Samples were then consolidated under the same pressure and subjected to cyclic shear stress at about 2/3 cycles per sec. With a cyclic shear stress of ± 0.6 kg. per sq. cm. the shear strains increased slightly and then fell away, while the pore pressure rose slightly (see Fig. 45a). With a cyclic shear stress of ± 1.0 kg. per sq. cm. the deflections increased at an increasing rate to failure. During this time the pore pressure rose to 0.47 kg. per sq. cm. (see Fig. 45b). After failure and switching off cyclic load, pore pressure continued to rise. At first, the rate was the same as at the end of the test, but it gradually levelled off at about 3.7 kg. per sq. cm. after some 40 minutes. The interesting points about these tests on clay are:

a) There appears to be a threshold cyclic stress below which shear strains are ultimately reduced by strain hardening but above which failure will ultimately occur.

b) A cyclic stress which leads to failure may be less than the static shear strength, measured in the manner already described. However, as the "static" shear strength appears to depend on the rate of
Fig. 45a. Cyclic Simple Shear Tests on Haney Clay. Normal load 5.28 kg/cm². Shear load ± 0.6 kg/cm².

Fig. 45b. Cyclic Simple Shear on Haney Clay. Normal Load 5.28 kg/cm². Shear Load ± 1.0 kg/cm².
strain at failure\textsuperscript{18}, this is, perhaps, a premature conclusion.

c) The mechanism of failure for clay under cyclic shear appears to be one of progressive remoulding.

The conclusions above are based on very few tests and are therefore only tentative, especially as they do not agree with Seed and Chan who concluded from stress controlled tests\textsuperscript{19}, or Thiers and Seed who concluded from strain controlled tests\textsuperscript{20}, that cyclic shear would not reduce the strength of clay below its normally measured static strength. Seed and Chan, in the same reference, did note, however, a rise of pore pressure after cessation of cyclic stress, which might lead to failure.

\textsuperscript{18} Snead, D.E., Ph.D. Thesis, University of British Columbia, pending.
**LIST OF SYMBOLS**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1, A_2$</td>
<td>Functions of elastic constants.</td>
</tr>
<tr>
<td>$B$</td>
<td>Average compressibility of soil and water.</td>
</tr>
<tr>
<td>$C$</td>
<td>Function of elastic constants.</td>
</tr>
<tr>
<td>$C_s$</td>
<td>Matrix of anisotropic elastic constants.</td>
</tr>
<tr>
<td>$C_w$</td>
<td>Compressibility of soil skeleton.</td>
</tr>
<tr>
<td>$E_x, E_y$</td>
<td>Compressibility of water.</td>
</tr>
<tr>
<td>$E_{x^<em>}, E_{y^</em>}$</td>
<td>Young's modulus in the $x$-direction.</td>
</tr>
<tr>
<td>$e$</td>
<td>Young's modulus in the $y$-direction.</td>
</tr>
<tr>
<td>$F$</td>
<td>Void ratio.</td>
</tr>
<tr>
<td>$F_{xx}$</td>
<td>Intake factor.</td>
</tr>
<tr>
<td>$G_{xx}$</td>
<td>Shear modulus in a plane at right angles to the $y$-axis.</td>
</tr>
<tr>
<td>$G_{xy}$</td>
<td>Shear modulus in an $x$-$y$ plane.</td>
</tr>
<tr>
<td>$H, h$</td>
<td>Height of simple shear sample.</td>
</tr>
<tr>
<td>$K_0$</td>
<td>Half of HT.</td>
</tr>
<tr>
<td>$K_1, K_2, K_3$</td>
<td>Compliance of pore pressure measuring apparatus.</td>
</tr>
<tr>
<td>$K_1, K_2, K_3$</td>
<td>Co-efficient of earth pressure at rest.</td>
</tr>
<tr>
<td>$K_1, K_2, K_3$</td>
<td>Functions of elastic constants.</td>
</tr>
<tr>
<td>$k$</td>
<td>Permeability.</td>
</tr>
<tr>
<td>$l$</td>
<td>Length of simple shear sample.</td>
</tr>
<tr>
<td>$M_{x^*}, \mu_{xx}$</td>
<td>Poisson's ratio in a plane at right angles to the $y$-axis.</td>
</tr>
<tr>
<td>$M_{x^*}, \mu_{xy}$</td>
<td>Poisson's ratio in an $x$-$y$ plane.</td>
</tr>
<tr>
<td>$n$</td>
<td>Porosity. Term number in a series.</td>
</tr>
<tr>
<td>$p$</td>
<td>Pressure.</td>
</tr>
<tr>
<td>$S$</td>
<td>Change in $V$, due to unit pressure change.</td>
</tr>
<tr>
<td>$\sigma_{x, y}$</td>
<td>Abbreviation for $\sigma_x, \sigma_y$.</td>
</tr>
<tr>
<td>$t$</td>
<td>Time.</td>
</tr>
<tr>
<td>$u$</td>
<td>Elastic displacement in $x$-direction.</td>
</tr>
<tr>
<td>$V$</td>
<td>Pore pressure.</td>
</tr>
<tr>
<td>$V$</td>
<td>Compliance of a piezometer cell.</td>
</tr>
<tr>
<td>$V_1$</td>
<td>Volume of pore pressure measuring device.</td>
</tr>
<tr>
<td>$V$</td>
<td>Volume.</td>
</tr>
<tr>
<td>$v$</td>
<td>Elastic displacement in $y$-direction.</td>
</tr>
<tr>
<td>$x, y, z$</td>
<td>Directions of co-ordinate axes in space.</td>
</tr>
<tr>
<td>Symbol</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
</tr>
<tr>
<td>$Z_1, Z_2$</td>
<td>Functions of elastic constants.</td>
</tr>
<tr>
<td>$\alpha_1, \alpha_2$</td>
<td>Functions of elastic constants.</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>Shear strain.</td>
</tr>
<tr>
<td>$\rho$</td>
<td>Density.</td>
</tr>
<tr>
<td>$\varepsilon$</td>
<td>Linear strain.</td>
</tr>
<tr>
<td>$\varepsilon_A$</td>
<td>Axial linear strain.</td>
</tr>
<tr>
<td>$\eta$</td>
<td>Function of elastic constants.</td>
</tr>
<tr>
<td>$\theta, \theta'$</td>
<td>Angle between normal to plane of applied shear stress and resultant stress on same plane, same quantity measured in terms of effective stress.</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>A coefficient of stress uniformity.</td>
</tr>
<tr>
<td>$\mu$</td>
<td>Poisson's ratio. Also see $\mu_{xx}, \mu_{xy}$.</td>
</tr>
<tr>
<td>$\xi$</td>
<td>Function of elastic constants.</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>Normal stress.</td>
</tr>
<tr>
<td>$\sigma_1, \sigma_2, \sigma_3$</td>
<td>Major, intermediate and minor principal stresses.</td>
</tr>
<tr>
<td>$\bar{\sigma}$</td>
<td>Effective normal stress.</td>
</tr>
<tr>
<td>$\tau$</td>
<td>Shear stress.</td>
</tr>
<tr>
<td>$\phi$</td>
<td>An Airey stress function. Angle of max. stress obliquity.</td>
</tr>
<tr>
<td>$\chi_1, \chi_2$</td>
<td>Functions of elastic constants.</td>
</tr>
<tr>
<td>$\omega$</td>
<td>Angle of deformation of simple shear sample.</td>
</tr>
</tbody>
</table>
LIST OF REFERENCES


