RESONANCE IN PRESSURIZED PIPING SYSTEMS

by

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A B S T R A C T

A new approach, based on the transfer matrix method used in the theory of vibrations, is presented to analyze the steady-oscillatory flows, and to determine the resonating characteristics of piping systems. By linearizing the friction loss term, considering the system as distributed and assuming the discharge and pressure head fluctuations as sinusoidal, field matrices for a simple pipeline and for a parallel system are derived. A numerical technique is presented to determine the field matrix for a pipe having variable characteristics along its length. Point matrices for orifices, and for oscillating valves are obtained by linearizing the gate equation. Point matrices for the junction of the main and a branch having various boundary conditions, e.g., dead end, reservoir, orifice, oscillating valve, are also derived.

A numerical procedure is outlined for computing the resonant frequencies of piping systems. Expressions are developed to determine the frequency response of systems having periodic forcing functions, such as fluctuating pressure head; fluctuating discharge; and oscillating valve.

A number of systems commonly used in waterpower development and water supply schemes are analyzed. The results obtained by the method presented herein are in close agreement with those obtained experimentally by earlier investigators; or determined by using the method of characteristics, impedance theory, or energy concepts.

TABLE OF CONTENTS

<u>Chapter</u>	<u>Description</u>	<u>Page</u>
	ABSTRACT	ii
	LIST OF TABLES	vi
	LIST OF FIGURES	vii
	NOTATION	ix
	ACKNOWLEDGMENT	xiii
I	INTRODUCTION	1
II	TERMINOLOGY AND BLOCK DIAGRAMS	3
	2.1 Terminology	11
	2.2 Block diagrams	15
III	DERIVATION OF TRANSFER MATRICES	
	3.1 Field matrices	19
	3.2 Point matrices	32
IV	RESONANT FREQUENCIES AND FREQUENCY RESPONSE OF PIPING SYSTEMS	
	4.1 Resonant frequencies	44
	4.2 Frequency response	46
	4.3 Pressure and discharge variation along pipeline	54
V	VERIFICATION OF TRANSFER MATRIX METHOD	
	5.1 Experimental results	61
	5.2 Method of characteristics	67
	5.3 Impedance method	79
	5.4 Energy concepts	87

VI	SUMMARY AND CONCLUSIONS	98
	BIBLIOGRAPHY	100
	APPENDICES	
	A. Summary of transfer matrices	106
	B. Example	108

LIST OF TABLES

<u>No.</u>	<u>Description</u>	<u>Page</u>
I	Calculated and measured periods	66
II	Resonant frequencies of piping system of Fig. 5.4 _g	82
III	Phase angles	86
IV	Verification by energy concepts	94
V	Resonant frequencies of piping system of Fig. 5.17a	97
VI	Scheme for multiplication of transfer matrices	111

LIST OF FIGURES

<u>No.</u>	<u>Description</u>	<u>Page</u>
2.1	Instantaneous and mean discharge.	13
2.2	Single pipeline.	16.
2.3	Series connection.	16
2.4	Block diagram.	17
3.1	Pipe having variable characteristics along its length.	26
3.2	Actual and substitute pipe.	26
3.3	Parallel system.	29
3.4	Valves.	35
3.5	Branch system.	38
4.1	Suction and discharge pipeline.	49
4.2	Designation of k^{th} section on i^{th} pipe.	56
4.3	Plot of $\tan \omega x/a_2 \sim \omega x/a_2$.	59
5.1	Longitudinal profile.	62
5.2	Plot of Residual $\sim \omega$ for Toulouse system.	65
5.3	Series system with dead end.	69
5.4	Series system.	70
5.5	Branch system (side branch having reservoir).	71
5.6	Branch system (dead end side branch).	72
5.7	Branch system (side branch having orifice).	73
5.8	Branch system (side branch having oscillating valve).	74
5.9	Parallel system.	78

5.10	Effect of friction loss on frequency response of series system of Fig. 5.4a.	80
5.11	Effect of mean discharge on frequency response of series system of Fig. 5.4a.	81
5.12	$h_r^* \omega t$, $q_r^* \omega t$, and $\tau^* \omega t$ curves.	83
5.13	Series system.	88
5.14	Branch system (side branch having reservoir).	89
5.15	Branch system (dead end side branch).	90
5.16	Impedance diagram for parallel system of Fig. 5.9a.	91
5.17	Pipeline having variable characteristics along its length.	95

NOTATION

The following symbols are used in this thesis:

A	= cross-sectional area of pipe, in sq. ft;
A_v	= gate opening of valve, in sq. ft;
a	= celerity of water hammer wave, in ft/sec;
b	= constant for pipe = ℓ/a , in sec;
B	= matrix given in Eq. 3.22;
C	= constant for pipe = $a/(gA)$, in sec/ft ² ;
C_d	= coefficient of discharge;
c_1, c_2	= constants in Eq. 3.14;
c_3, c_4	= constants in Eq. 3.27;
D	= inside diameter of pipe, in ft;
E	= energy, in lb-ft;
e	= base in the natural logarithms;
F	= field matrix;
F_p	= field matrix for parallel pipes;
f	= Darcy-Weisbach friction factor;
$f_{11}, f_{12}, f_{21}, f_{22}$	= elements of field matrix;
g	= acceleration due to gravity, in ft/sec ² ;
I	= identity or unit matrix;
j	= constant = $\sqrt{-1}$;
K	= amplitude of fluctuating pressure head forcing function;
k	= amplitude of fluctuation of relative gate opening;
H	= instantaneous pressure head, in ft;
H_o	= mean or average pressure head, in ft;
h	= amplitude of pressure head fluctuation;

h^* = deviation of pressure head from mean, in ft;

h_r = $2|h_{n+1}^L|/H_o$, i.e., ratio of twice the amplitude of pressure fluctuation (twice the maximum deviation from mean) at valve and static head;

h_r^* = h^*/H_o ;

ℓ = length of pipe, in ft;

n = exponent of velocity in friction loss term;

P = point matrix;

P_{sc} = point matrix for a series connection;

P_v = point matrix for a valve or orifice having constant gate opening;

P'_{ov} = extended point matrix for oscillating valve;

P_{bde} = point matrix for dead end side branch;

P_{bres} = point matrix for side branch having constant head reservoir;

P_{borf} = point matrix for side branch with orifice;

$P_{11}, P_{12}, P_{21}, P_{22}$ = elements of point matrix;

Q = instantaneous discharge, in cu. ft/sec;

Q_o = mean or average discharge, in cu. ft/sec;

q = amplitude of discharge fluctuation;

q^* = deviation of instantaneous discharge from mean, in cu. ft/sec;

q_r = $2|q_{n+1}^L|/Q_o$, i.e., ratio of twice the amplitude of discharge fluctuation at valve and mean discharge;

q_r^* = q^*/Q_o ;

R = constant in Eq. 3.7

T_F	= period of the forcing function, in sec;
T_{th}	= theoretical period of piping system;
t	= time, in seconds;
U	= overall transfer matrix;
U'	= overall extended transfer matrix;
u_{11}, u_{12}	= elements of overall transfer matrix;
x	= distance along pipeline (positive in the downstream direction), in ft;
Z	= hydraulic impedance;
Z_c	= characteristic impedance;
z	= state vector;
z'	= extended state vector;
z_r	= normalized impedance at value = $ Z /Z_c$.

Greek letters:

γ	= specific weight of fluid, in lbs/ft ³ ;
ζ, ξ, η	= constants in Eq. 3.38;
λ_1, λ_2	= eigenvalues of matrix $B\lambda$;
μ	= constant in Eq. 3.11;
ν	= kinematic viscosity of fluid, in ft ² /sec;
τ	= relative gate opening of valve or orifice;
τ_0	= mean relative gate opening of valve or orifice;
τ^*	= deviation of τ from τ_0 ;
ω	= frequency, in radians/sec;
ω_r	= frequency ratio = ω/ω_{th} ;
ω_{th}	= theoretical frequency, in radians/sec;
ψ_m	= phase angle of the m^{th} harmonic of pump discharge curve;
	and
ϕ_m	= phase angle of the m^{th} harmonic of pressure head

Subscripts and superscripts:

1,2, ... n, n+1 = number of a section on a pipe;

in = input quantities;

L = left hand side of a section;

o = mean, or average, value;

out = output quantities;

R = right hand side of a section;

s = steady state values; and

~ = quantities for side branch.

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CHAPTER ONE

INTRODUCTION

If a piping system is subjected to a periodic excitation having a period equal to one of the natural periods of the system, severe oscillations of the pressure head and discharge develop. This may result in serious damage or failure of the system. Several incidents and accidents caused by resonance have been reported [1, 27, 28, 32, 52]¹ in the literature.

Resonance in pipelines have been studied by a number of investigators. Allievi was the first to present, in Note V of the Theory of Water Hammer [3], the analysis of resonating conditions in a simple pipeline caused by the rythmical opening and closing of a valve at the downstream end of the pipeline. By using the general solution

$$H - H_o = F(t - \frac{x}{a}) + f(t + \frac{x}{a}) \quad (1.1)$$

and

$$V - V_o = - \frac{g}{a} \{F(t - \frac{x}{a}) - f(t + \frac{x}{a})\} \quad (1.2)$$

of the simplified partial differential equations governing the unsteady frictionless flow, namely

¹ Numerals in brackets refer to corresponding items in Bibliography.

$$H_x = -\frac{1}{g} V_t \quad (1.3)$$

$$H_t = -\frac{a^2}{g} V_x \quad (1.4)$$

he proved that the relative pressure head at the valve, H_i/H_o , at odd and even periods approaches two limits Z_1^2 and Z_2^2 . In the above equations, x = distance along the pipeline, measured positive in the downstream direction; t = time; H = pressure head; H_o = static head; V = velocity of the fluid; V_o = steady state velocity of the fluid; a = celerity of the pressure wave; F, f are integration functions; and g = acceleration due to gravity. The subscripts x and t denote partial differentiation. For rythmical valve movements such that

$$\tau_o = \tau_T = \tau_{2T} = \dots = C' \quad (1.5)$$

$$\tau_{T/2} = \tau_{3T/2} = \tau_{5T/2} = \dots = 1 \quad (1.6)$$

Z_1^2 and Z_2^2 are related by the equation

$$Z_1^2 + Z_2^2 = 2 \quad (1.7)$$

The theoretical period of the pipe, $T = 4l/a$; and the relative gate opening of the valve, $\tau = (C_d A_v)/(C_d A_v)_o$. In these expressions, l = length of pipeline; C_d = coefficient of discharge; A_v = area of the gate opening; and the subscript "o" denotes initial steady state condition.

It was demonstrated by Allievi that as the amplitude of the valve movement increases, i.e., as $C' \rightarrow 0$, $Z_1^2 \rightarrow 0$, and $Z_2^2 \rightarrow 2$. Thus pressure variation in extreme conditions is between 0 and $2H_o$. Later on, however, Bergeron [5-7] proved graphically that for large values of Allievi's parameter, $\rho = (aV_o)/(2gH_o)$, it is possible to have cavitation at the valve and to reach superpressures in excess of the ordinarily considered maximum of double the static head. Also, Camichel [8] demonstrated that doubling of pressure head is not possible unless $H_o > aV_o/g$.

Camichel conducted a number of experiments at the laboratory of Toulouse and Eydous performed field tests on high head installations. In most of these experiments, the resonating conditions were produced by a rotating cock driven by a motor which opened and closed the valve rhythmically. Their experimental results, without detailed theoretical analysis, are summarized in two reports [8]. Wylie [58, 59], and the author [12] have used these results for the verification of the theoretical analyses.

If a pressure wave is followed in a compound pipeline, then the period of the system, which Camichel et al. called the theoretical period, is

$$T_{th} = 4 \sum_{i=1}^n \frac{l_i}{a_i} \quad (1.8)$$

However, the period of the maximum pressure surge, termed as the apparent period, was observed to be different from the theoretical period; the former being generally shorter. The experiments performed by Camichel et al. demonstrated the existence of the higher odd harmonics. It was found that the period of the higher odd harmonics are integer fractions of the theoretical period, and the period of the fundamental is equal to the apparent period. These results are valid for a simple pipeline but cannot be generalized for complex systems. It was also reported that through flow is a minimum during resonating conditions—a result which will be used in the following chapters to determine the resonant frequencies.

Wilkins [56] was the first to report severe pressure oscillations combined with machine vibrations in hydroelectric power plants in 1923. Den Hartog [16] concluded from theoretical analysis that objectionable vibrations are to be expected in penstocks of Francis turbine installations if the number of vanes on the turbine runner is one less than the number of guide vanes. Although a number of simplifying assumptions are made in his analysis, observations on eight different installations show good agreement

with this theory.

By assuming a sinusoidal pressure variation, using the general solution of the differential equations, and making use of the fact that during resonance $v \equiv 0$, Jaeger [26, 28] derived the following equation for the fundamental frequency of two pipes in series having a constant head reservoir at the upstream end:

$$\frac{A_1}{a_1} \tan \frac{\omega l_1}{a_1} = \frac{A_2}{a_2} \cot \frac{\omega l_2}{a_2} \quad (1.9)$$

The subscripts "1" and "2" refer to the number of the pipe. The solution of this equation for ω gives the frequency of the fundamental. Similarly, the following equation was obtained for a typical hydropower development scheme:

$$\frac{A_2 a_1}{a_2 A_1} \cot \frac{\pi}{2} \frac{l_2 a_1}{a_2 l_1} \frac{1}{1+\epsilon} + \frac{A_3 a_1}{a_3 A_1} \cot \frac{\pi}{2} \frac{l_3 a_1}{a_3 l_1} \frac{1}{1+\epsilon} = \tan \frac{\pi}{2} \frac{1}{1+\epsilon} \quad (1.10)$$

in which the subscripts "1", "2" and "3" refer to, respectively, the penstock, surge tank and tunnel. After calculating ϵ from Eq. 1.10, the apparent period of the system can be determined from the equation

$$m = \frac{2a_1 \pi}{4l_1(1+\epsilon)} \quad (1.11)$$

While analysing the higher harmonics Jaeger [26, 28] implicitly assumes that only odd harmonics are possible. He considers a point A in a complex system such that the cross-section of the pipeline is constant between the point A and the gate. The conditions at point A are represented by

$$f_n = -\alpha_{n-1} F_{n-1} \quad (1.12)$$

in which α_{n-1} is an unknown function which depends on the geometry of the conduit and on the movement of the gate. He shows that if $\alpha \equiv 1$, resonance will occur. When α is equal to one, a nodal point exists at A with a corresponding loop at the valve. In the derivation of this theory, it is assumed

that the half period of each pipe in the system is an integer multiple of the half period of the pipe from the valve to the point A. Since by this method only those periods are predicted which are related to the theoretical period by integers, Jaeger concluded that the higher periods of a complex system are related to the theoretical period while the fundamental is associated with the apparent period. This is true in a simple pipe but cannot be generalized for complex systems.

Jaeger [27, 28] has compiled data of a number of hydroelectric power plants and pumping schemes in which extensive damage or failure of the system occurred. Out of twenty such cases, twelve were caused by resonance or self-excited vibrations.

Bergeron [5-7] introduced the graphical method to analyze resonant conditions. In his latest work [7], graphical analysis of the resonant conditions caused by reciprocating pumps is also included.

Angus [4] reported that valves having a loose connection between the stem or operating spindle and the piece that performs the function of changing the area, or having a slight play of the spindle or stem, or having spindles which bend under pressure, are subjected to chattering when partially closed and, quite frequently, when nearly closed. He demonstrated graphically the amplification of the pressure head due to chattering of valves. It is shown that the shape of the relative pressure head versus relative velocity diagram depends upon the value of the Allievi's parameter and on the setting of the valve at which the chattering occurs.

Schnyder [44] showed, by the graphical method, the amplification of the pressure head at a leaking valve.

Rocard [43] was the first to introduce the concept of hydraulic impedance, later used by Paynter [38], Waller [53-55], and Wylie [58-60], to

analyze auto-oscillations in the Lac Blanc-Lac Noir pumped storage scheme. His solution yielded lengthy equations which were practically impossible to solve by hand. To overcome this difficulty, he used a pipe having equivalent diameter instead of the actual piping system.

Evangelisti [18] derived equations for resonant frequencies of three system types, namely, a simple pipeline; two pipes in series; and two branch pipes by applying the admittance method commonly used in electrical engineering. He did not, however, solve any example or compare the results with experimental values.

By using the general solution of Allievi's equations, assuming a sinusoidal pressure variation and considering that a discharge node is formed at the valve during resonance, Favre [19] developed the following expression for the period of the fundamental and of the higher harmonics of a pipe having linearly variable characteristics—diameter and wave velocity—along its length:

$$\tan \frac{\omega \ell}{a_m} = - \frac{\omega \ell}{a_m} \frac{1}{\sigma} \quad (1.13)$$

in which $\sigma = [1 + \nu/2] [\mu(1 + \nu/2) + \nu]$; $\nu = (a_o - a_m)/a_m$; and $\mu = (D_A - D_o)/D_o$. The subscripts o, m and A refer to values at the valve, at the mid-point, and at the reservoir end of the pipe. As the conical pipe approaches a cylinder, σ approaches zero. Favre also demonstrated that for continuous periodic valve motion, the apparent period, and for rapidly closing and opening a valve at regular intervals, the theoretical period produces severe resonating conditions.

Paynter [38] introduced the idea of electrical-hydraulic analogy. By using the concept of hydraulic impedance, he studied the surges caused by rythmical motion of a valve at the downstream end of a pipeline and

presented the results in a graphical form showing the period and amplitude of gate movement, theoretical period of the pipeline, Allievi's parameter, and amplitude and phase angle of pressure fluctuations. The results are, however, valid only for small pressure fluctuations.

Surges in oil pipelines connected to reciprocating pumps were studied by Waller [53-55] by using the impedance concepts. He decomposed the pump discharge curve (which he derived by neglecting the compressibility of the liquid in the pump) into a set of harmonics and then determined the surge pressures from the computed values of the impedances. He also presented the design of a few corrective devices for the system.

Abbott, Gibson, and McCaig [1] and McCaig and Gibson [32] reported measurements of self-excited vibrations. In the first example a slight leak in the twelve foot diameter penstock valve, caused by a reduction of the seal pressure, generated self-sustained vibrations of the valve. The valve vibrations and the pressure oscillations were found to be sinusoidal. Opening a bypass valve reduced the vibrations to zero. The second incident occurred when a ten-inch diameter spring-cushioned check valve in a pump discharge line leaked under static head. Measurements showed that the resultant pressure oscillations were approximately sinusoidal with sharp impulses of large magnitude imposed every third cycle. The vibrations were prevented by installing a weaker cushioning spring in the check valve and removing the air valve from the pipeline.

D'Souza and Oldenburger [17] presented transfer functions relating the pressure and velocity variables at two cross sections of a pipeline. The viscosity and the compressibility of the fluid were taken into consideration in the analysis. However, only laminar flow and a single pipe of constant area were considered and the elasticity of the pipe walls was

neglected. The theoretical results were verified by comparing them with the experimental values obtained by Roberts [42].

Oldenburger and Goodson [34] in their analysis of the dynamics of hydraulic lines replaced the transcendental functions representing the transfer functions by the infinite products of linear factors and their approximations. It was shown that satisfactory results are obtained by retaining only a few terms. The longitudinal vibrations and the viscosity effects were, however, neglected in the analysis.

Oldenburger and Donelson [35] conducted frequency response tests at the Apalachia hydroelectric power plant to check the validity of expressions derived by Paynter. These tests confirmed that the expressions in general are valid under all major operating conditions.

Although the idea of impedance has been used since 1937, Wylie [50, 58-60] was the first to extend and systematize this concept for solving complex systems, such as, series, parallel, and branch systems with the side branch having an orifice, reservoir or a dead end. He assumed a sinusoidal pressure and flow variation, linearized the nonlinear terms and computed the ratio of the deviation of the discharge, q , and pressure head, h (both expressed as complex numbers), from the mean. By using the known boundary conditions he computed the sending end impedance, Z_s , and then plotted an impedance diagram, ω versus $|Z_s|$. Frequencies at which $|Z_s|$ is a maximum are the resonant frequencies of the system. Because of the lengthy algebraic equations involved, the method is suitable for computer analysis only. For a parallel piping system a procedure is suggested in which a number of simultaneous equations are to be solved. This becomes cumbersome if there are many pipes in parallel. For example, for two pipes in parallel eight simultaneous equations have to be solved.

The impedance theory was verified by Wylie by comparing the results with experimental values or with those obtained by the method of characteristics. The resonating conditions at the Bersimis No. 2 system [60] were studied and close agreement was found between the measured and the computed values.

Streeter and Wylie [49] studied the resonating characteristics of a system having a reciprocating pump by using the impedance theory. The kinematics and dynamics of flow through reciprocating pumps were discussed. The pump discharge was decomposed into a set of harmonics and the expression for pressure fluctuations on the suction side of a pump having a simple suction line was developed. In the analysis, compressibility of the water in the pump was taken into consideration. An excellent agreement is shown to exist between the measured values and those calculated by the impedance theory and by the method of characteristics.

Streeter and Wylie [48] were the first to use the impedance theory for the stability analysis of governed hydro-systems. Other methods [13, 15, 23] are available for this type of analysis.

Recently Zielke, Wylie, and Keller [61] used the impedance theory to study the forced and self-excited oscillations in the propellant feed systems of rocket engines. The fluid system consisted of a suction pipe, centrifugal pump, and a discharge pipe. The adequacy of the analytical model as well as the usefulness of the linearization procedure was verified experimentally.

Holley [21, 22] investigated, experimentally and theoretically, the surging in pipelines. He considered the water as incompressible and the walls of the pipeline as rigid. After normalization the governing differential equation was solved by the fourth order Runge-Kutta method. While

discussing Ref. 27, the writer [10] normalized the governing equation in terms of physically interpretable parameters and elucidated some of the limitations of the theoretical analysis presented by the author.

In this thesis a new approach, based on the transfer matrix method of the theory of mechanical vibrations [40, 41, 51] or four pole-parameters of electrical engineering, is presented for determining the frequency response and resonating characteristics of piping systems with various boundary conditions and having one or more forcing functions. The method presented herein is suitable for both hand and digital computations. For using this method, the reader should have some background in the theory of complex variables, and an elementary knowledge of the ordinary matrix operations.

Terminology is first introduced; then the block diagrams are discussed. This is followed by a derivation of field matrices for a simple pipe, for a pipe having variable characteristics, and for a system of parallel pipes. Thereafter the point matrices for valves, orifices, series connections, and the junction of a branch and a main are presented. For determining the resonating frequencies, a numerical procedure for complex or simple systems and expressions for simple systems are presented in Chapter 4. Then equations are developed for frequency response of piping systems having various forcing functions, e.g. fluctuating discharge, fluctuating pressure and oscillating valve. This is followed by a derivation of equations for determining the location of pressure nodes and antinodes. The validity of the method is demonstrated by comparing the results with those obtained by experiment, the method of characteristics, the impedance method and energy concepts.

C H A P T E R T W O

TERMINOLOGY AND BLOCK DIAGRAMS

In this chapter, the terms frequently used in this thesis are defined, block diagrams are introduced and the advantages of using these diagrams are discussed.

2.1 TERMINOLOGY

The terminology established by Camichel et al. [8] and later used by Jaeger [24 - 29] and Wylie [58 - 60] is followed herein.

STEADY-OSCILLATORY FLOW

A flow in which a permanent regime is established such that the conditions at a point, e.g., pressure, discharge, are periodic functions of time is called steady-oscillatory flow. In the theory of vibrations, steady state oscillations refer to oscillations of constant amplitude. However, the term steady-oscillatory is used herein to avoid confusion with steady flow in which conditions at a point are constant with respect to time.

INSTANTANEOUS AND MEAN DISCHARGE AND PRESSURE HEAD

In a steady-oscillatory flow, the instantaneous discharge, Q , and the instantaneous pressure head, H , can be divided into two parts:

$$Q = Q_0 + q^* \quad (2.1)$$

$$H = H_0 + h^* \quad (2.2)$$

in which Q_0 = average, or mean, discharge; q^* = discharge deviation from the mean, (see Fig. 2.1); H_0 = average, or mean, pressure head; and h^* = pressure head deviation from the mean. Both h^* and q^* are functions of time, t , and distance, x . It is assumed that h^* and q^* are sinusoidal in time which, in practice, is often true or a satisfactory approximation [1, 26, 28, 32]. Hence, by using complex algebra, one can write

$$q^* = \text{Re} [q(x) e^{j \omega t}] \quad (2.3)$$

$$h^* = \text{Re} [h(x) e^{j \omega t}] \quad (2.4)$$

in which ω = frequency, in rad/sec., $j = \sqrt{-1}$; h and q are complex variables and are functions of x only; and "Re" stands for real part of the complex variable.

THEORETICAL PERIOD

For a series piping system

$$T_{th} = 4 \sum_{i=1}^n \frac{\ell_i}{a_i} \quad (2.5)$$

and

$$\omega_{th} = \frac{2\pi}{T_{th}} \quad (2.6)$$

in which T_{th} = theoretical period, in sec; ω_{th} = theoretical frequency, in rad/sec; n = number of pipes; and a = celerity of water hammer wave. The subscript i denotes quantities for the i^{th} pipe. In a branch system, ℓ_i/a_i is calculated along the main pipeline.

RESONANT FREQUENCY

The frequency corresponding to the fundamental or one of the higher harmonics of the system is called the resonant frequency.

STATE VECTORS

The quantities of interest at a section, i , of a pipeline are h and

Fig. 2-1 Instantaneous and mean discharge.

q which can be combined in matrix notation as

$$\underline{z}_i = \begin{Bmatrix} q \\ h \end{Bmatrix}_i \quad (2.7)$$

The column vector $\underline{z}_i$ is known as the state vector at the section i. The state vectors just to the left and to the right of a section are designated by the superscripts "L" and "R" respectively. For example, $\underline{z}_i^L$ refers to the state vector just to the left of the i^{th} section (Fig. 2.2).

To combine the matrix terms in some cases (see Section 3.2-2) the state vector is defined as

$$\underline{z}'_i = \begin{Bmatrix} q \\ h \\ 1 \end{Bmatrix}_i \quad (2.8)$$

Because of the additional element with unit value, the column vector $\underline{z}'_i$ is called the extended state vector. A prime is used to designate an extended state vector.

TRANSFER MATRICES

A matrix relating two state vectors is called a transfer matrix. The upper case letters F, P, and U are used to designate transfer matrices; the corresponding lower case letters with double subscripts refer to the elements of the matrix: the first subscript represents the row, and the second, the column of the element. For example, the element in the second row and the first column of the matrix U is represented by u_{21} .

Transfer matrices are of three types:

(1) Field transfer matrix, or field matrix, F. This matrix relates the state vectors at two adjacent sections. For example in Fig. 2.2,

$$\underline{z}_{i+1}^L = F_i \underline{z}_i^R \quad (2.9)$$

in which F_i = field matrix for the i^{th} pipe.

(2) Point transfer matrix, or Point matrix, P. The state vectors just

to the left and to the right of a discontinuity, such as at a series connection (Fig. 2.3) or at a valve, are related by this matrix. The type of the discontinuity is designated by specifying a subscript with the letter, "P". For example, in Fig. 2.3

$$\underline{z}_{i+1}^R = P_{sc} \underline{z}_{i+1}^L \quad (2.10)$$

in which P_{sc} is the point matrix for a series connection.

(3) Overall transfer matrix, U. This matrix relates the state vector at one end of a system, or a side branch, to that at the other end. For example if $n+1$ is the last section, then

$$\underline{z}_{n+1}^L = U \underline{z}_1^R \quad (2.11)$$

in which U = overall transfer matrix. This is obtained by an ordered multiplication of all the intermediate field and point matrices.

2.2 BLOCK DIAGRAMS

A block diagram is a schematic representation of a system in which each component, or a combination of components, of the system is represented by a "black box". The box representing a pipeline of constant cross-sectional area, wall thickness, and wall material is characterized by a field matrix, while that representing a discontinuity in the system geometry, by a point matrix. The block diagram for a system can be simplified by representing a block of individual boxes by a single box. This procedure is illustrated in the following sections by a number of typical examples.

A section on a block diagram is shown by a small circle on the line joining the two boxes. The number of the section is written below the circle and the left and right hand side of the section are designated by writing the letters "L" and "R" above the circle. For example, in Fig. 2.4,

Fig. 2·2 Single pipeline.

Fig. 2·3 Series connection.

Fig. 2.4 Block diagram.

i and $i+1$ denote the number of the sections; and "L" and "R", left and right hand of the section. In the case of a branch pipe, the number of the section is written to the right of the circle and the left and right hand sections are identified by writing the letters "BL" and "BR" to the left of the circle (see Fig. 3.5).

The block diagrams are of great help for: (i) the concise and orderly formulation and analysis of problems involving complex systems; (ii) an easy understanding of the interaction of different parts of the system; and (iii) determining the sequence of multiplication of transfer matrices while doing the calculations by hand or writing a computer program.

CHAPTER THREE

DERIVATION OF TRANSFER MATRICES

To analyze the steady-oscillatory flows and to determine the resonating characteristics of a piping system by the method presented herein, it is necessary that the transfer matrices of the elements of the system be known. In this chapter field matrices for a simple pipeline and for a system of parallel loops are derived. A numerical procedure is presented to determine the field matrix for a pipe having variable characteristics along its length. Then point matrices for a series connection, for valves and orifices, and for the junction of a branch (branch having various boundary conditions) and the main are developed. All the transfer matrices derived in this chapter are summarized in Appendix A for easy reference.

3.1 FIELD MATRICES²

1. SINGLE PIPELINE

The field matrix for a pipeline having a constant cross-sectional

² Most of the material presented in Section 3.1-1, 3.1-3, and 3.2 has been reported by the writer in two papers [11, 12]. To facilitate cross-reference, the notation used herein is the same as that used in these papers except as otherwise indicated by a footnote.

area, constant wall thickness and the same wall material is derived in this section. In the derivation, the system is considered as distributed and the friction loss term is linearized.

Flow through pipes is governed by the following two equations which in a simplified form [50] are:

Continuity equation:

$$Q_x + \frac{gA}{a^2} H_t = 0 \quad (3.1)$$

Momentum equation:

$$H_x + \frac{1}{gA} Q_t + \frac{fQ^n}{2gDA^n} = 0 \quad (3.2)$$

in which A = the cross-sectional area of the pipeline; g = acceleration due to gravity; D = the inside diameter of the pipeline; f = Darcy-Weisbach friction factor; n = exponent of velocity in the friction loss term; x = distance along the pipeline measured positive in the downstream direction (see Fig. 2.2); and t = time. The subscripts x and t denote partial differentiation with respect to x and t .

Since $(Q_o)_x = (Q_o)_t = (H_o)_t = 0$, it follows from Eqs. 2.1 and 2.2 that

$$\left. \begin{aligned} Q_x &= q_x^* & Q_t &= q_t^* \\ H_t &= h_t^* & H_x &= (H_o)_x + h_x^* \end{aligned} \right\} \quad (3.3)$$

Note that due to friction losses $(H_o)_x \neq 0$. For turbulent flow

$$(H_o)_x = - \frac{f Q_o^n}{2gDA^n} \quad (3.4a)$$

and for laminar flow

$$(H_o)_x = - \frac{32 \nu Q_o}{gAD^2} \quad (3.4b)$$

in which ν = kinematic viscosity of the fluid. If $q^* \ll Q_o$, then

$$Q^n = (Q_o + q^*)^n \approx Q_o^n + n Q_o^{n-1} q^* \quad (3.5)$$

in which higher order terms are neglected.

It follows from Eqs. 3.1 through 3.5 that

$$q_x^* + \frac{gA}{a^2} h_t^* = 0 \quad (3.6)$$

$$h_x^* + \frac{1}{gA} q_t^* + R q^* = 0 \quad (3.7)$$

in which $R = (nfQ_o^{n-1})/(2gDA^n)$ for turbulent flow and $R = (32 \nu)/(gAD^2)$ for laminar flow.

The field matrix for a pipe can be derived by either of the following two methods (both the methods give identical field matrix):

(a) By using the separation of variable technique:

In this method, Eqs. 3.6 and 3.7 are solved by the technique of separation of variables [57] assuming a sinusoidal solution in time and substituting the boundary conditions at the i^{th} , and at the $(i+1)^{\text{th}}$ section.

The elimination of h^* from Eqs. 3.6 and 3.7 yields

$$q_{xx}^* = \frac{1}{a^2} q_{tt}^* + \frac{gAR}{a^2} q_t^* \quad (3.8)$$

Now, if it is assumed that the variation of q^* is sinusoidal with respect to t , then on the basis of Eq. 2.3, Eq. 3.8 takes the form

$$\frac{d^2 q}{dx^2} = \left(-\frac{\omega^2}{a^2} + \frac{jgA \omega R}{a^2} \right) q \quad (3.9)$$

or

$$\frac{d^2 q}{dx^2} - \mu^2 q = 0 \quad (3.10^3)$$

in which

$$\mu^2 = -\omega^2/a^2 + jgA \omega R/a^2 \quad (3.11)$$

The solution of Eq. 3.10 is

$$q = c_1 \sinh \mu x + c_2 \cosh \mu x \quad (3.12)$$

in which c_1 and c_2 are arbitrary constants.

³ In Ref. 12, $\mu^2 = -\omega^2 \ell^2/a^2 + jgA \omega \ell^2 R/a^2$.

If h^* is also assumed sinusoidal in t , then by substituting Eqs. 3.12 and 2.4 into Eq. 3.6 and solving for h , one obtains

$$h = - \frac{a^2 \mu}{j g A_i \omega} (c_1 \cosh \mu x + c_2 \sinh \mu x) \quad (3.13)$$

The field matrix relating the state vectors at the i^{th} and at the $(i+1)^{\text{th}}$ section on the i^{th} reach (see Fig. 2.2) of length ℓ_i is to be derived. It is known that at the i^{th} section (at $x = 0$), $h = h_i^R$ and $q = q_i^R$. Hence, it follows from Eqs. 3.11 and 3.13 that

$$\left. \begin{aligned} c_1 &= - \frac{j g A_i \omega}{a_i^2 \mu_i} h_i^R \\ \text{and} \\ c_2 &= q_i^R \end{aligned} \right\} \quad (3.14)$$

In addition, at the $(i+1)^{\text{th}}$ section (at $x = \ell_i$), $h = h_{i+1}^L$ and $q = q_{i+1}^L$.

The substitution of these values of h and q , and c_1 and c_2 from Eq. 3.14 into Eqs. 3.12 and 3.13 yields

$$\left. \begin{aligned} q_{i+1}^L &= (\cosh \mu_i \ell_i) q_i^R - (1/Z_c) (\sinh \mu_i \ell_i) h_i^R \\ h_{i+1}^L &= -Z_c (\sinh \mu_i \ell_i) q_i^R + (\cosh \mu_i \ell_i) h_i^R \end{aligned} \right\} \quad (3.15)$$

in which characteristic impedance [50] for the pipe, $Z_c = (\mu_i a_i^2) / (j \omega g A_i)$.

These equations can be expressed in matrix notation as

$$\begin{Bmatrix} q \\ h \end{Bmatrix}_{i+1}^L = \begin{bmatrix} \cosh \mu_i \ell_i & - (1/Z_c) \sinh \mu_i \ell_i \\ - Z_c \sinh \mu_i \ell_i & \cosh \mu_i \ell_i \end{bmatrix} \begin{Bmatrix} q \\ h \end{Bmatrix}_i^R \quad (3.16)$$

or

$$\underline{z}_{i+1}^L = F_i \underline{z}_i^R$$

Hence, field matrix for the i^{th} pipe is

$$F_i = \begin{bmatrix} \cosh \mu_i \ell_i & - (1/Z_c) \sinh \mu_i \ell_i \\ - Z_c \sinh \mu_i \ell_i & \cosh \mu_i \ell_i \end{bmatrix} \quad (3.17)$$

If friction is neglected, i.e., $R_i = 0$, then F_i becomes

$$F_i = \begin{pmatrix} \cos b_i \omega & -\frac{j}{C_i} \sin b_i \omega \\ -j C_i \sin b_i \omega & \cos b_i \omega \end{pmatrix} \quad (3.18)$$

in which $b_i = \ell_i/a_i$ and $C_i = a_i/(gA_i)$. Note that b_i and C_i are constants for a pipe and are not functions of ω , and that C_i is the characteristic impedance [58, 59] for the i^{th} pipe if friction is neglected.

(b) By using the Cayley-Hamilton Theorem

Substitution of Eqs. 2.3 and 2.4 into Eqs. 3.6 and 3.7 yields

$$\frac{dq}{dx} + \frac{jgA \omega}{a^2} h = 0 \quad (3.19)$$

$$\frac{dh}{dx} + \left(R + \frac{j \omega}{gA}\right) q = 0 \quad (3.20)$$

Eqs. 3.19 and 3.20 can be written in matrix form as

$$\frac{dz}{dx} = Bz \quad (3.21)$$

in which $\underline{z}$ is the column vector as given by Eq. 2.7 and

$$B = \begin{pmatrix} 0 & -jgA\omega/a^2 \\ -\left(R + \frac{j \omega}{gA}\right) & 0 \end{pmatrix} \quad (3.22)$$

The solution of the set of differential equations, Eq. 3.21, is given by

$$\underline{z} = e^{Bx} \underline{z}_0 = F \underline{z}_0 \quad (3.23)$$

in which $\underline{z}_0$ = initial state vector at $x = 0$; and F = field matrix. F can be expanded in an infinite series as

$$F(x) = e^{Bx} = I + Bx + \frac{1}{2!} (Bx)^2 + \dots \quad (3.24a)$$

in which I = identity or unit matrix. It follows from the above equation that the transfer matrix for a pipe of length ℓ is

$$F = e^{B\ell} = I + B\ell + \frac{1}{2!} (B\ell)^2 + \dots \quad (3.24b)$$

The characteristic equation for the matrix $B\ell$ is given by the determinant

$$\begin{vmatrix} \lambda & \frac{jgA \omega \ell}{a^2} \\ R + \frac{j \omega}{gA} & \lambda \end{vmatrix} = 0$$

which upon simplification becomes

$$\lambda^2 = -\left(\frac{\omega \ell}{a}\right)^2 + \frac{jgA \omega \ell^2 R}{a^2} \quad (3.25)$$

Hence

$$\lambda_1, \lambda_2 = \pm \mu \ell$$

in which μ is as defined in Eq. 3.11. Since $B\ell$ is a 2×2 square matrix, Eq. 3.24b can be written [50] as

$$F = e^{B\ell} = c_3 I + c_4 B\ell \quad (3.26)$$

in which c_3 and c_4 are constants. By substituting the eigenvalues λ_1 and λ_2 for $B\ell$ into the above equation, and solving for c_3 and c_4 , one obtains

$$\begin{aligned} c_3 &= (e^{\mu \ell} + e^{-\mu \ell})/2 = \cosh \mu \ell \\ c_4 &= \frac{1}{2\mu \ell} (e^{\mu \ell} - e^{-\mu \ell}) = \frac{\sinh \mu \ell}{\mu \ell} \end{aligned} \quad (3.27)$$

By substituting these values of c_3 and c_4 into Eq. 3.26 and simplifying the resulting equation yield the field matrix for the i^{th} pipe as

$$F_i = \begin{bmatrix} \cosh \mu_i \ell_i & -\frac{jg \omega A_i}{\mu_i a_i^2} \sinh \mu_i \ell_i \\ -\frac{1}{\mu_i} \left(R_i + \frac{j \omega}{gA_i}\right) \sinh \mu_i \ell_i & \cosh \mu_i \ell_i \end{bmatrix} \quad (3.28)$$

It can be seen that the elements of matrix F_i in Eq. 3.17 and 3.28 are identical except f_{21} . By multiplying f_{21} in Eq. 3.17 by μ_i in the numerator and denominator and then substituting for μ_i^2 from Eq. 3.11, it can be shown that the element f_{21} in Eq. 3.17 and 3.28 are also identical.

If $\omega \ell_i / a_i \ll 1$ then the system may be analyzed as a lumped system.

In this case, for a frictionless system, field matrix F_i becomes

$$F_i = \begin{pmatrix} 1 & -\frac{gA_i \ell_i \omega j}{a_i^2} \\ -\frac{\ell_i \omega j}{gA_i} & 1 \end{pmatrix} \quad (3.29)$$

which follows from Eq. 3.18 since $\cos(\omega \ell_i / a_i) \sim 1$ and $\sin(\omega \ell_i / a_i) \sim \omega \ell_i / a_i$ for small values of $\omega \ell_i / a_i$.

While doing the analysis one first calculates the elements of the field matrix for each pipe. A comparison of the field matrices of Eqs. 3.18 and 3.29 shows that this idealization of a distributed system does not result in much simplification of the computations.

2. PIPE HAVING VARIABLE CHARACTERISTICS ALONG ITS LENGTH

The continuity, and momentum equation for the flow in a pipeline having variable characteristics, e.g., A , a , wall thickness, wall material, along its length are [39],

$$\begin{aligned} Q_x + \frac{gA(x)}{a^2(x)} H_t &= 0 \\ H_x + \frac{1}{gA(x)} Q_t &= 0 \end{aligned} \quad (3.30a)$$

in which $A(x)$ and $a^2(x)$ denote that A and a^2 are functions of x . In these equations nonlinear terms of higher order and friction are neglected. By substituting Eqs. 2.1 through 2.4 into the above equations and simplifying, one obtains

$$\begin{aligned} q_x + \frac{jgA(x)\omega}{a^2(x)} h &= 0 \\ h_x + \frac{j\omega}{gA(x)} q &= 0 \end{aligned} \quad (3.30b)$$

These equations can be expressed in the matrix notation as

$$\frac{dz}{dx} = \underline{Bz} \quad (3.31)$$

Fig. 3-1 Pipe having variable characteristics along its length.

Fig. 3-2 Actual and Substitute pipe.

in which $\underline{z}$ is the column vector as defined in Eq. 2.7 and

$$B = \begin{bmatrix} 0 & -\frac{jgA(x)\omega}{a^2(x)} \\ -\frac{j\omega}{gA(x)} & 0 \end{bmatrix} \quad (3.32)$$

Since the elements of the matrix B are functions of x, the procedure outlined in the last section to determine the field matrix for the pipe fails. To analyze such cases, recourse is made to either of the following procedures:

(i) The actual pipeline is replaced by a substitute pipeline having piecewise constant elements (see Fig. 3.2), and the system is analyzed by using the field matrices derived in the last section. This gives satisfactory results at low frequencies (see Chapter 5).

(ii) A numerical procedure is adopted to determine the elements of the field matrix.

The determination of the field matrix for a pipeline having variable characteristics is equivalent to integrating the differential equation, Eq. 3.31. This may be done by the Runge-Kutta method as follows.

The pipeline is divided into n reaches as shown in Fig. 3.1. First the field matrix for each reach is computed and then the field matrix for the pipeline is determined by multiplying these matrices in a proper sequence. If the length of the reach between the sections i and i+1 is s, then the fourth-order Runge-Kutta method gives [40]

$$\underline{z}_{i+1} = \underline{z}_i + \frac{1}{6} (\underline{k}_0 + 2\underline{k}_1 + 2\underline{k}_2 + \underline{k}_3) \quad (3.33)$$

in which

$$\left. \begin{aligned} \underline{k}_0 &= s B(x_i) \underline{z}_i \\ \underline{k}_1 &= s B(x_i + s/2) (\underline{z}_i + \underline{k}_0/2) \\ \underline{k}_2 &= s B(x_i + s/2) (\underline{z}_i + \underline{k}_1/2) \\ \underline{k}_3 &= s B(x_{i+1}) (\underline{z}_i + \underline{k}_2) \end{aligned} \right\} \quad (3.34)$$

in which $B(x_i)$, $B(x_{i+1})$, and $B(x_i + s/2)$ are, respectively, the values of the matrix $B(x)$ at section i , at $i+1$, and at the section between i and $i+1$.

By substituting Eqs. 3.34 into Eq. 3.33, one obtains

$$\underline{z}_{i+1} = F_{vc} \underline{z}_i \quad (3.35)$$

in which the field matrix for a pipe having variable characteristics along its length is

$$\begin{aligned} F_{vc} = I &+ \frac{s}{6} [B(x_i) + 4 B(x_i + s/2) + B(x_{i+1})] \\ &+ \frac{s^2}{6} [B(x_i + s/2)B(x_i) + B(x_{i+1})B(x_i + s/2) + B^2(x_i + s/2)] \\ &+ \frac{s^3}{12} [B^2(x_i + s/2)B(x_i) + B(x_{i+1})B^2(x_i + s/2)] \\ &+ \frac{s^4}{24} [B(x_{i+1})B^2(x_i + s/2)B(x_i)] \end{aligned} \quad (3.36)$$

3. PARALLEL SYSTEM

Let there be n loops in parallel (Fig. 3.3) whose overall transfer matrices are

$$U^{(m)} = F_{n'_m}^{(m)} P_{n'_m}^{(m)} \dots P_{2'_m}^{(m)} F_{1'_m}^{(m)} \quad m = 1, 2, \dots, n$$

The superscript in the parenthesis refers to the number of the loop. The matrix $U^{(m)}$ relates the state vectors at the $1'_m{}^{st}$ and at the $(n'_m+1)^{th}$ section of the m^{th} loop (see Fig. 3.3b), i.e.,

$$\underline{z}_{n'_m+1}^{(m)L} = U^{(m)} \underline{z}_{1'_m}^{(m)R}$$

A prime on the subscript denotes a section on the parallel loop.

The elements of the field matrix, F_p , for the parallel loops relating the state vectors $\underline{z}_{i+1}^R$ and $\underline{z}_i^L$ (Fig. 3.3b) can be determined from the following equations [33]:

Longitudinal section of n^{th} loop.

(a) Piping system.

Fig. 3-3 Parallel system.

(b) Block diagram.

Fig. 3-3 Parallel system.

$$\left. \begin{aligned} f_{11} &= \xi/\eta \\ f_{12} &= (\xi\zeta/\eta) - \eta \\ f_{21} &= 1/\eta \\ f_{22} &= \zeta/\eta \end{aligned} \right\} \quad (3.37)$$

in which

$$\left. \begin{aligned} \xi &= \sum_{m=1}^n u_{11}^{(m)} / u_{21}^{(m)} \\ \eta &= \sum_{m=1}^n 1/u_{21}^{(m)} \\ \zeta &= \sum_{m=1}^n u_{22}^{(m)} / u_{21}^{(m)} \end{aligned} \right\} \quad (3.38)$$

In deriving the above expressions use has been made of the relation that

$$|U^{(m)}| = 1 \quad m=1,2,\dots,n$$

i.e.,

$$u_{11}^{(m)} u_{22}^{(m)} - u_{12}^{(m)} u_{21}^{(m)} = 1 \quad m=1,2,\dots,n \quad (3.39)$$

Note that Eq. 3.37 is valid only if the elements of the overall transfer matrix for each parallel loop satisfy Eq. 3.39. It is known from the theory of matrices that, for square matrices, the determinant of the product of matrices is equal to product of the determinants of matrices. Hence if $|P_k^{(m)}| = 1$, $k = 2,3,\dots,n_m$ and $|F_k^{(m)}| = 1$, $k = 1,2,\dots,n_m$, for $m = 1,2,\dots,n$, then $|U^{(m)}| = 1$. It is clear from Eqs. 3.17 and 3.18 that $|F| = 1$. Furthermore, the determinants of the point matrices for series connections, and for valves and orifices are also unity (see the point matrices derived in the following sections). Thus if there is a discontinuity other than a series connection, valve, or orifice in any of the parallel loops the determinant of the point matrix for the discontinuity should be checked to ensure it has unit value before using Eq. 3.37.

3.2 POINT MATRICES

At a section where there is a discontinuity in the geometry of the system, e.g., series connection, orifice, valve, branch connection, one has to derive a point matrix relating the state vector to the left of the discontinuity with that to the right. This point matrix is required in the calculation of the overall transfer matrix for the system which is then used to determine the resonant frequencies and/or frequency response of the system.

Point matrices for various boundary conditions usually found in hydro-power, and in water supply schemes are derived in the following sections.

1. SERIES CONNECTION

A junction of two pipes having different diameters (see Fig. 2.3), different wall thickness, different wall material, or any combination of these variables is called a series connection.

It follows from the continuity equation that

$$q_i^R = q_i^L \quad (3.40)$$

In addition

$$h_i^R = h_i^L \quad (3.41)$$

if losses at the junction are neglected. These two equations can be expressed in matrix notation as

$$\underline{z}_i^R = P_{sc} \underline{z}_i^L \quad (3.42)$$

in which the point matrix for the series connection is

$$P_{sc} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.43)$$

Since P_{sc} is a unit matrix, it can be incorporated into the field matrix while doing the calculations.

2. VALVES AND ORIFICES

The point matrices for valves and orifices are derived by linearizing the gate equation. As will be clear after completing the derivation, this linearization does not introduce serious errors if the pressure rise at the valve is much less than the static head. For an oscillating valve, a sinusoidal valve motion is assumed. It is possible, however, to analyze non-sinusoidal periodic valve motions by this method. The periodic motion is reduced to a set of harmonics by Fourier analysis [57], and the system response is determined for each harmonic. Then the individual responses are superimposed to determine the total response for the given valve motion (since all the equations are linear, the principle of superposition can be applied).

(a) Oscillating valve discharging into atmosphere

The instantaneous and mean discharge through a valve (Fig. 3.4a) are given by the equations

$$Q_{n+1}^L = C_d A_v (2gH_{n+1}^L)^{1/2} \quad (3.44)$$

$$Q_o = (C_d A_v)_o (2gH_o)^{1/2} \quad (3.45)$$

in which C_d = coefficient of discharge; and A_v = area of the valve opening. Division of Eq. 3.44 by Eq. 3.45 yields

$$\frac{Q_{n+1}^L}{Q_o} = \frac{\tau}{\tau_o} \left[\frac{H_{n+1}^L}{H_o} \right]^{1/2} \quad (3.46)$$

in which the instantaneous relative gate opening, $\tau = (C_d A_v)/(C_d A_v)_s$; and the mean relative gate opening, $\tau_o = (C_d A_v)_o/(C_d A_v)_s$. The subscript "s" denotes steady state reference, or index, values.

The relative gate opening may be considered to be made up of two parts, i.e.,

$$\tau = \tau_o + \tau^* \quad (3.47)$$

in which τ^* = deviation of the relative gate opening from the mean (Fig. 3.4b). Substitution of Eqs. 2.1, 2.2, and 3.47 into Eq. 3.46 yields

$$1 + \frac{q_{n+1}^L}{Q_o} = \left(1 + \frac{\tau^*}{\tau_o}\right) \left(1 + \frac{h_{n+1}^L}{H_o}\right)^{1/2} \quad (3.48)$$

If the valve motion is assumed sinusoidal, then

$$\tau^* = \text{Re}(ke^{j\omega t}) \quad (3.49)$$

in which k = amplitude of valve motion. The phase angle between any other forcing function in the system and the oscillating valve can be taken into consideration by making k a complex number; otherwise k is real.

By expanding Eq. 3.48, neglecting terms of higher order (this is valid only if $|h_{n+1}^L| \ll H_o$), and substituting Eqs. 2.3, 2.4, and 3.49 into the resulting equation, one obtains

$$h_{n+1}^L = \frac{2H_o}{Q_o} q_{n+1}^L - \frac{2H_o k}{\tau_o} \quad (3.50)$$

Since $h_{n+1}^R = 0$, on the basis of Eq. 3.50, one can write

$$h_{n+1}^R = h_{n+1}^L + \frac{2H_o k}{\tau_o} - \frac{2H_o}{Q_o} q_{n+1}^L \quad (3.51a)$$

In addition, from the continuity equation it follows that

$$q_{n+1}^R = q_{n+1}^L \quad (3.51b)$$

Eqs. 3.51a and 3.51b may be expressed in matrix notation as

$$\begin{Bmatrix} q \\ h \end{Bmatrix}_{n+1}^R = \begin{bmatrix} 1 & 0 \\ -\frac{2H_o}{Q_o} & 1 \end{bmatrix} \begin{Bmatrix} q \\ h \end{Bmatrix}_{n+1}^L + \begin{Bmatrix} 0 \\ \frac{2H_o k}{\tau_o} \end{Bmatrix}$$

The two matrix terms on the right hand side may be combined as follows

$$\begin{Bmatrix} q \\ h \\ 1 \end{Bmatrix}_{n+1}^R = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2H_o}{Q_o} & 1 & \frac{2H_o k}{\tau_o} \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} q \\ h \\ 1 \end{Bmatrix}_{n+1}^L \quad (3.53)$$

(a) Valve at downstream end of pipeline.

(b) Sinusoidal valve motion.

(c) Orifice at intermediate section.

Fig. 3.4 Valves.

Note that expansion of Eq. 3.53 yields Eqs. 3.51a, 3.51b, and $1 = 1$.

Thus the additional element one in the column vector aids in writing the right hand side of Eq. 3.52 in a compact form. As defined in Chapter 2, Eq. 2.8, the column vector with one as an additional element is called an extended state vector, $\underline{z}'$. The extended state vectors and extended transfer matrices are denoted by a prime.

On the basis of Eq. 2.8, Eq. 3.53 may be written as

$$\underline{z}'_{n+1}^R = P'_{ov} \underline{z}'_{n+1}^L \quad (3.54)$$

in which P'_{ov} = the extended point matrix for an oscillating valve and is given by

$$P'_{ov} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{2H_o}{Q_o} & 1 & \frac{2H_o k}{\tau_o} \\ 0 & 0 & 1 \end{pmatrix} \quad (3.55)$$

(b) Valve having constant gate opening discharging into atmosphere

In this case, $k = 0$. Hence, Eq. 3.52 takes the form

$$\begin{Bmatrix} q \\ h \end{Bmatrix}_{n+1}^R = \begin{pmatrix} 1 & 0 \\ -\frac{2H_o}{Q_o} & 1 \end{pmatrix} \begin{Bmatrix} q \\ h \end{Bmatrix}_{n+1}^L \quad (3.56)$$

or

$$\underline{z}_{n+1}^R = P_v \underline{z}_{n+1}^L \quad (3.57)$$

in which P_v = the point matrix for a valve or orifice discharging into atmosphere, and is given by

$$P_v = \begin{pmatrix} 1 & 0 \\ -\frac{2H_o}{Q_o} & 1 \end{pmatrix} \quad (3.58)$$

Note that P_v is not an extended point matrix.

If a valve of constant gate opening, or an orifice, is at an intermediate section (Fig. 3.4c) then Eq. 3.58 becomes

$$P_{vi} = \begin{pmatrix} 1 & 0 \\ -\frac{2\Delta H_o}{Q_o} & 1 \end{pmatrix} \quad (3.59)$$

in which ΔH_o = the mean head loss across the valve corresponding to the mean discharge, Q_o .

3. BRANCH PIPELINES

In the piping systems shown in Fig. 3.5a, pipeline abc is the main and bd, the side branch. The transfer matrix for the pipeline ab can be computed by using the field and point matrices derived above. To calculate the overall transfer matrix for abc, the point matrix at the junction b, relating the state vectors to the left and to the right of the junction, must be known. This matrix can be obtained if the boundary conditions at point d are specified.

Point matrices for the junction of the main and the branch having various boundary conditions are derived in this section.

Let $\tilde{U}$ be the overall transfer matrix for the branch (refer to Fig. 3.5b), i.e.,

$$\tilde{z}_{n+1}^L = \tilde{U} \tilde{z}_1^R \quad (3.60)$$

or

$$\begin{Bmatrix} \tilde{q} \\ \tilde{h} \end{Bmatrix}_{n+1}^L = \begin{pmatrix} \tilde{u}_{11} & \tilde{u}_{12} \\ \tilde{u}_{21} & \tilde{u}_{22} \end{pmatrix} \begin{Bmatrix} \tilde{q} \\ \tilde{h} \end{Bmatrix}_1^R \quad (3.61)$$

in which $\tilde{U} = \tilde{F}_n \tilde{P}_n \dots \tilde{P}_3 \tilde{F}_2 \tilde{P}_2 \tilde{F}_1$. The quantities relating to the branch are designated by a tilde⁴.

Expansion of Eq. 3.61 yields

$$\tilde{q}_{n+1}^L = \tilde{u}_{11} \tilde{q}_1^R + \tilde{u}_{12} \tilde{h}_1^R \quad (3.62)$$

⁴ In Re. 12, superscript B is used to designate quantities relating to the branch.

(b) Block diagram.

Fig. 3.5 Branch system.

$$\tilde{h}_{n+1}^L = \tilde{u}_{21} \tilde{q}_1^R + \tilde{u}_{22} \tilde{h}_1^R \quad (3.63)$$

If the flow direction is assumed positive as shown in Fig. 3.5a and the losses at the junction are neglected, the following equations can be written

$$\tilde{q}_i^L = \tilde{q}_i^R + \tilde{q}_1^R \quad (3.64)$$

$$\tilde{h}_i^L = \tilde{h}_i^R = \tilde{h}_1^R \quad (3.65)$$

By substituting appropriate boundary conditions in Eqs. 3.62 and 3.63 and making use of Eqs. 3.64 and 3.65, the point matrices at the junction of the main and the branch can be derived. The following examples illustrate the procedure.

(a) Dead end branch

In this case, $\tilde{q}_{n+1}^L = 0$. Hence, it follows from Eqs. 3.62 and 3.65 that

$$\tilde{q}_1^R = - \frac{\tilde{u}_{12}}{\tilde{u}_{11}} \tilde{h}_i^L \quad (3.66)$$

Substitution of this equation into Eq. 3.64 yields

$$\tilde{q}_i^R = \tilde{q}_i^L + \frac{\tilde{u}_{12}}{\tilde{u}_{11}} \tilde{h}_i^L \quad (3.67)$$

Eq. 3.65 can be written as

$$\tilde{h}_i^R = 0 \tilde{q}_i^L + \tilde{h}_i^L \quad (3.68)$$

Eqs. 3.67 and 3.68 can be expressed in matrix notation as

$$\begin{Bmatrix} \tilde{q} \\ \tilde{h} \end{Bmatrix}_i^R = \begin{bmatrix} 1 & \tilde{u}_{12}/\tilde{u}_{11} \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \tilde{q} \\ \tilde{h} \end{Bmatrix}_i^L \quad (3.69)$$

or

$$\underline{\tilde{z}}_i^R = P_{bde} \underline{\tilde{z}}_i^L \quad (3.70)$$

in which P_{bde} = the point matrix for the side branch with dead end and is given by

$$P_{bde} = \begin{pmatrix} 1 & \tilde{u}_{12}/\tilde{u}_{11} \\ 0 & 1 \end{pmatrix} \quad (3.71)$$

(b) Branch with constant head reservoir

In this case, $\tilde{h}_{n+1}^L = 0$. Hence, it follows from Eqs. 3.63 through 3.65 that

$$q_i^R = q_i^L + \frac{\tilde{u}_{22}}{\tilde{u}_{21}} h_i^L \quad (3.72)$$

Eqs. 3.72 and 3.68 can be expressed in matrix notation as

$$\begin{Bmatrix} q \\ h \end{Bmatrix}_i^R = \begin{pmatrix} 1 & \tilde{u}_{22}/\tilde{u}_{21} \\ 0 & 1 \end{pmatrix} \begin{Bmatrix} q \\ h \end{Bmatrix}_i^L \quad (3.73)$$

or

$$\underline{z}_i^R = P_{bres} \underline{z}_i^L \quad (3.74)$$

in which P_{bres} = the point matrix for the branch with constant head reservoir and is given by

$$P_{bres} = \begin{pmatrix} 1 & \tilde{u}_{22}/\tilde{u}_{21} \\ 0 & 1 \end{pmatrix} \quad (3.75)$$

(c) Branch having oscillating valve at the downstream end

If the frequency of the oscillating valve on the branch is the same as that of the forcing function on the main (these may not be in phase), the system can be analyzed by using the point matrix derived in this section. In case the frequencies of the forcing functions are not the same, then the system is analyzed considering each forcing function at a time and the results are superimposed to determine the total response.

Since an extended overall transfer matrix relating the state vector at the first, and at the last section on the branch is required, the extended transfer matrices will also have to be used for the main. Let the extended overall transfer matrix for the branch be

$$\tilde{U}' = \begin{pmatrix} \tilde{u}_{11} & \tilde{u}_{12} & 1 \\ \tilde{u}_{21} & \tilde{u}_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (3.76)$$

In writing $\tilde{U}'$ in Eq. 3.76, it is assumed that there is no other forcing function on the branch; if this is not the case, then at least one of the elements $\tilde{u}_{13}$, $\tilde{u}_{23}$, $\tilde{u}_{31}$, and $\tilde{u}_{32}$ is not equal to zero and the point matrix derived in this section should be modified accordingly.

For the branch pipeline

$$\tilde{z}_{n+1}^R = \tilde{P}'_{ov} \tilde{z}_{n+1}^L \quad (3.77)$$

and

$$\tilde{z}_{n+1}^L = \tilde{U}' \tilde{z}_1^R \quad (3.78)$$

By substituting $\tilde{P}'_{ov}$ from Eq. 3.55 and $\tilde{z}_{n+1}^L$ from Eq. 3.78 into Eq. 3.77, and expanding the resulting equation, one obtains

$$\tilde{q}_{n+1}^R = \tilde{u}_{11} \tilde{q}_1^R + \tilde{u}_{12} \tilde{h}_1^R \quad (3.79)$$

$$\tilde{h}_{n+1}^R = (\tilde{u}_{21} - \frac{2\tilde{H}_o}{\tilde{Q}_o} \tilde{u}_{11}) \tilde{q}_1^R + (\tilde{u}_{22} - \frac{2\tilde{H}_o}{\tilde{Q}_o} \tilde{u}_{12}) \tilde{h}_1^R + \frac{2\tilde{H}_o \tilde{k}}{\tilde{\tau}_o} \quad (3.80)$$

All the notations defined in the previous sections apply except that the tilde ($\sim$) refers to the branch. For example, $\tilde{\tau}_o$ = the mean relative gate opening of the valve on the branch. Any phase shift between the valve on the branch and the forcing function on the main can be taken into consideration by making $\tilde{k}$ a complex number; otherwise $\tilde{k}$ is real.

Since $\tilde{h}_{n+1}^R = 0$, and $\tilde{h}_1^R = \tilde{h}_1^L$, it follows from Eq. 3.80 that

$$\tilde{q}_1^R = -\tilde{p}_{12} \tilde{h}_1^L - \tilde{p}_{13} \quad (3.81)$$

in which

$$\tilde{p}_{12} = \frac{\tilde{u}_{22} - \frac{2\tilde{H}_o}{\tilde{Q}_o} \tilde{u}_{12}}{\tilde{u}_{21} - \frac{2\tilde{H}_o}{\tilde{Q}_o} \tilde{u}_{11}} \quad (3.82)$$

and

$$p_{13} = \frac{2\tilde{H}_o \tilde{k} / \tilde{\tau}_o}{\tilde{u}_{21} - 2\tilde{H}_o \tilde{u}_{11} / \tilde{Q}_o} \quad (3.83)$$

By substituting $\tilde{q}_1^R$ from Eq. 3.81 into Eq. 3.64, one obtains

$$q_i^R = q_i^L + p_{12} h_i^L + p_{13} \quad (3.84)$$

Moreover, one can write

$$1 = 0 q_i^L + 0 h_i^L + 1 \quad (3.85)$$

Eqs. 3.84, 3.68, and 3.85 can be expressed in matrix notation as

$$\begin{Bmatrix} q \\ h \\ 1 \end{Bmatrix}_i^R = \begin{pmatrix} 1 & p_{12} & p_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{Bmatrix} q \\ h \\ 1 \end{Bmatrix}_i^L \quad (3.86)$$

or

$$\underline{z}_i^R = \underline{P}'_{bov} \underline{z}_i^L \quad (3.87)$$

in which $\underline{P}'_{bov}$ = the transfer matrix at the junction of the side branch having an oscillating valve and is given by

$$\underline{P}'_{bov} = \begin{pmatrix} 1 & p_{12} & p_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (3.88)$$

If there is an orifice, or a valve having constant gate opening at the downstream end of the branch, then $\tilde{k} = 0$. Hence, $p_{13} = 0$, and the point matrix for the branch can be written as

$$\underline{P}_{borf} = \begin{pmatrix} 1 & p_{12} \\ 0 & 1 \end{pmatrix} \quad (3.89)$$

Note that this is not an extended point matrix.

CHAPTER FOUR

RESONANT FREQUENCIES AND FREQUENCY RESPONSE OF PIPING SYSTEMS

This chapter deals with determining the resonant frequencies and frequency response of frictionless piping systems. First a numerical procedure is presented to determine the resonant frequencies. Then, the expressions for the resonant frequencies of a number of simple systems are given. This is followed by derivation of equations for determining the frequency response of piping systems having fluctuating pressure, oscillating valves, and fluctuating discharge as the forcing function.

4.1 RESONANT FREQUENCIES

In a frictionless system having a constant head reservoir at the upstream end and an oscillating valve at the downstream end, the amplitude of discharge fluctuation at the valve is zero during resonating conditions at the fundamental or one of the higher odd harmonics. This was observed by Camichel et al. [8], and reported to be true by Jaeger [26, 28, 29]. The frequency response diagrams obtained by theoretical analysis of a number of series, parallel, and branch systems (a branch system with a side branch having an orifice or an oscillating valve being an exception) done by Wylie [58-60], and by the author [12] confirm this result. Expressions

for the resonant frequencies of simple frictionless systems and their numerical values for simple or complex systems can be determined by using the result as follows.

Let U be the overall transfer matrix for a system having a constant head reservoir at the upstream end (section 1) and an oscillating valve at the downstream end (section $n+1$), i.e.,

$$\underline{z}_{n+1}^L = U \underline{z}_1^R \quad (4.1)$$

By expanding Eq. 4.1 and noting that $h_1^R = 0$ (constant head reservoir), and $q_{n+1}^L = 0$ (discharge fluctuation node) at a resonant frequency, one obtains

$$u_{11} q_1^R = 0 \quad (4.2)$$

Recall that u_{11} is the element in the first row and the first column of the matrix U . Since for non-trivial solution, $q_1^R \neq 0$, therefore

$$u_{11} = 0 \quad (4.3)$$

To determine the resonant frequencies, the value of u_{11} is computed for different trial values of ω , and the $u_{11} \sim \omega$ curve is plotted. The points of intersection of this curve and the ω -axis are the resonant frequencies (see Fig. 5.2). If the chosen value of ω is equal to one of the resonant frequency, $u_{11} = 0$. This requirement is not normally met by the first guess for ω and the resulting numerical value of u_{11} is referred to as the residual.

The expressions for u_{11} for simple systems are derived and then by using Eq. 4.3, equations for the resonant frequencies are obtained. The following example illustrates the procedure:

For a series system having two pipes (Fig. 2.3)

$$U = F_2 P_2 F_1 \quad (4.4)$$

By substituting F_2 and F_1 from Eq. 3.18 and P_2 from Eq. 3.43, multiplying

the matrices and using Eq. 4.3, one obtains

$$\cos b_1\omega \cos b_2\omega - \frac{a_1}{a_2} \left(\frac{D_2}{D_1} \right)^2 \sin b_1\omega \sin b_2\omega = 0 \quad (4.5a)$$

in which b_1 and b_2 are constants as defined in Eq. 3.18.

Proceeding similarly, the following equation for three pipes in series is obtained.

$$\begin{aligned} \cos b_1\omega \cos b_2\omega \cos b_3\omega - \frac{a_1}{a_2} \left(\frac{D_2}{D_1} \right)^2 \sin b_1\omega \sin b_2\omega \cos b_3\omega \\ - \frac{a_2}{a_3} \left(\frac{D_3}{D_2} \right)^2 \cos b_1\omega \sin b_2\omega \sin b_3\omega \\ - \frac{a_1}{a_3} \left(\frac{D_3}{D_1} \right)^2 \sin b_1\omega \cos b_2\omega \sin b_3\omega = 0 \end{aligned} \quad (4.5b)$$

Solution of Eqs. 4.5 for ω gives the resonant frequencies of the system. Note that this equation is only valid for a frictionless system.

By proceeding in a similar manner, equations for four or more pipes in series, for branch systems, and for parallel systems can be derived. These equations are cumbersome and it is better to follow the numerical procedure outlined above rather than to derive the equations and then solve them.

4.2 FREQUENCY RESPONSE

The method presented herein may be used to determine the frequency response of a system having one or more than one periodic forcing function. If all the forcing functions are sinusoidal and have the same frequency then the equations derived in this chapter can be used without any modification. However, if the frequencies are different, then the system is analyzed considering each forcing function at a time and the results are superimposed to determine the total response. Non-harmonic periodic functions are decomposed into different harmonics by Fourier analysis [57]. By considering each harmonic at its particular frequency, the system response is determined. The results are then superimposed to find the total response.

Expressions to determine the frequency response of typical piping systems having three common types of forcing functions—fluctuating pressure head, oscillating valve, and fluctuating discharge—are derived in the following sections. By proceeding in a similar manner, expressions for other types of systems can be derived.

1. FLUCTUATING PRESSURE HEAD

Consider the system shown in Fig. 5.3a having a dead end at the right end. A wave on the surface of the reservoir produces pressure oscillations in the system. Due to the wave, the pressure head at section 1 fluctuates sinusoidally about the mean pressure head. Let this pressure head variation be given by

$$h_1^{*R} = \text{Re} [h_1^R e^{j \omega t}] = K \cos \omega t = \text{Re} [K e^{j \omega t}] \quad (4.6)$$

and U be the transfer matrix relating the state vectors at the 1st and $(n+1)^{\text{th}}$ section, i.e.,

$$\underline{z}_{n+1}^L = U \underline{z}_1^R \quad (4.7)$$

It is assumed that there is no other forcing function in the system; otherwise the extended transfer matrix, U' will have to be used. Expansion of Eq. 4.7 yields

$$q_{n+1}^L = u_{11} q_1^R + u_{12} h_1^R \quad (4.8)$$

$$h_{n+1}^L = u_{21} q_1^R + u_{22} h_1^R \quad (4.9)$$

Since $q_{n+1}^L = 0$ (dead end), it follows from Eq. 4.8 that

$$q_1^R = - \frac{u_{12}}{u_{11}} h_1^R \quad (4.10)$$

which on the basis of Eq. 4.6 becomes

$$q_1^R = - u_{12} K / u_{11} \quad (4.11)$$

Substitution of Eq. 4.11 into Eq. 4.9 and simplification of the resulting

equation gives

$$h_{n+1}^L = (u_{22} - \frac{u_{12}u_{21}}{u_{11}})K \quad (4.12)$$

Hence the amplitude of the pressure head fluctuation at the dead end is

$$h_a = |h_{n+1}^L| = |(u_{22} - \frac{u_{12}u_{21}}{u_{11}})K| \quad (4.13)$$

The amplitude of the pressure head at the dead end may be nondimensionalized by dividing the amplitude of pressure fluctuations at the reservoir end, i.e.,

$$h_r = |h_a/K| = |u_{22} - \frac{u_{12}u_{21}}{u_{11}}| \quad (4.14)$$

2. FLUCTUATING DISCHARGE

Periodic flows are encountered on the suction, and on the discharge side of a reciprocating pump. These fluctuations can be decomposed into a set of harmonics. Severe pressure oscillations may develop if any of these harmonics has a period equal to one of the natural periods of the suction or discharge pipeline.

Expressions are derived in this section to determine, by the transfer matrix method, the frequency response of systems having a reciprocating pump. The suction and the discharge pipeline may have stepwise changes in diameter and/or wall thickness and may have branches with reservoirs, dead ends, or orifices.

(a) Suction line.

Let the transfer matrix relating the state vectors at the 1st and (n+1)th section of the suction line (Fig. 4.1) be U, i.e.,

$$\underline{z}_{n+1}^L = U \underline{z}_1^R \quad (4.15)$$

By expanding Eq. 4.15 and noting that $h_1^R = 0$, one obtains

$$q_{n+1}^L = u_{11}q_1^R \quad (4.16)$$

Fig. 4-1 Suction and discharge pipelines.

and

$$h_{n+1}^L = u_{21} q_1^R \quad (4.17)$$

Hence,

$$h_{n+1}^L = \frac{u_{21}}{u_{11}} q_{n+1}^L \quad (4.18)$$

The inflow~time curve for one period can be decomposed into a set of harmonics by Fourier analysis [57]. Let the discharge for the m^{th} harmonic be

$$q_{n+1}^{*L} = A'_m \sin(m \omega t + \psi_m) \quad (4.19)$$

or

$$q_{n+1}^{*L} = \text{Re} [A_m e^{jm \omega t}] \quad (4.20)$$

in which $A_m = A'_m \exp [j(\psi_m - \frac{\pi}{2})]$; A'_m and ψ_m are the amplitude and the phase angle for the m^{th} harmonic; and ω = frequency of the fundamental. It follows from Eqs. 2.3 and 4.20 that $q_{n+1}^L = A_m$ in which A_m is a complex constant. Substitution of this relationship into Eq. 4.18 yields

$$h_{n+1}^L = \frac{u_{21}}{u_{11}} A_m \quad (4.21)$$

Hence the amplitude of pressure head fluctuation at the suction flange is

$$h_m = |h_{n+1}^L|_m = |u_{21} A_m / u_{11}| \quad (4.22)$$

and the phase angle for the pressure head is

$$\phi_m = \tan^{-1} \left[\text{Im}(h_{n+1}^L)_m / \text{Re}(h_{n+1}^L)_m \right] \quad (4.23)$$

The pressure head~time curve may be obtained by vectorially adding the pressure~time curve for each harmonic. For the m^{th} harmonic

$$h_{n+1}^{*L} = \text{Re} \left[h_m e^{j(m \omega t + \phi_m)} \right] \quad (4.24)$$

or

$$h_{n+1}^{*L} = h_m \cos(m \omega t + \phi_m) \quad (4.25)$$

Hence the pressure head~time curve may be computed from the equation

$$h_{n+1}^{*L} = \sum_{m=1}^M h_m \cos(m \omega t + \phi_m) \quad (4.26)$$

in which M = number of harmonics into which inflow time curve for the pump is decomposed.

(b) Discharge line.

By proceeding in a similar manner and noting that $h_{n+1}^L = 0$, the following equation is obtained for the pressure head time curve at the discharge side of the pump:

$$h_1^{*R} = \sum_{m=1}^M h'_m \cos(m \omega t + \phi'_m) \quad (4.27)$$

in which

$$h'_m = |h_1^R|_m = \frac{|u_{21} A_m|}{|u_{12} u_{21} - u_{11} u_{22}|} \quad (4.28)$$

$$\phi'_m = \tan^{-1} \left[\text{Im}(h^R)_m / \text{Re}(h^R)_m \right] \quad (4.29)$$

and A_m = complex amplitude of m^{th} harmonic of the discharge time curve of the pump.

3. OSCILLATING VALVE

In this case the area of the opening of the gate or valve is varied periodically. Since the gate equation, Eq. 3.44, relating the head, discharge, and the gate area is nonlinear, this case is more difficult to analyze than the preceding ones. However, as discussed in section 3.2-2, this equation can be linearized if $h \ll H_0$. In the derivation of expressions in this section, the point matrix of Eq. 3.55 is used. This matrix is derived by assuming the valve movement as sinusoidal and linearizing the gate equation. Thus to use the expressions derived herein for the nonharmonic periodic valve movements, the valve motion is decomposed into a set of harmonics by Fourier analysis, the system response is determined considering each harmonic at its frequency, and then the total system response is calculated by superimposing the individual responses.

Let U' be the extended overall transfer matrix relating the state vectors at the 1st and the $(n+1)$ th section of the system, i.e.,

$$\underline{z}_{n+1}'^L = U' \underline{z}_1'^R \quad (4.30)$$

In addition

$$\underline{z}_{n+1}'^R = P'_{ov} \underline{z}_{n+1}'^L \quad (4.31)$$

Hence

$$\underline{z}_{n+1}'^R = P'_{ov} U' \underline{z}_1'^R \quad (4.32)$$

By substituting P'_{ov} from Eq. 3.55, multiplying the matrices P'_{ov} and U' , expanding, and noting that $h_1^R = 0$, $h_{n+1}^R = 0$, and $q_{n+1}^L = q_{n+1}^R$, one obtains

$$q_1^R = - \frac{u_{23} - (2H_o/Q_o)u_{13} + (2H_o k/\tau_o)u_{33}}{u_{21} - (2H_o/Q_o)u_{11} + (2H_o k/\tau_o)u_{31}} \quad (4.33)$$

$$q_{n+1}^L = u_{11} q_1^R + u_{13} \quad (4.34)$$

in which u_{11} , u_{12} ,, u_{33} are the elements of the matrix, U' . By expanding Eq. 4.30 and noting that $h_1^R = 0$ one obtains

$$h_{n+1}^L = u_{21} q_1^R + u_{23} \quad (4.35)$$

Extended field and point matrices are first computed. Then the extended overall transfer matrix is determined by multiplying the field and point matrices starting at the downstream end, i.e.,

$$U' = F'_n P'_n \dots P'_2 F'_1 \quad (4.36)$$

The value of q_1^R is determined from Eq. 4.33 and q_{n+1}^L and h_{n+1}^L are computed from Eqs. 4.34 and 4.35. The absolute values of h_{n+1}^L and q_{n+1}^L are the amplitudes of pressure head and discharge fluctuations at the valve, and their arguments are, respectively, the phase angles between head and τ^* , and between discharge and τ^* .

If there is no other forcing function except the oscillating valve at the downstream end of the system, ordinary field and point matrices may be used instead of the extended ones. In this case $u_{13} = u_{23} = u_{31} = 0$ and $u_{33} = 1$ in Eqs. 4.33-4.35.

4. PROCEDURE FOR DETERMINING THE FREQUENCY RESPONSE

The frequency response of piping systems at a particular point may be determined as follows:

(i) Draw the block diagram and then the simplified block diagram for the system. In the case of simple systems, this step may be omitted.

(ii) In the case of a non-harmonic periodic forcing function, decompose it into a set of harmonics by Fourier analysis and consider one harmonic at a time. For the specified frequency, compute the point and field matrices. A summary of field and point matrices derived in Chapter 3 is presented in Appendix A for an easy reference. To write an extended transfer matrix, simply add the following elements to the transfer matrix listed in Appendix A: $u_{13} = u_{23} = u_{31} = u_{32} = 0$ and $u_{33} = 1$. Note that extended transfer matrices are used only if there is more than one forcing function in the system.

(iii) Calculate the overall transfer matrix by an ordered multiplication of the point and field matrices. For this calculation, the block diagram of step (i) is very helpful. For multiplication of matrices the scheme outlined in appendix B may be followed if the calculation is done by hand, slide rule or desk calculator. This scheme will considerably reduce the amount of computations.

(iv) Use the expressions developed in Sections 4.2-1 through 4.2-3 to determine the frequency response.

(v) If a frequency response diagram is to be plotted repeat the above procedure by taking different frequencies.

Computations to determine the frequency response at the valve end of a branch system having an oscillating valve at the downstream end are presented in Appendix B to illustrate the above procedure.

4.3 PRESSURE AND DISCHARGE VARIATION ALONG PIPELINE

The previous sections dealt with the determination of the pressure and discharge oscillations at the end sections of a system. Sometimes it is, however, necessary to determine the amplitudes of the discharge and pressure fluctuations along the length of the pipeline. In this section, a procedure to determine the amplitudes of discharge and pressure fluctuations along the length of the pipeline is first outlined. Then expressions to find the location of nodes and antinodes are derived.

To analyze a piping system two of the four quantities—discharge and pressure or their relationship at either end of the system—must be known. The other two quantities can then be calculated by using the equations derived in the last section. The amplitudes of the discharge and pressure fluctuations at the upstream end being known, their amplitudes along the pipeline may be determined as follows. The procedure is illustrated by discussing a system having a reservoir at the upstream end and an oscillating valve at the downstream end. Similarly, equations for other systems having different boundary conditions can be developed.

Suppose that the amplitudes of discharge and pressure at the k^{th} section on the i^{th} pipe (see Fig. 4.2a) are to be determined. Let the transfer matrix relating the state vectors at the first section of the first and of the i^{th} pipe be designated by W , i.e.,

$$(\underline{z}_1^R)_i = W(\underline{z}_1^R)_i \quad (4.37)$$

and the field matrix relating the state vectors at the first and the k^{th} section of the i^{th} pipe by F_x , i.e.,

$$(\underline{z}_k^L)_i = F_x(\underline{z}_1^R)_i \quad (4.38)$$

The matrix W is computed by multiplying the point and field matrices for the first $(i-1)$ pipes in a proper sequence (see the block diagram of Fig. 4.2b), i.e.,

$$W = P_i F_{i-1} P_{i-1} \dots F_1 \quad (4.39)$$

and the matrix F_x is calculated by replacing ℓ with x in Eq. 3.18. Note that the elements of the matrix W for a specified frequency are constants while those of the matrix F_x depend upon the value of x as well.

It follows from Eqs. 4.37 and 4.38 that

$$(z_k^L)_i = S(z_1^R)_1 \quad (4.40)$$

in which

$$S = F_x W = F_x P_i F_{i-1} P_{i-1} \dots F_1 \quad (4.41)$$

The value of $(q_1^R)_1$ is calculated from Eq. 4.33. Furthermore it is known that $(h_1^R)_1 = 0$. Substitution of these values into the expanded form of Eq. 4.40 yields

$$(q_k^L)_i = s_{11} (q_1^R)_1 \quad (4.42)$$

and

$$(h_k^L)_i = s_{21} (q_1^R)_1 \quad (4.43)$$

The amplitudes of the discharge and pressure fluctuations at any other section can be determined by proceeding in a similar manner.

LOCATION OF PRESSURE NODES AND ANTINODES

The location of pressure nodes and antinodes is an important aspect of the analysis of resonance in pipelines at higher harmonics. The amplitude of the pressure fluctuation is a minimum at a node and a maximum at an antinode. For a frictionless system, the amplitude of pressure fluctuations at the node is zero.

At the antinodes the pipelines may be subjected to severe pressure fluctuations. Thus the pipe may burst due to pressure in excess of the design pressure or may collapse due to subatmospheric pressure. A surge tank becomes inoperative in preventing the appearance of pressure waves upstream of the tank if a node is formed at its base. Jaeger explained

(a) Piping system.

(b) Block diagram.

Fig. 4.2 Designation of k^{th} section on i^{th} pipe.

the development of fissures in the Kandergrund tunnel [27, 28] due to the establishment of a pressure node at the tank which made the tank inoperative although it was over-designed.

The locations of the nodes and antinodes may be determined as follows:

Eq. 4.43 gives the amplitude of the pressure fluctuation at a point. By making use of the fact that for a frictionless system, the amplitude of pressure fluctuations at a nodal point is zero and $q_1^R \neq 0$ for non-trivial solutions, one obtains

$$s_{21}(x) = 0 \quad (4.44)$$

The solution of this equation for x gives the location of the nodes on the i^{th} pipe.

The amplitude of pressure fluctuation is a maximum at the antinodes. The location of these points may be determined by differentiating Eq. 4.44 with respect to x , equating the result to zero and then solving for x , i.e., the roots of the equation

$$\frac{d}{dx} s_{21}(x) = 0 \quad (4.45)$$

give the location of the antinodes.

By making use of Eqs. 4.44 and 4.45 expressions for the location of nodes and antinodes in simple systems can be derived. The procedure is illustrated below by deriving expressions for a simple pipe and for two pipes in series. Expressions for complex systems may be derived in a similar manner. However, it is better to solve Eq. 4.44 and 4.45 numerically rather than to derive the expressions and then solve them.

On the basis of the Eq. 3.18, for a frictionless single pipeline having constant cross-sectional area, Eq. 4.44 becomes

$$-j C_1 \sin(\omega x/a_1) = 0 \quad (4.46)$$

or

$$\sin (\omega x/a_1) = 0 \quad (4.47)$$

whose solution gives

$$x = n\pi a_1/\omega \quad (n = 0, 1, 2, \dots) \quad (4.48)$$

The values of $x > \ell_1$ represent the locations of the imaginary nodes which are discarded. It follows from Eqs. 4.45 and 4.47 that

$$\cos (\omega x/a_1) = 0 \quad (4.49)$$

The solution of this equation gives the locations of antinodes, i.e.,

$$x = (n + 1/2)\pi a_1/\omega \quad (n = 0, 1, 2, \dots) \quad (4.50)$$

Again the values of $x > \ell_1$ are the locations of imaginary nodes and are discarded.

Eqs. 4.47 and 4.49 show that a standing wave is formed along the length of the pipeline. It is clear from Eq. 4.46 that the pressure variation is sinusoidal.

Series system:

In a series system having two pipes (Fig. 2.3), the locations of nodes and antinodes in the pipe leading from the reservoir are given by Eqs. 4.47 and 4.49. However, their location in the second pipe can be determined by using Eqs. 4.44 and 4.45.

By substituting the expressions for F_x , F_1 , and p_2 into Eq. 4.39, multiplying the matrices, and using Eq. 4.44, one obtains

$$-C_2 \sin(\omega x/a_2) \cos(\omega \ell_1/a_1) - C_1 \cos(\omega x/a_2) \sin(\omega \ell_1/a_1) = 0 \quad (4.51)$$

which upon simplification becomes

$$\tan \frac{\omega x}{a_2} = - \frac{a_1 A_2}{a_2 A_1} \tan \frac{\omega \ell_1}{a_1} \quad (4.52)$$

Note that Eqs. 4.51 and 4.52 are valid for a frictionless system only.

Since a_1 , a_2 , A_1 , and A_2 are positive constants, the right hand side of Eq. 4.50 is positive or negative if $\tan(\omega \ell_1/a_1)$ is negative or positive respectively. Hence, it follows from Fig. 4.3 that no node is established

Fig. 4·3 Plot of $\tan \frac{\omega x}{a_2} \sim \frac{\omega x}{a_2}$

in the intervals $0 \leq x \leq \pi a_2/2\omega$, $\pi a_2/\omega \leq x \leq 3\pi a_2/2\omega$, ... if $0 < \omega l_1/a_1 < \pi/2$ or $\pi < \omega l_1/a_1 < 3\pi/2$; and in the intervals $\pi a_2/2\omega \leq x \leq \pi a_2/\omega$, $3\pi a_2/2\omega \leq x \leq 2\pi/\omega$, ..., if $\pi/2 < \omega l_1/a_1 < \pi$ or $3\pi/2 < \omega l_1/a_1 < 2\pi$.

CHAPTER FIVE

VERIFICATION OF TRANSFER MATRIX METHOD

To demonstrate the validity of the method presented herein, the results obtained by the transfer matrix method are compared with experimental values and with those determined by the method of characteristics, by the impedance theory, and by energy concepts. A brief description of each method, its advantages and limitations are also presented

5.1 EXPERIMENTAL RESULTS

Except for the laboratory and field tests reported by Camichel and his collaborators [8], few experimental results on the resonating characteristics of pipes are available in the literature. In the tests reported by Camichel et al., resonating conditions in series pipes were established by a rotating cock located at the downstream end of the pipeline. Each system had a reservoir of constant head at the upstream end. The data for these systems are given in Fig. 5.1.

The values of the periods of the fundamental and higher harmonics determined experimentally and by the procedure outlined in section 4.1 are given in Table I. As can be seen, close agreement is found between the experimental values and those determined by the transfer matrix method.

(a) Toulouse pipeline.

(b) Fully pipeline.

Fig. 5-1 Longitudinal profile.

Fig. 5-1 Longitudinal profile.

(d) Pipeline P_3 .

Fig. 5-1 Longitudinal profile.

Fig. 5.2 Plot of residual $\sim \omega$ for Toulouse system .

TABLE I. CALCULATED AND MEASURED PERIODS

System (1)	No. of Pipes (2)	Theoret- ical period in seconds (3)	Periods in Seconds											
			Fundamental		3rd Harmonic		5th Harmonic		7th Harmonic		9th Harmonic		11th Harmonic	
			calc. (4)	meas. (5)	calc. (6)	meas. (7)	calc. (8)	meas. (9)	calc. (10)	meas. (11)	calc. (12)	meas. (13)	calc. (14)	meas. (15)
Toulouse	2	0.932	0.708	0.69	0.311	0.31	0.198	0.19	-	-	-	-	-	-
Fully	2	15.96	13.719	13.50	-	-	-	-	-	-	-	-	-	-
C ₄	15	2.008	1.887	1.882	-	-	-	-	-	-	-	-	-	-
F ₃	9	1.464	1.405	1.368	0.502	0.505	0.296	0.310	0.2117	0.2150	0.1650	0.1667	0.1338	0.1420

For purposes of illustration the plot of residual h_w used to determine the resonant frequencies of the Toulouse system is presented in Fig. 5.2.

5.2 METHOD OF CHARACTERISTICS

The method of characteristics [9, 31, 45-47, 50] is well established for the analysis of the transient state, and steady-oscillatory flows in piping systems. The unsteady flow is represented by nonlinear, hyperbolic partial differential equations. Without neglecting or linearizing the nonlinear terms, these partial differential equations are converted into ordinary differential equations which are then solved by a finite-difference technique. To ensure convergence and stability of the numerical procedure [9, 50], the time interval, Δt , must be selected such that $\Delta t < \Delta x/a$, in which Δx is the length of one of the reaches into which a pipe is divided.

To analyze the steady-oscillatory flows by this method, the initial steady state velocity and pressure head are assumed equal to their mean values. The prescribed forcing function is then imposed as a boundary condition and the system is analyzed by considering one frequency at a time. The amplitudes of the pressure head and of the discharge fluctuations are determined when a steady-oscillatory regime is established, i.e., when the initial transients have vanished. The process of convergence to steady-oscillatory conditions is slow and requires a considerable amount of computer time thus making it uneconomical for general studies. In addition, the "round off" error introduced in the computations during each time interval might outweigh the accuracy of the method resulting from the inclusion of the nonlinear terms.

A number of systems—series, parallel and branch systems with the side branch having various boundary conditions—are analyzed using the transfer matrix method and the method of characteristics. The data for these systems

and the frequency response diagrams are presented in Fig. 5.3 through 5.9.

The frequency response diagrams are presented in non-dimensional form. The frequency ratio, ω_r , is defined as ω/ω_{th} ; the pressure head ratio, h_r , as $2|h_{n+1}^L|/H_0$; and the discharge ratio, q_r , as $2|q_{n+1}^L|/Q_0$. The values of h_r and q_r determined by the method of characteristics represent the amplitude of the swing from the minimum to the maximum value. The frequency of the forcing function is designated by ω .

The oscillating valves are the forcing functions in all the systems except the dead end series system of Fig. 5.3 in which the fluctuating pressure head at the upstream end is the forcing function. The valve movement is taken as sinusoidal with $\tau_0 = 1.0$ and $k = 0.2$. The fluctuating pressure head in Fig. 5.3 is also sinusoidal with $K = 1.0$. In the branch systems of Fig. 5.8, $\tau_0 = 1.0$ and $k = 0.2$.

The series system of Fig. 5.4 is analyzed by the transfer matrix method to study the effect of the friction losses and of the variation in the mean discharge on the frequency response. Frequency response diagrams for the system having friction losses about 35% of the static head and having no friction losses is shown in Fig. 5.10. The data are the same as given above except that $Q_0 = 9.42$ ft/sec. For calculating the system response by the method of characteristics the friction losses are taken proportional to the square of velocity. The resonant frequencies determined by considering the system as frictionless and by taking the friction losses into consideration are recorded in Table II. It is clear from Fig. 5.10 and Table II that the friction losses, even as high as 35%, have a negligible effect on the values of resonant frequencies. Hence, in the analysis of other systems presented in this study, friction losses are neglected. This simplifies the programming and reduces the amount of

$l = 1000\text{ft.}$	$l = 2000\text{ft.}$	$l = 1600\text{ft.}$	$l = 2150\text{ft.}$
$D = 4.0\text{ft.}$	$D = 3.5\text{ft.}$	$D = 3.0\text{ft.}$	$D = 2.5\text{ft.}$
$a = 4000\text{ft/sec.}$	$a = 3000\text{ft/sec.}$	$a = 2000\text{ft/sec.}$	$a = 4300\text{ft/sec.}$

(a) Piping system.

(b) Frequency response diagram.

Fig. 5.3 Series system with dead end.

(a) Piping system .

(b) Frequency response diagram .

Fig. 5-4 Series system .

(b) Frequency response diagram.

Fig. 5.5 Branch system(side branch having reservoir).

(a) Piping system .

(b) Frequency response diagram .

Fig. 5.6 Branch system(side branch having dead end).

(a) Piping system.

(b) Frequency response diagram.

Fig. 5.7 Branch system (side branch having orifice).

(a) Piping system.

Fig. 5-8 Branch system(side branch having oscillating valve).

(b) Frequency response diagram (valve No.1 and No.2 are in phase).

Fig. 5-8 Branch system (side branch having oscillating valve).

(c) Frequency response diagram (valve No.2 lagging No.1 by 90°) .

Fig.5-8 Branch system(side branch having oscillating valve) .

(d) Frequency response diagram (valve No.2 lagging No.1 by 180°)

Fig. 5·8 Branch system (side branch having oscillating valve).

(a) Piping system.

(b) Frequency response diagram.

Fig. 5.9 Parallel system.

computer time required. Frequency response diagram for the series system for two different discharges— $Q_0 = 9.42$, and 0.314 cu. ft/sec—is shown in Fig. 5.11. Friction losses are neglected in calculating the response.

Analysis of the frictional system by the method of characteristics showed that the amplitudes of the positive swing of pressure head and negative swing of the discharge are larger than the corresponding negative and positive swings. This is caused by the nonlinear terms, especially the friction loss term, of the governing differential equations. In the transfer matrix method, however, amplitudes of the positive and negative oscillations are equal because a sinusoidal solution is assumed.

To check the values of the phase angles between different quantities of interest, the oscillatory discharge and pressure head at the valve are computed by using the method of characteristics. The $q_r^* \sim t$, $h_r^* \sim t$, and $\tau^* \sim t$ curves are plotted in Fig. 5.12. In this diagram, $h_r^* = h^*/H_0$ and $q_r^* = q^*/Q_0$. The phase angles determined by the transfer matrix method and by the method of characteristics are presented in Table III. Close agreement is found between the results obtained by the two methods.

5.3 IMPEDANCE METHOD

The concept of impedance was introduced by Rocard [43] and later used by Paynter [38], Waller [53-55] and Wylie [58-60]. The pressure head and the discharge fluctuations are assumed sinusoidal, nonlinear relationships are linearized and the ratio of the oscillatory pressure head and discharge is termed the hydraulic impedance. For the given boundary conditions of the system, the terminal impedance is calculated and the impedance diagram is plotted. Those frequencies at which terminal impedance is a maximum are the natural frequencies of the system. If the value of either h or q is known, the other can be calculated from the computed value of the

Fig. 5·10 Effect of friction losses on frequency response of series system of Fig. 5·4 a .

Fig. 5.11 Effect of mean discharge on frequency response of series system of Fig. 5.4 a .

TABLE II.

RESONANT FREQUENCIES OF PIPING SYSTEM OF FIG. 5.4a

System (1)	$\omega_r = \omega/\omega_{th}$					
	Funda- mental (2)	3rd Harm. (3)	5th Harm. (4)	7th Harm. (5)	9th Harm. (6)	11th Harm. (7)
Friction considered	1.259	3.000	4.741	7.259	9.000	10.741
Friction neglected	1.260	3.000	4.740	7.259	9.000	10.741

(a) Series system of Fig. 5.4a.

Fig. 5.12 $h_r^* \sim t$, $q_r^* \sim t$, and $\tau^* \sim t$ curves.

(i) $\omega_r = 2.5$

(ii) $\omega_r = 3.0$

(b) Branch system of Fig. 5.6a.
Fig. 5.12

(i) $\omega_r = 2.5$

(ii) $\omega_r = 3.0$

(c) Branch system of Fig. 5.8a (Valve No.1 and 2 in phase).

Fig. 5.12 $h_r^* \sim t$, $q_r^* \sim t$, and $zeta^* \sim t$ curves.

TABLE III. PHASE ANGLES

System (1)	Frequency ratio, ω_r (2)	Phase Angles, ϕ , in degrees			
		between h and τ^*		between q and τ^*	
		Transfer Matrix Method (3)	Method of Character- istics (4)	Transfer Matrix Method (5)	Method of Character- istics (6)
Series (Fig. 5.4a)	2.5	-110.99	-110.50	-20.99	-20.5
	3.0	-180.01	-180.00	-270.01	--
Branch (Fig. 5.6a) (Side branch with dead end)	2.5	-117.90	-119.00	-27.90	-29.50
	3.0	-180.01	-180.00	-270.01	--
Branch (Fig. 5.8a) (Side branch with oscillating valve)	2.5	-117.13	-118.00	-18.17	-18.00
	3.0	-180.01	-180.00	-270.01	--

(a) Piping system.

(b) Impedance diagram.

Fig. 5.13 Series system.

(a) Piping system.

(b) Impedance diagram.

Fig. 5-14 Branch system (side branch having reservoir).

(a) Piping system.

(b) Impedance diagram.

Fig. 5-15 Branch system(side branch having dead end).

Fig. 5.16 Impedance diagram for parallel system of Fig. 5.9a .

Energy entering the system during the time interval Δt , is

$$E_{in} = \gamma Q H \Delta t \quad (5.1)$$

in which γ = specific weight of the fluid and the subscript "in" refers to the input quantities. Substitution of Eqs. 2.1 and 2.2 into Eq. 5.1 and expansion of the resulting equation yield

$$\Delta E_{in} = \gamma (Q_o H_o + q_{in}^* H_o + h_{in}^* Q_o + q_{in}^* h_{in}^*) \Delta t \quad (5.2)$$

Let q_{in}^* and h_{in}^* be sinusoidal, i.e.,

$$h_{in}^* = h_{in}' \cos \omega t \quad (5.3)$$

$$q_{in}^* = q_{in}' \cos(\omega t - \phi_{in}) \quad (5.4)$$

in which ϕ_{in} = phase angle between q_{in}^* and h_{in}^* and h_{in}' and q_{in}' are the amplitudes of pressure and discharge fluctuations. Note that both h_{in}' and q_{in}' are real quantities. The energy input during one cycle may be calculated by substituting Eqs. 5.3 and 5.4 into Eq. 5.2 and integrating the resulting equation over one period, T . This process gives

$$E_{in} = \gamma Q_o H_o T + \gamma q_{in}' h_{in}' \int_0^T \cos \omega t \cos(\omega t - \phi_{in}) dt \quad (5.5)$$

If there is a reservoir of constant level at the upstream end, then

$h_{in}' = 0$. Hence, Eq. 5.5 becomes

$$E_{in} = \gamma Q_o H_o T \quad (5.6)$$

By proceeding in a similar manner

$$E_{out} = \gamma Q_o H_o T + \gamma h_{out}' q_{out}' \int_0^T \cos \omega t \cos(\omega t - \phi_{out}) dt \quad (5.7)$$

The subscript "out" designates output quantities.

If the losses in the system are neglected, then $E_{in} = E_{out}$ for establishment of steady-oscillatory conditions. Hence, it follows from Eqs. 5.6 and 5.7 that

$$\int_0^T \cos \omega t \cos(\omega t - \phi_{out}) dt = 0 \quad (5.8)$$

which yields

$$\phi_{out} = 90^\circ \quad (5.9)$$

For all the systems analyzed in this study, ϕ_{out} was 90° . The only exceptions were the branch systems with the side branch having an orifice or an oscillating valve. A close examination of the above derivation reveals that Eqs. 5.8 and 5.9 do not hold in these cases because there is energy output at more than one point. By following a similar procedure as above, one obtains

$$\begin{aligned} E_{out} = & \gamma Q_{om} H_{om} T + \gamma h'_{out} q'_{out} \int_0^T \cos \omega t \cos(\omega t - \phi_{out}) dt \\ & + \gamma \tilde{Q}_o \tilde{H}_o T + \gamma \tilde{h}'_{out} \tilde{q}'_{out} \int_0^T \cos \omega t \cos(\omega t - \phi_{out}) dt \end{aligned} \quad (5.10)$$

in which a tilde designates quantities for the side branch and the subscript "m", the downstream end of the main.

For a steady-oscillatory condition to exist in a frictionless system, $E_{in} = E_{out}$. Hence, by comparing Eqs. 5.6 and 5.10, noting that $\gamma Q_o H_o T = \gamma Q_{om} H_{om} T + \gamma \tilde{Q}_o \tilde{H}_o T$ and simplifying the resulting equation, one obtains

$$h'_{out} q'_{out} \cos \phi_{out} + \tilde{h}'_{out} \tilde{q}'_{out} \cos \tilde{\phi}_{out} = 0 \quad (5.11)$$

This equation is used herein to verify the numerical values of different quantities determined by using the transfer matrix method. The computations for the branch system of Fig. 5.8 are recorded in Table IV.

5.5 STUDIES ON PIPELINE HAVING VARIABLE CHARACTERISTICS

The resonating characteristics of a pipeline having linearly variable characteristics—A and a—along its length, a constant head reservoir at the upstream end, and an oscillating valve at the downstream end, Fig.

5.17a, are studied by using the transfer matrix method. Frequency response

TABLE IV. VERIFICATION BY ENERGY CONCEPTS

Branch system of Fig. 5.8

System	ω_r	h'_{out} ft.	q'_{out} cfs	ϕ_{out} degrees	$\tilde{h}'_{out}$ ft.	$\tilde{q}'_{out}$ cfs	$\tilde{\phi}_{out}$ degrees	$\cos \phi_{out}$	$\cos \tilde{\phi}_{out}$	$h'q' \cos \phi_{out} + \tilde{h}'\tilde{q}' \cos \tilde{\phi}_{out}$
Valve No. 1 and No. 2 in phase	2.0	4.4	.0620	98.92	2.2	.0637	72.57	-.1552	.3000	-.0425 + .0425 $\approx$ 0.0
	3.2	23.0	.0430	104.69	9.9	.0669	67.83	-.2536	.3773	-.2520 + .2510 $\approx$ 0.0
Valve No. 1 leading No. 2 by 90°	2.0	9.3	.0672	66.94	4.5	.0559	169.00	.3909	-.9816	.2445 - .2465 $\approx$ 0.0
	3.5	18.5	.0458	112.44	9.5	.0688	60.53	-.3816	.4919	-.3230 + .3220 $\approx$ 0.0
Valve No. 1 leading No. 2 by 180°	2.0	12.5	.0605	86.85	5.9	.0609	96.62	.0548	-.1161	.0416 - .0417 $\approx$ 0.0
	3.6	20.6	.0562	85.72	9.8	.0587	98.65	.0746	-.1504	.0865 - .0866 $\approx$ 0.0

(a) Piping system.

(b) Frequency response diagram.

Fig. 5.17 Pipeline having variable characteristics along its length.

is determined by using the transfer matrix derived in Section 3.1-2. Then the actual pipe is replaced by a substitute pipe having stepwise changes in characteristics as shown in Fig. 5.17a. The expressions presented in Section 4.2-3 are used to determine the frequency response. To compute ω_r , the theoretical period is calculated from the equation

$$T_{th} = 4L/a_m \quad (5.12)$$

in which a_m = velocity of water hammer wave at the midpoint of the pipeline. The results for both the cases are presented in Fig. 5.17b.

Resonant frequencies for the system of Fig. 5.17a were determined by considering the pipe per se and then replacing it with a substitute pipe (shown dotted in Fig. 5.17a). Favre's expression, Eq. 1.13, was used to compute the resonant frequencies. The results are tabulated in Table V. Close agreement is found between the results obtained in these cases up to the fifth harmonic. The higher harmonics can be predicted to a reasonable degree of accuracy by increasing the number of reaches into which the pipeline is divided.

TABLE V
 RESONANT FREQUENCIES OF PIPING SYSTEM OF FIG. 5.17a

Mode (1)	Resonant Frequencies, in rad/sec		
	Favre's expression (2)	Transfer Matrix Method	
		Actual Pipe (3)	Substitute Pipe (4)
Fundamental	15.127	15.075	14.905
Third	35.683	35.702	35.001
Fifth	57.647	57.856	56.375
Seventh	79.963	74.130	77.749

SUMMARY AND CONCLUSIONS

A new method which is known as the transfer matrix method has been presented to analyze resonating characteristics of piping systems. This method is suitable for hand or digital computations. Following the initial derivation of transfer matrices, one does not have to deal with differential equations or lengthy algebraic equations or take into consideration reflection and transmission coefficients in complex systems. In the derivation of transfer matrices, the friction loss term and the gate equation, relating the head and the discharge through a valve, are linearized and the pressure head and discharge fluctuations are assumed sinusoidal. The validity of the method presented herein has been verified by comparing the results with those obtained experimentally, by using the method of characteristics, by the impedance theory, and by the energy concepts.

The following conclusions can be drawn from the analysis of a number of system types commonly found in waterpower development and water supply schemes:

1. The friction losses have a negligible effect on the values of resonant frequencies of a system. Analysis of the frictional system by the method of characteristics shows that the amplitude of the positive swing of pressure head and negative swing of discharge are larger than the corresponding negative or positive swings. This is due to the nonlinear

friction term of the governing differential equation. In the transfer matrix method, however, amplitudes of the positive and negative oscillations are equal because a sinusoidal solution is assumed in the derivation of transfer matrices.

2. The values of the resonant frequencies are independent of the mean discharge or mean pressure head. A change in any of these variables results in a quantitative, not qualitative, modification of the frequency response diagram.
3. The pressure and discharge nodes and antinodes are not necessarily located at the points of geometric changes in the system.
4. The linearization of the gate equation is valid only if h_r is small. For example, for $h_r = 0.8$ (amplitude of pressure oscillations in $0.4 H_0$), this linearization resulted in an overestimation of h_r by 3.5%.
5. The periods of higher harmonics of a system are not necessarily integral fractions of the theoretical period.
6. The frequency response of pipelines having variable characteristics along its length can be calculated by replacing the actual pipe by a substitute pipe having stepwise changes. To predict the higher harmonics accurately, the pipeline should be divided into a larger number of reaches.

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APPENDIX A

SUMMARY OF TRANSFER MATRICES

The transfer matrices derived in Chapter 3 are summarized herein. The notation and equation numbers of Chapter 3 are used here to facilitate cross-reference.

A.1 FIELD MATRICES

(1) For i^{th} pipe having constant properties along its length.

a. Friction taken into consideration

$$F_i = \begin{pmatrix} \cosh \mu_i \ell_i & -\frac{1}{Z_c} \sinh \mu_i \ell_i \\ -Z_c \sinh \mu_i \ell_i & \cosh \mu_i \ell_i \end{pmatrix}, \quad (3.17)$$

b. Friction neglected

$$F_i = \begin{pmatrix} \cos b_i \omega & -\frac{j}{C_i} \sin b_i \omega \\ -jC_i \sin b_i \omega & \cos b_i \omega \end{pmatrix}. \quad (3.18)$$

(2) For pipe having variable characteristics along its length

$$\begin{aligned} F_{vc} = & I + \frac{s}{6} [B(x_i) + 4 B(x_i+s/2) + B(x_{i+1})] \\ & + \frac{s^2}{6} [B(x_i+s/2)B(x_i) + B(x_{i+1})B(x_i+s/2) + B^2(x_i+s/2)] \\ & + \frac{s^3}{12} [B^2(x_i+s/2)B(x_i) + B(x_{i+1})B^2(x_i+s/2)] \\ & + \frac{s^4}{24} [B(x_{i+1}) B^2(x_i+s/2)B(x_i)] \end{aligned} \quad (3.36)$$

(3) Parallel pipes

$$F_p = \begin{pmatrix} \xi/\eta & (\xi\zeta/\eta) - \eta \\ 1/\eta & \zeta/\eta \end{pmatrix} \quad (3.37)$$

A.2 POINT MATRICES

(1) Series connection

$$P_{sc} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.43)$$

(2) Valves and orifices

a. Oscillating valve discharging into atmosphere

$$P'_{ov} = \begin{pmatrix} 1 & 0 & 0 \\ -2H_o/Q_o & 1 & 2H_o k/\tau_o \\ 0 & 0 & 1 \end{pmatrix} \quad (3.53)$$

b. Orifice discharging into atmosphere

$$P_v = \begin{pmatrix} 1 & 0 \\ -2H_o/Q_o & 1 \end{pmatrix} \quad (3.58)$$

c. Orifice at intermediate section

$$P_{vi} = \begin{pmatrix} 1 & 0 \\ -2 \Delta H_o/Q_o & 1 \end{pmatrix} \quad (3.59)$$

(3) Junction of branch and main

a. Dead end branch

$$P_{bde} = \begin{pmatrix} 1 & \tilde{u}_{12}/\tilde{u}_{11} \\ 0 & 1 \end{pmatrix} \quad (3.71)$$

b. Branch with constant head reservoir

$$P_{bres} = \begin{pmatrix} 1 & \tilde{u}_{22}/\tilde{u}_{21} \\ 0 & 1 \end{pmatrix} \quad (3.75)$$

c. Branch having oscillating valve

$$P'_{\text{bov}} = \begin{pmatrix} 1 & P_{12} & P_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (3.88)$$

d. Branch having orifice

$$P_{\text{borf}} = \begin{pmatrix} 1 & P_{12} \\ 0 & 1 \end{pmatrix} \quad (3.89)$$

APPENDIX B

EXAMPLE

The branch system shown in Fig. 5.6a is analyzed herein to illustrate the procedure of calculating the frequency response by using a slide rule or a desk calculator. By taking different values of ω_r and following the procedure outlined below, the frequency response diagram shown in Fig. 5.6b can be plotted.

DATA

$$\begin{aligned}\omega_r &= 2.0 & T_{th} &= 3.0 \text{ sec.} \\ R &= 0.0 & k &= 0.2 \\ \tau_o &= 1.0 & H_o &= 100 \text{ ft} \\ Q_o &= 0.314 \text{ cu ft/sec}\end{aligned}$$

Dimensions of the system are shown in Fig. 5.6a.

Components of transfer matrices

$$\begin{aligned}\omega_{th} &= 2 \pi/3. = 3.094 \text{ rad/sec} \\ \omega &= \omega_r \omega_{th} = 2. \times 3.094 = 6.189 \text{ rad/sec}\end{aligned}$$

Pipe 1:

$$\begin{aligned}b_1 &= \ell_1/a_1 = 2000/4000 = 0.5 \text{ sec} \\ A_1 &= \pi D_1^2/4 = \pi(2)^2/4 = 3.1416 \text{ sq ft} \\ C_1 &= a_1/(gA_1) = 4000/(32.2 \times 3.1416) = 39.542 \text{ sec/ft}^2\end{aligned}$$

Substitution of these values into Eq. 3.18 yields

$$\begin{aligned}f_{11} &= f_{22} = \cos(0.5 \times 6.189) = -0.5 \\ f_{21} &= -39.542 \sin(0.5 \times 6.189) j = -34.244 j \\ f_{21} &= -j \sin(0.5 \times 6.189)/39.542 = -0.022 j\end{aligned}$$

Proceeding in a similar manner, the following field matrix, F_2 , for pipe 2 is obtained

$$F_2 = \begin{pmatrix} 0.500 & - 0.007 j \\ - 102.732 j & 0.500 \end{pmatrix}$$

Branch pipe:

Since the branch pipe is made up of a single pipe

$$\tilde{U} = \tilde{F}$$

Proceeding similarly as above, the following values of the elements of the field matrix for the branch are obtained:

$$\tilde{u}_{11} = 0.5$$

$$\tilde{u}_{12} = - 0.0154 j$$

Substitution of these values into Eq. 3.71 yields the following point matrix for the junction of the branch and the main

$$P_{bde} = \begin{pmatrix} 1.0 & - 0.0308 j \\ 0.0 & 1.0 \end{pmatrix}$$

It is clear from the block diagram shown in Fig. B that

$$U = F_2 P_{bde} F_1$$

These matrices may be multiplied in a schematic manner as shown in Table VI. Since $h_1^R = 0$ (constant head reservoir) the second column in the matrices F_1 , $P_{bde} F_1$, and $F_2 P_{bde} F_1$ is multiplied by zero. Thus the elements in the second column of these matrices are unnecessary and therefore may be dropped. The unnecessary elements in Table VI are indicated by a horizontal dash.

Note that ordinary transfer matrices have been used because there is only one forcing function. Hence, u_{13} , u_{23} , and u_{31} are zero and u_{33} is

Fig.B. Block diagram for branch system of Fig.5-6a .

unity in Eqs. 4.33 through 4.35. Substitution of these values and those for u_{11} and u_{21} calculated in Table VI into Eq. 4.33 yields

$$q_1^R = -0.0584 + 0.0127 j$$

Hence, it follows from Eqs. 4.34 and 4.35 that

$$q_3^L = 0.0600 - 0.0131 j$$

and

$$h_3^L = -1.8134 - 8.3215 j$$

Hence

$$h_r = 2 | h_3^L | / H_o = 0.170$$

$$q_r = 2 | q_3^L | / Q_o = 0.390$$

The phase angle between head and the relative gate opening

$$= \tan^{-1} [(-8.3215)/(-1.8134)]$$

$$= -102.29^\circ$$

The phase angle between discharge and relative gate opening

$$= \tan^{-1} [(-0.0131)/0.0600]$$

$$= -12.29^\circ$$

TABLE VI. SCHEME FOR MULTIPLICATION OF TRANSFER MATRICES

$$\begin{array}{l}
 \begin{array}{c} \\ \\ F_2 \end{array} \begin{array}{c} \\ P_{bde} \\ \\ \end{array} \begin{array}{c} \\ \begin{pmatrix} 1.000 & -0.031 j \\ 0.000 & 1.000 \end{pmatrix} \\ \begin{pmatrix} 0.500 & -0.0073j \\ -102.732j & 0.500 \end{pmatrix} \end{array} \begin{array}{c} \\ \begin{pmatrix} -0.500 & - \\ -34.244 j & - \end{pmatrix} \\ \begin{pmatrix} -1.555 & - \\ -34.244 j & - \end{pmatrix} \\ \begin{pmatrix} -1.0273 & - \\ 142.595 j & - \end{pmatrix} \end{array} \begin{array}{c} \\ \begin{Bmatrix} q \\ 0 \end{Bmatrix}_1^R \\ \begin{Bmatrix} q \\ h \end{Bmatrix}_1^R \\ \begin{Bmatrix} q \\ h \end{Bmatrix}_1^R \end{array} \begin{array}{c} \\ = \underline{z}_2^L \\ = \underline{z}_2^R \\ = \underline{z}_3^L \end{array}
 \end{array}$$