STABILITY ANALYSIS OF A SPACEFRAME STRUCTURE

by

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ABSTRACT

Two theories for the stability analysis of a spaceframe structure are presented: the first uses only Livesley stability functions, the second includes in addition the effects due to chord rotation and flexural end shortening of a member.

The critical condition is defined by the load which makes the tangent stiffness matrix of the structure become singular.

Three methods for obtaining the critical load are presented: determinant plot, Southwell plot, and load-deflection curves.

The analysis is carried out for the plexiglass model of an actual conical spaceframe, made of glulam timber, and built for the storage of potash.

The overall critical load for this structure is found to be in satisfactory agreement with the experimental results obtained in previous model tests.

Some additional effects, such as geometric imperfections in the joint coordinates and different member end-fixity conditions are investigated.

The concept of effective length for a member is introduced to present the results obtained by varying the height/span ratio of the structure. Finally some design suggestions are given for structures of this type.

The analyses were made using spaceframe programs based on the stiffness method, modified to include stability effects. An IBM 360/67 computer was used for the calculations.
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NOTATION

x, y, z = member axes;
X, Y, Z = structure axes;

{P} = external load vector;
{U} = structure displacement vector;
[K] = structure stiffness matrix;
[K_t] = structure tangent stiffness matrix;
{f} = member force vector in member axes;
{F} = member force vector in structure axes;
{u} = member displacement vector in member axes;
[k]^m = member stiffness matrix in member axes;
[k] = member stiffness matrix in structure axes;
[k_G] = member geometric stiffness matrix in structure axes;
[k_t] = member tangent stiffness matrix in structure axes;
[R] = matrix of direction cosines of a member;
[T] = transformation matrix for a member;
N = axial force in a member;
λ = load factor;

{P}_O = external load vector at λ = 1;
N_o = linear axial force in a member at λ = 1;
Δ_N = joint displacement component normal to the surface of the cone;
ω_N = joint rotation component about an axis normal to the surface of the cone;
α = height/span ratio;
NOTATION (Continued)

$L$ = member joint-to joint length;
$A$ = member cross-sectional area;
$I$ = moment of inertia;
$J$ = torsional constant;
$r$ = radius of gyration of member cross-section;
$E, G$ = elastic moduli of material;
$L_e$ = effective length of a member;
$P$ = total applied external load in vertical direction;
$\rho$ = chord rotation.
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INTRODUCTION

Various structural systems may be used to cover large areas for recreational or industrial purposes, e.g., systems of trusses and beams, continuous shells, or spaceframes. When the geometrical configuration presents an axis of rotational symmetry, the most widely used systems are either continuous shells, generally in concrete, or framed domes.

In countries where the cost of labour is high with respect to that of materials, the use of spaceframe structures may show an economical advantage over the use of continuous shells; the former may also be preferred from the point of view of speed of erection and general construction quality, since field work is reduced to a minimum with respect to shop prefabrication.

Notable examples of such domes are R. B. Fuller's geodesic domes, the Astrodome of Houston, Texas, the Schwedler Dome of Berlin.

In the design of these structures it is relatively easy to find the linear internal forces and moments due to a specified external load, by means of a linear stiffness analysis (1). At present, however, questionable rules are used to find the allowable compressive loads in the members.

Nonlinearity in the behaviour of these structures under static loads, excluding material nonlinearities, arises from changes in the

*Numbers in parentheses refer to references listed in Appendix 4.*
geometry of the structure due to its deformation: these effects are particularly important in those spaceframes where members meeting at a joint are almost coplanar.

The object of this thesis is to determine the effect of some geometric nonlinearities, namely chord rotation and bowing, as shown in fig. 1:1, on the evaluation of the elastic critical load for a spaceframe structure, and to give some design suggestions for this type of structure, particularly about the concept of effective length of a member. In addition, an investigation is made about the applicability of the Southwell plot for the evaluation of the critical load of the structure.

This will be achieved by analysing an elastic spaceframe having the shape of a conical shell, with 16 straight ribs, 4 main rings, and bracing diagonals, as shown in fig. 3:1. A structural model of such a cone has been used in the past to predict the critical load (2). The present work will be carried out analytically using two different structural stiffness matrices, namely

(1) considering the effect of the axial force on the bending stiffness of a member,

(2) same as (1), but adding the effects of the change in axial force due to chord rotation and flexural end-shortening (bowing), as from fig. 1:1.

![Initial position, chord rotation, and bowing](image-url)

**FIG. 1:1 - GEOMETRICALLY NONLINEAR EFFECTS**
Since the results of the two analyses were always very close, it was not deemed necessary to analyse separately the effect of chord rotation only.

In the stability analysis of framed structures the importance of the change in a member bending stiffness, due to the presence of an axial force, has been shown to be essential by several investigators (e.g. Livesley (3), Merchant & Horne (4)).

In the first analysis the axial force in a member is determined only by the orthogonal projections of the end-displacements on the initial position of the member, while the second analysis takes into account the actual difference in length between the final deformed shape and the initial configuration. Thus, with reference to fig. 1:1, the first analysis would give zero axial force in the member, whereas the second would show the presence of a tensile axial force.

The additional effects considered in theory 2 could be significant for structures of shell-like configuration, which do not allow inextensional bending, and have been shown to be of essential importance in the post-buckling domain (5).

Among other methods to predict the critical load of a spaceframe dome, there is the possibility of establishing an equivalent homogeneous shell, whose critical load may be obtained from the available analytical and experimental results. This method may be useful when the number of members in the spaceframe is so large as to make a computer analysis impossible or uneconomical. The method, however, fails if the mesh size of the spaceframe is too coarse, or if the lateral stiffness of the ribs is much smaller than their stiffness normal to the shell surface, in
which case buckling may occur because of lateral bending of the members of the spaceframe.

The nonlinear equations governing the deformation of the structure may be solved by various methods (6,7), among which the Newton-Raphson and the step-by-step incremental method have been widely used. In the present study a modified version of this second method will be used, since the Newton-Raphson method has been reported to exhibit difficulties in calculation stability and convergence (8). Other advantages of the incremental method are the possibility of following the progressive deformation of the structure and the fact that the determinant of the tangent stiffness matrix provides at each step a measure of relative stability of an equilibrium configuration.

The following additional conditions will also be considered in this investigation:

(a) changing members' end conditions so as to consider the two cases:

i) Possibility of buckling about both axes of member cross-section admitted.

ii) Possibility of buckling about the weak axis (i.e. "in plane" buckling) excluded.

In the first case a member provides bending resistance at a joint with respect to both principal axes of the cross-section; in the second case no bending stiffness exists for rotations about the weak axis of the cross-section;

(b) a random variation of joint coordinates, to simulate possible construction imperfections;

(c) changes of geometrical configuration, i.e. conical and doubly curved domes of different height/span ratio.
CHAPTER II

GOVERNING EQUATIONS AND SOLUTION PROCEDURE

2.1 Governing Equations

As mentioned earlier, two methods for stability analysis are used. The first makes use of Livesley stability functions, as outlined in (3) and (4), to take into account the variations in the bending stiffness of a member due to the presence of an axial force. In this case the stiffness matrix of the structure \([K]\) is a function only of the axial loads \(N_i\) in the members.

The governing load-displacement relations will be:

\[
\{P\} = [K(N_i)] \{U\} \ 
\]

where \(\{P\}\) is the external load vector, and \(\{U\}\) is the structure displacement vector. Generally a few iterations, at each load level, are required to achieve convergence for the elements of \([K]\).

For stability analysis we can rewrite [1] in incremental form (assuming that the member axial loads do not change for a small increment of the external loads):

\[
\{\Delta P\} = [K(N_i)] \{\Delta U\} \ 
\]

Then, by definition, at the critical load we have \(\{\Delta P\} = \{0\}\) for some non-zero \(\{\Delta U\}\), i.e. \([K]\) must become singular, or

\[|K| = 0.\]

In other words, at the critical load the equilibrium configuration of the structure is not unique.
The second analysis, including the effects due to chord rotation and bowing, is based on the derivation described in references (9) and (10).

The assumptions on which this theory rests are:
1. The material is linear elastic.
2. Each member is prismatic and homogeneous.
3. Loads are applied only at the ends of a member.
4. Shear deformations are neglected.
5. Linear strains and squares of the rotations are of the same order of magnitude, and small compared to one.
6. Torsion-flexure coupling and warping restraint are neglected.

It can be noted that assumptions 4 and 5 correspond exactly to those made by Von Kármán in his "large deflection" theory of plate bending (11).

An outline of the derivation of the load-displacement relations for a member, in the two-dimensional case, is shown in Appendix 1.

For a general spaceframe member, the member coordinate system, end-forces and end-displacements are shown in fig. 2.1.

The member reference frame does not follow the deformation of the member, but is fixed to its undeformed position.

FIG. 2.1 - MEMBER NOTATION
The complete set of force-displacement relations for a member can be written, in matrix form, as:

\[
\{f\}_{12x1} = [k]^m_{12x12} \{u\}_{12x1} - \{f_G\}_{12x1} \ldots \ldots \ldots \quad [3]
\]

where

\[
\{f\} = \{F_{A1}', F_{A2}', F_{A3}', M_{A1}', M_{A2}', M_{A3}', F_{B1}', F_{B2}', F_{B3}', M_{B1}', M_{B2}', M_{B3}'\}^T
\]

and

\[
\{u\} = \{u_{A1}', u_{A2}', u_{A3}', \omega_{A1}', \omega_{A2}', \ldots \ldots \ldots \omega_{B3}'\}^T
\]

are the column-vectors of member end-forces and end-displacements; \([k]^m\) is the member stiffness matrix, including Livesley stability functions; \(\{f_G\}\) is the vector of the nonlinear geometric terms, arising from bowing and chord rotation. In the present analysis it reduces to only two axial terms, since the sway terms (of the type \(\rho F_1\), where \(\rho = \frac{u_{B2}' - u_{A2}'}{L}\) ), which appear in (10), are included in Livesley stability functions.

It should also be noted that finite rotations do not obey the addition law for vectors, but the error is of the order of the square of a rotation, which is negligible with the assumptions used in the present derivation.

To obtain the system equations in structure, or global, coordinates, let \([R]_{3x3}\) be the matrix of the direction cosines for the undeformed member directions \((x,y,z)\) with respect to the global reference frame \((X,Y,Z)\). Then it can be shown (1) that the transformation matrix \([T]\) for a spaceframe member results:
\[ [T]_{12 \times 12} = \begin{bmatrix} [R] & [0] & [0] & [0] \\ [0] & [R] & [0] & [0] \\ [0] & [0] & [R] & [0] \\ [0] & [0] & [0] & [R] \end{bmatrix} \]

Then, for each member, we have, in structure coordinates:

\[ \{U\}^m = [T]^T \{u\} \]
\[ \{F\} = [T]^T \{f\} \]
\[ \{F_G\} = [T]^T \{f_G\} \]
\[ [k] = [T]^T [k]^m [T] \]

Adding up the contributions of all the members connected to a joint, the system of joint equilibrium equations is obtained:

\[ \{P\} = [K] \{U\} - \{P_G\} \]

where \( \{P\} \) is the external load vector;

\( [K] \) is the structure stiffness matrix, which again is a function only of the axial loads in the members;

\( \{U\} \) is the structure displacement vector;

\( \{P_G\} \), produced by \( \{F_G\} \), is equivalent to a load vector, due to geometric nonlinearities, and is a function both of the members' axial loads and end-displacements.

2.2 Solution Procedure

To solve system [4] of nonlinear equations, a modified incremental method is used, whereby the external loads are applied by finite increments, and an iteration cycle by successive substitutions is carried out after a number of steps, that can be varied at choice, using the complete, "exact", set of nonlinear equations [4].
To obtain the incremental part of the solution, the expressions for the differentials of the end-actions of a member are needed. Following reference (10), we can write, in member coordinates:

\[ \{\Delta f\} = [k]^{m} \{\Delta u\} + [k_G]^{m} \{\Delta u\} \]

where \([k_G]^{m} \{\Delta u\} = - \{\Delta f_G\} \]

and \([k_G]^{m}\) may be interpreted as a geometric stiffness matrix, which, with our assumptions, happens to be symmetric: \([k_G]^{m}\) is given in Appendix 1.

Then we can write:

\[ \{\Delta f\} = [k_t]^{m} \{\Delta u\} \ldots \ldots \ldots \ldots \ldots [5] \]

where \([k_t]^{m} = [k]^{m} + [k_G]^{m}\)

is the member tangent stiffness matrix. Assembling the contributions of each member we obtain the system equations in incremental form:

\[ \{\Delta P\} = [K_t] \{\Delta U\} \ldots \ldots \ldots \ldots \ldots \ldots [6] \]

where \([K_t]\) is now the structure tangent stiffness matrix.

Again instability is reached when equation [6] has a non-trivial solution for \{\Delta P\} = \{0\}, i.e. when \(|K_t| = 0\).

A second stability criterion can be established by the failure in achieving convergence in the cycle of successive substitutions: for this method a small increment size is necessary in the neighborhood of the critical load, in order to achieve reasonable accuracy.

An additional method of determining the critical load is the use of Southwell plots in connection with selected disturbing force systems. It has been shown (12) that in a plane frame, as long as a disturbance, either a load or a geometric imperfection, excites any component of the
first critical mode shape, the corresponding Southwell plot will yield the relative critical load. Thus it was decided to check the applicability of this method to the present spaceframe structure.

2.3 Computer Program Outline

A standard spaceframe stiffness program, including Livesley stability functions, has been modified in order to build the tangent stiffness matrix of the structure at each load level, using the axial forces and joint displacements from the previous step, since \([k^m]\), from the assemblage of which \([K_\ell]\) is made up, is a function both of the axial load and the end-displacements of a member.

Every few steps a cycle of successive substitutions is performed, keeping the external loads constant, as shown in fig. 2:2.

![FIG. 2:2 - STEP-BY-STEP SOLUTION](image-url)
Segment AB corresponds to the incremental step along the tangent to the load-displacement curve, using equation [6], while BC represents the corrective iteration cycle at constant load. In this cycle equation [4] is used repeatedly: at each iteration the new values of displacements and axial loads are used to build \([K]\) and \(\{P_G\}\); then the equation can be written as
\[
\{P\} + \{P_G\} = [K] \{U\}
\]
and solved to obtain a better solution for the displacement vector \(\{U\}\).

The step size is reduced in geometric progression, as the total load is increased, to allow greater accuracy in the determination of the critical point.

Only a proportional load system is considered, where at each level the external load distribution is a multiple of the initial, basic loading pattern, which can be represented by a vector \(\{P\}_0\), so that at each stage we have
\[
\{P\} = \lambda \{P\}_0
\]
where \(\lambda\) is a numerical parameter. Then the critical load will be given by the corresponding value \(\lambda_{cr}\) of the load factor \(\lambda\).

Finally, to check the program, fig. 2:3 shows an example (a portal frame) presented in (10), which was used as the basic reference for the present investigation.

It is to be noted that the solution of ref. (10) allowed a 5% convergence tolerance, whereas the procedure used by the writer permitted full convergence, which may explain the slight gap between the two curves.

As another simple example of a structure where geometric non-linearities are important, fig. 2:4 shows a two-bar structure with very low rise/span ratio. With the given cross-sectional properties, geometric instability will occur before Euler buckling.
LOAD–DISPLACEMENT CURVES.

FIG. 2:3 - CHECK OF COMPUTER PROGRAM FOR THEORY 2.
FIG. 2.4 - TWO-BAR STRUCTURE.
The two computer analyses are compared with the exact solution, developed from strain energy considerations: the relative expressions are derived in Appendix 2. The structure was analysed as a plane frame with 4 members, as shown below.

It can be seen from fig. 24 that the results given by theory 2 are very close to the exact solution; this is another check for the correctness of the computer program written for theory 2.
CHAPTER III

DESCRIPTION OF THE STRUCTURE AND LOADING SYSTEMS

The first large structure analysed was the plexiglass model (1:28 lineal ratio) of an actual conical spaceframe, made of glulam timber and built in Esterhazy, Saskatchewan, for International Minerals and Chemical Corporation (Canada) Ltd. (2). The geometric and elastic properties of the structure are shown in figs. 3:1 and 3:2. The angle $\beta$ is measured from a vertical plane through the axis of a member to a plane containing the weak principal axis of the cross-section.

The structure consists of 16 ribs, 64 ring members, and 64 diagonals, which develop the function of the web members of a truss: altogether there are 80 joints and 192 members. Thus the structure is statically determinate in its primary stress system, since we have: 80 joints yielding 240 equations of equilibrium, 192 bars providing 192 unknown axial forces, and $16 \times 3 = 48$ unknown foundation reaction components. This yields 240 unknowns and 240 equations, and by inspection it can be seen that there are no redundant members.

To simulate the rigid foundation ring, each rib was considered perfectly fixed to the respective foundation joint.

In the model the members meeting at a joint were held in place by two aluminum plates (2), one at the top and one at the bottom, tightly fastened by bolts, as sketched in fig. 3:3.
ORIGINAL CONE (Plexiglass Model)

FIG. 3:1 - THE ORIGINAL CONE
<table>
<thead>
<tr>
<th>Member</th>
<th>Size (in. x in.)</th>
<th>Area A (in.²)</th>
<th>$I_z$ (in.⁴)</th>
<th>$I_y$ (in.⁴)</th>
<th>$J$ (in.⁴)</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ribs:</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1/4 x 1</td>
<td>.25</td>
<td>.02083</td>
<td>.00130</td>
<td>.00439</td>
<td>0°</td>
</tr>
<tr>
<td>2</td>
<td>3/16 x 3/4</td>
<td>.1406</td>
<td>.00659</td>
<td>.000412</td>
<td>.00139</td>
<td>0°</td>
</tr>
<tr>
<td>3</td>
<td>3/16 x 11/16</td>
<td>.1289</td>
<td>.00508</td>
<td>.000378</td>
<td>.00125</td>
<td>0°</td>
</tr>
<tr>
<td>4</td>
<td>3/16 x 9/16</td>
<td>.1055</td>
<td>.00278</td>
<td>.000309</td>
<td>.000976</td>
<td>0°</td>
</tr>
<tr>
<td>Rings:</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1/2 x 1/2</td>
<td>.25</td>
<td>.00521</td>
<td>.00521</td>
<td>.00879</td>
<td>34°</td>
</tr>
<tr>
<td>2</td>
<td>3/8 x 3/8</td>
<td>.1406</td>
<td>.00165</td>
<td>.00165</td>
<td>.00278</td>
<td>34°</td>
</tr>
<tr>
<td>3</td>
<td>1/4 x 1/4</td>
<td>.0625</td>
<td>.000325</td>
<td>.000325</td>
<td>.00055</td>
<td>34°</td>
</tr>
<tr>
<td>4</td>
<td>3/16 x 1/2</td>
<td>.0938</td>
<td>.00195</td>
<td>.000275</td>
<td>.000837</td>
<td>34°</td>
</tr>
<tr>
<td>Diagonals:</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>3/16 x 11/16</td>
<td>.1289</td>
<td>.00508</td>
<td>.000378</td>
<td>.00125</td>
<td>28.2°</td>
</tr>
<tr>
<td>2</td>
<td>3/16 x 1/2</td>
<td>.0938</td>
<td>.00195</td>
<td>.000275</td>
<td>.000837</td>
<td>25.23°</td>
</tr>
<tr>
<td>3</td>
<td>3/16 x 3/8</td>
<td>.0703</td>
<td>.000824</td>
<td>.000206</td>
<td>.000566</td>
<td>20.07°</td>
</tr>
<tr>
<td>4</td>
<td>3/16 x 9/16</td>
<td>.1055</td>
<td>.00278</td>
<td>.000309</td>
<td>.000976</td>
<td>11.41°</td>
</tr>
</tbody>
</table>

$E = 450$ ksi  \quad G = 187$ ksi

FIG. 3:2 - MEMBER PROPERTIES
In addition, glue and sand were sprinkled on the edges of the members to increase friction between the members and the plates, so as to try and obtain a rigid connection with respect to both cross-sectional axes of a member.

In the computer analysis two fixity conditions were examined: in the first, which will be called condition (a), all the members were considered perfectly fixed to the joints, except the diagonals which were coded as fixed about the strong axis, but hinged about the weak axis of the cross-section. Because of the way a member stiffness matrix is built by the computer program, a member coded as hinged at the joints at its ends, with no extra degrees of freedom along its length, behaves as a simple strut, with no buckling considered. Thus the possibility of lateral
buckling of the long diagonals, which in the model were held in place by strings stretching from rib to rib, and in the actual structure by purlins, was excluded.

In the second condition, called condition (b), all members were considered fixed to the joints for bending about the strong axis, i.e. in a direction normal to the surface of the cone, and hinged about the weak axis, i.e. for bending in the surface of the cone. This second condition, perhaps, represents more closely the actual behaviour of the joints of the model, and certainly allows a better stability analysis of the actual structure, where purlins and decking provide adequate lateral bracing to prevent "in plane" buckling.

The following fig. 3:4 shows the actual coding of the structure for the computer analysis.

Because of symmetry, only half of the structure was analysed: the joints lying in the plane of symmetry were permitted only displacements in that plane, and the members cut by the plane of symmetry were entered with half their stiffnesses.

This procedure still allowed the analysis of partial loadings and of buckling modes not possessing rotational symmetry, as sketched in fig. 3:5.

FIG. 3:5 (i) - HALF SNOW LOAD
FIG. 3-4 - CODING OF THE STRUCTURE
For each joint the computer analysis provides displacements and rotations in the X, Y, Z directions (structure axes), and for each member the internal forces shown in fig. 3:6.
The basic loading condition examined was a uniformly distributed snow load, which was simulated with concentrated loads at each joint according to the pressure at and the surface area tributary to the joint.

In all the following graphs the value of $P_0$ shown will represent the total vertical load due to such U.D.L. when the load factor $\lambda$ is equal to one.

The different disturbing force systems used ("trigger systems") are shown in fig. 3:7. The magnitude of these perturbation loads was contained in a range from 4% to 8% of the given external load, acting at the same joint.
FIG. 3-7 - PERTURBATION FORCE SYSTEMS (TRIGGER SYSTEMS)
PERTURBATION FORCE SYSTEMS ("TRIGGER SYSTEMS")

- Downward Trigger
- Upward

Plan View

FIG. 37 (Cont. d)
CHAPTER IV

RESULTS FOR THE ORIGINAL CONE

4.1 Introduction

The model tests carried out in the past (2) yielded a value of 904 ± 80 lb. for the total structure critical load, determined by Southwell and Lundquist plots, using joint deflections normal to the surface of the cone.

In the present investigation, determinant plots, Southwell plots, and load-deflection curves were used for the evaluation of the critical load.

4.2 Determinant Plot

Table 4.1 contains the results from the determinant plots, which are shown in figs. 4:1 and 4:2 for fixity conditions (a) and (b) respectively.

The random variation of joint coordinates, simulating geometric imperfections, was contained in a range of ±.5%. The same "random geometry" was used in all runs thus labelled.

The tolerance shown in the values of critical load depends essentially on the accuracy of the analysis in the neighborhood of the critical load, i.e. on the magnitude of the incremental step and on the number of iterations performed at each load level.

The values obtained by the determinant graphs are seen to be in good agreement with the previous experimental determination of the
TABLE 4.1. DETERMINATION OF STRUCTURE CRITICAL LOAD
BY DETERMINANT PLOT

(a) All members, except bracing diagonals, perfectly fixed to the joints at their ends.

<table>
<thead>
<tr>
<th></th>
<th>Theory 1</th>
<th>Theory 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact geometry</td>
<td>920 ± 10 lb.</td>
<td>915 ± 10 lb.</td>
</tr>
<tr>
<td>Random geometry</td>
<td>927 ± 10 lb.</td>
<td>928 ± 10 lb.</td>
</tr>
</tbody>
</table>

Theory 1: Stability functions only.
Theory 2: Stability functions plus chord rotation and bowing.

(b) All members fixed to the joints about the strong axis, but hinged about the weak axis.

<table>
<thead>
<tr>
<th></th>
<th>Theory 1</th>
<th>Theory 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact geometry</td>
<td>1048 ± 15 lb.</td>
<td>1035 ± 15 lb.</td>
</tr>
<tr>
<td>Random geometry</td>
<td>996 ± 15 lb.</td>
<td>972 ± 15 lb.</td>
</tr>
</tbody>
</table>
FIG. 4:1 - ORIGINAL CONE DEFORMABILITY PLOTS.
FIXITY CONDITION (A)
FIG. 4 - ORIGINAL CONE DETERMINANT PLOTS.
FIXITY CONDITION (B)
critical load, especially considering the likely differences between the model and the structure here analysed, owing to material non-linearities (creep of plexiglass), actual joint behaviour, application of the external loads, and actual construction imperfections.

The higher critical load for fixity condition (b) is due to the fact that in this case the lateral buckling of individual members about their weak axis is excluded from the analysis. It is as if the members were actually restrained against motion in the plane of the shell, since no degrees of freedom are considered along the length of a member between panel points. It can also be argued that if member buckling about the weak axis of the cross-section is allowed for, as in condition (a), above a certain load level some members bring negative contributions to the stiffness matrix of the structure.

The influence of geometric imperfections and of different perturbation force systems on the zeros of the stiffness matrix determinant appears to be minor as shown in fig. 4:3. This could be explained by the fact that the structure is statically determinate in its primary stress system: hence the distribution of the axial loads in the members should not change much for small imperfections, except probably in the neighborhood of the critical load.

In effect table 4.2 points out some of the biggest differences in axial loads between the structure with exact geometry and that with random geometry, at about 1/3 of the critical load and near buckling. It can be seen that the differences are magnified as $\lambda \rightarrow \lambda_{cr}$, but since our geometric imperfections are distributed at random, their overall effect on the structure stiffness determinant probably tends to zero.
ORIGINAL CONE DETERMINANT PLOTS

FIG. 4: 3 COMPARISON DETERMINANT PLOTS

Fixity condition (b)
Exact geometry
Theory 1

Trigger system 3
7
9

Load Factor \( \lambda \) \( \left( P_0 = 120 \text{lb.} \right) \)

Determinant, \( |K_{ij}| \times 10^{32} \)
### TABLE 4.2. INFLUENCE OF GEOMETRIC IMPERFECTIONS ON AXIAL LOADS

(a) Members perfectly fixed to the joints.

(Theory 1. Trigger system 3). $\lambda_{cr} = 7.70$. $P_o = 120$ lb.

<table>
<thead>
<tr>
<th>Member No.</th>
<th>$X = 2.67$</th>
<th>$X = 7.52$</th>
<th>$X = 7.60$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>16.4 lb.</td>
<td>16.2 lb.</td>
<td>48.9 lb.</td>
</tr>
<tr>
<td>5</td>
<td>16.4</td>
<td>13.6</td>
<td>48.8</td>
</tr>
<tr>
<td>6</td>
<td>32.3</td>
<td>33.7</td>
<td>90.4</td>
</tr>
<tr>
<td>13</td>
<td>5.65</td>
<td>7.7</td>
<td>14.9</td>
</tr>
<tr>
<td>34</td>
<td>5.46</td>
<td>5.9</td>
<td>15.9</td>
</tr>
<tr>
<td>39</td>
<td>4.1</td>
<td>5.4</td>
<td>10.2</td>
</tr>
<tr>
<td>63</td>
<td>1.4</td>
<td>.77</td>
<td>4.18</td>
</tr>
<tr>
<td>65</td>
<td>1.4</td>
<td>1.87</td>
<td>4.17</td>
</tr>
<tr>
<td>76</td>
<td>2.1</td>
<td>2.75</td>
<td>5.6</td>
</tr>
<tr>
<td>85</td>
<td>-1.3</td>
<td>-2.</td>
<td>-3.4</td>
</tr>
</tbody>
</table>

Axial loads: +ve compression, -ve tension.

(b) Members fixed about strong axis, hinged about weak axis.

(Trigger system 9) $\lambda_{cr} = 8.50$. $P_o = 120$ lb.

<table>
<thead>
<tr>
<th>Member No.</th>
<th>$\lambda = 2.67$</th>
<th>$\lambda = 8.15$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>31.9 lb.</td>
<td>32.8 lb.</td>
</tr>
<tr>
<td>3</td>
<td>16.0</td>
<td>15.9</td>
</tr>
<tr>
<td>6</td>
<td>32.5</td>
<td>34.3</td>
</tr>
<tr>
<td>8</td>
<td>33.3</td>
<td>32.6</td>
</tr>
<tr>
<td>15</td>
<td>6.0</td>
<td>5.7</td>
</tr>
<tr>
<td>23</td>
<td>20.9</td>
<td>20.0</td>
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<tr>
<td>34</td>
<td>5.6</td>
<td>6.1</td>
</tr>
<tr>
<td>42</td>
<td>4.5</td>
<td>3.7</td>
</tr>
<tr>
<td>87</td>
<td>-1.5</td>
<td>-.75</td>
</tr>
</tbody>
</table>
4.3 Southwell Plot

The second method used for the evaluation of the critical load was the Southwell plot: the results from the graphs are summarized in table 4.3.

The deflection components plotted were displacements and rotations in a direction normal to the surface of the cone: \( \Delta_N \) and \( \omega_N \) respectively, as sketched below.

A few typical Southwell plots are shown in figs. 4.4 and following.

For these deflection components (i.e. \( \Delta_N \) and \( \omega_N \)) it was observed that in all graphs, the results of which are shown in table 4.3, if the Southwell plot was a straight line over a range of deflections of amplitude equal to at least 60% of the value of the largest deflection used, on the side of the largest deflection, then the corresponding critical load was within 15% of the value given by the determinant plot for the same case.

This condition could then be taken as a practical criterion for the use of the Southwell plot. Actually, in the majority of the plots using \( \Delta_N \) and \( \omega_N \), if the above condition was met, the Southwell and determinant plots gave values in agreement within 10%.
TABLE 4.3. DETERMINATION OF STRUCTURE CRITICAL LOAD
BY SOUTHWELL PLOT

(i) Plotting joint deflections normal to the surface of the cone ($A_N$)

A) Members' end fixity condition (a) (fully fixed)

<table>
<thead>
<tr>
<th></th>
<th>Exact geometry</th>
<th></th>
<th>Random geometry</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Critical load (lb.)</td>
<td>Joint</td>
<td>Trigger System</td>
<td>Critical load (lb.)</td>
</tr>
<tr>
<td>Theory 1</td>
<td></td>
<td></td>
<td></td>
<td>Theory 2</td>
</tr>
<tr>
<td>2500</td>
<td>28</td>
<td>9</td>
<td>1360*</td>
<td>19</td>
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</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>5240</td>
<td>23</td>
</tr>
<tr>
<td>Theory 2</td>
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<td>28</td>
<td>1552</td>
<td>18</td>
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<td>3680</td>
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<td>18</td>
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<td></td>
<td></td>
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</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>2256</td>
<td>23</td>
</tr>
</tbody>
</table>

Theory 1: Stability functions only.
Theory 2: Stability functions plus chord rotation and bowing.

*Torsional rigidity of members put equal to zero (GJ = 0).

(cont'd ...)
B) Members' fixity condition (b): fixed about strong axis, hinged about weak axis.

<table>
<thead>
<tr>
<th></th>
<th>Exact geometry</th>
<th>Random geometry</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Critical load (lb.)</td>
<td>Joint</td>
</tr>
<tr>
<td>Theory 1</td>
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<tr>
<td>2750</td>
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</tr>
<tr>
<td>1665</td>
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<tr>
<td>Theory 2</td>
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<td>1230</td>
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<td>5</td>
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<td>5</td>
</tr>
<tr>
<td>1648</td>
<td>38</td>
<td>5</td>
</tr>
</tbody>
</table>

Theory 1: Stability functions only.
Theory 2: Stability functions plus chord rotation and bowing.

(cont'd ...)
(ii) Plotting joint rotations about an axis normal to the surface of the cone ($\omega_N$)

A) Members' fixity condition (a)

<table>
<thead>
<tr>
<th></th>
<th>Exact geometry</th>
<th></th>
<th>Random geometry</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
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<td>Critical load (lb.)</td>
<td>Joint</td>
<td>Trigger System</td>
<td>Critical load (lb.)</td>
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<td>960</td>
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<td>8</td>
<td>930</td>
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<td>8</td>
<td>1000</td>
</tr>
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<td></td>
<td>922</td>
<td>18</td>
<td>8</td>
<td>1000*</td>
</tr>
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<tr>
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<td>920</td>
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<td>1096</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1168</td>
</tr>
</tbody>
</table>

Theory 1: Stability functions only.

Theory 2: Stability functions plus chord rotation and bowing.

*Torsoinal rigidity of members equal zero (GJ = 0).
JOINT No. 18

$\Delta_N$

\[ \frac{\Delta_N \times 10^3}{\lambda} \]

FIG. 4:4 - SOUTHWELL PLOT, EXACT GEOMETRY.
Fixity Condition (a)
Trigger System 8
Theory 1
$P_0 = 320\, \text{lb.}$

\[
\lambda_{cr} = \frac{1.44 \times 10}{1 \times 5} = 2.88
\]

\[
P_{cr} = 320 \times 2.88 = 922\, \text{lb.}
\]

FIG. 4:5—SOUTHWELL PLOT. EXACT GEOMETRY.
Joint No. 18

$\Delta_N$

**FIG. 4-6 - SOUTHWELL PLOT. EXACT GEOM.**
Joint No. 18

$\Delta_N$

$\frac{\Delta N (x10^3)}{\lambda}$

Fixity \( (b) \)
Theory \( 1 \)
Trigger System \( 3 \)
\( P_0 = 320 lb \).

\[ \lambda_{cr} = \frac{0.81 \times 40}{1 \times 10} = 3.24 \]

\[ P_{cr} = 320 \times 3.24 = 1038 lb. \]

FIG. 4 : 7 - SOUTHWELL PLOT. RANDOM GEOM.
JOINT No. 13

$\Delta_N$

Note: $\Delta_N$+ve downwards

Fixity Condition (b)
Trigger System 9
Theory (1)
$P_0 = 320\text{lb.}$

$\lambda_{cr} = \frac{158 \times 40}{1 \times 20} = 3.16$

$P_{cr} = 320 \times 3.16 = 1010\text{lb.}$

FIG. 4-8 - SOUTHWELL PLOT. RANDOM GEOMETRY
JOINT No. 28

$\Delta_N$

Fixity Condition (b)  
Trigger System 9  
Theory (1)

$P_c = 320 \text{lb.}$

$\lambda_{cr} = \frac{1.23 \times 50}{1 \times 20} = 3.075$

$P_{cr} = 320 \times 3.075 = 985 \text{lb.}$

FIG. 4:9 – SOUTHWELL PLOT. RANDOM GEOMETRY.
Other deflection components were investigated, namely the meridional and circumferential components, but for these the above criterion does not apply, i.e. some Southwell plots, straight over a range of deflections as specified above, yielded a value of critical load more than 30% higher than the correct value.

This may be explained by the fact that these latter deflections were not components of the lowest critical mode shape. Actually, for the structure with exact geometry and members perfectly fixed to the joints (condition (a)), it was possible to find a disturbance force system, namely system 8, consisting of two couples about the Y-axis acting at joints 18 and 28, exciting a mode shape in which all the joint rotations plotted (see table 4.3 (ii)) yielded practically the same value of critical load. The corresponding mode shape is given below in fig. 4:10.

![Critical Mode Shape](image_url)

**FIG. 4:10 - CRITICAL MODE SHAPE. FIXITY CONDITION (A)**
The fact that the joint rotations showed a large rate of increase at approximately the same critical load indicates that buckling is a global phenomenon, and the structure as a whole distorts.

In this case (i.e. fixity condition (a)) buckling seems to be initiated by bending of the main members about the weak axis of the cross-section, which may be called "in plane" buckling. This may explain why for this condition the normal deflection $\Delta_N$ is not applicable, in the sense that, in the range of loads applied to the structure (below the first critical load), it does not satisfy the previously mentioned criterion, i.e. Southwell plots are not straight lines over a wide range. The values of critical load shown in table 4:3 for these cases were obtained by drawing a straight line interpolating through the last few points of the graph, as shown in fig. 4:4.

There is the possibility that, if the analysis could be carried beyond the first critical load, these graphs would also become straight lines over a wide range, indicating a higher critical mode. It should be possible to perform this check by determining the first few eigenvectors of the structure stiffness matrix; the process, however, would be lengthy given the size of the matrix to be analysed.

In a few cases the torsional rigidity of the members was put equal to zero, to see whether this affected the graphs, since the Southwell method was developed from the eigenvalue formulation of the beam-column problem (4, 12), where the torsional rigidity of a member, $GJ$, does not intervene. This had a certain effect on one plot with $\Delta_N$, no practical effect for $\omega_N$. 
4.4 Load-deflection curves

Figs. 4:11 and following show some typical load-deflection curves for the two fixity conditions (a) and (b).

These curves are consistent with the previous results, because in the first case (perfectly rigid-jointed spaceframe) the deflection component $\Delta_N$ appears to be largely unaffected by the first critical load, whereas the rotation $\omega_N$ clearly indicates the onset of buckling.

Under condition (b) instead, $\Delta_N$ becomes a good indication of buckling, and a change of sign for the deflection, probably depending on the trigger system used, can be noted as the load approaches the critical level.

Fig. 4:15 shows the pattern of the displacements $\Delta_N$ along the middle ring, near buckling, for fixity condition (b) and various trigger systems. It is remarkable that the same wavy pattern, essentially due to the particular arrangement of the diagonals, was also observed in the experimental tests.

It can also be noted that a small disturbance system can cause a large distortion in the displacement pattern.

4.5 Partial loading

To conclude the results for the original cone, an investigation was made of a U. D. L. covering only half of the structure.

Fig. 4:16 shows that the value of $\lambda_{cr}$ is slightly less than for the corresponding case of a U. D. L. over the whole structure, namely
JOINT No. 18

$\Delta_N$

FIG. 4: II - LOAD - DEFLECTION CURVE. FIXITY (A).
FIG. 4: 12 - LOAD-DEFLECTION CURVE. FIXITY (A).
FIG. 4: LOAD DEFLECTION CURVE. FIXITY (B).

Random Geometry
Trigger System 9

1. Stability Functions only
2. + Chord Rotation and Bowing

Note: $\Delta_N^+$ ve downward
FIG. 4-14 - LOAD-DEFLECTION CURVE. FIXITY (B).

JOINT No. 28

$\Delta_N$

Random Geometry Trigger System 9

1. Stability Functions only
2. Chord Rotation and Bowing

Critical loads

Load Factor $\lambda (P = 120$ lb. $)$

Displacement, inches

$\Delta_N (x10^3)$
Fixity (b)
Theory
Stability Functions only
$P_0 = 120$ lb

$\lambda = 8.16$ ($\lambda_{cr} = 8.30$)

$\lambda = 8.45$ ($\lambda_{cr} = 8.72$)

Angular Position

Displacement, inches

Note: $\Delta_N$ Positive Downward

FIG. 4.15 - DISPLACEMENT $\Delta_N$ ALONG MIDDLE RING.
Fixity Condition (b)
Exact Geometry

FIG. 4: 16-ORIGINAL CONE, HALF SNOW LOAD.
DETERMINANT PLOTS
\[ \lambda_{cr} = 8.20 \text{ vs. } 8.62 \text{ for theory 2} \text{, and} \]
\[ \lambda_{cr} = 8.31 \text{ vs. } 8.72 \text{ for theory 1} \]

In conclusion then, the structure buckles at about the same load per square foot whether on the whole or half dome.

4.6 Conclusions about Southwell plot

The load-deflection curves allow another explanation of the different results shown in table 4.3, depending on the use of \( \Delta_N \) or \( \omega_N \): in fact, it is shown in Appendix 3 that the Southwell plot being a straight line implies the corresponding load-deflection curve being a rectangular hyperbola having the critical load for horizontal asymptote. This is seen to occur to \( \omega_N \) for fixity condition (a), and to \( \Delta_N \) for fixity condition (b) only.

This shape of load-deflection curve (i.e. rectangular hyperbola) is also consistent with the results obtained (Horne and Merchant (4), Timoshenko (17)) by using the critical modes for the series expansion of any deflection component. Then each term of the series is independently magnified as the external load approaches the corresponding critical load.

By analogy with the load-deflection curves relative to the Euler column problem, it could also be said that the Southwell plot is well applicable to instability cases of the bifurcation type. Instead, from the two-bar structure example (Appendix 2), it appears that when geometric instability (snap-through) dominates, the Southwell plot would not be a straight line and should not be applied (see fig. 4:17).
In fact the equations governing geometric instability differ from those governing Euler buckling precisely for the presence of the additional terms due to chord rotation and bowing. As long as these additional effects are negligible, it must be possible to use the Southwell plot for the evaluation of the critical load, provided the previously mentioned criterion is satisfied, and a correct set of deflections and disturbance systems is used.

However, for this type of structure, there may be difficulties
in detecting the lowest critical mode and corresponding critical load, as "in plane" and "out of plane" buckling are coupled, because of the curvature of the cone surface.
CHAPTER V

RESULTS FOR SPACEFRAMES WITH DIFFERENT HEIGHT/SPAN RATIO

5.1 Conical shapes

To check the importance of geometric nonlinearities, the critical load level for a U. D. L. over the whole structure was determined for two other cones of lower height/span ratio but with the same geometric configuration. Only the determinant plot was used in these cases.

Let $\frac{h}{s}$ be the height/span ratio for the original cone, which will be called FC0; the other two structures analysed, called FC1 and FC2 respectively, will have height/span ratios $\frac{h}{s_1} = \frac{1}{2}\frac{h}{s}$ and $\frac{h}{s_2} = \frac{1}{4}\frac{h}{s}$.

Table 5.1 shows the values of critical load for the three structures, obtained by determinant plots, which are presented in figs. 5:1 and 5:2. Only fixity condition (b) was analysed, since it was assumed that lateral buckling of the members would be prevented by purlins and decking.

Again it can be noted that the differences in the values obtained by theories 1 and 2 are negligible. However the influence of nonlinearities is appreciable, since the overall critical load drops faster than the sine of the slope angle, as it would happen for a linear theory, since the sizes of the members, hence the maximum load they could carry, have been kept constant throughout.

From a dimensional analysis point of view it can be said that for cones having the same geometric configuration and arrangement of web members, the critical axial load in a member, $N_{cr}$, will be a function
TABLE 5.1. DETERMINATION OF CRITICAL LOAD BY DETERMINANT PLOT FOR DIFFERENT HEIGHT/SPAN RATIOS

Fixity condition (b): members fixed to the joints in the strong direction, hinged in the weak direction.

Exact geometry: U. D. L. over the whole structure.

<table>
<thead>
<tr>
<th></th>
<th>Theory 1</th>
<th>Theory 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>FCO (α_o)</td>
<td>1048 lb.</td>
<td>1035 lb.</td>
</tr>
<tr>
<td>FC1 (βα_o)</td>
<td>494 lb.</td>
<td>486 lb.</td>
</tr>
<tr>
<td>FC2 (γα_o)</td>
<td>218 lb.</td>
<td>214 lb.</td>
</tr>
</tbody>
</table>

$$\alpha_o = \frac{2.21}{7.43} = \frac{1}{3.35}$$

Theory 1: Livesley stability functions.

Theory 2: Livesley stability functions plus chord rotation and bowing.
DETERMINANT PLOT.

Fixity condition (b)
Exact geometry
No Triggers

Theory 1: Stability Functions only
Theory 2: + Chord Rotation and Bowing

FIG. 5:1 - FC1 CONE · 1/2 ORIGINAL HEIGHT
DETERMINANT PLOT.

No Triggers
Fixity condition (b)
Exact geometry

FIG. 5:2 - FC2 CONE • 1/4 ORIGINAL HEIGHT
only of its stiffness, its length and the height/span ratio of the cone, \( \alpha \). Thus we can write:

\[ N_{cr} = f[AE, EI, L, \alpha] \quad \ldots \ldots \ldots \ldots \quad [*] \]

where \( L = \) joint-to-joint length of a member,

\( \alpha = \) height/span ratio of the cone.

Having a relationship among 5 variables, by Buckingham's \( \Pi \)-theorem, this can be reduced to a relationship involving only 3 dimensionless parameters; for instance we may choose:

\[ \frac{N_{cr}L^2}{EI} , \frac{AEI^2}{EI} , \alpha , \quad \text{obtaining} \]

\[ \frac{N_{cr}L^2}{EI} = g\left(\frac{L}{r}, \alpha\right) \quad \ldots \ldots \ldots \ldots \quad [**] \]

where \( \frac{L}{r} \) is the slenderness ratio of the member, being \( Ar^2 = I \).

Now we can define an effective length of a member, \( L_e = kL \), such as

\[ N_{cr} = \frac{\pi^2 EI}{(L_e)^2} \quad \text{Substituting into [**] we obtain the simple} \]

relationship

\[ \frac{L_e}{L} = g'\left(\frac{L}{r}, \alpha\right) \quad \text{which can be used to present the effects of} \]

a change in height/span ratio for similar cones.

This is shown numerically in tables 5.2 to 5.4 and graphically in fig. 5:4.

It is to be noted that rib members behave in two different ways, because of the arrangement of the diagonals. These would be
TABLE 5.2. ORIGINAL CONE FCO. \( \alpha = \frac{1}{3.35} \)

Fixity condition (b). Exact geometry. Theory 1.

<table>
<thead>
<tr>
<th>Member Number</th>
<th>( \frac{L}{r} )</th>
<th>( N_{cr} ) (lb.)</th>
<th>( K = \frac{L_e}{L} )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Rib Members:</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Type 1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>41.6</td>
<td>105.8</td>
<td>2.46</td>
</tr>
<tr>
<td>27</td>
<td>55.5</td>
<td>47.0</td>
<td>2.08</td>
</tr>
<tr>
<td>52</td>
<td>60.9</td>
<td>11.8</td>
<td>3.63</td>
</tr>
<tr>
<td>77</td>
<td>74.4</td>
<td>4.66</td>
<td>4.28</td>
</tr>
<tr>
<td>Type 2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>41.6</td>
<td>51.5</td>
<td>3.52</td>
</tr>
<tr>
<td>28</td>
<td>55.5</td>
<td>34.8</td>
<td>2.41</td>
</tr>
<tr>
<td>53</td>
<td>60.9</td>
<td>24.25</td>
<td>2.53</td>
</tr>
<tr>
<td>78</td>
<td>74.4</td>
<td>13.4</td>
<td>2.52</td>
</tr>
<tr>
<td><strong>Ring Members:</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>93.5</td>
<td>67.9</td>
<td>1.37</td>
</tr>
<tr>
<td>43</td>
<td>88.3</td>
<td>58.1</td>
<td>1.17</td>
</tr>
<tr>
<td>68</td>
<td>78.5</td>
<td>35.45</td>
<td>1.125</td>
</tr>
<tr>
<td><strong>Diagonals:</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>98.4</td>
<td>18.92</td>
<td>1.77</td>
</tr>
<tr>
<td>38</td>
<td>114.5</td>
<td>14.61</td>
<td>1.47</td>
</tr>
<tr>
<td>63</td>
<td>130.0</td>
<td>4.05</td>
<td>2.13</td>
</tr>
</tbody>
</table>

\( N_{cr} = \lambda_{cr} N_0 \), where \( N_0 \) = axial force in a member at load level \( \lambda = 1 \), determined by linear spaceframe analysis.

\( L = \text{member length (joint-to-joint)} \); \( L_e = \pi \sqrt{\frac{EI}{N_{cr}}} \)
TABLE 5.3. CONE FC1. \( a_1 = \frac{1}{2} a_o = \frac{1}{6.7} \)

Fixity condition (b). Exact geometry. Theory 1.

<table>
<thead>
<tr>
<th>Member Number</th>
<th>( \frac{L}{r} ) (in.)</th>
<th>( N_{cr} ) (lb.)</th>
<th>( K = \frac{L_e}{L} )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Rib Members:</strong></td>
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<td></td>
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<tr>
<td><strong>Type 1</strong></td>
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<tr>
<td>2</td>
<td>36.5</td>
<td>85.0</td>
<td>3.12</td>
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<tr>
<td>27</td>
<td>48.6</td>
<td>39.8</td>
<td>2.57</td>
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<td>53.3</td>
<td>8.6</td>
<td>4.85</td>
</tr>
<tr>
<td>77</td>
<td>65.2</td>
<td>4.33</td>
<td>5.05</td>
</tr>
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<td><strong>Type 2</strong></td>
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<td></td>
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<tr>
<td>3</td>
<td>36.5</td>
<td>37.0</td>
<td>4.75</td>
</tr>
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<td>28</td>
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<td>53.3</td>
<td>19.5</td>
<td>3.22</td>
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<td>12.3</td>
<td>3.0</td>
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<td><strong>Ring Members:</strong></td>
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<td></td>
</tr>
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<td>93.5</td>
<td>49.95</td>
<td>1.6</td>
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<td>88.3</td>
<td>55.5</td>
<td>1.2</td>
</tr>
<tr>
<td>68</td>
<td>78.5</td>
<td>32.4</td>
<td>1.18</td>
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<tr>
<td><strong>Diagonals:</strong></td>
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<td>94.0</td>
<td>15.27</td>
<td>2.06</td>
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<td>38</td>
<td>107.5</td>
<td>14.7</td>
<td>1.56</td>
</tr>
<tr>
<td>63</td>
<td>118.7</td>
<td>4.6</td>
<td>2.19</td>
</tr>
</tbody>
</table>
TABLE 5.4. CONE FC2. \( a_2 = \frac{1}{4} a_0 = \frac{1}{13.4} \)

Fixity condition (b). Exact geometry. Theory 1.

<table>
<thead>
<tr>
<th>Member Number</th>
<th>( \frac{L}{r} )</th>
<th>( N_{cr} ) (lb.)</th>
<th>( K = \frac{L_e}{L} )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Rib Members:</strong></td>
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<td></td>
<td></td>
</tr>
<tr>
<td><strong>Type 1</strong></td>
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<td></td>
</tr>
<tr>
<td>2</td>
<td>35.15</td>
<td>64.6</td>
<td>3.73</td>
</tr>
<tr>
<td>27</td>
<td>46.9</td>
<td>33.72</td>
<td>2.9</td>
</tr>
<tr>
<td>52</td>
<td>51.3</td>
<td>8.36</td>
<td>5.11</td>
</tr>
<tr>
<td>77</td>
<td>62.6</td>
<td>4.69</td>
<td>5.05</td>
</tr>
<tr>
<td><strong>Type 2</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>35.15</td>
<td>26.1</td>
<td>5.85</td>
</tr>
<tr>
<td>28</td>
<td>46.9</td>
<td>21.92</td>
<td>3.6</td>
</tr>
<tr>
<td>53</td>
<td>51.3</td>
<td>16.11</td>
<td>3.68</td>
</tr>
<tr>
<td>78</td>
<td>62.6</td>
<td>11.0</td>
<td>3.3</td>
</tr>
<tr>
<td><strong>Ring Members:</strong></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>93.5</td>
<td>27.76</td>
<td>2.14</td>
</tr>
<tr>
<td>43</td>
<td>88.3</td>
<td>43.9</td>
<td>1.35</td>
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<td>68</td>
<td>78.5</td>
<td>27.7</td>
<td>1.28</td>
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<tr>
<td><strong>Diagonals:</strong></td>
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<td></td>
</tr>
<tr>
<td>13</td>
<td>92.8</td>
<td>9.85</td>
<td>2.6</td>
</tr>
<tr>
<td>38</td>
<td>105.5</td>
<td>11.73</td>
<td>1.79</td>
</tr>
<tr>
<td>63</td>
<td>115.6</td>
<td>4.41</td>
<td>2.3</td>
</tr>
</tbody>
</table>
stressless and every rib (e.g. AB and CD from fig. 5:3) would behave identically, if the structure were analysed as a space truss, because of the rotational symmetry of the structure and of the loading system (UDL).

If this is done, it can be seen that the joints of a rib like AB move outward from the surface of the cone, while the joints of adjacent ribs like CD move inward. It is to be noted that, even for small strains, as must occur in the elastic range, the joint displacements normal to the cone surface are quite large, because adjacent ring members are almost collinear.

To restore slope continuity of the ring members (frame action), shear forces, acting normal to the surface of the cone, are required at the ends of these members. The magnitude of these forces is about 25% of
EFFECT OF VARYING HEIGHT/SPAN RATIO.

(For L/r shown in Tables 5.2 to 5.4)

\[ k = \frac{L_e}{L} \]

\[ \lambda_2 = \frac{\lambda_0}{4} \]
\[ \lambda_1 = \frac{1}{2} \lambda_0 \]
\[ \lambda_0 \]

Height/Span Ratio, \( \lambda \)

\[ L_e = \pi \sqrt{\frac{EI}{N_{cr}}} \]
\[ N_{cr} = \lambda_{cr} N_0 \]
\[ L_0 = \frac{1}{3.35} \]
\[ L = \text{member length} \]

FIG. 5: 4 - CONICAL SPACEFRAMES
the corresponding external joint load. Because of the small angle between the plane determined by two diagonals meeting at a joint (say No. 22) and the plane tangent to the cone at the same joint, these shear forces arising from frame action produce rather large compressive loads in the diagonals of the two lower arrays. These loads carried by the diagonals, in turn, modify the distribution of axial loads in two adjacent ribs like AB and CD.

In summary then, the secondary stresses set up in this continuous space truss are much larger than the secondary stresses in an ordinary plane truss because members at a joint are almost coplanar.

This accounts for the two types of rib members shown in the following tables and figures: type 1 for ribs like CD and type 2 for ribs like AB.

The high values of k obtained indicate that, under the given conditions of loading and member fixity, buckling of the structure is a global phenomenon, involving the whole structure rather than individual members.

In some cases, e.g. for the upper rib members of type 1, the high value of k is due to overdesign, i.e. the member is not stressed to its full capacity. This happens because the actual cone had to support a superstructure housing mechanical systems for the handling of potash (2).

Since it has been shown that the effect of geometric nonlinearities is always minor for these conical spaceframes, an eigenvalue analysis could be carried out to detect the actual critical mode shapes: this could help explain the results obtained for the effective lengths.

5.2 Spherical shape

To investigate the structural advantages offered by a doubly
curved shape with respect to a singly curved one, an analysis was carried out for a spherical spaceframe dome, having the same height and span as the conical spaceframe FC1 (i.e. \( \alpha = \frac{1}{6.7} \)) as shown in fig. 5:5.

![Diagram showing spherical and conical spaceframes](image)

**FIG. 5:5 - SPHERICAL SPACEFRAME**

This spherical spaceframe will be called FS1.

The critical load was determined by determinant plots, shown in fig. 5:6. Again theory 1 and theory 2 yield practically the same value.

Table 5.5 compares the results obtained by theory 1 for the conical and spherical dome respectively.

Again it can be noted that \( k \) is generally greater than one, indicating that buckling involves the whole structure and not individual members.

For the rib members the reduction in the values of \( k \) for the spherical dome with respect to the conical would permit a reduction in member sizes of the order of 50%.
DETERMINANT PLOT.

Fixity Condition (b)
Exact Geometry
No Triggers
U.D.L.

1. Livesley Stability Functions only
2. \[ - - - - - - - + \]
   Chord Rotation and Bowing

FIG. 5 : 6 - SPHERICAL DOME SF1
**TABLE 5.5. COMPARISON OF A CONICAL AND A SPHERICAL SPACEFRAME**

Fixity condition (b). Exact geometry. U. D. L.

<table>
<thead>
<tr>
<th>Critical load factor, $\lambda_{cr}$ ($P_0 = 120$ lb.)</th>
<th>Conical dome FC1</th>
<th>Spherical dome FS1</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>4.11</strong></td>
<td><strong>10.81</strong></td>
<td></td>
</tr>
</tbody>
</table>

| Overall critical load (vertical)                       | **494 lb.**      | **1300 lb.**      |

<table>
<thead>
<tr>
<th>Member Number</th>
<th>$\frac{L}{r}$</th>
<th>$N_{cr}$ (lb.)</th>
<th>$K = \frac{L_e}{L}$</th>
<th>$\frac{L}{r}$</th>
<th>$N_{cr}$ (lb.)</th>
<th>$K = \frac{L_e}{L}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rib Members:</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>36.5</td>
<td>85.0</td>
<td>3.12</td>
<td>37.2</td>
<td>150.2</td>
<td>2.3</td>
</tr>
<tr>
<td>27</td>
<td>48.6</td>
<td>39.8</td>
<td>2.57</td>
<td>48.75</td>
<td>96.0</td>
<td>1.65</td>
</tr>
<tr>
<td>53</td>
<td>53.3</td>
<td>19.5</td>
<td>3.22</td>
<td>53.5</td>
<td>51.8</td>
<td>1.98</td>
</tr>
<tr>
<td>78</td>
<td>65.2</td>
<td>12.3</td>
<td>3.0</td>
<td>66.5</td>
<td>35.5</td>
<td>1.73</td>
</tr>
</tbody>
</table>

| Ring Members: |               |                |                     |               |                |                     |
| 18            | 93.5          | 49.95          | 1.6                 | 95.4          | 42.0           | 1.71                |
| 43            | 88.3          | 55.5           | 1.2                 | 91.5          | 72.6           | 1.02                |
| 68            | 78.5          | 32.4           | 1.18                | 82.1          | 61.8           | .82                 |

| Diagonals:    |               |                |                     |               |                |                     |
| 13            | 94.0          | 15.27          | 2.06                | 95.2          | 25.6           | 1.57                |
| 38            | 107.5         | 14.7           | 1.56                | 109.1         | 17.15          | 1.43                |
| 63            | 118.7         | 4.6            | 2.19                | 120.3         | 7.37           | 1.71                |

**height/span ratio** $\alpha = \frac{1}{6.7}$

$N_{cr} = \lambda_{cr} N_o$; $N_o = $ axial force in a member at $\lambda=1$ determined by linear spaceframe analysis

$L_e = \Pi \sqrt{\frac{E I}{N_{cr}}}$
However, this advantage from a structural point of view would be partially offset by fabrication complications due to the fact that the ribs would no longer be straight throughout.

Also, partial loadings may cause tension in the main members, increasing the complexity of the connections.

Finally a spherical shape would require a greater quantity of decking material and longer members than the corresponding conical shape (but it would provide a greater volume of covered, available space).

In conclusion, the possibility of using a spherical dome instead of a conical one seems worthy of detailed investigation.
CHAPTER VI

CONCLUSIONS AND RECOMMENDATIONS

The overall critical load for the original cone determined in this analysis was found to be in satisfactory agreement with the value obtained in the previous model tests.

The results of this investigation show also that, for a space-frame of this type and for practical values of height/span ratios, a stability analysis in the elastic range up to the first critical load can be successfully carried out using only Livesley stability functions: the additional effects due to chord rotation and bowing are generally negligible. Actually it should be noted that Livesley's version of stability functions (3, 4) contains sway terms of the type \( N \rho \), where \( N \) is the axial load in a member and \( \rho \) is the chord rotation, as shown in Appendix 1. Strictly speaking, these terms should be included in the geometric part of the stiffness matrix, as it is done in ref. (10). This may explain partly the closeness of results obtained by theories 1 and 2, since Livesley stability functions were used for theory 1.

An investigation of the post-buckling behaviour of this type of spaceframes (statically determinate in their primary stress system) seems to be of little practical interest, since the structure does not possess any adequate reserve strength beyond the first critical load.

For other spaceframes, namely those with redundant members, it may be interesting to follow the deformation of the structure past the lowest critical load: then displacements and rotations may become very
large and the present analysis would no longer be applicable.

In such a case, as well as in the general case of structures exhibiting truly large deflections while remaining elastic, the best way to tackle the problem seems to be to resort to a finite element formulation with an incremental step-by-step solution by means of Taylor's expansion theorem, as shown in references (13) and (14).

If the problem requires investigation, the lateral-torsional stability of an individual member can be checked separately in a second stage of the analysis, using for instance the procedure outlined in references (15) and (16).

From the present investigation it appears also that, when performing a numerical stability analysis, the Southwell plot should only be used together with the determinant plot, since it can be misleading, especially for the difficulty, in a complex structure, of finding a correct disturbing force system and a corresponding set of deflections, truly representative of the lowest buckling mode. For this purpose, a set of geometric imperfections together with several different perturbation force systems should be used for a reliable investigation. If the Southwell plots thus obtained are straight lines upward from about 40% of the value of the largest deflection used, and yield consistent values for the critical load, then such a value can be relied upon. In these cases the Southwell plot can give useful information about the sensitivity of the structure to different perturbations, and can be regarded as an alternate method of finding the eigenvector corresponding to the lowest eigenvalue of the structure stiffness matrix.

It can be noted that the difficulty mentioned above of finding
a correct disturbance system, does not occur in an actual model test, since natural imperfections in the construction of the model or in the loading system usually provide a sufficient perturbation for the use of the Southwell plot.

The concept of effective length of a member with associated allowable stress can be useful in the preliminary design of a structure of this type. However, since buckling is generally not due to failure of an individual member, but rather is a global phenomenon involving the whole structure, it cannot be safely assumed that the unsupported length of a member is the joint-to-joint length, as it is often taken for plane trusses. For instance, for the conical spaceframes analysed in the present work, the ratio \( \frac{L_e}{L} \) is seen to vary approximately from 2 to 4 for the rib members and from 1 to 2 for the ring members, according to the height/span ratio. Thus for each given structure of this type (spaceframe domes) it is advisable to carry out a stability analysis with Livesley stability functions.

An investigation should also be made of partial loadings and, possibly, of the effect of geometric imperfections, since they can cause a considerable change in the axial load distribution in the members, especially close to the critical load level.

To conclude, with reference to fig. 6:1, and quoting from reference (4), p. 45, "Since, in structures subject to instability, the margin of safety as measured by a load factor may be markedly and dangerously smaller than the 'factor of safety' measured by a stress
ratio, it is important that load factors, not stress factors, should be used in design."
The assumptions made in this derivation, as listed in Chapter 2, are:

1. The material is linear elastic.
2. Each member is prismatic and homogeneous.
3. Loads are applied only at the ends of a member.
4. Shear deformations are neglected.

5. Linear strains and squares of the rotations are of the same order of magnitude and small compared to one.

6. Torsion-flexure coupling and warping restraint are neglected.

With these assumptions it can be shown (ref. (11), page 466) that the Eulerian and Lagrangian description of the deformation differ only by higher order infinitesimals.

Then we can write, for the two-dimensional case:

\[
\begin{align*}
(\varepsilon)_t &= \varepsilon - \zeta \kappa \\
\varepsilon &= \frac{\partial u}{\partial x} + \frac{1}{2} \omega^2 \\
\kappa &= \frac{\partial \omega}{\partial x} \\
\omega &= \frac{\partial v}{\partial x}
\end{align*}
\]

where

\(\varepsilon\) = the longitudinal strain of the center-line of the member,

\(u, v\) = translation components with respect to \(x, y\) directions,

\(\omega\) = rotation of a cross-section about the \(z\)-axis,

\(\kappa\) = curvature of the center-line of the member.

With our assumptions \(\frac{\partial u}{\partial x}, \omega^2 \ll 1\).

Then the axial force \(N\) and bending moment \(M\) result: (in the absence of initial strains)

\[
\begin{align*}
N &= AE \varepsilon \\
M &= EI \kappa
\end{align*}
\]

and the transverse shear force \(V\) is:

\[
V = - \frac{dM}{dx}
\]
We can now apply the principle of virtual displacements, in the form:

$$\delta W_1 = \delta W_e$$, where $\delta W$ denotes virtual work.

In our case:

$$\int_0^L (N\delta e + M\delta \kappa) \, dx = F_{Bx} \delta u_B + F_{By} \delta v_B + M_B \delta \omega_B$$

$$+ F_{Ax} \delta u_A + F_{Ay} \delta v_A + M_A \delta \omega_A \ldots \text{ [4]}$$

Note that, using the Lagrangian description, where all variables are referred to the initial state, we can perform the integration at the L.H.S. of [4] along the undeformed position of the member. Moreover we can substitute everywhere total derivatives for partial derivatives, since $u,v,w$, being displacement components of points of the center-line, are now functions of $x$ only.

Now let $\eta(x)$ and $\alpha(x)$ be (small) variations of $u(x)$ and $w(x)$; then $\delta \epsilon = \delta \left( \frac{du}{dx} + \frac{1}{2} \omega^2 \right) = \frac{d\eta}{dx} + \omega \alpha$.

We also have:

$$\delta \kappa = \delta \left( \frac{d\omega}{dx} \right) = \frac{d\alpha}{dx} \quad \text{and letting}$$

$$\beta(x) \text{ be the variation of } v(x), \text{ we have:}$$

$$\delta \omega = \delta \left( \frac{dv}{dx} \right) = \frac{d\beta}{dx} \quad .$$

Substituting in the L.H.S of equation [4] we obtain:
\[ \int_0^L (N \delta \varepsilon + M \delta \kappa) \, dx = \int_0^L \left[ N \left( \frac{dn}{dx} + \omega \alpha \right) + M \frac{dM}{dx} \right] \, dx \]

integrating by parts

\[ = \left. N \eta \right|_0^L + M \alpha \left|_0^L \right. - \int_0^L (\eta \frac{dN}{dx} - N \omega \alpha + \alpha \frac{dM}{dx}) \, dx \]

... [5]

We now notice that \( \eta(0) = \delta u_A \), \( \eta(L) = \delta u_B \), \( \alpha(0) = \delta \omega_A \), \( \alpha(L) = \delta \omega_B \), \( \beta(0) = \delta v_A \), \( \beta(L) = \delta v_B \), and that

\[ \int_0^L \alpha (N \omega - \frac{dM}{dx}) \, dx = \int_0^L \beta \left( \frac{d \omega}{dx} \right) \, dx \]

Thus we get: (substituting back into [5])

\[ \int_0^L (N \delta \varepsilon + M \delta \kappa) \, dx = N_B \delta u_B - N_A \delta u_A + M(L) \delta \omega_B - M(0) \delta \omega_A \]

\[ + (N \omega - \frac{dM}{dx}) B \delta v_B - (N \omega - \frac{dM}{dx}) A \delta v_A - \int_0^L \eta \frac{dN}{dx} \, dx \]

\[ - \int_0^L \left[ \beta \frac{d}{dx} \left( N \omega - \frac{dM}{dx} \right) \right] \, dx \] .......................... [6]

Comparing [6] with [4], for arbitrary virtual displacements, and by Lagrange's lemma (e.g. ref. (11), p. 273), we must have:

\[ \frac{dN}{dx} = 0 \] .......................... [7]

\[ \frac{d}{dx} (N \omega - \frac{dM}{dx}) = 0 \] .......................... [8]
and the boundary conditions:

\[
\begin{align*}
F_{Ax} &= -N_A \\
F_{Ay} &= -(N\omega)_A + \left( \frac{dM}{dx} \right)_A = -(N\omega)_A - V_A \\
M_A &= -M(0)
\end{align*}
\]

and

\[
\begin{align*}
F_{Bx} &= N_B \\
F_{By} &= (N\omega)_B - \left( \frac{dM}{dx} \right)_B = (N\omega)_B + V_B \\
M_B &= M(L)
\end{align*}
\]

at \( x=0 \)

It should be noted that the term involving the shear force \( V \) is not present in the expressions for \( F_{Ax} \) and \( F_{Bx} \). It can be shown that this is a consequence of the initial assumptions:

\[
\epsilon = \frac{du}{dx} + \frac{1}{2}\omega^2, \quad \kappa = \frac{d^2v}{dx^2} \quad \text{and} \quad V = -\frac{dM}{dx}
\]

or that \( \omega^2 \ll 1 \).

If the full expressions for \( \epsilon \) and \( \kappa \) are used, and taking \( V = -\frac{dM}{ds} \), then a shear force component will appear in the horizontal components of the end-forces.

The above derivation shows one of the advantages of the variational methods in obtaining automatically the differential equations and the boundary conditions for a given problem.

We can now follow ref. (10) in obtaining:

\[
N = \text{const.} = F_{Bx} \quad \text{[from [7]]}
\]

\[
\epsilon = \frac{N}{AE} = \frac{du}{dx} + \frac{1}{2}\omega^2
\]

Hence:

\[
F_{Bx} = \frac{AE}{L} (u_B - u_A) + \frac{AE}{2L} \int_0^L \omega^2 \, dx \quad \ldots \ldots \ldots \ldots \ldots \quad [9]
\]
The second term at the right-hand side of [9] includes the effects of chord rotation and bowing.

Now [8] becomes:

\[
\frac{d^2M}{dx^2} - N \frac{d\omega}{dx} = 0
\]

or

\[
\frac{d^2M}{dx^2} - N \frac{d^2v}{dx^2} = 0,
\]

where \( M = EI \frac{d^2v}{dx^2} \),

which has for solution:

\[
\frac{v}{L} = -\frac{1}{\phi} \left( D_{21} \sin \phi \frac{x}{L} + D_{22} \cos \phi \frac{x}{L} \right) + D_{23} \frac{x}{L} + D_{24} \ldots [10]
\]

where:

\[
\phi^2 = \frac{-F_{Bx} L^2}{EI}
\]

\[
D_{22} = -\kappa_{22} \omega_A - \kappa_{22} \omega_B + \kappa_{23} \rho
\]

\[
D_{21} = \frac{\phi}{1-C} \left( \omega_B - \omega_A \right) - \frac{S}{1-C} D_{22}
\]

\[
D_{23} = \omega_A + \frac{D_{21}}{\phi}
\]

\[
D_{24} = \frac{v_A}{L} + \frac{D_{22}}{\phi^2}, \text{ in which}
\]

\[
\kappa_{21} = \frac{\phi (S-C \phi)}{2 (1-C) - S \phi} \quad ; \quad \kappa_{22} = \frac{\phi (\phi - S)}{2 (1-C) - S \phi}
\]

\[
\kappa_{23} = \kappa_{21} + \kappa_{22}
\]

and

\[
C = \cos \phi \quad \quad \quad \rho = \frac{v_B - v_A}{L}
\]

\[
S = \sin \phi
\]

Note that \( \rho \) represents the "chord rotation" of a member.

If \( F_{Bx} \) is positive, i.e. the member is under tensile axial load, \( \phi \) becomes imaginary, and hyperbolic functions substitute the trigonometric functions.
Using equation [10] it is now possible to perform the integration appearing in [9], namely

\[
\frac{AE}{2L} \int_0^L \omega^2 \, dx = \frac{AE}{2L} \int_0^L \left(\frac{dy}{dx}\right)^2 \, dx
\]

\[
= \frac{AE}{2} \left\{ D_{23}^2 + 2 \frac{D_{23}}{\phi} \left[ -SD_{21} + (1-C)D_{22} \right] \right.
\]

\[
+ \frac{D_{21}D_{22}}{\phi} \left[ -\frac{S^2}{\phi^2} + \frac{D_{21}^2}{2\phi^2} \left[ 1 + \frac{SC}{\phi} \right] \right]^2
\]

\[
+ \frac{D_{22}^2}{2\phi^2} \left[ 1 - \frac{SC}{\phi} \right] \right\}.
\]

Then we can express the end-forces on a member as functions only of the end-displacements, i.e.

\[
M_B = M(L) = EI \frac{d^2 \phi}{dx^2} \bigg|_{x=L} = \frac{EI}{L} \left[ \kappa_{22} \omega_A + \kappa_{21} \omega_B - \kappa_{23} \rho \right]
\]

\[
M_A = M(0) = \frac{EI}{L} \left[ \kappa_{21} \omega_A + \kappa_{22} \omega_B - \kappa_{23} \rho \right]
\]

\[
F_{By} = \frac{EI}{L^2} \kappa_{23} \left[ -\omega_A - \omega_B + 2\rho \right] + \rho F_{Bx}
\]

\[
F_{Ay} = -F_{By}
\]

\[
F_{Bx} = \frac{AE}{L} (u_B - u_A) + \frac{AE}{2L} \int_0^L \omega^2 \, dx
\]

\[
F_{Ax} = -F_{Bx}
\]

The relations for the three-dimensional case are obtained associating bending in the (x,y) plane, bending in the (x,z) plane and unrestrained torsion. These relationships can be put in matrix form,
in member coordinates, as follows:

\[ \{f\} = [k]^m \{u\} - \{f_G\} \quad \ldots \ldots \ldots \ldots \ldots \quad [12] \]

where \( \{f\} \) = vector of member end-forces,

\( \{u\} \) = vector of member end-displacements,

\( [k]^m \) = member stiffness matrix, including the stability functions,

\( \{f_G\} \) = vector of geometrically nonlinear terms

\[ = \left[ \frac{AE}{2} \cdot \frac{\Delta}{L}, 0,0,0,0,0, -\frac{AE}{2} \cdot \frac{\Delta}{L}, 0,0,0,0 \right]^T \]

where \( \Delta = \int_0^L (\omega_y^2 + \omega_z^2) \, dx \)

if Livesley's form of stability functions is used in \( [k]^m \). In fact it should be noted that \( \kappa_{21} \) and \( \kappa_{22} \) previously defined correspond exactly to the stability functions \( s \) and \( s_c \) described in ref. (4), page 52.

In incremental form, equation [12] becomes

\[ \{\Delta f\} = [k]^m \{\Delta u\} + [k_G]^m \{\Delta u\} = [k_t]^m \{\Delta u\} \]

being \( [k_G]^m \{\Delta u\} = -\{\Delta f_G\} \),

\( [k_t]^m = [k]^m + [k_G]^m \),

and

\[ [k_G]^m_{12x12} = \begin{bmatrix} [k_{1G}]_{3x3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

where

\[ [k_{1G}]_{3x3} = \begin{bmatrix} 0 & \frac{AE \rho_3}{L} & -\frac{AE \rho_2}{L} \\ \frac{AE \rho_3}{L} & \frac{AE \rho_2}{L} & -\frac{AE \rho_{2,3}}{L} \\ -\frac{AE \rho_2}{L} & -\frac{AE \rho_{2,3}}{L} & \frac{AE \rho_2}{L} \end{bmatrix} \]
\[ \rho_3 = \frac{v_B - v_A}{L}, \quad \rho_2 = \frac{w_B - w_A}{L} \]
Because of the symmetry, the problem has one degree of freedom, say $u$.

By the principle of virtual work:

$$W\delta u = \delta U$$

where $U$ is the strain energy of the structure.

But now

$$\delta U = \frac{dU}{du}\delta u$$

hence

$$W = \frac{dU}{du}$$

For the two bars:

$$U = \frac{AE}{L_o} (\Delta L)^2$$

where

$$\Delta L = L_o - L$$

Therefore

$$U = \frac{AE}{L_o} \left[ 2L_o^2 + u^2 - 2uh - 2L_o \sqrt{L_o^2 + u^2} - 2uh \right]$$
Hence

\[ W = \frac{dU}{du} = \frac{2AE}{L_o} \left[ u - h - L_o \frac{u - h}{\sqrt{L_o^2 + u^2 - 2uh}} \right] \] ....... [2]

Therefore

\[ W = 0 \text{ for } u = 0, h, 2h \]

\[ \frac{dW}{du} = 0 \text{ for } u = u_{cr}, \text{ where} \]

\[ \frac{u_{cr}}{h} = 1 \pm \frac{a}{h} \sqrt{\frac{L_o^2}{a^2} - 1} \] ......... [3]

We see that indeed there are two critical points, symmetric with respect to \( u = h \).
To find $W_{cr}$, we substitute [3] back into [2], obtaining:

$$W_{cr} = 2AE\left[1 - \left(\frac{a}{L_0}\right)^2\right]^{3/2}$$

For $\frac{h}{a} = 0.10$, we have

$$\frac{u_{cr}}{h} = 0.4236 \quad \frac{1000 W_{cr}}{2AE} = 0.19054$$
The Southwell plot will be a straight line iff:

\[ \frac{d(\frac{\delta}{P})}{d\delta} = \text{a constant} = c \]

Expanding the differentiation:

\[ \frac{1}{P} \cdot 1 + \delta \cdot \frac{d\left(\frac{1}{P}\right)}{d\delta} = c \]

or \[ \frac{1}{P} - c = - \delta \cdot \frac{d\left(\frac{1}{P}\right)}{d\delta} \]

Separating the variables:

\[ - \frac{d\delta}{\delta} = \frac{d\left(\frac{1}{P}\right)}{\frac{1}{P} - c} \]
Integrating:

\[-\ln \delta = \ln \left(\frac{1}{P} - c\right) + k_1\]

where \(k_1\) is a constant of integration.

Therefore

\[\ln \delta \left(\frac{1}{P} - c\right) = -k_1\]

or

\[\delta \left(\frac{1}{P} - c\right) = e^{-k_1} = c_1 \text{ (a constant)} \] \[\] \[
\]

which in the plane \((P, \delta)\) represents a rectangular hyperbola with a horizontal asymptote for \(P = P_{cr} = \frac{1}{c}\), since then

\[\delta = \frac{c_1}{\frac{1}{P} - c} \to \infty\]

To see this better, consider the translation of axes:

\[
\begin{align*}
\delta' &= \delta' - \frac{c_1}{c} \\
P &= -P' + \frac{1}{c}
\end{align*}
\]

and substitute back into [1]:

\[\left(\delta' - \frac{c_1}{c}\right)\left(-\frac{1}{-P' + \frac{1}{c}} - c\right) = c_1\]

or

\[\left(\delta' - \frac{c_1}{c}\right)(1 + cP' - 1) = -c_1P' + \frac{c_1}{c}\]

Whence

\[\delta'cP' - c_1P' = -c_1P' + \frac{c_1}{c}\]

or

\[\delta'P' = \frac{c_1}{c^2} = \text{a constant}.\]
The above equation is the usual form for a rectangular hyperbola, in the plane \((P', \delta')\), referred to its asymptotes.

Note that \(c\) was the slope of the Southwell line, and we have thus shown that

\[
c = \frac{1}{P_{cr}}.
\]
APPENDIX 4

LIST OF REFERENCES


