FINITE ELEMENT METHOD - A GALERKIN APPROACH

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ABSTRACT

This study is concerned with defining the mathematical framework in which the finite element procedure can most advantageously be considered. It is established that the finite element method generates an approximate solution to a given equation which is defined in terms of assumed co-ordinate functions and unknown parameters. The advantages of determining the parameters by Galerkin's method are discussed and the convergence characteristics of this method are reviewed using functional analysis principles. Comparisons are made between the Galerkin and Rayleigh-Ritz procedures and the connection between virtual work and Galerkin's method is illustrated. The convergence results presented for the Galerkin procedure are used to provide sufficient conditions that ensure the convergence of a finite element solution of a general system of time independent linear differential equations. Application of the principles developed is illustrated with a convergence proof for a finite element solution of a non-symmetric eigenvalue problem and by developing a computer program for the finite element analysis of the two-dimensional steady state flow of an incompressible viscous fluid.

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NOTATION

The specific usuage and meaning of symbols is defined in the text where they are introduced.

The summation convention holds for subscripted variables with repeated lower case indices; it does not apply to repeated upper case indices. The range of summation is indicated where the variables are first introduced.

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CHAPTER 1

INTRODUCTION

1.1 Background

The equations that are encountered in engineering practice are, in general, of such a nature that no closed form solution is available. In order to obtain answers to such problems recourse must be made to approximate solution techniques. The finite element method was developed, on the basis of physical intuition, as such a tool for application in the analysis of complex structural systems. The development of the method is well documented and Zienkiewcz (20) provides a comprehensive list of references in his review of the method.

In recent years the works of Melosh (8) and Keys (6) in particular have served to associate the method with the Rayleigh-Ritz procedure. Such an association has enabled rigorous mathematical arguments to be used to justify the use of finite elements and to provide sufficient conditions that ensure convergence of the approximate

solution to the correct one. The work of Oliveira (13) is notable in this respect.

1.2 Purpose and Scope

The purpose of this thesis is to extend the work of previous investigators in this field and to define precisely the mathematical framework in which the finite element procedure can most advantageously be considered.

It will be established that the finite element procedure generates an approximate solution for a given equation that is defined in terms of assumed co-ordinate functions and unknown parameters. This approximation may be used in conjunction with a number of methods for the determination of the unknown parameters. The relative advantages of the Galerkin, Rayleigh-Ritz, and virtual work procedures are presented and the Galerkin method is judged preferable. Convergence results established by Mikhlin (9) for the Galerkin procedure are reviewed using functional analysis principles, and it is demonstrated that they can be applied when a finite element method is used to generate the approximate solution. Particular attention is paid to the boundary conditions that must be satisfied by the assumed co-ordinate functions. It is shown that by suitably formulating the Galerkin procedure it is often sufficient to satisfy only the principal ones.

The convergence results presented by Mikhlin are extended to include problems with non-homogeneous boundary conditions and the equations, and corresponding convergence criteria, that govern a finite element solution of a general system of linear differential equations is presented. This system of equations includes problems to which the Rayleigh-Ritz procedure is not applicable.

Application of the principles developed is illustrated with a convergence proof for a finite element solution of a non-symmetric eigenvalue problem and by developing a computer program for the finite element analysis of the two-dimensional steady state flow of an incompressible viscous fluid.

1.3 Limitations

Attention will be confined to the consideration of problems that are characterized by time independent linear differential equations.

CHAPTER 2

MATHEMATICAL PRELIMINARIES

The subsequent definitions and later proofs are based upon those presented by Mikhlin (9,10) and further mention of these references will be omitted.

2.1 Basic Concepts and Definitions

In attempting to predict the behaviour of a physical system by means of mathematical analysis it is necessary to idealize the system in a manner that renders the analysis tractable. This idealization is known as the mathematical model of the system and in many cases it is a differential equation. The finite element method will be presented as a means of generating an approximate solution form to such an equation and thus to the physical system. In order that the finite element procedure may be utilized to its full potential a complete understanding of its mathematical basis is desirable. With this in mind a number of relevant mathematical concepts and definitions will be introduced.

The problem at hand is the determination of some function that satisfies a given differential equation, within some region, and certain conditions on the boundary of that region. The region will be a surface if the function sought depends upon two independent variables or a volume if the number of independent variables is three. If the function depends upon four or more variables then the domain in which the function is defined is a hyperspace. The concept of region, or domain, can be formalized by the following two properties:

(i) If some point P belongs to the domain, then all points sufficiently close to P belong to the domain;

(ii) Any two points in the domain can be joined by a line lying entirely within the domain.

The first property is equivalent to saying the domain is open, or consists of only interior points; and the second property specifies the domain be connected. The boundary of the domain is defined as that set of points in any neighbourhood of which there are both points belonging to and not belonging to the domain. Attention will be confined to those problems where the curve or surface forming the boundary is either smooth i.e. it has a continuously turning tangent or tangent plane, or is piecewise smooth,

i.e. it consists of a finite number of smooth pieces. The domain will be denoted by Ω and the boundary by S . Note that the domain does not include the boundary. The set of points that are obtained by combining the domain and its boundary is known as the closed domain and will be denoted by $\bar{\Omega}$. Only finite domains will be considered, i.e. domains that can be included in a sufficiently large sphere.

The solution of the given equation is accomplished by finding that function which when acted upon by a given operator yields a known function. Attention will be restricted to thoses functions that are square summable over the domain i.e. to those functions u_i such that

$$\int_{\Omega} u_j u_j \, d\Omega < K$$

where K is a finite constant and Lebesgue integration is employed, and the repeated lower case indices are summed. The functions considered will, in general, be vector valued and u_i thus represents a column vector with components $u_1, u_2, \ldots\ldots u_c$. Thus

$$u_i = \langle u_1, u_2, \ldots\ldots u_c \rangle^T$$

The class of square summable functions over Ω , which will be denoted by $L_2(\Omega)$, constitutes a vector space over

the field of real numbers. Thus if v_i and ω_i are members of $L_2(\Omega)$ and c is a real number, then

$$c(v_i + \omega_i) = cv_i + c\omega_i$$

and the vector with components $c(v_i + \omega_i)$ is also a member of $L_2(\Omega)$. The notation $u_i \in L_2(\Omega)$ will be employed to mean u_i is a member of the set $L_2(\Omega)$.

When discussing approximate solution of a given equation it becomes necessary to compare different approximate answers. It is thus necessary to be able to measure the "distance" between functions. The structure for this can be obtained by introducing the concept of a norm into $L_2(\Omega)$. This is accomplished by first introducing an inner product into the space. An inner product of two functions u_i, v_i is a function (u_i, v_i) defined such that

(i) $(u_i, v_i) = (v_i, u_i)$

(ii) $(u_i, bv_i + c\omega_i) = b(u_i, v_i) + c(u_i, \omega_i)$

(iii) $(u_i, u_i) \geq 0$ where the equality only holds if $u_i \equiv 0$

For functions in $L_2(\Omega)$ define

$$(u_i, v_i) = \int_{\Omega} u_i v_i \, d\Omega \tag{2.1}$$

The norm of a function is any function $\| u_i \|$ satisfying the axioms

(i) $\| u_i \| \geq 0$, where the equality only holds if $u_i \equiv 0$

(ii) $\| u_i + v_i \| \leq \| u_i \| + \| v_i \|$

(iii) $\| c\, u_i \| = |c| \, \| u_i \|$, where c is a constant

It can readily be seen that defining

$$\| u_i \| = \sqrt{(u_i, u_i)} \tag{2.2}$$

satisfies these axioms. The distance between two functions may then be characterized by the norm of their difference, i.e. $\| u_i - v_i \|$. A complete vector space, such as $L_2(\Omega)$, with an associated inner product is known as a Hilbert space. It is possible to define alternative inner products and corresponding norms to the ones presented, as will be discussed in subsequent chapters.

Consider now more specifically the type of equations that will be considered. Attention will be primarily confined to equations that can be expressed in the form

$$A_{ij}\, \omega_j = f_i \qquad \epsilon \ \Omega \tag{2.3}$$

where ω_i and f_i are members of $L_2(\Omega)$ and f_i is known. A_{ij} is a linear differential operator or matrix of such operators. The linearity of A_{ij} implies

$$\text{(i)} \quad A_{ij}(u_j + \omega_j) = A_{ij}u_j + A_{ij}\omega_j$$

$$\text{(ii)} \quad \langle A_{i1}, A_{i2}, \cdots A_{ic}\rangle \langle c_1 u_1, \cdots c_c u_c\rangle^T = c_1 A_{i1}u_1 + \cdots + c_c A_{ic}u_c \tag{2.4}$$

where the c are constants.

Together with reducing Eq. 2.3 to an identity within Ω, ω will also be required to satisfy certain boundary conditions. The class of functions that satisfy all the boundary conditions of the problem and possess the required continuity properties to make the evaluation of $A_{ij}u_j$ possible is known as the field of definition of A_{ij} and is denoted by D_A. For example, if A_{ij} were a differential operator of the fourth order then functions in D_A must be continuous fourth order derivatives at every point in Ω. In general, A_{ij} will be considered defined for a dense set of some Hilbert space H. A set M is said to be dense in H if every element in H can be obtained as the limit of a sequence of functions from M.

The following types of operators will play an important role in the subsequent discussions. An operator A_{ij} is

(i) symmetric, if $(A_{ij} u_j, v_i) = (A_{ij} v_j, u_i)$ for all $u_i, v_i \in D_A$

(ii) positive definite, if $(A_{ij} u_j, u_i) \geqslant 0$ for all $u_i \in D_A$ where the equality sign holds only if $u_i = 0$

(iii) positive bounded below, if $(A_{ij} u_j, u_i) \geqslant \gamma^2 (u_j, u_j)$ for all $u_i \in D_A$ where γ is a positive constant.

The class of operators that are known as completely continuous operators will also be of importance in the following presentation, and therefore a definition of such operators will be given. An operator T', defined in some Hilbert space H, is said to be degenerate if it can be represented in the form

$$T'_{ij} u_j = \sum_{k=1}^{N} (u_i, \psi_k) \phi_k$$

where N is finite and both $\psi_k, \phi_k \in H$. An operator T_{ij} is then completely continuous if for any $\epsilon > 0$ it can be represented in the form

$$T_{ij} u_j = T'_{ij} u_j + T''_{ij} u_j$$

where T'_{ij} is degenerate and the norm of T''_{ij} is less than ϵ $(\| T''_{ij} u_j \| \leqslant \epsilon \| u_i \|)$.

Together with Eq. 2.3 the problem of determining the eigenvalues and eigenvectors of the operator A_{ij} will also be considered. That is, the solution of the equation

$$A_{ij}\omega_j = \lambda\omega_i \tag{2.5}$$

will be discussed. In this equation λ is a numerical parameter and the solution of the equation entails the determination of those values of λ say λ_n for which there exist corresponding non trivial solutions for u_i say u_i^n. Such λ_n are called the eigenvalues and the corresponding u_i^n the eigenvectors of the operator A_{ij}.

In the following developments the use of subscripted variables will be abandoned, except where necessary for clarification, and all the functions employed assumed to be vector valued. Thus for example Eq. 2.3 will be written

$$A\,\omega = f$$

2.2 Variational Formulation of the Problem

The purpose of this section is to introduce concepts that are necessary for the later development of the Galerkin procedure and at the same time to discuss the range of applicability of the Rayleigh-Ritz procedure. In

order to do this the conditions under which the solution of a given differential equation coincides with the function that minimizes a known functional will be discussed. It will further be shown that it is possible to obtain a solution via the variational formulation even when no solution of the original equation exists in D_A .

Consider the equation

$$A\,\omega = f \qquad \in \Omega \tag{2.6}$$

where ω and f are members of some Hilbert space H , and ω is prescribed to satisfy certain homogeneous boundary conditions.

Theorem 2.1. Let A be a symmetric positive bounded below operator defined for some dense linear set D_A of H . If Eq. 2.6 has a solution (in D_A) then this solution minimizes the functional

$$F(u) = (Au,u) - 2(u,f) \qquad u \in D_A \tag{2.7}$$

Conversely, if there exists in D_A a function which minimizes $F(u)$ then this function satisfies Eq. 2.6.

proof:-

Assume that $u, \eta \in D_A$ and set $u - \omega = \eta$, where ω is the solution of Eq. 2.6. Thus $u = \omega + \eta$

$$\therefore \quad F(u) = (A(\omega+\eta), \omega+\eta) - 2(\omega+\eta, f)$$

$$= F(\omega) + 2(A\omega - f, \eta) + (A\eta, \eta)$$

But $A\omega - f = 0$ by hypothesis, hence

$$F(u) = F(\omega) + (A\eta, \eta)$$

Now A is positive bounded below whence

$$(A\eta, \eta) \geq \gamma^2(\eta, \eta) \geq 0$$

Thus $F(u) \geq F(\omega)$ with the equality only valid if $\eta \equiv 0$. Hence the function attains its minimum value when $u \equiv \omega$.

Conversely, assume the functional attains its minimum value when $u \equiv \omega$. Let $\eta \in D_A$ and thus $\omega + \lambda\eta \in D_A$ where λ is a constant. Then by hypothesis

$$F(\omega + \lambda\eta) \geq F(\omega)$$

which reduces to

$$\lambda^2(A\eta, \eta) + 2\lambda(A\omega - f, \eta) \geq 0$$

The left-hand side is a non negative quadratic function for the real parameter λ. Thus

$$4(A\omega - f, \eta)^2 \leq 0$$

$$\therefore \quad (A\omega - f, \eta) = 0$$

But η is an arbitrary function from the dense set D_A, and the only function orthogonal to all such functions is the zero function. Therefore

$$A\,\omega = f$$

That ω is the only function that satisfies Eq. 2.6 can be seen by assuming that ω_0 is also a solution. This implies $A(\omega - \omega_0) = 0$ whence $(A(\omega-\omega_0), \omega-\omega_0) = 0$ The positive bounded below nature of A then requires $\omega - \omega_0 \equiv 0$ Thus $\omega_0 \equiv \omega$.

It can occur that for some functions $f \in H$ there does not exist a function ω in the field of definition of A that will satisfy Eq. 2.6. As an example of this situation consider the problem of predicting the deflection of a uniform cantilevered beam under the action of distributed load $q(x)$. The equations characterizing the deflection of the beam are

$$EI\,y'''' = q(x) \qquad x \in [0, L] \tag{2.8a}$$

$$y(0) = y'(0) = y''(L) = y'''(L) = 0 \tag{2.8b}$$

The differential equation is derived by considering the equilibrium of an infinitesimal length of the beam under the assumption that the loading is continuous across such a section.

In this case the operator is

$$A = EI \frac{d^4}{dx^4} \tag{2.9}$$

and its field of definition D_A is the totality of those functions defined over $[0, L]$ with continuous fourth derivatives that satisfy the boundary conditions of the problem. Thus if $q(x)$ is continuous then there exists a solution in D_A but if $q(x)$ is discontinuous no solution can be found in D_A. This difficulty can be overcome by considering limits of functions that lie in D_A, and it is then possible to formulate the functional $F(u)$ in such a manner that a generalized solution of Eq. 2.6 is obtained. Just as a discontinuous load may be considered as the limit of a sequence of continuous loads, so functions with discontinuous fourth derivatives are introduced that are the limits of sequences of functions with continuous fourth derivatives. It can then be asserted that amongst the new set of functions lies the solution (or generalized solution if it is not in D_A) of Eq. 2.6 for any $f \in H$. For the example considered H is taken as the set of functions square summable over $[0, L]$. The formal development of these ideas follows.

A new inner (or scalar) product, called the energy product, is introduced for the set D_A. Recalling

that the operator A is symmetric it is possible, using integration by parts, to write

$$(Au, v) = \int_{\Omega} (Bu)(Bv)\, d\Omega , \qquad u, v \in D_A$$

in which B is a differential operator. The energy product, which will be denoted by square brackets, is then defined as

$$[u, v]_A = \int_{\Omega} (Bu)(Bv)\, d\Omega , \qquad u, v \in D_A \tag{2.10}$$

The energy norm, which is denoted by bold vertical lines, is defined as

$$\| u \|_A = \sqrt{[u, u]_A} \tag{2.11}$$

The energy product and energy norm satisfy the axioms defining an inner product and a norm presented in the previous section.

It may be that the space D_A is incomplete with respect to the energy norm i.e. not all Cauchy sequences in D_A converge to a function in D_A. If this is so then D_A is completed by defining u to be a member of the space if

$$\| u_n - u \|_A \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty \tag{2.12}$$

where u_n is a typical member of a sequence $\{u_n\}$ each member of which is in D_A. The completed space so obtained is a Hilbert space and is denoted by H_A to emphasize its dependence upon A. The energy product in Eq. 2.10 is only defined for functions in D_A but may in an obvious fashion be defined for all functions in H_A:

$$[u, v]_A = \lim_{n \to \infty} \int_{\Omega} (Bu_n)(Bv_n)\, d\Omega , \quad u_n, v_n \in D_A \tag{2.13}$$

Thus the energy product and energy norm have meaning for any function in H_A. Their definition ensures that they satisfy all the required properties of inner products and norms.

The field of definition of the functional $F(u)$ of Eq. 2.7 can now be extended from D_A to H_A and Theorem 2.1 becomes:

Theorem 2.1A. If A is a symmetric positive bounded below operator, then of all of the functions in H_A the one that minimizes the functional

$$F(u) = [u, u]_A - 2(u, f) , \quad u \in H_A \tag{2.14}$$

is the solution of Eq. 2.6.

proof:

Theorem 2.1 demonstrates that if Eq. 2.6 has a solution ω in D_A this solution uniquely minimizes $F(u)$ in the class of functions constituting the field of definition of A . It will be shown that the minimum of $F(u)$ in the wider class H_A is not altered and that the function ω only gives the minimum value.

Denote by d the minimum value of $F(u)$ in D_A and by $\bar{d}$ the minimum in H_A . Then as H_A includes D_A $\bar{d} \leq d$.

Assume $\bar{d} < d$. Then there exists a function $\bar{u} \in H_A$ such that $F(\bar{u}) < d$, i.e.

$$[\bar{u}, \bar{u}] - 2(\bar{u}, f) = |\bar{u}|^2 - 2(\bar{u}, f) < d$$

But as $\bar{u} \in H_A$ it follows that there exists a sequence of functions $\{u_n\} \in D_A$ such that $|u_n - \bar{u}| \to 0$, which also implies $\|u_n - \bar{u}\| \to 0$ as A is positive bounded below. Thus $|u_n| \to |\bar{u}|$ and $(u_n, f) \to (\bar{u}, f)$ Therefore for sufficiently large n $F(\bar{u})$ and $F(u_n)$ differ by an arbitrary small amount and it follows that $F(u_n) < d$ This however is impossible as $u_n \in D_A$. The contradiction shows $\bar{d} = d$.

To show that the minimum of the functional is unique assume that $\bar{\omega} \in H_A$ also minimizes the functional.

From the proof of Theorem 2.1 $(A\omega - f, \eta) = 0$ for any function $\eta \in D_A$. This relation may be written

$$[\omega, \eta] = (f, \eta) \tag{2.15}$$

In particular setting $\eta = \omega$ gives

$$[\omega, \omega] = (f, \omega) \tag{2.16}$$

Eq. 2.15 is also valid for any function in H_A . In fact if $\eta \in H_A$ then there exists a sequence of functions $\{\eta_n\} \in D_A$ such that $|\eta - \eta_n| \to 0$, $\|\eta - \eta_n\| \to 0$ and $[\omega, \eta_n] = (f, \eta_n)$. Preceeding to the limit gives Eq. 2.15 which is the valid for arbitrary functions in H_A . Thus

$$[\omega, \bar{\omega}] = (f, \bar{\omega}) \tag{2.17}$$

By repeating the proof of Theorem 2.1 with $F(u)$ expressed as in Eq. 2.14 the identity $[\bar{\omega}, \eta] = (f, \eta)$ is obtained where η is an arbitrary function in H_A . Putting $\eta = \bar{\omega}, \omega$ gives

$$[\bar{\omega}, \bar{\omega}] = (f, \bar{\omega}) \tag{2.18}$$

$$[\bar{\omega}, \omega] = (f, \omega) \tag{2.19}$$

Subtracting Eq. 2.18 from Eq. 2.17 and Eq. 2.19 from Eq. 2.16 gives

$$[\omega - \bar{\omega}\,,\ \bar{\omega}] = 0 \quad ; \quad [\omega - \bar{\omega},\ \omega] = 0$$

Finally subtracting these two equations gives $[\omega-\bar{\omega},\ \omega-\bar{\omega}] = 0$ whence $\bar{\omega} = \omega$.

If the minimum of the functional expressed in Eq. 2.14 is given by a function that is not in D_A , then this function is known as a generalized solution of Eq. 2.6.

As an illustration of these concepts consider again the problem of the bending of a beam defined by Eqs. 2.8.

$$(Au, v) = \int_0^L v\, EI\, u^{IIII}\, dx$$

$$= \left(v EI u''' - v' EI u''\right)_0^L + \int_0^L E\bar{I}\, u'' v''\, dx$$

Thus

$$[u, v] = \int_0^L u'' EI\, v''\, dx \tag{2.20}$$

The operator is thus symmetric and is in fact also positive bounded below. Thus the functional $F(u)$ is

$$F(u) = \int_0^L u'' EI\, u''\, dx - 2\int_0^L u\, q\, dx \tag{2.21}$$

which is twice the potential energy of the system. In this case functions, $\tilde{u}$, in H_A are defined such that

$$\sqrt{\int_0^L (\bar{u}'' - u_n'')^2 \, dx} \to 0 \quad \text{as } n \to \infty$$

in which the $u_n \in D_A$ and therefore have continuous fourth derivatives and satisfy all the boundary conditions of the problem. Such a definition means that the functions in H_A have generalized second derivatives which in this case implies that they have continuous first derivatives. These functions must therefore satisfy the same boundary conditions involving the first derivative of the function or the function itself as the functions u_n . The definition does not imply that the functions must satisfy those boundary conditions involving second or third derivatives.

Theorem 2.1A states that the function that minimizes the functional given in Eq. 2.14 is the solution of Eq. 2.6 and that this function can be found uniquely amonst the elements of H_A . In this context it is convenient to introduce the concept of a complete set of functions in H_A .

A set of functions $\{\phi_k\}$ $k = 1, \ldots$ is said to be complete in H_A (with respect to the energy norm) if for every $v \in H_A$ and $\varepsilon > 0$ there is an integer M and constants $a_1, \ldots a_M$ such that

$$\| v - \sum_{k=1}^{M} a_k \phi_k \| < \varepsilon \tag{2.22}$$

In other words, any function in H_A can be approximated arbitrarily closely, in energy norm, by a linear combination of members of a complete set in H_A .

Consider the general case where the symmetric operator A is of order $2m$. Expressing the energy product in its symmetric form would then involve derivatives of maximum order m and functions in H_A would be the limit in energy of functions with $2m$ derivatives that satisfy all the boundary conditions. Such functions possess generalized m th order derivatives and must satisfy all those boundary conditions that do not involve derivatives of the m th order and higher. The boundary conditions that involve derivatives of order greater than or equal to m are known as natural for A . The remaining boundary conditions are called the forced or principal boundary conditions. In an equivalent way those derivatives of order less than m are known as principal derivatives. Thus functions in H_A necessarily satisfy the forced boundary conditions but need not satisfy the natural. This is an important consideration when choosing trial solutions for the approximate solution techniques that will be discussed in the next chapter.

The eigenvalue problem that is represented by the equation

$$A w = \lambda w \tag{2.23}$$

can also be expressed in a variational manner if A is symmetric and positive bounded below. The developments are presented by Mikhlin (9) and will not be repeated here as those concepts necessary for the further development of this thesis have already been introduced in the preceeding discussion.

CHAPTER 3

APPROXIMATE SOLUTION TECHNIQUES

The purpose of this chapter is to present methods that can be used to obtain an approximate solution for Eq. 2.6. Again it is convenient to assume that A is a differential operator of order $2m$ and that the boundary conditions are homogeneous. The developments presented herein follow those given by Mikhlin (9).

Many approximate methods are based upon the concept of assuming a solution in the form

$$\bar{\omega} = \sum_{k=1}^{M} a_k \phi_k \tag{3.1}$$

in which the a_k are unknown parameters and the ϕ_k are known co-ordinate functions. This form is valid if ω is a single function. However, as indicated in Chapter 2 ω may be considered as a vector quantity with more than one component. In this case an approximation of the form of Eq. 3.1 must be assumed for each component of ω . Thus, in general, if ω is a vector quantity it will be denoted by ω_i where

$$\bar{\omega}_i = \langle \bar{\omega}_1, \bar{\omega}_2, \ldots \bar{\omega}_c \rangle^T$$

and where each component is then approximated by

$$\bar{\omega}_T = \sum_{k=1}^{L^T} a_{Tk} \phi_k^T \qquad T = 1, \ldots c.$$

Once again in the interests of algebraic simplicity the following developments will be in terms of Eq. 3.1.

Treatment of such approximate methods can be found in the works of Crandall (2) and Finylason and Scriven (4). Attention in this thesis will be confined to the discussion of two related methods: Rayleigh-Ritz and Galerkin.

3.1 Rayleigh-Ritz Method

The Rayleigh-Ritz method is applicable only if the equation to be solved has a solution that corresponds to the stationary value of some known functional. The method then calculates the unknown a_k in such a manner that the approximate solution $\bar{\omega}$ renders the given functional stationary in the M dimensional subspace spanned by the co-ordinate functions $\phi_k \quad k = 1, \ldots M$.

Theorem 2.1A states that if A is symmetric and positive bounded below then the solution of Eq. 2.6 is that function in H_A which minimizes the functional

$$F(u) = [u, u]_A - 2(u, f)$$

The Rayleigh-Ritz procedure is to substitute the approximate solution $\bar{\omega}$ into the functional and then minimize $F(\bar{\omega})$ with respect to the a_k . Hence

$$F(\bar{\omega}) = \left[\sum_{k=1}^{M} a_k \phi_k, \sum_{j=1}^{M} a_j \phi_j\right] - 2\left(\sum_{k=1}^{M} a_k \phi_k, f\right)$$

Now, if the ϕ_k are linearly independent $F(\bar{\omega})$ is stationary when

$$\frac{\partial F(\bar{\omega})}{\partial a_k} = 0 \qquad k = 1, \ldots M$$

Thus

$$\left[\phi_k, \sum_{j=1}^{M} a_j \phi_j\right] + \left[\sum_{j=1}^{M} a_j \phi_j, \phi_k\right] - 2(\phi_k, f) = 0$$

Using the symmetry of the energy product gives

$$\sum_{k=1}^{M} a_k \left[\phi_k, \phi_j\right]_A = (\phi_j, f) \qquad j = 1, \ldots M \qquad (3.2)$$

which is a system of linear equations for the a_k which has a unique solution as the ϕ_k have been assumed to be linearly independent.

If the set of co-ordinate functions $\{\phi_k\}$, where the ϕ_k are linearly independent, is complete in H_A the approximate solution obtained by the Rayleigh-Ritz method can be made arbitrarily close in energy norm to the exact solution by increasing M sufficiently.

proof:

The minimum value of $F(u)$ is obtained when $u \equiv \omega$, the solution of $A\omega = f$.

$$F(\omega) = [\omega, \omega] - 2(\omega, A\omega) = -|\omega|^2$$

Let $d = -|\omega|^2$ which is the exact lower bound of the functional $F(u)$. Hence if ε is an arbitrary small positive number, then there exists in H_A a function v such that

$$d \leq F(v) \leq d + \varepsilon/2$$

Further as the set $\{\phi_k\}$ is complete in energy it is possible to show

$$F(\bar{v}) - F(v) < \varepsilon/2$$

where

$$\bar{v} = \sum_{k=1}^{M} b_k \phi_k$$

To see this note

$$F(u) = [u, u] - 2(u, f) = [u-\omega, u-\omega] - [\omega, \omega]$$

$$= |u-\omega|^2 - |\omega|^2$$

Whence

$$F(\bar{v}) - F(v) = |\bar{v}-\omega|^2 - |v-\omega|^2$$

$$= (|\bar{v}-\omega| + |v-\omega|)(|\bar{v}-\bar{\omega}| - |v-\omega|)$$

The triangle inequality gives

$$|\bar{v}-\omega| - |v-\omega| \leq |\bar{v}-v|$$

Thus

$$F(\bar{v}) - F(v) \leq (|\bar{v}-\omega| + |v-\omega|)\,|\bar{v}-v|$$

By the completeness of the set $\{\phi_k\}$ M can be chosen such that

$$|\bar{v} - v| < \varepsilon/c$$

where c is to be chosen

$$\therefore |\bar{v}| < |v| + \varepsilon/c$$

and

$$F(\bar{v}) - F(v) < (2|v| + 2|\omega| + \varepsilon/c)\,\varepsilon/c$$

Choose c such that

$$(2|v| + 2|\omega| + \varepsilon/c)\,1/c < 1/2$$

Thus

$$F(\bar{v}) - F(v) < \varepsilon/2$$

$$\therefore d \leq F(\bar{v}) \leq F(v) + \varepsilon/2 < d + \varepsilon$$

Let $\bar{\omega}$ be a function constructed by the Rayleigh-Ritz method. Then

$$d \leq F(\bar{\omega}) \leq F(\bar{v}) \quad \text{OR} \quad d \leq F|\bar{\omega}) \leq d+\varepsilon$$

Letting $\varepsilon \to 0$ implies $F(\bar{\omega}) \to d = -|\omega|^2$

Thus

$$F(\bar{\omega}) = |\bar{\omega} - \omega|^2 - |\omega|^2 \to -|\omega|^2$$

Therefore

$$| \bar{\omega} - \omega |_A \rightarrow 0 \qquad \text{as } M \rightarrow \infty \tag{3.3}$$

Thus the Rayleigh-Ritz approximation converges, in the sense of the energy norm, to the exact solution if the co-ordinate functions are complete in H_A .

In practice it is not strictly necessary for the co-ordinate functions to be complete in H_A . It is sufficient that they be complete with respect to any subset of H_A that contains the exact solution. In this respect recall that consideration in this thesis has been restricted to those functions that are square summable. Specifically, the right hand side of Eq. 2.6 must be such a function. Thus

$$\int_{\Omega} (A\omega)^2 \, d\Omega = \int_{\Omega} f^2 \, d\Omega < \infty \tag{3.4}$$

If A has order $2m$ and if f is a bounded function then Eq. 3.4 implies that the $2m-1$ derivatives of ω are continuous. Therefore it may be concluded that the space of functions that have continuous $2m-1$ derivatives and further, satisfy all the boundary conditions, contains the exact solution. Thus it is sufficient that the co-ordinate functions be complete with respect to this space which will be known as H_A^o . Convergence, however, is still only ensured in the sense of the energy norm.

The Rayleigh-Ritz procedure may also be employed for the determination of the eigenvalues of Eq. 2.23. The necessary development is presented in Mikhlin (9) but will not be repeated here. It is worthy of note that the approximate eigenvalues so obtained are bounded below by their respective exact values.

3.2 Galerkin's Method

Galerkin's method specifies that the residual obtained by substituting the approximate solution $\bar{\omega}$ into Eq. 2.6 is made orthogonal, throughout the domain, to each of the co-ordinate functions ϕ_k . This procedure is applicable to any operator that is positive definite. Thus it is required that

$$\int_{\Omega} (A\bar{\omega} - f)\,\phi_j \, d\Omega = 0 \qquad j = 1, \ldots M \tag{3.5a}$$

which may be written

$$\int_{\Omega} \left(A\left(\sum_{k=1}^{M} a_k \phi_k\right) - f \right) \phi_j \, d\Omega = 0 \qquad j = 1, \ldots M \tag{3.5b}$$

This system of linear equations has a unique solution for the a_k if the ϕ_k are linearly independent throughout Ω . Note, however, that the evaluation of this equation in the form given is only possible if the co-ordinate functions have

continuous $2m-1$ derivatives. In fact convergence is ensured, as will be proved in this section, if the co-ordinate functions are complete in H_A^o , i.e. they have continuous $2m-1$ derivatives and satisfy all the boundary conditions. However, as will also be demonstrated, it may be possible to express Eq. 3.5b in a form such that the co-ordinate functions need only be complete in a space equivalent to H_A to ensure convergence.

The general characteristics of a class of problems for which the Galerkin procedure is known to converge will first be discussed. Theorems will then be presented that form the basis of the subsequent convergence investigation. In this discussion the conditions that the co-ordinate functions must satisfy to ensure convergence, the type of convergence obtained, and the relationship between the Rayleigh-Ritz and Galerkin procedures are specifically dealt with.

A class of equations for which the Galerkin procedure is known to converge is characterized by the equation

$$\omega - \lambda T\omega = f \qquad \epsilon \Omega \tag{3.6}$$

where ω is the required element and f is the given element of some Hilbert space H . T is some completely

continuous operator in H , and λ is a numerical parameter. The general properties of such equations will first be established.

Assume that λ can assume any fixed value with modulus not exceeding some constant R , so that $|\lambda| \leq R$ Then, as T is completely continuous it may be expressed in the form

$$T = T' + T'' \tag{3.7}$$

where T' is degenerate and

$$\| T'' \| \leq 1/2R \tag{3.8}$$

It can then be shown (9, p. 463) that the linear operator $(E - \lambda T'')^{-1}$, where E is the identity operator, exists and is bounded. Eq. 3.6 can be written in the form

$$\omega - \lambda T''\omega - \lambda T'\omega = f \tag{3.9}$$

which when multiplied by $(E-\lambda T'')^{-1}$ gives

$$\omega - \lambda (E - \lambda T'')^{-1} T'\omega = f(E - \lambda T'')^{-1} = F_\lambda \tag{3.10}$$

Further as T' is degenerate it is possible to write

$$T'\omega = \sum_{k=1}^{N} (\omega, \psi_k)\, \phi_k \tag{3.11}$$

where the set of elements ϕ_k and also the set ψ_k can be considered as linearly independent. Then

$$(E - \lambda T'')^{-1} T' \omega = \sum_{k=1}^{N} (\omega, \psi_k)\, u_{\lambda,k} \tag{3.12}$$

where

$$u_{\lambda,k} = (E - \lambda T'')^{-1} \phi_k \tag{3.13}$$

are linearly independent. Eq. 3.10 may now be written in the form

$$\omega - \lambda \sum_{k=1}^{N} C_k u_{\lambda,k} = F_\lambda \tag{3.14}$$

where $C_k = (\omega, \psi_k)$ (3.15)

Forming the inner product of each term in Eq. 3.14 with ψ_m gives

$$(\omega, \psi_m) - \lambda \sum_{k=1}^{N} C_k (u_{\lambda,k}, \psi_m) = (F_\lambda, \psi_m)$$

$$C_m - \lambda \sum_{k=1}^{N} a_{mk}(\lambda)\, C_k = b_m(\lambda), \quad m = 1, \ldots N \tag{3.16}$$

where

$$a_{mk} = (u_{\lambda,k}, \psi_m) \quad ; \quad b_m(\lambda) = (F_\lambda, \psi_m)$$

Thus the solution of Eq. 3.6 can be obtained from Eq. 3.14 if Eq. 3.16 can be solved for the unknown C_k .

The conditions under which Eq. 3.16 will have a solution can be examined by investigation of the matrix of coefficients of the C_k . Writing the equation in matrix form gives

$$\begin{bmatrix} 1-\lambda a_{11}, & -\lambda a_{12}, \ldots\ldots\ldots, & -\lambda a_{1N} \\ -\lambda a_{21}, & 1-\lambda a_{22}, \ldots & -\lambda a_{2N} \\ \vdots & & \vdots \\ -\lambda a_{N1}, & -\lambda a_{N2}, \ldots\ldots\ldots, & 1-\lambda a_{NN} \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{Bmatrix} = \begin{Bmatrix} b_1(\lambda) \\ b_2(\lambda) \\ \vdots \\ b_N(\lambda) \end{Bmatrix} \tag{3.17}$$

Denote the determinant of the above matrix by $D_R(\lambda)$. The coefficients a_{mk} and consequently $D_R(\lambda)$ are continuous functions of λ in the circle $|\lambda| \leq R$ of the complex plane. This continuity, together with the fact that $D_R(0) = 1$ implies $D_R(\lambda) \neq 0$ and that the determinant has only a finite number of roots in the circle $|\lambda| \leq R$.

If $D_R(\lambda) = 0$ then the homogeneous system obtained from Eq. 3.16 by replacing the right hand side by zeros has a non trivial solution. Then it follows that the homogeneous equation

$$\omega - \lambda T \omega = 0 \tag{3.18}$$

has a non-trivial solution and the λ considered is a quantity which is the reciprocal of the eigenvalue of T . Such λ are also known as characteristic values.

If $D_R(\lambda) \neq 0$ then Eq. 3.16, and hence Eq. 3.6, has a unique solution. In this case therefore the operator $(E - \lambda T)^{-1}$ exists and will be denoted by Γ_λ . Those values of λ for which Γ_λ exists are known as regular values.

Thus the existence of solutions of equations of the form of Eq. 3.6 containing a completely continuous operator is proven. The following alternative (called Fredholm's alternative) holds: either the non-homogeneous equation is soluble and uniquely so for any independent term f and then the corresponding homogeneous equation has only the trivial solution, or the non-homogeneous equation is not soluble for some value of f and then the corresponding homogeneous equation has a non-trivial solution . The first part of the alternative holds if λ is a regular value and the second if λ is a characteristic value.

The following theorems which form the basis for the discussion of the convergence of the Galerkin procedure can now be proven.

Theroem 3.1. Let $\{T_n\}$ be a set of completely continuous operators in some Hilbert space H which tend to some completely continuous operator T in the sense that

$$\| T - T_n \| \to 0 \quad \text{as} \quad n \to \infty \tag{3.19}$$

Further let $\{f_n\}$ be a set of elements of the same space which tend to some element f. If λ is a regular value of the equation

$$\omega - \lambda T \omega = f \tag{3.20}$$

then for sufficiently large n, λ will also be a regular value for the equation

$$\omega_n - \lambda T_n \omega_n = f_n \tag{3.21}$$

and the solution of Eq. 3.21 will tend to the solution of Eq. 3.20 as $n \to \infty$.

proof:

Consider the equation

$$v - \lambda T_n v = g \tag{3.22}$$

where g is an arbitrary element from H. This equation

may be written

$$v - \lambda T v - \lambda (T_n - T) v = g \tag{3.23}$$

By hypothesis $\Gamma_\lambda = (E - \lambda T)^{-1}$ exists and applying it to both sides of Eq. 3.23 gives

$$v - \lambda \Gamma_\lambda (T_n - T) v = \Gamma_\lambda g \tag{3.24}$$

Now

$$\| \lambda \Gamma_\lambda (T_n - T) \| \leq |\lambda| \, \| \Gamma_\lambda \| \, \| T_n - T \| \tag{3.25}$$

which from the conditions of the theorem can be made as small as required for sufficiently large n . Choose n so large that

$$\| \lambda \Gamma_\lambda (T_n - T) \| < 1/2$$

It then follows (9, p. 467) that the operator $\left(E - \lambda \Gamma_\lambda (T_n - T)\right)^{-1}$ exists, is defined for the whole space and its norm does not exceed 2.

From Eq. 3.24 the solution of Eq. 3.22 is

$$v = (E - \lambda \Gamma_\lambda (T_n - T)^{-1} \Gamma_\lambda g$$

Thus the operator

$$\Gamma_{n\lambda} = (E - \lambda T_n)^{-1} = (E - \lambda \Gamma_\lambda (T_n - T))^{-1} \Gamma_\lambda \qquad (3.26)$$

exists for the given value of n . Therefore the given value of λ is regular for Eq. 3.22.

To establish the second part of the theorem it will first be shown that $\| \Gamma_{n\lambda} - \Gamma_\lambda \| \to 0$ as $n \to \infty$

Define $B_n = \lambda \Gamma_\lambda (T_n - T)$

Note that

$$\sum_{k=0}^{\infty} \lambda^k (T_n)^k (E - \lambda T_n) \omega = (E - \lambda T_n) \sum_{k=0}^{\infty} \lambda^k (T_n)^k \omega = \omega$$

whence

$$(E - \lambda T_n)^{-1} \omega = \sum_{k=0}^{\infty} \lambda^k T_n^k \omega$$

Thus making use of Eq. 3.26

$$\Gamma_{n\lambda} \omega = \sum_{k=0}^{\infty} B_n^k \Gamma_\lambda \omega$$

$$\therefore \quad \Gamma_{n\lambda} \omega - \Gamma_\lambda \omega = \sum_{k=1}^{\infty} B_n \Gamma_\lambda \omega$$

and making use of Eq. 3.25

$$\| \Gamma_{n\lambda} - \Gamma_\lambda \| \leqslant \sum \| B_n \| \| \Gamma_\lambda \| = \frac{\| B_n \| \| \Gamma_\lambda \|}{1 - \| B_n \|} \to 0 \text{ as } n \to \infty \qquad (3.27)$$

Thus

$$\omega_n - \omega = \Gamma_{n\lambda} f_n - \Gamma_\lambda f = (\Gamma_{n\lambda} - \Gamma_\lambda) f_n + \Gamma_\lambda (f_n - f)$$

hence

$$\| \omega_n - \omega \| \leq \| \Gamma_{n\lambda} - \Gamma_\lambda \| \, \| f_n \| + \| \Gamma_\lambda \| \, \| f_n - f \|$$

By virtue of Eq. 3.27 the first term on the right hand side tends to zero and by postulation so does the second term. Thus

$$\| \omega_n - \omega \| \to 0 \qquad \text{as} \quad n \to \infty \tag{3.28}$$

and the theorem has been proved.

Theorem 3.2. If $\| T_n - T \| \to 0$ where T and T_n are completely continuous operators then the eigenvalues of the equation

$$\omega - \lambda T \omega = 0 \tag{3.29}$$

are obtained by the limit process as $n \to \infty$ from the eigenvalues of the equation

$$\omega_n - \lambda T_n \omega_n = 0 \tag{3.30}$$

proof:

The proof utilizes the fact that as $T_n \to T$ the coefficient matrix of Eq. 3.17 corresponding to T_n whose determinant is denoted by $D_R^n(\lambda)$ converges to the matrix shown in Eq. 3.17 whose determinant is denoted by $D_R(\lambda)$.

Let λ_o be any root of $D_R(\lambda)$ which lies within the circle $|\lambda| \leq R$ and let p be its multiplicity. Surround λ_o by a circle of radius ε such that there is no root of $D_R(\lambda)$ besides λ_o within or on a circle with this radius. In particular $D_R(\lambda)$ is non-zero on the circle $|\lambda - \lambda_o| = \varepsilon$. Define

$$q = \min_{|\lambda-\lambda_o|=\varepsilon} |D_R(\lambda)| \quad , \quad q > 0$$

Now select N such that for $n > N$

$$|D_R(\lambda) - D_R^n(\lambda)| \quad q \quad , \quad |\lambda-\lambda_o| = \varepsilon$$

By Rouche's theorem (16, p. 89) $D_R^n(\lambda)$ has p equal roots in the circle $|\lambda - \lambda_o| < \varepsilon$. Denote them by $\lambda_{n1}, \lambda_{n2} \cdots \lambda_{np}$ whence

$$|\lambda_{nj} - \lambda_o| < \varepsilon \quad , \quad j = 1, \cdots p \quad , \quad n > N$$

Since ε can be chosen as small as required, this is equivalent to the statement

$$\lim_{n\to\infty} \lambda_{nj} = \lambda_0 \qquad j = 1, \cdots P \tag{3.31}$$

Theorems 3.1 and 3.2 enable the question of the convergence of the Galerkin procedure to be considered when applied to equations of the form of Eqs. 3.6 and 3.18. Consider first the Galerkin equations that are applicable to Eq. 3.6.

$$\omega - \lambda T \omega = f$$

where T is completely continuous in some Hilbert space H. Assuming an approximate solution in the usual manner the Galerkin equations become

$$\int_{\Omega} (\bar{\omega} - \lambda T \bar{\omega}) \phi_j \, d\Omega = \int_{\Omega} f \phi_j \, d\Omega \quad , \quad j = 1, \ldots M \tag{3.32}$$

where

$$\bar{\omega} = \sum_{k=1}^{M} a_k \phi_k$$

Eq. 3.32 may be rewritten

$$\sum_{k=1}^{M} a_k \left((\phi_k, \phi_j) - \lambda (T\phi_k, \phi_j) \right) = (f, \phi_j) \qquad j = 1, \ldots M$$

Also without any loss of generality the co-ordinate functions ϕ_k, which are linearly independent, may be considered to be orthonormalized in the space H. That is

$$(\phi_k, \phi_j) = \delta_{kj} = 1 \quad k=j$$
$$= 0 \quad k \neq j$$

whence

$$a_j - \lambda \sum_{k=1}^{M} a_k (T\phi_k, \phi_j) = (f, \phi_j) \quad j = 1, \ldots M \qquad (3.33)$$

The convergence of the approximate solution so obtained is governed by the following theorem.

Theorem 3.3. The approximate solution of Eq. 3.6 constructed by the Galerkin procedure converges to the exact solution (in the norm of H) if

a) Eq. 3.6 has only one solution in H

b) the operator T is completely continuous in H

c) the co-ordinate functions form a complete set in H

proof:

As the set of co-ordinate functions ϕ is complete in H it is possible to write

$$T\omega = \sum_{k=1}^{\infty} (T\omega, \phi_k)\, \phi_k \quad ; \quad f = \sum_{k=1}^{\infty} (f, \phi_k)\, \phi_k$$

Define

$$T_n \omega = \sum_{k=1}^{n} (T\omega, \phi_k)\, \phi_k \quad ; \quad f_n = \sum_{k=1}^{n} (f, \phi_k)\, \phi_k \qquad (3.34)$$

Then $\| f_n - f \| \to 0$ as $n \to \infty$. Also $\| T_n - T \| \to 0$ as will now be proven.

As T is completely continuous in H it can be expanded into a sum $T = T' + T''$, where T' is degenerate and $\| T'' \| < \varepsilon/2$, where ε is an arbitrarily small positive number. Then

$$(T - T_n)\omega = T\omega - T_n\omega = \sum_{k=1}^{\infty} (T\omega, \phi_k)\, \phi_k$$

$$\therefore (T - T_n)\omega = \sum_{k=n+1}^{\infty} (T'\omega, \phi_k)\, \phi_k + \sum_{k=n+1}^{\infty} (T''\omega, \phi_k)\, \phi_k \tag{3.35}$$

Recall

$$\| \omega \|^2 = (\omega, \omega)$$

whence

$$\| \sum_{k=n+1}^{\infty} (T''\omega, \phi_k)\, \phi_k \|^2 = \sum_{k=n+1}^{\infty} |(T''\omega, \phi_k)|^2 \tag{3.36}$$

as the ϕ_k are assumed normalized in H . Now by Bessel's inequality the right hand side of Eq. 3.36 does not exceed

$$\| T''\omega \|^2 \leq \| T'' \|^2 \| \omega \|^2 < \varepsilon/2 \, \| \omega \|$$

Hence

$$\| \sum_{k=n+1}^{\infty} (T''\omega, \phi_k)\, \phi_k \| < \varepsilon/2 \, \| \omega \| \tag{3.37}$$

Consider now the first term in Eq. 3.35. The degenerate operator $T'\omega$ may be written in the form

$$T'\omega = \sum_{j=1}^{s} (\omega, \psi_j)\, u_j$$

where s is a finite number and ψ_j, u_j are members of H. Thus

$$\| \sum_{k=n+1}^{\infty} (T'\omega, \phi_k)\, \phi_k \| = \| \sum_{k=n+1}^{\infty} \sum_{j=1}^{s} (\omega, \psi_j)(u_j, \phi_k)\, \phi_k \|$$

$$= \| \sum_{j=1}^{s} (\omega, \psi_j) \sum_{k=n+1}^{\infty} (u_j, \phi_k)\, \phi_k \| \leq \sum_{j=1}^{s} |(\omega, \psi_j)| \, \| \sum_{k=n+1}^{\infty} (u_j, \phi_k)\, \phi_k \|$$

$$\leq \| \omega \| \sum_{j=1}^{s} \| \psi_j \| \sqrt{\left(\sum_{k=n+1}^{\infty} |(u_j, \phi_k)|^2 \right)} \tag{3.38}$$

The series $\sum_{k=1}^{\infty} |(u_j, \phi_k)|^2$ converges. Hence the coefficient of $\| \omega \|$ will be less than $\varepsilon/2$ for sufficiently large n, say $n > N$. Thus from Eqs. 3.35, 3.37 and 3.38

$$\| (T - T_n)\omega \| < \varepsilon \, \| \omega \| \qquad n > N$$

whence

$$\| T - T_n \| \longrightarrow 0 \qquad \text{as} \quad n \longrightarrow \infty$$

which was to be proved.

The operator T_n is degenerate and hence completely continuous. Thus from Theorem 3.1 the equation

$$\omega_n - \lambda T_n \omega_n = f_n \tag{3.39}$$

has a unique solution for sufficiently large n and

$$\| \omega_n - \omega \| \longrightarrow 0 \quad \text{as } n \to \infty$$

Substituting for T_n and f_n in Eq. 3.39 gives

$$\omega_n = \sum_{k=1}^{n} \left((f, \phi_k) \phi_k + \lambda (T\omega_n, \phi_k) \phi_k \right) = \sum_{k=1}^{n} A_k \phi_k \tag{3.40}$$

where $$A_k = (f, \phi_k) + \lambda (T\omega_n, \phi_k) \tag{3.41}$$

Substituting the value of ω_n from Eq. 3.40 into Eq. 3.41 gives

$$A_j - \lambda \sum_{k=1}^{M} A (T\phi_k, \phi_j) = (f, \phi_j) \quad j = 1, \ldots M \tag{3.42}$$

From the preceeding statements the constants calculated from this equation ensure that the approximate solution as given by Eq. 3.40 converges to the correct solution. But Eq. 3.42 is identical to Eq. 3.33 obtained from the Galerkin procedure. Thus the Galerkin approximation converges to the exact solution in the norm of H.

Similarly by repeating the above arguments the application of the Galerkin method to the problem of finding the eigenvalues of the equation

$$\omega - \lambda T \omega = 0 \tag{3.43}$$

can be shown to be equivalent to finding the eigenvalues of the equation

$$\omega_n - \lambda T_n \omega_n = 0 \tag{3.44}$$

From which it follows by Theorem 3.2 that the eigenvalues of Eq. 3.43 are the limits of the corresponding eigenvalues of Eq. 3.44. Thus the following theorem may be stated.

Theorem 3.4. The application of the Galerkin method to the problem of seeking eigenvalues of equations of the form

$$\omega - \lambda T \omega = 0$$

leads to a convergent process if

a) T is completely continuous in H

b) the co-ordinate functions form a complete set in H.

Note that the above theorems have been developed with respect to a given Hilbert space H. In general, when dealing with differential equations, this space will coincide with the H_A space developed in the previous section.

The basic equations considered in this thesis have been assumed to be expressible in the form of Eq. 2.6.

$$A\,\omega = f$$

In order to see how this corresponds to equations of the form of Eq. 3.20, which have been central to the preceeding discussion, and at the same time to illustrate the relationship between the Rayleigh-Ritz and Galerkin procedures, consider that the operator A has the form

$$A = R + K \tag{3.45}$$

In this equation R is a symmetric and positive bounded below operator of order $2m$ and K is any operator such that its field of definition encompasses that of R, that is Ku has a meaning whenever Ru is meaningful. Thus in general A need not be symmetric. Substituting for A in Eq. 2.6 gives

$$R\omega + K\omega = f \tag{3.46}$$

which can be expressed in the form of Eq. 3.20 as

$$\omega + R^{-1}K\omega = R^{-1}f$$

or

$$\omega + T\omega = f' \tag{3.47}$$

where $T = R^{-1}K$ and $f' = R^{-1}f$.

Define a space H_R as in Chapter 3.1 in which the inner product is given by

$$[u, v]_R = (Ru, v) = \int_{\Omega} (Du)(Dv)\, d\Omega \tag{3.48}$$

in which D is a differential operator of order m .

Theorem 3.3 ensures the convergence of the approximate solution of Eq. 3.47 if $T = R^{-1}K$ is completely continuous in some Hilbert space H and if the co-ordinate functions are complete in H . Thus if T is completely continuous in H_R repeating the proof of Theorem 3.3 in terms of the energy product in H_R gives the equivalent of Eq. 3.42 as

$$A_j + \sum_{k=1}^{M} A_k [T\phi_k, \phi_j]_R = [f', \phi_j]_R \quad j = 1, \ldots M \tag{3.49}$$

in which

$$\bar{\omega} = \sum_{k=1}^{M} A_k \phi_k$$

and the ϕ_k have been assumed orthonormalized in H_R , i.e.

$$[\phi_k, \phi_j]_R = 1 \qquad k = j$$
$$= 0 \qquad k \neq j$$

Further the theorem states

$$\| \bar{\omega} - \omega \|_R \to 0 \quad \text{as} \quad M \to \infty$$

Consider now applying the Galerkin procedure directly to Eq. 3.46. The required equations are

$$\int_{\Omega} (R\bar{\omega} + K\bar{\omega} - f)\, \phi_j \, d\Omega = 0 \qquad j = 1, \ldots M$$

which may be written

$$\sum_{k=1}^{M} a_k \{ (R\phi_k, \phi_j) + (K\phi_k, \phi_j) \} = (f, \phi_j) \tag{3.50}$$

K has been specified to be such that its field of definition encompasses that of R. Assume further that it is defined for every element of H_R. Then Eq. 3.50 can be written

$$\sum_{k=1}^{M} a_k \{ [\phi_k, \phi_j]_R + (K\phi_k, \phi_j) \} = (f, \phi_j) \tag{3.51}$$

which then has meaning for any $\phi_k \in H_R$. Eq. 3.51 may be rewritten by noting first that the ϕ_k have been assumed orthonormalized in H_R and secondly that the following relations hold

$$(K\phi_k, \phi_j) = (RR^{-1}K\phi_k, \phi_j) = [T\phi_k, \phi_j]_R$$

$$(f, \phi_j) = (RR^{-1}f, \phi_j) = [f', \phi_j]_R$$

Thus Eq. 3.51 may be written

$$a_j + \sum_{k=1}^{M} a_k [T\phi_k, \phi_j]_R = [f', \phi_j]_R \quad j = 1, \ldots M \quad (3.52)$$

This equation coincides with Eq. 3.49. Thus the application of the Galerkin procedure to Eq. 3.46 provides an approximate solution that converges in the norm of H_R to the exact solution of Eq. 3.47. Clearly any solution of Eq. 3.46 is a solution of Eq. 3.47. However it may be that no solution exists in D_R but a solution does exist in H_R. In this case the solution of Eq. 3.47 is the generalized solution of Eq. 3.46 and in this way any solution of Eq. 3.47 will be considered a solution of Eq. 3.46.

Thus Eq. 3.51 obtained by applying the Galerkin procedure to Eq. 3.46 ensures convergence to the exact solution if $T = R^{-1}K$ is completely continuous in H_R and the co-ordinate functions are complete in H_R. The co-ordinate functions therefore need not satisfy the natural boundary conditions of the problem i.e. those boundary conditions involving derivatives of order m or higher.

If K is not defined for all functions in H_R convergence can be obtained by choosing co-ordinate functions from H_R^o i.e. functions with continuous $2m-1$ derivatives that satisfy all the boundary conditions.

A particular case of Eq. 3.46 is where K is the null operator, in which situation $A \equiv R$ and is symmetric and positive bounded below. Eq. 3.51 then reduces to

$$\sum_{k=1}^{M} a_k [\phi_k, \phi_j]_R = (f, \phi_j) \qquad j = 1, \ldots M \tag{3.53}$$

This equation is identical to Eq. 3.2 obtained by the Rayleigh-Ritz procedure and converges under identical conditions. Thus for a symmetric, positive bounded below operator the two methods lead to the same equations and are governed by the same convergence criteria. The Galerkin procedure thus appears as a generalization of the Rayleigh-Ritz procedure.

The Galerkin procedure can thus ensure convergence in energy norm for equations of the type given in Eq. 3.46 when the co-ordinate functions do not satisfy the natural boundary conditions of the problem. This convergence involves the derivatives of $\bar{w}$ up to order m but provides no information as to the manner of convergence of higher order derivatives. The following presentation illustrates the manner in which these higher order derivatives converge.

Assume g is some fixed element from H_R .

$$(R\bar{\omega} + K\bar{\omega} - f, g) = (R(\bar{\omega}-\omega) + K(\bar{\omega}-\omega), g)$$

Thus

$$|(R\bar{\omega} + K\bar{\omega} - f, g)| \leq |\bar{\omega} - \omega|_R |g|_R + (K(\bar{\omega}-\omega), g) \quad (3.54)$$

The second term on the right hand side of the equation is a bounded linear function in H_R and may therefore be expressed as an inner product on this space (9). Thus

$$(K(\bar{\omega}-\omega), g) = [\bar{\omega}-\omega, \psi] \leq |\bar{\omega}-\omega|_R |\psi|_R$$

where ψ is a fixed element of H_R . Thus Eq. 3.54 may be written

$$|(R\bar{\omega} + K\bar{\omega} - f, g)| \leq |\bar{\omega}-\omega|_R (|g| + |\psi|) \quad (3.55)$$

However the Galerkin procedure ensures

$$|\bar{\omega} - \omega|_R \rightarrow 0$$

Thus Eq. 3.55 implies

$$\int_{\Omega} (R\bar{\omega} + K\bar{\omega} - f)\, g \, d\Omega \longrightarrow 0 \qquad g \in H_R$$

If A is symmetric and positive bounded below i.e. $K \equiv 0$ a stronger convergence can be proven. To see

this assume $g \in L_2(\Omega)$. Then since H_A is dense in $L_2(\Omega)$ there exists $g' \in H_A$ such that $\| g' - g \| < \varepsilon$. Now

$$(A\bar{\omega} - f, g) = (A\bar{\omega} - f, g') + (A\bar{\omega} - f, g - g')$$

Further

$$|(A\bar{\omega} - f, g')| \leq \{ \| A\bar{\omega} \| + \| f \| \} \varepsilon < C\varepsilon$$

$C = const.$

and for M sufficiently large

$$|(A\bar{\omega} - f, g')| \leq |\bar{\omega} - \omega|_A \, |g'|_A < \varepsilon$$

Hence

$$|(A\bar{\omega} - f, g)| \leq (C+1)\varepsilon$$

Whence as ε can be made arbitrarily small

$$\int_{\Omega} (A\bar{\omega} - f) g \, d\Omega \to 0 \qquad g \in L_2(\Omega)$$

CHAPTER 4

THE FINITE ELEMENT PROCEDURE

The finite element procedure was originally developed by engineers on the basis of physical intuition for application in the analysis of complex structural systems. The review by Zienkiewcz (19) outlines the development of the method and contains a comprehensive list of references. Recently the mathematical framework of the procedure has come under close scrutiny and it is the purpose of this chapter to discuss this aspect of the method.

In 1969 Oden (11,12) pointed out that the formulation of a finite element model of a function is a purely topological construction and has nothing to do with variational principles. The finite element method, as will be shown in this chapter, is in fact a means of constructing an approximate solution form to a given equation. This approximation is expressed in terms of known co-ordinate functions and unknown parameters. This approximate solution form may be used in conjunction with a number of techniques to determine the unknown parameters.

The problem considered is that of obtaining an approximate solution to Eq. 2.6 under homogeneous boundary conditions. As was discussed in Chapter 3 there are a number of techniques available that are based upon the idea of assuming an approximate solution in the form

$$\bar{w} = \sum_{k=1}^{M} a_k \phi_k$$

It will first be demonstrated that the finite element procedure generates such an approximation, and then the convergence results that have been presented for the Galerkin procedure will be interpreted in terms of a finite element approximation.

4.1 Generation of a Finite Element Approximation

The basic steps that characterize the finite element procedure will be presented. A rigorous discussion of the following points has been presented by Oden (11).

The first step is to replace the domain of definition of the problem $\bar{\Omega}$ by $\bar{\Omega}^*$ such that $\bar{\Omega}^*$ may be exactly subdivided into a number, say E, of non overlapping subdomains called elements. The domain of a typical element will be denoted by Ω^e and such domains are generally chosen to have a simple geometrical form. Adjacent elements are specified to have a common boundary. Thus

$$\Omega^m \cap \Omega^n = \theta \,, \quad m \neq n \,, \quad m,n = 1,\ldots E \tag{4.1}$$

where θ is the empty set, and

$$\bar{\Omega}^* = \bigcup_{e=1}^{E} \bar{\Omega}^e \tag{4.2}$$

The elements are chosen, if possible, such that $\bar{\Omega}^*$ coincides with $\bar{\Omega}$, but if not, in such a manner that the error involved is acceptable. It will be assumed that the $\bar{\Omega}^e$ have been thus chosen and the notation $\bar{\Omega}$, Ω will be used to represent $\bar{\Omega}^*$, Ω^* .

The second step in the method involves the assumption of an approximate solution for ω in each of the elements that can be expressed in the form

$$\bar{\omega}^e = \sum_{k=1}^{N} a_k^e \, \phi_k^e \qquad e = 1,\ldots E \tag{4.3}$$

where the ϕ_k^e are co-ordinate functions defined only in $\bar{\Omega}^e$ and the a_k^e are the values of $\bar{\omega}^e$ or one of its derivatives at certain nodal points generally situated on the boundary of $\bar{\Omega}^e$. For example if a_n^e $n = 1,\ldots N$ corresponds to the value of $\bar{\omega}^e$ at the node with co-ordinates x_i^n then

$$\phi_k^e(x_i^n) = 1 \qquad k = n$$
$$= 0 \qquad k \neq n \;;\; k, n = 1, \ldots N \tag{4.4}$$

Such a definition ensures that the ϕ_k^e are linearly independent throughout $\bar{\Omega}^e$.

It is possible, by a linear transformation, to express any $\bar{\omega}^e$ containing N linearly independent terms in the form of Eq. 4.3. In particular a polynomial may be so expressed, which means that the approximate solution may be expressed in polynomial form, which is often convenient, and then transformed into the form of Eq. 4.3.

In the application of the finite element procedure it is only necessary to assume co-ordinate functions defined over individual elements to obtain a solution. However, in order to demonstrate that such approximations can be considered to be of the form of Eq. 3.1 it is convenient to introduce other functions which are defined in terms of the ϕ_k^e in the following manner.

Consider functions γ_k^e defined over the whole domain $\bar{\Omega}$ such that

$$\gamma_k^e(x_i) = \phi_k^e(x_i) \qquad x_i \in \bar{\Omega}^e$$
$$= 0 \qquad \text{otherwise} \tag{4.5}$$

where the X_i represents a point in the domain. Then the assumed approximation for ω throughout the whole domain $\bar{\Omega}$ may be written

$$\bar{\omega} = \sum_{e=1}^{E} \sum_{k=1}^{N} a_k^e \gamma_k^e \tag{4.6}$$

On interelement boundaries where nodes of adjacent elements coincide it is natural to specify that these nodal values should be the same. Assume that there are M independent global degrees of freedom in $\bar{\Omega}$ which will be denoted by a_k. Then the element degrees of freedom are related to the global degrees of freedom by the relationship

$$a_k^e = \sum_{j=1}^{M} G_{jk}^e a_j \tag{4.7}$$

where

$$\begin{aligned} G_{jk}^e &= 1 \qquad \text{if node } a_k^e \text{ coincides with } a_j \\ &= 0 \qquad \text{otherwise} \end{aligned}$$

Then

$$\bar{\omega} = \sum_{e=1}^{E} \sum_{k=1}^{N} \sum_{j=1}^{M} a_j G_{jk}^e \gamma_k^e$$

define

$$\phi_j = \sum_{e=1}^{E} \sum_{k=1}^{N} G_{jk}^e \gamma_k^e \tag{4.8}$$

Eq. 4.6 may then be written

$$\bar{\omega} = \sum_{j=1}^{M} a_j \phi_j$$

in which the ϕ_j are linearly independent throughout the domain.

Thus a finite element approximation has the form of Eq. 3.1. The essential feature of the method lies in formulating an approximate solution that is defined over the whole domain in terms of approximations that are non-zero only over subdomains.

A refined approximation is obtained by resubdividing the domain $\bar{\Omega}$ into a larger number of elements. The same approximate solution is assumed in each of the new elements and therefore the final approximation is again in the form of Eq. 3.1. The co-ordinate functions ϕ_j are refined in such a way that they have the same shape but are defined to be non-zero over a smaller region of $\bar{\Omega}$ than their predecessors.

It is also possible to refine the approximation by leaving the number of elements constant and increasing the number of co-ordinate functions per element.

The unknown parameters in the approximate solution may be evaluated, for example, by solving the equations given by the Galerkin procedure:

$$\sum_{k=1}^{M} a_k \left[\phi_k , \phi_j \right]_A = (f, \phi_j) = f_j \quad , \quad j = 1, \ldots M \qquad (4.9)$$

This equation is in practice generated by assembling the relations obtained from individual elements. Writing Eq. 4.9 for each of the elements in turn gives

$$\sum_{k=1}^{N} a_k^e \left[\phi_k^e , \phi_j^e \right]_A^e = (f, \phi_j^e)^e = f_j^e \quad , \quad \begin{matrix} e = 1, \ldots E \\ j = 1, \ldots N \end{matrix} \qquad (4.10)$$

where the superscript e indicates that the inner products are evaluated over the subdomains Ω^e . Solving Eqs. 4.10 gives the relationship between a_k^e and f_j^e which can then be employed to determine the relationship between a_k and f_j by making use of Eqs. 4.7 and 4.8.

An important feature of the finite element procedure that follows from the above construction of the governing equation is the banded nature of the coefficient matrix that may be obtained by suitably ordering the a_k . Such a feature is important in the numerical solution of problems with a large number of degrees of freedom.

4.2 General Remarks

It is worthwhile to note the analogy between a finite element approximation of a function and a Fourier

series approximation of a function. One important difference, however, lies in the fact that in a Fourier series the co-ordinate functions have continuous derivatives to any order throughout the whole domain, whereas a finite element co-ordinate function ϕ_j, generally has discontinuities in its lowest derivatives at element boundaries. Another difference is that refinement of a finite element approximation is effected by a redefinition of the co-ordinate functions as opposed to simply adding extra functions as is common in Fourier series approximations. One advantage of a given finite element approximation is its facility to approximate various boundary conditions. This is possible as the boundary conditions are handled by prescribing values to the generalized co-ordinates a_j located on the boundary.

A central question in the application of a finite element approximation concerns the conditions that the assumed solution within each element must satisfy to ensure convergence as more and more elements are taken. In particular, two related questions must be answered:

(i) on what basis should the approximation in each element be chosen, and

(ii) what continuity of $\bar{\omega}$ and its derivatives should be ensured at the nodes and across element boundaries?

These questions can be answered by investigating the conditions under which the particular method employed to evaluate the unknown parameters is known to converge. It is clear that such evaluation can be effected by a number of different techniques. Thus it is also clear that the finite element "method" need not be associated with any particular technique. Specifically it is not accurate to state that the finite element method is a Rayleigh-Ritz procedure. The finite element method simply generates an approximate solution form, defined in terms of unknown parameters, that may be used in conjunction with a number of techniques to obtain an approximate answer to the given equation.

It is natural to investigate the relative advantages of the different solution techniques available. Traditionally virtual work or Rayleigh-Ritz have been used in finite element work. The Galerkin procedure has been used in a number of specific cases (17,18,20). These cases could, however, have been analysed using the Rayleigh-Ritz procedure. The possibility of applying the Galerkin procedure to a class of problems to which the Rayleigh-Ritz method is not applicable, and at the same time ensuring convergence, does not appear to have been previously explored.

As was pointed out in Chapter 3 the Galerkin procedure is a generalization of the Rayleigh-Ritz method

and hence in general preferable. In particular, problems that are characterized by non-symmetric operators may be amenable to the Galerkin procedure, whereas they cannot be handled by Rayleigh-Ritz. Also, as is illustrated in Chapter 6, the Galerkin procedure is applicable to all those problems of structural mechanics that virtual work can be used for, with the added advantage that unlike virtual work Galerkin has proven convergence criteria.

On the basis of these remarks the Galerkin procedure will be chosen for the determination of the generalized co-ordinates a_j and for the investigation of those sufficient conditions that the element approximation $\bar{w}^e$ must satisfy in order to ensure convergence of $\bar{w}$ to the correct answer.

4.3 Convergence Criteria

In Chapter 3 conditions that ensured convergence of the Galerkin procedure for a wide class of problems were presented. In this section these results will be utilized to provide sufficient convergence criteria for a finite element approximation.

Theorem 3.3 asserts that the convergence of the Galerkin approximation is ensured when applied to equations of the form

$$\omega - \lambda T \omega = f$$

if the solution is unique in some Hilbert space H, the operator T is completely continuous in H , and if the co-ordinate functions are complete in H . The interpretation of these conditions in terms of a finite element approximation will be presented by means of a particular example.

Consider the problem of determining the equilibrium configuration of a uniform beam when subjected to both a normal load and a load that is proportional to the slope of the beam. Such non conservative loads are encountered in the study of aeroelasticity. The governing equations are

$$EI\, y'''' + c\, y' = f \qquad x \in [0, L] \tag{4.11a}$$

$$y(0) = y(L) = y''(0) = y''(L) = 0 \tag{4.11b}$$

where c is a constant and f is the normal force. The operator in this equation can readily be seen to be unsymmetric. This equation corresponds to Eq. 3.46 in which

$$R = EI\frac{d^4}{dx^4} \quad ; \quad K = c\frac{d}{dx} \tag{4.12}$$

As in Chapter 3.2 construct a space H_R in which

$$\|u\|_R = \sqrt{[u,u]_R} = \sqrt{\int_0^L EI\, u''u''\, dx} \tag{4.13}$$

Functions u that are in H_R must then satisfy the condition

$$\sqrt{\int_0^L EI\,(u''-u_n'')^2 dx} \to 0 \quad \text{as } n\to\infty\,;\ u_n \in D_R \tag{4.14}$$

The space H_R contains the exact solution and, as is verified in Chapter 7, $T = R^{-1}K$ is completely continuous in this space. Thus Theorem 3.3 ensures convergence if the co-ordinate functions are complete in H_R.

A finite element approximation is obtained by dividing the beam into E sections and within each section assuming a solution of the form

$$\bar{q}^e = \sum_{k=1}^{N} a_k^e\, \phi_k^e \qquad e = 1, \ldots E \tag{4.15}$$

The element stiffness equations are, on the basis of Eq. 4.10, given by

$$\sum_{k=1}^{N} a_k^e \int_{L^e} \left(EI \left(\phi_k^e\right)'' \left(\phi_j^e\right)'' + c \left(\phi_k^e\right)' \phi_j^e \right) dx = \int_{L^e} f \phi_j^e \, dx \quad j = 1, \ldots N \tag{4.16}$$

where the relation

$$\left[\phi_k^e, \phi_j^e \right]_A^e = \left[\phi_k^e, \phi_j^e \right]_R^e + \left(K \phi_k^e, \phi_j^e \right)^e \tag{4.17}$$

has been used. Eqs. 4.16 can then be assembled into the form

$$\sum_{k=1}^{M} a_k \int_0^L \left(EI \, \phi_k'' \, \phi_j'' + c \, \phi_k' \, \phi_j \right) dx = \int_0^L f \, \phi_j \, dx \quad j = 1, \ldots M \tag{4.18}$$

and the approximate solution

$$\bar{q} = \sum_{k=1}^{M} a_k \phi_k \tag{4.19}$$

converges to the correct answer if the ϕ_k are complete in H_R. The conditions that must be satisfied by the assumed solution within each element in order to ensure that the co-ordinates defining the total solution be complete in H_R must therefore be established.

In this example, Eq. 4.14 shows that functions in H_R must have generalized second derivatives and hence continuous first derivatives. Thus the finite element approximation must ensure the continuity of slope across element boundaries. Further it must also satisfy the forced boundary conditions as all functions in H_R are required to satisfy these conditions.

Before considering the conditions under which completeness can be obtained the definition of completeness will be repeated in terms of a finite element approximation. A finite element approximation is complete in H in some stated norm if for arbitrary function $v \in H$ and any $\epsilon > 0$ there exists a subdivision of the domain corresponding to M degrees of freedom such that

$$\left| v - \sum_{j=1}^{M} a_j \phi_j \right| < \epsilon$$

The conditions that a finite element approximation must satisfy for completeness have been provided by Oliveira (13). They will be quoted for a function ω_i that has c components $\omega_1, \omega_2, \ldots \omega_c$. It is assumed that the energy norm is based upon a symmetric energy product that contains derivatives of each component

ω_T of maximum order m_T . It is further assumed that the exact solution is such that the derivatives of its components of order $m_T + 1$ are continuous within each element. Discontinuities of the m_T and $m_T + 1$ derivatives are still allowed at points which always remain on element boundaries as the size of the element is progressively reduced.

Oliveira then proves that completeness will be obtained if continuity of the $m_T - 1$ derivatives is ensured throughout the domain and if the approximation $\bar{\omega}_T^e$ for ω_T within each element is based upon a polynomial of degree not less than m_T , all the terms of which are affected by independent arbitrary coefficients. These conditions are often expressed by stating that the elements must be conforming and that they must be able to represent constant strain.

Thus in the example being considered the completeness requirement means that $\bar{y}^e$ must be based upon a complete polynomial of order not less than two. Satisfaction of both the conforming and completeness conditions can be obtained by assuming an approximation within each element of the form

$$\bar{y}^e = a_0^e + a_1^e x + a_2^e x^2 + a_3^e x^3 \qquad (4.20)$$

and choosing degrees of freedom corresponding to the displacement and rotation at each end of the element.

Note that it was possible to choose an approximation in H_R as $K = c\, d/dx$ is defined for all the co-ordinate functions in H_R. Co-ordinate functions in H_R have continuous first derivatives and K is defined for all such functions.

Satisfaction of the indicated conditions thus ensures energy convergence of the finite element approximation to the correct solution y_0. Specifically

$$\int_0^L EI\,(\bar{y}'' - y_0'')^2\, dx \rightarrow 0 \quad \text{as} \quad M \rightarrow \infty \tag{4.21}$$

It should be noted that conformity is only a sufficient condition for proving the convergence of a finite element approximation. Oliveira (13) has also studied the question of non-conforming elements in the context of the Rayleigh-Ritz procedure and concluded that under certain conditions they too can ensure convergence to the correct solution. Non-conforming elements will not be discussed in this thesis. However it is possible to introduce a well defined norm into the space of functions that satisfy the forced boundary conditions and have continuous principal derivatives only at nodal points of the

domain. In this way the work of this thesis can be extended to provide a systematic study of the convergence properties of non-conforming elements.

The developments that have been presented with respect to Eq. 4.11 can be paralleled for any equation that can be expressed in the form of Eq. 3.20. Consider, in general, the problem of obtaining a finite element solution for Eq. 3.46. Writing this equation out in indicial notation gives

$$R_{mn}\,\omega_n + K_{mn}\,\omega_n = f_n \qquad m,n = 1,\ldots c \tag{4.22}$$

Assume a solution

$$\bar{\omega}_n = \langle\, \bar{\omega}_1 , \bar{\omega}_2 , \ldots\ldots \bar{\omega}_c \,\rangle^T \tag{4.23a}$$

where

$$\bar{\omega}_T = \sum_{k=1}^{L^T} a_{Tk}\,\phi_k^T \qquad T = 1,\ldots c \tag{4.33b}$$

The Galerkin equations for Eq. 4.22 then become

$$\int_{\Omega} \left(R_{Tm}\,\bar{\omega}_m + K_{Tm}\,\bar{\omega}_m - f_T \right) \phi_k^T \, d\Omega = 0 \qquad \begin{matrix} T = 1,\ldots c \\ k = 1,\ldots L^T \end{matrix} \tag{4.24}$$

If K is defined for all the co-ordinate functions in this equation may be expressed in the form

$$\left[\bar{\omega}_T, \phi_k^T\right]_R + \left(K_{Tm}\bar{\omega}_m, \phi_k^T\right) = \left(f_T, \phi_k^T\right) \tag{4.25}$$

where $\left[\bar{\omega}_T, \phi_k^T\right]_R$ is the symmetric form of $\int_{\Omega}(R_{Tm}\bar{\omega}_m)\phi_k^T\, d\Omega$. Writing Eq. 4.25 in element form gives

$$\left[\bar{\omega}_T^e, \phi_k^{eT}\right]_R^e + \left(K_{Tm}\bar{\omega}_m^e, \phi_k^{eT}\right)^e = \left(f_T, \phi_k^{eT}\right)^e \quad \begin{matrix} T=1,\dots C \\ k=1,\dots N^T \\ e=1,\dots E \end{matrix} \tag{4.26}$$

where

$$\bar{\omega}_T^e = \sum_{k=1}^{N^T} a_{Tk}\,\phi_k^{eT} \tag{4.27}$$

Similarly eigenvalue problems that are expressible in the form of Eq. 3.18 may be so treated. Consider, for example, the problem

$$R_{mn}\omega_n + \lambda K_{mn}\omega_n = 0 \tag{4.28}$$

The required finite element equations are

$$\left[\bar{\omega}_T^e, \phi_k^{eT}\right]_R^e + \lambda\left(K_{Tn}\bar{\omega}_n^e, \phi_k^{eT}\right)^e = 0 \quad \begin{matrix} T=1,\dots C \\ k=1,\dots N^T \\ e=1,\dots E \end{matrix} \tag{4.29}$$

Thus application of the Galerkin procedure enables the equations and corresponding convergence criteria of a finite element solution to be set down for a wide class of problems.

CHAPTER 5

APPLICATION OF THE GALERKIN PROCEDURE TO PROBLEMS WITH MIXED AND NONHOMOGENEOUS BOUNDARY CONDITIONS

Attention has been confined so far in this thesis to problems with unmixed homogeneous boundary conditions. In this chapter a problem with mixed homogeneous boundary conditions will be treated and the modifications necessary to deal with nonhomogeneous boundary conditions presented. For illustration purposes the equations governing the equilibrium configuration of a linear elastic continuum will be considered.

5.1 Homogeneous Mixed Boundary Conditions

The governing equations in this case are

$$-\sigma_{ij,j} = \rho X_i \qquad \in \Omega \tag{5.1a}$$

$$u_i = 0 \qquad \in S_u \tag{5.1b}$$

$$\sigma_{ij} n_j = 0 \qquad \in S_T \tag{5.1c}$$

$$\sigma_{ij} n_j + c u_i = 0 \qquad \in S_M \tag{5.1d}$$

where σ_{ij} is the stress tensor, ρ the density, X_i the body force per unit mass, u_i the displacement, n_j the component of unit outward normal to the boundary, and c is a constant. S_u , S_T , and S_M represent respectively those portions of the boundary on which the displacements, stresses, and mixed conditions, are specified.

In order to formulate this problem in terms of the displacement vector u_i the gradient of the stress tensor in Eq. 5.1a is related to the displacement by means of the constitutive relation

$$\sigma_{ij} = 2\mu \varepsilon_{ij} + \lambda \varepsilon_{kk} \delta_{ij} \tag{5.2}$$

and the kinematic equation

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{5.3}$$

where μ and λ are Lamé's constants, δ_{ij} is the Kronecker

delta, and ε_{ij} is the strain tensor. Carrying out the indicated substitutions gives

$$\mu u_{i,jj} + (\lambda + \mu) u_{j,ji} = \rho X_i \tag{5.4}$$

which may be placed into correspondence with the general equation

$$A_{ij} \omega_j = f_i \tag{5.5}$$

In this case A_{ij} is a matrix of differential operators of the second order and is symmetric. For present purposes it is convenient not to express $A_{ij}\omega_j$ by means of Eq. 5.4 but instead to use the relation

$$A_{ij} \omega_j = - \sigma_{ij,j} \tag{5.6}$$

Eq. 5.6 can now be used to illustrate the symmetry of A_{ij}. Thus

$$[\, u_i', u_i'' \,] = \int_{\Omega} - \sigma_{ij,j}' \, u_i'' \, d\Omega \qquad u_i', u_i'' \in D_A$$

where σ_{ij}' denotes the stress tensor corresponding to the displacement vector u_i'. Now

$$[\, u_i', u_i'' \,] = \int_{\Omega} \left(-(\sigma_{ij}' u_i'')_{,j} + \sigma_{ij}' u_{i,j}'' \right) d\Omega$$

$$= \int_{\Omega} \sigma_{ij}' u_{i,j}'' \, d\Omega - \int_{S} \sigma_{ij}' n_j u_i'' \, ds$$

As the variables considered are in the field of definition of the operator they must satisfy all the boundary conditions of Eqs. 5.1. Therefore

$$[u_i', u_i''] = \int_{\Omega} \tfrac{1}{2} \sigma_{ij}' (u_{i,j}'' + u_{j,i}'') \, d\Omega + \int_{S_M} c u_i' u_i'' \, ds$$

$$= \int_{\Omega} \sigma_{ij}' \varepsilon_{ij}'' \, d\Omega + \int_{S_M} c u_i' u_i'' \, ds$$

$$= \int_{\Omega} (2\mu \varepsilon_{ij}' \varepsilon_{ij}'' + \lambda \varepsilon_{kk}' \varepsilon_{jj}'') \, d\Omega + \int_{S_M} c u_i' u_i'' \, ds$$

$$= [u_i'', u_i']$$

Whence the operator A_{ij} is symmetric and the energy product in the space H_A is given by

$$[u_i', u_i'']_A = \int_{\Omega} \sigma_{ij}' u_{i,j}'' \, d\Omega + \int_{S_M} c u_i' u_i'' \, ds \qquad (5.7)$$

In this case the mixed boundary condition gives rise to a surface integral in the energy product. The Galerkin equations governing the solution of this problem can be obtained from Eq. 4.25 by equating A_{ij} and R_{ij} and assuming K_{ij} is the null operator. The equations are

$$\left[\bar{u}_T , \phi_j^T \right]_A = \left(f_T , \phi_j^T \right) \qquad \begin{matrix} T = 1,2,3 \\ j = 1, \ldots L^T \end{matrix} \tag{5.8}$$

where an approximate solution has been assumed in the form

$$\begin{aligned} \bar{u}_i &= \langle \bar{u}_1 , \bar{u}_2 , \bar{u}_3 \rangle^T \\ \bar{u}_T &= \sum_{k=1}^{L^T} a_{Tk} \phi_k^T \qquad T = 1,2,3 \end{aligned} \tag{5.9}$$

Writing Eq. 5.8 out in full gives

$$\int_{\Omega} \bar{\sigma}_{Tk} \phi_{k,j}^T \, d\Omega + \int_{S_M} c \bar{u}_T \phi_j^T \, ds = \int_{\Omega} \rho X_T \phi_j^T \, d\Omega \tag{5.10}$$

in which $\bar{\sigma}_{Tk}$ is the stress tensor derived from the approximate solution $\bar{u}_i$. This equation ensures convergence of the approximate solution if the co-ordinate functions are complete in H_A . By noting that the energy product defined by Eq. 5.7 contains first derivatives of the displacement it is sufficient that the assumed displacement field be continuous throughout Ω . Thus a finite element approximation may give use to stress discontinuities at element boundaries. Further it can be noted that the boundary conditions prescribed on S_T and S_M contain first order derivatives and are therefore natural. The co-ordinate functions therefore need not satisfy these boundary conditions.

If the co-ordinate functions are chosen from H_A^o i.e. if they have continuous first derivatives, which ensures a continuous stress field, and satisfy all the boundary conditions, then

$$\int_{\Omega} -\bar{\sigma}_{Tk,k}\,\phi_j^T\,d\Omega = \int_{\Omega} \bar{\sigma}_{Tk}\,\phi_{j,k}^T\,d\Omega + \int_{S_m} c\,\bar{u}_T\,\phi_j^T\,ds \qquad (5.11)$$

Eq. 5.10 can then be written

$$\int_{\Omega} -\bar{\sigma}_{Tk,k}\,\phi_j^T\,d\Omega = \int_{\Omega} \rho\,X_T\,\phi_j^T\,d\Omega \qquad \begin{matrix} T = 1,2,3 \\ j = 1,\ldots L^T \end{matrix} \qquad (5.12)$$

Thus if the co-ordinate functions are in H_A^o it is not necessary to express the energy product in its symmetric form.

In general Eq. 5.12 may be written

$$\left[\,\bar{u}_T\,,\,\phi_j^T\,\right]_A^o = \left(\,f_T\,,\,\phi_j^T\,\right) \qquad \begin{matrix} T = 1,2,3 \\ j = 1,\ldots L^T \end{matrix} \qquad (5.13a)$$

where

$$\left[\,\bar{u}_T\,,\,\phi_j^T\,\right]_A^o = \int_{\Omega} A_{Tn}\,\bar{u}_n\,\phi_j^T\,d\Omega \qquad (5.13b)$$

5.2 Nonhomogeneous Boundary Conditions

The governing equations are

$$-\sigma_{ij,j} = \rho X_i \qquad \in \Omega \tag{5.14a}$$

$$u_i = u_i^\circ \qquad \in S_u \tag{5.14b}$$

$$\sigma_{ij} n_j = T_i^\circ \qquad \in S_T \tag{5.14c}$$

$$\sigma_{ij} n_j + c u_i = C_i^\circ \qquad \in S_M \tag{5.14d}$$

The approach in this case is to change variables in such a way that the problem is reduced to one with homogeneous boundary conditions. Thus, assume that there exists a function u_i', whose components have continuous first derivatives, such that

$$u_i' = u_i^\circ \tag{5.15a}$$

$$\sigma_{ij}' n_j = T_i^\circ \tag{5.15b}$$

$$\sigma_{ij}' n_j + c u_i' = C_i^\circ \tag{5.15c}$$

Define a new variable u_i'' such that

$$u_i'' = u_i - u_i' \tag{5.16}$$

Substituting Eq. 5.16 into Eqs. 5.14 and taking note of Eqs. 5.15 gives

$$-\sigma''_{ij,j} = \rho X_i + \sigma'_{ij,j} = f''_i \qquad \in S_2 \qquad (5.17a)$$

$$u''_i = 0 \qquad \in S_u \qquad (5.17b)$$

$$\sigma''_{ij} n_j = 0 \qquad \in S_T \qquad (5.17c)$$

$$\sigma''_{ij} n_j + c u''_i = 0 \qquad \in S_M \qquad (5.17d)$$

Thus in terms of u''_i the problem has homogeneous boundary conditions, and the only difference from Eqs. 5.1 is the introduction of the term $\sigma'_{ij,j}$. If this term were known it would be possible to obtain an approximate solution for u''_i and convergence would be ensured if the co-ordinate functions were complete with respect to the H_A space corresponding to the energy product given in Eq. 5.7. Assuming

$$\bar{u}''_T = \sum_{k=1}^{L^T} q''_{Tk} \phi^T_k \qquad (5.18)$$

the required Galerkin equations, from Eq. 4.25, are

$$\left[\bar{u}''_T, \phi^T_j \right]_A = \left(f''_T, \phi^T_j \right) \qquad \begin{matrix} T = 1,2,3 \\ j = 1, \ldots L^T \end{matrix} \qquad (5.19)$$

Thus a solution for u''_i could be obtained such that

$$\left| \bar{u}''_i - u''_i \right|_A \longrightarrow 0 \qquad (5.20)$$

However, a solution may be obtained directly for u_i without actually knowing u_i' . This is accomplished by assuming an approximation for the components of u_i in the form

$$\bar{u}_T = \sum_{k=1}^{L^T} a_{Tk} \psi_k^T \qquad T = 1,2,3 \qquad (5.21)$$

where the set of functions $\{\psi_k\}$ is complete in the H_A space considered with respect to the nonhomogeneous boundary conditions. Then as the function $\bar{u}_i'' + u_i'$ satisfies such nonhomogeneous boundary conditions it follows that there exist a_{Tk} such that

$$\left| \bar{u}_i - (\bar{u}_i'' + u_i') \right|_A \longrightarrow 0 \quad \text{as} \quad L^T \to \infty \qquad (5.22)$$

Further the function $\bar{u}_i - (\bar{u}_i'' + u_i')$ has homogeneous forced boundary values and thus Eq. 5.22 implies

$$\left[\bar{u}_i - (\bar{u}_i'' + u_i'), \theta_i \right]_A = 0 \qquad (5.23)$$

where $\theta_i = \langle \theta_1, \theta_2, \theta_3 \rangle^T$ is an arbitrary function in the space considered, with homogeneous forced boundary values. Thus in particular

$$\left[\bar{u}_T - (\bar{u}_T'' + u_T'), \phi_k^T \right]_A = 0 \qquad \begin{matrix} T = 1,2,3 \\ k = 1, \ldots L^T \end{matrix}$$

or

$$\left[\bar{u}_T'', \phi_k^T \right]_A = \left[\bar{u}_T - u_T', \phi_k^T \right]_A \tag{5.24}$$

Eq. 5.19 can therefore be written

$$\left[\bar{u}_T, \phi_j^T \right]_A = \left(f_T'', \phi_j^T \right) + \left[u_T', \phi_j^T \right]_A \tag{5.25}$$

Now from Eq. 5.17a

$$\left(f_T'', \phi_j^T \right) = \int_\Omega \left(\rho X_T + \sigma'_{Tk,k} \right) \phi_j^T \, d\Omega$$

$$= \int_\Omega \left(\rho X_T \phi_j^T - \sigma'_{Tk} \phi_{j,k}^T \right) d\Omega + \int_S \sigma'_{Tk} n_k \phi_j^T \, ds$$

Using Eqs. 5.15 and the fact that $\phi_j^T = 0$ on S_u gives

$$\left(f_T'', \phi_j^T \right) = \int_\Omega \left(\rho X_T \phi_j^T - \sigma'_{Tk} \phi_{j,k}^T \right) d\Omega + \int_{S_T} T_T^\circ \phi_j^T \, ds + \int_{S_M} \left(C_T^\circ - c u_T' \right) \phi_j^T \, ds \tag{5.26}$$

Thus Eqs. 5.26 and 5.7 enable Eq. 5.25 to be written

$$\left[\bar{u}_T, \phi_j^T \right]_A = \left(\rho X_T, \phi_j^T \right) + \int_{S_T} T_T^\circ \phi_j^T \, ds + \int_{S_M} C_T^\circ \phi_j^T \, ds \tag{5.27}$$

which are the Galerkin equations that govern a solution in H_A when the boundary conditions are nonhomogeneous.

Written out in full using Eq. 5.7 these equations become

$$\int_{\Omega} \bar{\sigma}_{Tk}\, \phi^T_{j,k}\, d\Omega \;+\; \int_{S_M} c\, \bar{u}_T\, \phi^T_j\, ds \;=$$

$$\int_{\Omega} \rho X_T\, \phi^T_j\, d\Omega \;+\; \int_{S_T} T^{\circ}_T\, \phi^T_j\, ds \;+\; \int_{S_M} C^{\circ}_T\, \phi^T_j\, ds \qquad \begin{matrix} T = 1,2,3 \\ j = 1 \ldots L^T \end{matrix} \tag{5.28}$$

in which it has been required that the approximate solution for u_i given by Eq. 5.21 should satisfy the nonhomogeneous forced boundary conditions and that the co-ordinate functions ϕ^T_j should satisfy homogeneous forced boundary conditions. The forced boundary conditions are those occuring on S_u.

In generating a finite element solution there is no necessity to introduce different approximations corresponding to the ψ and ϕ co-ordinate functions. For a finite element approximation in H_A the co-ordinate functions associated with the degreesof freedom that do not lie on S_u satisfy homogeneous conditions on S_u . Eqs. 5.28 are solved by first specifying the values of a_{Tk} that lie on S_u and eleminating these a_{Tk} from the equations. The remaining co-ordinate functions satisfy homogeneous forced boundary conditions and thus only one set of co-ordinate functions need be introduced.

A solution in H°_A can be achieved by choosing co-ordinate functions that satisfy all the nonhomogeneous boundary conditions and have continuous

first derivatives. In this case Eq. 5.28 may be written

$$\int_{\Omega} -\bar{\sigma}_{Tk,k}\,\phi_j^T\,d\Omega \;=\; \int_{\Omega} \rho X_T\,\phi_j^T\,d\Omega \qquad \begin{matrix} T=1,2,3 \\ j=1,\ldots L^T \end{matrix} \tag{5.29}$$

or

$$[\,\bar{u}_T\,,\,\phi_j^T\,]_A^{\circ} \;=\; (f_T\,,\,\phi_j^T) \tag{5.30}$$

which is the same as Eq. 5.13a obtained for the homogeneous boundary condition problem.

It remains to demonstrate that the approximate solution $\bar{u}_i$ obtained from Eq. 5.28 converges to the correct answer. Using Eq. 5.23 it is possible to write

$$[\,\bar{u}_i\,,\,\theta_i\,]_A \;=\; [\,\bar{u}_i^{\,\prime\prime} + u_i^{\prime}\,,\,\theta_i\,]_A \tag{5.31}$$

Also Eq. 5.16 may be used to write

$$[\,u_i\,,\,\theta_i\,]_A \;=\; [\,u_i^{\,\prime\prime} + u_i^{\prime}\,,\,\theta_i\,]_A \tag{5.32}$$

Subtracting Eq. 5.31 from Eq. 5.32 gives

$$[\,u_i - \bar{u}_i\,,\,\theta_i\,]_A \;=\; [\,u_i^{\,\prime\prime} - \bar{u}_i^{\,\prime\prime}\,,\,\theta_i\,]_A$$

Using the Cauchy Bunyakovsky inequality gives

$$[\,u_i^{\,\prime\prime} - \bar{u}_i^{\,\prime\prime}\,,\,\theta_i\,]_A \;\leq\; \|\,u_i^{\,\prime\prime} - \bar{u}_i^{\,\prime\prime}\,\|_A\;\|\,\theta_i\,\|_A$$

Eq. 5.20 then implies

$$[\, u_i - \bar{u}_i \,,\, \theta_i \,]_A \longrightarrow 0$$

But as θ_i is an arbitrary function with homogeneous forced boundary values, it may be set equal to $u_i - \bar{u}_i$. Then

$$[\, u_i - \bar{u}_i \,,\, u_i - \bar{u}_i \,] \longrightarrow 0$$

whence

$$| \, u_i - \bar{u}_i \, |_A \longrightarrow 0 \qquad (5.33)$$

i.e. the approximate solution to the nonhomogeneous boundary condition problem converges in the norm of H_A to the exact solution.

The equations governing a finite element solution in H_A may, on the basis of Eq. 5.27, be written as

$$[\, \bar{u}_T^e \,,\, \phi_j^{eT} \,]_A^e = (\rho X_T \,,\, \phi_j^{eT})^e + \int_{S_T^e} T_T^o \, \phi_j^T \, ds + \int_{S_M^e} C_T^o \, \phi_j^{eT} \, ds \quad \begin{matrix} T=1,2,3 \\ j=1,\ldots N^T \\ e=1,\ldots E \end{matrix} \qquad (5.34)$$

in which S_T^e , S_M^e represent that portion of the boundary of element e that coincides with S_T , S_M respectively.

CHAPTER 6

BOUNDARY RESIDUAL CONCEPT AND VIRTUAL WORK

In this chapter the equations generated by the Galerkin procedure as have been presented herein are compared with those obtained from the boundary residual concept as applied to the Galerkin procedure and to those obtained using virtual work principles.

6.1 Boundary Residual Concept

An alternate interpretation of the Galerkin procedure that has been presented by Finlayson and Scriven (4) considers that a boundary residual term is included with the domain residual in the governing equation. The unknown parameters are then determined such that the sum of these two residuals is minimized. Specifically, the boundary residual considered is that occuring on that portion of the boundary on which the natural boundary conditions are specified. Thus if the equation and natural boundary conditions are

$$A_{ij}\, u_j = f_i \qquad \epsilon\ S\Omega\ ;\ i,j = 1,2,3 \tag{6.1a}$$

$$B_{ij}\, u_j = p_i^{\circ} \qquad \epsilon\ S_N \tag{6.1b}$$

the required equations become

$$\int_{S\Omega} (A_{Tj}\, \bar{u}_j - f_T)\, \phi_k^T\, dS\Omega + \int_{S_N} (B_{Tj}\, \bar{u}_j - p_T^{\circ})\, \phi_k^T\, dS = 0, \quad \begin{matrix} T=1,2,3 \\ k=1,\ldots L^T \end{matrix} \tag{6.2}$$

where the components of the approximate solution $\bar{u}_i$ have been assumed to be

$$\bar{u}_T = \sum_{k=1}^{L^T} a_{Tk}\, \phi_k^T \tag{6.3}$$

and this approximation satisfies the forced boundary conditions.

It will be shown that such an approach leads to the same equations as have been developed herein for a solution in H_A° but that different equations are obtained in H_A if the forced boundary conditions are not homogeneous.

Consider again the nonhomogeneous problem defined by Eqs. 5.14. The equations obtained by the application of the boundary residual concept are

$$\int_{\Omega} (-\bar{\sigma}_{Tj,j} - \rho X_T)\, \phi_k^T \, d\Omega + \int_{S_T} (\bar{\sigma}_{Tj} n_j - T_T^o)\, \phi_k^T \, ds$$

$$+ \int_{S_M} (\bar{\sigma}_{Tj} n_j + c\,\bar{u}_T - C_T^o)\, \phi_k^T \, ds = 0 , \quad \begin{matrix} T = 1,2,3 \\ k = 1, \ldots L^T \end{matrix} \tag{6.4}$$

First consider the generation of a solution in H_A^o . As was mentioned in Chapter 5 the co-ordinate functions must then be such that the stresses are continuous throughout the domain and all the boundary conditions are satisfied. Under these conditions Eq. 6.4 reduces to

$$\int_{\Omega} -\bar{\sigma}_{Tj,j}\, \phi_k^T \, d\Omega = \int_{\Omega} \rho X_T \, \phi_k^T \, d\Omega \tag{6.5}$$

which coincides with Eq. 5.29. Thus the boundary residual concept gives the same equations as does the Galerkin procedure if the solution is sought in H_A^o .

If a solution is required in H_A the required equations derived on the basis of the boundary residual concept can be obtained by applying Gauss' theorem to Eq. 6.4. This gives

$$\int_{\Omega} (\bar{\sigma}_{Tj}\, \phi_{k,j}^T - \rho X_T \phi_k^T)\, d\Omega - \int_S \bar{\sigma}_{Tj} n_j \phi_k^T \, ds$$

$$+ \int_{S_T} (\bar{\sigma}_{Tj} n_j - T_T^o)\, \phi_k^T \, ds + \int_{S_M} (\bar{\sigma}_{Tj} n_j - c\,\bar{u}_T - C_T^o)\, \phi_k^T \, ds = 0 \tag{6.6}$$

Therefore

$$\int_{\Omega} \bar{\sigma}_{Tj}\, \phi^{T}_{k,j}\, d\Omega \;+\; \int_{S_M} c\, \bar{u}_T\, \phi^{T}_{k}\, ds \;=\; \int_{\Omega} \rho\, X_T\, \phi^{T}_{k}\, d\Omega \tag{6.7}$$

$$+\; \int_{S_T} T_T^{\circ}\, \phi^{T}_{k}\, ds \;+\; \int_{S_M} C_T^{\circ}\, \phi^{T}_{k}\, ds \;+\; \int_{S_u} \bar{\sigma}_{Tj}\, n_j\, u_T^{\circ}\, ds$$

which corresponds to Eq. 5.28 only if the forced boundary conditions are homogeneous.

The boundary residual concept, as presented herein, would appear to be an attempt to express the Galerkin equations in a form in which co-ordinate functions from H_A are admissable by a more intuitive approach than has been presented in this thesis. However the approach is not entirely correct and should be modified somewhat. The required modifications would be to express the assumed solution in Eq. 6.2 in the form

$$\bar{u}_T \;=\; \sum_{k=1}^{L^T} a_{Tk}\, \psi^{T}_{k} \tag{6.8}$$

This approximation is required to satisfy the nonhomogeneous forced boundary conditions, whereas the co-ordinate functions ϕ^{T}_{k} should satisfy homogeneous forced boundary conditions. Then integrating Eq. 6.2 by parts results in the correct equation as given by Eq. 5.28. In this case the physical interpretation of Eq. 6.2 is that the domain residual plus the residual on that portion of the boundary corresponding

to the natural boundary conditions are made orthogonal to a finite number of displacements that satisfy homogeneous forced boundary conditions.

6.2 Virtual Work

The principle of virtual work states that for any system in equilibrium the internal and the external work performed by the existing stress system as the body moves through any compatible virtual displacement should be equal. A compatible displacement is one that satisfies homogeneous forced boundary conditions and ensures continuity of the principal derivatives i.e. those derivatives of order less than m if the governing equation has derivatives of maximum order $2m$. Such a displacement function is allowable in H_A , and further, as was presented in Chapter 4.3, a set of such functions can be complete in H_A .

Applying the virtual work principle to Eqs. 5.14 gives

$$\int_{\Omega} \frac{1}{2} \sigma_{Tj} (\delta u_{T,j} + \delta u_{j,T}) \, d\Omega = \int_{\Omega} \rho X_T \, \delta u_T \, d\Omega + \int_{S_T} T_T^{\circ} \, \delta u_T \, ds + \int_{S_M} \sigma_{Tj} \, n_j \, \delta u_T \, ds \qquad T = 1, 2, 3 \tag{6.9}$$

where the body has been given a compatible virtual displacement δu_T whence it is continuous throughout the whole domain and $\delta u_T = 0$ on S_u . In this equation σ_{Tj} represents the equilibrium stress field i.e. that stress field that satisfies Eqs. 5.14. The principle of virtual work states that Eq. 6.9 must hold for any δu_T .

Note that as σ_{Tj} is symmetric

$$\int_{\Omega} \frac{1}{2}\left(\sigma_{Tj}\,\delta u_{T,j} + \sigma_{Tj}\,\delta u_{j,T}\right) d\Omega = \int_{\Omega} \frac{1}{2}\left(\sigma_{Tj}\,\delta u_{T,j} + \sigma_{Tj}\,\delta u_{T,j}\right) d\Omega$$

Thus Eq. 6.9 can be written

$$\int_{\Omega}\left(\sigma_{Tj}\,\delta u_{T,j} - \rho X_T\,\delta u_T\right) d\Omega - \int_{S_T} T_T^{\circ}\,\delta u_T\, ds - \int_{S_m}\left(-c u_T + C_T^{\circ}\right)\delta u_T\, ds = 0 \tag{6.10}$$

where the fact that the stress field satisfies Eq. 5.14d has been used.

As δu_T is arbitrary let

$$\delta u_T = \phi_k^T \quad , \quad \begin{matrix} T = 1,2,3 \\ k = 1, \ldots L^T \end{matrix}$$

whence

$$\delta u_{T,j} = \phi_{k,j}^T$$

and the ϕ_k^T must be continuous and satisfy homogeneous forced boundary conditions. If the co-ordinate functions ϕ_k^T $k = 1, \ldots L^T$ represented all possible degrees of freedom of the system then in Eq. 6.10 it would be allowable to

replace the δu_T with ϕ_k^T and the stress system would still correspond to the exact one. However as the co-ordinate functions in general do not represent all possible degrees of freedom of the system an approximate stress field $\bar{\sigma}_{Tj}$ is obtained which is defined by

$$\int_{\Omega} \bar{\sigma}_{Tj}\, \phi^T_{k,j}\, d\Omega + \int_{S_M} c\, \bar{u}_T\, \phi^T_k\, ds = \int_{\Omega} \rho X_T\, \phi^T_k\, d\Omega + \int_{S_T} T^\circ_T\, \phi^T_k\, ds + \int_{S_M} C^\circ_T\, \phi^T_k\, ds \qquad \begin{matrix} T = 1,2,3 \\ k = 1,\ldots L^T \end{matrix} \tag{6.11}$$

This equation is the same as Eq. 5.28. Thus for the type of equations considered the equations developed by virtual work correspond to those generated by the Galerkin procedure.

CHAPTER 7

FINITE ELEMENT SOLUTION OF A NON-SYMMETRIC PROBLEM

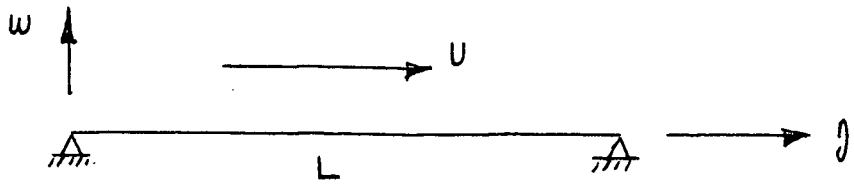
The problem considered is that of panel flutter as analysed by Olson (14) using finite elements. This analysis was achieved using virtual work principles and although no convergence proof was presented the accuracy of the results was justified by the agreement obtained in comparison with known solutions. The concepts presented herein enable the convergence of his analysis to be proven, as will be demonstrated in the following discussion.

The problem concerns the behaviour of a panel over which a supersonic air stream flows in the positive η direction. The specific question is that of determining the conditions under which the motion of the panel becomes unstable. The panel is assumed to be a stress free plate and only one dimensional deformations are considered. Neglecting the effect of the air entrapped below the panel

the differential equation and boundary conditions governing infinitesimal motion of a simply supported panel are (14)

$$\frac{\partial^4 Y}{\partial x^4} + \frac{B\,\partial Y}{\partial x} + B\,\frac{M_\infty^2 - 2}{M_\infty^2 - 1}\,\frac{L}{U}\,\frac{\partial Y}{\partial t} + \frac{m L^4}{D}\,\frac{\partial^2 Y}{\partial t^2} = 0 \qquad (7.1a)$$

$$Y(0) = Y(1) = Y''(0) = Y''(1) = 0 \qquad (7.1b)$$



where m is the panel mass per unit area, q the dynamic pressure, U the freestream velocity, M_∞ the freestream Mach number, D the plate bending rigidity and $Y = w/L$, $x = \mathcal{I}/L$, $B = 2qL^3/D(M_\infty^2 - 1)^{1/2}$ are the non dimensional deflection, streamwise co-ordinate, and aerodynamic parameter, respectively.

Assuming a solution in the form

$$Y = y(x)\, e^{\alpha t} \qquad (7.2)$$

where, in general, α is a complex number $\alpha = \beta + i\theta$, Eq. 7.1 becomes

$$y'''' + B y' - \lambda y = 0 \qquad (7.3a)$$

$$y(0) = y(1) = y''(0) = y''(1) = 0 \tag{7.3b}$$

where λ is an eigenvalue of the form

$$\lambda = -B(L/U)[(M_\infty^2 - 2)/(M_\infty - 1)]\alpha - (mL^2/D)\alpha^2 \tag{7.4}$$

Thus the problem reduces to the determination of the eigenvalues of the non-symmetric Eqs. 7.3 using a finite element approximation.

First consider the simplified equation that is obtained if $B = 0$. Eqs. 7.3 then become

$$y'''' - \lambda y = 0 \tag{7.5a}$$

$$y(0) = y(1) = y''(0) = y''(1) = 0 \tag{7.5b}$$

In this case the operator A is given by

$$A = \frac{d^4}{dx^4} \tag{7.6}$$

Using Eq. 7.5b in can readily be verified that

$$(Au, v) = [u, v] = \int_0^1 u''v'' \, dx = [v, u] \tag{7.7}$$

whence the operator is symmetric. It is also positive bounded below as can be seen from the following development.

$$u'(x_1) - u'(x_2) = \int_{x_2}^{x_1} u''(t)\, dt$$

If $x_2 \leq x_1$,

$$(u'(x_1) - u'(x_2))^2 = \left(\int_{x_2}^{x_1} 1 \cdot u''(t)\, dt \right)^2$$

$$\leq (x_1 - x_2) \int_{x_2}^{x_1} (u''(t))^2 dt \leq \int_0^1 (u''(t))^2 dt$$

The same inequality results if $x_1 \leq x_2$. Thus

$$u'(x_1)^2 + u'(x_2)^2 - 2\, u'(x_1) u'(x_2) \leq \int_0^1 u''(t)^2 dt$$

$$\int_0^1 2\, u'(x)^2 dx - 2 \left(\int_0^1 u'(x)\, dx \right)^2 \leq \int_0^1 \int_0^1 dx_1\, dx_2 \int_0^1 u''(t)^2 dt$$

As the function u is in D_A $u(0) = u(1) = 0$, whence

$$\int_0^1 2\, u'(x)^2 dx \leq \int_0^1 u''(t)^2 dt \qquad (7.8)$$

Now

$$u(x) = \int_0^x u'(t)\, dt$$

whence

$$u(x)^2 = \left(\int_0^x 1 \cdot u'(t)\, dt \right)^2$$

$$\leq x \int_0^x u'(t)^2 dt \leq x \int_0^1 u'(t)^2 dt$$

$$\therefore \qquad \int_0^1 u(x)^2\,dx \;\leq\; \int_0^1 x\,dx \int_0^1 u'(t)^2\,dt$$

$$\therefore \qquad \int_0^1 u(x)^2\,dx \;\leq\; \frac{1}{2}\int_0^1 u'(t)^2\,dt \tag{7.9}$$

Eqs. 7.8 and 7.9 imply

$$\int_0^1 u(x)^2\,dx \;\leq\; \frac{1}{4}\int_0^1 u''(x)^2\,dx$$

Recall $(u,u) = \int_0^1 u^2\,dx$ and from Eq. 7.7 $(Au,u) = \int_0^1 (u'')^2\,dx$

Thus

$$(u,u) \;\leq\; \frac{1}{4}(Au,u)$$

and the operator is seen to be positive bounded below.

The equations governing a finite element solution of Eqs. 7.5 are given by Eq. 4.29 if K is the identity operator. For this problem they become

$$\sum_{k=1}^{N} a_k^e \int_0^1 \left(\phi_k^{e\,\prime\prime}\,\phi_j^{e\,\prime\prime} - \lambda\,\phi_k^e\,\phi_j^e \right) dx = 0, \qquad \begin{matrix} j = 1,\ldots N \\ e = 1,\ldots E \end{matrix} \tag{7.10}$$

in which an approximate solution for y within each element has been assumed to be

$$\bar{y}^e = \sum_{k=1}^{N} a_k^e\,\phi_k^e \tag{7.11}$$

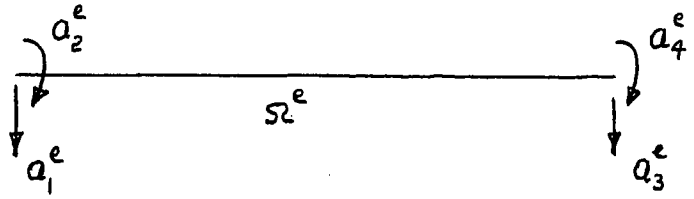
Eq. 7.10 corresponds to that developed in (14) using virtual work principles. The required eigenvalues are obtained by setting the coefficient matrix of Eq. 7.10 equal to zero.

From the results of Chapter 4 convergence of the eigenvalues and eigenvectors in H_A is ensured if the co-ordinate functions are complete in H_A . The energy product in H_A is given by Eq. 7.7 and involves derivatives of order two. Thus functions in H_A must have generalized second derivatives and hence continuous first derivatives. The finite element approximation must therefore ensure slope continuity across element boundaries and be such that the forced boundary conditions are satisfied. Such elements will then give a solution that converges to the correct answer if they are complete in H_A . A sufficient condition for completeness (see Chapter 4) is that the element approximations be based upon a complete polynomial of order not less than two.

In the above mentioned paper these conditions are all satisfied by choosing an element approximation that is based upon

$$\bar{y}^e = b_0^e + b_1^e x + b_2^e x^2 + b_3^e x^3 \qquad (7.12)$$

and by choosing degrees of freedom that correspond to deflections and rotations at each end of the element.



Thus convergence in H_A is ensured and the results obtained illustrating the monotonic convergence actually found are reproduced in Table 7.2. Such convergence could have been predicted from the convergence properties of the Rayleigh-Ritz procedure.

However in the case where the aerodynamic parameter B is non-zero the operator in Eq. 7.3 becomes

$$A = \frac{d^4}{dx^4} + B\frac{d}{dx} \tag{7.13}$$

which is non-symmetric and the Rayleigh-Ritz procedure is not applicable, whereas the Galerkin procedure may still be used. In order to be ensured of convergence Eq. 7.3a can be expressed in a form that corresponds to Eq. 3.43, for which convergence criteria are known, in the following manner.

$$y - \lambda T y = 0 \tag{7.14}$$

where

$$T = \left[\frac{d^4}{dx^4} + B \frac{d}{dx} \right]^{-1} \tag{7.15}$$

Further, in order to apply the results developed with respect to Eq. 3.43 it is necessary to show that T is completely continuous. To do this consider the equation

$$u'''' + B u' = f \tag{7.16}$$

under the boundary conditions 7.3b. f is some arbitrary function of x. Now if

$$u = \int_0^1 G(x, \xi) f(\xi) \, d\xi \tag{7.17}$$

then $G(x, \xi)$ is known as the Greèn's function corresponding to Eq. 7.16. If $G(x, \xi)$ is such that

$$\int_0^1 \int_0^1 G(x, \xi)^2 \, dx \, d\xi < \infty \tag{7.18}$$

then T is completely continuous in $L_2(\Omega)$ (9) since

$$u = Tf \tag{7.19}$$

It will be demonstrated that Eq. 7.18 is valid and thus that T is completely continuous.

The existence of a Green's function is ensured if Eq. 7.16 has a unique solution for arbitrary f, which is the case if the homogeneous equation has a unique solution. The general solution of the homogeneous equation can be found by assuming

$$u = C_m e^{mx} \tag{7.20}$$

which when substituted into the equation gives

$$C_m (m^3 + B) m = 0$$

$$m = 0, \ \frac{a}{2}(1 + i\sqrt{3}), \ -a, \ \frac{a}{2}(1 - i\sqrt{3})$$

where

$$a = \sqrt[3]{B} \quad , \quad i = \sqrt{-1}$$

Thus

$$u = C_1 + C_2 e^{-ax} + C_3 e^{a/2(1+i\sqrt{3})x} + C_4 e^{a/2(1-i\sqrt{3})x}$$

which may be written

$$u = D_1 + D_2 e^{-ax} + D_3 e^{ax/2} \cos\sqrt{3}\frac{ax}{2} + D_4 e^{ax/2} \sin\sqrt{3}\frac{ax}{2} \tag{7.21}$$

where the D's are constants and are combinations of the C's. Applying the boundary conditions leads to a set of four simultaneous equations for the D, which was shown numerically to have a unique solution for all B of interest. Table 7.1 illustrates the relationship between

AERODYNAMIC PARAMETER B	DETERMINANT VALUE
614.12	-69,121.
512.00	-41,487.
421.87	-24,690.
343.00	-14,573.
274.62	- 8,556.
216.00	- 5,034.
166.37	- 3,006.
125.00	- 1,850.
91.12	- 1,190.
64.00	- 800.
42.87	- 556.
27.00	- 388.
8.00	- 167.
1.00	- 42.
0.12	- 10.

TABLE 7.1
DETERMINANT VALUE VERSUS AERODYNAMIC PARAMETER

the determinant of the matrix and B . Thus Eq. 7.16 has a unique solution for arbitrary f ; whence the Green's function exists. Further the Green's function for the problem is constructed, by definition, such that it satisfies the following conditions.

$$\frac{\partial^4 G(x,\xi)}{\partial x^4} + B\frac{\partial G(x,\xi)}{\partial x} = 0, \quad 0 < x < \xi \tag{7.22a}$$

$$G(0,\xi) = G''(0,\xi) = 0 \tag{7.22b}$$

$$\frac{\partial^4 G(x,\xi)}{\partial x^4} + B\frac{\partial G(x,\xi)}{\partial x} = 0, \quad \xi < x < 1 \tag{7.22c}$$

$$G(1,\xi) = G''(1,\xi) = 0 \tag{7.22d}$$

Further $G(x,\xi)$, $G'(x,\xi)$, $G''(x,\xi)$ must be continuous at $x = \xi$ and

$$\lim_{\epsilon \to 0}\left[\left.\frac{\partial^3 G(x,\xi)}{\partial x^3}\right|_{x=\xi+\epsilon} - \left.\frac{\partial^3 G(x,\xi)}{\partial x^3}\right|_{x=\xi-\epsilon}\right] = 1 \tag{7.22e}$$

The continuity of the general solution given in Eq. 7.21 and the above definitions ensure the continuity of the Green's function for the problem. Therefore as a continuous function defined on a closed bounded interval is bounded it follows that Eq. 7.18 holds. Thus the operator T is completely continuous in the space of functions that are

square summable over $[0,1]$, and thus it is completely continuous in H_A.

Choosing co-ordinate functions that are complete in H_A, as Olson did, thus ensures the convergence of the eigenvalues and eigenvectors of Eqs. 7.3 when the governing equations are generated by the Galerkin or virtual work procedures.

The solution of the problem is characterized by the fact that the two lowest eigenvalues are real and distinct for all values of B less than the critical value B_{CR} which first causes flutter; but they approach each other and coalesce to λ_{CR} when $B = B_{CR}$. Table 7.3 reproduces the results obtained and demonstrates the convergence obtained for B_{CR} and λ_{CR}. In this case it is noted that the convergence is not monotonic.

The required Galerkin equations are obtained directly from application of the procedure to Eq. 7.3a. In terms of a finite element approximation the equations are

$$\sum_{k=1}^{N} a_k^e \int_0^1 \left(\phi_k^{e\,\prime\prime} \phi_j^{e\,\prime\prime} + B \phi_k^{e\,\prime} \phi_j^e - \lambda \phi_k^e \phi_j^e \right) dx = 0, \quad \begin{matrix} k=1,\ldots N \\ e=1,\ldots E \end{matrix} \qquad (7.23)$$

which correspond to those derived by Olson on the basis of virtual work principles.

Total Number of Elements	First Eigenvalue		Second Eigenvalue	
	λ_1	% Error	λ_2	% Error
1	120.00	23.6	2520.00	61.5
2	98.18	0.79	1920.00	23.1
3	97.57	0.16	1595.61	2.38
4	97.46	0.05	1570.87	0.79
Exact Value	97.41		1558.55	

TABLE 7.2, EIGENVALUE RESULTS WHEN B=0

Total Number of Elements	B_{CR}		λ_{CR}	
	B_{CR}	% Error	λ_{CR}	% Error
1	453.56	32.10	1320.00	25.50
2	398.54	16.07	1206.31	14.69
3	340.72	- 0.77	1027.85	- 2.28
4	342.34	- 0.30	1043.46	- 0.79
Exact Value	343.36		1051.80	

TABLE 7.3, EIGENVALUE COALESCENCE RESULTS

CHAPTER 8

A FINITE ELEMENT SOLUTION OF THE LINEAR VISCOUS FLOW PROBLEM

In this chapter a finite element solution will be developed for the two dimensional flow of an incompressible viscous fluid in which inertial effects may be neglected in comparison to viscous effects. The development further illustrates the application of the Galerkin procedure to problems involving more than one dependent variable.

A programme is written that generates a solution in the H_A space, which is based upon minimum convergence requirements. Comparison with known solutions are presented that illustrate the convergence obtained.

8.1 Generation of Equations Governing a Finite Element Solution

The equations governing the flow of an incompressible viscous fluid are

(i) Equilibrium

$$-\sigma_{ij,j} = \rho X_i - \rho \frac{Dv_i}{Dt} \quad \in \Omega \tag{8.1a}$$

$$\sigma_{ij} n_j = T_i^{\circ} \quad \in S_T \tag{8.1b}$$

(ii) Constitutive

$$\sigma_{ij} = 2\mu d_{ij} - p\delta_{ij} \quad \in \Omega \tag{8.2}$$

(iii) Compatibility

$$d_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}) \quad \in \Omega \tag{8.3a}$$

$$-v_{i,i} = 0 \quad \in \Omega \tag{8.3b}$$

$$v_i = v_i^{\circ} \quad \in S_v \tag{8.3c}$$

in which v_i is the velocity, D/Dt the material derivative, ρ the density, d_{ij} the strain rate tensor, μ the coefficient of viscosity, p the hydrostatic pressure, and S_v that portion of the boundary on which the velocity is specifies. Eq. 8.3b specifies that the flow be incompressible.

Formulating the problem in terms of the velocity and pressure gives the well known Navier-Stokes equations:-

$$-\mu(v_{i,jj} + v_{j,ij}) + p_{,i} = \rho X_i - \rho \frac{Dv_i}{Dt} \tag{8.4}$$

together with the constraint of incompressibility. If the ratio of the inertial forces to the viscous forces is small compared to unity then the system of equations to be solved can be reduced to

$$-\mu\left(V_{i,jj} + V_{j,ij}\right) + p_{,i} = \rho X_i \quad \epsilon\ \Omega \tag{8.5a}$$

$$-V_{j,j} = 0 \quad \epsilon\ \Omega \tag{8.5b}$$

$$\sigma_{ij}\, n_j = T_i^{o} \quad \epsilon\ S_T \tag{8.5c}$$

$$V_i = V_i^{o} \quad \epsilon\ S_V \tag{8.5d}$$

A detailed discussion of the range of applicability of these equations may be found in Schlichting (15).

Eqs. 8.5 may be placed into correspondence with the general system of equations.

$$A_{mn}\, \omega_n = f_m \qquad m, n = 1, 2, 3 \tag{8.6}$$

where $$\omega_n = \langle \omega_1, \omega_2, \omega_3 \rangle^T = \langle V_1, V_2, p \rangle^T \tag{8.7}$$

$$f_n = \langle f_1, f_2, f_3 \rangle^T = \langle \rho X_1, \rho X_2, 0 \rangle^T \tag{8.8}$$

Throughtout this section repeated subscripts n, m will be summed 1, 2, 3 and repeated subscripts i, j will be summed 1, 2.

Note that Eq. 8.5b is included as one of the equations to which the Galerkin procedure is to be applied. In this case the residual is a vector V_m with three components which are

$$V_m = A_{mn}\,\omega_n - f_m \qquad m = 1, 2, 3 \tag{8.9}$$

The Galerkin procedure then specifies that this residual vector should be made orthogonal to arbitrary variations of the assumed vector $\bar{\omega}_n$. The required equations are

$$(V_m, \bar{\omega}_m) = (V_1, \bar{\omega}_1) + (V_2, \bar{\omega}_2) + (V_3, \bar{\omega}_3) = 0$$

where $\bar{\omega}_m$ is arbitrary and therefore this equation may be written

$$(V_T, \bar{\omega}_T) = \int_{\Omega} (A_{Tn}\,\bar{\omega}_n - f_T)\,\bar{\omega}_T \, d\Omega = 0 \qquad T = 1, 2, 3 \tag{8.10}$$

where $\bar{\omega}_T$ is an arbitrary variation of the Tth component of $\bar{\omega}_n$.

Once again it is convenient not to interpret $A_{mn}\omega_n$ in terms of Eqs. 8.5 but instead to use

$$A_{Tn}\,\omega_n = -\sigma_{Tj,j} \qquad T = 1, 2 \tag{8.11a}$$

$$A_{Tn}\,\omega_n = -V_{j,j} \qquad T = 3. \tag{8.11b}$$

The energy product for this matrix of operators with respect to homogeneous boundary conditions is

$$(A_{mn}\,\omega_n\,,\,\omega_m') \;=\; \int_{\Omega} (-\sigma_{ij,j}\,v_i' - p'\,v_{j,j})\,d\Omega \tag{8.12}$$

in which σ_{ij} is the stress tensor corresponding to ω_n and

$$\omega_m' \;=\; \langle\, v_1',\; v_2',\; p' \,\rangle^T \tag{8.13}$$

Rearranging, applying Gauss' theorem and making use of the symmetry of σ_{ij} gives

$$(A_{mn}\,\omega_n\,,\,\omega_m') \;=\; \int_{\Omega} (\sigma_{ij}\,d_{ij}' - p'\,v_{j,j})\,d\Omega \;-\; \int_{S} \sigma_{ij}\,n_j\,v_i'\,dS$$

$$= \int_{\Omega} (2\mu\,d_{ij}\,d_{ij}' - p\,v_{j,j}' - p'\,v_{j,j})\,d\Omega$$

$$= \;(A_{mn}\,\omega_n'\,,\,\omega_m)$$

Thus the operator is symmetric and therefore a variational formulation exists, as has been developed by Johnson (5). However the introduction of such a functional is unnecessary as the following derivation illustrates.

From the symmetry proof the energy product of the operator is

$$[\omega_n, \omega_n']_A = \int_{\Omega} (2\mu d_{ij} d_{ij}' - p v_{j,j}' - p' v_{j,j}) d\Omega \qquad (8.14)$$

The appropriate finite element equations can be derived from Eq. 5.27 by assuming that the approximate solution within each element is

$$\bar{\omega}_n^e = \langle \bar{\omega}_1^e, \bar{\omega}_2^e, \bar{\omega}_3^e \rangle^T = \langle \bar{v}_1^e, \bar{v}_2^e, \bar{p}^e \rangle^T$$

where

$$\bar{v}_T^e = \sum_{k=1}^{N} a_{Tk}^e \phi_k^{eT} \qquad T = 1, 2. \qquad (8.15a)$$

$$\bar{p}^e = \sum_{k=1}^{P} a_{3k}^e \phi_k^{e3} \qquad (8.15b)$$

Then using Eq. 8.14 it can be seen that

$$[\bar{\omega}_T^e, \phi_k^{eT}]_A^e = \int_{\Omega^e} \left(\mu (\bar{v}_{T,j}^e + \bar{v}_{j,T}^e) \phi_{k,j}^{eT} - \bar{p}^e \phi_{k,T}^{eT} \right) d\Omega \qquad T = 1, 2. \qquad (8.16a)$$

$$[\bar{\omega}_3^e, \phi_k^{e3}]_A^e = \int_{\Omega^e} - \bar{v}_{j,j}^e \phi_k^{e3} d\Omega \qquad (8.16b)$$

The required equations, from Eq. 5.27, may be written

$$\int_{\Omega^e} \left(\mu (\bar{v}_{T,j}^e + \bar{v}_{j,T}^e) \phi_{k,j}^{eT} - \bar{p}^e \phi_{k,T}^{eT} \right) d\Omega = \int_{\Omega^e} \rho X_T \phi_k^{eT} d\Omega + \int_{S_T} T_T^o \phi_k^{eT} ds \qquad (8.17a)$$

$T = 1, 2$
$e = 1, \ldots E$
$k = 1, \ldots N$

$$\int_{\Omega^e} -\bar{v}^e_{j,j}\, \phi^{e3}_k \, d\Omega = 0 \qquad \begin{matrix} e = 1,\ldots E \\ k = 1,\ldots P \end{matrix} \qquad (8.17b)$$

Convergence of the approximate solution generated by Eqs. 8.17 is ensured if the co-ordinate functions are complete in H_A . From Eq. 8.14 it can be seen that the energy product involves derivatives of v_1 and v_2 of order one, and zero order derivatives of p . Denote the maximum order derivative of ω_τ (the τ component of the unknown function ω), occuring in the energy product by m_τ ; then $m_1 = m_2 = 1$, $m_3 = 0$. Oliveira's work proves completeness if the elements ensure continuity of the $m_\tau - 1$ derivatives and the approximation for each component within each element is based upon a complete polynomial of degree not less then m_τ . Thus it is sufficient that the co-ordinate functions for $\omega_1 = v_1$ and $\omega_2 = v_2$ be based upon a first order polynomial and ensure continuity of the velocity. The co-ordinate functions for $\omega_3 = p$ need only be based upon a zero order polynomial. Thus the pressure approximation within each element may be a constant, and need not ensure any continuity between elements.

Further it can be noted that boundary conditions that involve the pressure or first derivative of the velocity are the natural boundary conditions. Hence the boundary

condition on S_V is the only forced boundary condition. The approximate solution $\bar{\omega}_n$, or more precisely the components $\bar{V}_1$ and $\bar{V}_2$, must therefore satisfy this forced boundary condition.

If a complete set of functions is chosen from H_A^o then as in Chapter 5.2 it is not necessary to express the energy product in its symmetric form. In this case from Eq. 8.12 it can be seen

$$\left[\omega_n , \omega_n' \right]_A^o = \int_{\Omega} \left(- \sigma_{ij,j} v_i' - p' v_{j,j} \right) d\Omega \tag{8.18}$$

whence

$$\left[\bar{\omega}_T^e , \phi_k^{eT} \right]_A^{eo} = \int_{\Omega^e} \bar{\sigma}_{Tj,j}^e \phi_k^{eT} \, d\Omega \qquad T = 1, 2 \tag{8.19a}$$

$$\left[\bar{\omega}_3^e , \phi_k^{e3} \right]_A^{eo} = \int_{\Omega^e} - \bar{V}_{j,j} \phi_k^{e3} \, d\Omega \tag{8.19b}$$

The required equations for a solution in H_A^o are then obtained from Eq. 5.30:

$$\int_{\Omega^e} \bar{\sigma}_{Tj,j}^e \phi_k^{eT} \, d\Omega = \int_{\Omega^e} \rho X_T \phi_k^{eT} \, d\Omega \qquad \begin{matrix} T = 1, 2 \\ e = 1, \ldots E \\ k = 1, \ldots N \end{matrix} \tag{8.20a}$$

$$\int_{\Omega^e} - \bar{V}_{j,j}^e \phi_k^{e3} \, d\Omega = 0 \qquad \begin{matrix} e = 1, \ldots E \\ k = 1, \ldots P \end{matrix} \tag{8.20b}$$

For a finite element solution in H_A^o the elements must ensure continuity of the pressure and of the first derivatives of the velocity. The development of an element to satisfy the pressure requirement presents no problem; to derive an element that satisfies the velocity requirements is somewhat more complicated but such an element has been developed for use in the analysis of plate bending problems (1). Thus the generation of a solution is certainly feasible in this case. However such a solution will not be generated in this thesis.

8.2 Development of a Finite Element Model

In this section a finite element model is developed in H_A assuming zero body forces. The required equations are given in Eqs. 8.17, omitting the body force term.

The minimum conditions for completness presented in the previous section are satisfied by assuming

$$\bar{V}_T^e = b_{T1}^e + b_{T2}^e x + b_{T3}^e y \qquad T = 1,2 \qquad (8.21a)$$

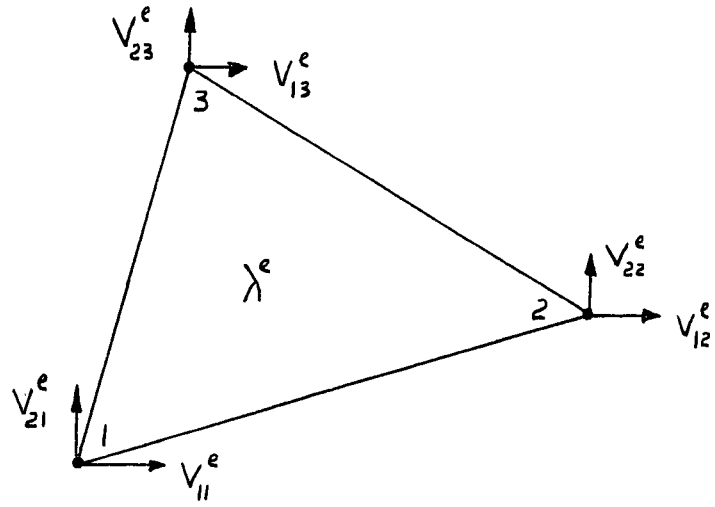
$$\bar{P}^e = c^e \ (const.) \qquad (8.21b)$$

Which in the notation of Eqs. 8.15 is

$$\bar{V}_T^e = \sum_{k=1}^{3} V_{Tk}^e \, \phi_k^{eT} \qquad T = 1, 2$$

$$\bar{p}^e = -\lambda^e \quad (\text{const.})$$

where V_{Tk}^e is the velocity in direction T at node k and λ^e is the pressure in element e, assumed constant. The required condition of continuity can then be satisfied by assuming a triangular element with six degrees of freedom for the velocity and one degree of freedom for the pressure. This element is shown below.

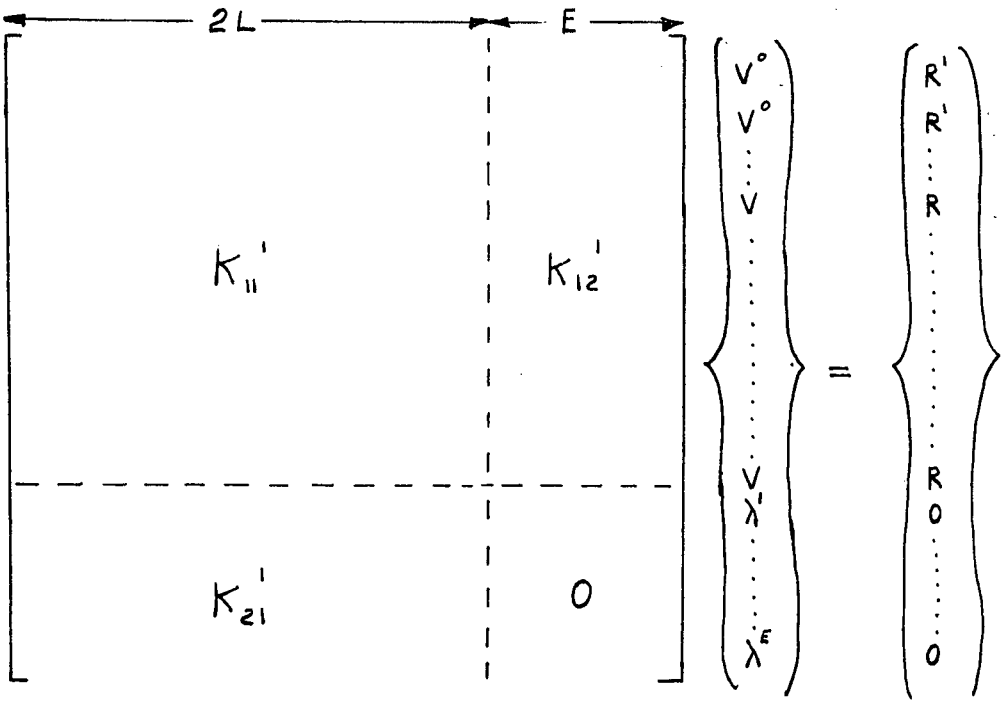


Note that the location of the nodes and the linearity of the assumed velocity field ensure that the velocity is continuous across element boundaries and thus throughout the whole domain as required.

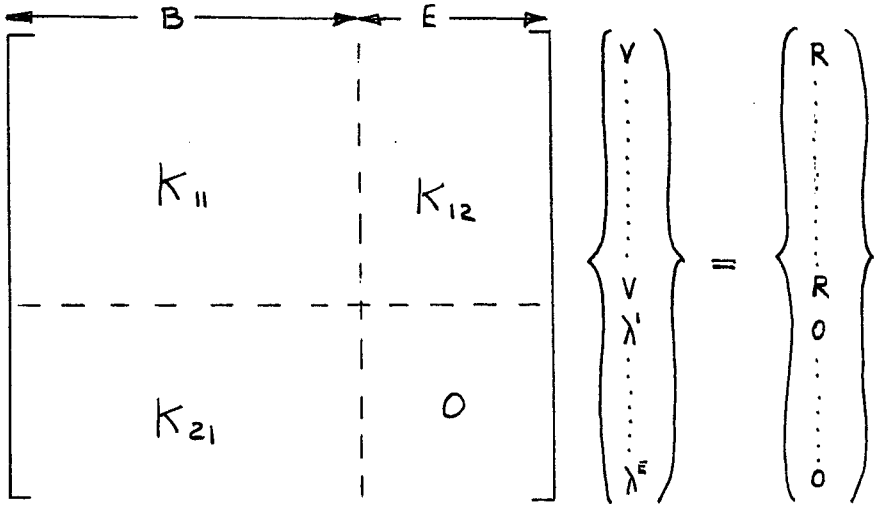
Although the element would appear satisfactory, it is in fact not suitable for problems in which a

relatively large number of velocities are prescribed. The difficulty stems from the incompressibility condition which constrains the nodal velocities in each element. If the domain is subdivided into E elements there are then E constraint equations for the 2(E+2) velocity degrees of freedom. Thus there are E+4 degrees of freedom remaining to satisfy the equilibrium condition. Therefore if the number of prescribed velocities exceeds (E+4) no solution can exist.

Equivalently, the difficulty can be seen by considering the equations that must be solved to obtain the required nodal velocities. The assembled equations are obtained from Eqs. 8.17 and they may be expressed in the form



where the R' correspond to prescribed velocities and are unknown. If F velocities must be prescribed to satisfy the forced boundary conditions then this system of equations may be reduced to



(8.22)

where B = 2L-F. In order for a unique solution to exist for the nodal velocities and element pressures the matrix K defined by

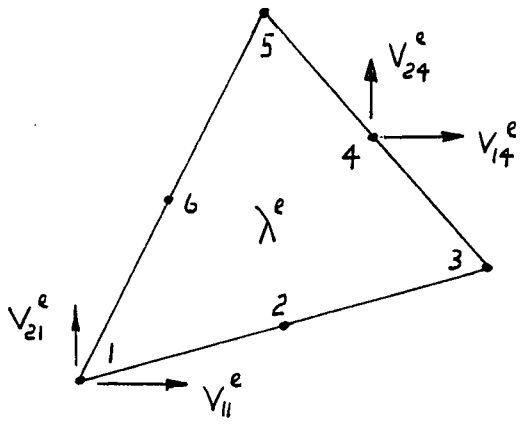
$$K = \left[\begin{array}{c:c} K_{11} & K_{12} \\ \hdashline K_{21} & 0 \end{array}\right]$$

must have a non-zero determinant. Thus all its rows must be linearly independent, which means $B \geq E$ i.e. $2L-F \geq E$. But $L=E+2$ whence $E+4 \geq F$ becomes a necessary condition for a unique solution to exist. Which of course is the same conclusion that was obtained by the more physical argument presented above.

To avoid this difficulty it is advisable to modify the element. To do this assume a velocity field that is based upon a second order polynomial, i.e.

$$\bar{V}_T^e = \sum_{k=1}^{6} V_{Tk}^e \, \phi_k^{2T} \qquad T = 1, 2 \tag{8.23}$$

and a triangular element with nodes at the mid point of the sides and at the corners. The pressure assumption remains unchanged.



In this case E elements result in 6(E+1) velocity degrees of freedom and the number of prescribed velocities must exceed 5E+1 for no solution to exist. Once again continuity of velocity is ensured, as the three nodal velocities on each side of the element uniquely determine the quadratic distribution assumed, but continuity of velocity gradient is not obtained.

The assumed velocity field is expressed directly in terms of area co-ordinates (3) which are characterized by the property that they automatically satisfy Eq. 4.4. Thus

$$\phi_k^{eT} = \langle \xi_1(2\xi_1-1),\ 4\xi_1\xi_3,\ \xi_3(2\xi_3-1),\ 4\xi_3\xi_5,\ \xi_5(\xi_5-1),\ 4\xi_5\xi_1 \rangle \quad (8.24)$$

where

$$\begin{Bmatrix} \xi_1 \\ \xi_3 \\ \xi_5 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} x_3y_5 - x_5y_3 & , & y_3-y_5 & , & x_5-x_3 \\ x_5y_1 - x_1y_5 & , & y_5-y_1 & , & x_1-x_5 \\ x_1y_3 - x_3y_1 & , & y_1-y_3 & , & x_3-x_1 \end{bmatrix} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} \quad (8.25)$$

in which A is the area of the triangle and (x_N, y_N) are the co-ordinates of node N. Fig. 8.1 further defines these co-ordinates.

Eqs. 8.17 therefore take the form

$$\int_{\Omega^e} \left(\mu \sum_{n=1}^{6} \left(V_{Tn}^e \phi_{n,j}^{eT} + V_{jn}^e \phi_{n,T}^{eT} \right) \phi_{k,j}^{eT} + \lambda^e \phi_{k,T}^{eT} \right) d\Omega = \int_{S_T} T_T^o \phi_k^{eT} ds \quad \begin{matrix} T=1,2 \\ k=1,\ldots 6 \\ e=1,\ldots E \end{matrix} \quad (8.26a)$$

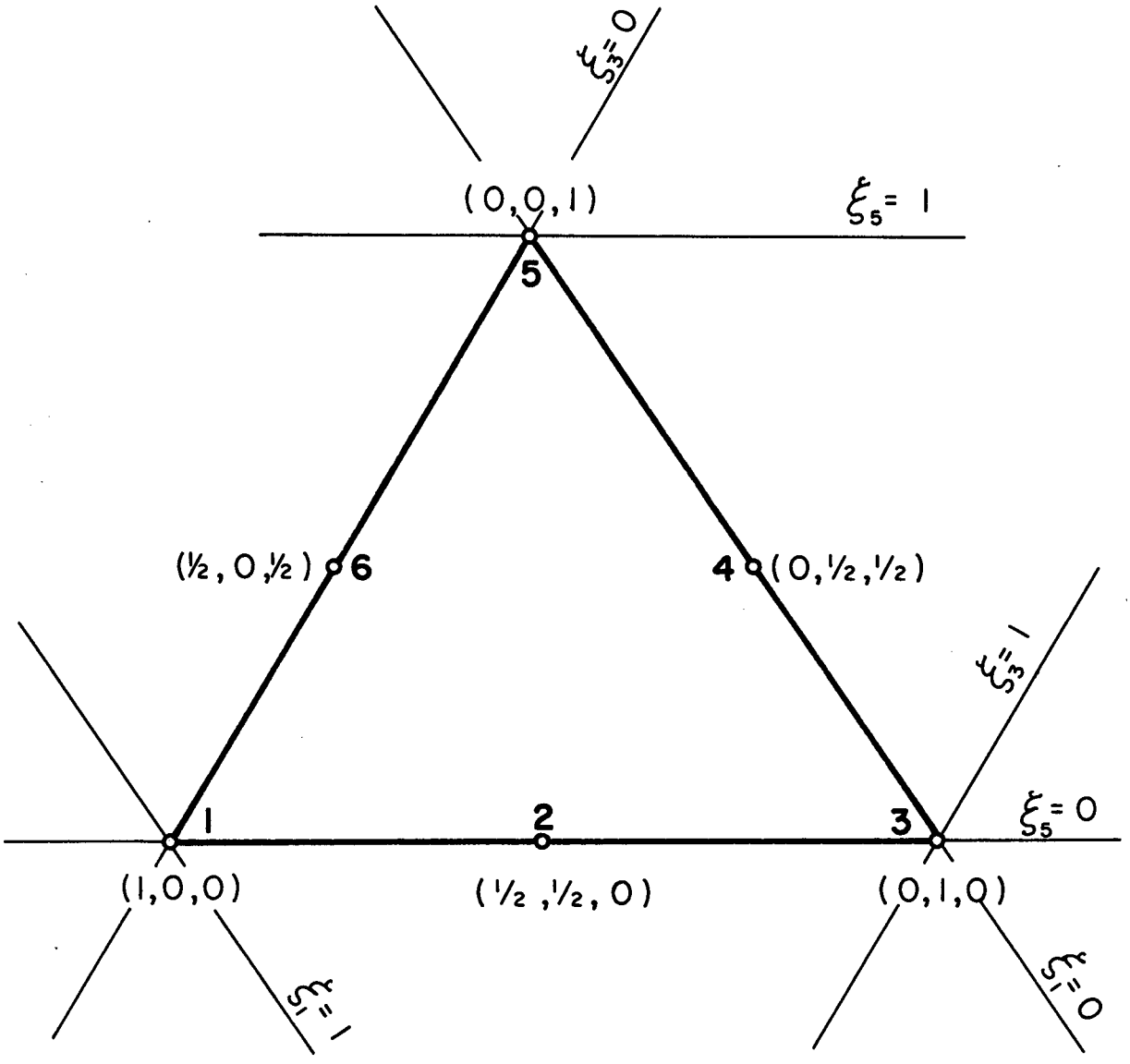


Fig. 8.1 DEFINITION OF AREA COORDINATES.

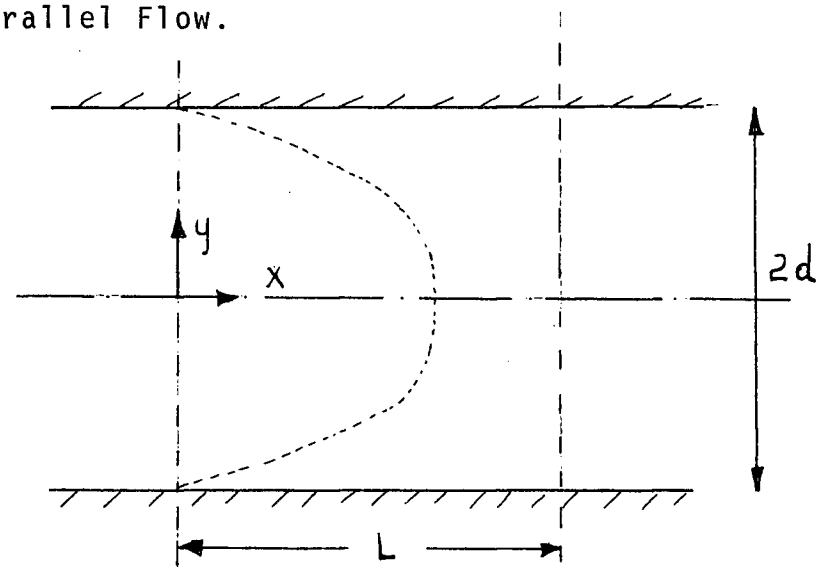
$$\int_{\Omega^e} \left(\sum_{n=1}^{6} V_{jn}^{e} \, \phi_{n,j}^{ej} \right) d\Omega = 0 \qquad e = 1, \ldots E \qquad (8.26b)$$

in which the $-\lambda^e$ has been cancelled from Eq. 8.26b. Eqs. 8.26 may now be assembled and solved for the unknown velocities and pressures. The programme used for this is included in Appendix 1.

8.3 Comparison with Known Solutions

Two problems are solved for which the exact solutions are known and the convergence of the finite element solution is illustrated. The first example concerns the flow of a viscous fluid between parallel stationary plates of infinite extent that are a distance $2d$ apart and the second, that situation when one of the plates is moving with respect to the other at constant velocity (Coueette flow). The exact solution for both these cases is presented by Schlichting (15) and these solutions are summarized below. In both cases the motion is such that the acceleration term in Eq. 8.4 is identically zero.

(1) Parallel Flow.



The boundary conditions are

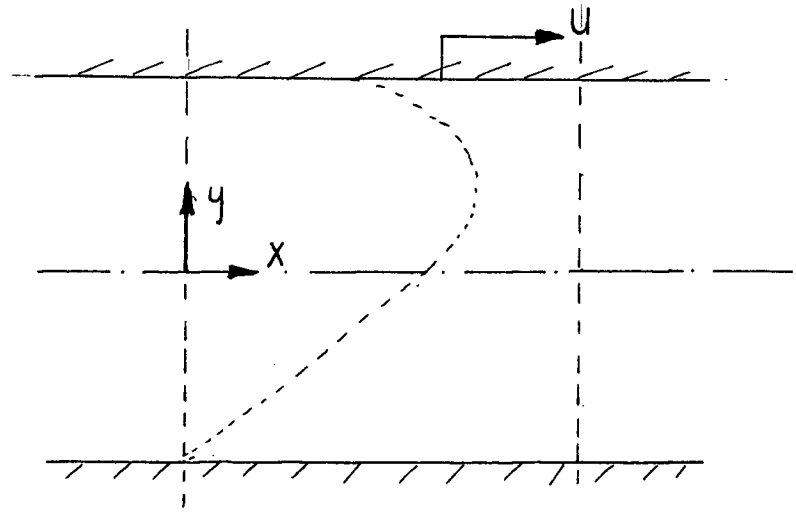
$$p(0, y) = p_1 \;, \quad p(L, y) = p_2 \;, \quad v(x, \pm d) = 0$$

The solution is

$$v(x,y) = -\frac{1}{2\mu}\,\frac{p_2 - p_1}{L}\,(d^2 - y^2)$$

$$\frac{dp}{dy} = 0 \qquad \frac{dp}{dx} = \frac{p_2 - p_1}{L}$$

(ii) Couette Flow.



The boundary conditions are

$$P(0,y) = P_1 \qquad P(L,y) = P_2$$

$$V(x,-d) = 0 \qquad V(x,d) = U$$

The solution is

$$V(x,y) = \frac{U}{2d}(y+d) - \frac{1}{2\mu}\frac{P_2-P_1}{L}(d^2-y^2)$$

$$\frac{dP}{dy} = 0 \qquad \frac{dP}{dx} = \frac{P_2-P_1}{L}$$

In the example considered the following values were used:

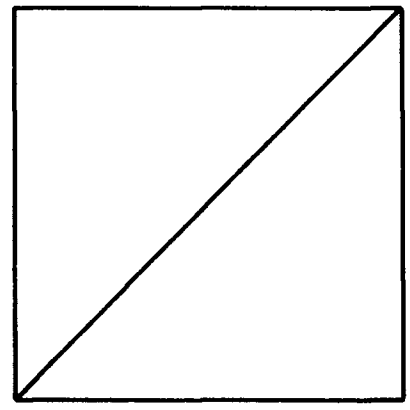
$$\mu = 20 \times 10^{-6} \text{ lbf sec/ft}^2$$
$$L = 2d = 2 \text{ ft.}$$
$$U = 5 \text{ f.p.s}$$
$$P_1 = 6 \times 10^{-4} \text{ p.s.f} \qquad P_2 = 2 \times 10^{-4} \text{ p.s.f.}$$

The approximate solutions corresponding to three different subdivisions of the domain were calculated. The subdivisions employed are illustrated in Fig. 8.2 and

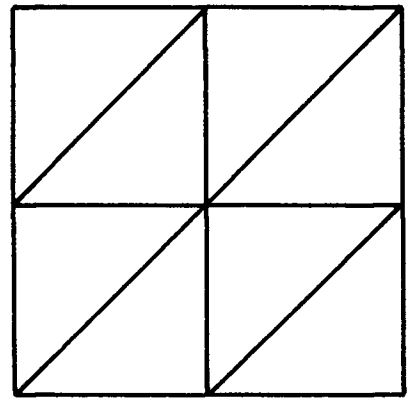
involve the use of 2, 8, and 32 elements. Figs. 8.3, 8.4, and 8.5 show the results obtained for the parallel flow problem for the different subdivisions of the domain. The resulting velocities are compared with the exact solution in Fig. 8.6. Two values of velocity are plotted for each subdivision corresponding to the different values obtained from nodes at the same height of the same triangle. In each case it can be seen that these values span the exact solution and with increasing number of elements both values converge to the exact solution. Similar results were obtained for the Couette flow problem and Fig. 8.7 illustrates the convergence obtained.

In both cases the exact velocity solution is a quadratic distribution. Thus as the assumed velocity field within each element is quadratic it would be expected that an accurate answer would be obtained. However, assuming the pressure to be a constant within each element whereas the exact solution is a linear distribution would appear a more realistic test of the convergence of the solution. Fig. 8.8 compares the exact solution with the pressures obtained on the assumption that the pressure acts at the centroid of its triangle. The pressure distribution for the parallel and Couette flow was found to be the same,

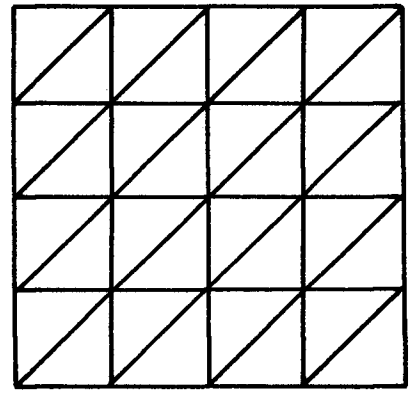
and excellent agreement was obtained with the exact solution even with the two element subdivision.



a) 2 Elements



b) 8 Elements



c) 32 Elements

Fig. 8.2 ASSUMED DOMAIN SUB-DIVISIONS.

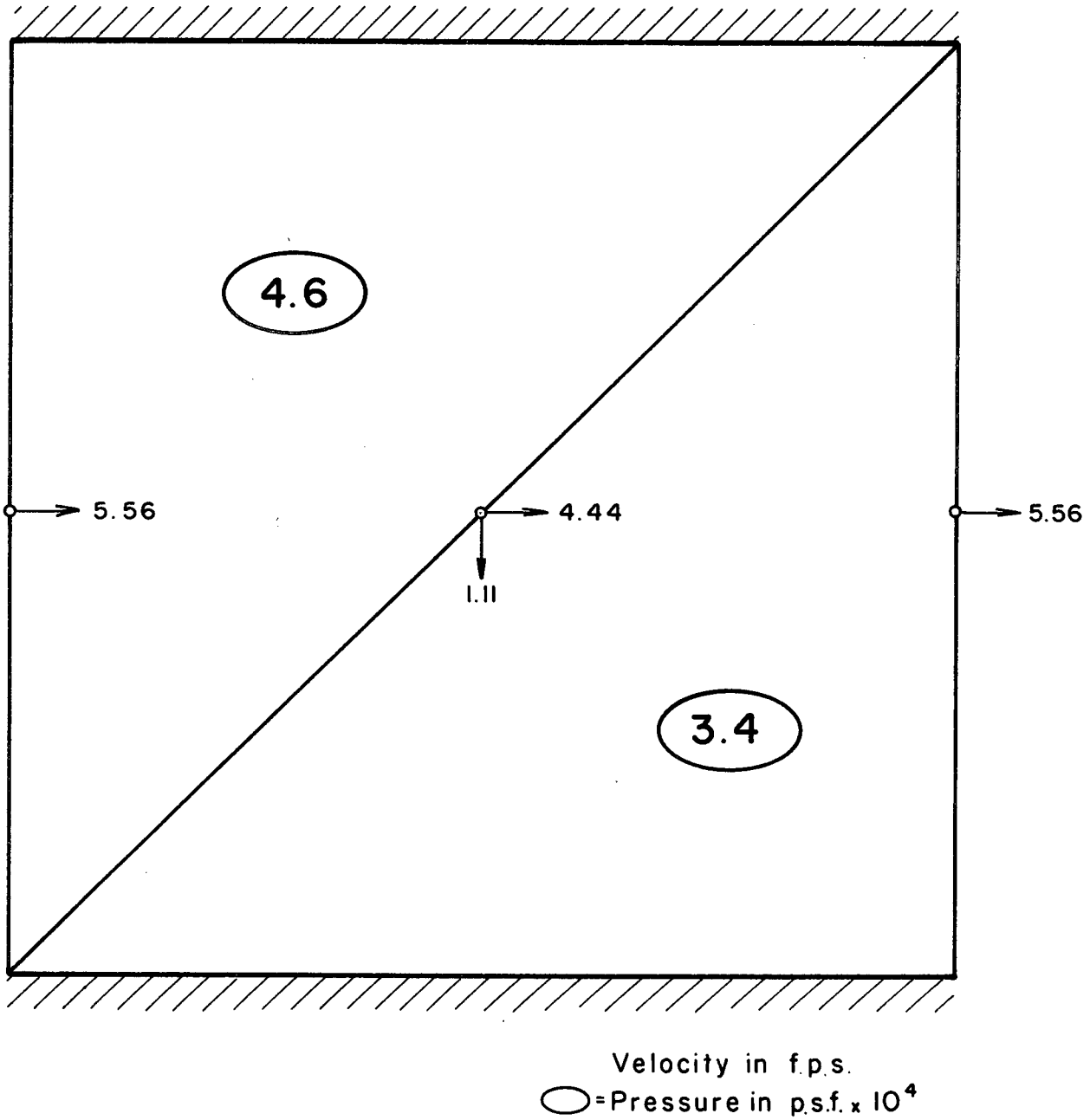


Fig. 8.3 2 ELEMENT PARALLEL FLOW SOLUTION.

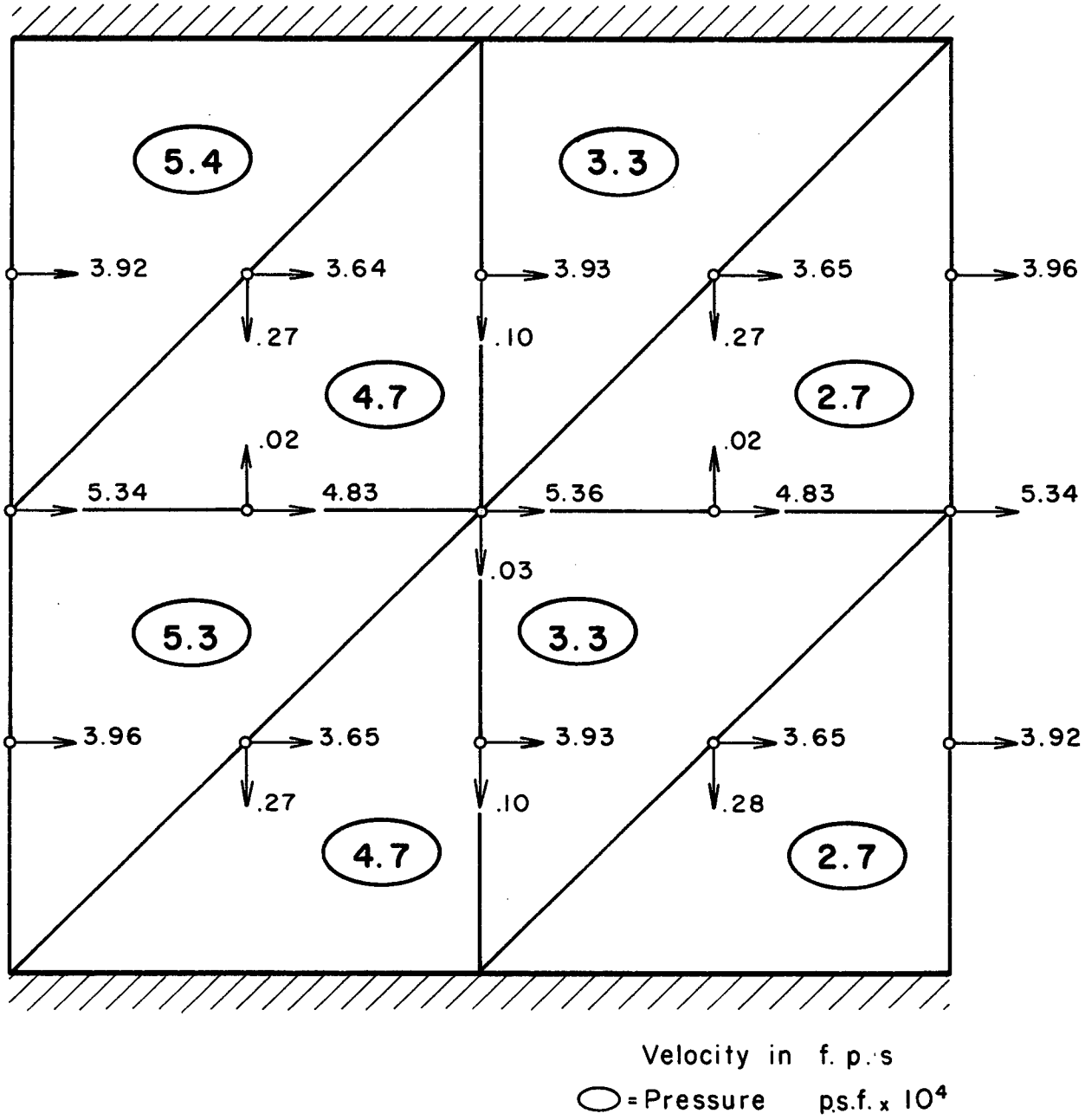


Fig. 8.4 8 ELEMENT PARALLEL FLOW SOLUTION.

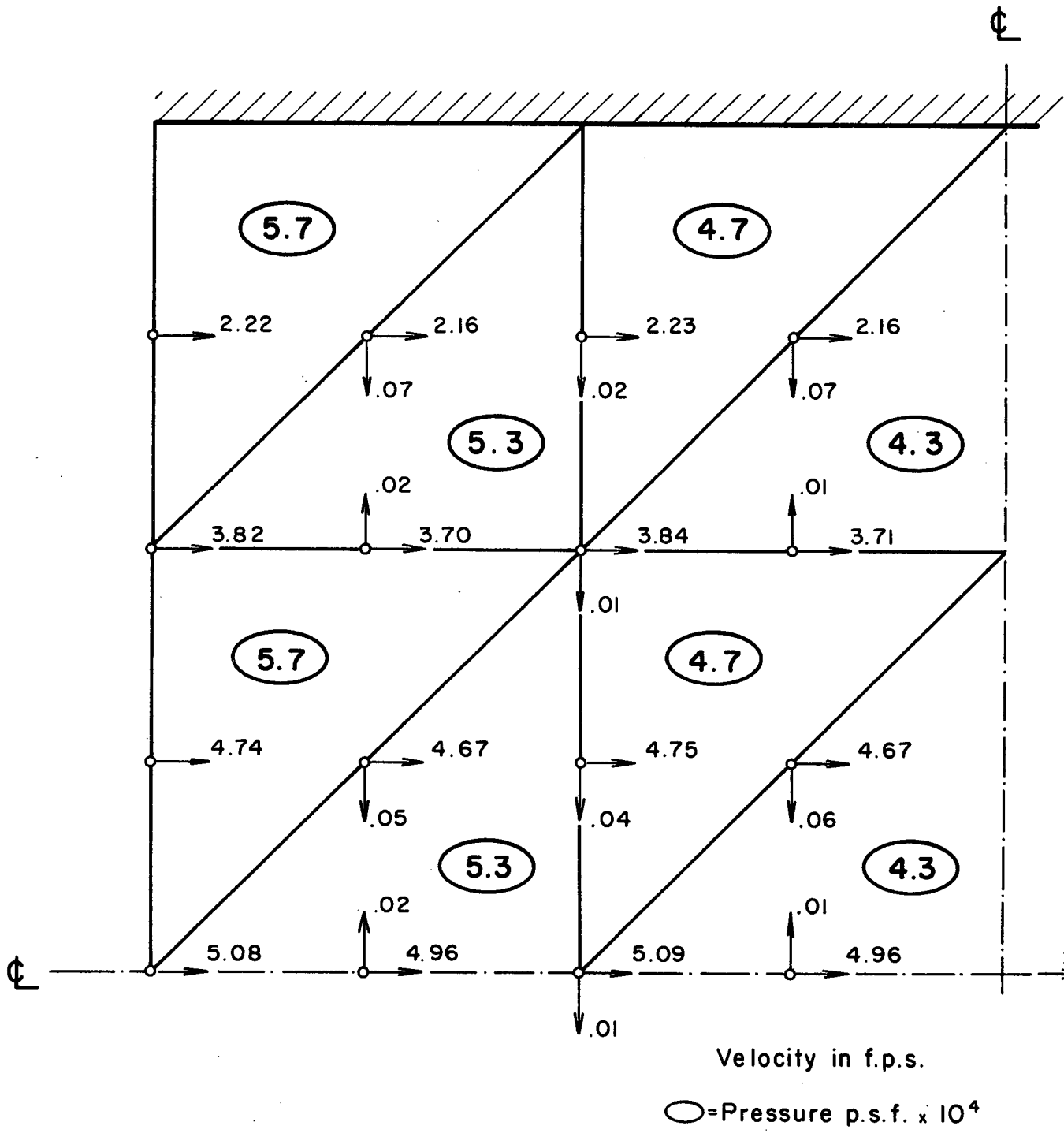


Fig. 8.5 32 ELEMENT PARALLEL FLOW SOLUTION.

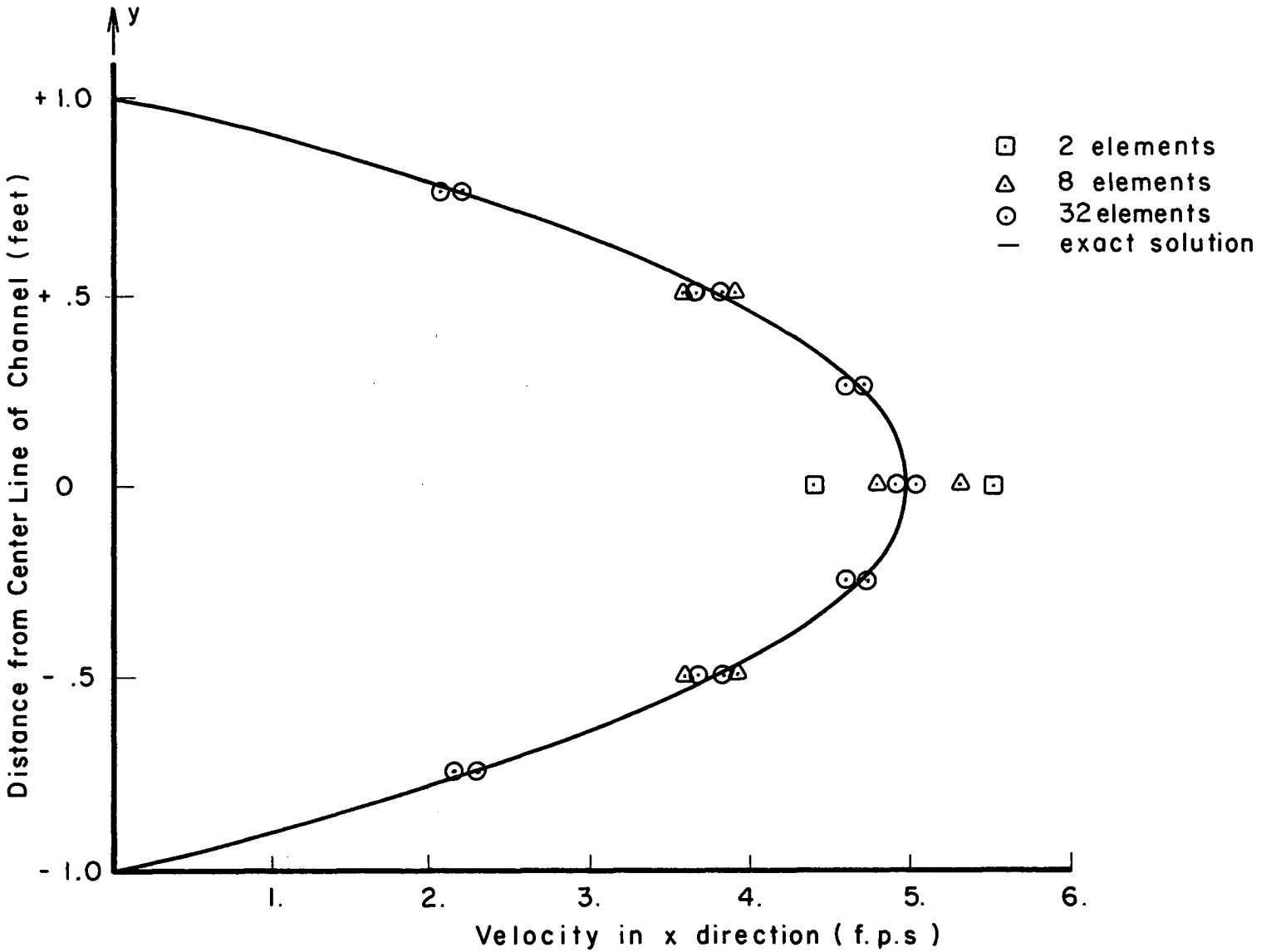


Fig. 8.6 VELOCITY DISTRIBUTION IN CHANNEL FOR PARALLEL FLOW.

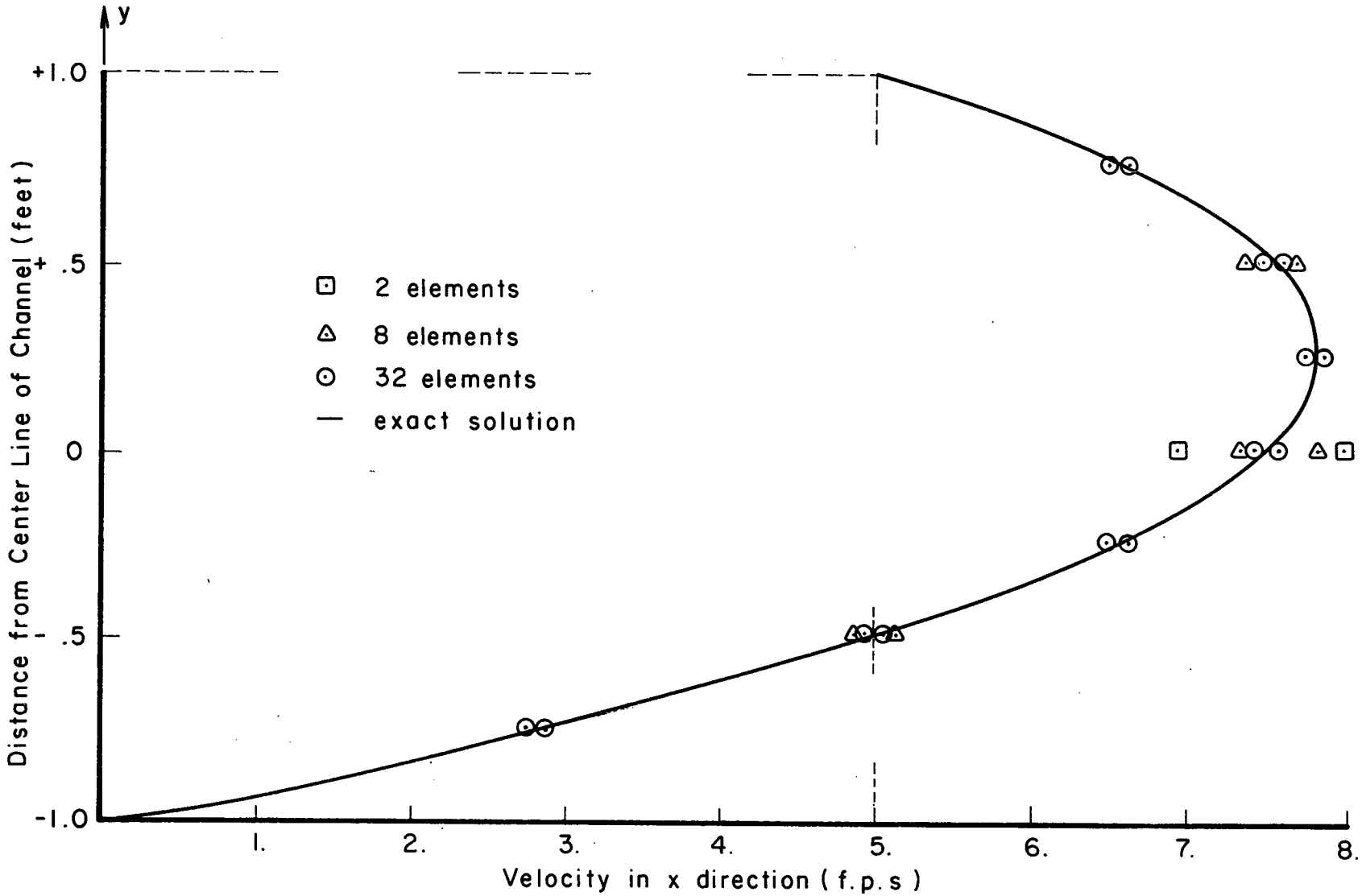


Fig. 8.7 VELOCITY DISTRIBUTION IN CHANNEL FOR COUETTE FLOW.

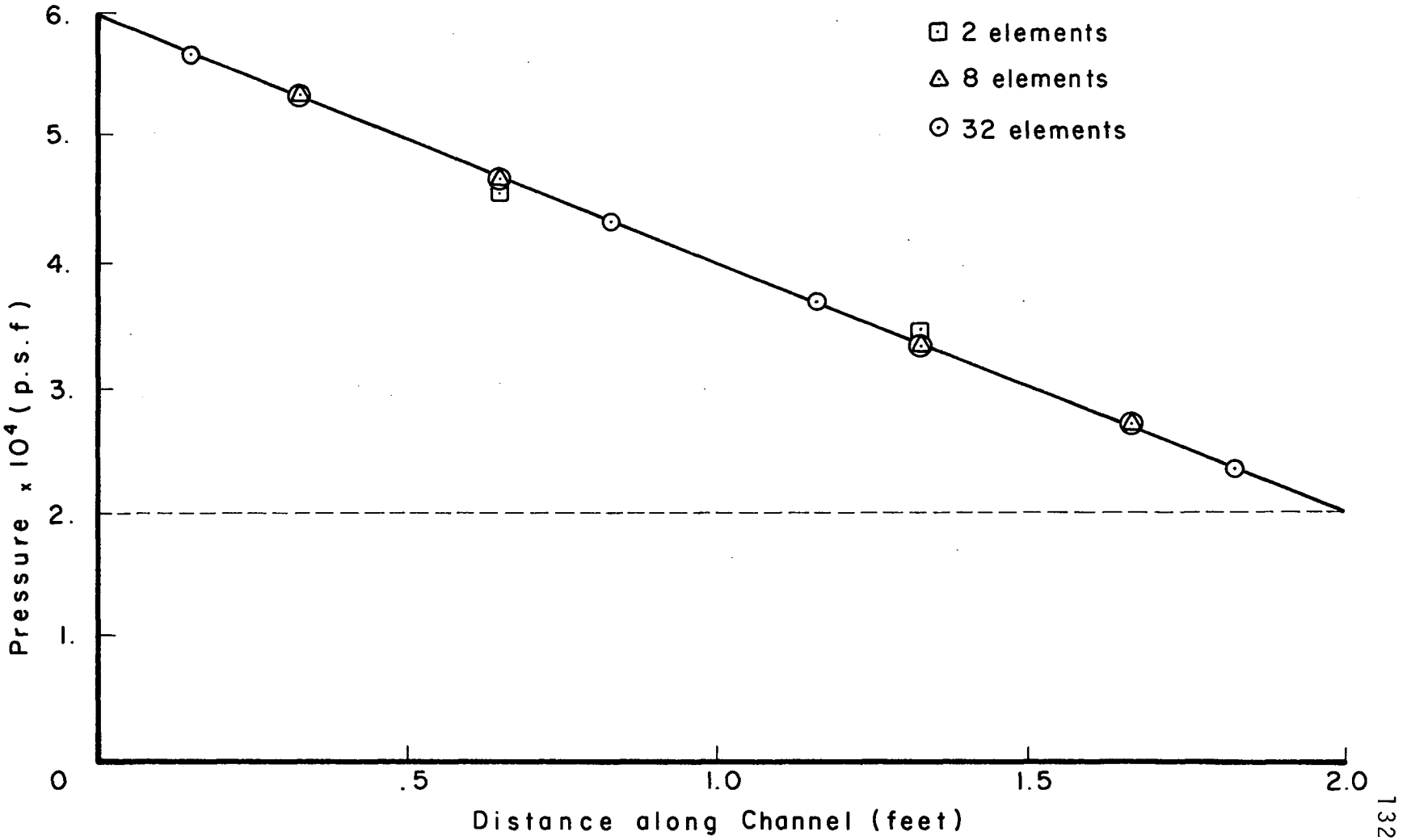


Fig. 8.8 PRESSURE DISTRIBUTION IN CHANNEL.

CHAPTER 9

SUMMARY

A mathematical framework for the finite element procedure has been presented that shows the method to be a technique for generating an approximate solution for a given equation in terms of known functions piece-wise defined over the domain, and unknown parameters. The unknown parameters, which are usually nodal values of the required functions or one of its derivatives, may be solved for in a number of different ways. Any technique that gives a solution that converges to the correct answer with increasing number of elements is equally valid. The Rayleigh-Ritz procedure and virtual work principles have commonly been used to generate the required equations. However, the Rayleigh-Ritz method is limited to those problems whose solution corresponds to the stationary value of a known functional and the virtual work approach does not yield convergence criteria.

Using results obtained by Mikhlin (9), it has been shown that the Galerkin procedure enables convergence

criteria to be stated for a wide class of problems. For those problems to which the Rayleigh-Ritz method is applicable the Galerkin method coincides with the Rayleigh-Ritz procedure, but there exist many problems for which Galerkin is applicable but Rayleigh-Ritz is not.

It is often assumed that Galerkin's method requires that the assumed co-ordinate functions satisfy all the boundary conditions; however, it is shown that by suitably formulating the problem it is often only necessary to satisfy the principal boundary conditions.

The boundary residual concept has been shown to lead to equations that are the same as those developed by the Galerkin procedure if the forced boundary conditions are homogeneous. If this condition is not satisfied the concept leads to incorrect equations. A corrected version of the boundary residual concept is presented which shows it to be equivalent to application of the virtual work principle. It is further demonstrated that deriving the equations governing the equilibrium configuration of a linear continuum using virtual work leads to the same equations as are obtained by solving the equilibrium equations using Galerkin's method.

The convergence proof of Olson's (14) finite element analysis of the panel flutter problem has been

presented. This proof illustrates the application of the Galerkin procedure to a problem for which the Rayleigh-Ritz method is inapplicable. The application of the Galerkin procedure to problems involving more than one dependent variable has been illustrated by generating a finite element solution for the linear viscous flow problem. The development indicates that even if a variational formulation of the problem does exist the Rayleigh-Ritz procedure is not necessarily the most advantageous way to develop the required equations. Numerical results obtained show excellent agreement with known solutions.

Attention in this thesis has been confined to the consideration of conforming elements, however the approach adopted enables the study of non-conforming elements to be undertaken in a systematic manner. Further, adopting the point of view proposed enables the convergence of a finite element approximation to be investigated for a far wider class of problems than have been considered herein. For example, the work of applied mathematicians concerning the convergence of Galerkin's method for non-linear equations becomes of immediate interest to those concerned with the finite element solution of non-linear problems.

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********** THIS JOB SUBMITTED THROUGH FRONT DESK READER **********
G OSGH
AST SIGNON WAS: 14:33:27   08-18-71
SER "OSGH" SIGNED ON AT  14:37:28 ON 08-18-71
ST *SOURCE*
    1       C     PROGRAM H2 FOR SLOW VISCOUS FLOW OF INCOMPRESSIBLE NEWTONIAN FLUID
    2       C    1USING QUADRATIC DISPLACEMENT FIELD AND 6 NODE TRIANGULAR ELEMENTS
    3             DIMENSION NP1(150),NP2(150),X(150),Y(150),NODE1(150),NODE2(150),NO
    4            1DE3(150),NODE4(150),NODE5(150),NODE6(150),TOA(100),SE(13,13),SK(90
    5            1,250), S(9000), VK(99),MU(100),MC(13),DELTA(160),X1(150),X2(150),X
    6            13(150),X4(150),X5(150),X6(150),Y1(150),Y2(150),Y3(150),Y4(150),Y5(
    7            1150),Y6(150),NODE7(150),A1(100),A5(100),A3(100),B1(100),B3(100),B5
    8            1(100),R(100),RK(300),RK1(150),RK2(150)
    9             REAL MU
   10             DOUBLE PRECISION S,DELTA,RATIO,SUM,DE
   11             COMMON/ZDET/DE,NCN
   12             COMMON/ZCON/COND
   13       C     WHEN NUMBERING SRTUCTURE 1)FIRST OR LAST DEGREE OF FREEDOM SHOULD
   14       C     NOT CORRESPOND TO PRESSURE,2)PRESSURE DEGREE OF FREEDOM SHOULD NOT
   15       C     BE THE SMALLEST NUMBER IN ITS TRIANGLE .GT. MFIX .BOTH LEAD TO A
   16       C     BLOW UP OF BANDI
   17       C     NODES NO. 123456 NOT 142536
   18       C     NO. FOR MIN. BAND WIDTH IGNORING FIXED DEGREES OF FREEDOM
   19       C     NFIX=NO.OF PRESCRIBED VEL. NOT = TO ZERO+ANY PRESCRIBED PRESSURE
   20       C     NFIXO=NO. OF VELOCITIES PRESCRIBED= TO 0
   21       C     IF NO PRESCRIBED FORCES NFORCE=0,OTHERWISE NFORCE=1.UNKNOWN FORCES
   22       C     READ IN AS ZERO FORCES
   23             READ(5,2)NELEM,NNODE,NFIX,NFIXO,NFORCE
   24       2     FORMAT(5I10)
   25             WRITE(6,4)NELEM,NNODE,NFIX,NFIXO,NFORCE
   26       4     FORMAT('NELEM = ',I10 ,'NNODE = ',I10,'NFIX = ',I10,' NFIXO ',I10,
   27            1'  NFORCE  ',I10)
   28             DO6 I=1,NELEM
   29             READ(5,5)NODE1(I),NODE2(I),NODE3(I),NODE4(I),NODE5(I),NODE6(I),NOD
   30            1E7(I)
   31       5     FORMAT(7I10)
   32       6     CONTINUE
   33       C     NODE 7 = PRESSURE IN ELEMENT (CANNOT BE SPECIFIED AS 1)
   34       C     SPECIFYING A NEGATIVE PRESSURE CORRESPONDS TO A COMPRESSIVE STRESS
   35             WRITE(6,8)
   36       8     FORMAT('ELEMENT NO.  NODE(1)  NODE(2)  NODE(3)  NODE(4)  NODE(5)
   37            1NODE(6),NODE7(I) ')
   38             DO10 I=1,NELEM
   39             WRITE(6,9)I,NODE1(I),NODE2(I),NODE3(I),NODE4(I),NODE5(I),NODE6(I),
   40            1NODE7(I)
   41       9     FORMAT(I7,I11,6I9)
   42       10    CONTINUE
   43       C     ONLY INTEGER VELOCITIES ALLOWED
   44             DO12 J=1,NNODE
   45             READ(5,11) NP1(J),NP2(J),X(J),Y(J),RK1(J),RK2(J)
   46       11    FORMAT(2I10,2F10.3,2E15.4)
   47       12    CONTINUE
   48       C     ONLY NEED SPECIFY X,Y FOR NODES 1,3,5 AS2,4,6 ARE MID POINTS
   49             WRITE(6,14)
   50       14    FORMAT(' NODE NO.      NP1          NP2      X        Y          X-FO
   51            1RCE        Y-FORCE')
   52             DO16 J=1,NNODE
```

```
      WRITE(6,15)J,NP1(J),NP2(J),X(J),Y(J),RK1(J),RK2(J)
15    FORMAT(I6,I11,I11,F8.2,F8.2,2E15.4)
16    CONTINUE
C     IF NODE IS FIXED AT 0 NP=0, IF NODE VEL.=V NP=V,IF NODE IS FREE
C    1NP=1
C     FOR PRESSURE DEGREES OF FREEDOM NP1=1,NP2=-1
      MFIX=NFIXO+NFIX
      N=0
      NFIX1=NFIX+1
      NU=NFIX+NFIXO
      NUU=NFIXO
      DO25 J=1,NNODE
      IF(NP1(J).EQ.0)GO TO 161
      IF(NP1(J).EQ.1)GO TO 18
      GO TO 163
161   N=N+1
      VK(N)=NP1(J)
      NP1(J)=N
      WRITE(6,162)J,N
162   FORMAT(' NP1(',I3,')=',I4)
      GO TO 20
163   NUU=NUU+1
      VK(NUU)=NP1(J)
      NP1(J)=NUU
      WRITE(6,17)J,NUU
17    FORMAT(' NP1(',I3,')=',I4)
      GO TO 20
18    NU=NU+1
      NP1(J)=NU
      WRITE(6,19)J,NU
19    FORMAT(' NP1(',I3,')=',I4)
20    IF(NP2(J).EQ.-1) GO TO 25
      IF(NP2(J).EQ.0)GO TO 201
      IF(NP2(J).EQ.1)GO TO 22
      GO TO 203
201   N=N+1
      VK(N)=NP2(J)
      NP2(J)=N
      WRITE(6,202)J,N
202   FORMAT(' NP2(',I3,')=',I4)
      GO TO 25
203   NUU=NUU+1
      VK(NUU)=NP2(J)
      NP2(J)=NUU
      WRITE(6,21)J,NUU
21    FORMAT(' NP2(',I3,')=',I4)
      GO TO 25
22    NU=NU+1
      NP2(J)=NU
       WRITE(6,23)J,NU
23    FORMAT(' NP2(',I3,')=',I4)
25    CONTINUE
      IF(MFIX.EQ.0)GO TO 291
      WRITE(6,27)
27    FORMAT('        VK ')
      DO 29 K=1,NUU
      WRITE(6,28)VK(K)
28    FORMAT(1X,F12.2)
29    CONTINUE
291   WRITE(6,30)NU
```

```
      30    FORMAT('TOTAL NO. OF DEGREES OF FREEDOM =',I6)
            DO 301 J=1,9000
            S(J)=0
      301   CONTINUE
            DO 32 I=1,90
            DO 31 J=1,250
            SK(I,J)=0
      31    CONTINUE
      32    CONTINUE
            IF(MFIX-90)325,325,323
      325   IF(NU-250)326,326,323
      323    WRITE(6,324)
      324   FORMAT( 'DIMENSION OF SK EXCEEDED ')
            GO TO 70
      326   CONTINUE
      C     MBAND IS ASSUMED MAX. BAND WIDTH -IF EXCEEDED PROGRAM PRINTS OUT
      C     THE FACT AND STOPS
            MBAND=50
            FACTOR=.00001
      C     FACTOR= COEFF. OF VISCOSITY DIVIDED BY 2
            DO50 I=1,NELEM
            MAX=0
            MIN=1000
            DO 322 L=1,13
            DO 321 K=1,13
            SE(K,L)=0
      321   CONTINUE
      322   CONTINUE
            X1(I)=X(NODE1(I))
            X2(I)=X(NODE2(I))
            X3(I)=X(NODE3(I))
            X4(I)=X(NODE4(I))
            X5(I)=X(NODE5(I))
            X6(I)=X(NODE6(I))
            Y1(I)=Y(NODE1(I))
            Y2(I)=Y(NODE2(I))
            Y3(I)=Y(NODE3(I))
            Y4(I)=Y(NODE4(I))
            Y5(I)=Y(NODE5(I))
            Y6(I)=Y(NODE6(I))
            A1(I)=X5(I)-X3(I)
            A3(I)=X1(I)-X5(I)
            A5(I)=X3(I)-X1(I)
            B1(I)=Y3(I)-Y5(I)
            B3(I)=Y5(I)-Y1(I)
            B5(I)=Y1(I)-Y3(I)
            TOA(I)=Y1(I)*(X5(I)-X3(I))+Y3(I)*(X1(I)-X5(I))+Y5(I)*(X3(I)-X1(I))
            MU(I)=FACTOR/TOA(I)
      C     MU = COEFF OF VISC. DIVIDED BY 4A
            SE(1,1)=(2*B1(I)**2+A1(I)**2)*MU(I)
            SE(1,2)=A1(I)*B1(I)*MU(I)
            SE(1,3)=4./3.*(2*B1(I)*B3(I)+A1(I)*A3(I))*MU(I)
            SE(1,4)=     A1(I)*B3(I)*MU(I)*4./3.
            SE(1,5)=     -(2*B1(I)*B3(I)+A1(I)*A3(I))*MU(I)*1./3.
            SE(1,6)=     -A1(I)*B3(I)*MU(I)*1./3.
            SE(1,7)=0
            SE(1,8)=0
            SE(1,9)=      -(2*B1(I)*B5(I)+A1(I)*A5(I))*MU(I)*1./3.
            SE(1,10)=     -A1(I)*B5(I)*MU(I)*1./3.
            SE(1,11)=     (2*B1(I)*B5(I)+A1(I)*A5(I))*MU(I)*4./3.
```

```
      SE(1,12)=     A1(I)*B5(I)*MU(I)*4./3.
      SE(2,2) =(2*A1(I)**2+B1(I)**2)*MU(I)
      SE(2,3) =     B1(I)*A3(I)*MU(I)*4./3.
      SE(2,4) =     (2*A1(I)*A3(I)+B1(I)*B3(I))*MU(I)*4./3.
      SE(2,5) =     -A3(I)*B1(I)*MU(I)*1./3.
      SE(2,6) =     -(2*A3(I)*A1(I)+B3(I)*B1(I))*MU(I)*1./3.
      SE(2,7) =0
      SE(2,8) =0
      SE(2,9) =     -A5(I)*B1(I)*MU(I)*1./3.
      SE(2,10)=     -(2*A1(I)*A5(I)+B1(I)*B5(I))*MU(I)*1./3.
      SE(2,11)=     B1(I)*A5(I)*MU(I)*4./3.
      SE(2,12)=     (2*A1(I)*A5(I)+B1(I)*B5(I))*MU(I)*4./3.
      SE(3,3)=    (2*(B3(I)**2+B1(I)*B3(I)+B1(I)**2)+A3(I)**2+A3(I)*A1(I
     1)+A1(I)**2)*MU(I)*8./3.
      SE(3,4) =     (A3(I)*B3(I)+0.5*(A1(I)*B3(I)+A3(I)*B1(I))+A1(I)*B1(I
     1))*MU(I)*8./3.
      SE(3,5) =     (2*B3(I)*B1(I)+A3(I)*A1(I))*MU(I)*4./3.
      SE(3,6) =     B3(I)*A1(I)*MU(I)*4./3.
      SE(3,7) =     (2*(B3(I)*B5(I)+B3(I)**2+2*B1(I)*B5(I)+B1(I)*B3(I))+
     1A3(I)*A5(I)+A3(I)**2+2*A1(I)*A5(I)+A1(I)*A3(I))*MU(I)*4./3.
      SE(3,8) =     (A3(I)*B5(I)+2*A1(I)*B5(I)+A3(I)*B3(I)+A1(I)*B3(I))*M
     1U(I)*4./3.
      SE(3,9) =0
      SE(3,10)=0
      SE(3,11)=     (2*(2*B3(I)*B5(I)+B1(I)*B5(I)+B1(I)*B3(I)+B1(I)**2)+
     12*A3(I)*A5(I)+A1(I)*A5(I)+A1(I)*A3(I)+A1(I)**2)*MU(I)*4./3.
      SE(3,12)=     (2*A3(I)*B5(I)+A1(I)*B5(I)+A3(I)*B1(I)+A1(I)*B1(I))*M
     1U(I)*4./3.
      SE(4,4) =     (2*(A3(I)**2+A1(I)*A3(I)+A1(I)**2)+B3(I)**2+B3(I)*B1(
     1I)+B1(I)**2)*MU(I)*8./3.
      SE(4,5) =     A3(I)*B1(I)*MU(I)*4./3.
      SE(4,6) =     (2*A3(I)*A1(I)+B3(I)*B1(I))*MU(I)*4./3.
      SE(4,7) =     (B3(I)*A5(I)+2*B1(I)*A5(I)+A3(I)*B3(I)+B1(I)*A3(I))*M
     1U(I)*4./3.
      SE(4,8) =     (2*(A3(I)*A5(I)+A3(I)**2+2*A1(I)*A5(I)+A1(I)*A3(I))+
     1B3(I)*B5(I)+B3(I)**2+2*B1(I)*B5(I)+B1(I)*B3(I))*MU(I)*4./3.
      SE(4,9) =0
      SE(4,10)=0
      SE(4,11)=     (2*B3(I)*A5(I)+B1(I)*A5(I)+B3(I)*A1(I)+A1(I)*B1(I))*M
     1U(I)*4./3.
      SE(4,12)=     (2*(2*A3(I)*A5(I)+A1(I)*A5(I)+A1(I)*A3(I)+A1(I)**2)+
     12*B3(I)*B5(I)+B1(I)*B5(I)+B1(I)*B3(I)+B1(I)**2)*MU(I)*4./3.
      SE(5,5) =(2*B3(I)**2+A3(I)**2)*MU(I)
      SE(5,6) =B3(I)*A3(I)*MU(I)
      SE(5,7)=    (2*B3(I)*B5(I)+A3(I)*A5(I))*MU(I)*4./3.
      SE(5,8)=      A3(I)*B5(I)*MU(I)*4./3.
      SE(5,9) =     -(2*B3(I)*B5(I)+A3(I)*A5(I))*MU(I)*1./3.
      SE(5,10)=     -A3(I)*B5(I)*MU(I)*1./3.
      SE(5,11)=0
      SE(5,12)=0
      SE(6,6) =(2*A3(I)**2+B3(I)**2)*MU(I)
      SE(6,7) =     B3(I)*A5(I)*MU(I)*4./3.
      SE(6,8) =     (2*A3(I)*A5(I)+B3(I)*B5(I))*MU(I)*4./3.
      SE(6,9)=    -A5(I)*B3(I)*MU(I)*1./3.
      SE(6,10)=     -(2*A3(I)*A5(I)+B3(I)*B5(I))*MU(I)*1./3.
      SE(6,11)=0
      SE(6,12)=0
      SE(7,7) =     (2*(B5(I)**2+B3(I)*B5(I)+B3(I)**2)+A5(I)**2+A3(I)*A5(
     1I)+A3(I)**2)*MU(I)*8./3.
      SE(7,8) =     (A5(I)*B5(I)+0.5*(A3(I)*B5(I)+A5(I)*B3(I))+A3(I)*B3(I
```

```
     1))*MU(I)*8./3.
      SE(7,9) =     (2*B3(I)*B5(I)+A5(I)*A3(I))*MU(I)*4./3.
      SE(7,10)=     B5(I)*A3(I)*MU(I)*4./3.
      SE(7,11)=     (2*(B5(I)**2+B3(I)*B5(I)+B1(I)*B5(I)+2*B1(I)*B3(I))+A
     15(I)**2+A3(I)*A5(I)+A1(I)*A5(I)+2*A1(I)*A3(I))*MU(I)*4./3.
      SE(7,12)=     (B5(I)*A5(I)+B5(I)*A3(I)+B1(I)*A5(I)+2*B1(I)*A3(I))*M
     1U(I)*4./3.
      SE(8,8) =     (2*(A5(I)**2+A3(I)*A5(I)+A3(I)**2)+B5(I)**2+B3(I)*B5(
     1I)+B3(I)**2)*MU(I)*8./3.
      SE(8,9) =     B3(I)*A5(I)*MU(I)*4./3.
      SE(8,10)=     (2*A5(I)*A3(I)+B5(I)*B3(I))*MU(I)*4./3.
      SE(8,11)=     (A5(I)*B5(I)+A5(I)*B3(I)+A1(I)*B5(I)+A1(I)*B3(I)*2)*M
     1U(I)*4./3.
      SE(8,12)=     (2*(A5(I)**2+A3(I)*A5(I)+A1(I)*A5(I)+2*A1(I)*A3(I))+B
     15(I)**2+B3(I)*B5(I)+B1(I)*B5(I)+2*B1(I)*B3(I))*MU(I)*4./3.
      SE(9,9) =(2*B5(I)**2+A5(I)**2)*MU(I)
      SE(9,10)=B5(I)*A5(I)*MU(I)
      SE(9,11)=     (2*B5(I)*B1(I)+A5(I)*A1(I))*MU(I)*4./3.
      SE(9,12)=     A5(I)*B1(I)*MU(I)*4./3.
      SE(10,10)=(2*A5(I)**2+B5(I)**2)*MU(I)
      SE(10,11)=     B5(I)*A1(I)*MU(I)*4./3.
      SE(10,12)=     (2*A5(I)*A1(I)+B5(I)*B1(I))*MU(I)*4./3.
      SE(11,11)=     (2*(B5(I)**2+B1(I)*B5(I)+B1(I)**2)+A5(I)**2+A1(I)*A5
     1(I)+A1(I)**2)*MU(I)*8./3.
      SE(11,12)=     (A5(I)*B5(I)+0.5*(A1(I)*B5(I)+A5(I)*B1(I))+A1(I)*B1(
     1I))*MU(I)*8./3.
      SE(12,12)=     (2*(A5(I)**2+A1(I)*A5(I)+A1(I)**2)+B5(I)**2+B1(I)*B5
     1(I)+B1(I)**2)*MU(I)*8./3.
C     ASSEMBLE INCOMPRESSIBILITY CONDITION MULTIPLIED BY .00006
      A=.00001
      SE(13,1)=B1(I)*A
      SE(13,2)=A1(I)*A
      SE(13,3)=4*(B1(I)+B3(I))*A
      SE(13,4)=4*(A1(I)+A3(I))*A
      SE(13,5)=B3(I)*A
      SE(13,6)=A3(I)*A
      SE(13,7)=4*(B3(I)+B5(I))*A
      SE(13,8)=4*(A3(I)+A5(I))*A
      SE(13,9)=B5(I)*A
      SE(13,10)=A5(I)*A
      SE(13,11)=4*(B1(I)+B5(I))*A
      SE(13,12)=4*(A1(I)+A5(I))*A
      SE(13,13)=0
      DO 33 K=1,12
      SE(K,13)=SE(13,K)
33    CONTINUE
      DO 35 K=1,12
      DO 34 L=1,12
      SE(L,K)=SE(K,L)
34    CONTINUE
35    CONTINUE
      DO 341 K=1,13
      MC(K)=0
341   CONTINUE
      MC(1)=NP1(NODE1(I))
      MC(2)=NP2(NODE1(I))
      MC(3)=NP1(NODE2(I))
      MC(4)=NP2(NODE2(I))
      MC(5)=NP1(NODE3(I))
      MC(6)=NP2(NODE3(I))
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293             MC(7)=NP1(NODE4(I))
294             MC(8)=NP2(NODE4(I))
295             MC(9)=NP1(NODE5(I))
296             MC(10)=NP2(NODE5(I))
297             MC(11)=NP1(NODE6(I))
298             MC(12)=NP2(NODE6(I))
299             MC(13)=NP1(NODE7(I))
300             DO 37 K=1,13
301             IF(MC(K)-MFIX)37,37,351
302     351     DO 36 L=1,13
303             IF(MC(L)-MFIX)36,36,352
304     352     IF(MC(K)-MC(L))353,36,354
305     353     IF(MC(L).GT.MAX)MAX=MC(L)
306             GO TO 36
307     354     IF(MC(L).LT.MIN)MIN=MC(L)
308     36      CONTINUE
309     37      CONTINUE
310             NAND=MAX-MIN+1
311             IF(MBAND.GT.NAND) GOTO 372
312             WRITE(6,371)I,NAND
313     371     FORMAT(' BAND WIDTH EXCEEDED IN ELEMENT ',I4,' BAND WIDTH= ',I4)
314             GO TO 70
315     C       GENERATE S((NU-MFIX)**2),SK(MFIX*(NU-NFIXO))
316     372     DO 48 L=1,13
317             IF(MC(L)-NFIXO)48,48,38
318     38      IF(MC(L)-MFIX)42,42,39
319     39      DO 41 K=1,13
320             IF(MC(K)-MFIX)401,401,40
321     40      IF(MC(L).LT.MC(K)) GO TO 41
322             KK=MC(K)-MFIX
323             LL=MC(L)-MFIX
324             M=(KK-1)*(MBAND-1)+LL
325             S(M)=S(M)+SE(K,L)
326             GO TO 41
327     401     MCL=MC(L)-NFIXO
328             SK(MC(K),MCL)=SK(MC(K),MCL)+SE(K,L)
329     41      CONTINUE
330             GO TO 48
331     42      DO 44 K=1,13
332             IF(MC(K)-MFIX)43,43,44
333     43      MCL=MC(L)-NFIXO
334             SK(MC(K),MCL)=SK(MC(K),MCL)+SE(K,L)
335     44      CONTINUE
336     48      CONTINUE
337     50      CONTINUE
338             DO 49 I=1,300
339             RK(I)=0
340     49      CONTINUE
341             IF(NFORCE.EQ.0)GO TO 502
342             DO 501 J=1,NNODE
343             RK(NP1(J))=RK1(J)
344             IF(NP2(J).EQ.-1) GO TO 501
345             RK(NP2(J))=RK2(J)
346     501     CONTINUE
347             DO 504 J=1,NU
348             WRITE(6,503)RK(J)
349     504     CONTINUE
350     503     FORMAT(1X,E16.7)
351     502     MFIX1=MFIX+1
352             NUFO=NU-NFIXO
```

```
353             IF(MFIX.EQ.0)GO TO 541
354             IF(NFIX.EQ.0)GO TO 541
355             DO 54 L=NFIX1,NUFO
356             SUM=0
357              NFIXO1=NFIXO+1
358             DO 51 K=NFIXO1,MFIX
359             SUM=SUM + SK(K,L)*VK(K)
360       51    CONTINUE
361             M=L-NFIX
362             J=L+NFIXO
363             DELTA(M)=RK(J)-SUM
364       54    CONTINUE
365             GO TO 544
366       541   DO 543 J=1,NUFO
367             M=J+NFIXO
368             DELTA(J)=RK(M)
369       543   CONTINUE
370             DO 546 K=1,NUFO
371             WRITE(6,545)DELTA(K)
372       545   FORMAT(1X,E16.7)
373       546   CONTINUE
374       544   NET=NU-MFIX
375             RATIO=1.E-6
376             CALL CBANDI(S,DELTA,NET,MBAND,1,RATIO)
377             RNET=NET
378             D=COND/RNET
379             WRITE(6,55)D
380       55    FORMAT( 'RATIO OF CONDITICN NO. TO ORDER OF S =',D16.7)
381             WRITE(6,56)DE,NCN
382       56    FORMAT( 'DETERMINANT='D16.7,'*1.E'I3)
383             WRITE(6,58)
384       58    FORMAT(' NODE          VEL-PRESS.')
385       C     IF PRESSURE TERM =-A THEN ACTUAL PRESSURE IS V.0006*A COMPRESSIVE
386             DO60 I=1,NET
387             L=MFIX+I
388             WRITE(6,59)L,DELTA(I)
389       59    FORMAT(I5,D18.8)
390       60    CONTINUE
391             IF(MFIX.EQ.0)GO TO 70
392             DO 63 K=1,MFIX
393             SUM=0
394             IF(NFIX.EQ.0) GO TO 611
395             DO 61 L=1,NFIX
396             M=L+NFIXO
397             SUM=SUM+SK(K,L)*VK(M)
398       61    CONTINUE
399       611   NFIX1=NFIX+1
400             NETO=NU-NFIXO
401             DO 62 L=NFIX1,NETO
402             M=L-NFIX
403             SUM=SUM+SK(K,L)*DELTA(M)
404       62    CONTINUE
405             R(K)= SUM
406       63    CONTINUE
407             WRITE(6,64)
408       64    FORMAT(' NODE           FORCE ')
409             DO 66 I=1,MFIX
410             WRITE(6,65)I,R(I)
411       65    FORMAT(I5,E18.8)
412       66    CONTINUE
```

```
 413      70     STOP
 414             END
D OF FILE
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