BUCKLING OF THIN PLATES USING THE FRAMEWORK METHOD

by

RAJAN SEN

B. Tech (Hons.) Indian Institute of Technology
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ABSTRACT

Finite element method involving rectangular bar cells capable of imitating elastic action in plane stress and flexure of plates with any value of the Poisson's ratio, is extended to investigation of stability of rectangular plates. This requires formulation of the stability matrix used for solution of the eigenvalue problem, which gives the magnitude of the critical load.

Four different examples are solved and the results, compared with the exact values and the available no bar solutions, are found to be good.

A brief study is also made of the effect of negative extensional and flexural stiffnesses of the members of the cell and suggestion is made on selection of the desirable range for the values of the aspects ratio of the cell as related to the values of μ.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF SYMBOLS

A, A', A''	cross-sectional areas of short side-bars of cells, total, primary, secondary.
A_1, A_1', A_1''	cross-sectional areas of longer side bars of cells, total, primary, secondary.
A_2, A_3	cross-sectional areas of diagonal and corner bars.
D	elastic flexural constant of plate.
E	modulus of elasticity.
F, F_1, F_2	additional stresses in members of secondary bar system.
$[\underline{k}]$, $[\underline{K}]_m$	stiffness matrices of cell and of model.
$[\underline{k}_s]$, $[\underline{K}_s]_m$	stability matrices of cell and of model.
P	plane stress loads.
S	with and without subscripts, bar stresses, direct.
T	increment of total energy of model on buckling.
U	increment of energy of deformation on buckling.
V	increment of potential energy on buckling.
X,Y	components of forces.
X,Y	with subscripts--members of plane stress stiffness matrix.
a	smaller dimension of cell.
b	dimension of plate along y axis
$\underline{e}$, e_i	displacement of point of application of load.
f	critical intensity of load.
k	aspect ratio of cell.
ℓ	length

m	with subscripts--elements of flexural stiffness matrix.
t	thickness of plate.
w	deflection of plate and of bar.
w	with subscripts--node and joint displacements.
x, y	rectangular coordinaty.
z	with subscripts--elements of flexural stiffness matrix.
α	angle of diagonal bar.
β	parameter defining critical load.
γ	elements of stability matrix.
λ	with subscripts--shortening of projection of bar length caused by flexure
$\underline{\delta}, \delta_m, \delta_n$	movements of the nodes.
σ_{cr}	critical normal unit stress.
τ_{cr}	critical unit shear stress.
θ^x, θ^y	with and without subscripts--nodal rotations.
μ	Poisson's Ratio.

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CHAPTER I

INTRODUCTION

Any rigorous mathematical solution of a stability problem must satisfy simultaneously the equilibrium, constitutive relations and the strain compatibility. Unfortunately these requirements can be met fully only in a few buckling problems and this has led to the use of numerical methods which yield only approximate solutions. Of these, the variational or energy methods and the finite difference approach have been the most widely used. Lately, however, a new approximate method called the finite element method has been proposed for the investigation of stability.

This method was introduced in 1941 by Hrennikoff (1)* for determination of stresses and displacements in plates subjected to plane stress and flexure. The method consists of replacing the structure under consideration by an assembly of units or cells composed of bars and conforming almost exactly to the outline of the structure. The cells have a definite repeating pattern and are joined to each other at the corners. The bars of the cells are endowed with elastic properties which are determined from the conditions of equality of deformability of the structure and its model under any arbitrary uniform stress condition. The use of the framework method requires the knowledge of the

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*Numbers signify the references, listed in Bibliography.

relationship between the corner forces and the corner displacements. The corner forces holding the cell in equilibrium, when one of its corners is given a unit movement along one of the coordinate axes, are termed "the distribution factors." The assembly of the distribution factors corresponding to all unit movements of all joints is arranged in the form of a matrix and is called the stiffness matrix of the cell.

Cells not involving bars have also been proposed (2). Unlike the framework cells, the "no bar" cells are a mathematical abstraction but they still allow formation of distribution factors and generation of the stiffness matrix like the bar cells.

The model obtained by replacing the plate with bar or no bar cells must be solved for the movement of the nodes and this involves numerous linear simultaneous equations. With the advent of high speed digital computers this method has become practical and has rapidly gained favor for solving plane stress (3) and plate bending (4) problems. Recently (5) this method has been extended also to the stability problems. For this purpose development of another matrix, the stability matrix is required, as will be discussed in detail in the later chapters.

CHAPTER II

DERIVATION OF PLANE STRESS AND FLEXURE STIFFNESS MATRICES

Cell Properties

The bar model used in stability problems must imitate the action of the plate both in conditions of plane stress while the load is still below the critical intensity, and in flexure as the structure becomes unstable and begins to buckle. The model is formed of equal rectangular cells interconnected at the corners of the rectangles. Different rectangular cells are possible, but the type which will be used in this work is particularly convenient for the instability studies. The plane stress and flexural properties of the cell may be described and determined independently of each other.

Plane Stress Cell

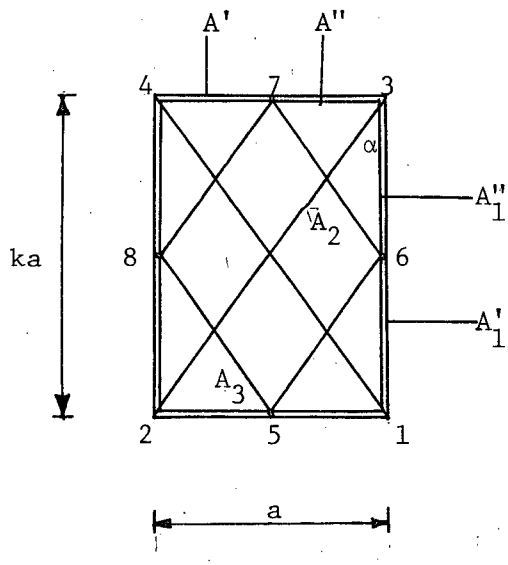


Fig. 2.1

The cell (fig. 2.1) of the dimensions a and ka is made up of two systems of bars, the primary and the secondary. The primary system consists of three kinds of bars: the side bars of the lengths a and ka and the cross sectional areas A' and A_1' respectively and of the diagonals of the areas A_2. The primary system is quite satisfactory to imitate

the deformability of the plate with one particular value of Poisson's ratio. The secondary system is added on to make the model suitable for any value of Poisson's ratio. This system is composed of subdivided side bars of the areas A" and A_1'', two on each edge, joined at the corners to the main system and at the mid-edges to each other and to the corner bars of the areas A_3.

These areas are determined with the assistance of some assumptions, as explained later, from the conditions of equal deformability, as the cell and the plate are subjected to an arbitrary uniform stress.

Cell requirements:

A general condition of an arbitrary uniform stress in a plate may be achieved by a combination of the following strain conditions:

$$\begin{aligned} &\varepsilon x \quad ; \quad \varepsilon y = \gamma xy = 0 \\ &\varepsilon y \quad ; \quad \varepsilon x = \gamma xy = 0 \\ &\gamma xy \quad ; \quad \varepsilon x = \varepsilon y = 0 \end{aligned} \tag{2.1}$$

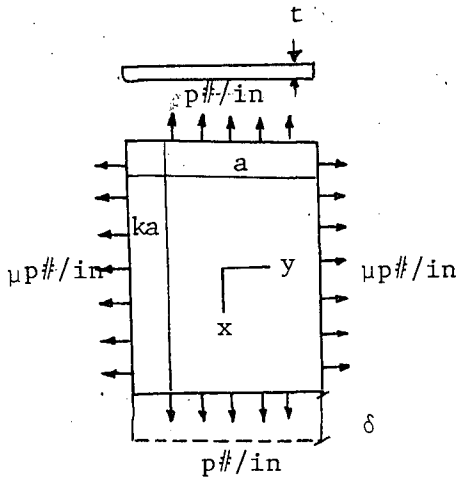
CONDITION 1 (Fig. 2.2)

Uniform stresses p and μp are applied to the plate in the x and y directions respectively. The piece of plate of the size ka by a is elongated in x direction an amount

$$\delta = \frac{p}{tE}(1-\mu^2)\,ka \tag{2.2}$$

while remaining unchanged in length in y direction. The cell of the same size as the plate, must imitate its deformation while acted upon

by the corner forces statically equivalent to the ones in the plate.



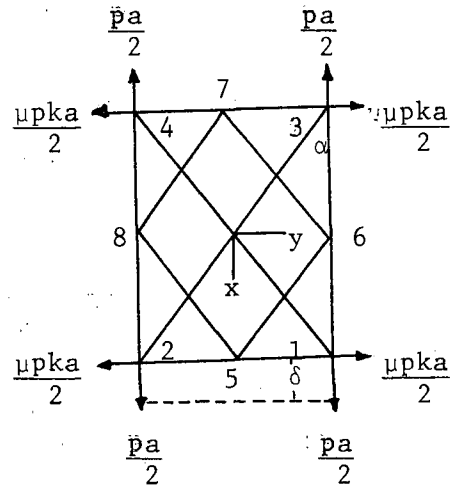


PLATE Fig. 2.2 CELL

These corner forces evidently are $\frac{pa}{2}$ and $\frac{\mu pka}{2}$ respectively in x and y directions. The corner bars of the secondary system do not work: the loading is symmetrical and if one of the corner bars, for example 7-8 is found to be under tension, so should also be the bar 6-7. Since however there is no member at the mid length of the edge 4-3 to resist the transverse component of the stresses in the corner bars these bars must be unstressed.

The subdivided side bars 4-2 and 1-3 simply join the primary bars in carrying the appropriate stresses in spite of slight transverse displacements occurring at the mid-sides of the cells. Their combined area is

$$A_1 = A_1'' + A_1'$$

The bars 1-2 and 4-3, both in the primary and the secondary systems are undeformed, and for this reason their stress X is zero. The elongation

of the diagonal bar (Fig. 2.2) is

$$\delta_2 = \delta \cos\alpha$$

$$= \frac{p\ (1-\mu^2)k^2 a}{tE\ (k^2+1)^{1/2}} \qquad (2.3)$$

The free body diagram for corner 1 is shown in Fig. 2.3. From its equilibrium in Y direction

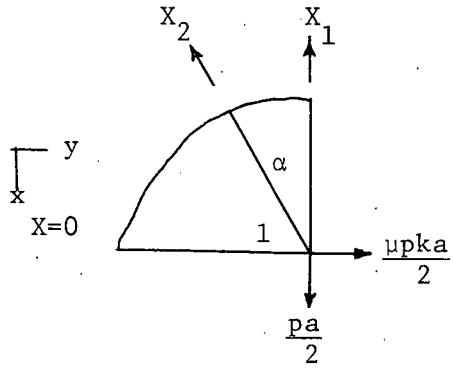


Fig. 2.3

$$X_2 \sin\alpha = \frac{\mu pka}{2} \qquad (2.4)$$

The general relation between the stresses X and the elongation of an elastic bar of the area A and the length is

$$X = \frac{AE\Delta}{\ell} \qquad (2.5)$$

On substituting into Eqn(2.5) the expression for X_2 and δ_2 from the equation (2.4) and (2.3) the expression for the cross section area of the diagonal bar is found

$$A_2 = \frac{\mu at(k^2+1)^{3/2}}{2k(1-\mu^2)} \qquad (2.6)$$

Equilibrium of the corner in X direction gives

$$X_1 = \frac{pa}{2} - X_2 \cos\alpha \qquad (2.7)$$

On replacing X_2 by its expression from (2.4) and X_1 by the equation (2.5), with substitution for the quantity Δ of the elongation δ (Eqn 2.2) of the

bar, the cross sectional area of the latter is found

$$A_1 = \frac{(1-\mu k^2)\ at}{2(1-\mu^2)} \tag{2.8}$$

CONDITION 2 $\varepsilon y,\ \varepsilon x = \gamma xy = 0$

By a similar analysis the total area of the side bar 1-2 is found

$$A = \frac{(k^2-\mu)\ at}{2k(1-\mu^2)} \tag{2.9}$$

where $A = A' + A''$ (2.10)

CONDITION 3 $\gamma xy,\ \varepsilon x = \varepsilon y = 0$

The framework cell (Fig. 2.4) is subjected to the stress condition equivalent to uniform shear in the plate $\tau xy = \tau yx = p^{\#}/in$. The primary side bars are unstressed and so they have been omitted from Fig. 2.4. The diagonal bars are evidently stressed as well as the corner bars and the subdivided side bars.

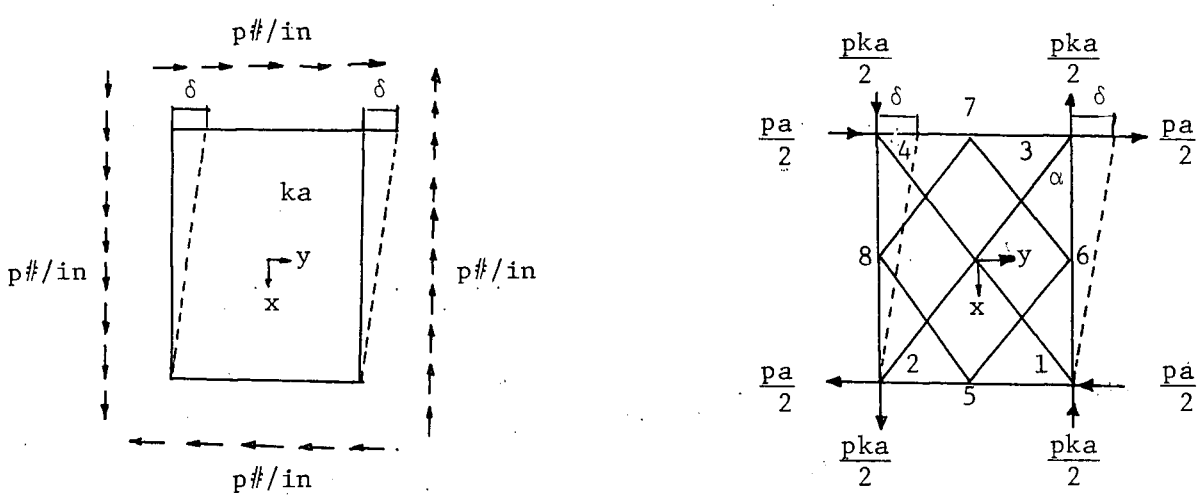


PLATE Fig. 2.4 CELL

The free body diagram for forces acting in different bars at the corner 3 is shown in Fig. 2.5. The resultant of the forces $\frac{pka}{2}$ and $\frac{pa}{2}$ at the corner 3 is

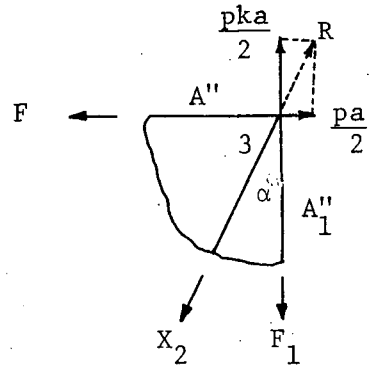


Fig. 2.5

$$R = \frac{pa}{2}(k^2+1)^{1/2} \tag{2.11}$$

Evidently it acts in the direction of the diagonal. The unit shear strain in the plate is

$$\gamma = \frac{2p}{tE}(1+\mu) \tag{2.12}$$

So the horizontal displacement δ, of the corner 3 equal to the shear displacement in the plate is

$$\delta = \frac{2p\ (1+\mu)\ ka}{tE} \tag{2.13}$$

The corresponding change in length of the diagonal bar is

$$\delta_2 = \delta \sin \alpha$$

$$= \frac{2p\ (1+\mu)}{tE} \frac{ka}{(k^2+1)^{1/2}} \tag{2.14}$$

This is an elongation in the bar 2-3 and a shortening in the bar 1-4. Using Eqn 2.5 and substituting for A, Δ, and ℓ their appropriate values it is found that the stress in the diagonal bar is

$$X_2 = \frac{\mu a\ (k^2+1)^{1/2}}{1-\mu} \tag{2.15}$$

The difference R_1, of the corner force R and X_2 must be carried by the subdivided bars (3-7) and (3-6)

Then

$$R_1 = \frac{a\ (k^2+1)^{1/2}(1-3\mu)}{2(1-\mu)} \tag{2.16}$$

The stresses F and F_1 in the bars 3-7 and 3-6 are

$$F = R_1 \operatorname{Sin}\alpha = \frac{a(1-3\mu)}{2(1-\mu)} \tag{2.17}$$

and

$$F_1 = R_1 \operatorname{Cos}\alpha = \frac{ka(1-3\mu)}{2(1-\mu)} \tag{2.18}$$

It may be observed that at all four corners the forces R_1, F and F_1 are of the same magnitude but different in sign.

Now consider one of the intermediate joints, such as 7. Let F_3 be the stress in each of the corner bars. For equilibrium we have (Fig. 2.6)

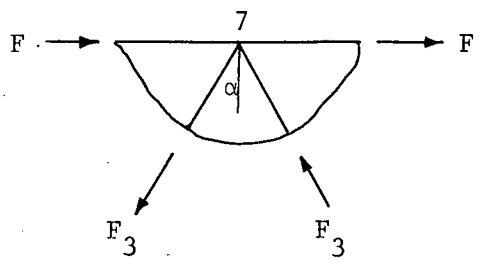


Fig. 2.6

$$2F_3 \operatorname{Sin}\alpha = 2F$$

This gives

$$F_3 = \frac{a(1-3\mu)(k^2+1)^{1/2}}{2(1-\mu)} \tag{2.19}$$

Note, that each corner bar carries a stress opposite in sign to the stresses in edge bars of the same corner.

Deformation of each corner triangle, such as 6-3-7 (Fig. 2.7) may be visualized separately from the other corner triangles. Let the sides 3-7 and 3-6 elongate under their respective stresses F and F_1 by the amounts 7-7' = $\frac{1}{2}$ da and 6-6' = $\frac{1}{2}$ d(ka). The corner bar 7-6 is subjected to compression F_3, and so the points 7' and 6' move

transversely through the distances dx and dy respectively into the positions 7" and 6" as the corner bar shortens by the amount

$$(7\text{-}6) - (7'' - 6'') = d\ell$$

The stresses in the members of the adjacent corner triangle 7-4-8 are equal and opposite to the ones in the triangle 7-3-6 and so their corners 7 and 8 move similarly to the corresponding corners in the previously discussed triangle. The behaviour of the two remaining triangles in Fig. 2.7 is similar to the ones just considered.

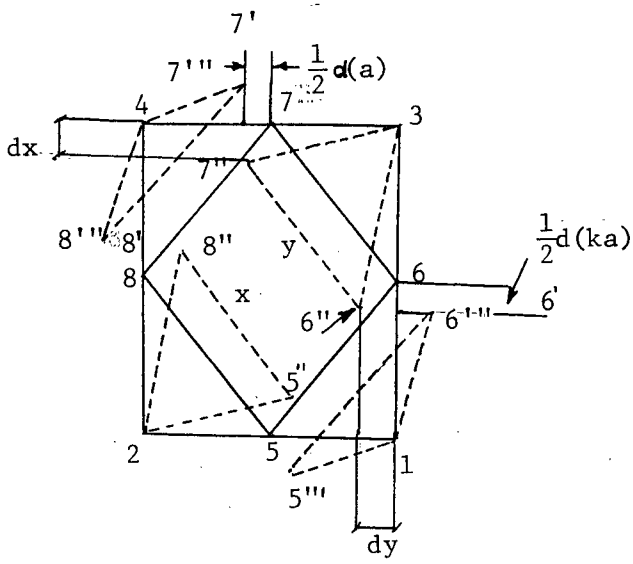


Fig. 2.7

Projecting the displacements 7-7', 7'-7", 6-6' and 6'-6" on the line 7-6 it is found that

$$d\ell = dx\text{Cos}\alpha + dy\text{Sin}\alpha - \frac{1}{2}d(ka)\text{Cos}\alpha - \frac{1}{2}da\text{Sin}\alpha$$

From this

$$(k^2+1)^{1/2}d\ell = kdx + dy - K[\frac{d(ka)}{2}] - \frac{1}{2}da \qquad (2.20)$$

It may be observed in Fig. 2.7 that the deformed portions of the two perpendicular sides of all corner triangles are respectively parallel. to each other. By moving the triangle 8'"-4-7'" to the right a distance 2dy, the triangle 5'"-1-6'" in the negative direction of the x axis a distance 2dx, and the triangle 7"-3-6" the distances 2dx and 2dy along the y and minus x axes, the deformed cell becomes the

parallelogram 2abc (Fig. 2.8). The deviations of the corner angles of this parallelogram from 90° represent the unit shear strain

$$\gamma xy = \frac{2(1+\mu)p}{tE}$$

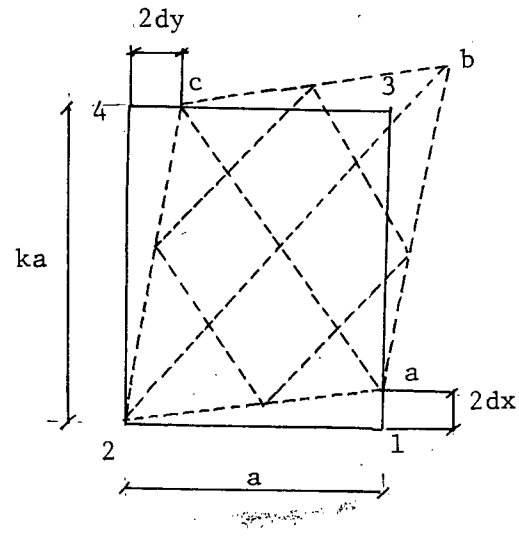


Fig. 2.8

Expressing γxy from Fig. 2.8 and dividing by two

$$(1+\mu)\frac{p}{tE} = \frac{dy}{ka} + \frac{dx}{a} \tag{2.21}$$

Multiplying this equation by ka and subtracting from it Eqn (2.20) we find

$$(k^2+1)\ d\ell + \frac{k}{2}\ d(ka) + \frac{da}{2} = \frac{(1+\mu)pka}{tE} \tag{2.22}$$

The length changes of the three bars $d\ell$, $\frac{1}{2}d(ka)$ and $\frac{1}{2}da$ are expressed through their stresses F_3, F_1 and F and the cross sectional areas A_3, A_1'' and A" still unknown.

This gives

$$\frac{(k^2+1)^{3/2}a}{A_3} + \frac{k^3 a}{A_1''} + \frac{a}{A''} = \frac{4k(1-\mu^2)}{t(1-3\mu)} \tag{2.23}$$

This equation represents the basic relation serving for determination of the three bar areas. It is, of course, insufficient for finding these three values and since no additional equations are available, it will be assumed for simplicity that

$$A_1'' = kA''$$

and

$$A_3 = (k^2+1)^{1/2}A'' \tag{2.24}$$

Substituting Eqn (2.24) into Eqn (2.23) we obtain

$$A'' = \frac{at(1-3\mu)(k^2+1)}{2k(1-\mu^2)} \tag{2.25}$$

$$A_1'' = \frac{at(1-3\mu)(k^2+1)}{2(1-\mu^2)} \tag{2.26}$$

and

$$A_3 = \frac{at(1-3\mu)(k^2+1)^{3/2}}{2k(1-\mu^2)} \tag{2.27}$$

The primary system bars A_1' and A' are found by subtracting the expressions (2.25) and (2.26) from the combined side bar areas Eqn (2.8) and (2.9)

$$A_1' = \frac{at(2\mu k^2+3\mu-k^2)}{2(1-\mu^2)} \tag{2.28}$$

$$A' = \frac{at(3\mu k^2+2\mu-1)}{2(1-\mu^2)} \tag{2.29}$$

Plane Stress Stiffness Matrix

With the bar areas known, the derivation of expressions for the distribution factors is a matter of conventional structural analysis. However, the easiest way of obtaining them is by the use of an auxiliary condition in combination with the three main conditions discussed earlier.

AUXILIARY CONDITION 4

The cell is subjected to a state of deformation in which the four corners are moved in X direction in a flexure-like manner through the distances δ as shown in Fig. 2.9. The members 1-3 and 2-4 are thus stressed, the first in tension and the second in compression with the stresses

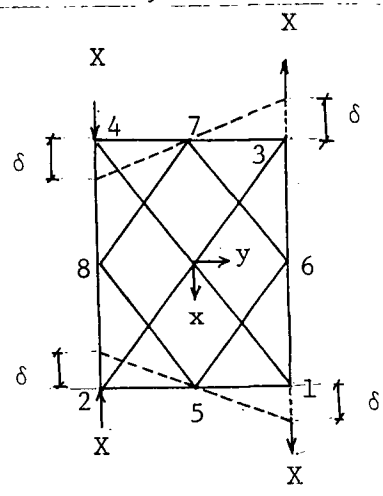


Fig. 2.9

$$X_1 = \frac{A_1 E 2\delta}{ka}$$

$$= \frac{(1-\mu k^2)Et\delta}{k(1-\mu^2)} \qquad (2.30)$$

The symmetry of the loading condition leaves the corner members unstressed. Equally unstressed are the remaining two side members and the diagonals of the cell since they undergo no changes in length. Thus the cell is held in equilibrium by the corner forces X equal to X_1 as shown in Fig. 2.9, with no Y forces contributing to the equilibrium.

ACTION 1

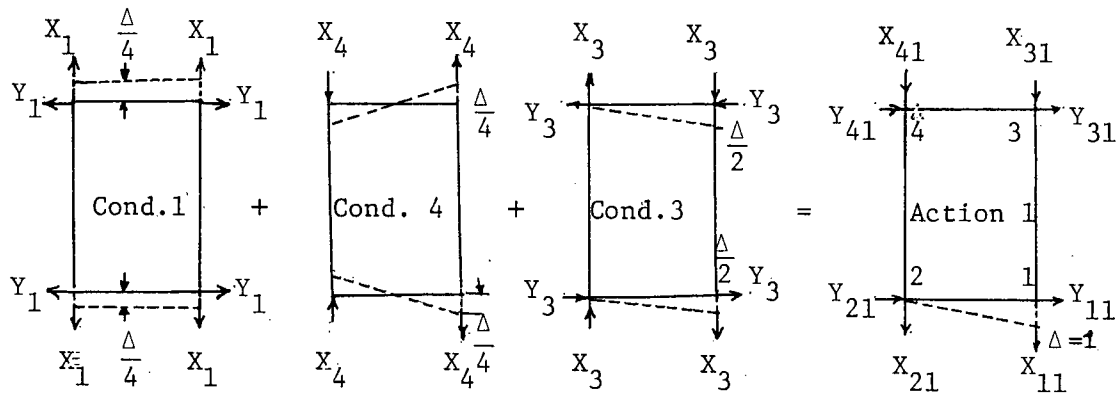


Fig. 2.10

The condition of the corner 1 of the cell being moved a distance Δ_1^x in the x direction, while the other three corners remain in their original positions may be obtained by combination of the three elementary conditions 1, 4 and 3 as is shown in Fig. 2.10. Then the distribution factors corresponding to the displacement Δ_1^x may be found by adding up the known corner forces in the three component conditions.

We obtain

$$X_1 = \frac{tE\Delta}{4k(1-\mu^2)} \qquad Y_1 = \frac{\mu tE\Delta}{4(1-\mu^2)}$$

$$X_3 = \frac{ktE\Delta}{8(1+\mu)} \qquad Y_3 = \frac{tE\Delta}{8(1+\mu)}$$

$$X_4 = \frac{(1-\mu k^2)Et\Delta}{4k(1-\mu^2)} \tag{2.31}$$

It is interesting to observe that these distribution factors are affected by the areas of the bars only to the extent of contribution by the condition 4, since the corner forces in the conditions 1 and 3 are independent of the bar areas.

The distribution factors for this action are given in Table 1. Each term is obtained as

$$X_{11} = X_1 + X_4 + X_3$$

$$X_{21} = X_1 - X_3 - X_4$$

$$X_{31} = X_3 - X_1 - X_4$$

$$X_{41} = X_4 - X_1 - X_3$$

$$Y_{11} = Y_1 + Y_3$$

$$Y_{21} = Y_3 - Y_1$$

$$Y_{31} = Y_1 - Y_3$$

$$Y_{41} = -Y_1 - Y_3$$

The first subscript, in the symbols used here for the distribution factors, describes the number of the joint and the second the number of the action, the vertical displacement of joint 1 being arbitrarily called Action 1 and the horizontal displacement of the same joint, Action 2.

The distribution factors corresponding to the displacements Δ_1^y of the corner 1 may be derived directly by a procedure similar to the one just employed. However if is more convenient to find them by simple manipulations with the Δ_1^x distribution factors as indicated in Fig. 2.11.

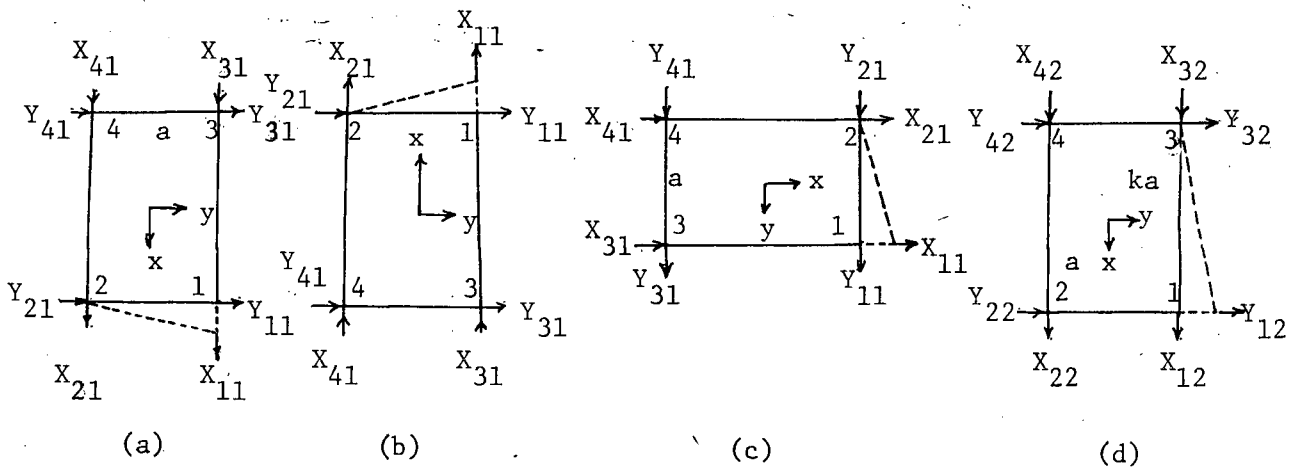


Fig. 2.11

The cell in Fig. 2.11(a) with all its corner forces and displacements, is rotated 180° about the y axis to the position shown in (b). This figure is then turned through 90° about the z axis into the position (c) and is compared with Fig.(d) representing the cell with the displacement Δ_1^y.

The corner forces corresponding the Fig.(d) i.e., the required distribution factors, may be evidently copied from Fig.(c) making allowance for the different placement of the cell. The ratio k of the two sides of the rectangle in Fig.(c) becomes $\frac{1}{k}$ in Fig.(d), and the dimension a in the former figure becomes ka in the latter. With these substitutions the distribution factors for Δ_1^x are transferred into those for Δ_1^y.

The distribution factors for Action 1 and Action 2 given in Table 1 comprise the first two columns of the 8 x 8 stiffness matrix of the cell. The terms of the other columns of the matrix are equal or equal and opposite in sign to some terms in the first two columns and may be obtained from the first two columns by appropriate reversals about some coordinate planes. Thus in order to find $\Delta_2^x = 1$, i.e.

column 3 of the matrix, the cell in Fig. 2.11(a) with all its corner forces and displacements is rotated 180° about the x axis to the position shown in Fig. 2.12(a). The corner forces corresponding to Fig. 2.12(b), i.e. the required distribution factors, may be copied from Fig. 2.12(a).

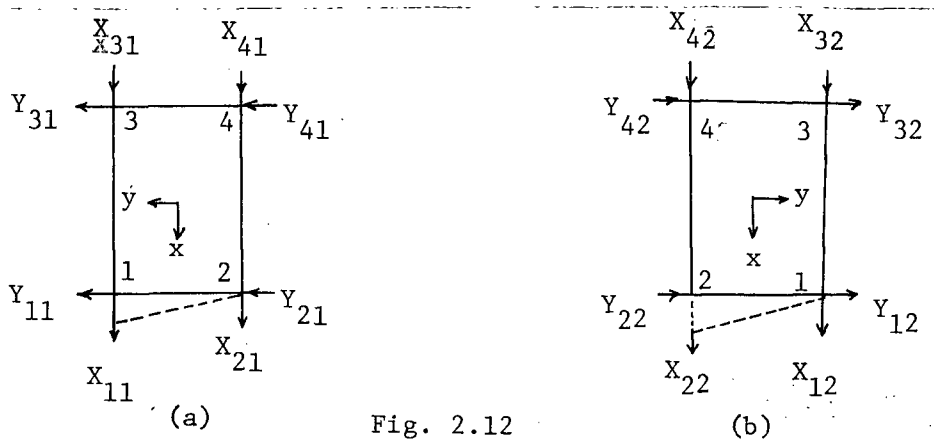


Fig. 2.12

The complete equation $\{\underline{P}\} = [\underline{K}]\{\underline{\Delta}\}$ is

$$
\begin{vmatrix} P_1^x \\ P_1^y \\ P_2^x \\ P_2^y \\ P_3^x \\ P_3^y \\ P_4^x \\ P_4^y \end{vmatrix}
=
\begin{vmatrix}
X_{11} & X_{12} & X_{21} & -X_{22} & X_{31} & -X_{32} & X_{41} & X_{42} \\
Y_{11} & Y_{12} & -Y_{21} & Y_{22} & -Y_{31} & Y_{32} & Y_{41} & Y_{42} \\
X_{21} & X_{22} & X_{11} & -X_{12} & X_{41} & -X_{42} & X_{31} & X_{32} \\
Y_{21} & Y_{22} & -Y_{11} & Y_{12} & -Y_{41} & Y_{42} & Y_{31} & Y_{32} \\
X_{31} & X_{32} & X_{41} & -X_{42} & X_{11} & -X_{12} & X_{21} & X_{22} \\
Y_{31} & Y_{32} & -Y_{41} & Y_{42} & -Y_{11} & Y_{12} & Y_{21} & Y_{22} \\
X_{41} & X_{42} & X_{31} & -X_{32} & X_{21} & -X_{22} & X_{11} & X_{12} \\
Y_{42} & Y_{42} & -Y_{31} & Y_{32} & -Y_{21} & Y_{22} & Y_{11} & Y_{12}
\end{vmatrix}
\times
\begin{vmatrix} \Delta_1^x \\ \Delta_1^y \\ \Delta_2^x \\ \Delta_2^y \\ \Delta_3^x \\ \Delta_3^y \\ \Delta_4^x \\ \Delta_4^y \end{vmatrix}
$$

TABLE I

FIRST TWO COLUMNS OF PLANE

STRESS STIFFNESS MATRIX

Action 1 $\Delta_1^x = 1$	Action 2 $\Delta_1^y = 1$
$X_{11} = \dfrac{4+k^2\ (1-3\mu)}{8k\ (1-\mu^2)}\ Et$	$X_{12} = \dfrac{Et}{8(1-\mu)}$
$X_{21} = -\dfrac{(1-3\mu)\ k}{8(1-\mu^2)}\ Et$	$X_{22} = -\dfrac{(1-3\mu)}{8(1-\mu^2)}\ Et$
$X_{31} = \dfrac{-4+k^2\ (1+\mu)}{8k(1-\mu^2)}\ Et$	$X_{32} = \dfrac{(1-3\mu)}{8(1-\mu^2)}\ Et$
$X_{41} = -\dfrac{k}{8(1-\mu)}\ Et$	$X_{42} = \dfrac{-Et}{8(1-\mu)}$
$Y_{11} = \dfrac{Et}{8(1-\mu)}$	$Y_{12} = \dfrac{1-3\mu+4k^2}{8k(1-\mu^2)}\ Et$
$Y_{21} = \dfrac{1-3\mu}{8(1-\mu^2)}\ Et$	$Y_{22} = \dfrac{1+\mu-4k^2}{8k(1-\mu^2)}\ Et$
$Y_{31} = -\dfrac{(1-3\mu)}{8(1-\mu^2)}\ Et$	$Y_{32} = \dfrac{-(1-3\mu)}{8k(1-\mu^2)}\ Et$
$Y_{41} = -\dfrac{Et}{8(1-\mu)}$	$Y_{42} = -\dfrac{Et}{8k(1-\mu)}$

FLEXURE CELL

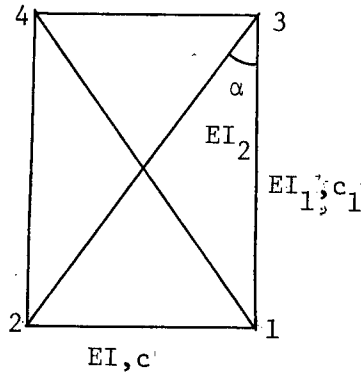


Fig. 2.13

The rectangular flexural bar cell (Fig. 2.13) which may be used in conjunction with the described plane stress cell consists of the side members and the diagonal members, endowed with flexural stiffnesses EI, EI_1 and EI_2 for bending out of the plane of the cell and zero stiffnesses for bending in its plane. Since the bar cell of the kind described here was found capable of imitating flexural action of a plate only with one particular value of Poisson's ratio, it was found necessary for generality to endow the side members of the cell with torsional stiffnesses C and C_1 in addition to their flexural stiffnesses. Torsional stiffness is not required in the diagonal members.

The flexural cell must of course be consistent in its geometry with the plane stress cell and for this reason is must possess the corner members of the latter. Since however flexural action does not require these members, they are simply assumed to be of the stiffness zero, although they will be flexurally deformed as explained later, they will not contribute to the flexural equilibrium of the joints on their ends.

REQUIREMENTS OF THE CELL [6]

The bar stiffnesses are determined in such a way that the cell deforms in bending in exactly the same manner as the plate in conditions of any arbitrary uniform flexure. The easiest way to comply with this necessary condition is to make the cell behave as the plate in the following three states:

(1) Constant curvature $\frac{1}{r}$ in the x direction, no curvature in the y direction, no torsion in planes x and y, i.e. $Mxy = 0$.

(2) Constant curvature $\frac{1}{r}$ in the y direction, no curvature in the x direction, $Mxy = 0$.

(3) Uniform torsional condition in x and y planes with no flexure in x and y directions.

CONDITION 1 Bending in X direction (Fig. 2.14)

Uniform moments m_y for bending in x direction and μmy for bending in y direction are applied to the plate. The plate bends to the curvature $\frac{1}{r} = \frac{\Phi}{ka}$ where Φ is the deflection angle on the length ka. The cell of the same size as the plate must imitate its deformation while acted upon by the corner moments statically equivalent to the ones in the plate. These are

$$Mx = \frac{1}{2} \mu kam_y$$

and

$$My = \frac{1}{2} am_y \qquad (2.32)$$

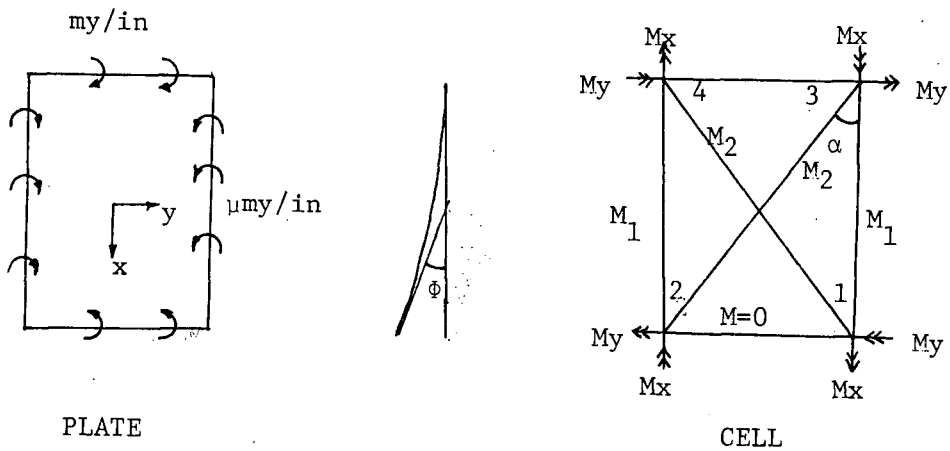


Fig. 2.14

By the well known relation

$$m_y = \frac{D}{r} = \frac{D\Phi}{ka}$$

and so the Eqns (2.32) become

$$Mx = \frac{1}{2}\mu D\Phi$$

and

$$My = \frac{1}{2}\frac{D\Phi}{k} . \qquad (2.33)$$

where

$$D = \frac{Et^3}{12(1-\mu^2)} \qquad (2.34)$$

the quantity t being the thickness of the plate. The bending moments in the members of the cell are expressed in terms of their stiffnesses and angles of deflection by the relation

$$\frac{M}{EI} = \frac{1}{R} \qquad (2.35)$$

Since the members 1-2 and 3-4 remain straight their moments are zero $M = 0$

The bending moments in the side members 1-3 and 2-4 are

$$M_1 = \frac{EI_1\Phi}{ka} \qquad (2.36)$$

In order to find the moment in the diagonal it is necessary to determine its angle of flexural deformation. The vector Φ in Fig. 2.15 represents the angle of rotation of the end 1 of the diagonal 1-4 in Fig. 2.14 in relation to the end 4. The components $\Phi\text{Cos}\alpha$ and $\Phi\text{Sin}\alpha$ of this vector are respectively the angles of flexure and twist developed on the length of the diagonal. The bending moment in the diagonal then is

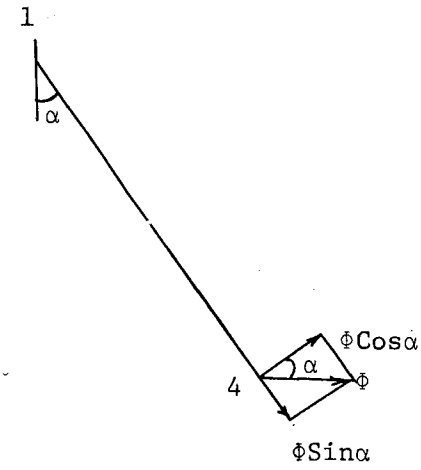


Fig. 2.15

$$M_2 = \frac{EI_2\Phi\ \text{Cos}\alpha}{a(k^2+1)^{1/2}} \qquad (2.37)$$

The diagonal offers no torsional resistance to its twist. Equilibrium of any joint in the cell such as 3 (Fig. 2.16) gives

$$M_2\ \text{Sin}\alpha = Mx$$

$$M_2\ \text{Cos}\alpha + M_1 = My$$

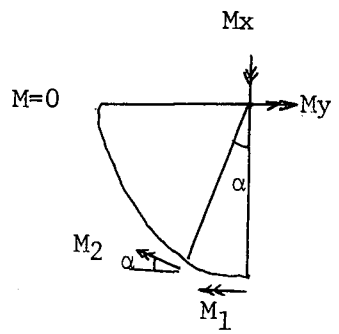


Fig. 2.16

Substitution of appropriate values for M_1, M_2, Mx and My gives

$$I_1 = \frac{(1-\mu k^2)at^3}{24(1-\mu^2)} \qquad (2.38)$$

$$I_2 = \frac{\mu(k^2+1)^{3/2}at^3}{24k(1-\mu^2)} \qquad (2.39)$$

CONDITION 2 Bending in y direction

By a similar analysis

$$I = \frac{(k^2-\mu)at^3}{24k(1-\mu^2)} \qquad (2.40)$$

and I_2 = expression (2.39)

CONDITION 3

The plate is subjected to uniform torsional moments m_{xy} in planes x and y. Since no bending moments exist in the planes x and y, the cross-sections of the plate parallel to these dimensions remain straight and for this reason the side bars in the cell develop no bending moments. Leaving the x and y cross-sections, passing through the centre O, in undisplaced position the corners of the plate deflect the amount δ as shown in Fig. 2.17. The equivalent moments at the corners of the cell are

$$Mx = \frac{1}{2} m_{xy} a$$

$$My = \frac{1}{2} m_{xy} ka \qquad (2.40a)$$

The angles of torsional rotation of the straight edges of the plate in relation to the lines parallel to them and passing through the centre 0 are

$$\Phi_y = \frac{2\delta}{ka} \quad \text{for the edges 1-3 and 2-4}$$

$$\text{and} \quad \Phi_x = \frac{2\delta}{a} \quad \text{for the edges 1-2 and 3-4} \qquad (2.41)$$

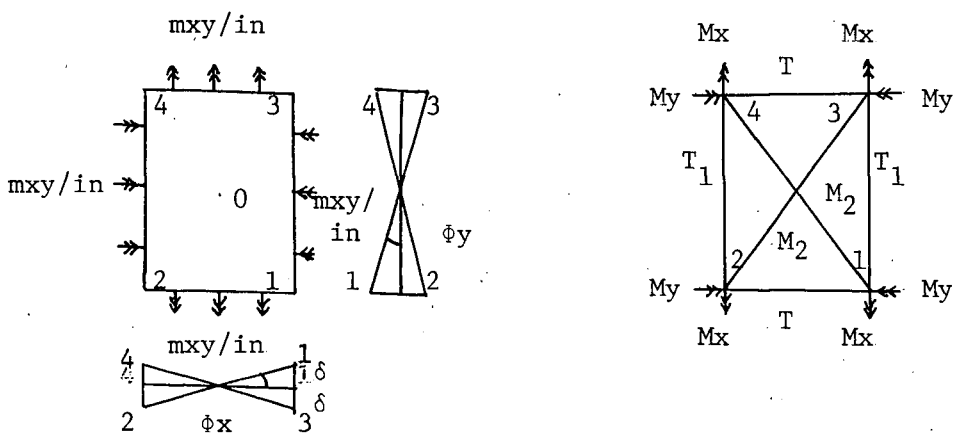


Fig. 2.17

These quantities represent the angles of twist of the side members on the lengths $\frac{a}{2}$ and $\frac{ka}{2}$ respectively. Then by the familiar relations between the torque, torsional stiffness and the angle of twist

$$T = \frac{4\delta}{ka^2} C$$

$$\text{and} \quad T_1 = \frac{4\delta}{ka^2} C_1 \qquad (2.42)$$

The diagonal members develop flexural moments which may be expressed through their angular deformations. As may be observed in Fig. 2.18,

the corner 2 (as any other corner of the cell) rotates in relation to the centre 0 through the angles indicated by the vectors $\frac{2\delta}{ka}$ and $\frac{2\delta}{a}$

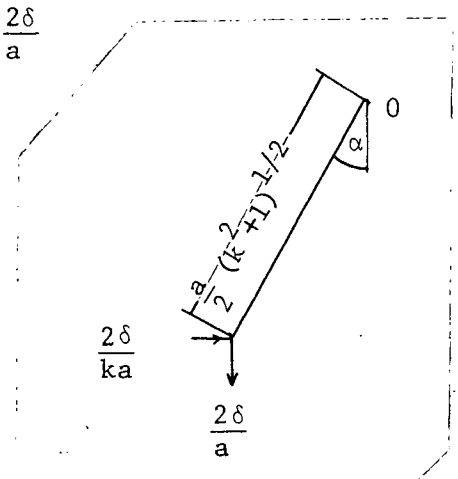


Fig. 2.18

The flexural angle of rotation of the diagonal is

$$\Phi = \frac{2\delta}{ka}\,\mathrm{Cos}\alpha + \frac{2\delta}{a}\,\mathrm{Sin}\alpha$$

$$= \frac{4\delta}{a(k^2+1)^{1/2}} \qquad (2.43)$$

Then the bending moment of the diagonal is

$$M_2 = \frac{EI_2\ \Phi 2}{a(k^2+1)^{1/2}} = \frac{8EI_2\ \delta}{a^2(k^2+1)} \qquad (2.44)$$

The equation relating the torsional moment and the deflection in the plate is

$$m_{xy} = D(1-\mu)\frac{\partial^2 W}{\partial x \partial y}$$

Since $\frac{\partial^2 W}{\partial x \partial y} = \frac{\partial}{\partial x}\frac{\partial W}{\partial y}$ is the increment of the slope of the deformed section in y direction per unit length in x direction.

$$\frac{\partial^2 W}{\partial x \partial y} = \frac{1}{ka}\left(\frac{2\delta}{a} + \frac{2\delta}{a}\right) = \frac{4\delta}{ka^2} \qquad (2.45)$$

Then $$m_{xy} = D(1-\mu)\frac{4\delta}{ka^2}$$

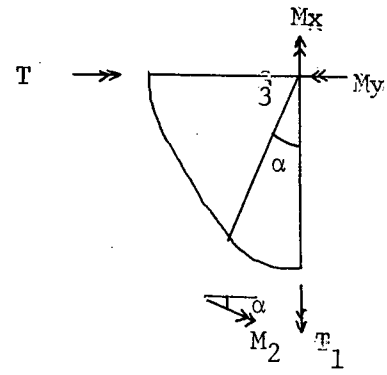


Fig. 2.19

Equilibrium of any joint such as 3 (Fig. 2.19) gives

$$M_2 \sin\alpha + T_1 = Mx$$

$$M_2 \cos\alpha + T = My$$

Substitution into these equations of the expressions for Mx and My (Eqn 2.40a) with mxy replaced from Eqn 2.45, and of the expressions (2.44) and (2.42) for M_2, T and T_1 respectively results in

$$C = \frac{(1-3\mu)K\ Eat^3}{24(1-\mu^2)} \qquad (2.46)$$

$$C_1 = \frac{(1-3\mu)\ Eat^3}{24(1-\mu^2)} \qquad (2.47)$$

Derivation of Stiffness Matrix

A flexural cell possesses three degrees of freedom at each node: the displacement perpendicular to the plane of the cell and the rotations about the x and y axes.

Distribution Factors Action 3

In this action node 1 is moved upwards without rotation through a distance Δ_1^z, and the other nodes and the ends of the members meeting there are restrained from any movement. The corner forces and moments holding the cell in the displaced position are made up of the forces and bending moments on the ends of the bars 1-2, 1-3 and 1-4 meeting

at the corner in question. These members carry no torsion. The remaining three members of the cell are unstressed.

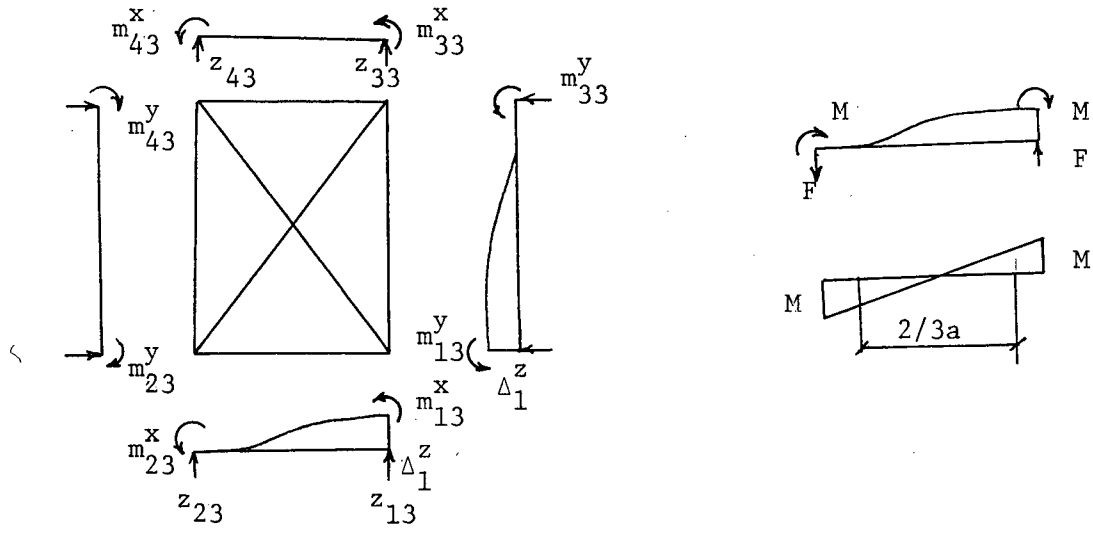


Fig. 2.20

The equal bending moments M on the end of the member 1-2 (Fig. 2.20) are related to the end deflection as follows

$$M = \frac{6EI}{a^2} \Delta^z_1$$

Substituting for I from Eqn (2.40)

$$M = \frac{(k^2-\mu)\ Et^3}{4k(1-\mu^2)a} \Delta^z_1 \qquad (2.48)$$

The transverse reactions on the ends of the member are

$$F = \frac{(k^2-\mu)\ Et^3}{2k(1-\mu^2)a^2} \Delta^z_1 \qquad (2.49)$$

Similarly for the member 1-3 and the diagonal member

$$M_1 = \frac{6EI_1\Delta_1^z}{k^2a^2} = \frac{(1-\mu k^2)\ Et^3\ \Delta_1^z}{4k^2(1-\mu^2)a} \tag{2.50}$$

$$F_1 = \frac{(1-\mu k^2)\ Et^3\ \Delta_1^z}{2k^3(1-\mu^2)a^2} \tag{2.51}$$

$$M_2 = \frac{6EI_2\ \Delta_1^z}{a^2(k^2+1)} = \frac{\mu(k^2+1)^{1/2}\ Et^3\ \Delta_1^z}{4(1-\mu^2)ka} \tag{2.52}$$

$$F_2 = \frac{\mu Et^3\ \Delta_1^z}{2k(1-\mu^2)a^2}$$

The bending moment M_2 on the ends of the diagonal member 1-4 is resolved into the components parallel to the coordinate axes, and these provide contributions to the distribution factors at the nodes 1 and 4. In writing down the expressions for the distribution factors close attention should be paid to the signs of the forces and moments.

The corner forces are designated by the symbol z because they act in the direction of this axis, positive if directed upward. The corner moments are given the symbol m^x or m^y depending on the axis about which they tend to rotate the node. Two subscripts are used in each distribution factor, the first indicating the number of the corner where the function is applied and the second, the digit 3, describing arbitrarily the transverse movement of the node 1 as the action 3. Rotations of the joint 1 about the axes x and y, the subject of the

subsequent investigation will be designated as the actions 4 and 5 respectively. All distribution factors are assembled in Table 2. In this table

$$L = \frac{Et^3}{12a(1-\mu^2)} \tag{2.54}$$

Rotation Factors due to θ_1^x

The node 1 is rotated through a positive angle θ_1^x about the x axis while the other nodes remained fixed. The positive rotation is assumed clockwise looking in positive direction along the axis of rotation. The members 1-2, 1-4 are subjected to flexure and 1-3 to torsion. The three other members remain unstressed. The relation between the angle of rotation θ_1^x and the bending moment on the end 1 of the member 1-2 is

$$M = \frac{4EI\theta_1^x}{a}$$

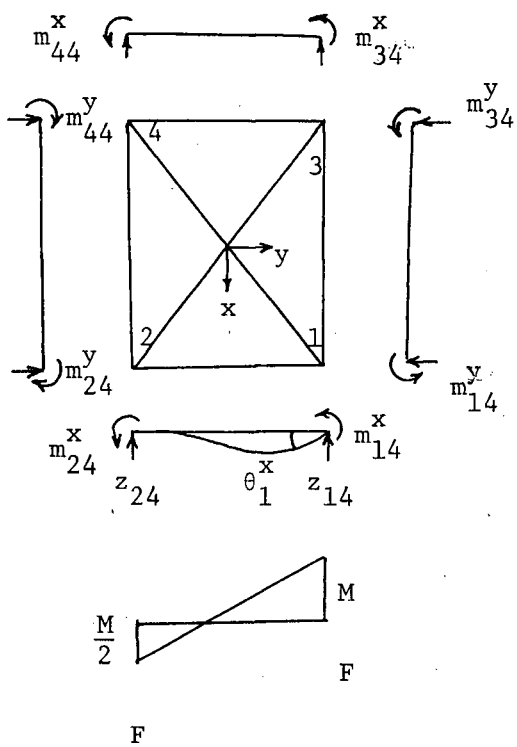


Fig. 2.21

Substituting I from Eqn (2.40)

$$M = \frac{(k^2-\mu)\ Et^3\ \theta_1^x}{6k(1-\mu^2)} \tag{2.55}$$

On the other end of this same member the bending moment is only half as great. The reactions on the ends of the members are

$$F = \frac{(k^2-\mu)\ Et^3\ \theta_1^x}{4(1-\mu^2)ka} \tag{2.56}$$

The angle of flexure at the end 1 of the diagonal member 1-4 is $\frac{\theta_1^x}{(k^2+1)^{1/2}}$ and the bending moment on the same end

$$M_2 = \frac{4EI_2\theta_1^x}{a(k^2+1)} = \frac{\mu(k^2+1)^{1/2}Et^3\theta_1^x}{6k(1-\mu^2)} \tag{2.57}$$

The reactions on the ends of the member are

$$F_2 = \frac{\mu Et^3\theta_1^x}{4(1-\mu^2)ka} \tag{2.58}$$

The torque T_1 in the member 1-3 using Eqn (2.47) for its torsional stiffness is

$$T_1 = \frac{c_1\theta_1^x}{ka} = \frac{(1-3\mu)Et^3\ \theta_1^x}{24(1-\mu^2)k} \tag{2.59}$$

The rotation distribution factors derived here, by adding the necessary member reaction and moment components at different corners, are incorporated in Table 2 and so are the factors of the action 5 (rotation θ_1^y) determined in a similar manner.

TABLE II

DISTRIBUTION FACTORS FOR FLEXURE CELL

$$L = \frac{Et^3}{12a(1-\mu^2)}$$

Action 3 $\Delta_1^z = 1$	Action 4 $\theta_1^x = 1$	Action 5 $\theta_1^y = 1$
$z_{13} = \frac{6}{a}(K - \frac{\mu}{K} + \frac{1}{k^3})L$	$z_{14} = -3kL$	$z_{15} = \frac{3L}{k^2}$
$z_{23} = -\frac{6}{a}(k - \frac{\mu}{k})L$	$z_{24} = 3(k - \frac{\mu}{k})L$	$z_{25} = 0$
$z_{33} = -\frac{6}{a}(\frac{1}{k^3} - \frac{\mu}{k})L$	$z_{34} = 0$	$z_{35} = -3(\frac{1}{k^2} - \mu)L$
$z_{43} = -\frac{6\mu}{ak}L$	$z_{44} = \frac{3\mu L}{k}$	$z_{45} = -3\mu L$
$m_{13}^x = -3kL$	$m_{14}^x = 2a(k + \frac{1-3\mu}{4k})L$	$m_{15}^x = -2a\mu L$
$m_{23}^x = -3(k - \frac{\mu}{k})L$		$m_{25}^x = 0$
$m_{33}^x = 0$	$m_{24}^x = a(k - \frac{\mu}{k})L$	$m_{35}^x = 0$
		$m_{45}^x = -a\mu L$
$m_{43}^x = -\frac{3\mu L}{k}$	$m_{34}^x = -a(\frac{1-3\mu}{2k})L$	

		$m^y_{15} = 2a(\frac{1}{k} + \frac{k}{4}[1-3\mu])L$
$m^y_{13} = \frac{3L}{k^2}$	$m^x_{44} = a\frac{\mu}{k} L$	
$m^y_{23} = 0$		$m^y_{25} = -\frac{1}{2}ak(1-3\mu)L$
$m^y_{33} = 3(\frac{1}{k^2} - \mu)L$	$m^y_{14} = -2a\mu L$	
	$m^y_{24} = 0$	$m^y_{35} = a(\frac{1}{k} - \mu k)L$
$m^y_{43} = 3\mu L$	$m^y_{34} = 0$	
	$m^y_{44} = -a\mu L$	$m^y_{45} = a\mu kL$

Stiffness Matrix

The contents of Table 2 represent the first 3 columns of the 12 x 12 stiffness matrix [K] relating the nodal forces and moments of the cell with the modal movements. The other 9 columns of the matrix corresponding to nodal forces, brought about by unit displacements of the corners 2, 3 and 4 may be found in a similar manner, but it is much easier to determine them from the considerations of symmetry. This is illustrated on the example of the distribution factors of the fifth column of the matrix produced by the unit rotation $\theta_2^x = 1$ of the node 2.

Symmetrical reversal about the x axis of Fig. 2.21 including all deformations and stresses involved in it, is presented in Fig. 2.22(a). The picture is self explanatory. Since the direction of θ^x rotation of the joint 2 in Fig. 2.22(a) is negative it is further reversed in Fig. 2.22(b). The corner forces and moments in (b) are thus relocated, and sometimes reversed in sign distribution factors of Fig. 2.21.

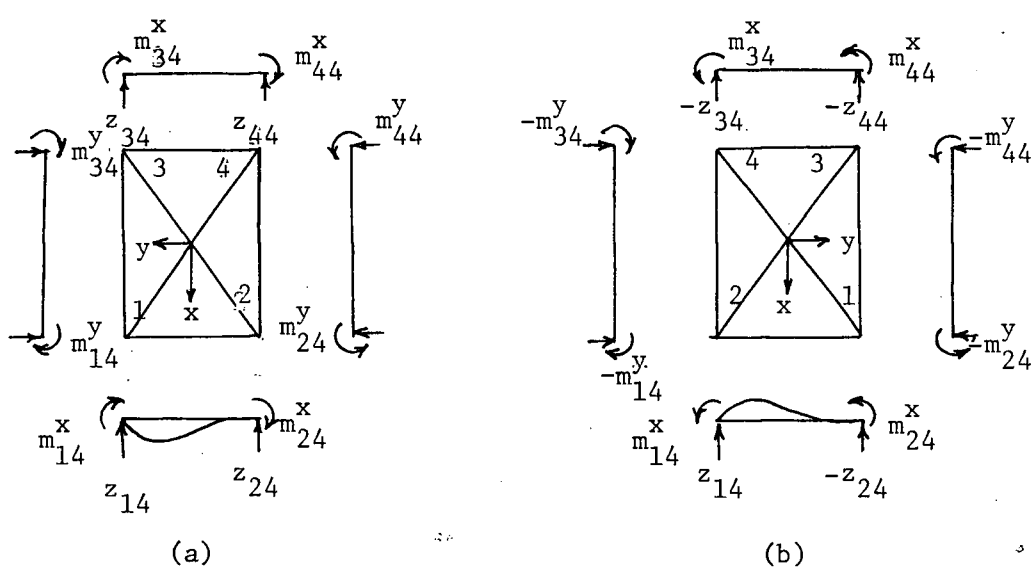


Fig. 2.22

When taken in proper sequence they represent the fifth column of the stiffness matrix. The complete equation $\{\underline{P}\} = [\underline{K}]\{\underline{\Delta}\}$ is

$$
\begin{bmatrix}
z_{13} & z_{14} & z_{15} & z_{23} & -z_{24} & -z_{25} & z_{33} & z_{34} & -z_{35} & z_{43} & -z_{44} & -z_{45} \\
m^x_{13} & m^x_{14} & m^x_{15} & -m^x_{23} & m^x_{24} & -m^x_{25} & m^x_{33} & m^x_{34} & -m^x_{35} & -m^x_{43} & m^x_{44} & m^x_{45} \\
m^y_{13} & m^y_{14} & m^y_{15} & m^y_{23} & -m^y_{24} & m^y_{25} & -m^y_{33} & -m^y_{34} & m^y_{35} & -m^y_{43} & m^y_{44} & m^y_{45} \\
z_{23} & z_{24} & z_{25} & z_{13} & -z_{14} & z_{15} & z_{43} & z_{44} & -z_{45} & z_{33} & -z_{34} & -z_{35} \\
m^x_{23} & m^x_{24} & m^x_{25} & -m^x_{13} & m^x_{14} & -m^x_{15} & m^x_{43} & m^x_{44} & -m^x_{45} & -m^x_{33} & m^x_{34} & m^x_{35} \\
m^y_{23} & m^y_{24} & m^y_{25} & m^y_{13} & -m^y_{14} & m^y_{15} & -m^y_{43} & -m^y_{44} & m^y_{45} & -m^y_{33} & m^y_{34} & m^y_{35} \\
z_{33} & z_{34} & z_{35} & z_{43} & -z_{44} & -z_{45} & z_{13} & z_{14} & -z_{15} & z_{23} & -z_{24} & -z_{25} \\
m^x_{33} & m^x_{34} & m^x_{35} & -m^x_{43} & m^x_{44} & -m^x_{45} & m^x_{13} & m^x_{14} & -m^x_{15} & -m^x_{23} & m^x_{24} & m^x_{25} \\
m^y_{33} & m^y_{34} & m^y_{35} & m^y_{43} & -m^y_{44} & m^y_{45} & -m^y_{13} & -m^y_{14} & m^y_{15} & -m^y_{23} & m^y_{24} & m^y_{25} \\
z_{43} & z_{44} & z_{45} & z_{33} & -z_{34} & -z_{35} & z_{23} & z_{24} & -z_{25} & z_{13} & -z_{14} & -z_{15} \\
m^x_{43} & m^x_{44} & m^x_{45} & -m^x_{33} & m^x_{34} & m^x_{35} & -m^x_{23} & m^x_{24} & m^x_{25} & -m^x_{13} & m^x_{14} & m^x_{15} \\
m^y_{43} & m^y_{44} & m^y_{45} & m^y_{33} & -m^y_{34} & m^y_{35} & -m^y_{23} & -m^y_{24} & m^y_{25} & -m^y_{13} & m^y_{14} & m^y_{15}
\end{bmatrix}
\times
\begin{bmatrix}
\Delta^z_1 \\ \theta^x_1 \\ \theta^y_1 \\ \Delta^z_2 \\ \theta^x_2 \\ \theta^y_2 \\ \Delta^z_3 \\ \theta^x_3 \\ \theta^y_3 \\ \Delta^z_4 \\ \theta^x_4 \\ \theta^y_4
\end{bmatrix}
=
\begin{bmatrix}
P^z_1 \\ M^x_1 \\ M^y_1 \\ P^z_2 \\ M^x_2 \\ M^y_2 \\ P^z_3 \\ M^x_3 \\ M^y_3 \\ P^z_4 \\ M^x_4 \\ M^y_4
\end{bmatrix}
$$

It may be seen that the matrix [K] is symmetrical about its principal diagonal, which is in agreement with the Betti's reciprocal theorem.

CHAPTER III

BASIC THEORY OF INSTABILITY

In a mathematical sense, stability implies a configuration where infinitesimal disturbances will cause only infinitesimal departures from the given equilibrium configuration. The criterion of stability of conservative holonomic systems can be formulated as follows (7). "A conservative holonomic system is in a configuration of stable equilibrium if and only if, the value of the potential energy is a relative minimum." In the system to be investigated here, it is assumed that the bar model is subjected to a conservative set of inplane loads P which give rise to a set of axial forces S in the bars of the model. The loads P are increased proportionally by a common multiplier f to a critical value f_{cr} where instability of the system occurs.

It is assumed that the bar forces do not change during the buckling deformation, which is in agreement with the theory of plate instability. Thus if T represents the change in potential energy during the buckling deformation we can write

$$T = U + V$$

where U is the strain energy due to flexure caused by the buckling deformation and V is the potential energy of the external loads measured from the unbuckled position.

For structural systems made up of linear elastic bars it is shown (10) that the change in potential energy T is a quadratic function of the displacements W, θ^x, θ^y that describe the buckled deformation. Since the first variation of T must vanish to satisfy equilibrium, a sufficient condition that T be a relative minimum is $T \geq 0$ for all possible buckling deformation configurations. A criterion for determining f_{cr} can then be that $T = 0$ for some configuration. This is the familiar Timoshenko (8) criterion for stability of elastic systems.

Let

$$U = \frac{1}{2} \underline{\delta}^T \underline{K} \underline{\delta}$$

$$V = -\frac{1}{2} \underline{\delta}^T f\underline{K}_s \underline{\delta}$$

where $\underline{\delta}$ represents collectively the flexural nodal displacements.

$\underline{K}$ is the flexural stiffness matrix of the model.

$\underline{K}_s$ is a new matrix--the stability matrix of the model.

Then

$$T = \frac{1}{2} \underline{\delta}^T [\underline{K} - f\underline{K}_s] \underline{\delta} = 0$$

As $T = 0$ for $\underline{\delta} \neq 0$ the matrix of the quadratic form $[\underline{K} - f\underline{K}_s]$ is positive semi-sefinite; therefore the critical load is obtained as the lowest root of the determinantal equation

$$|\underline{K} - f\underline{K}_s| = 0 \qquad (3.1)$$

To calculate K_s we note that

$$V = -f\,\underline{P}^T\,\underline{e} \quad \text{or} \quad -\Sigma\; f\,P_i\,e_i$$

where $\underline{e}$ or e_i represent in plane displacements of the points of application of the loads P_i in the direction of P_i, and are functions of the buckling displacements δ. A convenient method of calculating V is to use a virtual work principle, which states that

$$\underline{P}^T\,\underline{e} = \underline{S}^T\,\underline{\lambda}$$

where $\underline{P}$ and $\underline{S}$ are the plane stress equilibrium system and $\underline{e}$ and $\underline{\lambda}$ are a compatible displacement system caused by the buckling displacements. If S is taken as positive for compression, λ represents the inplane shortening of each bar due to the out of plane buckling deformation δ Thus

$$V = -\frac{1}{2}\,\underline{\delta}^T\,f\underline{K}_s\,\underline{\delta} = -f\,\underline{S}^T\,\underline{\lambda}$$

i.e. $$\underline{\delta}^T\,f\underline{K}_s\,\underline{\delta} = 2f\,\underline{S}^T\,\underline{\lambda} = 2\Sigma\; S_i\,\lambda_i \qquad (3.2)$$

CHAPTER IV

DERIVATION OF STABILITY MATRIX

As follows from Eqn (3.2) derivation of the stability matrix of a cell involves evaluation of the quantities making up the sum $\Sigma(S_i \lambda_i)$ extended to all bars of the plane stress cell, the primary as well as the secondary. The bar stresses S corresponding to the load of unit intensity are found by the plane stress analysis, the compression being considered positive and the tension negative. Each of the quantities λ represents the difference in length between the bar and its projection in the plane of the model as the corners of the cell undergo their flexural movements. Only some of these affect each λ.

Formation of the stability matrix of a cell thus reduces to the derivation of expressions for λ of different bars and these require the equations for the transverse deflections z along the length of the bars resulting from the flexural nodal movements. With z known, λ is found as the following integral, extended over the length of the bar ℓ.

$$\lambda = \frac{1}{2} \int_o^{\ell} \left(\frac{dz}{dx} \right)^2 dx \qquad (4.1)$$

Derivation of expressions for z and λ may be illustrated on the example of the bar 1-2. Once these have been found, similar expressions for λ of the three other side bars and of the diagonal bars may be written down by analogy. The expression z for the through side bars is utilized also for λ in the subdivided side bars when these are

effective. Some additional explanation will be given further in connection with the corner bars.

LENGTH CHANGE λ IN BARS (1-2) AND (4-3)

The quantity λ_{12}, is affected only by the corner movements W_1, W_2, θ_1^x and θ_2^x of the nodes 1 and 2. The flexural conditions in these bars corresponding to these movements, are presented in Figs. 4.1(a) to (d) and the deflections z may be found by making use of the moment area relation. Thus from Fig. 4.1(a)

$$z_1 = \frac{1}{2}x \frac{M}{\frac{a}{2}EI} x \cdot \frac{2}{3} x - \frac{M}{\frac{a}{2}} \frac{(x - \frac{a}{2})}{EI} x \cdot \frac{x}{2}$$

$$= \frac{Mx^2}{aEI} \left(\frac{a}{2} - \frac{x}{3} \right)$$

Since $$\frac{M}{EI} = \frac{6W_1}{a^2}$$

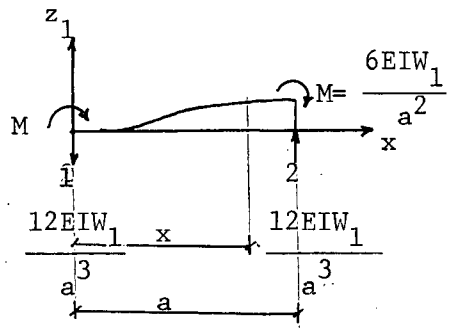
$$z_1 = W_1 \left(\frac{3x^2}{a^2} - 2 \frac{x^3}{a^3} \right) \tag{4.2}$$

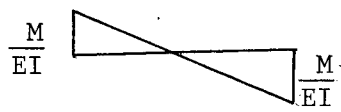
The deflections z_2, z_3 and z_4 in the three other cases are found similarly as follows

$$z_2 = W_2 \left(\frac{2x^3}{a^3} - \frac{3x^2}{a^2} \right) \tag{4.3}$$

$$z_3 = \theta_1^x \left(\frac{x^3}{a^2} - \frac{x^2}{a} \right) \tag{4.4}$$

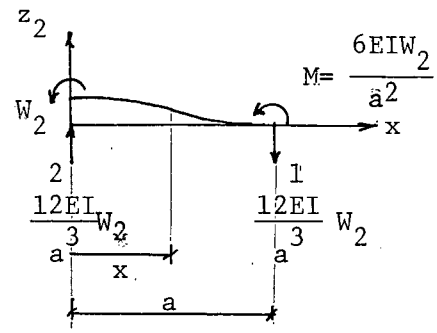
$$z_4 = \theta_2^x \left(x - \frac{2x^2}{a} + \frac{x^3}{a^2} \right) \tag{4.5}$$

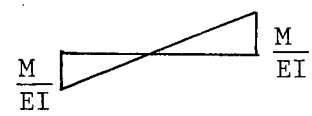




$\frac{M}{EI}$ diagram

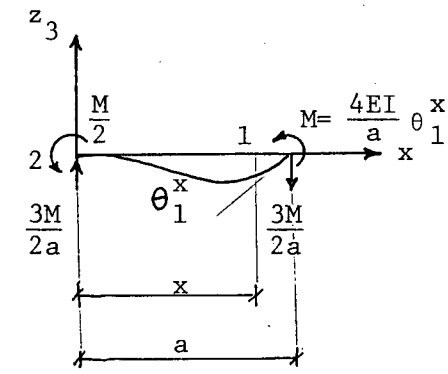
(a)

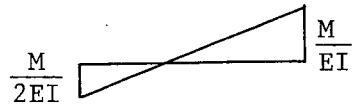




$\frac{M}{EI}$ diagram

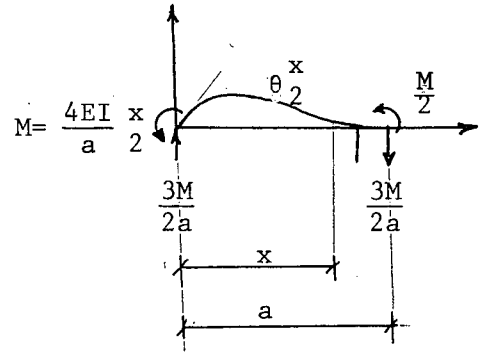
(b)

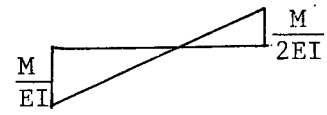




$\frac{M}{EI}$ Diagram

(c)





$\frac{M}{EI}$ Diagram

(d)

Fig. 4.1

The deflection corresponding to the combined action of the four nodal movements is determined by superposition

$$z = W_1 \left(\frac{3x^2}{a^2} - \frac{2x^3}{a^3} \right) + W_2 \left(\frac{2x^3}{a^3} - \frac{3x^2}{a^2} \right) + \theta_1^x \left(\frac{x^3}{a^2} - \frac{x^2}{a} \right)$$

$$+ \theta_2^x \left(x - \frac{2x^2}{a} + \frac{x^3}{a^2} \right) \qquad (4.6)$$

Its derivative

$$\frac{dz}{dx} = W_1 \left(\frac{6x}{a^2} - \frac{6x^2}{a^3} \right) + W_2 \left(\frac{6x^2}{a^3} - \frac{6x}{a^2} \right) + \theta_1^x \left(\frac{3x^2}{a^2} - \frac{2x}{a} \right)$$

$$+ \theta_2^x \left(1 - \frac{4x}{a} + \frac{3x^2}{a^2} \right) \qquad (4.7)$$

This expression is squared and substituted into the integral for λ_{12}, Eqn (4.1) which gives

$$\lambda_{12} = 0.6 \frac{W_1^2}{a} + 0.6 \frac{W_2^2}{a} + \frac{1}{15} (\theta_1^x)^2 a + \frac{1}{15} (\theta_2^x)^2 a - 1.2 \frac{W_1 W_2}{a}$$

$$-0.1\, W_1\, \theta_1^x - 0.1\, W_1\, \theta_2^x + 0.1\, W_2\, \theta_1^x + 0.1\, W_2\, \theta_2^x - \frac{1}{30} \theta_1^x\, \theta_2^x\, a$$

$$(4.8)$$

The values of λ in the subdivided bars (2-5) and (5-1) are found by integrating the squares of $\left(\frac{dz}{dx} \right)$ in Eqn (4.1) between the limits 0 and $\frac{\ell}{2}$ and again $\frac{\ell}{2}$ and ℓ respectively.

Thus (4.9)

$$\lambda_{25} = 0.3 \frac{W_1^2}{a} + 0.3 \frac{W_2^2}{a} + \frac{17}{960} (\theta_1^x)^2 a + \frac{47}{960} (\theta_2^x)^2 a - 0.6 \frac{W_1 W_2}{a}$$

$$- \frac{23}{160} W_1\, \theta_1^x + \frac{7}{160} W_1\, \theta_2^x + \frac{23}{160} W_2\, \theta_1^x - \frac{7}{160} W_2\, \theta_2^x - \frac{1}{60} \theta_1^x\, \theta_2^x\, a$$

and

$$\lambda_{15} = 0.3\,\frac{W_1^{\,2}}{a} + 0.3\,\frac{W_2^{\,2}}{a} + \frac{47}{960}\,(\theta_1^x)^2 a + \frac{17}{960}\,(\theta_2^x)^2 a - 0.6\,\frac{W_1 W_2}{a}$$

$$+ \frac{7}{160}\,W_1\,\theta_1^x - \frac{23}{160}\,W_1\,\theta_2^x - \frac{7}{160}\,W_2\,\theta_1^x + \frac{23}{160}\,W_2\,\theta_2^x - \frac{1}{60}\,\theta_1^x\,\theta_2^x\,a \tag{4.10}$$

All these expressions for λ are composed of ten quadratic terms, involving the squares of the displacements W_1, W_2, θ_1^x, θ_2^x, and all their possible products.

Expressions (4.8), (4.9), and (4.10) may evidently be used for the λ values of the through bar (4-3) and its subdivisions (4-7) and (7-3) by simply replacing the indices of the nodal displacements 2, 1 and 5 by 4, 3 and 7 respectively (see page 43).

LENGTH CHANGE λ IN SIDE BARS (1-3) AND (2-4)

Only minor modifications are needed for extension of the same formulae to the two other side bars Fig. (4.2).

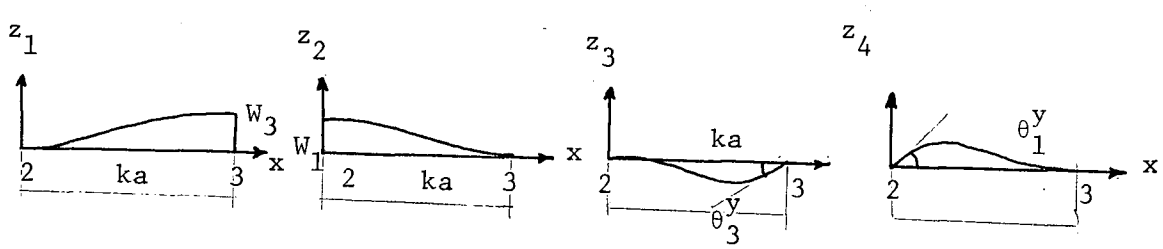


Fig. 4.2

In order to preserve complete similarity with Figs. 4.1, the positive angle changes θ_1^y and θ_3^y must appear above or below the plane of the cell in the same way as θ_2^x and θ_1^x appear in the previous figure. With this provision the required expression for the λ values in the bars 1-6-3 and 2-8-4 may be copied from the corresponding expressions for the bars 2-5-1 and 4-7-3 respectively with the following substitution in their formulae: the bar length a is replaced everywhere by ka and the indices 2, 5, 1 are replaced by 1, 6, 3 and 4-7-3 by 2-8-4 respectively with concurrent substitution of the angles θ^y for the angles θ^x.

This gives:

<u>For Side Bars</u>

$$\lambda_{13} = 0.6\,\frac{W_3^{\,2}}{ka} + 0.6\,\frac{W_1^{\,2}}{ka} + \frac{1}{15}\,(\theta_3^y)^2 ka + \frac{1}{15}\,(\theta_1^y)^2 ka - 1.2\,\frac{W_1 W_3}{ka}$$

$$-0.1\,W_3\,\theta_3^y - 0.1\,W_3\,\theta_1^y + 0.1\,W_1\,\theta_1^y + 0.1\,W_1\,\theta_3^y - \frac{1}{30}\,\theta_1^y\,\theta_3^y\,ka \tag{4.11}$$

$$\lambda_{34} = 0.6\,\frac{W_3^{\,2}}{a} + 0.6\,\frac{W_4^{\,2}}{a} + \frac{1}{15}\,(\theta_3^x)^2 a + \frac{1}{15}\,(\theta_4^x)^2 a - 1.2\,\frac{W_3 W_4}{a}$$

$$-0.1\,W_3\,\theta_3^x - 0.1\,W_3\,\theta_4^x + 0.1\,W_4\,\theta_3^x + 0.1\,W_4\,\theta_4^x - \frac{1}{30}\,\theta_3^x\,\theta_4^x\,a \tag{4.12}$$

$$\lambda_{24} = 0.6\,\frac{W_4^{\,2}}{ka} + 0.6\,\frac{W_2^{\,2}}{ka} + \frac{1}{15}\,(\theta_4^y)^2 ka + \frac{1}{15}\,(\theta_2^y)^2 ka - 1.2\,\frac{W_2 W_4}{ka}$$

$$-0.1\,W_4\,\theta_4^y - 0.1\,W_4\,\theta_2^y + 0.1\,W_2\,\theta_4^y + 0.1\,W_2\,\theta_2^y - \frac{1}{30}\,\theta_2^y\,\theta_4^y\,ka \tag{4.13}$$

Subdivided Side Bars

$$\lambda_{16} = 0.3\,\frac{W_1^{\,2}}{ka} + 0.3\,\frac{W_3^{\,2}}{ka} + \frac{47}{960}\,(\,\theta_1^y\,)^2 ka + \frac{17}{960}\,(\,\theta_3^y\,)^2 ka - 0.6\,\frac{W_1 W_3}{ka}$$

$$- \frac{7}{160}\,W_1\,\theta_1^y + \frac{23}{160}\,W_1\,\theta_3^y - \frac{7}{160}\,W_3\,\theta_1^y - \frac{23}{160}\,W_3\,\theta_3^y - \frac{1}{60}\,\theta_1^y\,\theta_3^y\,ka \tag{4.14}$$

$$\lambda_{36} = 0.3\,\frac{W_1^{\,2}}{ka} + 0.3\,\frac{W_3^{\,2}}{ka} + \frac{17}{960}\,(\,\theta_1^y\,)^2 ka + \frac{47}{960}\,(\,\theta_3^y\,)^2 ka - 0.6\,\frac{W_1 W_3}{ka}$$

$$+ \frac{23}{160}\,W_1\,\theta_1^y - \frac{7}{160}\,W_1\,\theta_3^y - \frac{23}{160}\,W_3\,\theta_1^y + \frac{7}{160}\,W_3\,\theta_2^y - \frac{1}{60}\,\theta_1^y\,\theta_3^y\,ka \tag{4.15}$$

$$\lambda_{37} = 0.3\,\frac{W_3^{\,2}}{a} + 0.3\,\frac{W_4^{\,2}}{a} + \frac{47}{960}\,(\,\theta_3^x\,)^2 a + \frac{17}{960}\,(\,\theta_4^x\,)^2 a - 0.6\,\frac{W_3 W_4}{a}$$

$$+ \frac{7}{160}\,W_3\,\theta_3^x - \frac{23}{160}\,W_3\,\theta_4^x - \frac{7}{160}\,W_4\,\theta_3^x + \frac{23}{160}\,W_4\,\theta_4^x - \frac{1}{60}\,\theta_3^x\,\theta_4^x\,a \tag{4.16}$$

$$\lambda_{47} = 0.3\,\frac{W_3^{\,2}}{a} + 0.3\,\frac{W_4^{\,2}}{a} + \frac{17}{960}\,(\,\theta_3^x\,)^2 a + \frac{47}{960}\,(\,\theta_4^x\,)^2 a - 0.6\,\frac{W_3 W_4}{a}$$

$$- \frac{23}{160}\,W_3\,\theta_3^x + \frac{7}{160}\,W_3\,\theta_4^x + \frac{23}{160}\,W_4\,\theta_3^x - \frac{7}{160}\,W_4\,\theta_4^x - \frac{\theta_3^x\,\theta_4^x\,a}{60} \tag{4.17}$$

$$\lambda_{48} = 0.3\,\frac{W_2^{\,2}}{ka} + 0.3\,\frac{W_4^{\,2}}{ka} + \frac{17}{960}\,(\,\theta_2^y\,)^2 ka + \frac{47}{960}\,(\,\theta_4^y\,)^2 ka - 0.6\,\frac{W_2 W_4}{ka}$$

$$+ \frac{23}{160}\,W_2\,\theta_2^y - \frac{7}{160}\,W_2\,\theta_4^y - \frac{23}{160}\,W_4\,\theta_2^y + \frac{7}{160}\,W_4\,\theta_4^y - \frac{1}{60}\,\theta_2^y\,\theta_4^y\,ka \tag{4.18}$$

$$\lambda_{28} = 0.3\,\frac{W_2^{\,2}}{ka} + 0.3\,\frac{W_4^{\,2}}{ka} + \frac{47}{960}\,(\,\theta_2^y\,)^2 ka + \frac{17}{960}\,(\,\theta_4^y\,)^2 ka - 0.6\,\frac{W_2 W_4}{ka}$$

$$- \frac{7}{160}\,W_2\,\theta_2^y + \frac{23}{160}\,W_2\,\theta_4^y + \frac{7}{160}\,W_4\,\theta_2^y - \frac{23}{160}\,W_4\,\theta_4^y - \frac{1}{60}\,\theta_2^y\,\theta_4^y\,ka \tag{4.19}$$

LENGTH CHANGE λ IN DIAGONAL BARS 4-1 AND 2-3

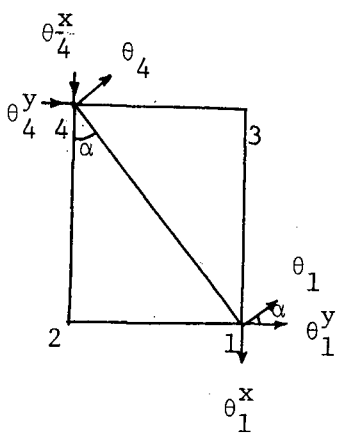


Fig. 4.3

Flexural rotations of the ends of this diagonal θ_1 and θ_4 (Fig. 4.3) are made up of the nodal rotations about both coordinate axes. Choosing the positive directions of these rotations as indicated in the figure

$$\theta_1 = \theta_1^y \cos\alpha - \theta_1^x \sin\alpha$$

$$= (k^2+1)^{-1/2}(K\theta_1^y - \theta_1^x) \tag{4.20}$$

$$\theta_4 = \theta_4^y \cos\alpha - \theta_4^x \sin\alpha$$

$$= (k^2+1)^{-1/2}(K\theta_4^y - \theta_4^x) \tag{4.21}$$

The positive directions of θ_1 and θ_4 are so chosen that the nodes 1 and 4 in Fig. 4.3 correspond respectively to the nodes 2 and 1 in Fig. 4.1. Now the value of λ_{14} may be found from Eqn (4.8) by using the length of the diagonal $a(k^2+1)^{1/2}$ in place of a and replacing W_2 by W_1, W_1 by W_4, θ_2^x by θ_1 (Eqn. 4.20) and θ_1^x by θ_4 (Eqn. 4.21).

Thus

$$\lambda_{14} = \frac{0.6}{a} W_4^{\,2} \operatorname{Sin}\alpha + \frac{0.6}{a} W_1^{\,2} \operatorname{Sin}\alpha + \frac{a}{15} (\theta_4^y)^2 \frac{\operatorname{Cos}^2\alpha}{\operatorname{Sin}\alpha} - \frac{2a}{15} \theta_4^x \theta_4^y \operatorname{Cos}\alpha$$

$$+ \frac{a}{15} (\theta_4^x)^2 \operatorname{Sin}\alpha + \frac{a}{15} (\theta_1^y)^2 \frac{\operatorname{Cos}^2\alpha}{\operatorname{Sin}\alpha} - \frac{2a}{15} \theta_1^x \theta_1^y \operatorname{Cos}\alpha$$

$$+ \frac{a}{15} (\theta_1^x)^2 \operatorname{Sin}\alpha - 1.2 \frac{W_1 W_4}{a} \operatorname{Sin}\alpha - 0.1\, W_4 \theta_4^y \operatorname{Cos}\alpha + 0.1\, W_4 \theta_4^x \operatorname{Sin}\alpha$$

$$- 0.1\, W_4 \theta_1^y \operatorname{Cos}\alpha + 0.1\, W_4 \theta_1^x \operatorname{Sin}\alpha + 0.1\, W_1 \theta_4^y \operatorname{Cos}\alpha - 0.1\, W_1 \theta_4^x \operatorname{Sin}\alpha$$

$$+ 0.1\, W_1 \theta_1^y \operatorname{Cos}\alpha - 0.1\, W_1 \theta_1^x \operatorname{Sin}\alpha - \frac{a}{30} \theta_1^y \theta_4^y \frac{\operatorname{Cos}^2\alpha}{\operatorname{Sin}\alpha}$$

$$+ \frac{a}{30} \theta_1^x \theta_4^y \operatorname{Cos}\alpha + \frac{a}{30} \theta_4^x \theta_1^y \operatorname{Cos}\alpha - \frac{a}{30} \theta_1^x \theta_4^x \operatorname{Sin}\alpha. \qquad (4.22)$$

The same expression may be used for the diagonal 2-3 replacing in it the indices 1 and 4 by 2 and 3 respectively

$$\lambda_{23} = \frac{0.6}{a} W_3^{\,2} \operatorname{Sin}\alpha + \frac{0.6}{a} W_2^{\,2} \operatorname{Sin}\alpha + \frac{a}{15} (\theta_3^x)^2 \operatorname{Sin}\alpha + \frac{2a}{15} \theta_3^x \theta_3^y \operatorname{Cos}\alpha$$

$$+ \frac{a}{15} (\theta_3^y)^2 \frac{\operatorname{Cos}^2\alpha}{\operatorname{Sin}\alpha} + \frac{a}{15} (\theta_2^x)^2 \operatorname{Sin}\alpha + \frac{2a}{15} \theta_2^x \theta_2^y \operatorname{Cos}\alpha + \frac{a}{15} (\theta_2^y)^2 \frac{\operatorname{Cos}^2\alpha}{\operatorname{Sin}\alpha}$$

$$- \frac{1.2}{a} W_2 W_3 \operatorname{Sin}\alpha - 0.1\, W_3 \theta_3^x \operatorname{Sin}\alpha - 0.1\, W_3 \theta_2^y \operatorname{Cos}\alpha - 0.1\, W_3 \theta_2^x \operatorname{Sin}\alpha$$

$$- 0.1\, W_3 \theta_2^y \operatorname{Cos}\alpha + 0.1\, W_2 \theta_3^x \operatorname{Sin}\alpha + 0.1\, W_2 \theta_3^y \operatorname{Cos}\alpha + 0.1\, W_2 \theta_2^x \operatorname{Sin}\alpha$$

$$+ 0.1\, W_2 \theta_2^y \operatorname{Cos}\alpha - \frac{a}{30} \theta_2^x \theta_3^x \operatorname{Sin}\alpha - \frac{a}{30} \theta_2^x \theta_3^y \operatorname{Cos}\alpha - \frac{a}{30} \theta_3^x \theta_2^y \operatorname{Cos}\alpha$$

$$- \frac{a}{30} \theta_2^y \theta_3^y \frac{\operatorname{Cos}^2\alpha}{\operatorname{Sin}\alpha} \quad . \qquad (4.23)$$

LENGTH CHANGE λ IN CORNER BARS

The corner bar 5-6, situated near the node 1, is chosen for illustration of the method of determination of the length changes of the corner bars. As was pointed out earlier the corner bars are assumed to possess negligible flexural stiffness approaching zero in the limit; in other words, they deflect flexurally in accordance with the beam theory without affecting the equilibrium of the joints on their ends, where they simply comply with the deflections and slopes of the side members of the cell. Inspection of Fig. 4.4 shows that flexure of bar 5-6 is affected by all three corner movements of the corners 1, 2 and 3 while being uninfluenced by the movements of the diagonally opposite corner 4. The deflections of the ends 5 and 6 are found from the corresponding deflection equations of the side bars, and the end slopes from the slopes $\frac{dz}{dx}$, making allowance for the angular connection of the side and the corner bars.

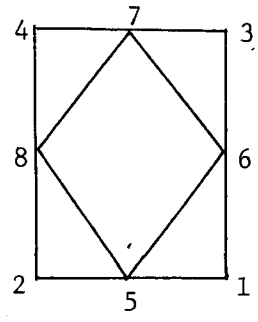


Fig. 4.4

Fig. 4.5(a) to (i) present the deflected shapes of the bar 5-6 in the 9 deflection fields causing them. Here is how some of the end values in these figures have been calculated:

In Fig. 4.5(a), the end slope θ_5 is found by substituting $x = \frac{a}{2}$ in the W_1 term for $\frac{dz}{dx}$ (Eqn 4.7) and multiplying the result by $\sin\alpha$ in order to

change the slope in the bar 1-2 into the end slope of the bar 5-6. This makes $\theta_5 = \frac{3W_1}{2a}\,\text{Sin}\alpha$. The slope θ_6 in Fig. 4.5(a) happens to be equal to the slope θ_5. Deflections of the joints 5 and 6 in the side bars are equal, and so the relative deflection of these points is zero. Thus the equation for the deflection z of the bar 5-6 may be obtained from Eqn (4.6) for the bar 1-2 using in it $W_1 = W_2 = 0$, $\theta_2^x = \frac{3W_1}{2a}\,\text{Sin}\alpha$ and $\theta_1^x = -\frac{3W_1}{2a}\,\text{Sin}\alpha$ and replacing a by the length $\frac{a}{2\text{Sin}\alpha}$ of the bar 5-6.

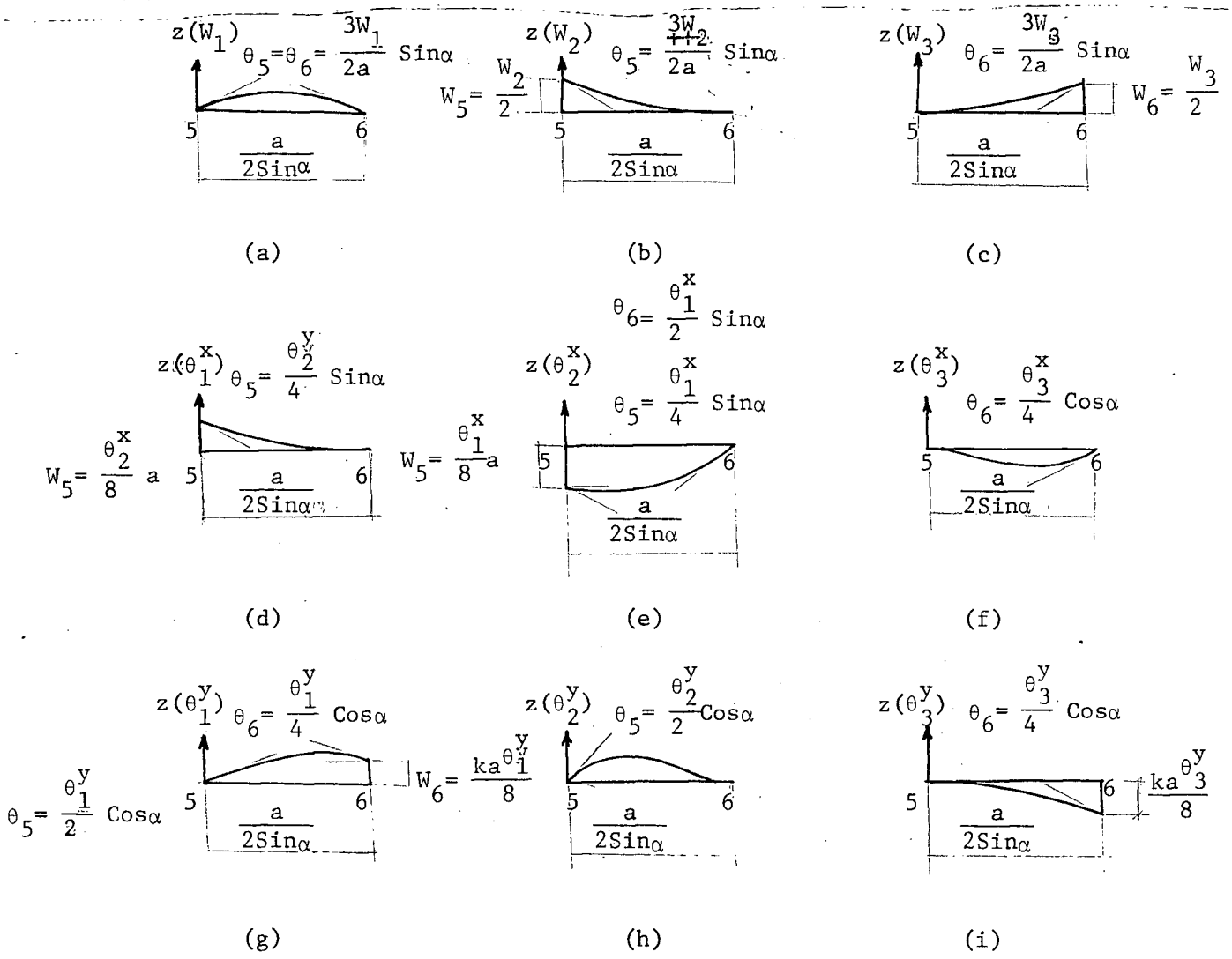


Fig. 4.5

As a further illustration consider the deformation of the same bar in the field θ_1^x. From Eqns (4.4) and (4.7), with further multiplication of the latter by Sinα, the downward deflection and slope of the end 5 of the bar are found as shown in Fig. 4.7(e). As the node 1 rotates through the angle θ_1^x, the mid point 6 of the bar 1-3 undergoes a torsional rotation $\frac{1}{2}\theta_1^x$, which becomes the flexural rotation $\theta_6 = \frac{1}{2}\theta_1^x \operatorname{Sin}\alpha$ of the end 6 of the corner bar. The deflection equation of the bar 5-6 is thus combined with proper substitutions of the equations (4.3), (4.4) and (4.5). The deflection curves corresponding to the seven other fields in Fig. 4.7 are derived in a similar way.

The complete set of these deflections z, described by the suggestive symbols $z(W_1)$, $z(\theta_1^x)$ etc. is

$$z(W_1) = \frac{3W_1}{2a} x \operatorname{Sin}\alpha - \frac{3W_1}{a^2} x^2 \operatorname{Sin}^2\alpha$$

$$z(W_2) = W_2 \left[-\frac{3}{2a} x \operatorname{Sin}\alpha + \frac{2x^3}{a^3} \operatorname{Sin}^3\alpha \right]$$

$$z(W_3) = W_3 \left[\frac{3x^2}{a^2} \operatorname{Sin}^2\alpha - \frac{2x^3}{a^3} \operatorname{Sin}^3\alpha \right]$$

$$z(\theta_1^x) = \theta_1^x \left[-\frac{x}{4} \operatorname{Sin}\alpha + \frac{3x^2}{2a} \operatorname{Sin}^2\alpha - \frac{x^3}{a^2} \operatorname{Sin}^3\alpha \right]$$

$$z(\theta_2^x) = \theta_2^x \left[-\frac{x}{4} \operatorname{Sin}\alpha - \frac{x^2}{2a} \operatorname{Sin}^2\alpha + \frac{x^3}{a^2} \operatorname{Sin}^3\alpha \right]$$

$$z(\theta_3^x) = \theta_3^x \left[-\frac{x^2}{a} \operatorname{Sin}^2\alpha + \frac{2x^3}{a^2} \operatorname{Sin}^3\alpha \right]$$

$$z(\theta_1^y) = K\theta_1^y \left[\frac{x}{2} \operatorname{Sin}\alpha - \frac{x^3}{a^2} \operatorname{Sin}^3\alpha \right]$$

$$z(\theta_2^y) = K\theta_2^y \left[\frac{x}{2} \operatorname{Sin}\alpha - \frac{2x^2}{a} \operatorname{Sin}^2\alpha + \frac{2x^3}{a^2} \operatorname{Sin}^3\alpha \right]$$

$$z(\theta_3^y) = K\theta_3^y \left[- \frac{x^2}{a} \operatorname{Sin}^2\alpha + \frac{x^3}{a^2} \operatorname{Sin}^3\alpha \right]$$

Their derivatives $\frac{dz}{dx}$ are

$$\frac{dz(W_1)}{dx} = W_1 \left(\frac{3}{2a} \operatorname{Sin}\alpha - 6 \operatorname{Sin}^2\alpha \frac{x}{a^2} \right)$$

$$\frac{dz(W_2)}{dx} = W_2 \left(- \frac{3}{2a} \operatorname{Sin}\alpha + \frac{6x^2}{a^3} \operatorname{Sin}^3\alpha \right)$$

$$\frac{dz(W_3)}{dx} = W_3 \left(\frac{6x}{a^2} \operatorname{Sin}^2\alpha - \frac{6x^2}{a^3} \operatorname{Sin}^3\alpha \right)$$

$$\frac{dz(\theta_1^x)}{dx} = \theta_1^x \left(- \frac{1}{4} \operatorname{Sin}\alpha + \frac{3x}{a} \operatorname{Sin}^2\alpha - \frac{3x^2}{a^2} \operatorname{Sin}^3\alpha \right)$$

$$\frac{dz(\theta_2^x)}{dx} = \theta_2^x \left(- \frac{1}{4} \operatorname{Sin}\alpha - \frac{x}{a} \operatorname{Sin}^2\alpha + \frac{3x^2}{a^2} \operatorname{Sin}^3\alpha \right)$$

$$\frac{dz(\theta_3^x)}{dx} = \theta_3^x \left(- \frac{2x}{a} \operatorname{Sin}^2\alpha + \frac{6x^2}{a^2} \operatorname{Sin}^3\alpha \right)$$

$$\frac{dz(\theta_1^y)}{dx} = K\theta_1^y \left(\frac{1}{2} \operatorname{Sin}\alpha - 3 \operatorname{Sin}^3\alpha \frac{x^2}{a^2} \right)$$

$$\frac{dz(\theta_2^y)}{dx} = K\theta_2^y \left(\frac{1}{2} \operatorname{Sin}\alpha - \frac{4x}{a} \operatorname{Sin}^2\alpha + \frac{6x^2}{a^2} \operatorname{Sin}^3\alpha \right)$$

$$\frac{dz(\theta_3^y)}{dx} = K\theta_3^y \left(-\frac{2x}{a} \operatorname{Sin}^2\alpha + \frac{3x^2}{a^2} \operatorname{Sin}^3\alpha \right)$$

Substituting the square of the sum of these expressions into the integral Eqn (4.1) yields the following expression for λ_{56}.

$$\begin{aligned}
\lambda_{56} = \operatorname{Sin}\alpha \Big[& \frac{3W_1^2}{16a} + \frac{3W_2^2}{10a} + \frac{3W_3^2}{10a} + \frac{9}{320}(\theta_1^x)^2 a + \frac{17a}{960}(\theta_2^x)^2 + \frac{1}{120}(\theta_3^x)^2 a \\
& + \frac{9}{320}K^2(\theta_1^y)^2 a + \frac{K^2}{120}(\theta_2^y)^2 a + \frac{17k^2}{960}(\theta_3^y)^2 a - \frac{3}{16a}W_1W_2 \\
& - \frac{3}{16a}W_1W_3 - \frac{3}{32}W_1\theta_1^x - \frac{W_1\theta_2^x}{32} - \frac{W_1\theta_3^x}{16} + \frac{3}{32}W_1k\theta_1^y + \frac{W_1k\theta_2^y}{16} \\
& + \frac{W_1k\theta_3^y}{32} - \frac{33}{80a}W_2W_3 - \frac{13}{160}W_2\theta_1^x + \frac{23}{160}W_2\theta_2^x + \frac{3}{80}W_2\theta_3^x \\
& - \frac{7}{40}W_2K\theta_1^y - \frac{1}{40}KW_2\theta_2^y + \frac{9}{80}KW_2\theta_3^y + \frac{7}{40}W_3\theta_1^x - \frac{9}{80}W_3\theta_2^x \\
& + \frac{W_3\theta_3^x}{40} + \frac{13}{160}KW_3\theta_1^y - \frac{3}{80}KW_3\theta_2^y - \frac{23}{160}KW_3\theta_3^y - \frac{1}{40}\theta_1^x\theta_2^x a \\
& + \frac{\theta_1^x\theta_3^x a}{80} + \frac{3}{320}K\theta_1^x\theta_1^y a - \frac{3}{160}K\theta_1^x\theta_2^y a - \frac{13}{320}K\theta_1^x\theta_3^y a \\
& + \frac{1}{120}\theta_2^x\theta_3^x a - \frac{13}{320}K\theta_2^x\theta_1^y a - \frac{1}{480}K\theta_2^x\theta_2^y a + \frac{29}{960}K\theta_2^x\theta_3^y a \\
& - \frac{3}{160}K\theta_3^x\theta_1^y a - \frac{1}{240}K\theta_3^x\theta_2^y a - \frac{1}{480}K\theta_3^x\theta_3^y a + \frac{1}{80}K^2\theta_1^y\theta_2^y a \\
& - \frac{1}{40}K^2\theta_1^y\theta_3^y a + \frac{1}{120}K^2\theta_2^y\theta_3^y a \Big]
\end{aligned} \tag{4.24}$$

λ VALUES IN CORNER MEMBERS 5-8, 7-6 AND 7-8

The easiest way to determine the expressions for the changes in length in the three other corner members is by making use of the general symmetry of the cell. The cell is rotated through 180° about one of the coordinate axes x, y or z so that the corner member in question will take the position occupied in Fig. 4.4 by the member 5-6. This reverses the positive directions of the two other coordinate axes and with them the signs of two of the three vectors W, θ^x and θ^y. At the same time the numbers of the corresponding corners and the mid edge points change also. This is illustrated in Table 3. Making the changes shown in this table the λ for the remaining corner bars are obtained from Eqn (4.24).

$$\lambda_{58} = \text{Sin}\alpha \Big[\frac{3W_2^2}{16a} + 0.3\,\frac{W_1^2}{a} + 0.3\,\frac{W_4^2}{a} + \frac{9}{320}(\theta_2^x)^2\,a + \frac{17}{960}(\theta_1^x)^2\,a$$

$$+ \frac{1}{120}(\theta_4^x)^2\,a + \frac{9}{320}K^2(\theta_2^y)^2\,a + \frac{1}{120}K^2(\theta_1^y)^2\,a$$

$$+ \frac{17}{960}k^2(\theta_4^y)^2\,a - \frac{3}{16a}W_1W_2 - \frac{33}{16a}W_2W_4 + \frac{3}{32}W_2\theta_2^x + \frac{1}{32}W_2\theta_1^x$$

$$+ \frac{1}{16}W_2\theta_4^x + \frac{3}{32}kW_2\theta_2^y + \frac{1}{16}kW_2\theta_1^y + \frac{1}{32}kW_2\theta_4^y - \frac{33}{80a}W_1W_4$$

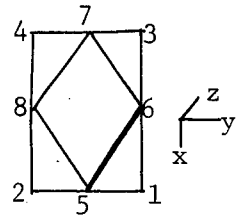
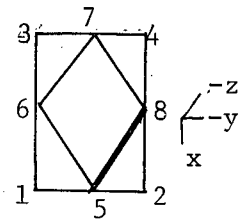
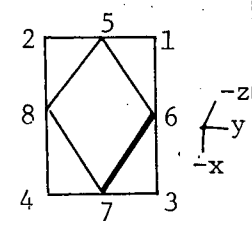
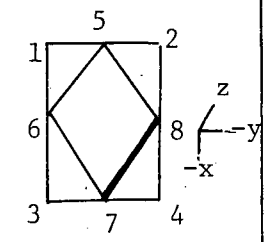
$$+ \frac{13}{160}W_1\theta_2^x - \frac{23}{160}W_1\theta_1^x - \frac{3}{80}W_1\theta_4^x - \frac{7}{40}kW_1\theta_2^y - \frac{K}{40}W_1\theta_1^y$$

$$+ \frac{9K}{80}W_1\theta_4^y - \frac{7}{40}W_4\theta_2^x + \frac{9}{80}W_4\theta_1^x - \frac{1}{40}W_4\theta_4^x + \frac{13k}{160}W_4\theta_2^y$$

TABLE III

ANALOGY BETWEEN THE TERMS AND LOCATIONS IN λ_{56}

AND λ OF THREE OTHER CORNER BARS

			
$\underline{\lambda_{56}}$	$\underline{\lambda_{58}}$	$\underline{\lambda_{76}}$	$\underline{\lambda_{78}}$
W_1	$-W_2$	$-W_3$	W_4
W_2	$-W_1$	$-W_4$	W_3
W_3	$-W_4$	$-W_1$	W_2
θ_1^x	θ_2^x	$-\theta_3^x$	$-\theta_4^x$
θ_2^x	θ_1^x	$-\theta_4^x$	$-\theta_3^x$
θ_3^x	θ_4^x	$-\theta_1^x$	$-\theta_2^x$
θ_1^y	$-\theta_2^y$	θ_3^y	$-\theta_4^y$
θ_2^y	$-\theta_1^y$	θ_4^y	$-\theta_3^y$
θ_3^y	$-\theta_4^y$	θ_1^y	$-\theta_2^y$

$$- \frac{3}{80} kW_4\theta_1^y - \frac{23}{160} KW_4\theta_4^y - \frac{1}{40} \theta_1^x\theta_2^x a + \frac{1}{80} \theta_2^x\theta_4^x a$$

$$- \frac{3k}{320} \theta_2^x\theta_2^y a + \frac{3ka}{160} \theta_2^x\theta_1^y + \frac{13k}{320} \theta_2^x\theta_4^y a + \frac{1}{120} \theta_1^x\theta_4^x a$$

$$+ \frac{13}{320} K\theta_1^x\theta_2^y a + \frac{K}{480} \theta_1^x\theta_1^y a - \frac{29}{960} K\theta_1^x\theta_4^y a + \frac{3K}{160} \theta_4^x\theta_2^y a$$

$$+ \frac{K}{240} \theta_4^x\theta_1^y a + \frac{K}{480} \theta_4^x\theta_4^y a + \frac{K^2}{80} \theta_1^y\theta_2^y a - \frac{K^2}{40} \theta_2^y\theta_4^y a$$

$$+ \frac{K^2}{120} \theta_1^y\theta_4^y a \quad] \qquad (4.25)$$

$$\lambda_{76} = \text{Sin}\alpha \quad [\frac{3}{16} W_3^{\,2} + \frac{0.3}{a} W_4^{\,2} + \frac{0.3}{a} W_1^{\,2} + \frac{9}{320} (\theta_3^x)^2 a + \frac{17}{960} (\theta_4^x)^2 a$$

$$+ \frac{1}{120} (\theta_1^x)^2 a + \frac{9k^2}{320} (\theta_3^y)^2 a + \frac{K^2}{120} (\theta_4^y)^2 a + \frac{17k^2}{960} (\theta_1^y)^2 a$$

$$- \frac{3}{16a} W_3W_4 - \frac{3}{16a} W_1W_3 - \frac{3}{32} W_3\theta_3^x - \frac{1}{32} W_3\theta_4^x - \frac{1}{16} W_3\theta_1^x$$

$$- \frac{3}{32} W_3K\theta_3^y - \frac{1}{16} W_3k\theta_4^y - \frac{1}{32} W_3k\theta_1^y - \frac{33}{80a} W_1W_4 - \frac{13}{160} W_4\theta_3^x$$

$$+ \frac{23}{160} W_4\theta_4^x + \frac{3}{80} W_4\theta_1^x + \frac{7}{40} W_4k\theta_3^y + \frac{k}{40} W_4\theta_4^y - \frac{9k}{80} W_4\theta_1^y$$

$$+ \frac{7}{40} W_1\theta_3^x - \frac{9}{80} W_1\theta_4^x + \frac{1}{40} W_1\theta_1^x - \frac{13}{160} kW_1\theta_3^y + \frac{3k}{80} W_1\theta_4^y$$

$$+ \frac{23}{160} KW_1\theta_1^y - \frac{1}{40} \theta_3^x\theta_4^x a + \frac{1}{80} \theta_3^x\theta_1^x a - \frac{3k}{320} \theta_3^x\theta_3^y a$$

$$+ \frac{3k}{160} \theta_3^x\theta_4^y a + \frac{13k}{320} \theta_3^x\theta_1^y a + \frac{1}{120} \theta_4^x\theta_1^x a + \frac{13k}{320} \theta_4^x\theta_3^y a$$

$$+ \frac{K}{480}\,\theta_4^x\theta_4^y\,a - \frac{29k}{960}\,\theta_4^x\theta_1^y\,a + \frac{3}{160}\,k\theta_1^x\theta_3^y\,a + \frac{Ka}{240}\,\theta_1^x\theta_4^y$$

$$+ \frac{Ka}{480}\,\theta_1^x\theta_1^y + \frac{K^2}{80}\,\theta_3^y\theta_4^y\,a - \frac{K^2}{40}\,\theta_1^y\theta_3^y\,a + \frac{K^2}{120}\,\theta_4^y\theta_1^y\,a \quad]$$

(4.26)

$$\lambda_{78} = \mathrm{Sin}\alpha \quad [\ \frac{3}{16a}\,W_4^{\,2} + \frac{0.3}{a}\,W_3^{\,2} + \frac{0.3}{a}\,W_2^{\,2} + \frac{9}{320}\,(\theta_4^x)^2 a + \frac{17}{960}\,(\theta_3^x)^2 a$$

$$+ \frac{1}{120}\,(\theta_2^x)^2 a + \frac{9K^2}{320}\,(\theta_4^y)^2 a + \frac{K^2}{120}\,(\theta_3^y)^2 a + \frac{17}{960}\,k^2(\theta_2^y)^2 a$$

$$- \frac{3}{16a}\,W_3W_4 - \frac{3}{16a}\,W_4W_2 + \frac{3}{32}\,W_4\theta_4^x + \frac{1}{32}\,W_4\theta_3^x + \frac{1}{16}\,W_4\theta_2^x$$

$$- \frac{3k}{32}\,W_4\theta_4^y - \frac{k}{16}\,W_4\theta_3^y - \frac{k}{32}\,W_4\theta_2^y - \frac{33}{80a}\,W_2W_3 + \frac{13}{160}\,W_3\theta_4^x$$

$$- \frac{23}{160}\,W_3\theta_3^x - \frac{3}{80}\,W_3\theta_2^x + \frac{7K}{40}\,W_3\theta_4^y + \frac{K}{40}\,W_3\theta_3^y - \frac{9K}{80}\,W_3\theta_2^y$$

$$- \frac{7}{40}\,W_2\theta_4^x + \frac{9}{80}\,W_2\theta_3^x - \frac{1}{40}\,W_2\theta_2^x - \frac{13K}{160}\,W_2\theta_4^y + \frac{3K}{80}\,W_2\theta_3^y$$

$$+ \frac{23K}{160}\,W_2\theta_2^y - \frac{1}{40}\,\theta_3^x\theta_4^x\,a + \frac{1}{80}\,\theta_2^x\theta_4^x\,a + \frac{3K}{320}\,\theta_4^x\theta_4^y\,a$$

$$- \frac{3K}{160}\,\theta_4^x\theta_3^y\,a - \frac{13K}{320}\,\theta_4^x\theta_2^y\,a + \frac{1}{120}\,\theta_3^x\theta_2^x\,a - \frac{13K}{320}\,\theta_3^x\theta_4^y\,a$$

$$- \frac{K}{480}\,\theta_3^x\theta_3^y\,a + \frac{29K}{960}\,\theta_3^x\theta_2^y\,a - \frac{3k}{160}\,\theta_2^x\theta_4^y\,a - \frac{K}{240}\,\theta_2^x\theta_3^y\,a$$

$$- \frac{K}{480}\,\theta_2^x\theta_2^y\,a + \frac{K^2}{80}\,\theta_4^y\theta_3^y\,a - \frac{K^2}{40}\,\theta_2^y\theta_4^y\,a + \frac{K^2}{120}\,\theta_2^y\theta_3^y\,a \quad]$$

(4.27)

Calculation of Bar Forces

It was shown in Chapter II that the state of plane stress in a cell corresponding to displacement of one or more corners may be combined of several strain conditions, oriented in the directions of the x and y axes, including the uniform normal strains, the flexure like strains and the shear strains. Of these in the first two types of conditions the corner bars remain inactive and the side bars of the secondary system join the through bars as if they were parts of them. Only the shear strain conditions produce stresses F_2 in the corner bars, and equal and opposite in sign stresses F and F_1 in the half lengths of all side bars. Thus the mean stresses in the two halves of each combined side bar are independent of the stresses F and F_1, and may be determined from the change in length of the total member.

Under plane stress action the model is assumed to be completely free at the edges. Fig. 4.6 represents a typical cell of the model in plane stress action analysed for displacements, under a load of unit intensity. The displacements U and V of its four corners and the corner forces X and Y are all known. With the corner displacements available, the elongations of the side and the diagonal bars are as stated in Table 4, and the mean stresses in the side bars S_{12}, S_{34}, S_{13}, S_{24} and the

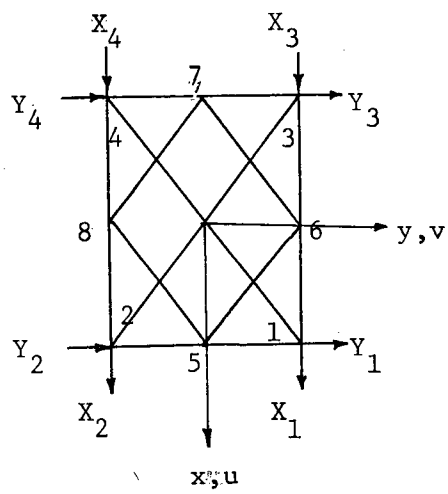


Fig. 4.6

stresses in the diagonals S_{14}, S_{23} are easily found knowing the cross sectional areas of the bars (Eqns. 2.8, 2.7, 2.5).

TABLE IV

ELONGATION OF PRIMARY BARS

Bar	Elongation
1-2	$V_1 - V_2$
1-3	$U_1 - U_3$
3-4	$V_3 - V_4$
2-4	$U_2 - U_4$
1-4	$\sin\alpha\ [\ k(U_1-U_4) + V_1-V_4\ ]$
2-3	$\sin\alpha\ [\ k(U_2-U_3) + V_3-V_2\ ]$

Should the stress condition in the cell include also some shear action, additional stresses F in the halves of the side bars 1-2, and 3-4, F_1 in the halves of the two other side bars, and F_2 in the corner bars are also present. The numerical values of these stresses stand in relation

$$\frac{F}{\sin\alpha} = \frac{F_1}{\cos\alpha} = F_2 \qquad (4.28)$$

In all four corners the stresses in the corner bars are opposite in sign to the stresses in the adjacent halves of the side bars. The signs of all these stresses are respectively the same in the diagonally opposite corners like 2 and 3 and respectively opposite to each other at the corners adjacent the same side of the cell like 1 and 2.

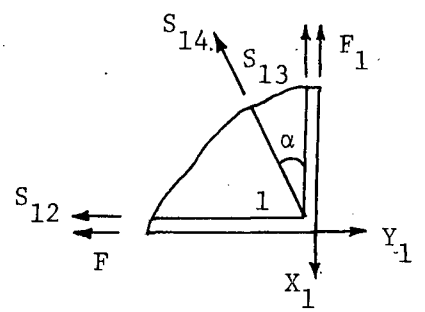


Fig. 4.7

It is sufficient to determine either stresses F or F_1 from equilibrium of one of the nodes like node 1 in Fig. 4.7. With the stresses S_{12} and S_{14} known

$$F = Y_1 - S_{12} - S_{14} \sin\alpha \qquad (4.29)$$

The stresses F_1 and F_2 may be found by the relation (4.28). Equilibrium of the other corners must produce the same numerical values of F, F_1 and F_2.

Stability Matrix of a Cell

With the extensional bar stresses known and the flexural changes in the bar lengths λ expressed in terms of the nodal displacements W, θ^x and θ^y the elements of the stability matrix are found by Eqn. (3.2).

The bar stresses S in Eqn. (3.2) are positive for compression and negative for tension. The contribution of each half side member such as 1-5 (Fig. 4.9) may be taken in two parts: the full side member 1-2 with the mean stress S_{12} and the additional half side member 1-5 with the stress F. This treatment is more convenient than consideration of each half member under the action of its stress S + F or S-F, as the case may be.

Since the symbols F signify stresses of different signs in different members, it is necessary for the generality of the stiffness matrix to make a definite assumption at this stage as to the disposition

of signs of the stresses F in the cell. This is done in Fig. 4.9.

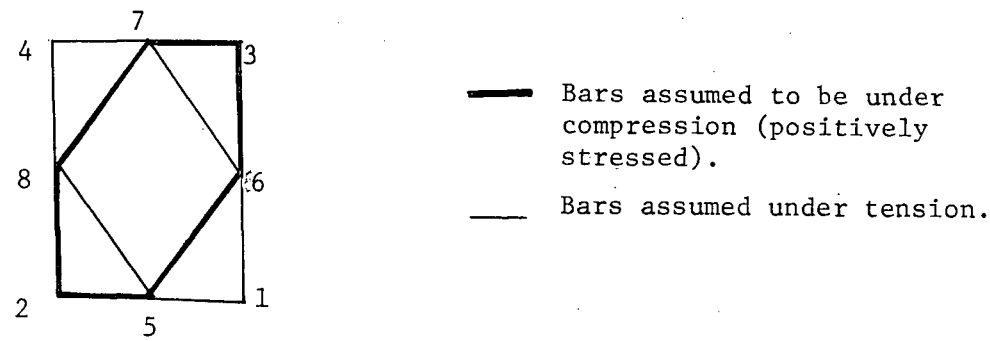


Fig. 4.9

Here are the examples of calculation of the terms of the stability matrix.

$\underline{\gamma W_1 W_1}$

The only expressions for λ (among the Eqns. 4.8 to 4.19, 4.22 - 4.27) which contain W_1^2 are λ_{12}, λ_{13}, λ_{14}, λ_{15}, λ_{25}, λ_{16}, λ_{36}, λ_{56}, λ_{76} and λ_{58}. The product formed according to Eqn. 3.2, with the expressions for λ being given by Eqns. 4.8, 4.11, 4.22, 4.10, 4.9, 4.14, 4.15, 4.24, 4.26 and 4.25 is

$$2\,(0.6\,S_{12}\frac{W_1^2}{a} + 0.6\,S_{13}\frac{W_1^2}{a} + 0.6\,S_{14}\frac{W_1^2}{a}\,\mathrm{Sin}\alpha - 0.3\,F_1\frac{W_1^2}{ka}$$

$$- 0.3\,F\frac{W_1^2}{a} + 0.3\,F\frac{W_1^2}{a} + 0.3\,F_1\frac{W_1^2}{ka} + \frac{3}{16a}W_1^2F_2\,\mathrm{Sin}\alpha - 0.3\,F_2\frac{W_1^2}{a}\,\mathrm{Sin}\alpha$$

$$- 0.3\,F_2\frac{W_1^2}{a}\,\mathrm{Sin}\alpha\,)$$

Using relation (4.28) and rewriting this equation in the quadratic form obtain

$$\gamma W_1 W_1 = \frac{1.2}{a} (S_{12} + \frac{S_{13}}{k} + S_{14} \text{Sin}\alpha) - 0.825 \frac{F}{a}$$

$\underline{\gamma\theta_3^x \theta_2^y}$

The only terms containing the product $\theta_3^x \theta_2^y$ are λ_{23}, λ_{56} and λ_{78} given by equations 4.23, 4.24, and 4.27.

Forming the product $\Sigma S\lambda$ obtain

$$2 (S_{23} (- \frac{a}{30} \text{Cos}\alpha) \theta_3^x\theta_2^y + F_2 (- \frac{ka}{240}) \text{Sin}\alpha \ \theta_3^x\theta_2^y + F_2 (\frac{29}{960} ka) \text{Sin}\alpha \ \theta_3^x\theta_2^y)$$

Again using relation 4.28 and rewriting in the quadratic form obtain

$$\gamma\theta_3^x\theta_2^y = - \frac{a}{30} \text{Cos}\alpha \ S_{23} + \frac{5}{192} Fka$$

All terms of the stability matrix are assembled in Table 5.

Symmetric

$\gamma W_1 W_1$											
$\gamma\theta_1^x W_1$	$\gamma\theta_1^x\theta_1^x$										
$\gamma\theta_1^y W_1$	$\gamma\theta_1^y\theta_1^x$	$\gamma\theta_1^y\theta_1^y$									
$\gamma W_2 W_1$	$\gamma W_2\theta_1^x$	$\gamma W_2\theta_1^y$	$\gamma W_2 W_2$								
$\gamma\theta_2^x W_1$	$\gamma\theta_2^x\theta_1^x$	$\gamma\theta_2^x\theta_1^y$	$\gamma\theta_2^x W_2$	$\gamma\theta_2^x\theta_2^x$							
$\gamma\theta_2^y W_1$	$\gamma\theta_2^y\theta_1^x$	$\gamma\theta_2^y\theta_1^y$	$\gamma\theta_2^y W_2$	$\gamma\theta_2^y\theta_2^x$	$\gamma\theta_2^y\theta_2^y$						
$\gamma W_3 W_1$	$\gamma W_3\theta_1^x$	$\gamma W_3\theta_1^y$	$\gamma W_3 W_2$	$\gamma W_3\theta_2^x$	$\gamma W_3\theta_2^y$	$\gamma W_3 W_3$					
$\gamma\theta_3^x W_1$	$\gamma\theta_3^x\theta_1^x$	$\gamma\theta_3^x\theta_1^y$	$\gamma\theta_3^x W_2$	$\gamma\theta_3^x\theta_2^x$	$\gamma\theta_3^x\theta_2^y$	$\gamma\theta_3^x W_3$	$\gamma\theta_3^x\theta_3^x$				
$\gamma\theta_3^y W_1$	$\gamma\theta_3^y\theta_1^x$	$\gamma\theta_3^y\theta_1^y$	$\gamma\theta_3^y W_2$	$\gamma\theta_3^y\theta_2^x$	$\gamma\theta_3^y\theta_2^y$	$\gamma\theta_3^y W_3$	$\gamma\theta_3^y\theta_3^x$	$\gamma\theta_3^y\theta_3^y$			
$\gamma W_4 W_1$	$\gamma W_4\theta_1^x$	$\gamma W_4\theta_1^y$	$\gamma W_4 W_2$	$\gamma W_4\theta_2^x$	$\gamma W_4\theta_2^y$	$\gamma W_4 W_3$	$\gamma W_4\theta_3^x$	$\gamma W_4\theta_3^y$	$\gamma W_4 W_4$		
$\gamma\theta_4^x W_1$	$\gamma\theta_4^x\theta_1^x$	$\gamma\theta_4^x\theta_1^y$	$\gamma\theta_4^x W_2$	$\gamma\theta_4^x\theta_2^x$	$\gamma\theta_4^x\theta_2^y$	$\gamma\theta_4^x W_3$	$\gamma\theta_4^x\theta_3^x$	$\gamma\theta_4^x\theta_3^y$	$\gamma\theta_4^x W_4$	$\gamma\theta_4^x\theta_4^x$	
$\gamma\theta_4^y W_1$	$\gamma\theta_4^y\theta_1^x$	$\gamma\theta_4^y\theta_1^y$	$\gamma\theta_4^y W_2$	$\gamma\theta_4^y\theta_2^x$	$\gamma\theta_4^y\theta_2^y$	$\gamma\theta_4^y W_3$	$\gamma\theta_4^y\theta_3^x$	$\gamma\theta_4^y\theta_3^y$	$\gamma\theta_4^y W_4$	$\gamma\theta_4^y\theta_4^x$	$\gamma\theta_4^y\theta_4^y$

(4.30)

TABLE V

ELEMENTS OF STABILITY MATRIX

$$\gamma W_1 W_1 = + \frac{1.2}{a} S_{12} + \frac{1.2}{ka} S_{13} + \frac{1.2}{a} S_{14} \sin\alpha - 0.825 \frac{F}{a}$$

$$\gamma \theta_1^x W_1 = - \frac{13F}{80} - 0.1 S_{14} \sin\alpha - 0.1 S_{12}$$

$$\gamma \theta_1^y W_1 = + 0.1 (S_{13} + S_{14} \cos\alpha) + \frac{13k}{80} F$$

$$\gamma W_2 W_1 = - \frac{1.2}{a} S_{12}$$

$$\gamma \theta_2^x W_1 = - 0.1 S_{12} + \frac{3F}{40}$$

$$\gamma \theta_2^y W_1 = + \frac{19k}{80} F$$

$$\gamma W_3 W_1 = - \frac{1.2}{ka} S_{13}$$

$$\gamma \theta_3^x W_1 = - \frac{19}{80} F$$

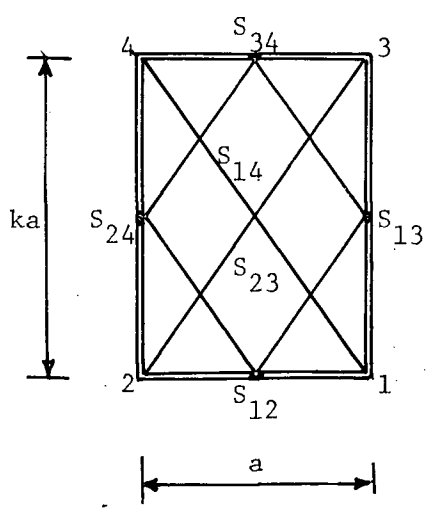
$$\gamma \theta_3^y W_1 = - \frac{3kF}{40} + 0.1 S_{13}$$

$$\gamma W_4 W_1 = + \frac{33F}{40a} - \frac{1.2}{a} S_{14} \sin\alpha$$

$$\gamma \theta_4^x W_1 = + \frac{3F}{20} - 0.1 S_{14} \sin\alpha$$

$$\gamma \theta_4^y W_1 = - \frac{3kF}{20} + 0.1 S_{14} \cos\alpha$$

$$\gamma \theta_1^x \theta_1^x = - \frac{7Fa}{120} + \frac{2a}{15} S_{12} + \frac{2a}{15} S_{14} \sin\alpha$$



$$\gamma\theta_1^y\theta_1^x = +\frac{Fka}{192} - \frac{2a}{15} S_{14} \cos\alpha$$

$$\gamma W_2\theta_1^x = +\frac{3F}{40} + 0.1 S_{12}$$

$$\gamma\theta_2^x\theta_1^x = -\frac{a}{30} S_{12}$$

$$\gamma\theta_2^y\theta_1^x = -\frac{19Fka}{320}$$

$$\gamma W_3\theta_1^x = +\frac{19F}{80}$$

$$\gamma\theta_3^x\theta_1^x = 0$$

$$\gamma\theta_3^y\theta_1^x = -\frac{19Fka}{320}$$

$$\gamma W_4\theta_1^x = -\frac{3F}{20} + 0.1 S_{14} \sin\alpha$$

$$\gamma\theta_4^x\theta_1^x = -\frac{aF}{60} - \frac{a}{30} S_{14} \sin\alpha$$

$$a\theta_4^y\theta_1^x = +\frac{5Fka}{192} + \frac{a}{30} \cos\alpha\ S_{14}$$

$$\gamma\theta_1^y\theta_1^y = +\frac{2}{15} ka\ S_{13} + \frac{2k^2a}{15} \sin\alpha\ S_{14} - \frac{7}{120} k^2aF$$

$$\gamma W_2\theta_1^y = -\frac{19kF}{80}$$

$$\gamma\theta_2^x\theta_1^y = -\frac{19Fka}{320}$$

$$\gamma\theta_2^y\theta_1^y = 0$$

$$\gamma W_3\theta_1^y = -\frac{3kF}{40} - 0.1 S_{13}$$

$$\gamma\theta_3^x\theta_1^y = -\frac{19Fka}{320}$$

$$\gamma\theta_3^y\theta_1^y = -\frac{ka}{30} S_{13}$$

$$\gamma W_4\theta_1^y = +\frac{3kF}{20} - 0.1\ S_{14}\ \text{Cos}\alpha$$

$$\gamma\theta_4^x\theta_1^y = +\frac{5Fka}{192} + \frac{a}{30} S_{14}\ \text{Cos}\alpha$$

$$\gamma\theta_4^y\theta_1^y = -\frac{Fk^2a}{60} - \frac{a}{30} k^2\ \text{Sin}\alpha\ S_{14}$$

$$\gamma W_2W_2 = +\frac{1.2}{a} S_{12} + \frac{1.2}{ka} S_{24} + \frac{1.2}{a} S_{23}\ \text{Sin}\alpha + 0.825\ \frac{F}{a}$$

$$\gamma\theta_2^xW_2 = -\frac{13F}{80} + 0.1\ S_{12} + 0.1\ S_{23}\ \text{Sin}\alpha$$

$$\gamma\theta_2^yW_2 = -\frac{13Fk}{80} + 0.1\ S_{24} + 0.1\ S_{23}\ \text{Cos}\alpha$$

$$\gamma W_3W_2 = -\frac{33F}{40a} - \frac{1.2}{a} S_{23}\ \text{Sin}\alpha$$

$$\gamma\theta_3^xW_2 = +\frac{3F}{20} + 0.1\ S_{23}\ \text{Sin}\alpha$$

$$\gamma\theta_3^yW_2 = +\frac{3kF}{20} + 0.1\ S_{23}\ \text{Cos}\alpha$$

$$\gamma W_4W_2 = -\frac{1.2}{ka} S_{24}$$

$$\gamma\theta_4^xW_2 = -\frac{19F}{80}$$

$$\gamma\theta_4^yW_2 = +\frac{3Fk}{40} + 0.1\ S_{24}$$

$$\gamma\theta_2^x\theta_2^x = +\frac{2a}{15} S_{12} + \frac{2a}{15} S_{23}\ \text{Sin}\alpha + \frac{7}{120}\ Fa$$

$$\gamma\theta_2^y\theta_2^x = +\frac{2a}{15} S_{23} \operatorname{Cos}\alpha + \frac{Fka}{192}$$

$$\gamma W_3\theta_2^x = -\frac{3F}{20} - 0.1\, S_{23} \operatorname{Sin}\alpha$$

$$\gamma\theta_3^x\theta_2^x = -\frac{a}{30} S_{23} \operatorname{Sin}\alpha + \frac{Fa}{60}$$

$$\gamma\theta_3^y\theta_2^x = -\frac{a}{30} S_{23} \operatorname{Cos}\alpha + \frac{5}{192} Fka$$

$$\gamma W_4\theta_2^x = +\frac{19F}{80}$$

$$\gamma\theta_4^x\theta_2^x = 0$$

$$\gamma\theta_4^y\theta_2^x = -\frac{19Fka}{320}$$

$$\gamma\theta_2^y\theta_2^y = +\frac{2ka}{15} S_{24} + \frac{2}{15} k^2 a \operatorname{Sin}\alpha\, S_{23} + \frac{7}{120} Fk^2 a$$

$$\gamma W_3\theta_2^y = -0.1 \operatorname{Cos}\alpha\, S_{23} - \frac{3Fk}{20}$$

$$\gamma\theta_3^x\theta_2^y = -\frac{a}{30} S_{23} \operatorname{Cos}\alpha + \frac{5}{192} Fka$$

$$\gamma\theta_3^y\theta_2^y = -\frac{k^2 a}{30} S_{23} \operatorname{Sin}\alpha + \frac{k^2 aF}{60}$$

$$\gamma W_4\theta_2^y = +\frac{3}{40} kF - 0.1\, S_{24}$$

$$\gamma\theta_4^x\theta_2^y = -\frac{19Fka}{320}$$

$$\gamma\theta_4^y\theta_2^y = -\frac{ka}{30} S_{24}$$

$$\gamma W_3 W_3 = +\frac{1.2}{a} S_{34} + \frac{1.2}{ka} S_{13} + \frac{1.2}{a} S_{23} \operatorname{Sin}\alpha + 0.825\,\frac{F}{a}$$

$$\gamma\theta_3^x W_3 = -0.1\, S_{34} - 0.1\, S_{23}\, \mathrm{Sin}\alpha + \frac{13F}{80}$$

$$\gamma\theta_3^y W_3 = -0.1\, S_{13} - 0.1\, S_{23}\, \mathrm{Cos}\alpha + \frac{13F}{80}\, k$$

$$\gamma W_4 W_3 = -\frac{1.2}{a}\, S_{34}$$

$$\gamma\theta_4^x W_3 = -0.1\, S_{34} - \frac{3F}{40}$$

$$\gamma\theta_4^y W_3 = +\frac{19Fk}{80}$$

$$\gamma\theta_3^x\theta_3^x = +\frac{7}{120}\, Fa + \frac{2a}{15}\, S_{34} + \frac{2a}{15}\, S_{23}\, \mathrm{Sin}\alpha$$

$$\gamma\theta_3^y\theta_3^x = +\frac{2a}{15}\, S_{23}\, \mathrm{Cos}\alpha + \frac{Fka}{192}$$

$$\gamma W_4\theta_3^x = +0.1\, S_{34} - \frac{3F}{40}$$

$$\gamma\theta_4^x\theta_3^x = -\frac{a}{30}\, S_{34}$$

$$\gamma\theta_4^y\theta_3^x = -\frac{19Fka}{320}$$

$$\gamma\theta_3^y\theta_3^y == +\frac{2ka}{15}\, S_{13} + \frac{2a}{15}\, k^2\, S_{23}\, \mathrm{Sin}\alpha + \frac{7Fk^2a}{120}$$

$$\gamma W_4\theta_3^y = -\frac{19Fk}{80}$$

$$\gamma\theta_4^x\theta_3^y = -\frac{19Fka}{320}$$

$$\gamma\theta_4^y\theta_3^y = 0$$

$$\gamma W_4 W_4 = \frac{1.2}{a}\, S_{34} + \frac{1.2}{ka}\, S_{24} + \frac{1.2}{a}\, S_{14}\, \mathrm{Sin}\alpha - 0.825\, \frac{F}{a}$$

$$\gamma\theta_4^x W_4 = 0.1\ S_{34} + 0.1\ S_{14}\ \text{Sin}\alpha + \frac{13F}{80}$$

$$\gamma\theta_4^y W_4 = -\ 0.1\ S_{24} - 0.1\ S_{14}\ \text{Cos}\alpha - \frac{13kF}{80}$$

$$\gamma\theta_4^x \theta_4^x = \frac{2a}{15}\ S_{34} + \frac{2a}{15}\ S_{14}\ \text{Sin}\alpha - \frac{7Fa}{120}$$

$$\gamma\theta_4^y \theta_4^x = -\ \frac{2a}{15}\ S_{14}\ \text{Cos}\alpha + \frac{Fka}{192}$$

$$\gamma\theta_4^y \theta_4^y = \frac{2}{15}\ S_{24}\ ka + \frac{2k^2a}{15}\ S_{14}\ \text{Sin}\alpha - \frac{7}{120}\ Fk^2a$$

<u>Note</u>: All bar stresses, S positive in compression.
For sign of F stresses see Fig. 4.9.

CHAPTER V

EXAMPLES

The finite element method presented here has been applied to several problems for which the exact solutions are available (8) and which have also been solved by the finite element method employing the no bar cells (9).

In bar cells both the stability and the stiffness matrices are affected by μ, while in the no bar cells the stability matrix does not depend on μ. In order to study the effect of varying μ several widely divergent values of μ have been used for the bar cells.

In the course of the solution of these examples a noticeable reduction in precision was observed in some cases. This was related to the negative and zero values of the extensional and flexural differences of some of the bars in the cells used in the analysis. This point was made the subject of a special investigation reported at the end of this section.

Example 1

A rectangular plate of the size $b \times \ell$ and the thickness t is compressed in the direction ℓ by the stress σ per unit area and is not stressed in the other direction. The plate is simply supported on all edges as far as the flexural deformation is concerned. The exact

critical stress is given by the expression

$$\sigma_{cr} = \frac{\beta\pi^2 D}{b^2 t} \tag{5.1}$$

in which D is found by Eqn (2.34). The comparison of the critical values is made on the basis of the coefficient β, which in this formula is independent of μ.

Three values of the Poisson's ratio μ are used and the aspect ratio $\frac{\ell}{b}$ is taken as 1 and 2.5. In all but one case, the bar model is formed by (3 x 3), (4 x 4) and (6 x 6) framework, each cell of which, a by ka in size, is geometrically similar to the plate. In one case (3 x 6), (4 x 8) and (6 x 12) models were employed.

Bar Forces (Fig. 5.1)

In view of the symmetry of loading the corner bars are inactive and the secondary side bars simply add their areas to the primary bars. With σ=1 the corner loads carried by each cell are $\frac{1}{2}$ at and the changes in the length and width of the cell are:

$$\delta_1 = \frac{ka}{E} \tag{5.2}$$

and

$$\delta_2 = \frac{\mu a}{E} \tag{5.3}$$

respectively since the edges of the plates are free to move unrestrained in plane stress. The shortening of the diagonal is

$$\delta_5 = \delta_1 \cos\alpha - \delta_2 \sin\alpha$$

$$= \frac{(k^2-\mu)a}{E(k^2+1)^{1/2}} . \tag{5.4}$$

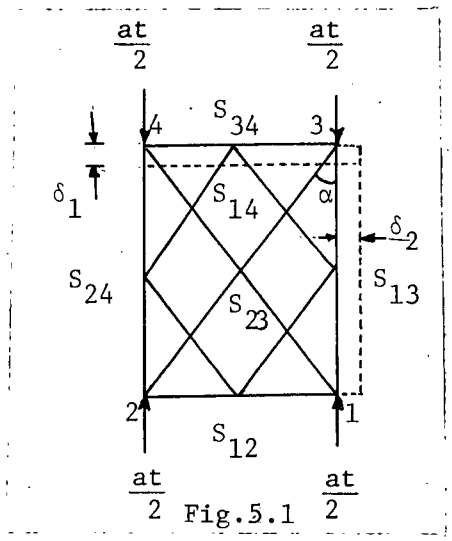


Fig.5.1

Using the Eqn (2.5) and the expressions (2.6), (2.8) and (2.9) for the cross sectional areas of the bars, the bar stresses caused by the load of unit intensity i.e. the one corresponding to $\sigma = 1$ are

$$S_{12} = S_{34} = -\frac{\mu a(k^2-\mu)}{2k(1-\mu^2)}$$

$$S_{13} = S_{24} = +\frac{a(1-\mu k^2)}{2(1-\mu^2)} \qquad (5.5)$$

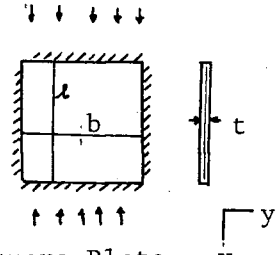
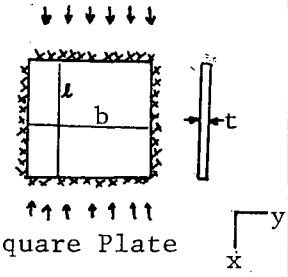
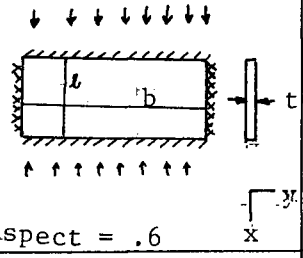
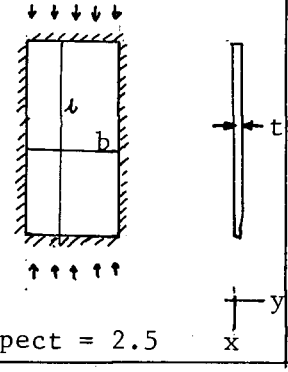
$$S_{14} = S_{23} = +\frac{\mu a(k^2-\mu)(k^2+1)^{1/2}}{2k(1-\mu^2)}$$

In these expressions the compressions are considered positive and the tensions negative in agreement with the basic equation (3.2). The critical buckling stress σ_{cr} of Eqn (5.1) is the lowest eigenvalue f of the Eqn (3.1) and it is found by a standard computer procedure.

The results of both bar and no bar solutions are presented in Table 6 in the form of percent errors in comparison with the exact values, with the minus sign indicating the approximate value being smaller than the exact. The same table also gives the results of both bar and no bar cells for a square plate with its edges fixed against flexure and for a rectangular plate ($\frac{\ell}{b} = 0.6$) whose non loaded edges are fixed for flexure while the loaded edges are simply supported.

TABLE VI

CRITICAL BUCKLING STRESSES IN PLATES
PERCENT ERRORS OF FINITE ELEMENT SOLUTIONS
WITH BAR AND NO BAR SOLUTIONS
UNIFORM COMPRESSION IN X DIRECTION

Problem	Exact Soln. β (Eqn 5.1)	Poisson's Ratio (μ)	Bar or No Bar	Percent Error Model Mesh 3 x 3	4 x 4	6 x 6
 Square Plate	4.0	-	No Bar	-8.88	-5.75	-2.80
	4.0	.3	Bar	-10.0	-5.75	-2.75
	4.0	.1	Bar	-9.5	-5.50	-1.75
	4.0	.45	Bar	-10.25	-6.50	-3.0
 Square Plate	10.07	-	No Bar	-	-7.80	-4.56
	10.07	.3	Bar	-13.0	-8.60	-5.17
	10.07	.1	Bar	-13.1	-8.60	-4.37
	10.07	.45	Bar	-13.1	-9.0	-5.17
 Aspect = .6	7.05	-	No Bar	-	-7.24	-3.74
	7.05	.3	Bar	-15.6	-10.8	-5.25
	7.05	.1	Bar	-16.6	-11.0	-6.0
	7.05	.45	Bar	-23.0	-	-
 Aspect = 2.5				3 x 6 K=1.25	4 x 8 K=1.25	6 x 12 K=1.25
	4.133	-	No Bar	-	-7.10	-3.73
	4.133	.3	Bar	-5.0	-4.05	-2.82
	4.133	.1	Bar	-11.2	-8.15	-5.10
	4.133	.45	Bar	-11.55	-1.97	-1.80

Example 2

The only difference of this example from example 1 is that the plate is subjected to equal stresses σ in the direction of both axes. The edges of the plate are again simply supported in flexure. The same Eqn (5.1) may be used to describe the buckling condition in the plate employing different appropriate theoretical values of the coefficient β.

The bar stresses may be found as the sums of the stresses produced by compression in X direction Eqn (5.5) and the stresses caused by the compression in y direction. The latter may be obtained from the former by replacing K by $\frac{1}{K}$ and a by ka with the $S_{12} = S_{34}$ stresses becoming $S_{13} = S_{24}$ stresses and vice versa. Following this procedure the total stresses are found

$$S_{12} = S_{34} = \frac{a(k^2-\mu)}{2k(1+\mu)}$$

$$S_{13} = S_{24} = \frac{a(1-\mu k^2)}{2(1+\mu)}$$

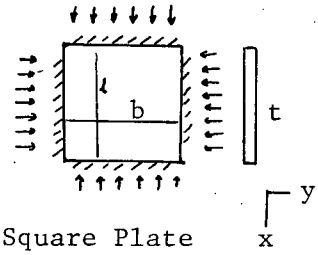
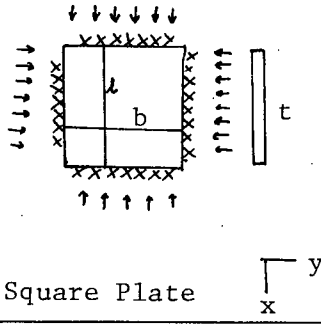
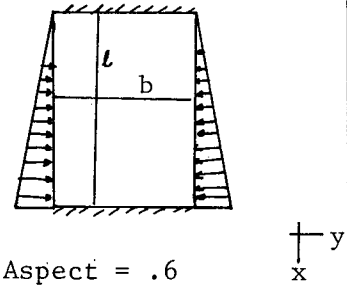
$$S_{14} = S_{23} = \frac{\mu a(k^2+1)^{3/2}}{2k(1+\mu)}$$

$$F = 0 \qquad (5.6)$$

The results of the solution are assembled in Table 7. Also given in Table 7 are the results for a plate subjected to uniform compression stress in the direction of both axes, but with the edges of the plate fixed against flexure.

TABLE VII

CRITICAL BUCKLING STRESSES IN PLATES
PERCENT ERRORS OF FINITE ELEMENT SOLUTIONS
WITH BAR AND NO BAR CELLS
UNIFORM COMPRESSION IN TWO DIRECTIONS
AND UNIFORMLY VARYING LOAD IN X DIRECTION

Problem	Exact Value of β (Eqn5.1)	Poisson's Ratio (μ)	Bar or No Bar	Percent Error Model Mesh 3 x 3	4 x 4	6 x 6
 Square Plate	2.0	-	No Bar	-12.95	-5.5	-2.75
	2.0	.3	Bar	-9.3	-5.85	-2.66
	2.0	.1	Bar	-8.96	-5.2	-2.32
	2.0	.45	Bar	-10.2	-6.5	-3.04
 Square Plate	5.315	-	No Bar	-	-6.38	-4.46
	5.315	.3	Bar	-16.0	-8.8	-4.9
	5.315	.1	Bar	-12.9	-8.7	-4.57
	5.315	.45	Bar	-11.4	-8.76	-5.26
 Aspect = .6	9.7	-	No Bar	-	-6.34	-2.84
	9.7	.3	Bar	-7.16	-5.36	-3.62
	9.7	.1	Bar	-8.45	-6.43	-4.34
	9.7	.45	Bar	-14.3	-77.1	-62.1

Simply supported

Clamped

in flexure

Example 3

The simply supported plate of the dimension (ℓ x b) and the thickness t (Fig. 5.2), is subjected to uniform shear stress τ along the edges.

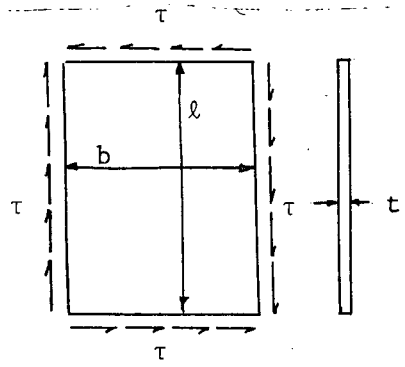


Fig. 5.2

The exact critical stress is given by the Eqn (5.1) as

$$\tau_{cr} = \beta \frac{\pi^2 D}{b^2 t} \qquad (5.7)$$

The model is again made up of the cells a xnkakgeometrically similar to the plate. With τ=1, the corner loads carried by each cell in the x and y directions are respectively $\frac{kat}{2}$ and $\frac{at}{2}$ and the displacement of corners 3 and 4 in the y direction is

$$\delta = \frac{ka}{tE} 2 (1+\mu) \qquad (5.8)$$

This causes an elongation in the diagonal bar (2-3) and an equal shortening in the diagonal bar (1-4) of the amount

$$\delta_5 = \delta \text{ Sin}\alpha$$

$$= \frac{2(1+\mu)}{tE} \frac{ka}{(k^2+1)^{1/2}} \qquad (5.9)$$

The stresses S_{14} and S_{23} are found using Eqn (2.5) with the expression for area being given by Eqn (2.6)

$$S_{14} = \frac{\mu a (k^2+1)^{1/2}}{1-\mu} \qquad (5.10)$$

$$= - S_{23} \qquad (5.11)$$

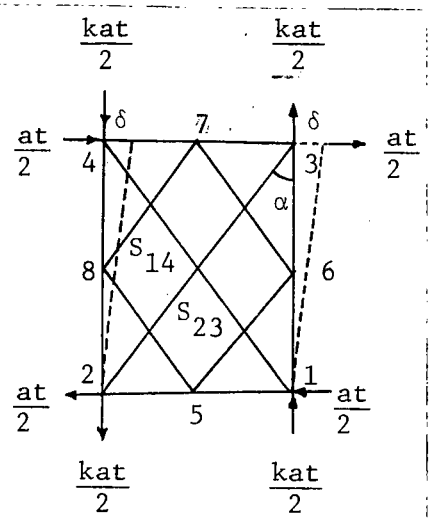


Fig. 5.3

The stresses in the subdivided side bars are found with the help of Eqn (4.29) as

$$F = \frac{a(3\mu-1)}{2(1-\mu)} \qquad (5.12)$$

$S_{12} = S_{34} = S_{13} = S_{24} = 0$ since this condition leaves these bars unstressed. The stress F is positive since it agrees with the sign convention adopted for positive stress in the secondary bars (Fig. 4.9).

The critical buckling stress τ_{cr} is found as before and is the lowest eigenvalue f of the Eqn (3.1). The results for both bar and no bar cell solutions are presented in Table 8 in the form of percent errors in comparison with the exact values. The structures analysed are:

(1) Simply supported square plate, in flexure

(2) Square plate with two opposite edges fixed and simply supported, in flexure

(3) Simply supported (in flexure) rectangular plate with $\frac{\ell}{b} = 1.25$,

all with the assumed values of μ: 0.1, 0.3 and 0.45.

Example 4

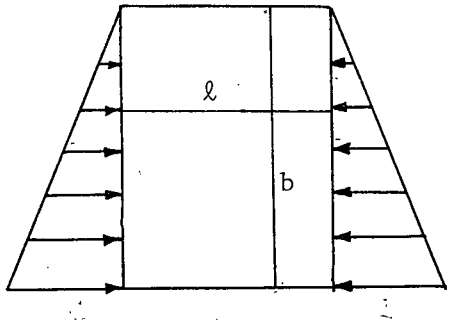
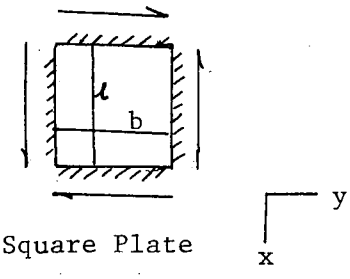
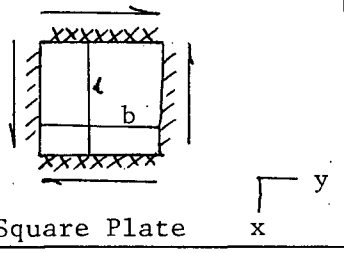
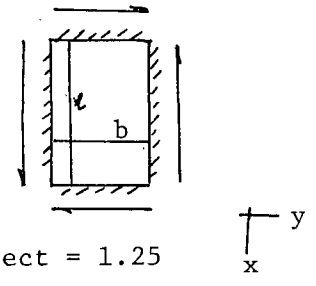


Fig. 5.4

A rectangular plate of the size (b x ℓ) simply supported against flexure, is subjected to a uniformly varying load along its longer sides as shown in Fig. 5.4. As before the critical stress is given by Eqn (5.1).

TABLE VIII

CRITICAL BUCKLING STRESSES IN PLATES
PERCENT ERRORS OF FINITE ELEMENT SOLUTIONS
WITH BAR AND NO BAR CELLS
PURE SHEAR

Problem	Exact Value of β (Eqn5.1)	Poisson's Ratio (μ)	Bar or No Bar	Percent Error Model Mesh 3 x 3	4 x 4	6 x 6
 Square Plate	9.34	-	No Bar	-12.1	-10.2	-6.47
	9.34	.3	Bar	-22.0	-11.6	-6.42
	9.34	.1	Bar	-17.95	-10.3	-6.07
	9.34	.45	Bar	-32.6	-15.1	-6.73
 Square Plate	12.28	-	No Bar	-	-10.2	-5.86
	12.28	.3	Bar	-17.2	-8.1	-4.92
	12.28	.1	Bar	-14.0	-8.34	-4.90
	12.28	.45	Bar	-37.6	-13.7	-5.06
 Aspect = 1.25	77.71	-	No Bar	-	- 9.92	-6.00
	7.71	.3	Bar	-23.1	-11.8	-5.95
	7.71	.1	Bar	-18.5	-10.35	-5.52
	7.71	.45	Bar	-34.8	-15.9	-6.3

Simply supported

Clamped in flexure

In order to calculate the bar stresses, the model is analysed for plane stress, with its edges restrained against rigid body movement only. The loads acting at the nodes of the model are the static equivalent of a load of unit intensity acting on the plate (Fig. 5.5). The bar forces are obtained using the relations given in Table 4 and Eqn 4.29.

With the bar forces known, the critical load is obtained as before and the results tabulated in Table 7.

Effect of Negative Stiffnesses

An inspection of Eqns (2.8 and (2.38) shows that for $k \geq 1$

A_1 and I_1 are zero if $\mu k^2 = 1$

A_1 and I_1 are negative if $\mu k^2 >> 1$

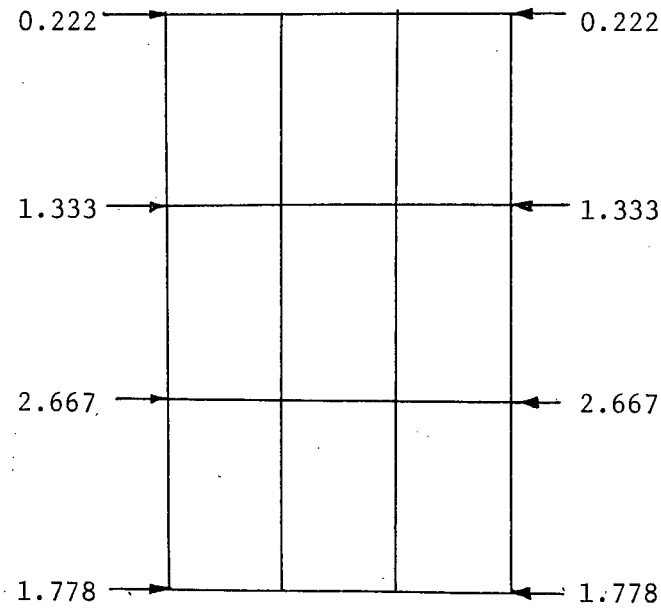
Eqn (2.46) and (2.47) reveal also that

C and C_1 are zero if $\hat{\mu} = \frac{1}{3}$

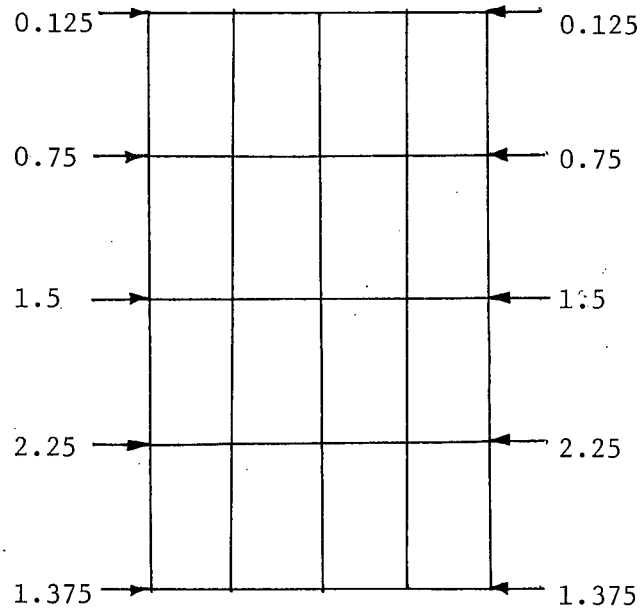
C and C_1 are negative if $\mu > \frac{1}{3}$

When the area A_1 of the longer side bar is zero the cell is no longer rigid. However the disappearance of the torsional stiffnesses C and C_1 does not make the structure non-rigid unless I_1 vanishes also.

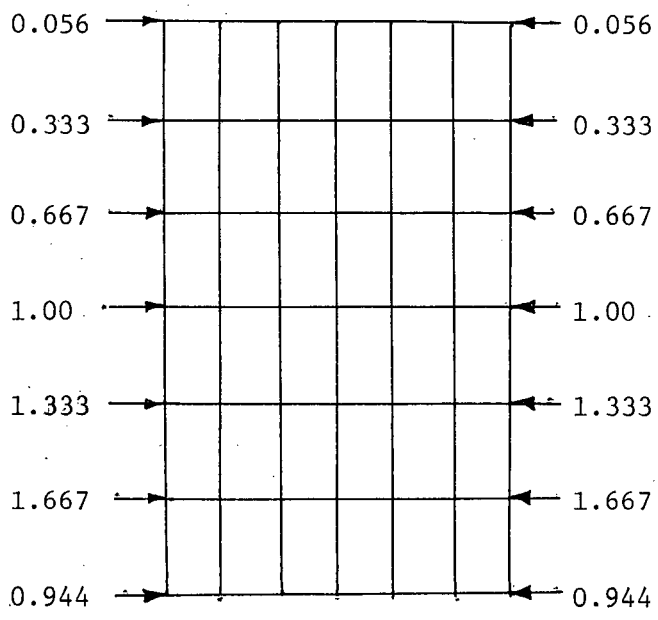
To investigate the effect of negative and zero stiffnesses the three examples solved earlier are solved again for different values of μ and k so that a wide range of the quantity μk^2 is covered, several values of which make A_1 and I_1 negative.



3 x 3 model



4 x 4 model



6 x 6 model

Loads (in lbs.) acting at nodes for various models (in plane stress) for determining bar forces in example 4. (Based on plate size 7.2" x 12" x 1"). Model is restrained only to prevent rigid body movement i.e. 3 degrees of freedom are suppressed.

Fig. 5.5

The results are summarised in Tables 9-13. As before their precision is expressed as percentage errors of the coefficient β in comparison with the elasticity solution. For certain values of μk^2 no result was obtained as indicated by the dash symbol. The examination of the tables reveals:

(1) The results are generally more accurate when $\mu k^2 \lesssim 0.75$

(2) On reduction of the mesh size the results tend to diverge for μk^2 close to and greater than unity

(3) In most cases no results may be obtained as μk^2 approaches the range of 1.15 to 1.35

(4) In the same plate accuracy decreases for a comparable mesh size as k is increased.

TABLE IX

PERCENT ERRORS OF FINITE ELEMENT SOLUTION FOR VARIOUS VALUES OF ASPECT RATIO AND POISSON'S RATIO

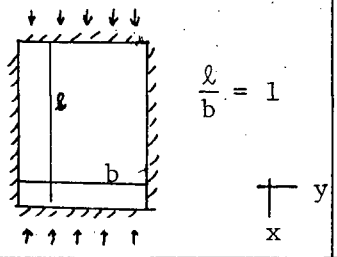
Problem	Exact Soln. β (Eqn5.1)	Poisson's Ratio (μ)	Aspect Ratio of Cell (k)	μk^2	Percent Error Model Mesh 4 x 6	8 x 12
 $\frac{\ell}{b} = 1$	4.0	.1	1.5	.225	-5.14	-2.88
	4.0	.333	1.5	.75	-2.66	-2.04
	4.0	.4	1.5	.900	-2.30	-2.00
	4.0	.45	1.5	1.025	-2.30	-2.58
					Percent Error Model Mesh 4 x 4	6 x 6
Same as Above but $\frac{\ell}{b} = 1.75$	4.071	.1	1.75	.306	-10.0	-7.6
	4.071	.333	1.75	1.020	-3.26	-4.7
	4.071	.4	1.75	1.225	-96.0	-
	4.071	.45	1.75	1.375	-	-
Same as above but $\frac{\ell}{b} = 2.0$	4.0	.1	2.0	.400	-11.15	-8.52
	4.0	.333	2.0	1.333	-56.0	-16.35
	4.0	.4	2.0	1.600	-	-
	4.0	.45	2.0	1.800	-	-
Same as above but $\frac{\ell}{b} = 2.25$	4.055	.1	2.25	.506	-13.6	-3.86
	4.055	.333	2.25	1.688	-	-
	4.055	.4	2.25	2.025	-	-
	4.055	.45	2.25	2.280	-	-
Same as above but $\frac{\ell}{b} = 2.50$	4.133	1.	2.50	.625	-20.2	-11.3
	4.133	.333	2.50	2.086	-	-
	4.133	.4	2.50	2.500	-	-
	4.133	.45	2.50	2.810	-	-

TABLE X

PERCENT ERRORS OF FINITE ELEMENT SOLUTION FOR

VARIOUS VALUES OF ASPECT RATIO AND POISSON'S RATIO

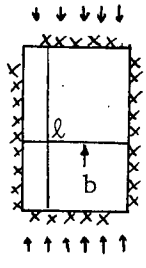
Problem	Exact Soln. β (Eqn5.1)	Poisson's Ratio (μ)	Aspect Ratio (k) of Cell	μk^2	Percent Error Model Mesh 4 x 4	6 x 6
 $\frac{\ell}{b} = 1.25$	9.25	.1	1.25	.156	-7.55	-3.55
	9.25	.333	1.25	.521	-2.46	+0.22
	9.25	.4	1.25	.625	-1.1	+1.1
	9.25	.45	1.25	.703	-0.05	+1.56
Same as above but $\frac{\ell}{b} = 1.50$	8.33	.1	1.50	.225	-9.85	±3.72
	8.33	.333	1.50	.750	-3.18	+1.06
	8.33	.4	1.50	.900	-1.71	+2.18
	8.33	.45	1.50	1.025	-50.5	-28.3
Same as above but $\frac{\ell}{b} = 1.75$	8.11	.1	1.75	.306	-15.7	-4.35
	8.11	.333	1.75	1.020	-10.8	+1.80
	8.11	.4	1.75	1.225	-75.0	-
	8.11	.45	1.75	1.375	-	-
Same as above but $\frac{\ell}{b} = 2.0$	7.88	.1	2.0	.400	-25.0	-6.25
	7.88	.333	2.0	1.333	-53.0	-32.2
	7.88	.4	2.0	1.600	-	-
	7.88	.45	2.0	1.800	-	-

TABLE XI

PERCENT ERRORS OF FINITE ELEMENT SOLUTION FOR VARIOUS VALUES OF ASPECT RATIO AND POISSON'S RATIO

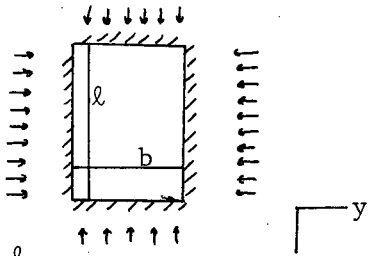
Problem	Exact Soln. β (Eqn5.1)	Poisson's Ratio (μ)	Aspect Ratio of Cell (k)	μk^2	Percent Error Model Mesh 4 x 4	6 x 6
 $\frac{\ell}{b} = 1.2$	2.44	.1	1.2	.144	-7.2	-4.76
	2.44	.333	1.2	.480	-3.5	-2.76
	2.44	.4	1.2	.576	-2.9	-2.40
	2.44	.45	1.2	.650	-2.64	-2.17
Same as above but $\frac{\ell}{b} = 1.4$	2.96	.1	1.4	.196	-6.6	-4.79
	2.96	.333	1.4	.653	-3.78	-3.1
	2.96	.4	1.4	.784	-3.38	-2.98
	2.96	.45	1.4	.882	-3.32	-3.2
Same as above but $\frac{\ell}{b} = 1.6$	3.56	.1	1.6	.256	-6.52	-4.40
	3.56	.333	1.6	.853	-4.14	-3.66
	3.56	.4	1.6	1.024	-4.27	-4.41
	3.56	.45	1.6	1.152	-	-
Same as above but $\frac{\ell}{b} = 1.8$	4.24	.1	1.8	.324	-6.4	-4.85
	4.24	.333	1.8	1.080	-4.67	-4.55
	4.24	.4	1.8	1.296	-	-
	4.24	.45	1.8	1.458	-	-

TABLE XII

PERCENT ERRORS OF FINITE SOLUTIONS FOR VARIOUS

VALUES OF ASPECT RATIO AND POISSON'S RATIO

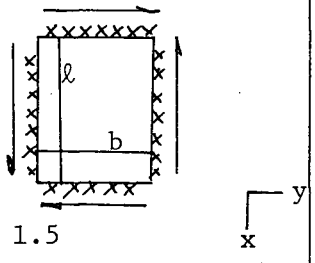
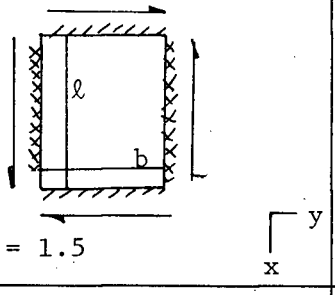
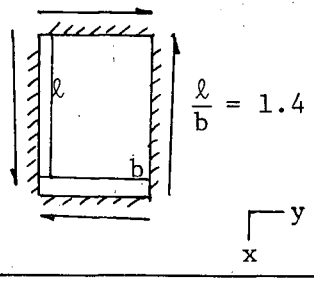
Problem	Exact Soln. β (Eqn5.1)	Poisson's Ratio (μ)	Aspect Ratio of Cell (k)	μk^2	Percent Error Model Mesh 4 x 4	6 x 6
 $\frac{\ell}{b} = 1.5$	11.50	.1	1.5	.225	+2.960	-1.01
	11.50	.333	1.5	.750	+3.220	+2.50
	11.50	.4	1.5	.900	-10.62	+1.40
	11.50	.45	1.5	1.025	-47.0	-14.0
Same as above but $\frac{\ell}{b} = 2.0$	10.34	.1	2.0	.400	-9.10	-1.45
	10.34	.333	2.0	1.333	-11.20	+0.46
	10.34	.4	2.0	1.600	-	-
	10.34	.45	200	1.800	-	-
 $\frac{\ell}{b} = 1.5$	11.12	.1	1.5	.225	-1.66	-4.3
	11.12	.333	1.5	..750	+1.68	0.0
	11.12	.4	1.5	.900	-19.20	-0.707
	11.12	.45	1.5	1.025	-63.5	-25.67
Same as above but $\frac{\ell}{b} = 2.0$	10.21	.1	2.0	.400	-12.8	-3.00
	10.21	.333	2.0	1.333	-30.6	-1.17
	10.21	.4	2.0	1.600	-	-
	10.21	.45	2.0	1.800	-	-

TABLE XIII

PERCENT ERRORS OF FINITE ELEMENT SOLUTION FOR VARIOUS VALUES OF ASPECT RATIO AND POISSON'S RATIO

Problem	Exact Soln. β (Eqn5.1)	Poisson's Ratio (μ)	Aspect Ratio of Cell (k)	μk^2	Percent Error Model Mesh 4 x 4	6 x 6
 $\frac{\ell}{b} = 1.4$	7.3	.1	1.4	.196	- .158	-3.17
	7.3	.333	1.4	.653	+2.07	-0.43
	7.3	.4	1.4	.784	-0.605	-0.268
	7.3	.45	1.4	.882	-11.0	-0.70
Same as above but $\frac{\ell}{b} = 1.6$	7.0	.1	1.6	.256	-2.23	-4.55
	7.0	.333	1.6	.853	-1.63	-2.22
	7.0	.4	1.6	1.024	-7.25	-2.88
	7.0	.45	1.6	1.152	-	-
Same as above but $\frac{\ell}{b} = 1.8$	6.8	.1	1.8	.324	-4.60	-5.19
	6.8	.33	1.8	1.080	-7.00	-3.88
	6.8	.4	1.8	1.296	-	-
	6.8	.45	1.8	1.458	-	-

CHAPTER VI

CONCLUSION

From an examination of the results the following conclusions can be drawn:

(1) Within the limits of applicability the framework method is a valid one for solving stability problems.

(2) Its comparison with the no bar cell method in plates when the results are available shows similar precision.

(3) The finite element values of the critical load are, as a rule, lower than the true values. The model thus behaves in relation to instability as a structure less rigid than the prototype.

(4) As the mesh size becomes finer, keeping the same aspect ratio of cells, the critical values of the finite element solutions advance monotonically towards the exact values, somewhat faster than in the inverse ratio of the linear cell sizes, perhaps closer to 3/2 power of this ratio.

(5) The cell proportions should preferably be kept within the limits of the aspect ratio $1 < k < 1.5$. Outside these limits precision is reduced.

(6) The cells should be so proportioned with reference to the value of the Poisson's ratio that the quantity μk^2 is restricted to remain under 0.75.

(7) With μk^2 exceeding this value and approaching the value of one, the solution at times tends to diverge on reduction of the mesh size, or the approximate critical load, normally smaller than the exact load, becomes greater than it. As μk^2 reaches the level of 1.15 - 1.3 the eigenvalue problem becomes insoluable by computer, possibly due to deficiency of matrices in the cells involved. The subject of the effects of high values of k and μk^2 requires further study.

(8) The necessary cell stability matrices cover all kinds of loading described by the stresses S. On the other hand the no bar cell analysis requires a multiplicity of stability matrices, only a few of which, applicable to certain simple types of cell loading are given explicitly. The complete generality of the proposed bar cell stability matrix makes it superior to its no-bar cell counterpart.

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