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THE LATERAL TORSIONAL BUCKLING OF OPEN THIN-WALLED BEAMS

by

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ABSTRACT

This thesis is concerned with the development of a stiffness matrix for the study of a large range of stability problems for beams of arbitrary, open, thin-walled cross sections.

This is done by first developing, using a consistent set of common engineering assumptions, a non-linear relation between the forces and the displacements of the beam. These relations are then substituted into the beam equilibrium equations to give a set of three differential equations of equilibrium in terms of the displacements. These differential equations are solved using an iteration technique. A member stiffness matrix is generated when the iterated solution is used with the non-linear deflection relations. The resulting fourteen by fourteen matrix includes the regular six forces plus a bi-moment at each end. The matrix is tested against known solutions and agreement is seen to be excellent in all cases. All the terms necessary for the building of the matrix are given in the Appendices.

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LIST OF SYMBOLS

Symbol	Description	Page of Definition
$'$	Denotes differentiation with respect to z	
$*$	Denotes differentiation with respect to z_o	
$\cdot$	Denotes differentiation with respect to s	
A	Area of cross section	44
BM_i	Bending moments about x, y axis	21
BW	Bi-moment in x, y axis	77
C_j^i	Constants in iteration scheme	63
ds	Differential length of s	6
dS, dS_o	Deformed, undeformed elemental areas	15
dT_i	Actual stress vector	15
E_{ij}	Green's strain tensor	13
e_{ij}	Almansi's strain tensor	14
E	Young's modulus	16
f_i	Generalized force	58
$f(z)$	Function of z	27
f_i^u, f_i^θ, f_i^v	Functions of M_i, θ, ϕ, C_j^i, z in iteration scheme	66

Symbol	Description	Page of Definition
f_j^i	Functions of f_i^u, f_i^β, f_i^v in iteration scheme	69
G, G', λ, λ'	Elastic constants	15
$G_i(x, y, z)$	Function of x, y, z	32
$I_{\bar{x}c}$, I_{yc}	Principal centroidal moment of inertia	46
I_p	Polar moment of inertia, later about the shear centre	50
K_i	Constants used to define section properties along with I_{xc}, I_{yc} $K_1 = \int_0^{\bar{s}} \sin\phi \int_0^s t(\omega_1 - \bar{\omega}_1)\, ds\, ds$ $K_2 = \int_0^{\bar{s}} \frac{t^3}{12} \sin^2\phi\, ds$ $\vdots$ $K_{24} = \int_0^{\bar{s}} \frac{t^3}{6} y^2 \sin\phi \cos\phi\, ds$	46
L	Length of section	
M	Secondary moment per unit length ds	16
M_i	Constants in dimensionless differential equations $M_1 = -PL^2/N_1$ $M_2 = -PL\bar{x}/N_1$ $\vdots$ $M_{16} = PI_pL^2/AN_3$	61

Page of Definition

N_i	Constants for cross section properties, consists of combinations of Φ, K_i	
	$N_1 = E(-\Phi_{xc} - K_7) = -EI_{xc}$	
	$N_2 = E(\Phi_{yc} + K_2) = EI_{yc}$	
	$N_3 = E(K_{13} + 2K_{16} - K_{17} - K_{18})$	
	$N_4 = JG = C$	
	$N_5 = E(\Phi_{xc} + K_7)$	
	$N_6 = -E(\Phi_{yc} + K_2)$	
	$N_7 = E(2K_{10} - K_{20} - K_{21} - K_{23} + K_{24})$	
	$N_8 = E(2K_{11} - K_{19} - 2K_9)$	
	$N_9 = E(2K_{12} - 2K_4 - K_{22})$	
	where Φ_{xc}, Φ_{yc}, C, K_{13}, K_{10}, K_{11}, K_{12} are due to membrane action, and all the rest are due to secondary or plate bending stresses	54
P	Axial load along z axis	21
q_o, q	Force flows on element $dz\,ds$	17
r	Perpendicular distance from origin to tangent at point s	6

		Page of Definition
s	Co-ordinate along mid-thickness of the cross section	6
$s_o, \bar{s}$	Initial and final values of s	6
S_{ij}	Kirchoff stress tensor	14
t	Thickness of cross section	6
$\bar{T}_1, \bar{T}_2, \bar{T}_3$	Elemental force resultants in x_2, y_2, z	17
T_1, T_2, T_3	Elemental force resultants in x, y, z	21
T	Torque along z	21
u, v, w	Displacements in fixed global system	4
u_1, v_1	Displacement of origin along x_1, y_1	11
u_2, v_2	Displacements of elements ds along x_2, y_2	11
u_s, v_s	Displacement of elements ds along x, y	11
$\tilde{u}, \tilde{v}, \tilde{z}$	Dimensionless values of u, v, z	61
$\tilde{u}_1, \tilde{v}_1$	First solution for $\tilde{u}, \tilde{v}$.	64
$\tilde{u}_2, \tilde{v}_2$	First iteration for $\tilde{u}, \tilde{v}$	66
w	Displacement along z	4
w_o	Displacement w of point $s = 0$	6

		Page of Definition
$\bar{w}$	Non-linear w	27
x, y, z	Fixed global co-ordinate system, used to define position of points on mid-thickness	4
x_o, y_o	Co-ordinates of points $s = 0$ in x, y, z	40
x_1, y_1	Fixed axis defined for each element ds; parallel to principal axes of ds; common origin with xy.	11
x_2, y_2	Fixed principal axes of element ds	11
$\bar{x}, \bar{y}$	Co-ordinates of centroid in x, y, z	46
x_D, y_D, z_D	Displaced axes similar to x, y, z	31
β	Rotational displacement about z	4
γ	$\phi - \beta$	22
δ	Generalized displacements	58
$\Delta\delta$	Variation in δ	74
λ	$-N_4 L^2 / N_3$	62
ν, ν_o	Deformed, undeformed normal	15
ρ, ρ_o	Deformed, undeformed density	15
ξ, η, γ	Axis that translates but does not rotate with section	22

		Page of Definition
ϕ	Angle between x axis and ds	6
Φ, Θ	Functions of z, $\tilde{v}_1 = \Phi C_1^1$, $\tilde{\beta}_1 = \Theta C_1^3$	63
$\Phi_{xc}, \Phi_{yc}, \Phi_{xyc}$	Quasi-moments of inertia	46
σ_{ij}	Actual stress tensor	14
$\tilde{\sigma} t$	Force flow	17
$\sigma_{zz} = \sigma$	Actual stress in z direction	19
ω_1	$\int_0^s r \, ds$	10
$\bar{\omega}_1$	$\frac{1}{A} \int_0^{\bar{s}} t \omega_1 \, ds$	44
$[\bar{\Phi}_1^1], [\bar{\Phi}_1^2]$, $[\bar{\Phi}_2^1], [\bar{\Phi}_2^2]$	Geometric matrix of Φ evaluated at $z = 0, L$	63
$[\bar{\Theta}_1^3], [\bar{\Theta}_2^3]$	Geometric matrix of Θ evaluated at $z = 0, L$	63
$[\widetilde{\widetilde{\Omega}}]$	Ω integrated twice	67
$[\check{\check{\Omega}}]$	Means $\tilde{\beta} = [\check{\check{\Omega}}]$ satisfies $\ddot{\tilde{\beta}} - \lambda^2 \tilde{\beta} = [\Omega]$	67

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CHAPTER 1

INTRODUCTION

The lateral torsional buckling of beams has been a subject of interest for many years. The first formal investigation began in 1899 when Prandtl and Michell independently developed the differential equations of lateral torsional buckling of thin rectangular sections. A few years later Timoshenko developed the equations for an I beam. From that time on, many authors contributed to the field and expanded the scope of the equations. Much work was done by Goodier, Timoshenko, Bleich and Vlassov [5], [7], [3], [9]. The result of this work has been to develop, by various methods, the differential equations of buckling for an arbitrary open cross section under various loadings. Solutions to these equations have been found for selected end conditions and loads but it was not until recently that Gallagher and Barsoum [2] published a matrix formulation based on an assumed displacement field and minimum potential energy. Barsoum [1] followed up with a dynamic approach using the Hamiltonian as the stationary functional to determine the parameters in the assumed displacement field, and this allowed the treatment of non-conservative loads.

It is the purpose of this thesis to develop, by direct treatment of the differential equations of equilibrium, a stiffness matrix to calculate the buckling load of various thin walled open cross sectioned structures. The differential equations will be developed under the application of a consistent set of common engineering assumptions. The resulting matrix will be valid under this range of assumptions.

The differential equations will be developed using equilibrium equations, strain equations and constitutive laws. The restrictions placed on the development will be the material remains elastic, the cross section retains its overall geometric shape, the rotations are small with respect to one, and their squares small with respect to themselves. As the equations are developed the effect of smaller order terms will be studied and in light of the restrictions on rotations, will be kept or discarded.

The differential equations will then be iterated to obtain a solution. This solution will be used to construct a force deflection relation. Several examples will then be treated and compared to known solutions. It will be seen that agreement is very good in all tested cases.

The advantage of this approach lies in the increase in the number of problems which become tractable. For instance, arbitrary boundary conditions, varying section properties and arbitrary loadings cease to be a problem and are easily treated by the stiffness matrix approach.

CHAPTER 2

PRELIMINARIES FOR DEVELOPMENT OF DIFFERENTIAL EQUATIONS

The development of the differential equations for the section will be done employing equilibrium equations, elastic constitutive laws and strain-displacement equations.

First the displacement and strain-displacement equations will be written. The relationship between strain and stress will then be examined. Finally the equilibrium equations will all be written. These separate sets of equations will all be assembled using suitable engineering approximations and constraints to obtain the governing differential equations for the problem of lateral torsional buckling. These differential equations will take the form of the overall equilibrium equations of a displaced element of beam length written in terms of displacement derivatives. For instance, the well-known Euler equation for column buckling,

$$EI\,u'''' + Pu'' = 0$$

is of this form.

The restrictions imposed by the assumed constraints will be studied to outline the domain of validity of the equations.

The actual steps of the development are as follows: The displacements will be used to calculate the strains. Constitutive laws will then be used to obtain the stresses in terms of the strains. The stresses will be integrated over the cross section to get the force resultants, which will now be in terms of the strains and hence the displacements. Substitution of these resultants into the overall equilibrium equations gives the desired result.

These steps require a detailed study of the displacements, stress-strain and equilibrium equations. The first group to be studied will be the displacement and strain equations.

1:1 Displacements

The main displacements involved are detailed in Figure 1, which shows the beam segment undeformed and in its initial position. The segment has constant physical dimensions along its length and carries no load between its ends. The axes x, y, z are the global axes and are fixed along the initial, undeformed position of the beam. The z axis lies along the length of the beam and the x and y axes define the co-ordinates of points on the mid-thickness of the cross section. The displacements u, v are in the directions of x, y respectively and are the displacements of the point on the cross section which lies on the z axis when undeformed. The displacement w is the displacement in the z direction of a point on the cross section. Since the cross section is assumed to retain its initial shape only one displacement β is required to define its rotation about the z axis.

The displacements u, v, w and β are not all independent, as the axial displacement w can be represented by u, v, β and the cross section properties. The next step is to develop this relation between u, v, β and w. This will be done subject to the conditions that the cross section retains its shape and the shear strains at mid-thickness are zero. This does not preclude out of plane warping of the cross section. Angular displacements are assumed small with respect to one and so their products are taken to be negligible compared to themselves. This allows the rotations to be treated as vectors. The consequences of this assumption will be examined later.

A co-ordinate system defining the position of points on the cross section is given in Figure 2. The axes x, y, z, with the associated displace-

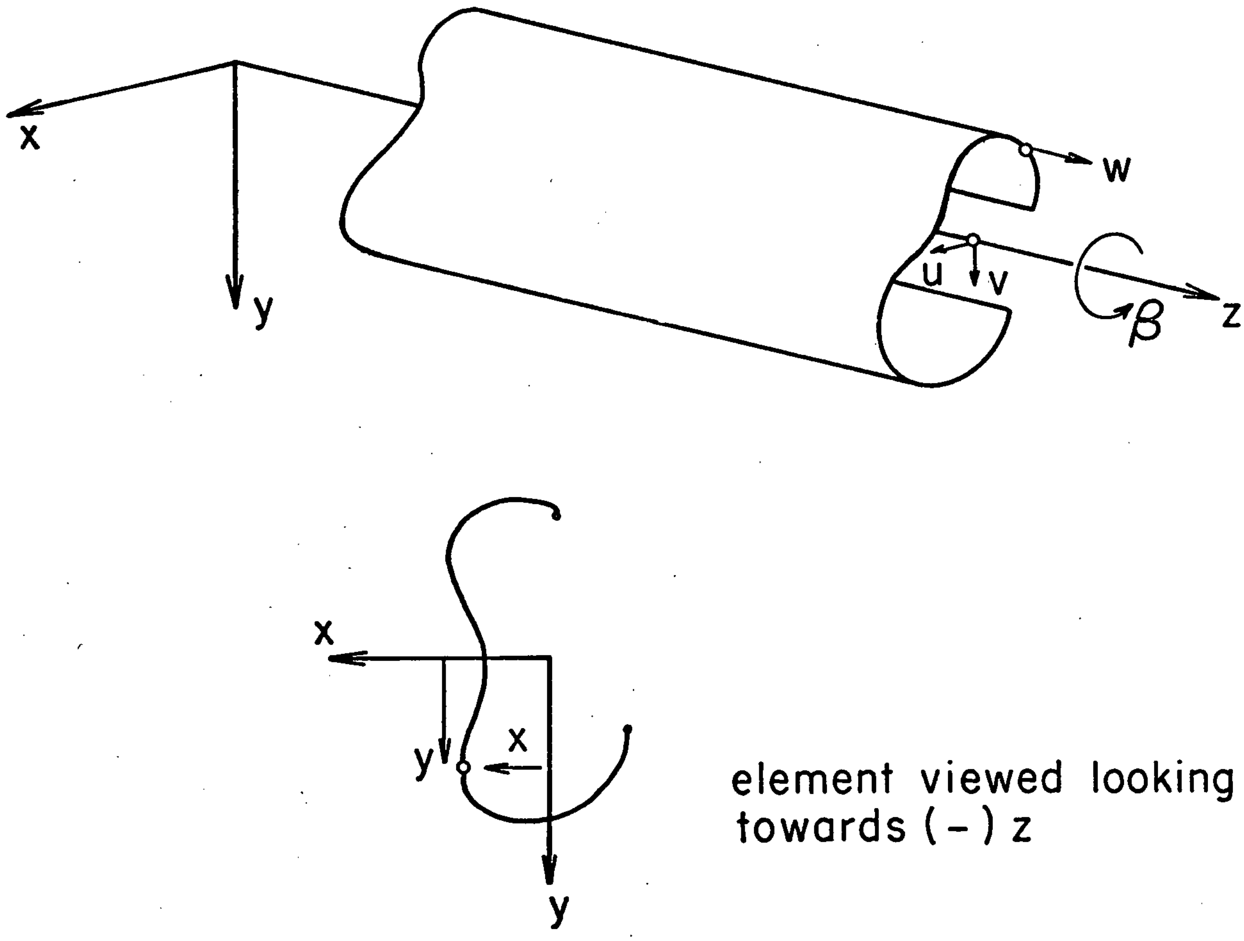


FIG. I UNDEFORMED BEAM SEGMENT IN x, y, z CO-ORDINATE SYSTEM.

ments u, v, w, β is the fixed global system previously defined. The co-ordinate s indicates distance along the centre-line of the cross section from the origin of s. This origin is taken at any convenient free edge. Figure 2a shows one possible origin and direction of s, while Figure 2b gives an alternate origin and shows s running in the opposite direction. The thickness t may vary with s. Both the systems x, y and s may be used to locate points on the cross section. The distance r is the distance perpendicular to the tangent line of the point of interest to an arbitrary origin, in this case the origin of x, y. The distance r has an associated sign: positive if the swept area $r\,ds$ is clockwise about the origin, negative if anticlockwise. The distance r may be written in terms of x, y and the angle ϕ and have the sign automatically accounted for if ϕ is defined the following way: The angle ϕ is taken to be the angle between the positive directed x axis and the positive directed element ds and is measured positive clockwise.

This means ϕ is the angle between the vectors x and ds, where ds is parallel to the tangent at s. With this definition of ϕ, r may be written as:

$$r = y\cos(\phi) + x\sin(\phi)$$

Using these quantities, the displaced shape w can be found from the following reasoning.

If the quantity $\frac{\partial w}{\partial s}$ were known, where w is a function of s, then w would be:

$$w = w_o + \int_0^s \frac{\partial w}{\partial s}\, ds$$

where w_o is the displacement of the point $s = 0$. To find $\frac{\partial w}{\partial s}$, three

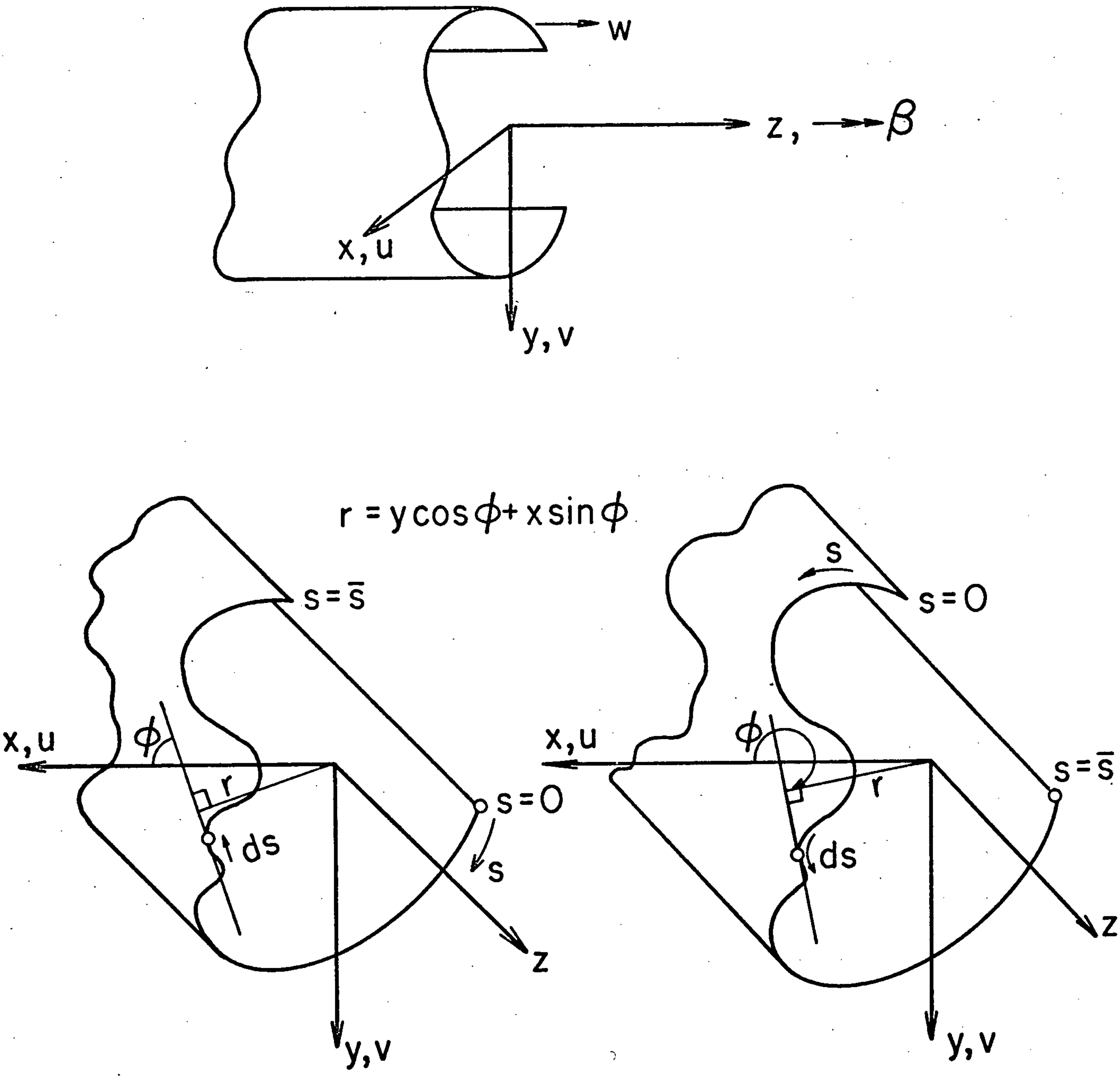


FIG. 2 CO-ORDINATES FOR POINTS ON THE CROSS SECTION.

displacements $\frac{du}{dz}$, $\frac{dv}{dz}$ and $\frac{d\beta}{dz}$ must be considered. First, consider a plane section rotation $\frac{du}{dz}$. This has a slope along s of $-\frac{du}{dz}\cos\phi$. Secondly, consider the plane section rotation $\frac{dv}{dz}$. This has a slope along s of $\frac{dv}{dz}\sin\phi$. The third deflection is slightly more complicated. Figure 3 shows a small strip of the beam ds in width deflected by a β varying with z.

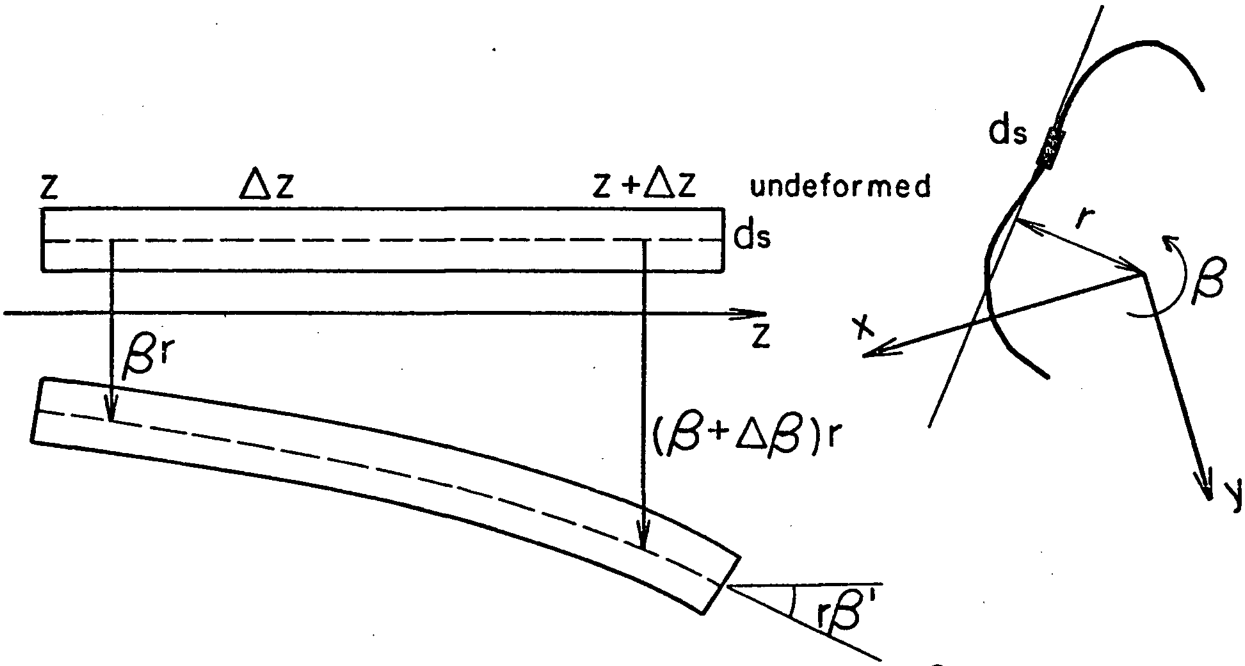


FIG.3 DEFLECTION DUE TO ROTATION β.

Clearly, the deflection at s due to β is βr and in the direction of the tangent at s. From this, the slope due to a β deflection varying with z is:

$$\frac{(\beta+\Delta\beta)r-\beta r}{(z+\Delta z)-z}=\frac{d\beta}{dz}r$$

Because of the limitations on the rotations, these three slopes may be added to obtain

$$\frac{\partial w}{\partial s}=-\frac{du}{dz}\cos\phi+\frac{dv}{dz}\sin\phi+\frac{d\beta}{dz}r$$

which is based on the shear strains being zero at mid-thickness.

Finally, this may be written as:

$$dw = -u' \cos\phi \, ds + v' \sin\phi \, ds + r\beta' ds \qquad (1)$$

where dw is the differential change of the displacement in the z direction at any point s on the centre line of the cross section for any change ds in the s co-ordinate. The primes denote differentiation with respect to z. Equation (1) is developed in several references. See for example [6] and [9].

The first two terms on the right-hand side of Equation (1) are the contributions of bending with plane sections remaining plane. The third term represents out of plane warping. It should be remembered that Equation (1) is valid only if terms such as $u'v'$ are negligible compared to u', v', thus allowing u', v', β to be treated as vectors.

The deflection w can be obtained by integration of (1) with respect to s.

$$w = w_o - u' \int_0^s \cos\phi \, ds + v' \int_0^s \sin\phi \, ds + \beta' \int_0^s r \, ds \qquad (2)$$

where w_o is the displacement of the point $s = 0$.

It may be noted here that $s = 0$ does not need to fall on the edge of the cross section, it may be placed anywhere on the section. In this development, however, it will be left at any arbitrary edge.

Equation (2) may be simplified by using the differential relations:

$$\frac{dy}{ds} = -\sin\phi$$
$$\qquad (3)$$
$$\frac{dx}{ds} = +\cos\phi$$

from Figure 2. This gives:

$$w = w_o - u' \int_o^s dx - v' \int_o^s dy + \beta' \int_o^s r\, ds \tag{4}$$

using

$$\int_o^s dx = x - x_o$$
$$\int_o^s dy = y - y_o \tag{5}$$

where x, y and x_o, y_o are the co-ordinates of the points s and $s = 0$ respectively and definining

$$\omega_1 = \int_o^s r\, ds \tag{6}$$

allows Equation (4) to be written

$$w = w_o + \beta' \omega_1 + u'(x_o - x) + v'(y_o - y) \tag{7}$$

Equation (7) describes the deformations along the z axis of the centreline of the cross section and can be used to get strains and therefore stresses. It should be remembered that terms of the order of $u'v'$ and $\beta v'$ were taken as negligible compared to terms such as u', v' in this development in order to add the angles vectorially. Therefore Equation (7) as it stands is purely linear. Since it is desired to treat a non-linear problem, terms representing axial foreshortening due to rotations should be included, but since this is a complicated procedure, and since these terms will be shown to be of no consequence for certain conditions, they will be introduced, treated and discarded at a later stage.

In addition to the deflection w there are some secondary deformations across the thickness of the cross section. They represent the effect of plate bending of the element of cross section ds. These deformations are due to deflections perpendicular to the direction of ds and produce plate bending stresses whereas the w deflection produces membrane type stresses. For most sections the plate bending stresses are ignored as their overall effect is negligible compared to the effect of membrane stress. However, for sections with certain geometrical properties, the plate bending stress may be the only stress available to resist applied load. For example, a thin rectangular beam behaves like a plate in the weak lateral direction. For this reason the plate bending deformations will be studied and kept.

To obtain values for these deformations some new co-ordinate systems will be introduced. In Figure 4, x, y and their associated deflections u, v are the fixed global system. The new system x_1, y_1 and the associated deflections u_1, v_1 are defined such that they are parallel to the principal axes of a small element ds of the cross section and share the same origin as x, y. The system x_1, y_1 also defines the position of points on the cross section, and u_1, v_1 represent deflections of the origin in this system. The second axis system x_2, y_2 and the associated deflections u_2, v_2 are defined to be the principal axes and the deflections respectively of the small element ds of the cross section. Therefore, x_1, y_1 and x_2, y_2 are parallel to each other but not to x, y. It is clear that since each element ds of the cross section must be parallel to the tangent to S at s, then x_1, y_1 and x_2, y_2 can be related to x, y by the angle ϕ, which the tangent makes with the x axis. The axes x_1, y_1, and x_2, y_2 always remain in the plane of x, y and do not displace with the section. These axes suffice to determine the plate bending deformation of interest, which is u_2''. The algebraic relation between the axes are as follows:

$$x_1 = x \sin\phi + y \cos\phi$$

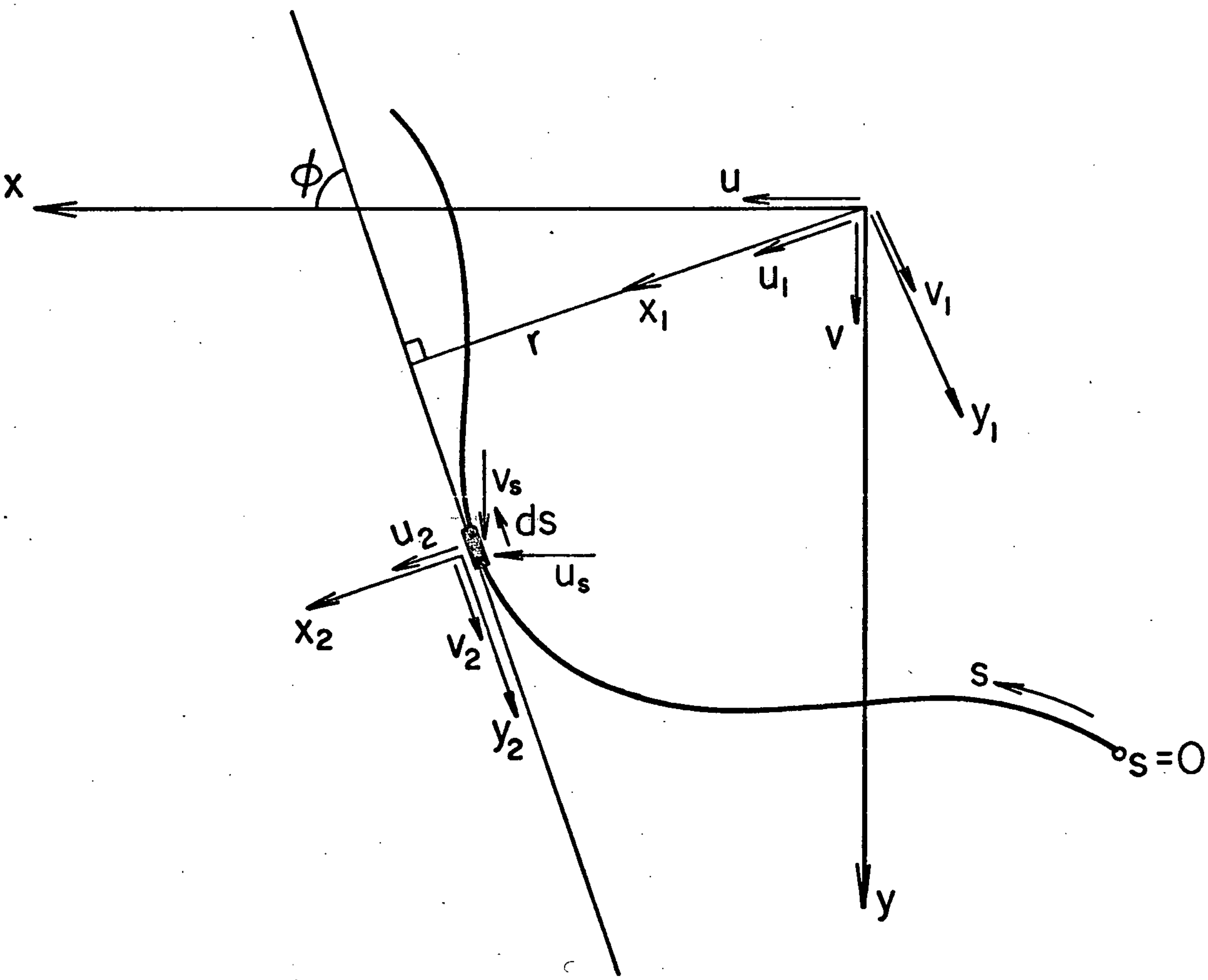


FIG. 4 PRINCIPAL AXIS x_2, y_2 OF ELEMENT ds OF CROSS SECTION.

$$y_1 = -x \cos\phi + y \sin\phi$$

$$u_1 = u \sin\phi + v \cos\phi \tag{8}$$

$$v_1 = -u \cos\phi + v \sin\phi$$

It is clear that since the section retains its geometric shape, the following relations hold:

$$u_2 = u_1 - \beta y_1 = u \sin\phi + v\cos\phi - \beta(-x\cos\phi + y\sin\phi)$$

$$v_2 = v_1 + \beta x_1 = -u\cos\phi + v\sin\phi + \beta(x\sin\phi + y\cos\phi) \tag{9}$$

$$u_s = u - \beta y \quad , \quad v_s = v + \beta x$$

where u_s, v_s are the displacements in the u and v directions of the point S.

Derivatives of u_2 and v_2 are easily found by differentiating (9) with respect to z and remembering that ϕ, x, y are independent of z for each elemental section.

This completes the development of the necessary displacements.

1:2 Strains

The strains necessary to calculate the required stresses can be found by differentiation of the displacements in accordance with the strain tensor chosen. Since this is a non-linear development, it would be wise to start out with non-linear strains and make any approximations later.

Two common strain tensors are Green's tensor E_{ij} and Almansi's tensor

e_{ij} . See, for example, Fung, Chapter 4 [4]. Both these are finite strain tensors and differ only in the co-ordinate system used to represent them. Green's tensor is written in terms of the undeformed body co-ordinates, Almansi's is written in terms of the deformed body co-ordinates. For example, the axial strain written in terms of the Green's tensor and using the co-ordinate system x, y, z defined previously becomes:

$$E_{zz} = \frac{1}{2}\left[2\frac{\partial w}{\partial z} + \left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial u_s}{\partial z}\right)^2 + \left(\frac{\partial v_s}{\partial z}\right)^2\right] \tag{10}$$

where the squared terms containing u_s, v_s and w in (10) can be evaluated by differentiating the relations given in Equations (7) and (9). The tensor used in this development will be the Almansi, as it relates to the actual stress and, for reasons which will become apparent later, it is the actual stress which will be desired. The Green's tensor was shown above merely to give some idea of the form of the strain of interest, and because the Green's tensor can be written in terms of co-ordinate systems already defined.

At this point, the strain and displacement equations are complete. The principal deflected shape has been developed and, using equations similar to (10), all the strains may be obtained. The secondary plate bending deformations have been introduced and detailed in Figure 4.

1:3 Constitutive Laws

It is now of interest to obtain the relations between the stresses in the cross section and the strains. Since the strains will be defined for finite-deformation, the stress tensors will have to be defined to match. To do this, Chapter 16, Fung [4], will be used to provide all the necessary definitions and relations.

Two stress tensors, S_{ij} , the Kirchoff stress, and σ_{ij}, the actual stress, are defined as related to the strain tensors E_{ij} and e_{ij} by the following relations:

$$\begin{aligned} \sigma_{ij} &= \lambda e_{\alpha\alpha} \delta_{ij} + 2G e_{ij} \\ S_{ij} &= \lambda' E_{\alpha\alpha} \delta_{ij} + 2G' E_{ij} \end{aligned} \tag{11}$$

where λ, λ', G, G' are elastic constants. These two stress tensors are related by the following transformation:

$$\sigma_{ij} = \frac{\rho}{\rho_o} \frac{\partial x_i}{\partial a_\alpha} \frac{\partial x_j}{\partial a_\beta} S_{\beta\alpha} \tag{12}$$

This expression is of course more complex than necessary for the problem. As with the strains, it will be discussed and reduced in a following section. In Equation (12), the x and a are measured in the same cartesian co-ordinate system, but x is in terms of the deformed position and a is in terms of the undeformed position. The densities ρ and ρ_o are the deformed and undeformed densities respectively.

The stress tensors are connected to the actual stress vector dT_i by the following relations:

$$\begin{aligned} \sigma_{ji} \nu_j \, dS &= dT_i \\ S_{ji} \nu_{oj} \, dS_o &= \frac{\partial a_i}{\partial x_\alpha} dT_\alpha \end{aligned} \tag{13}$$

where ν and dS are the normal vector and area of the deformed element and ν_o and dS_o are the normal vector and area of the undeformed element. Now that the actual stress vectors have been given in terms of the strains and hence indirectly the displacements, the secondary stresses and their force displacement relationship can be discussed. These stresses, due to the plate bending deformations, can be treated in a much more relaxed way than the stresses defined above. It will be assumed that the plate bending stresses produce a moment per unit length M that is given by:

$$M = \frac{Et^3}{12} u_2'' \tag{14}$$

where u_2 was defined in Figure 4. The moment acting on the element taken from the cross section will be defined as $M\,ds$ and will lie along the tangent to S in the negative S direction in the displaced position.

This completes the discussion of stress-strain or force deformation relations. Of the stresses defined above, only the actual stress σ_{ij} will be used. This will be due to certain problems which will arise, necessitating the writing of equilibrium equations in a displaced position, which require the use of the actual stress.

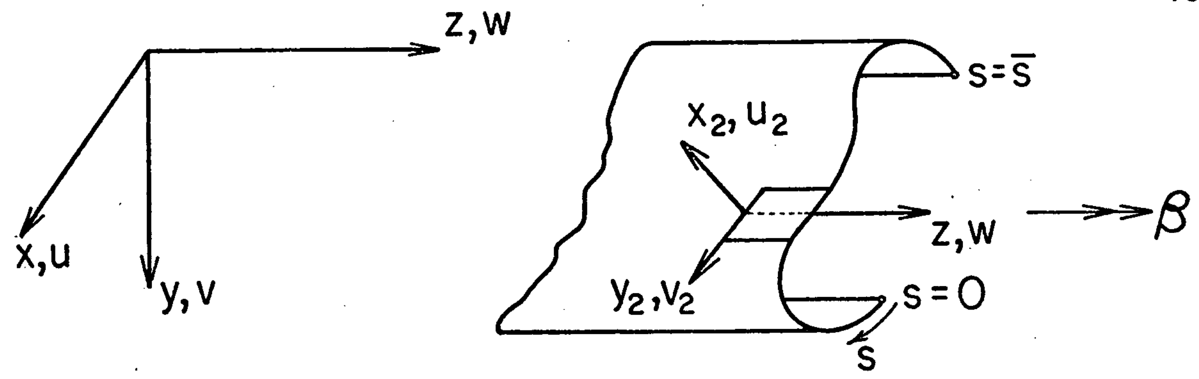
1:4 Equilibrium Equations

There are several sets of separate equilibrium equations to be used in the development. The main ones of interest are the equations relating the forces on the face of an element to the overall force resultants on the beam and the equations giving overall equilibrium of the beam segment in terms of these resultants.

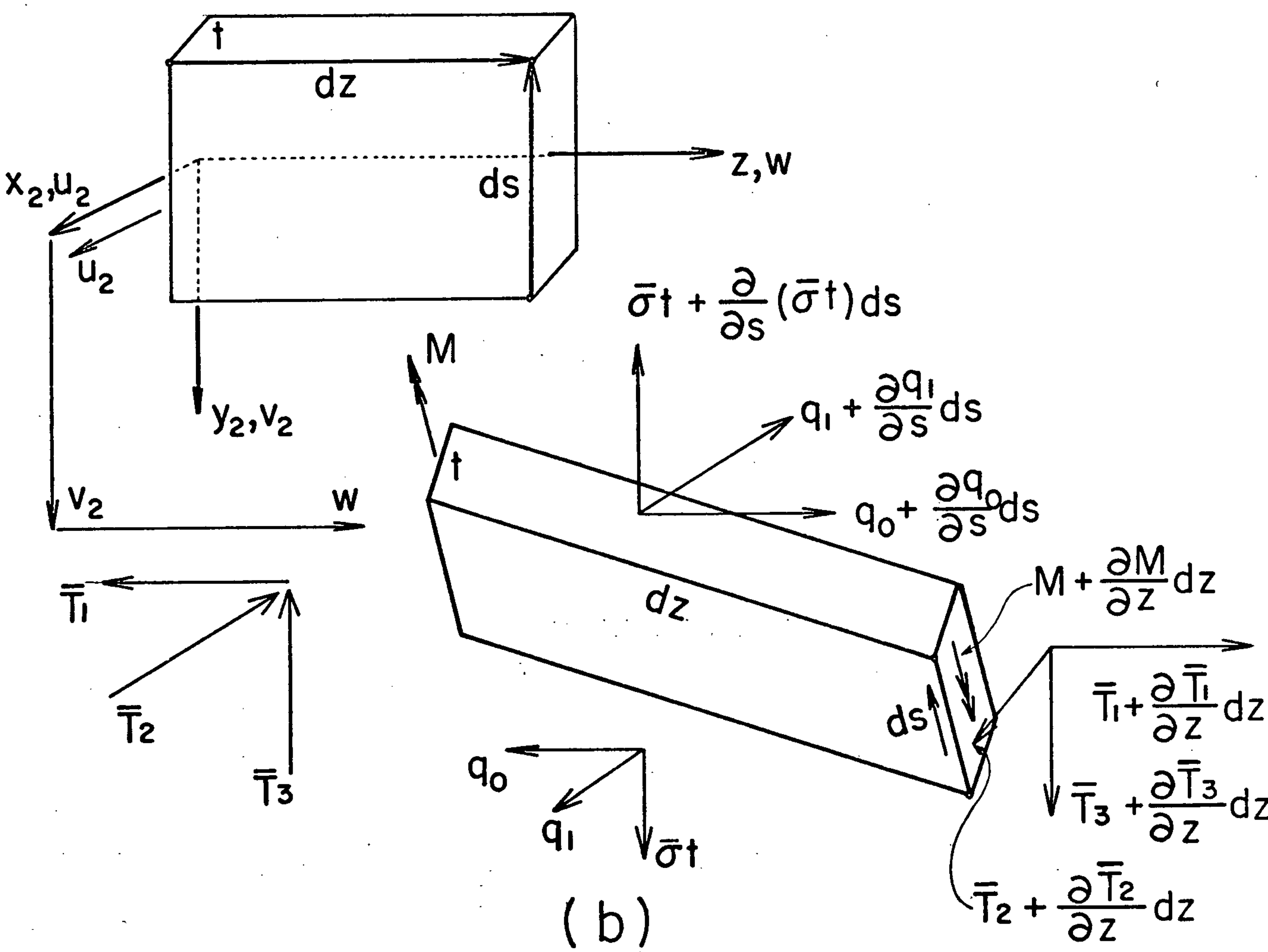
However, there is a problem that arises from the displacements that must be overcome before these sets of equilibrium equations can be used. Recall that the aim of this development is to obtain the overall equilibrium equations in terms of the displacements and their derivatives. This will be done by relating the force resultants on the beam to the stresses, which in turn can be found from the strains. Unfortunately, the assumptions on the displaced shape are that the shear strains are zero. Thus, only the normal stresses can be found from a stress-strain law and an alternative method must be found to find the shears. This can be done by writing equilibrium of a small displaced element of size ds by dz which will give relations between shear flows and normal forces. This will give the shears in terms of normal forces. A second set of equations will then be developed which will link the normal forces, and therefore indirectly, the shears, to the normal stress in the beam. Since the normal stress can be written in terms of the strains, this allows both the shear flows and normal forces to be written in terms of the displacements. This procedure successfully circumvents the problem of directly writing the shears in terms of the strains, and it will be developed before the overall equilibrium equations for the beam are developed.

To develop the equations between the shear and force flows, the small element of cross section shown in Figure 5 will be used. The quantities $\bar{\bar{T}}_1, \bar{\bar{T}}_2, \bar{\bar{T}}_3$ are elemental force resultant flows in the fixed axis x_2, y_2, z. They act on the cross section face. $\bar{\bar{T}}_1$ may be thought of as a normal force, $\bar{\bar{T}}_2$ and $\bar{\bar{T}}_3$ as shears. The quantities $\bar{\sigma}t$, q_o and q are also force resultant flows, and they act along planes of the element that were initially parallel to the z axis. These resultants are also defined along the fixed elemental axes x_2, y_2, z. The resultant M is taken as being along the tangent ds in the deformed shape. This is the secondary resultant due to plate bending effects. Any shears associated with M are automatically taken care of by $\bar{\bar{T}}_2$ and $\bar{\bar{T}}_3$.

Using the stress flows defined in Figure 5 and writing equilibrium gives the following equations:



(a)



(b)

(a) – element of interest shown in the undisplaced cross – section

(b) – enlarged view of element showing it undisplaced and displaced with respect to the element principal axis x_2, y_2, z .

FIG. 5 ELEMENT TAKEN FROM CROSS-SECTION .

$$\sum F_z = 0 \qquad \frac{\partial q_o}{\partial s} = -\frac{\partial \overline{T}_1}{\partial z} \tag{15}$$

$$\sum M_{x_2} = 0 \qquad \overline{T}_1 v_2' - \overline{T}_3 - q_o - (M\beta)' = 0 \tag{16}$$

$$\sum M_{y_2} = 0 \qquad -\overline{T}_1 u_2' + \overline{T}_2 + M' - q_o\beta = 0 \tag{17}$$

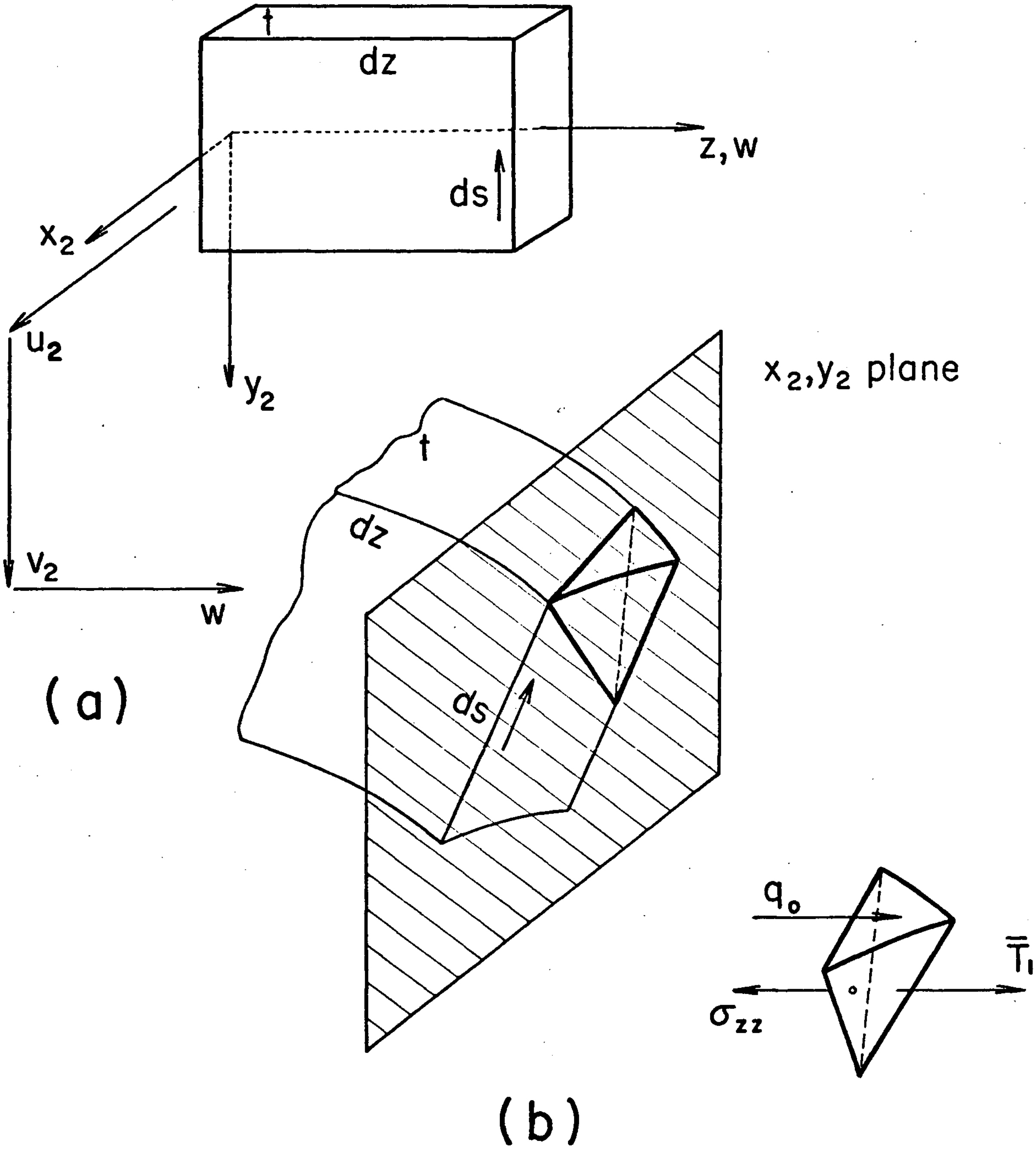
There are of course three other equilibrium equations, but they are not necessary for the development and so are not given. The three given equations relate the normal and shear force flows on the element. However, they do not as yet relate the force flows to the internal stresses and therefore the strains. Since it is desired to get the flows as a function of the strains, the following set of equations will be developed relating σ_{zz}, the actual stress in the z direction and $\overline{T}_1$, the force on the element in the z direction. The distinction between the two must be made as $\overline{T}_1$ is acting on a skewed surface once the element is deflected, and therefore need not coincide with σ_{zz} in value.

Using Figure 6, the axes x_2, y_2, z are the fixed elemental principal axes. The element is shown displaced with a small corner removed.

From equilibrium of the corner:

$$\overline{T}_1 = \sigma_{zz} t - q_o v_2' \tag{18}$$

where v_2' in the $q_o v'$ term arises from the ratio of areas of the little corner element and σ_{zz} is the actual internal stress acting in the displaced element in the direction of the original axes. In other words, it is the normal stress in the x_2, y_2 cut plane of the corner element. Equation (18) is similar to one of the equations in the well-known elasticity equation



(a) - element of interest shown in undisplaced and displaced position with respect to the element principal axis x_2, y_2, z. The x_2, y_2 plane is shown cutting the displaced element near a corner.

(b) - free body diagram showing forces and stresses of interest acting on the corner of the element truncated by the x_2, y_2 plane.

FIG. 6 ELEMENT RELATING $\bar{T}_1$ AND σ_{zz}.

$\sigma_{ij}\,\eta_j = T_i$ which gives the external stress vector T in terms of the internal stress σ_{ij} and the outward normal η_j. Equation (18) is enough to establish the relationship between internal stress and external force flows. Equation (18), along with Equations (15), (16) and (17) when coupled with the stress-strain laws, allows both shear and normal force $\bar{\bar{T}}_1$, $\bar{\bar{T}}_2$ and $\bar{\bar{T}}_3$ to be written in terms of the strains.

Now that the problem of the zero shear strains is overcome, the next step is to define overall stress resultants acting on the cross section. These are shown in Figure 7 where V are shears, P is an axial force, BM are moments and T is a torque and all are defined in the direction of the fixed axes. They are allowed to translate but must remain parallel to x, y, z.

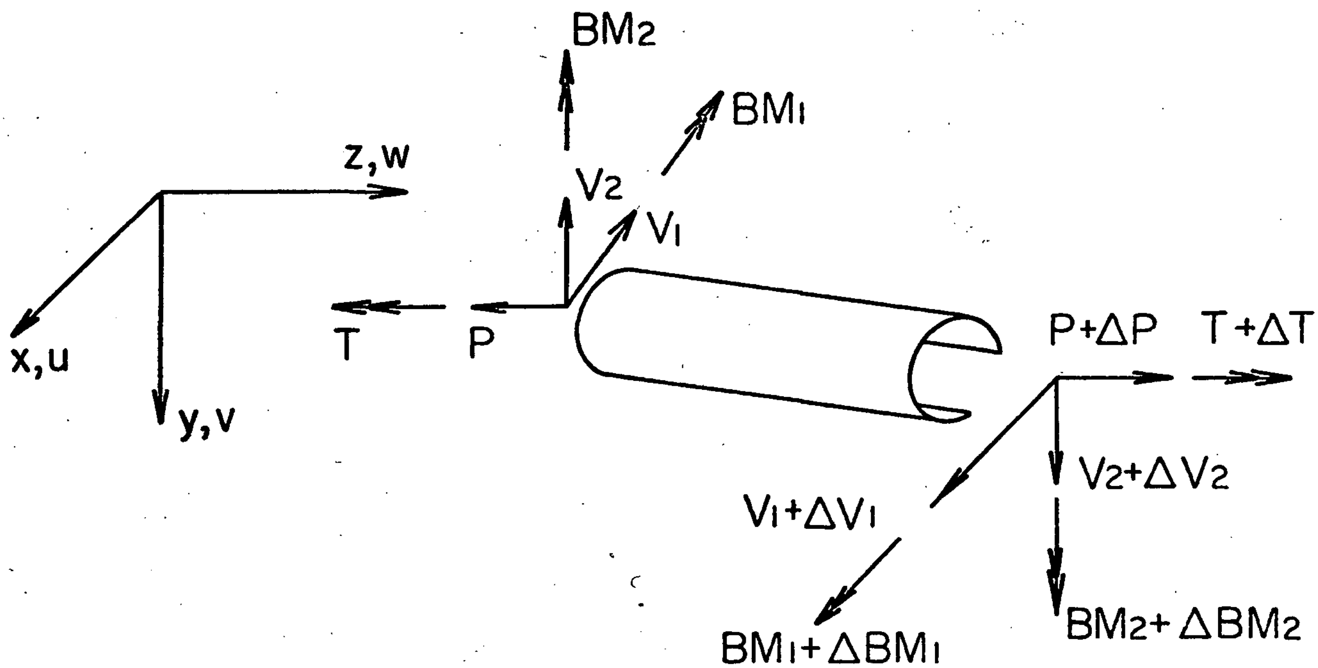


FIG.7 BEAM SECTION WITH RESULTANT FORCES.

Using equilibrium, the overall force resultants on the entire cross section may be defined in terms of $\bar{\bar{T}}_1$, $\bar{\bar{T}}_2$ and $\bar{\bar{T}}_3$. It is first convenient to define the force flows T_1, T_2 and T_3 as being analogous to $\bar{\bar{T}}_1$, $\bar{\bar{T}}_2$ and $\bar{\bar{T}}_3$ but parallel to z, x, y respectively rather than

z, x_2, y_2. From equilibrium and the geometry of Figure 4,

$$T_1 = \bar{\bar{T}}_1$$

$$T_2 = \bar{\bar{T}}_2 \sin\phi - \bar{\bar{T}}_3 \cos\phi \qquad (19)$$

$$T_3 = \bar{\bar{T}}_2 \cos\phi + \bar{\bar{T}}_3 \sin\phi$$

Now

$$\int_0^{\bar{s}} T_1 \, ds = P$$

$$\int_0^{\bar{s}} T_2 \, ds = V_1$$

$$\int_0^{\bar{s}} T_3 \, ds = V_2$$

$$\int_0^{\bar{s}} (T_1 \eta - T_3 w - M \cos\gamma) \, ds = BM_1 \qquad (20)$$

$$\int_0^{\bar{s}} (-T_1 \xi + T_2 w + M \sin\gamma) \, ds = BM_2$$

$$\int_0^{\bar{s}} (-T_2 \eta + T_3 \xi - M v_2') \, ds = T$$

where ξ, η and γ are shown in Figure 8. They are an axis that translates with the section but does not rotate. In the undeformed position of the beam, ξ, η and x, y are co-incident.

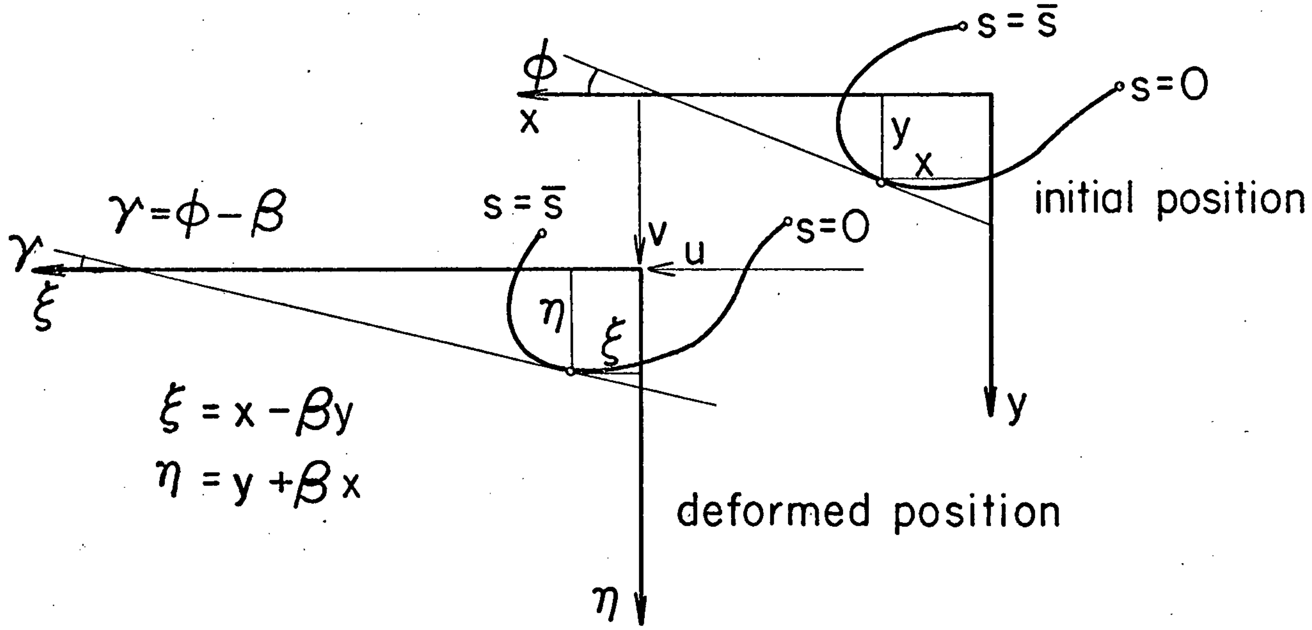


FIG.8 DEFORMED AND UNDEFORMED CROSS SECTION .

A final set of equilibrium equations can now be written relating the overall resultants BM, V, P, T. Using Figure 7, the equilibrium equations about the displaced position are:

$$\sum F_z = 0 \qquad -P + P + \Delta P = 0 \qquad P' = 0 \tag{21}$$

$$\sum F_y = 0 \qquad -V_2 + V_2 + \Delta V_2 = 0 \qquad V_2' = 0 \tag{22}$$

$$\sum F_x = 0 \qquad -V_1 + V_1 + \Delta V_1 = 0 \qquad V_1' = 0 \tag{23}$$

$$\sum M_z = 0 \qquad T' + V_2 u' - V_1 v' = 0 \tag{24}$$

$$\sum M_y = 0 \qquad BM_2' + V_1 - P u' = 0 \tag{25}$$

$$\sum M_x = 0 \qquad BM_1' - V_2 + P v' = 0 \tag{26}$$

There are now sufficient equations between the displacements, strains, constitutive laws and the equilibrium conditions to completely describe the problem. The task now is to assemble all these equations into a useful set of differential equations which compactly and completely describe the problem. This will be done in the next section where a number of simplifying engineering assumptions will be employed and their effects studied.

CHAPTER 3

ASSEMBLY OF DIFFERENTIAL EQUATIONS

The differential equations of buckling can now be developed by writing the overall equilibrium equations in terms of the overall force resultants which can be related to the displacements through the use of selected stress and strain equations. This gives the overall equilibrium equations in terms of displacements and their derivatives, and these will be taken as the governing differential equations.

This is easiest accomplished by working through the displacements, then the strains, then the stress-strain equations, getting the stress in terms of displacements, then getting the force resultants from the stresses and substituting them into the overall gross equilibrium equations. This can be done using the ideas and equations presented in the previous chapter.

Using this approach, the displacement and strain equations will be discussed first.

3:1 Discussion of Displacement and Strain

The constraint placed on the displacement at this point is u', v', β are small compared to one. This was used in the development of Equation (7) for w. When it comes to evaluating the strains it would be convenient if u', v' etc. were small compared to the linear strains, as this would take

$$E_{zz} = \frac{1}{2}\left[2w' + (w')^2 + (u'_s)^2 + (v'_s)^2\right] \tag{10}$$

and remove the squared terms from it. If this were the case, a condition for the case of bending in the y plane would be:

$$2|w'| \gg (v')^2 \tag{27}$$

But w' is of the order of $v''y + w_o'$. Therefore (27) becomes:

$$2\left| v'' y + W_o' \right| \gg (v')^2 \tag{28}$$

To get some physical idea for the result, postulate a beam deflecting in the following typical curve:

$$v = v_o \sin \frac{\pi z}{L}$$

Therefore,

$$v' = v_o \frac{\pi}{L} \cos \frac{\pi z}{L}$$

$$v'' = -v_o \left(\frac{\pi}{L}\right)^2 \sin \frac{\pi z}{L}$$

The maximum absolute values become:

$$v_{max} = \left| v_o \right|$$

$$v'_{max} = \left| v_o \frac{\pi}{L} \right|$$

$$v''_{max} = \left| v_o \frac{\pi^2}{L^2} \right|$$

Substituting these into Equation (28) and taking the case of $W_o' = 0$ gives:

$$2\left| y \right| \gg \left| v_o \right| \tag{29}$$

In other words, the width and depth of the section must be large compared to the deflection. Similar conclusions can be shown for deflections u and β.

This is undoubtedly satisfied by a great many actual cases, but unfortunately many problems of interest may violate Equation (29). A long thin cantilever for instance may deflect several times its depth before buckling. It is to be remembered that Equation (29) was developed under the conditions that the second order strain terms are small enough to be neglected. There is always the possibility that the second order terms may be large compared to w' and still have no effect. For instance, the long thin cantilever beam under end shear may deflect several times its depth and yet the strains are linear and the usual linear elastic analysis is adequate to obtain moments. It is worth pursuing this to see if it is possible to relax Equation (29).

To do this, Equation (7) will first of all be re-examined. Equation (7)

$$w = w_o + \beta'\omega_1 + u'(x_o - x) + v'(y_o - y) \tag{7}$$

as it stands is linear in u', v' and β'. This means that it is incapable of handling effects such as axial foreshortening, since this effect depends on non-linear combinations of u', v', β'. To include these effects, it is convenient to define the quantity $\overline{w}$, which is the axial displacement in the z direction of the point S and it includes such effects as axial foreshortening as well as the effect that continuity of the beam and boundary conditions have on axial foreshortening. Clearly $\overline{w}$ can be related to w by the following equation:

$$\overline{w} = w - \frac{1}{2}\int_0^z (u_s')^2 dz - \frac{1}{2}\int_0^z (v_s')^2 dz + f(z) \tag{30}$$

where w is defined by Equation (7), the squared terms are the effects of axial foreshortening if the rotations u_s', v_s' can be considered to take place as rigid body rotations and $f(z)$ is the term which represents the effect of constraints such as beam continuity or boundary conditions. For example, if the supports at either end of the beam prevent motion of the ends, then elongation takes place during lateral deflections and $f(z)$ is to look after effects similar to this, as the axial foreshortening in this case cannot be regarded as due to rigid body rotations alone. If the axial foreshortening can be treated as being due entirely to rigid body rotations only, then $f(z) = 0$. It should be noted that Equation (30) has meaning even under the original constraints that $u'v'$, $\beta v'$ etc. are negligible compared to u', v' as in Equation (30) the squared terms will be of the order $L \times (u')^2$ whereas the terms in u', v' in w are of the order $u'x$, $v'y$, and $u'v' \ll u', v'$ does not imply $L \times (u')^2 \ll xu', yv'$. Using the more precise value of $\bar{w}$ for w in Equation (10) for the strain E_{zz}, and using u_s and v_s from Equation (9) gives:

$$E_{zz} = \tfrac{1}{2}\left[2w' - (v' - \beta' y)^2 - (v' + \beta' x)^2 + f'(z) + (\bar{w}')^2 + (u' - \beta' y)^2 + (v' + \beta' x)^2 \right]$$

$$= w' + \tfrac{1}{2} f'(z) + \tfrac{1}{2}(\bar{w}')^2 \tag{31}$$

In other words, the linear strain w' is adequate if $f'(z) = 0$ as clearly $(\bar{w}')^2 \ll w'$. This will occur if $f(z) = 0$, which is the condition that the axial foreshortening can be viewed solely as the result of rigid body rotation foreshortening. It will now be discussed when this occurs.

First of all, assume $u', v' = 0$ and only $\beta' \neq 0$. Then the squared terms in Equation (30) for $\bar{w}$ are functions of β' and x, y only. Since β' is constant across the cross section, the axial foreshortening varies as x and y. This cannot be accomplished by any rigid body motion of the entire element cross section. This differential foreshortening means that, since adjacent fibres will be trying to change different lengths, continuity conditions of the material will cause adjacent fibres to constrain each other in some way. Because of this constraint, a rigid body movement is not possible for each fibre for a β deflection, and therefore $f(z) \neq 0$. Since $f(z)$ equals a constant is merely a rigid displacement in the z

direction, $f(z)$ will be some function of z other than a constant. This means that $f'(z) \neq 0$ and therefore the hoped for result of Equation (31)

$$E_{zz} = w'$$

does not materialize. Therefore, β will be constrained to values which render $(\beta' x)^2$ and $(\beta' y)^2$ small compared to w'. Some physical idea for this can be obtained by following a development similar to that of Equation (29).

The terms to be compared are $\beta'' \omega_1$ from w' and $\frac{1}{2}(\beta' x)^2$, $\frac{1}{2}(\beta' y)^2$ from $\overline{W}$ and Equation (10). Writing the desired condition, similar to Equation (28), gives:

$$|2\beta'' \omega_1| >> (\beta' x)^2, \; (\beta' y)^2 \qquad (32)$$

Assuming $\beta = \beta_0 \sin \frac{\pi z}{L}$ and placing the largest values for β', β'' into Equation (32) gives:

$$|2\omega_1| >> |\beta_0 x^2|, \; |\beta_0 y^2| \qquad (33)$$

The effect of the β that is being removed from the equations is easiest visualized by examining a solid right circular cylinder as shown in Figure 9. In Figure 9a, the undeformed cylinder is shown. In Figure 9b, the cylinder is shown with the fibres in the position they would assume under an end rotation β if they were unconstrained. The axial foreshortening would be proportional to R^2, and therefore the surface would not remain plane. Figure 9c shows the fibres as they actually appear along with the resulting stress. This final shape is assumed because of the constraint placed on the fibres to behave as a continuum. Also note the lack of axial

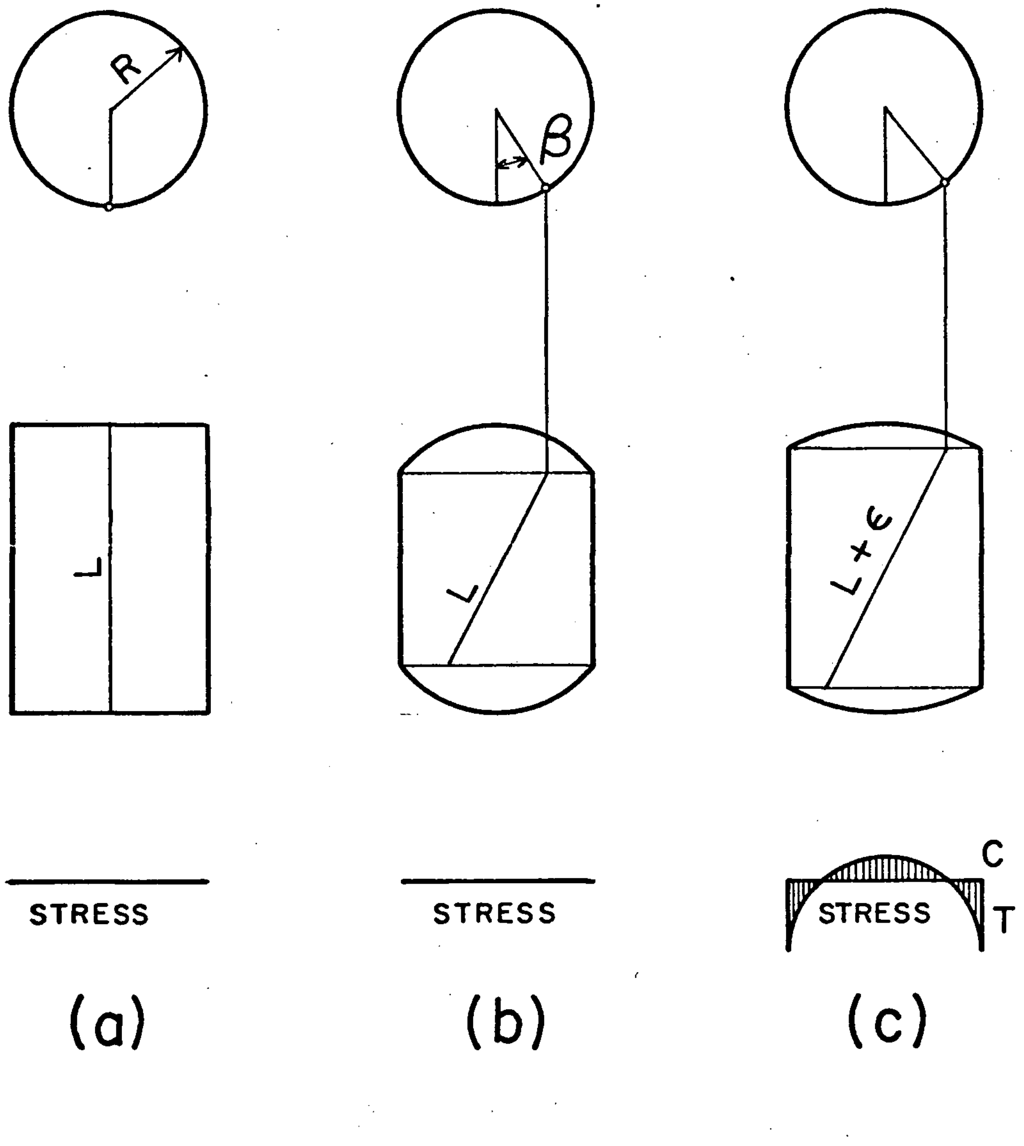


FIG.9 CYLINDER UNDER END ROTATION.

end restraint in this particular example. See Timoshenko [8], pp. 286-291.

Now that $\beta' \neq 0$ has been treated, take the case of $\beta' = 0$ and $u', v' \neq 0$. Since u', v' are constant across the cross section, the foreshortening of one fibre by Equation (30) is the same as all the others. Therefore there is no differential movement axially to require any constraint, as was the case with β'. However, if the beam is restrained axially at the boundarys, this will prevent the beam from shortening axially, as it tends to do if it has slopes anywhere. This would provide a constraint on the fibres as they rotate and so once again $f(z)$ would have a value in Equation (30) and $f'(z)$ would appear in the strain. But if this is not the case, and the end is free to move axially for shortening due to rotations, then $f(z) = 0$ and consequently $f'(z) = 0$. This then allows the desired result $E_{zz} = w'$.

This argument was based on the Green's tensor E_{ij}. This tensor is easy to use compared to the Almansi strain tensor e_{ij}, as the Almansi strain is based on the co-ordinate system defined by the displaced shape. However, the Almansi strain is related to the actual stress, and since it is the actual stress that will be used later, the tensor e_{ij} will be examined. This tensor lacks the physical interpretation that can be associated with the Green's tensor, so the Green's tensor was discussed first to provide some idea as to why the non-linear strain terms fall out.

The form of the strain tensor e_{ij} can be taken from Fung [4], Chapter 16, where the following relation between the displaced axis and the fixed axis is defined as:

$$x_D = x + u_s$$

$$y_0 = y + v_s$$

$$z_0 = z + \overline{w} \tag{34}$$

where once again $\overline{w}$ is given by Equation (30) and u_s , v_s are the displacements along the u , v directions at point S . The linear form of u_s , v_s can be found from Equation (9).

$$\begin{aligned} u_s &= u - \beta y \\ v_s &= v - \beta x \end{aligned} \tag{9}$$

Taking account of the change in u_s , v_s due to shortening because of slopes gives:

$$\begin{aligned} u_s &= u - \beta y + G_1(x, y, z) \\ v_s &= v - \beta x + G_2(x, y, z) \end{aligned} \tag{35}$$

where $G_1(x, y, z)$ is of the order $-\frac{1}{2}\left(u'_{s_{max}}\right)^2 x$

and $G_2(x, y, z)$ is of the order $-\frac{1}{2}\left(v'_{s_{max}}\right)^2 y$

The Almansi strain e_{zz} may be written as:

$$e_{zz} = \frac{\partial \overline{w}}{\partial z_0} - \frac{1}{2}\left[\left(\frac{\partial \overline{w}}{\partial z_0}\right)^2 + \left(\frac{\partial u_s}{\partial z_0}\right)^2 + \left(\frac{\partial v_s}{\partial z_0}\right)^2\right] \tag{36}$$

Now, the following differential relations can be written:

$$\frac{\partial u_s}{\partial z_0} = \frac{\partial u_s}{\partial x}\frac{\partial x}{\partial z_0} + \frac{\partial u_s}{\partial y}\frac{\partial y}{\partial z_0} + \frac{\partial u_s}{\partial z}\frac{\partial z}{\partial z_0}$$

$$\frac{\partial v_s}{\partial z_0} = \frac{\partial v_s}{\partial x}\frac{\partial x}{\partial z_0} + \frac{\partial v_s}{\partial y}\frac{\partial y}{\partial z_0} + \frac{\partial v_s}{\partial z}\frac{\partial z}{\partial z_0}$$

$$\frac{\partial \overline{w}}{\partial z_0} = \frac{\partial \overline{w}}{\partial x}\frac{\partial x}{\partial z_0} + \frac{\partial \overline{w}}{\partial y}\frac{\partial y}{\partial z_0} + \frac{\partial \overline{w}}{\partial z}\frac{\partial z}{\partial z_0} \tag{37}$$

$$\frac{\partial \beta}{\partial z_0} = \frac{\partial \beta}{\partial x}\frac{\partial x}{\partial z_0} + \frac{\partial \beta}{\partial y}\frac{\partial y}{\partial z_0} + \frac{\partial \beta}{\partial z}\frac{\partial z}{\partial z_0}$$

These are all that is required to transfer portions of Equation (36) from x_0, y_0, z_0 to x, y, z. This allows parts of Equation (36) to be written in terms of previously evaluated functions.

Denoting differentiation with respect to z_0 as $\frac{\partial \theta}{\partial z_0} = \overset{*}{\theta}$ and using Equation (34), Equation (37) becomes:

$$\overset{*}{u}_s = \frac{\partial u_s}{\partial x}(-\overset{*}{u}_s) + \frac{\partial u_s}{\partial y}(-\overset{*}{v}_s) + u_s'(1-\overset{*}{\overline{w}})$$

$$\overset{*}{v}_s = \frac{\partial v_s}{\partial x}(-\overset{*}{u}_s) + \frac{\partial v_s}{\partial y}(-\overset{*}{v}_s) + v_s'(1-\overset{*}{\overline{w}})$$

$$\overset{*}{\overline{w}} = \frac{\partial \overline{w}}{\partial x}(-\overset{*}{u}_s) + \frac{\partial \overline{w}}{\partial y}(-\overset{*}{v}_s) + \overline{w}'(1-\overset{*}{\overline{w}})$$

$$\overset{*}{\beta} = \frac{\partial \beta}{\partial x}(-\overset{*}{u}_s) + \frac{\partial \beta}{\partial y}(-\overset{*}{v}_s) + \beta'(1-\overset{*}{\bar{w}}) \tag{38}$$

By using Equations (35), the terms $\frac{\partial u_s}{\partial x}$, $\frac{\partial u_s}{\partial y}$, $\frac{\partial v_s}{\partial y}$ etc. may be evaluated:

$$\frac{\partial u_s}{\partial x} \approx -\frac{(u'_{s\,max})^2}{2} , \qquad \frac{\partial v_s}{\partial y} \approx -\frac{(v'_{s\,max})^2}{2}$$

$$\frac{\partial u_s}{\partial y} \approx -\beta - 2(u' - \beta' y)\beta'_{max} , \qquad \frac{\partial v_s}{\partial x} = +\beta - 2(v' + \beta' x)\beta'_{max}$$

Also,

$$\frac{\partial \beta}{\partial x} = 0 \qquad \frac{\partial \beta}{\partial y} = 0$$

Using these, Equations (38) become:

$$\overset{*}{u}_s \cong -\frac{(u'_{s\,max})^2}{2}(-\overset{*}{u}_s) + (-\beta - 2(u' - \beta' y)\beta'_{max})(-\overset{*}{v}_s) + u'_s(1-\overset{*}{\bar{w}})$$

$$\overset{*}{v}_s \cong (+\beta - 2(v' + \beta' x)\beta'_{max})(-\overset{*}{u}_s) - \frac{(v'_{s\,max})}{2}(-\overset{*}{v}_s) + v'_s(1-\overset{*}{\bar{w}}) \tag{39}$$

$$\overset{*}{\bar{w}} \cong \frac{\partial \bar{w}}{\partial x}(-\overset{*}{u}_s) + \frac{\partial \bar{w}}{\partial y}(-\overset{*}{v}_s) + \bar{w}'(1-\overset{*}{\bar{w}})$$

$$\overset{*}{\beta} = \beta'(1-\overset{*}{\bar{w}})$$

Since previously, the rotations have been restricted to be at least as small as $u'v' \ll u'$, $\beta u' \ll v'$ etc. it is clear that those terms containing three angles as a product can certainly be discarded compared to terms such as u_s', v_s' in the equations for $\overset{*}{u}_s$, $\overset{*}{v}_s$.

Therefore, Equations (39) reduce to (40):

$$
\begin{aligned}
\overset{*}{u}_s &= +\beta\,\overset{*}{v}_s + u_s'(1-\overset{*}{\bar{w}}) \\
\overset{*}{v}_s &= -\beta\,\overset{*}{u}_s + v_s'(1-\overset{*}{\bar{w}}) \\
\overset{*}{\bar{w}} &= \frac{\partial \bar{w}}{\partial x}(-\overset{*}{u}_s) + \frac{\partial \bar{w}}{\partial y}(-\overset{*}{v}_s) + \bar{w}'(1-\overset{*}{\bar{w}}) \\
\overset{*}{\beta} &= \beta'(1-\overset{*}{\bar{w}})
\end{aligned}
\tag{40}
$$

Now, from Equations (30) and (7):

$$
\begin{aligned}
\frac{\partial \bar{w}}{\partial x} &= -u' + \beta'\frac{\partial \omega_1}{\partial x} - \int_0^z 2(v'+\beta' x)\beta'\,dz \\
\frac{\partial \bar{w}}{\partial y} &= -v' + \beta'\frac{\partial \omega_1}{\partial y} - \int_0^z 2(u'-\beta' y)\beta'\,dz
\end{aligned}
\tag{41}
$$

Substitution of Equations (41) and the values for $\overset{*}{u}_s$, $\overset{*}{v}_s$ into the equation for $\overset{*}{\bar{w}}$ in Equation (39) gives:

$$
\begin{aligned}
\overset{*}{\bar{w}} = {} & \left(-u' + \beta'\frac{\partial \omega_1}{\partial x} - \int_0^z (v'+\beta' x)\beta'\,dz\right)\left(-\beta\,\overset{*}{v}_s - u'(1-\overset{*}{\bar{w}})\right) \\
& + \left(-v' + \beta'\frac{\partial \omega_1}{\partial y} - \int_0^z (u'-\beta' y)\beta'\,dz\right)\left(+\beta\,\overset{*}{u}_s - v'(1-\overset{*}{\bar{w}})\right) \\
& + \bar{w}'(1-\overset{*}{\bar{w}})
\end{aligned}
\tag{42}
$$

When this complicated expression is placed in Equation (36), the non-linear terms do not fall out of the strain equation. However, Equation (42) has many small high order terms in it. By neglecting terms containing three angles as products compared to terms such as $u'v'$, $u'\beta$ etc., realizing that β' is of the order β/L and that $\frac{\partial \omega_1}{\partial x} \cong y$ and $\frac{\partial \omega_1}{\partial y} \cong x$, Equation (42) becomes:

$$\overset{*}{w} = (u')^2(1-\overset{*}{w}) + (v')^2(1-\overset{*}{w}) + \overline{w}'(1-\overset{*}{w}) \tag{43}$$

When placed in Equation (36) for e_{zz}, and using

$$\overset{*}{u}_s \cong \beta \overset{*}{v}_s + u'_s(1-\overset{*}{w}) \cong u'_s(1-\overset{*}{w})$$

$$\overset{*}{v}_s \cong -\beta \overset{*}{u}_s + v'_s(1-\overset{*}{w}) \cong v'_s(1-\overset{*}{w})$$

since $\beta \overset{*}{v}_s, \beta \overset{*}{u}_s \ll u'_s, v'_s$, e_{zz} becomes:

$$e_{zz} \cong (u'_s)^2(1-\overset{*}{w}) + (v'_s)^2(1-\overset{*}{w}) + \overline{w}'(1-\overset{*}{w})$$
$$-\tfrac{1}{2}\left[\overset{*}{w}{}^2 + (u')^2(1-\overset{*}{w})^2 + (v')^2(1-\overset{*}{w})^2\right]$$

Since $\overset{*}{w} \ll 1$, then:

$$e_{zz} \cong (u'_s)^2 + (v'_s)^2 + \overline{w}' - \tfrac{1}{2}(u')^2 - \tfrac{1}{2}(v')^2$$

But $\overline{w}' \cong w' - \frac{1}{2}(v'_s)^2 - \frac{1}{2}(u'_s)^2 + f'(z)$

Therefore $e_{zz} \cong w' + f'(z)$

Once again, the conclusion has been reached that if $f'(z) = 0$, then the linear approximation to the strain is adequate. Recalling the previous discussion of $f(z)$, its assuming a zero value can be taken to imply $(\beta' x)^2$ and $(\beta' y)^2$ are small compared to w' and that there is no axial restraint on the beam.

This discussion has dealt with the axial strains E_{zz} and e_{zz} only. The shear strains are not necessary as an alternative method for finding a measure of the shear force was given in section 1:4 on Equilibrium Equations.

3:2 Stress-Strain Equations

Now that the strain-displacement relation has been decided upon, the constitutive relation

$$\sigma_{ij} = \lambda e_{\alpha\alpha} \delta_{ij} + 2 G e_{ij} \tag{11}$$

from section 1:3 can be used to relate stress and strain at point S. Using the usual beam assumption of Poisson's ratio being zero and the results of 3:1 changes Equation (11) to

$$\sigma_{zz} = E e_{zz} = E w' = \sigma \tag{44}$$

This gives σ in terms of the displacement w and Young's Modulus E.

Notice that the actual stress σ and the Almansi strain were used. This is because equilibrium equations were used which required actual stresses in displaced shapes to get Equation (13) and that only σ_{zz} was required to do this, whereas the Kirchoff stress has no easy physical representation and to write equilibrium using it would be difficult.

3:3 Relating Internal Stresses to External Stress Vectors

Now that the stresses have been related to displacements, the next step is to relate the internal stresses to the external stress vectors $\overline{T}_1, \overline{T}_2, \overline{T}_3$ on the beam face, as it is these vectors that are needed to get the overall beam resultants from Equation (20). This will require some manipulation, as these equations are full of non-linear high order terms that will be discarded because of the constraints on the rotations. The equations of interest from the section 1:4 on Equilibrium Equations are:

$$\frac{\partial q_o}{\partial s} = -\frac{\partial \bar{T}_1}{\partial z} \tag{15}$$

$$\bar{T}_1 v_2' - \bar{T}_3 - q_o - (M\beta)' = 0 \tag{16}$$

$$-\bar{T}_1 u_2' + \bar{T}_2 + M' - q_o \beta = 0 \tag{17}$$

$$\bar{T}_1 = \sigma_{zz} t - q_o v_2' \tag{18}$$

These equations give sufficient relations between $\bar{T}_1, \bar{T}_2, \bar{T}_3$; q_o and the one component σ of the stress tensor that can be found from the strain tensor to allow them all to be evaluated in terms of the displacements and their derivatives. Rewriting Equations (16), (17) and (18) and integrating (15):

$$\begin{aligned} \bar{T}_3 &= \bar{T}_1 v_2' - q_o - (M\beta)' \\ \bar{T}_2 &= \bar{T}_1 u_2' - M' + q_o \beta \\ \bar{T}_1 &= \sigma t - q_o v_2' \\ q_o &= \int_0^s -\bar{T}_1' \, ds \end{aligned} \tag{45}$$

The problem now is to reduce Equation (45) by removing all the terms rendered negligible by the constraints on the angles.

Obviously the term $q_o v_2'$ is important for $\bar{T}_1$, but in the equations for $\bar{T}_2$ and $\bar{T}_3$ it can be hopefully shown to be small, both in $\bar{T}_1 v_2'$, $\bar{T}_1 u_2'$ and q_o. Substituting for $\bar{T}_1$ in $\bar{T}_2$ and $\bar{T}_3$ gives:

$$\begin{aligned} \bar{T}_3 &= \sigma t v_2' - q_o v_2' v_2' - q_o - (M\beta)' \\ \bar{T}_2 &= \sigma t u_2' - q_o v_2' u_2' + q_o \beta - M' \end{aligned} \tag{46}$$

Now, the q_0 terms can be written as $q_0(v_2' v_2' - 1)$ and $q_0(u_2' v_2' + \beta)$. Previously, the equations were constrained to be valid only where $u'v'$ were negligible compared to u', v' or β, or where u', v', β could be discarded compared to one. Applying these conditions to the q_0 terms, it is obvious that $q_0 v_2' u_2'$ and $q_0 v_2' v_2'$ can be discarded. Now,

$$q_0 = -\int_0^s T_1' \, ds = -\int_0^s (\sigma t - q_0 v_2')' \, ds$$

This is a complicated non-linear equation. It will hopefully be shown that $q_0 v_2'$ is small when differentiated. When expanded, q_0 becomes:

$$q_0 = -\int_0^s (\sigma t)' \, ds + \int_0^s q_0' v_2' \, ds + \int_0^s q_0 v_2'' \, ds \tag{47}$$

Comparing the left-hand side of Equation (47) with the last term on the right-hand side gives:

$$q_0 \;\Big|\; \int_0^s q_0 v_2'' \, ds$$

Now,

$$\alpha q_{o_{max}} \;\Big|\; \int_0^s q_{o_{max}} v''_{2\,max} \, ds > \int_0^s q_0 v_2'' \, ds \qquad 0 \leq \alpha \leq 1$$

Therefore

$$\alpha q_{o_{max}} \;\Big|\; q_{o_{max}} v''_{2\,max} \, s \tag{48}$$

Invoking the previously used shape

$$v = v_0 \sin \frac{\pi z}{L}$$

results in Equation (48) becoming:

$$\alpha \;\Big|\; \frac{\pi^2 v_0 s}{L^2}$$

If $v_0 < \frac{L}{100}$, then Equation (48) finally becomes:

$$\alpha \Big| \quad \frac{\pi^2 s}{100\,L} \approx \frac{s}{10\,L}$$

If $\frac{s}{10L} < \frac{1}{20}$, then it will be assumed that it can safely be discarded with respect to α. This leads to the condition $s < \frac{L}{2}$. This is a reasonable condition, and it will be inferred from this simple example that if

$v_{max} < \frac{L}{100}$ and $\bar{s} < \frac{L}{2}$ then

$\int_0^s q_o v_2'' \, ds$ may be discarded.

It should be noted that this constraint tends to run counter to the constraints expressed in Equation (29).

Treatment of the term $q_o' v_2'$ is slightly more complicated, because as it stands q_o' would have to be compared to q_o, which is not very meaningful if done directly. Therefore, a discussion of q_o' will be helpful.

The term q_o is caused by two effects: shears due to the forces V and shears due to the torque T. However, for the linear case, that part of q_o due to V is constant and therefore the associated q_o' for the linear case is zero. This means that for the non-linear part q_o' due to V is second order and the product $q_o' v_2'$ is third and therefore can be discarded. The q_o portion not due to V is due to T and this has previously been taken as small because of the restraints on the β' terms, and because the section is being taken to act as a beam, not a shaft. This alone would imply that the q_o due to T is small and therefore, since q_o would be reasonably smooth with z, that q_o' is small.

However, even granting this, it is possible to show the unimportance of $q_o' v_2'$ from another argument.

The term q_o' can once again be broken into two parts: the linear and non-linear terms. The non-linear terms can be neglected, as they become third order terms in $q_o' v_2'$. This leaves the linear terms. They can be taken as $q_o' = \int_o^s (\sigma t)'' ds$

or $$q_o' = \int_o^s \left(w_o'' + u'''(x_o - x) + v'''(y_o - y) + \beta''' \omega_1 \right) t \, ds$$

The terms to be compared are q_o and $q_o' v_2'$. Invoking the approximations used previously gives:

$$v = v_o \sin \frac{\pi z}{L}$$

$$u = u_o \sin \frac{\pi z}{L}$$

$$\beta = \beta_o \sin \frac{\pi z}{L}$$

An inequality between $v'_{2\,max}$ and the β_o, v_o, u_o terms can be written:

$$v'_{2\,max} < v_o \frac{\pi}{L} + u_o \frac{\pi}{L} + \beta_o \frac{\pi}{L} r$$

Rewriting the comparison using the above gives:

$$\alpha \int_o^s \left(w_o' + u_o \frac{\pi^2}{L^2}(x_o - x) + v_o \frac{\pi^2}{L^2}(y_o - y) + \beta_o \omega_1 \frac{\pi^2}{L^2} \right) t \, ds$$

$$\left| \left(v_o \frac{\pi}{L} + u_o \frac{\pi}{L} + \beta_o \frac{\pi r}{L} \right) \int_o^s \left(w_o'' + u_o \frac{\pi^3}{L^3}(x_o - x) + v_o \frac{\pi^3}{L^3}(y_o - y) \right. \right.$$

$$+ \beta_o \frac{\pi^3}{L^3} \omega_1 \Big) t \, ds \qquad 0 \leq \alpha \leq 1 \tag{49}$$

Now, w_o' is equal to a constant in the linear case. For the remaining terms, the comparison becomes similar to:

$$\alpha\, u_o \frac{\pi^2}{L^2} \;\Big|\; v_o \frac{\pi}{L}\, u_o \frac{\pi^3}{L^3}$$

or

$$\alpha \;\Big|\; \frac{v_o \pi^2}{L^2} \qquad 0 \leq \alpha \leq 1 \tag{50}$$

Clearly, $\frac{v_o \pi^2}{L^2}$ will be in general negligible compared to α. If $v_o < \frac{L}{100}$, Equation (50) becomes:

$$\alpha \;\Big|\; \frac{\pi^2}{100 L} \tag{51}$$

This is merely a statement of the fact

q_o' is of the order q_o/L

v_2' is of the order v_2/L

This allows Equation (15) to be written:

$$q_o = -\int_0^s (\sigma t)'\, ds \tag{52}$$

which is the linear equation for q_o. Therefore, under the assumptions of $s < \frac{L}{2}$ and $v_o < \frac{L}{100}$ for an assumed sine shape, Equations (45) become:

$$\begin{aligned} \bar{T}_3 &= \sigma t v_2' - q_o - (M\beta)' \\ \bar{T}_2 &= \sigma t u_2' - M + q_o \beta \\ \bar{T}_1 &= \sigma t - q_o v_2' \end{aligned} \tag{53}$$

where q_o is given by Equation (52).

The limits $s < \frac{L}{2}$ and $v_{max} < \frac{L}{100}$ presented above are not definite limits but are only given as a guide to indicate that around these values the approximations are becoming doubtful. Remember that these were developed on the basis of a simple sine curve deflection and the arbitrary decision to discard terms of order $^1/20$ compared to values in the range of zero to one.

Also, these approximations took no account of distribution over the length of the section. For instance, some of the terms being compared were modified by sine curves, others by cosine curves. In some circumstances, this might allow a small term to dominate a larger term because at a certain point the sine or cosine terms are zero. However, since this occurs at certain points and the function is non-zero over large parts of the domain elsewhere, it was felt that comparison of maximum values was adequate. These limits therefore are based on possible orders of magnitude only and very rough physical reasoning and should not be construed as inviolable. It is still felt though that, despite the roughness of the approximations involved, the conclusions drawn are satisfactory.

From Equations (19), the stress vectors in the global fixed axes are:

$$
\begin{aligned}
T_1 &= \bar{T}_1 \\
T_2 &= \bar{T}_2 \sin\phi - \bar{T}_3 \cos\phi \\
T_3 &= \bar{T}_2 \cos\phi + \bar{T}_3 \sin\phi
\end{aligned}
\tag{19}
$$

Everything about Equations (19) is known except for W_o, the axial displacement of the origin of s on the cross section. To evaluate W_o, equilibrium along the z axis will be used.

Recalling the first equation of (20), which is $\int_0^{\bar{s}} T_1 \, ds = P$, and using the last equation of (53) gives:

$$P = \int_0^{\bar{s}} T_1 \, ds = \int_0^{\bar{s}} (\sigma t - q_0 v_2') \, ds \qquad (54)$$

This implies from Equations (44) and (7)

$$P = E \int_0^{\bar{s}} \left(t \left[w_0' + \beta'' \omega_1 + u''(x_0 - x) + v''(y_0 - y) \right] \right) ds + F(z)$$

where $\quad F(z) = -\int_0^{\bar{s}} q_0 v_2' \, ds$

Integrating with respect to s then z and rearranging, gives:

$$W_0 = \frac{Pz}{EA} - \beta \bar{\omega}_1 - u'(x_0 - \bar{x}) - v'(y_0 - \bar{y}) - \frac{\int F(z) \, dz}{EA} + K$$

where $\bar{x}, \bar{y}$ are the co-ordinates of the centroid in the x, y co-ordinate system, A is the area of the cross section, and $\bar{\omega}_1 = \frac{1}{A} \int_0^{\bar{s}} t \, \omega_1 \, ds$.
Substitution of W_0 into Equation (7) gives:

$$W = \beta'(\omega_1 - \bar{\omega}_1) + u'(\bar{x} - x) + v'(\bar{y} - y) + \frac{Pz}{EA} + K - \frac{\int F(z) \, dz}{EA} \qquad (55)$$

This has the unfortunate aspect of re-introducing the term $q_0 v_2'$, as it appears in $F(z)$. Therefore, since $\sigma = EW'$ and $q_0 = \int_0^s (\sigma t)' ds = \int_0^s ((EW')t)' \, ds$, the $F(z)$ term appears, in modified form in both σ and q_0. In q_0, it has the form $\frac{1}{A} \int_0^s (q_0 v_2')' \, ds$ which has already been shown to be negligible under the constraints of the problem. Its appearance in σ in the second and third of Equations (53) can also be neglected as σ appears in a second order term already, and so $\frac{1}{A}F(z)$ becomes a third order contribution to these equations. Therefore, for the purpose of evaluating $\overline{T}_2$, $\overline{T}_3$ and consequently T_2, T_3, w will be taken as:

$$w = \beta'(\omega_1 - \bar{\omega}_1) + u'(\bar{x} - x) + v'(\bar{y} - y) + \frac{Pz}{EA} + K \tag{56}$$

At this point, the shears and normal flows T_1, T_2, T_3 have been related to the displacements and then derivatives. This has been done through the use of stress-strain equations relating e_{zz}, σ and w and through equilibrium equations relating σ, T_1, T_2, T_3 and q_0. All this was done by neglecting terms in the equations which were rendered negligible by the conditions $u'v'$ can be discarded compared to u', v', β. There is now enough information to determine the resultant forces on the end of the section.

3:4 Relating the Surface Stress Vector to the Overall Beam Force Resultants

Using Equations (19), (20) and (53), the overall beam resultant can now be written in terms of σ and q_0 and therefore in terms of the displacements.

The resultants become:

$$V_1 = \int_0^{\bar{s}} T_2 \, ds = \int_0^{\bar{s}} (\overline{T}_2 \sin\varphi - \overline{T}_3 \cos\varphi)\, ds \tag{57}$$

Substitution of $\overline{T}_2$ and $\overline{T}_3$ gives:

$$V_1 = \int_0^{\bar{s}} (\sin\varphi(+q_0\beta + \sigma t u_2' - M') - \cos\varphi(-q_0 + \sigma t v_2' - (M\beta)'))\, ds \tag{58}$$

Substitution for q_0, σ and M, along with u_2', v_2' and integrating gives:

$$\begin{aligned} V_1 = E\Big[&(-K_6 - K_4 + K_5)\beta''' + (-\Phi_{yc} - K_2)u''' + (-\Phi_{xyc} - K_3)v''' \\ &- \beta(K_1 - K_8 + K_9)\beta''' + \beta(\Phi_{xyc} + K_3)u''' + \beta(\Phi_{xc} + K_7)v''' + \frac{P}{E}u' \\ &- \beta'\beta''(K_1 - K_8 + K_9) + \beta' u''(\Phi_{xyc} + K_3) + \beta' v''(\Phi_{xc} + K_7) - \frac{P}{E}\beta'\bar{y}\Big] \end{aligned} \tag{59}$$

where the following equalities are defined:

$$\int_0^{\bar{s}} t(\omega_1 - \bar{\omega}_1)\,ds = A\bar{\omega}_1 - A\bar{\omega}_1 = 0 \qquad \int_0^{\bar{s}} t\,ds = A$$

$$\int_0^{\bar{s}} t(\bar{x} - x)\,ds = A\bar{x} - A\bar{x} = 0 \qquad \int_0^{\bar{s}} x\,dA = A\bar{x}$$

$$\int_0^{\bar{s}} t(\bar{y} - y)\,ds = A\bar{y} - A\bar{y} = 0 \qquad \int_0^{\bar{s}} y\,dA = A\bar{y}$$

$$\int_0^{\bar{s}} ty(\omega_1 - \bar{\omega}_1)\,ds = K_1, \quad \int_0^{\bar{s}} txy\,ds = \Phi_{xy}, \quad \int_0^{\bar{s}} tx^2\,ds = \Phi_y \quad \text{etc.}$$

$$\int_0^{\bar{s}} ty(\bar{x} - x)\,ds = A\bar{y}\bar{x} - \Phi_{xy} = -\Phi_{xy\,centroid} = -\Phi_{xyc}$$

$$\int_0^{\bar{s}} ty(\bar{y} - y)\,ds = A\bar{y}^2 - \Phi_x = -\Phi_{x\,centroid} = -\Phi_{xc}$$

using the relations $\frac{dy}{ds} = \dot{y} = -\sin\phi, \ \frac{dx}{ds} = \dot{x} = \cos\phi$

$$\int_0^{\bar{s}} \cos\phi \int_0^{s} t(\omega_1 - \bar{\omega}_1)\,ds\,ds = x\int_0^{s} t(\omega_1 - \bar{\omega}_1)\,ds\Big|_0^{\bar{s}} - \int_0^{\bar{s}} xt(\omega_1 - \bar{\omega}_1)\,ds$$

$$= 0 - \int_0^{\bar{s}} xt(\omega_1 - \bar{\omega}_1)\,ds = K_6$$

$$\int_0^{\bar{s}} \cos\phi \int_0^{s} t(\bar{x} - x)\,ds\,ds = x\int_0^{s} t(\bar{x} - x)\,ds\Big|_0^{\bar{s}} - \int_0^{\bar{s}} xt(\bar{x} - x)\,ds$$

$$= 0 - \int_0^{\bar{s}} xt(\bar{x} - x)\,ds = +\Phi_{yc}$$

$$\int_0^{\bar{s}} \cos\phi \int_0^{s} t(\bar{y} - y)\,ds\,ds = x\int_0^{s} t(\bar{y} - y)\,ds\Big|_0^{\bar{s}} - \int_0^{\bar{s}} xt(\bar{y} - y)\,ds$$

$$= 0 - \int_0^{\bar{s}} xt(\bar{y} - y)\,ds = +\Phi_{xyc}$$

$$\int_0^{\bar{s}} \sin\phi \int_0^{s} t(\omega_1-\bar{\omega}_1)\,ds\,ds = -y\int_0^{s} t(\omega_1-\bar{\omega}_1)\,ds\Big|_0^{\bar{s}} + \int_0^{\bar{s}} y\,t(\omega_1-\bar{\omega}_1)\,ds$$

$$= 0 + \int_0^{\bar{s}} y\,t(\omega_1-\bar{\omega}_1)\,ds = K_1$$

$$\int_0^{\bar{s}} \sin\phi \int_0^{s} t(\bar{x}-x)\,ds\,ds = -y\int_0^{s} t(\bar{x}-x)\,ds\Big|_0^{\bar{s}} + \int_0^{\bar{s}} y\,t(\bar{x}-x)\,ds$$

$$= 0 + \int_0^{\bar{s}} y\,t(\bar{x}-x)\,ds = -\Phi_{xyc}$$

$$\int_0^{\bar{s}} \sin\phi \int_0^{s} t(\bar{y}-y)\,ds\,ds = -y\int_0^{s} t(\bar{y}-y)\,ds\Big|_0^{\bar{s}} + \int_0^{\bar{s}} y\,t(\bar{y}-y)\,ds$$

$$= 0 + \int_0^{\bar{s}} y\,t(\bar{y}-y)\,ds = -\Phi_{xc}$$

$$\int_0^{\bar{s}} \frac{t^3}{12}\sin^2\phi\,ds = K_2$$

$$\int_0^{\bar{s}} \frac{t^3}{12}\sin\phi\cos\phi\,ds = K_3$$

$$\int_0^{\bar{s}} \frac{t^3}{12}\,x\sin\phi\cos\phi\,ds = K_4$$

$$\int_0^{\bar{s}} \frac{t^3}{12}\,y\sin^2\phi\,ds = K_5$$

$$\int_0^{\bar{s}} \frac{t^3}{12}\cos^2\phi\,ds = K_7$$

$$\int_0^{\bar{s}} \frac{t^3}{12}\,x\cos^2\phi\,ds = K_8$$

$$\int_0^{\bar{s}} \frac{t^3}{12}\,y\sin\phi\cos\phi\,ds = K_9$$

(60)

Similarly

$$V_2 = \int_0^{\bar{s}} T_3\, ds = \int_0^{\bar{s}} (\bar{T}_2 \cos\phi + \bar{T}_3 \sin\phi)\, ds \tag{61}$$

By using similar operations as those used in V_1 , V_2 becomes:

$$\begin{aligned} V_2 = E\Big[&(K_1 - K_8 + K_9)\beta''' + (-\Phi_{xyc} - K_3)u''' + (-\Phi_{xc} - K_7)v''' \\ &-\beta(K_6 + K_4 - K_5)\beta''' - \beta(\Phi_{yc} + K_2)u''' - \beta(\Phi_{xyc} + K_3)v''' + \frac{P}{E}v' \\ &-\beta'\beta''(K_6 + K_4 - K_5) - \beta'u''(\Phi_{yc} + K_2) - \beta'v''(\Phi_{xyc} + K_3) + \frac{P}{E}\beta'\bar{x}\Big] \end{aligned} \tag{62}$$

For the torque T

$$T = \int_0^{\bar{s}} \left(-T_2(y + \beta x) + T_3(x - \beta y) - Mv_2'\right) ds \tag{63}$$

Expanding in terms of q_o, σ, M , u_2' and v_2' gives:

$$\begin{aligned} T = \int_0^{\bar{s}} \Big[&y\left(\sin\phi(-q_o\beta - \sigma t u_2' + M') - \cos\phi(q_o - \sigma t v_2' + (M\beta)')\right) \\ &+\beta y\left(\cos\phi(-q_o\beta - \sigma t u_2' + M') + \sin\phi(q_o - \sigma t v_2' + (M\beta)')\right) \\ &+x\left(-\cos\phi(-q_o\beta - \sigma t u_2' + M') - \sin\phi(q_o - \sigma t v_2' + (M\beta)')\right) \\ &+\beta x\left(\sin\phi(-q_o\beta - \sigma t u_2' + M') - \cos\phi(q_o - \sigma t v_2' + (M\beta'))\right) - Mv_2'\Big] ds \end{aligned} \tag{64}$$

By expanding, cancelling and throwing out terms of third order Equation (64) simplifies considerably. Introduction of the values for u_2' and v_2' then reduce it to:

$$T=\int_0^{\bar{s}}\Big[-q_o(y\cos\phi+x\sin\phi)+M'(y\sin\phi-x\cos\phi)-2\beta'M(x\sin\phi+y\cos\phi)$$

$$+\sigma t\left(-u'y+v'x+\beta'(y^2+x^2)\right)+M(u'\cos\phi+v'\sin\phi)\Big]ds \tag{65}$$

Introducing q_o, M, σ and integrating and adding $C\beta'$ for the torque due to plate twisting gives:

$$T=E\Big[(K_{13}+K_{16}-K_{17}-K_{18}+K_{16})\beta'''+(K_{14}+K_5-K_4)u'''+(K_{15}+K_9-K_8)v'''$$

$$+C\beta'+u'\left((-K_1+K_8-K_9)\beta''+(\Phi_{xyc}+K_3)u''+(\Phi_{xc}+K_7)v''-\frac{P}{E}\bar{y}\right)$$

$$+v'\left((-K_6-K_4-K_5)\beta''+(-\Phi_{yc}-K_2)u''+(-\Phi_{xyc}-K_3)v''+\frac{P}{E}\bar{x}\right)$$

$$+\beta'\Big((2K_{10}-K_{20}+K_{21}-K_{23}+K_{24})\beta''+(2K_{11}-K_{19}-2K_9)u''$$

$$+(2K_{12}-2K_4-K_{22})v''+\frac{PI_p}{AE}\Big] \tag{66}$$

where:

$$\int_0^{\bar{s}}\frac{1}{2}(x^2+y^2)t(\omega_1-\bar{\omega}_1)\,ds = K_{10}$$

$$\int_0^{\bar{s}}\frac{1}{2}(x^2+y^2)t(\bar{x}-x)\,ds = K_{11}$$

$$\int_0^{\bar{s}}\frac{1}{2}(x^2+y^2)t(\bar{y}-y)\,ds = K_{12}$$

$$\int_0^{\bar{s}}r\int_0^{s}t(\omega_1-\bar{\omega}_1)\,ds\,ds=\int_0^{s}r\,ds\int_0^{s}t(\omega_1-\bar{\omega}_1)\,ds\Big|_0^{\bar{s}}-\int_0^{\bar{s}}t(\omega_1-\bar{\omega}_1)\int_0^{s}r\,ds\,ds$$

$$=0-\int_0^{\bar{s}}t(\omega_1-\bar{\omega}_1)\omega_1\,ds=-\int_0^{\bar{s}}t(\omega_1-\bar{\omega}_1)^2\,ds=K_{13}=-\Gamma$$

$C = JG =$ St. Venant's torsion constant

$$\int_0^{\bar{s}} r \int_0^{s} t(\bar{x}-x)\,ds = K_{14}$$

$$\int_0^{\bar{s}} r \int_0^{s} t(\bar{y}-y)\,ds = K_{15}$$

$$\int_0^{\bar{s}} (x^2+y^2)\,t\,ds = I_p$$

$$\int_0^{\bar{s}} \frac{t^3}{12}\,xy \sin\phi\cos\phi\,ds = K_{16}$$

$$\int_0^{\bar{s}} \frac{t^3}{12}\,y^2 \sin^2\phi\,ds = K_{17}$$

$$\int_0^{\bar{s}} \frac{t^3}{12}\,x^2 \cos^2\phi\,ds = K_{18}$$

$$\int_0^{\bar{s}} \frac{t^3}{6}\,x \sin^2\phi\,ds = K_{19}$$

$$\int_0^{\bar{s}} \frac{t^3}{6}\,x^2 \sin\phi\cos\phi\,ds = K_{20}$$

$$\int_0^{\bar{s}} \frac{t^3}{6}\,xy \sin^2\phi\,ds = K_{21}$$

$$\int_0^{\bar{s}} \frac{t^3}{6}\,y \cos^2\phi\,ds = K_{22}$$

$$\int_0^{\bar{s}} \frac{t^3}{6}\,xy \cos^2\phi\,ds = K_{23}$$

$$\int_0^{\bar{s}} \frac{t^3}{6}\,y^2 \sin\phi\cos\phi\,ds = K_{24} \tag{67}$$

Summing up the stress vectors in the displaced position relative to the defined resultants gives for the moments:

$$BM_1 = \int_0^{\bar{s}} \left(T_1(y+\beta x) - T_3 w - M\cos(\phi-\beta)\right) ds \tag{68}$$

Recall that

$$T_1 = \sigma t - q_o v_2'$$

The term $q_o v_2'$ was shown hopefully to have no noticeable effect on the equations for T_2, T_3 and therefore on the shears and torques. However, it is of consequence now in defining the moment, as at first glance it is a term of importance because terms of similar magnitude have been kept in previous equations. Therefore $q_o v_2'$ will be kept in the equations and examined after Equation (68) has been expanded. Equation (68) becomes:

$$BM_1 = \int_0^{\bar{s}} \left((\sigma t - q_o v_2')(y + \beta x) \right) - T_3 w - M \cos(\phi - \beta)$$

$$= \sigma t (y + \beta x) - q_o v_2' y - M \cos(\phi - \beta) - T_3 w - q_o v_2' \beta x \qquad (69)$$

The last term on the righthand side of (69) can be neglected compared to the second term because of the restriction on the angles.

Substituting for T_3 gives:

$$BM_1 = \int_0^{\bar{s}} \Big[\sigma t (y + \beta x) - q_o v_2' y - w \big(\cos\phi \, (\sigma t u_2' - M' + q_o \beta)$$

$$+ \sin\phi \, (-q_o + \sigma t v_2' - (M\beta)') \big) - M \cos(\phi - \beta) \Big] ds \qquad (70)$$

Since w is not of the order of u, v but is of the order of $u'x$ etc. and since M' and $(M\beta)'$ are small, Equation (70) becomes:

$$BM_1 = \int_0^{s} \left(\sigma t (y + \beta x) - q_o v_2' y + q_o w \sin\phi + M \cos(\phi - \beta) \right) ds \qquad (71)$$

Examining the expanded form of the q_o terms in (71) gives:

$$\int_0^{\bar{s}} \left[q_0 u'(y\cos\phi - x\sin\phi) - 2q_0 y v'\sin\phi - q_0 y \beta' r + q_0 \beta' \omega_1 \sin\phi \right.$$

$$\left. + q_0 u' x_0 \sin\phi + q_0 v' y_0 \sin\phi + q_0 w_0 \sin\phi \right] ds \tag{72}$$

This is of the form:

$$\int_0^{\bar{s}} \left(-\underline{dT}\,(\alpha u' - \beta' y) - 2\,\underline{dV}_1 y v' + \underline{dV}_1 \beta' \omega_1 \right.$$

$$\left. + \underline{dV}_1 (u' x_0 + v' y_0 + w_0) \right) ds \tag{73}$$

where $\alpha \in \text{Real}$ and $\underline{dV}_1$, $\underline{dT}$ are very similar to the shears and torques previously developed. When integrated, (73) will be of the form:

$$\alpha T u' + V_1 w \qquad \alpha \in R \tag{74}$$

Since the section is open, thin walled and behaving as a beam rather than as a shaft, T will be small and therefore $\alpha T u'$ will be neglected. This means that any form of shaft buckling due to pure torque has been eliminated. The Vw term is significant with respect to the other terms in (70) and should be kept. However, when Equation (70) is placed in Equation (25), the $V_1 w$ term appears with V_1 as $V_1 \pm V_1 w'$. Compared to V_1 , it is inconsequential. Therefore it also will be discarded at this stage, even though its insignificance does not become apparent until substitution of the force resultants into the overall equilibrium equations. Therefore

$$BM_1 = \int_0^{\bar{s}} \left(\sigma t (y + \beta x) - M \cos(\phi - \beta) \right) ds \tag{75}$$

Integration and substitution of previously defined constants gives:

$$BM_1 = E\Big[(K_1 - K_8 + K_9)\beta'' + (-\Phi_{xyc} - K_3)u'' + (-\Phi_{xc} - K_7)v'' + \frac{P}{E}\bar{y}$$

$$-\beta\big((K_6 + K_4 - K_5)\beta'' + (\Phi_{yc} + K_2)u'' + (\Phi_{xyc} + K_3)v'' + \frac{P}{E}\bar{x}\big)\Big] \qquad (76)$$

Similarly, $BM_2 = -\int_0^{\bar{s}} \big(T_1(x - \beta y) + M\sin(\phi - \beta)\big)\,ds$

or:

$$BM_2 = E\Big[(K_6 + K_4 - K_5)\beta'' + (\Phi_{yc} + K_2)u'' + (\Phi_{xyc} + K_3)v'' - \frac{P}{E}\bar{x}$$

$$+\beta\big((K_1 - K_8 + K_9)\beta' + (-\Phi_{xyc} - K_3)u'' + (\Phi_{xc} - K_7)v'' + \frac{P}{E}\bar{y}\big)\Big] \qquad (77)$$

This completes the determination of the resultants to the order of accuracy of the rotations being negligible with respect to one, the squares of the rotations being negligible with respect to the axial strains if the section is axially restrained, $|\beta_0 x^2|, |\beta_0 y^2| \ll |2\omega_1|$ if the section is not axially restrained, u and v being roughly $< L/100$ and $\bar{s} < L/2$.

These force resultants are very complicated because of coupling between the unknown displacements and their derivations. This is partly due to the use of an arbitrary origin for the co-ordinate system. It can be shown, however, that the use of a co-ordinate system parallel to the principal axes of the beam and with its origin at the shear centre will uncouple the equations in the linear terms. This particular co-ordinate system will be used for x, y, z for the rest of the thesis. For example, see Vlassov [9] and Bleich [3].

Rewriting the resultants in this new axis system gives:

$$V_1 = -N_2 u''' - N_1 \beta v''' + P u' - P \beta' \bar{y} - N_1 \beta' v''$$

$$V_2 = +N_1 v''' - \beta N_2 u''' + P v' + P \beta' \bar{x} - N_2 \beta' u''$$

$$BM_1 = +N_1 v'' + P \bar{y} - N_2 \beta u'' + P \bar{x} \beta \tag{78}$$

$$BM_2 = +N_2 u'' - P \bar{x} + N_1 \beta v'' + P \bar{y} \beta$$

$$T = N_3 \beta''' + N_4 \beta + N_5 v'' u' + N_6 u'' v' - P \bar{y} u'$$

$$+ P \bar{x} v' + \beta' \left(N_7 \beta'' + N_8 u'' + N_9 v'' - P \frac{I_p}{A} \right)$$

where $N_1 = E(-\Phi_{xc} - K_7) = -E I_{xc}$

$$N_2 = E(\Phi_{yc} + K_2) = E I_{yc}$$

$$N_3 = E(K_{13} + 2K_{16} - K_{17} - K_{18}) \quad , K_{13} = -\Gamma^2$$

$$N_4 = JG = C$$

$$N_5 = E(\Phi_{xc} + K_7) = E I_{xc}$$

$$N_6 = -E(\Phi_{yc} + K_2) = -E I_{yc}$$

$$N_7 = E(2K_{10} - K_{20} - K_{21} - K_{23} + K_{24})$$

$$N_8 = E(2K_{11} - K_{19} - 2K_9)$$

$$N_9 = E(2K_{12} - 2K_4 - K_{22}) \tag{79}$$

and these are calculated in a co-ordinate system that is parallel to the principal axes and has its origin at the shear centre except I_{xc} and I_{yc} which are still the values about the principal centroidal axis.

3:5 The Differential Equations of Stability

Integration of the overall beam equilibrium equations (24), (25) and (26), and substituting in the moment resultants from Equation (78) gives:

$$
\begin{aligned}
&N_2 u'' - P\bar{x} + B = -N_1 \beta v'' - P\bar{y}\beta - V_1 z + Pu \\
&N_1 v'' + P\bar{y} + D = N_2 \beta u'' - P\bar{x}\beta + V_2 z - Pv \\
&N_3 \beta''' + N_4 \beta' + F = -V_2 u + V_1 v - N_5 v'' u' - N_6 u'' v' + P\bar{y} u' \\
&\qquad - P\bar{x} v' - N_7 \beta'' \beta' - N_8 u'' \beta' - N_9 v'' \beta' + P \frac{I_P}{A} \beta'
\end{aligned}
\tag{80}
$$

where B, D and F are constants of integration.

These are the final equations governing the behaviour of the section. They have been obtained by placing the equations relating the resultants to displacements into the overall equilibrium equations for the beam. Notice that all of the non-linear terms are of the form of a force times a displacement. For instance, in the first equation of (80) the term

$$-N_1 v'' \beta \cong -BM_1 \beta$$

Also note that all the non-linearities are due to equilibrium being taken about a displaced position. Any non-linearities due to the strain equations are assumed to be rendered negligible by the constraints on the rotations and axial boundary conditions. Equations similar to these are given in Bleich [3], Timoshenko [7], Oden [6] and, in particular, Vlassov [9]. However, in this development, the domain of validity of the equations is outlined, and the equations are more general, as they include the effect of coupling in a non-linear manner, whereas the above references give linear differential equations.

Finally, note that the constants Φ_{xc}, Φ_{yc}, Φ_{xyc} and I_ρ are defined by integrals of the type

$$\int_0^{\bar{s}} t y (\bar{x} - x) ds = \Phi_{xyc}$$

where the x, y terms are co-ordinates of the mid-thickness at point s. This means that, although the Φ have a form close to those of the moments of inertia, they are missing the terms which account for the effects of the moment of inertia of the element ds about its own centroidal axis at mid-thickness. However, these effects are accounted for by the constants K_7, K_2, K_3 due to secondary, or plate bending effects. Since they always appear coupled with the Φ in the equations for force resultants, it was a simple matter to define the values of I_{xc}, I_{yc} as $I_{xc} = \Phi_{xc} + K_7$ etc. as was done in Equations (79).

However, the constant I_ρ as defined lacks the necessary second order effects to make it the usual I_ρ. This definition will be kept though, as the nature of the sections considered renders the difference between the usual I_ρ and the I_ρ defined here to be negligible. This is not always true in the case for the difference between I_{xc} and Φ_{xc} or I_{yc} and Φ_{yc}. Considering a thin rectangular section it is clear that about the weak axis

$\Phi = 0$ and I consists entirely of the second order, or plate bending effects as represented by K_2 or K_7.

This is also a property of the constant N_3, as there exist sections such as thin rectangles where K_{13} (which is the negative of the warping constant Γ) is zero, and the only warping restraint is found from secondary, or plate bending effects, as given by K_{16}, K_{17}, K_{18}.

CHAPTER 4

SOLUTION OF DIFFERENTIAL EQUATIONS

Now that the equations have been developed, it remains to relate them to a matrix type formulation as this representation is the main object of the study. The desired form is:

$$K\delta = f \tag{81}$$

The δ are related to the deflections u, v, β etc. by the following equations and to the beam section by Figure 10.

$$
\begin{array}{lll}
\delta_1 = w_o & \delta_6 = -v' & \delta_{11} = \beta \\
\delta_2 = w_o & \delta_7 = u & \delta_{12} = \beta' \\
\delta_3 = v & \delta_8 = u' & \delta_{13} = \beta \\
\delta_4 = -v' & \delta_9 = u & \delta_{14} = \beta' \\
\delta_5 = v & \delta_{10} = u' &
\end{array} \tag{82}
$$

evaluated at respective ends.

Notice the presence of β' as a boundary condition at each end, making seven conditions per end to be satisfied. Their presence is necessary to account for warping at the ends. Absence of these terms would constrain plane sections to remain plane at the joints, obviously not the most general condition.

Equations (82) allow the solution of the differential equations in terms

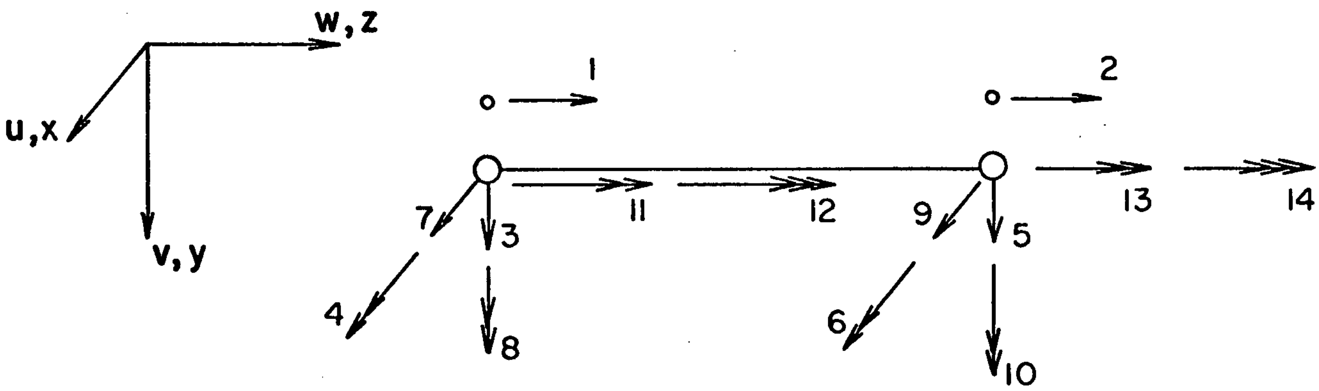


FIG. 10 GENERALIZED DISPLACEMENTS FOR BEAM.

of the vector of boundary conditions δ. The end forces may be found by substituting back into the force resultant equations evaluated at the boundaries. This will lead to an equation of the form:

$$\{f\} = [K(\delta)]\{\delta\} \tag{83}$$

where the matrix function $K(\delta)$ will be taken as the stiffness matrix.

The differential equations will be solved using an iteration technique that employs the linear solution as a first approximation to the deflected shape. In more general terms, if

$$Lq = F(q, q', q'', \cdots) \qquad \text{B.C.} = \delta \tag{84}$$

is the equation to be solved, the solution will be taken as:

$$q = q_1 + q_2 + q_3 + \cdots\cdots q_n \tag{85}$$

where q_1 is given by:

$$L(q_1) = 0 \qquad \text{B.C.} = \delta \tag{86}$$

and $q_n,\ n \neq 1$ is given by:

$$L(q_n) = F(q_{n-1}, q'_{n-1}, q''_{n-1}, \cdots) \qquad \text{B.C.} = 0 \tag{87}$$

For this problem, only one iteration is used, so:

$$q = q_1 + q_2 \tag{88}$$

To ease calculations and give some idea of the size and importance of the terms, the differential equations (78) are non-dimensionalised by the following substitutions.

$$z = L\tilde{z} \qquad u = L\tilde{u} \qquad v = L\tilde{v} \qquad \beta = \tilde{\beta} \tag{89}$$

and

$$\frac{df}{dz} = \frac{df}{d\tilde{z}}\frac{d\tilde{z}}{dz} = \frac{\dot{f}}{L} \qquad \frac{d^2f}{dz^2} = \frac{\ddot{f}}{L^2} \qquad \text{etc.}$$

Equations (78) become:

$$\begin{aligned}
\ddot{\tilde{v}} &= M_1\tilde{v} + (M_2 + M_3\ddot{\tilde{u}})\beta + A_1\tilde{z} + B_1 \\
\ddot{\tilde{u}} &= M_4\tilde{u} + (M_5 + M_6\ddot{\tilde{v}})\beta + C_1\tilde{z} + D_1 \\
\dddot{\tilde{\beta}} - \lambda^2\dot{\tilde{\beta}} &= -\dot{\tilde{u}}(M_9\ddot{\tilde{v}} + M_8) + M_7\tilde{u} - \dot{\tilde{v}}(M_{10}\ddot{\tilde{u}} + M_{11}) \\
&\quad + M_{12}\tilde{v} - \dot{\beta}(M_{13}\ddot{\tilde{\beta}} + M_{14}\ddot{\tilde{u}} + M_{15}\ddot{\tilde{v}} + M_{16}) + E_1
\end{aligned} \tag{90}$$

where A_1, B_1, C_1, D_1, and E_1 are arbitrary constants of integration and

$$M_1 = -PL^2/N_1 \qquad M_2 = -PL\bar{x}/N_1$$

$$M_3 = N_2/N_1 \qquad M_4 = PL^2/N_2$$

$$M_5 = -PL\bar{y}/N_2 \qquad M_6 = -N_1/N_2$$

$$M_7 = N_5L^2/N_3 \qquad M_8 = -PL^3\bar{y}/N_3$$

$$M_9 = -V_2 L^4/N_3 \qquad M_{10} = N_6 L^2/N_3$$

$$M_{11} = P\bar{x} L^3/N_3 \qquad M_{12} = V_1 L^4/N_3$$

$$M_{13} = N_7/N_3 \qquad M_{14} = N_8 L/N_3$$

$$M_{15} = N_9 L/N_3 \qquad M_{16} = P I_p L^2/A N_3$$

$$\lambda^2 = -N_4 L^2/N_3 \tag{91}$$

To solve these equations, the iteration previously mentioned will be employed.

Taking for the linear case

$$\overset{\cdot\cdot\cdot\cdot}{\tilde{v}}_1 = 0 \qquad \text{B.C.} = \delta$$

$$\overset{\cdot\cdot\cdot\cdot}{\tilde{u}}_1 = 0 \tag{92}$$

$$\ddot{\tilde{\beta}}_1 - \lambda^2 \tilde{\beta}_1 = A_2 \tilde{z} + B_2$$

where A_2, B_2 are constants of integration.

The solutions are:

$$\tilde{v}_1 = a_1 \tilde{z}^3 + b_1 \tilde{z}^2 + c_1 z + d_1$$

$$\tilde{u}_1 = e_1 \tilde{z}^3 + f_1 \tilde{z}^2 + g_1 z + h_1$$

$$\tilde{\beta}_1 = j_1 \sinh\lambda\tilde{z} + k_1 \cosh\lambda\tilde{z} + \ell_1\tilde{z} + m_1 \tag{93}$$

In matrix form, Equations (93) become

$$\begin{aligned}
\tilde{v}_1 &= \phi C_1^1 & & C_1^1 = (a_1, b_1, c_1, d_1)^T \\
& & \phi = (\tilde{z}^3, \tilde{z}^2, \tilde{z}, 1) & \\
\tilde{u}_1 &= \phi C_1^2 & & C_1^2 = (e_1, f_1, g_1, h_1)^T \\
& & \Theta = (\sinh\lambda\tilde{z}, \cosh\lambda\tilde{z}, \tilde{z}, 1) & \\
\tilde{\beta}_1 &= \Theta C_1^3 & & C_1^3 = (j_1, k_1, \ell_1, m_1)^T
\end{aligned} \tag{94}$$

Using Equations (82), (89) and (94) evaluated at the boundaries gives, in matrix form,

$$\left.\begin{matrix} \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \end{matrix}\right\} \delta^1,\quad
\left.\begin{matrix} \delta_7 \\ \delta_8 \\ \delta_9 \\ \delta_{10} \end{matrix}\right\} \delta^2,\quad
\left.\begin{matrix} \delta_{11} \\ \delta_{12} \\ \delta_{13} \\ \delta_{14} \end{matrix}\right\} \delta^3$$

$$\begin{Bmatrix} \delta^1 \\ \delta^2 \\ \delta^3 \end{Bmatrix} = \begin{bmatrix} \bar{\Phi}_1^1 & & \\ & \bar{\Phi}_1^2 & \\ & & \bar{\Phi}_1^3 \end{bmatrix} \begin{Bmatrix} C_1^1 \\ C_1^2 \\ C_1^3 \end{Bmatrix} \tag{95}$$

where

$$\bar{\Phi}_1^1 = \begin{pmatrix} 0 & 0 & 0 & L \\ 0 & 0 & -1 & 0 \\ L & L & L & L \\ -3 & -2 & -1 & 0 \end{pmatrix}, \quad
\bar{\Phi}_1^2 = \begin{pmatrix} 0 & 0 & 0 & L \\ 0 & 0 & 1 & 0 \\ L & L & L & L \\ 3 & 2 & 1 & 0 \end{pmatrix}, \quad
\bar{\Phi}_1^3 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ \lambda/L & 0 & 1/L & 0 \\ \sinh\lambda & \cosh\lambda & 1 & 1 \\ \frac{\lambda}{L}\cosh\lambda & \frac{\lambda}{L}\sinh\lambda & 1/L & 0 \end{pmatrix}$$

$$C_1^1 = \begin{Bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{Bmatrix}, \quad C_1^2 = \begin{Bmatrix} e_1 \\ f_1 \\ g_1 \\ h_1 \end{Bmatrix}, \quad C_1^3 = \begin{Bmatrix} j_1 \\ k_1 \\ \ell_1 \\ m_1 \end{Bmatrix}$$

Clearly,

$$C_1^1 = [\bar{\Phi}_1^1]^{-1}\delta^1$$

$$C_1^2 = [\bar{\Phi}_1^2]^{-1}\delta^2 \qquad (96)$$

$$C_1^3 = [\bar{\Phi}_1^3]^{-1}\delta^3$$

and therefore, from (94) and (96)

$$\tilde{v}_1 = \Phi C_1^1 = \Phi[\bar{\Phi}_1^1]^{-1}\delta^1$$

$$\tilde{u}_1 = \Phi C_1^2 = \Phi[\bar{\Phi}_1^2]^{-1}\delta^2 \qquad (97)$$

$$\tilde{\beta}_1 = \Theta C_1^3 = \Theta[\bar{\Phi}_1^3]^{-1}\delta^3$$

Equations (97) are the linear solution in terms of the boundary displacements δ. For the first iteration:

$$\ddot{\tilde{v}}_2 = M_1\tilde{v}_1 + (M_2 + M_3\ddot{\tilde{u}}_1)\tilde{\beta}_1 + A_3\xi + B_3$$

$$\ddot{\tilde{u}}_2 = M_4\tilde{u}_1 + (M_5 + M_6\ddot{\tilde{v}}_1)\tilde{\beta}_1 + C_3\xi + D_3 \qquad (98)$$

$$\dddot{\tilde{\beta}}_2 - \lambda^2\dot{\tilde{\beta}}_2 = -\dot{\tilde{u}}(M_7\ddot{\tilde{v}}_1 + M_8) + M_9\tilde{u}_1 - \dot{\tilde{v}}_1(M_{10}\ddot{\tilde{u}}_1 + M_{11})$$

$$+M_{12}\dot{\tilde{v}}_1 - \dot{\tilde{\beta}}_1(M_{13}\ddot{\tilde{\beta}}_1 + M_{14}\ddot{\tilde{u}}_1 + M_{15}\ddot{\tilde{v}}_1 + M_{16}) + E_3 \qquad \text{B.C.} = 0$$

where A_3, B_3, C_3, D_3 and E_3 are abitrary constants of integration.

Some of the constants M_i in Equations (98) contain the shears V_1, V_2 and the axial force P. According to Equations (21), (22) and (23), these forces are constant with respect to z. Therefore they will be left as is in the solution of the differential equations. However, during the solution of the stiffness matrix, they will be evaluated using the constitutive equations for forces previously developed.

Before Equation (98) can be solved for $\tilde{v}_2$, $\tilde{u}_2$ and $\tilde{\beta}_2$, the quantities $\tilde{u}_1, \tilde{v}_1, \tilde{\beta}_1$ must be replaced by the linear solution of Equation (97). This leads to terms of the type

$$M_i \; \phi [\bar{\Phi}]^{-1} \delta \; \theta [\bar{\Phi}]^{-1} \delta \tag{99}$$

Since the integration of Equation (98) is the next step, the presence of terms such as (99) are a difficulty. The separation of ϕ and θ, the two functions that depend on z, make integration difficult unless multiplied out in full. However, (99) can be viewed as:

$$M_i \; [\, \phi [\bar{\Phi}]^{-1} \delta \,] [\, \theta [\bar{\Phi}]^{-1} \delta \,]$$

where the terms in large brackets are scalars. Therefore, transposing the large brackets leaves the value unchanged. Doing this to the first bracket changes (99) to:

$$M_i \; \{\delta\}^T ([\bar{\Phi}]^{-1})^T (\phi)^T \theta \; [\bar{\Phi}]^{-1} \{\delta\} \tag{100}$$

Using this reduction, and substituting (97) into (98) gives:

$$\ddot{\tilde{v}}_2 = M_1 \, \phi [\bar{\phi}_1^1]^{-1} \delta^1 + M_2 \, \theta [\bar{\Phi}_1^3]^{-1} \delta^3 + M_3 \, \delta^{2^T} [\bar{\Phi}_1^2]^{-1^T} \ddot{\Phi}^T \theta \, [\bar{\Phi}_1^3]^{-1} \delta^3 + A_3 \, \tilde{z} + B_3$$

$$\ddot{\tilde{u}}_2 = M_4 \, \phi [\bar{\Phi}^2_i]^{-1} \delta^2 + M_5 \, \theta [\bar{\Phi}^3_i]^{-1} \delta^3$$

$$+ M_6 \, \delta^{1^T} [\bar{\Phi}^1_i]^{-1^T} \ddot{\Phi}^T \theta [\bar{\Phi}^3_i]^{-1} \delta^3 + C_3 \tilde{\xi} + D_3$$

$$\dddot{\tilde{\beta}}_2 - \lambda^2 \dot{\tilde{\beta}}_2 = -M_7 \, \delta^{1^T} [\bar{\Phi}^1_i]^{-1^T} \ddot{\Phi}^T \dot{\phi} [\bar{\Phi}^2_i]^{-1} \delta^2$$

$$- M_8 \, \mathring{\phi} [\bar{\phi}^2_i]^{-1} \delta^2 + M_9 \, \phi [\bar{\Phi}^2_i]^{-1} \delta^2$$

$$- M_{10} \delta^{2^T} [\bar{\Phi}^2_i]^{-1} \ddot{\phi} \dot{\phi} [\bar{\Phi}^1_i]^{-1} \delta^1 - M_{11} \dot{\phi} [\bar{\Phi}^1_i]^{-1} \delta^1 + M_{12} \dot{\phi} [\bar{\Phi}^1_i]^{-1} \delta^1$$

$$- M_{13} \, \delta^{3^T} [\bar{\Phi}^3_i]^{-1^T} \ddot{\Theta}^T \dot{\theta} [\bar{\Phi}^3_i]^{-1} \delta^3 - M_{14} \, \delta^{2^T} [\bar{\Phi}^2_i]^{-1^T} \ddot{\Phi}^T \dot{\theta} [\bar{\Phi}^3_i]^{-1} \delta^3$$

$$- M_{15} \, \delta^{1^T} [\bar{\Phi}^1_i]^{-1^T} \ddot{\Phi}^T \dot{\theta} [\bar{\Phi}^3_i]^{-1} \delta^3 - M_{16} \, \dot{\theta} [\bar{\Phi}^3_i]^{-1} \delta^3 + E_3 \qquad \text{B.C.} = 0 \qquad (101)$$

Solving (101) for the particular integral and adding the homogeneous solution gives:

$$\tilde{v}_2 = f^v_1 \delta^1 + f^v_2 \delta^3 + a_2 \tilde{\xi}^3 + b_2 \tilde{\xi}^2 + c_2 \tilde{\xi} + d_2$$

$$\tilde{u}_2 = f^u_1 \delta^2 + f^u_2 \delta^3 + e_2 \tilde{\xi}^3 + f_2 \tilde{\xi}^2 + g_2 \tilde{\xi} + h_2 \qquad (102)$$

$$\tilde{\beta}_2 = f^\beta_1 \delta^2 + f^\beta_2 \delta^1 + f^\beta_3 \delta^3 + j_2 \sinh \lambda \tilde{\xi} + k_2 \cosh \lambda \tilde{\xi} + \ell_2 \tilde{\xi} + m_2$$

where

$$f^v_1 = M_1 \, \widetilde{\widetilde{[\Phi]}} [\bar{\Phi}^1_i]^{-1}$$

$$f^v_2 = M_2 \, \widetilde{[\Theta]} [\bar{\Phi}^3_i]^{-1} + M_3 [\delta^2]^T [\bar{\Phi}^2_i]^{-1^T} \widetilde{\widetilde{[\ddot{\Phi}^T \theta]}} [\bar{\Phi}^3_i]^{-1}$$

$$f_1^{u} = M_4 \, \overset{\approx}{[\Phi]} \, [\bar{\Phi}_1^2]^{-1}$$

$$f_2^{u} = M_5 \, \overset{\approx}{[\Theta]} \, [\bar{\Phi}_1^3]^{-1} + M_6 \, S^{1^T} [\bar{\Phi}_1^1]^{-1^T} \, \overset{\approx}{[\ddot{\Phi}^T \Theta]} \, [\bar{\Phi}_1^3]^{-1}$$

$$f_1^{\beta} = - M_7 \, S^{1^T} \, \overset{\vee\vee}{[\ddot{\Phi}^T \dot{\Phi}]} [\bar{\Phi}_1^2]^{-1} - M_8 \, \overset{\vee\vee}{[\dot{\Phi}]} [\bar{\Phi}_1^2]^{-1} + M_9 \, \overset{\vee\vee}{[\Phi]} [\bar{\Phi}_1^2]^{-1}$$

$$f_2^{\beta} = - M_{10} \, S^{2^T} [\bar{\Phi}_1^2]^{-1^T} \overset{\vee\vee}{[\ddot{\Phi}^T \dot{\Phi}]} [\bar{\Phi}_1^1]^{-1} - M_{11} \, \overset{\vee\vee}{[\dot{\Phi}]} [\bar{\Phi}_1^1]^{-1} + M_{12} \, \overset{\vee\vee}{[\Phi]} [\bar{\Phi}_1^1]^{-1}$$

$$f_3^{\beta} = - M_{13} \, S^{3^T} [\bar{\Phi}_1^3]^{-1^T} \overset{\vee\vee}{[\ddot{\Theta}^T \dot{\Theta}]} [\bar{\Phi}_1^3]^{-1} - M_{14} \, S^{2^T} [\bar{\Phi}_1^2]^{-1^T} \overset{\vee\vee}{[\ddot{\Phi}^T \dot{\Theta}]} [\bar{\Phi}_1^3]^{-1}$$

$$- M_{15} \, S^{1^T} [\bar{\Phi}_1^1]^{-1^T} \overset{\vee\vee}{[\dddot{\Phi}^T \dot{\Theta}]} [\bar{\Phi}_1^3]^{-1} - M_{16} \, \overset{\vee\vee}{[\dot{\Theta}]} [\bar{\Phi}_1^3]^{-1}$$

where the symbol $\overset{\approx}{[\gamma]}$ means γ integrated twice w.r.t. $\tilde{\zeta}$ and $\overset{\vee\vee}{[\gamma]}$ means $\tilde{\beta} = \overset{\vee\vee}{[\gamma]}$ satisfies $\dddot{\tilde{\beta}} - \lambda^2 \dot{\tilde{\beta}} = [\gamma]$. The integrated values are given in Appendix 2.

Using a similar procedure to that used to get Equation (95) allows the boundary delfections, which are zero for this iteration, to be related to $\tilde{u}_2, \tilde{v}_2$ and $\tilde{\beta}_2$ by the following set of equations.

$$\begin{matrix}0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\end{matrix} = \left\{\begin{matrix} \left.\begin{matrix} L f^{v}_{1,0}\delta^1 + L f^{v}_{2,0}\delta^3 \\ - f^{v'}_{1,0}\delta^1 - f^{v'}_{2,0}\delta^3 \\ L f^{v}_{1,L}\delta^1 + L f^{v}_{2,L}\delta^3 \\ - f^{v'}_{1,L}\delta^1 - f^{v'}_{2,L}\delta^3 \end{matrix}\right\} f^1 \\ \left.\begin{matrix} L f^{u}_{1,0}\delta^2 + L f^{u}_{2,0}\delta^3 \\ f^{u'}_{1,0}\delta^2 + f^{u'}_{2,0}\delta^3 \\ L f^{u}_{1,L}\delta^2 + L f^{u}_{2,L}\delta^3 \\ f^{u'}_{1,L}\delta^2 + f^{u'}_{2,L}\delta^3 \end{matrix}\right\} f^2 \\ \left.\begin{matrix} f^{\beta}_{1,0}\delta^2 + f^{\beta}_{2,0}\delta^1 + f^{\beta}_{3,0}\delta^3 \\ \frac{1}{L}\left[f^{\beta'}_{1,0}\delta^2 + f^{\beta'}_{2,0}\delta^1 + f^{\beta'}_{2,0}\delta^3 \right] \\ f^{\beta}_{1,L}\delta^2 + f^{\beta}_{2,L}\delta^1 + f^{\beta}_{3,L}\delta^3 \\ \frac{1}{L}\left[f^{\beta'}_{1,L}\delta^2 + f^{\beta'}_{2,L}\delta^1 + f^{\beta'}_{3,L}\delta^3 \right] \end{matrix}\right\} f^3 \end{matrix}\right\} + \begin{bmatrix} [\bar{\Phi}^1_2] & & \\ & [\bar{\Phi}^2_2] & \\ & & [\bar{\Phi}^3_2] \end{bmatrix} \begin{matrix} \left.\begin{matrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{matrix}\right\} C^1_2 \\ \left.\begin{matrix} e_2 \\ f_2 \\ g_2 \\ h_2 \end{matrix}\right\} C^2_2 \\ \left.\begin{matrix} j_2 \\ k_2 \\ l_2 \\ m_2 \end{matrix}\right\} C^3_2 \end{matrix} \tag{103}$$

and $[\phi^1_2] = [\phi^1_1]$ $\quad [\phi^2_2] = [\phi^2_1]$ $\quad [\phi^3_2] = [\phi^3_1]$

where $f^{v'}_{1,0}$ is f^{v}_1 differentiated once w.r.t. $\tilde{z}$ and evaluated at $\tilde{z} = 0$ and $f^{v'}_{1,L}$ is f^{v}_1 differentiated once and evaluated at $\tilde{z} = 1$ (or $z = L$). Similar definitions hold for f^{β}_1 , f^{u}_1 etc.

This can be simplified by rewriting as follows:

$$\{0\} = f^1_1\delta^1 + f^1_2\delta^3 + [\bar{\Phi}^1_2]\, C^1_2$$

$$\{0\} = f^2_1\delta^2 + f^2_2\delta^3 + [\bar{\Phi}^2_2]\, C^2_2 \tag{104}$$

$$\{0\} = f^3_1\delta^2 + f^3_2\delta^1 + f^3_3\delta^3 + [\bar{\Phi}^3_2]\, C^3_2$$

where $$f_1^1 = \begin{array}{l} L f_{1,0}^{v} \\ -f_{1,0}^{v'} \\ L f_{1,L}^{v} \\ -f_{1,L}^{v'} \end{array}$$

and f_2^1, f_1^3, f_2^2, etc. are all similarly defined.

From Equations (104)

$$C_2^1 = -[\bar{\Phi}_2^1]^{-1}(f_1^1 \delta^1 + f_2^1 \delta^3)$$

$$C_2^2 = -[\bar{\Phi}_2^2]^{-1}(f_1^2 \delta^2 + f_2^2 \delta^3) \tag{105}$$

$$C_2^3 = -[\bar{\Phi}_2^3]^{-1}(f_1^3 \delta^2 + f_2^3 \delta^1 + f_3^3 \delta^3)$$

Writing $\tilde{u}_2$, $\tilde{v}_2$, $\tilde{\beta}_2$ as the sum of the particular and homogeneous solution, and using (105) for the constants in the homogeneous solution gives:

$$\tilde{v}_2 = f_1^{v} \delta^1 + f_2^{v} \delta^3 - \Phi[\bar{\Phi}_2^1]^{-1}(f_1^1 \delta^1 + f_2^1 \delta^3)$$

$$\tilde{u}_2 = f_1^{u} \delta^2 + f_2^{u} \delta^3 - \Phi[\bar{\Phi}_2^2]^{-1}(f_1^2 \delta^2 + f_2^2 \delta^3) \tag{106}$$

$$\tilde{\beta}_2 = f_1^{\beta} \delta^2 + f_2^{\beta} \delta^1 + f_3^{\beta} \delta^3 - \Theta[\bar{\Phi}_2^3]^{-1}(f_1^3 \delta^2 + f_2^3 \delta^1 + f_3^3 \delta^3)$$

This completes the first iteration. Since only one iteration is being used in this study, Equations (106) and (97) may be added to give the values of $\tilde{u}$, $\tilde{v}$ and $\tilde{\beta}$ that will be used.

$$\tilde{v} = \phi[\bar{\Phi}_1^1]^{-1}\delta^1 + f_1^v\delta^1 + f_2^v\delta^3 - \phi[\bar{\Phi}_2^1]^{-1}(f_1^1\delta^1 + f_2^1\delta^3)$$

$$\tilde{u} = \phi[\bar{\Phi}_1^2]^{-1}\delta^2 + f_1^u\delta^2 + f_2^u\delta^3 - \phi[\bar{\Phi}_2^2]^{-1}(f_1^2\delta^2 + f_2^2\delta^3) \tag{107}$$

$$\tilde{\beta} = \theta[\bar{\Phi}_1^3]^{-1}\delta^3 + f_1^\beta\delta^2 + f_2^\beta\delta^1 + f_3^\beta\delta^3 - \theta[\bar{\Phi}_2^3]^{-1}(f_1^3\delta^2 + f_2^3\delta^1 + f_3^3\delta^3)$$

These equations for $\tilde{u}, \tilde{v}, \tilde{\beta}$ consist of terms describing the linear behaviour of the structure plus terms handling the non-linear behaviour.

A typical linear term is $\phi[\bar{\Phi}_1^1]^{-1}\delta^1$ and it is of the form:

$$(z^3, z^2, z, 1)\begin{bmatrix} \\ \text{Geometric} \\ \\ \end{bmatrix}\begin{Bmatrix}\delta_3\\ \delta_4\\ \delta_5\\ \delta_6\end{Bmatrix}$$

The terms describing the non-linear behaviour may be linear in δ or non-linear in δ. Some typical terms describing non-linear behaviour are:

$$f_1^v\delta^1 = M_1\left(\iint(z^3, z^2, z, 1)\,dz\right)\begin{bmatrix} \\ \text{Geometric} \\ \\ \end{bmatrix}\begin{Bmatrix}\delta_3\\ \delta_4\\ \delta_5\\ \delta_6\end{Bmatrix}$$

and $f_2^v\delta^3 =$

$$M_2\left(\iint(z^3, z^2, z, 1)\,dz\right)\begin{bmatrix} \\ \text{Geometric} \\ \\ \end{bmatrix}\begin{Bmatrix}\delta_{11}\\ \delta_{12}\\ \delta_{13}\\ \delta_{14}\end{Bmatrix}$$

$$+\, M_3(\delta_7, \delta_8, \delta_9, \delta_{10})\begin{bmatrix} \text{Geometric} \end{bmatrix}\begin{bmatrix} f(z) \end{bmatrix}\begin{bmatrix} \text{Geometric} \end{bmatrix}\begin{Bmatrix}\delta_{11}\\ \delta_{12}\\ \delta_{13}\\ \delta_{14}\end{Bmatrix}$$

The terms of the form $\phi[\phi_1']^{-1}[f_2' \delta^3]$ are quite a bit more complicated and lengthy; however, they still follow the same general pattern.

$$\phi[\bar{\Phi}_1']^{-1}[f_2' \delta^3] = (z^3, z^2, z, 1)\begin{bmatrix}\text{Geometric}\end{bmatrix}\begin{bmatrix}
LM_2\,(\text{Geometric})\begin{bmatrix}\text{Geometric}\end{bmatrix}\left\{\begin{matrix}\delta_{11}\\ \delta_{12}\\ \delta_{13}\\ \delta_{14}\end{matrix}\right\} + LM_3\left\{\begin{matrix}\delta_{7}\\ \delta_{8}\\ \delta_{9}\\ \delta_{10}\end{matrix}\right\}^T\begin{bmatrix}\text{Geometric}\end{bmatrix}\begin{bmatrix}\text{Geometric}\end{bmatrix}\begin{bmatrix}\text{Geometric}\end{bmatrix}\left\{\begin{matrix}\delta_{11}\\ \delta_{12}\\ \delta_{13}\\ \delta_{14}\end{matrix}\right\} \\
M_2\,(\text{Geometric})\begin{bmatrix}\text{Geometric}\end{bmatrix}\left\{\begin{matrix}\delta_{11}\\ \delta_{12}\\ \delta_{13}\\ \delta_{14}\end{matrix}\right\} + M_3\left\{\begin{matrix}\delta_{7}\\ \delta_{8}\\ \delta_{9}\\ \delta_{10}\end{matrix}\right\}^T\begin{bmatrix}\text{Geometric}\end{bmatrix}\begin{bmatrix}\text{Geometric}\end{bmatrix}\begin{bmatrix}\text{Geometric}\end{bmatrix}\left\{\begin{matrix}\delta_{11}\\ \delta_{12}\\ \delta_{13}\\ \delta_{14}\end{matrix}\right\} \\
LM_2\,(\text{Geometric})\begin{bmatrix}\text{Geometric}\end{bmatrix}\left\{\begin{matrix}\delta_{11}\\ \delta_{12}\\ \delta_{13}\\ \delta_{14}\end{matrix}\right\} + LM_3\left\{\begin{matrix}\delta_{7}\\ \delta_{8}\\ \delta_{9}\\ \delta_{10}\end{matrix}\right\}\begin{bmatrix}\text{Geometric}\end{bmatrix}\begin{bmatrix}\text{Geometric}\end{bmatrix}\begin{bmatrix}\text{Geometric}\end{bmatrix}\left\{\begin{matrix}\delta_{11}\\ \delta_{12}\\ \delta_{13}\\ \delta_{14}\end{matrix}\right\} \\
M_2\,(\text{Geometric})\begin{bmatrix}\text{Geometric}\end{bmatrix}\left\{\begin{matrix}\delta_{11}\\ \delta_{12}\\ \delta_{13}\\ \delta_{14}\end{matrix}\right\} + M_3\left\{\begin{matrix}\delta_{7}\\ \delta_{8}\\ \delta_{9}\\ \delta_{10}\end{matrix}\right\}\begin{bmatrix}\text{Geometric}\end{bmatrix}\begin{bmatrix}\text{Geometric}\end{bmatrix}\begin{bmatrix}\text{Geometric}\end{bmatrix}\left\{\begin{matrix}\delta_{11}\\ \delta_{12}\\ \delta_{13}\\ \delta_{14}\end{matrix}\right\}
\end{bmatrix}$$

All the geometric matrices are functions of 0, 1, L and perhaps $\cosh\lambda L$, $\sinh\lambda L$. They result from various functions of z being evaluated at either $z = 0$ or $z = L$.

Note that in some of the terms, the deflections appear twice. This makes the equations non-linear in δ. Also notice that the constants may have shears V_1, V_2 and axial forces P in them.

The presence of these non-linear forms will be taken care of during the solution technique, where an iterative procedure will be used. During each iteration, the δ's, V's and P's will be evaluated and re-iterated.

CHAPTER 5

FORCE DEFLECTION EQUATION

The displacements found in the previous chapter can now be placed in the force resultant equations to give a relationship between the forces and the boundary displacement . If the relationship is evaluated at the boundaries, it gives the relations between the forces at the boundaries and the boundary displacements. As previously mentioned, this can be written as:

$$\{f\} = [K(\delta)]\{\delta\} \tag{83}$$

It is to be noted at this point that δ is a 1×14 vector and f is 1×12, as there are only six equilibrium equations at each end. However, because the element is not a line element but has a finite cross-section, it is possible to have a stress field existing that has no resultant but still has a gross overall effect. As it happens, this is the case here. This system of stresses is called the bi-moment. See, for example, [6], [9]. It is closely associated with warping and the rate of change of torsional angular displacement. Unfortunately, since the bi-moment has no resultant, it failed to show up in the overall equilibrium equations, but this very property allows it to be introduced now without any effect on the previous development. Since previously integrations of stress across the cross-section were performed in developing the non-linear equations, the effect of the bi-moment is already contained in the differential equations. It only remains to get some measure of its value to give 14 force equations to correspond with the 14 deflections at the boundary.

Although the bi-moment has no resultant, it is still capable of doing work under certain displacements. Because of this, the concept of generalized forces and displacements will be used to develop the f, δ relation rather than writing out the resultants as functions of the boundary displacements directly. This change, while significant, is one of technique and development only. It is introduced only to gain the measure of a stress field which has no resultant. It does not change the form of Equation (83).

Generalized forces and displacements are defined such that the work done by the generalized force through the generalized displacement is equivalent to the work of the actual stresses through the actual displacements. For this case, the generalized displacements will be taken as the δ already defined, and the generalized forces will be taken as acting at the displacements δ.

The virtual work of generalized forces at $z = L$ is:

$$f_2 \Delta\delta_2 + f_5 \Delta\delta_5 + f_9 \Delta\delta_9 + f_6 \Delta\delta_6 + f_{10} \Delta\delta_{10} + f_{13} \Delta\delta_{13} + f_{14} \Delta\delta_{14} \tag{108}$$

where $\Delta\delta$ is a variation in δ.

The virtual work of the actual external stresses is:

$$\int_0^{\bar{s}} \left(T_1 \Delta w + T_2 \Delta u_s + T_3 \Delta v_s + M \Delta u_2' \right) ds \tag{109}$$

where Δw, Δu_s, Δv_s and $\Delta u_2'$ are virtual displacements about a displaced position as T_1, T_2, T_3 and M are acting at the displaced position. In order to relate Δu_s, Δv_s to Δu, Δv, $\Delta\beta$, which are the virtual displacements of the origin, the co-ordinates of Figure 8 will be used. Also necessary will be the quantities given in Figure 11, which shows a side view of the deflected shape w viewed along the x axis:

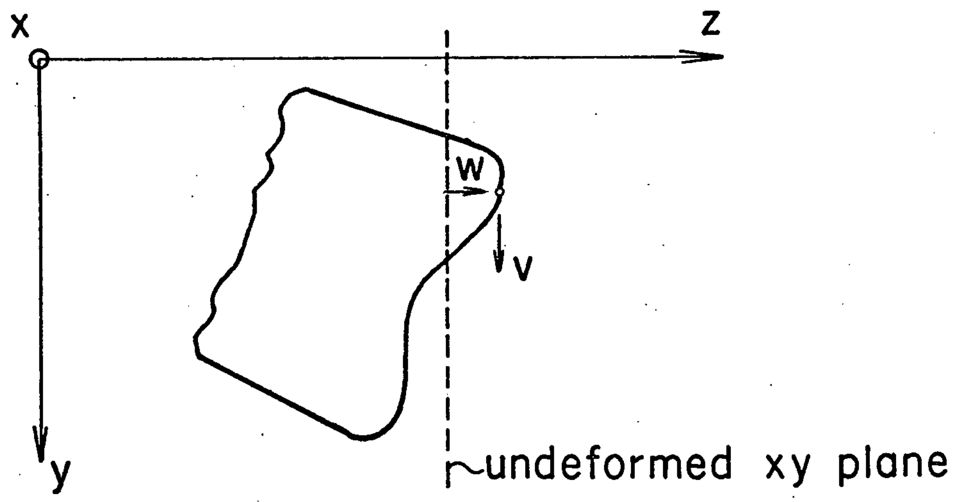


FIG.11 BEAM SEGMENT AND DISPLACEMENT W.

It is clear that Δv_s can be written as:

$$\Delta v_s = \Delta v - \Delta\beta\xi + \Delta v_s' W$$

where $\Delta v - \Delta\beta\xi$ comes from the quantities defined in Figure 8 and $\Delta v_s' W$ comes from quantities in Figure 11.

By using the same reasoning on Δu_s, and using the relations $\xi = x - \beta y$ and $\eta = y + \beta x$ given in Figure 8, Δu_s and Δv_s can be written as:

$$\Delta u_s = \Delta u - \Delta\beta(y + \beta x) + W(\Delta u' - \Delta\beta'(y + \beta x))$$

$$\Delta v_s = \Delta v + \Delta\beta(x - \beta y) + W(\Delta v' + \Delta\beta'(x - \beta y)) \qquad (110)$$

$$\Delta u_2' = \Delta u' \sin(\phi - \beta) + \Delta v' \cos(\phi - \beta) - \Delta\beta'(-x\cos\phi + y\sin\phi)$$

where $\Delta u_2'$ comes from manipulating Equation (9).

By analogy to Equation (2), ΔW may be written as:

$$\Delta W = \Delta W_o + \Delta\beta' \int_0^s r\, ds - \Delta u' \int_0^s \cos(\phi - \beta)\, ds + \Delta v' \int_0^s \sin(\phi - \beta)\, ds \tag{111}$$

Substituting Equations (110) and (111) into Equations (109) and using

$$\cos(\phi - \beta) = \cos\phi \cos\beta + \sin\phi \sin\beta \cong \cos\phi + \beta \sin\phi$$

$$\sin(\phi - \beta) = \sin\phi \cos\beta - \cos\phi \sin\beta \cong \sin\phi - \beta \cos\phi$$

in selected instances and also using Equations (82), gives for the virtual work:

$$\begin{aligned}
V.W. = {} & \Delta\delta_2 \int_0^{\bar{s}} T_1\, ds + \Delta\delta_{14} \int_0^{\bar{s}} \Big[T_1(\omega_1 - \bar{\omega}_1) + T_1 \bar{\omega}_1 - M(-x\cos\phi + y\sin\phi) \\
& + W\big(T_3(x - \beta y) - T_2(y + \beta x)\big)\Big] ds \\
& + \Delta\delta_{10} \int_0^{\bar{s}} \big[-T_1(x - \beta y) - T_1(-x_o + \beta y_o) + T_2 W + M\sin(\phi - \beta)\big] ds \\
& + \Delta\delta_6 \int_0^{\bar{s}} \big[T_1(y + \beta x) + T_1(-y_o - \beta x_o) - T_3 W - M\cos(\phi - \beta)\big] ds \\
& + \Delta\delta_9 \int_0^{\bar{s}} T_2\, ds + \Delta\delta_5 \int_0^{\bar{s}} T_3\, ds + \Delta\delta_{13} \int_0^{\bar{s}} \big[-T_2(y + \beta x) + T_3(x - \beta y)\big] ds
\end{aligned} \tag{112}$$

Using the previous definitions of V, M etc. given in the section on equilibrium equations, Equation (112) becomes:

$$\begin{aligned}
V.W. = {} & \Delta\delta_2 P + \Delta\delta_{14}[BW + P\bar{\omega}_1] + \Delta\delta_{10}[BM_2 + P(x_o - y_o\beta)] \\
& \Delta\delta_6[BM_1 - P(y_o + \beta x_o)] + \Delta\delta_9 V_1 + \Delta\delta_5 V_2 + \Delta\delta_{13} T
\end{aligned} \tag{113}$$

where P, BM, V etc. are evaluated at $z = L$ and where BW is the new term, the bi-moment, and is given by:

$$\int_0^{\bar{s}} \left[T_1(\omega_1 - \bar{\omega}_1) - M(-x\cos\phi + y\sin\phi) + w(T_3(x - \beta y) - T_2(y + \beta x)) \right] ds$$

The last term will be of the form Tw. Since T was previously constrained to be small, and since w is small, this term will be discarded. The bi-moment is then given by.

$$\int_0^{\bar{s}} \left[T_1(\omega_1 - \bar{\omega}_1) - M(-x\cos\phi + y\sin\phi) \right] ds \tag{114}$$

Equating coefficients between Equations (113) and (108) gives:

$$\begin{array}{lll} f_2 = P & f_5 = V_2 & f_6 = BM_1 - P(y_0 + \beta x_0) \\ f_9 = V_1 & f_{10} = BM_2 + P(x_0 - \beta y_0) & f_{13} = T \\ & f_{14} = BW + P\bar{\omega}_1 & \end{array} \tag{115}$$

Evaluating (114) in terms of previous constants gives:

$$BW = \beta'' \left[-K_{13} + K_{18} + K_{17} - 2K_{16} \right] \tag{116}$$

This will be the seventh "force" on the section and is a measure of the stress necessary to maintain out of plane warping. A corresponding development was done at $z = 0$, giving fourteen measures of force, corresponding to fourteen boundary deflections. This allows the connectivity matrix K to be square and to be the most general description of the problem.

All of the forces but P have been found in terms of displacements which can be written as functions of δ. However, P and W_0 are related as follows:

$$W_0 = \frac{Pz}{EA} - \frac{\beta\bar{\omega}_1}{E} - \frac{u'(x_0 - \bar{x})}{E} - \frac{v'(y_0 - \bar{y})}{E} + \gamma' \tag{54}$$

By definition $w_o = \delta_1$ @ $z = 0$ and $w_o = \delta_2$ @ $z = L$. Using this fact, and Equations (82), γ can be solved for and replaced and then P can be found. The result of these manipulations yields:

$$P = \frac{AE}{L} \Big[(\delta_2 - \delta_1) + (\delta_{14} - \delta_{12})\,\bar{\omega}_1 + (\delta_8 - \delta_{10})(\bar{x} - x_o)$$

$$+ (\delta_6 - \delta_4)(\bar{y} - y_o) \Big] \tag{117}$$

It is now possible to write out the force deflection relationship by utilizing Equations (117), (115), (107), (82), (78) and (89).

$$f_1 = -\frac{AE}{L}\Big[(\delta_2 - \delta_1) + (\delta_{14} - \delta_{12})\,\bar{\omega}_1 + (\delta_8 - \delta_{10})(\bar{x} - x_o) + (\delta_6 - \delta_4)(\bar{y} - y_o)\Big]$$

$$f_2 = +\frac{AE}{L}\Big[(\delta_2 - \delta_1) + (\delta_{14} - \delta_{12})\,\bar{\omega}_1 + (\delta_8 - \delta_{10})(\bar{x} - x_o) + (\delta_6 - \delta_4)(\bar{y} - y_o)\Big]$$

$$f_3 = -N_1 \frac{1}{L^2}\Big(\dddot{\Phi}_o [\bar{\Phi}^1_1]^{-1} \delta^1 + \dddot{f}^{\,v}_{1,0}\, \delta^1 + \dddot{f}^{\,v}_{2,0}\, \delta^3 - \dddot{\Phi}_o [\bar{\Phi}^1_2]^{-1} [f^1_1 \delta^1 + f^1_2 \delta^3]\Big)$$

$$+ \delta_{11} N_2 \frac{1}{L^2}\Big(\dddot{\Phi}_o [\bar{\Phi}^2_1]^{-1} \delta^2 + \dddot{f}^{\,u}_{1,0}\, \delta^2 + \dddot{f}^{\,u}_{2,0}\, \delta^3 - \dddot{\Phi}_o [\bar{\Phi}^2_2]^{-1} [f^2_1 \delta^2 + f^2_2 \delta^3]\Big)$$

$$+ \delta_4 P - \delta_{12} P \bar{x}$$

$$+ \delta_{14} N_2 \frac{1}{L}\Big(\ddot{\Phi}_o [\bar{\Phi}^2_1]^{-1} \delta^2 + \ddot{f}^{\,u}_{1,0}\, \delta^2 + \ddot{f}^{\,u}_{2,0}\, \delta^3 - \ddot{\Phi}_o [\bar{\Phi}^2_2]^{-1} [f^2_1 \delta^2 + f^2_2 \delta^3]\Big)$$

$$f_4 = -N_1 \frac{1}{L}\Big(\ddot{\Phi}_o [\bar{\Phi}^1_1]^{-1} \delta^1 + \ddot{f}^{\,v}_{1,0}\, \delta^1 + \ddot{f}^{\,v}_{2,0}\, \delta^3 - \ddot{\Phi}_o [\bar{\Phi}^1_2]^{-1} [f^1_1 \delta^1 + f^1_2 \delta^3]\Big)$$

$$+ \delta_{11} N_2 \frac{1}{L}\Big(\ddot{\Phi}_o [\bar{\Phi}^2_1]^{-1} \delta^2 + \ddot{f}^{\,u}_{1,0}\, \delta^2 + \ddot{f}^{\,u}_{2,0}\, \delta^3 - \ddot{\Phi}_o [\bar{\Phi}^2_2]^{-1} [f^2_1 \delta^2 + f^2_2 \delta^3]\Big)$$

$$- P(\bar{y} - y_o) - \delta_{11} P(\bar{x} - x_o)$$

$$f_5 = N_1 \frac{1}{L^2}\left(\dddot{\phi}_L [\bar{\phi}_1^1]^{-1}\delta^1 + \dddot{f}^{v}_{1,L}\delta^1 + \dddot{f}^{v}_{2,L}\delta^3 - \dddot{\phi}_L[\bar{\phi}_2^1]^{-1}[f_1^1\delta^1 + f_2^1\delta^3]\right)$$

$$-\delta_{13} N_2 \frac{1}{L^2}\left(\dddot{\phi}_L[\bar{\phi}_1^2]^{-1}\delta^2 + \dddot{f}^{u}_{1,L}\delta^2 + \dddot{f}^{u}_{2,L}\delta^3 - \dddot{\phi}_L[\bar{\phi}_2^2]^{-1}[f_1^2\delta^2 + f_2^2\delta^3]\right)$$

$$-\delta_{14} N_2 \frac{1}{L}\left(\ddot{\phi}_L[\bar{\phi}_1^2]^{-1}\delta^2 + \ddot{f}^{u}_{1,L}\delta^2 + \ddot{f}^{u}_{2,L}\delta^3 - \ddot{\phi}_L[\bar{\phi}_2^2]^{-1}[f_1^2\delta^2 + f_2^2\delta^3]\right)$$

$$-\delta_6 P + \delta_{14} P\bar{x}$$

$$f_6 = +N_1\frac{1}{L}\left(\ddot{\phi}_L[\bar{\phi}_1^1]^{-1}\delta^1 + \ddot{f}^{v}_{1,L}\delta^1 + \ddot{f}^{v}_{2,L}\delta^3 - \ddot{\phi}_L[\bar{\phi}_2^1]^{-1}[f_1^1\delta^1 + f_2^1\delta^3]\right.$$

$$-\delta_{13} N_2\frac{1}{L}\left(\ddot{\phi}_L[\bar{\phi}_1^2]^{-1}\delta^2 + \ddot{f}^{u}_{1,L}\delta^2 + \ddot{f}^{u}_{2,L}\delta^3 - \ddot{\phi}_L[\bar{\phi}_2^2]^{-1}[f_1^2\delta^2 + f_2^2\delta^3]\right.$$

$$+P(\bar{y}-y_0) + \delta_{13} P(\bar{x}-x_0)$$

$$f_7 = N_2\frac{1}{L^2}\left[\dddot{\phi}_0[\bar{\phi}_1^2]^{-1}\delta^2 + \dddot{f}^{u}_{1,0}\delta^2 + \dddot{f}^{u}_{2,0}\delta^3 - \dddot{\phi}_0[\bar{\phi}_2^2]^{-1}[f_1^2\delta^2 + f_2^2\delta^3]\right]$$

$$+\delta_{11} N_1\frac{1}{L^2}\left[\dddot{\phi}_0[\bar{\phi}_1^1]^{-1}\delta^1 + \dddot{f}^{v}_{1,0}\delta^1 + \dddot{f}^{v}_{2,0}\delta^3 - \dddot{\phi}_0[\bar{\phi}_2^1]^{-1}[f_1^1\delta^1 + f_2^1\delta^3]\right]$$

$$+\delta_{14} N_1\frac{1}{L}\left(\ddot{\phi}_0[\bar{\phi}_1^1]^{-1}\delta^1 + \ddot{f}^{v}_{1,0}\delta^1 + \ddot{f}^{v}_{2,0}\delta^3 - \ddot{\phi}_0[\bar{\phi}_2^1]^{-1}[f_1^1\delta^1 + f_2^1\delta^3]\right)$$

$$-\delta_8 P + \delta_{12} P\bar{y}$$

$$f_8 = -N_2\frac{1}{L}\left[\ddot{\phi}_0[\bar{\phi}_1^2]^{-1}\delta^2 + \ddot{f}^{u}_{1,0}\delta^2 + \ddot{f}^{u}_{2,0}\delta^3 - \ddot{\phi}_0[\bar{\phi}_2^2]^{-1}[f_1^2\delta^2 + f_2^2\delta^3]\right]$$

$$-\delta_{11} N_1\frac{1}{L}\left(\ddot{\phi}_0[\bar{\phi}_1^1]^{-1}\delta^1 + \ddot{f}^{v}_{1,0}\delta^1 + \ddot{f}^{v}_{2,0}\delta^3 - \ddot{\phi}_0[\bar{\phi}_2^1]^{-1}[f_1^1\delta^1 + f_2^1\delta^3]\right)$$

$$+P(\bar{x}-x_0) - \delta_{11} P(\bar{y}-y_0)$$

$$f_9 = -N_2 \frac{1}{L^2}\Big(\overset{\circ\circ\circ}{\Phi}_L [\bar{\Phi}^2_1]^{-1}\delta^2 + \overset{\circ\circ\circ}{f}{}^{u}_{1,L}\,\delta^2 + \overset{\circ\circ\circ}{f}{}^{u}_{2,L}\,\delta^3 - \overset{\circ\circ\circ}{\Phi}_L [\bar{\Phi}^2_2]^{-1}[f^2_1\delta^2 + f^2_2\delta^3]\Big)$$

$$-\delta_{13} N_1 \frac{1}{L^2}\Big(\overset{\circ\circ\circ}{\Phi}_L [\bar{\Phi}^1_1]^{-1}\delta^1 + \overset{\circ\circ\circ}{f}{}^{v}_{1,L}\,\delta^1 + \overset{\circ\circ\circ}{f}{}^{v}_{2,L}\,\delta^3 - \overset{\circ\circ\circ}{\Phi}_L [\bar{\Phi}^1_2]^{-1}[f^1_1\delta^1 + f^1_2\delta^3]\Big)$$

$$-\delta_{14} N_1 \frac{1}{L}\Big(\overset{\circ\circ}{\Phi}_L [\bar{\Phi}^1_1]^{-1}\delta^1 + \overset{\circ\circ}{f}{}^{v}_{1,L}\,\delta^1 + \overset{\circ\circ}{f}{}^{v}_{2,L}\,\delta^3 - \overset{\circ\circ}{\Phi}_L [\bar{\Phi}^1_2]^{-1}[f^1_1\delta^1 + f^1_2\delta^3]\Big)$$

$$+\delta_{10} P - \delta_{14} P \bar{y}$$

$$f_{10} = N_2 \frac{1}{L}\Big(\overset{\circ\circ}{\Phi}_L [\bar{\Phi}^2_1]^{-1}\delta^2 + \overset{\circ\circ}{f}{}^{u}_{1,L}\,\delta^2 + \overset{\circ\circ}{f}{}^{u}_{2,L}\,\delta^3 - \overset{\circ\circ}{\Phi}_L [\bar{\Phi}^2_2]^{-1}[f^2_1\delta^2 + f^2_2\delta^3]\Big)$$

$$+\delta_{13} N_1 \frac{1}{L}\Big(\overset{\circ\circ}{\Phi}_L [\bar{\Phi}^1_1]^{-1}\delta^1 + \overset{\circ\circ}{f}{}^{v}_{1,L}\,\delta^1 + \overset{\circ\circ}{f}{}^{v}_{2,L}\,\delta^3 - \overset{\circ\circ}{\Phi}_L [\bar{\Phi}^1_2]^{-1}[f^1_1\delta^1 + f^1_2\delta^3]\Big)$$

$$-P(\bar{x} - x_0) + \delta_{13} P(\bar{y} - y_0)$$

$$f_{11} = -N_3 \frac{1}{L^3}\Big(\overset{\circ\circ\circ}{\Theta}_0 [\bar{\Phi}^3_1]^{-1}\delta^3 + \overset{\circ\circ\circ}{f}{}^{\beta}_{1,0}\,\delta^2 + \overset{\circ\circ\circ}{f}{}^{\beta}_{2,0}\,\delta^1 + \overset{\circ\circ\circ}{f}{}^{\beta}_{3,0}\,\delta^3 - \overset{\circ\circ\circ}{\Theta}_0 [\bar{\Phi}^3_2]^{-1}[f^3_1\delta^2 + f^3_2\delta^1 + f^3_3\delta^3]\Big)$$

$$-N_4\delta_{12} + \delta_8 P\bar{y} + \delta_4 P\bar{x} + \delta_{12}\frac{P}{A} I_p$$

$$(-\delta_8 N_5 - \delta_{12} N_9)\frac{1}{L}\Big(\overset{\circ\circ}{\Phi}_0 [\bar{\Phi}^1_1]^{-1}\delta^1 + \overset{\circ\circ}{f}{}^{v}_{1,0}\,\delta^1 + \overset{\circ\circ}{f}{}^{v}_{2,0}\,\delta^3 - \overset{\circ\circ}{\Phi}_0 [\bar{\Phi}^1_2]^{-1}[f^1_1\delta^1 + f^1_2\delta^3]\Big)$$

$$(+\delta_4 N_6 - \delta_{11} N_8)\frac{1}{L}\Big(\overset{\circ\circ}{\Phi}_0 [\bar{\Phi}^2_1]^{-1}\delta^2 + \overset{\circ\circ}{f}{}^{u}_{1,0}\,\delta^2 + \overset{\circ\circ}{f}{}^{u}_{2,0}\,\delta^3 - \overset{\circ\circ}{\Phi}_0 [\bar{\Phi}^2_2]^{-1}[f^2_1\delta^2 + f^2_2\delta^3]\Big)$$

$$-\delta_{12} N_7 \frac{1}{L^2}\Big(\overset{\circ\circ}{\Theta}_0 [\bar{\Phi}^3_1]^{-1}\delta^3 + \overset{\circ\circ}{f}{}^{\beta}_{1,0}\,\delta^2 + \overset{\circ\circ}{f}{}^{\beta}_{2,0}\,\delta^1 + \overset{\circ\circ}{f}{}^{\beta}_{3,0}\,\delta^3 - \overset{\circ\circ}{\Theta}_0 [\bar{\Phi}^3_2]^{-1}[f^3_1\delta^2 + f^3_2\delta^1 + f^3_3\delta^3]\Big)$$

$$f_{12} = +N_3 \frac{1}{L^2}\Big(\ddot{\Theta}_0 [\bar{\Phi}^3_1]^{-1}\delta^3 + \ddot{f}{}^{\beta}_{1,0}\,\delta^2 + \ddot{f}{}^{\beta}_{2,0}\,\delta^1 + \ddot{f}{}^{\beta}_{3,0}\,\delta^3 - \ddot{\Theta}_0 [\bar{\Phi}^3_2]^{-1}[f^3_1\delta^2 + f^3_2\delta^1 + f^3_3\delta^3]\Big)$$

$$-P\,\bar{\omega}_1$$

$$f_{13} = N_3 \frac{1}{L^3}\left(\dddot{\Theta}_L [\bar{\Phi}^3_1]^{-1}\delta^3 + \dddot{f}^{\beta}_{1,L}\delta^2 + \dddot{f}^{\beta}_{2,L}\delta^1 + \dddot{f}^{\beta}_{3,L}\delta^3 - \dddot{\Theta}_L[\bar{\Phi}^3_2]^{-1}[f^3_1\delta^2 + f^3_2\delta^1 + f^3_3\delta^3]\right)$$

$$+ N_4\,\delta_{14} - \delta_{10}\,P_{\bar{y}} - \delta_6\,P_{\bar{x}} - \delta_{14}\frac{P}{A}I_p$$

$$+(\delta_{10}N_6 + \delta_{14}N_9)\frac{1}{L}\left(\ddot{\Phi}_L[\bar{\Phi}^1_1]^{-1}\delta^1 + \ddot{f}^{v}_{1,L}\delta^1 + \ddot{f}^{v}_{2,L}\delta^3 - \ddot{\Phi}_L[\bar{\Phi}^1_2]^{-1}[f^1_1\delta^1 + f^1_2\delta^3]\right)$$

$$+(-\delta_6N_6 + \delta_{14}N_8)\frac{1}{L}\left(\ddot{\Phi}_L[\bar{\Phi}^2_1]^{-1}\delta^2 + \ddot{f}^{u}_{1,L}\delta^2 + \ddot{f}^{u}_{2,L}\delta^3 - \ddot{\Phi}_L[\bar{\Phi}^2_2]^{-1}[f^2_1\delta^2 + f^2_2\delta^3]\right)$$

$$+\delta_{14}N_7\frac{1}{L^2}\left(\ddot{\Theta}_L[\bar{\Phi}^3_1]^{-1}\delta^3 + \ddot{f}^{\beta}_{1,L}\delta^2 + \ddot{f}^{\beta}_{2,L}\delta^1 + \ddot{f}^{\beta}_{3,L}\delta^3 - \ddot{\Theta}_L[\bar{\Phi}^3_1]^{-1}[f^3_1\delta^2 + f^3_2\delta^1 + f^3_3\delta^3]\right)$$

$$f_{14} = -N_3\frac{1}{L^3}\left(\ddot{\Theta}_L[\bar{\Phi}^3_1]^{-1}\delta^3 + \ddot{f}^{\beta}_{1,L}\delta^2 + \ddot{f}^{\beta}_{2,L}\delta^1 + \ddot{f}^{\beta}_{3,L}\delta^3 - \ddot{\Theta}_L[\bar{\Phi}^3_1]^{-1}[f^3_1\delta^2 + f^3_2\delta^1 + f^3_3\delta^3]\right)$$

$$+ P\bar{\omega}_1 \qquad (118)$$

Equations (118) are Equations (83) expanded where (83) is

$$f = [K(\delta)]\,\delta$$

The matrix $K(\delta)$ is given in Appendix I. The terms f^1_1, f^1_2, $f^{v}_{1,0}$ etc. are evaluated in the last part of Appendix II.

CHAPTER 6

NUMERICAL EXAMPLES

Since $K(\delta)$ is non-linear in δ as well as containing unknown forces V_1, V_2 and P, it is difficult to solve the equations $[K(\delta)]\delta = f$ in a direct manner. To overcome this problem the matrix was divided into two basic sub-matrices, one containing the linear structural behaviour, one containing the non-linear structural behaviour. A load increment procedure was then employed to solve the equations.

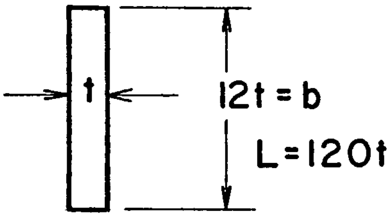
First, the complete linear structure matrix was generated and a found for a particular load level. This load level was then re-solved by constructing and adding the non-linear portion of $K(\delta)$ to the linear and solving again. The non-linear portion was constructed using the just-calculated linear deflections as well as the forces V_1, V_2 and P obtained by multiplying out the complete force deflection relation $[K(\delta)]\delta = f$ for the member. This procedure was repeated, each time constructing the non-linear portion of the matrix from the last previous-calculated deflections and forces.

This iteration at the given load level was terminated when δ became unchanged by any further iteration. This is a secant matrix approach. The load level was then increased and re-iterated, and so on. When a zero determinant for $[K(\delta)]$ was calculated, the structure was taken as buckled. Determinant plots were used to determine the load at which this occurred.

Several types of beam were studied, and the results are presented in the following section. The theoretical results for the channel section studied were taken from Vlassov [9], while all the others were taken from Timoshenko [7]. In the following section, critical loads from Timoshenko are subscripted with a T, from Vlassov with a V, and the results of the program are subscripted with a P .

There were four cross-sectional types analysed: a thin rectangle, a cruciform section, a wide flange and a channel.

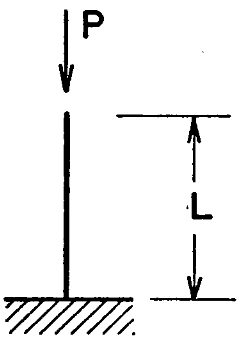
Thin Rectangular Section



$E/G = 3$

3 elements were used in all cases.

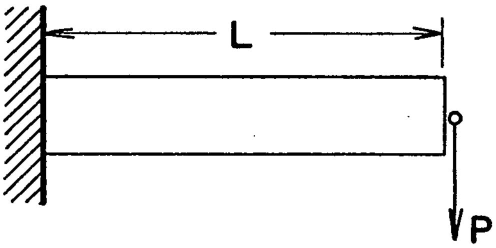
1. A Cantilever Column



$$P_T = \frac{4\pi^2 EI}{L^2}$$

$$P_P = 1.002\ P_T$$

2. Cantilever Beam Under End Shear, Warping Restrained At Wall

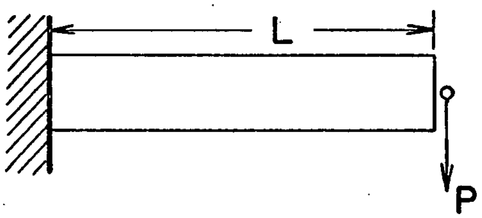


$$P_T = 4.013\sqrt{EIGJ}\ /L^2$$

$$P_P = 1.07\ P_T$$

The difference between P_P and P_T arises because the program assumed secondary warping restraint at the wall, whereas the Timoshenko solution ignored the warping effects entirely.

3. Cantilever Beam Under End Shear, Warping Allowed At Wall



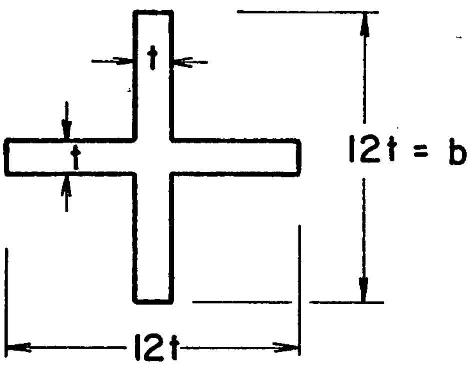
$$P_T = 4.013\sqrt{EIGJ}/L^2$$

$$P_p = 1.01\ P_T$$

The program assumed no warping restrain at the wall, but kept the warping terms internally in the beam. As before, the Timoshenko solution ignored the warping effects entirely.

<u>Cruciform Section</u>

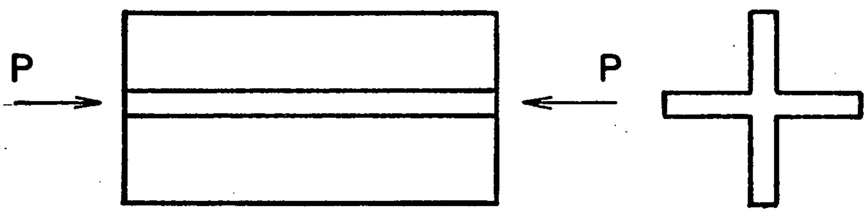
$L = 120t$



$$E/G = 2.6$$

3 elements used.

4. Torsional Buckling Under Pure Axial Load, Simply Supported In Both Planes



$$P_p = \begin{cases} 1.05\ P_T \\ .977\ P_{T_{PL}} \end{cases}$$

In the program, the ends were restrained from torsional rotation but free to warp. The program took account of internal warping constraints. The program' critical load was compared to two classical solutions:

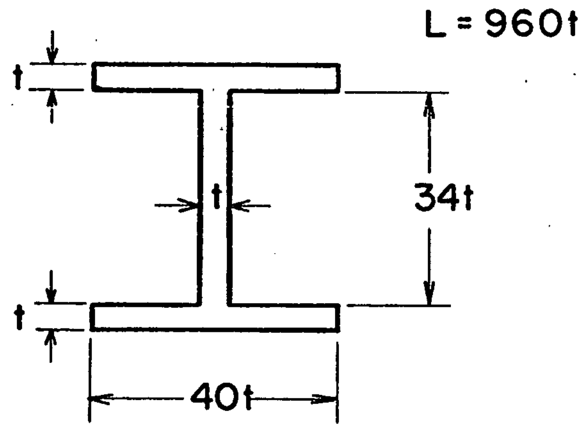
$$P_T = 8Gt^3/b$$

which ignores all warping restraint, and

$$P_{T_{PL}} = \left(.456 + \frac{b^2}{4L^2}\right) \frac{\pi^2}{6(1-\nu)} P_T$$

which is developed from plate theory, and consequently includes warping as well as other effects.

The Wide Flange Cross Section



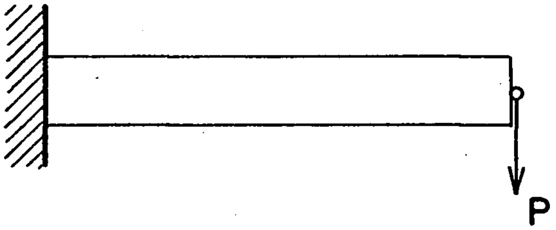
$$E/G = 2.71$$

$$P_T = \gamma\sqrt{EIGJ}\,/L^2$$

$$L^2 JG/E\Gamma = 4$$

γ is found from tables against $L^2JG/E\Gamma$ for various boundary conditions and loads. Six elements were used in all cases, except where noted.

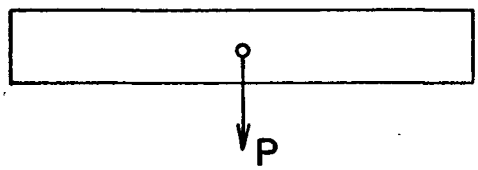
5. Cantilever Beam Under End Shear, Warping Restrained at Fixed End But Unrestrained At Free End



$$P_T = 9.76\sqrt{EIJG}/L^2$$

$$P_p = 1.006\,P_T$$

6. Beam Simply Supported In Both Planes, Torsional Rotation Restrained At Ends, Loaded At Centre, Warping Unrestrained At Ends



$$P_T = 31.9\sqrt{EIGJ}/L^2$$

$$P_p = 1.004\,P_T$$

7. Same As Example (6), But Only Two Elements Used

$$P_p = 1.008\,P_T$$

8. Same As Example (6), But Beam Fixed At Ends In Weak Lateral Direction Only, With Warping Restrained At Ends

$$P_T = 88.8\sqrt{EIGJ}/L^2$$

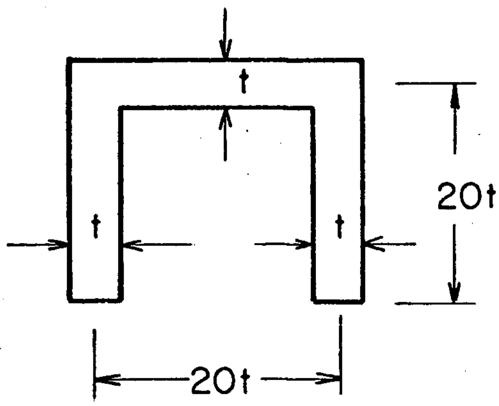
$$P_p = .998\,P_T$$

9. Same As Example (6), But With A Uniformly Distributed Load q At The Shear Centre

$$(qL)_T = 53\sqrt{EIGJ}/L^2$$

$$(qL)_P = 1.03\,(qL)_T$$

The Channel Cross Section

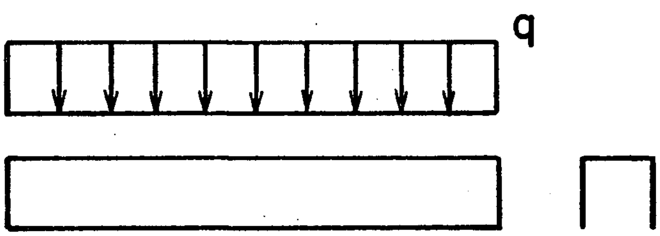


L = 480t

6 elements were used in all cases.

For critical loads, see Vlassov [9].

10. Channel Beam Simply Supported In Both Planes, Each End, Torsional Rotations Restrained At Ends, Warping Unrestrained, Uniformly Distributed Load q At Shear Centre



$$q_p = .985\,q_v$$

11. Same As Example (10), But Load In Opposite Direction

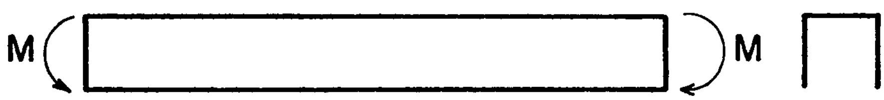
$$q_p = 1.01\, q_v$$

It is interesting to note that q_p for case 10 is 6.34 times that for case 11.

12. Same As Example (10), But Loaded With End Moments M Rather Than A Uniformly Distributed Load

$$M_p = 1.007\, M_v$$

13. Same As Example (12), But Moment M Reversed



$$M_p = 1.008\, M_v$$

It is interesting to note again that M_p for case 12 is 19.3 times that for case 13.

These examples were run on an IBM-360-67 machine using MTS. The channel section shown in example (10) was analysed using six load increments with three iterations per load increment. The total time taken was twenty-eight seconds, of which seventeen seconds were CPU time.

As can be seen from the above examples, agreement is excellent in all cases. Where there is some discrepancy it is in cases where the program contains secondary warping and the classical solutions do not. In these instances, the program should give higher results, which it does.

Good accuracy is also obtained even when a small number of elements is used, as for example structure (7) which with two elements gives very good agreement compared to the classical result.

All of the above tests are bifurcation type buckling. There is another type of buckling called amplification buckling and this occurs when the loads tend to cause a displacement in the direction the section wishes to buckle. This type of buckling is characterized by one or more of the deflections becoming unbounded. Unfortunately, there exist few classical solutions for amplification buckling and as a result no comparative tests were made.

However, a totally unsymmetric shape was studied for amplification buckling. The section was analyzed as a cantilever under end load, with the load being applied through the shear centre, and parallel to or at a small angle Ψ to the strong principal plane. This had the effect of causing displacements in the lateral and torsional modes, thus making the problem one of amplification, not bifurcation. The cross section properties and a plot of critical load versus Ψ is shown below.

$x_o = -0.326$ in , $y_o = 0.941$ in, $A = 0.0328$ in^2

$\bar{x} = -0.267$ in , $\bar{y} = 0.218$ in, $\bar{\omega}_1 = -0.1295$ in^2

$I_x = 0.483 \times 10^{-2}$ in^4

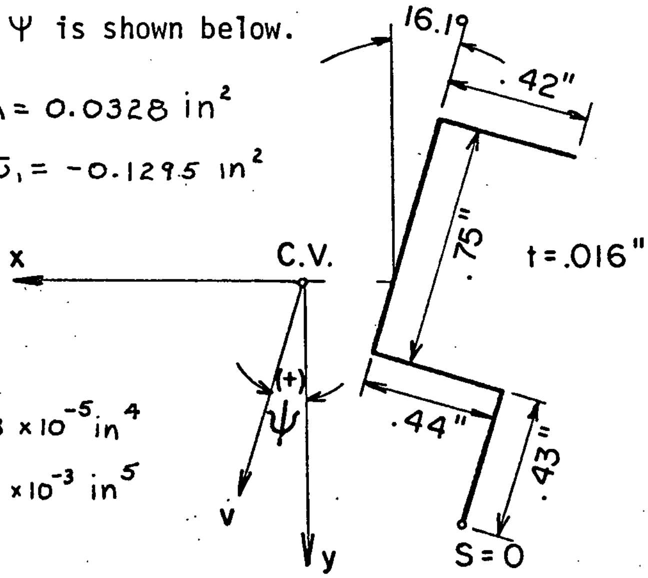
$I_y = 0.786 \times 10^{-3}$ in^4

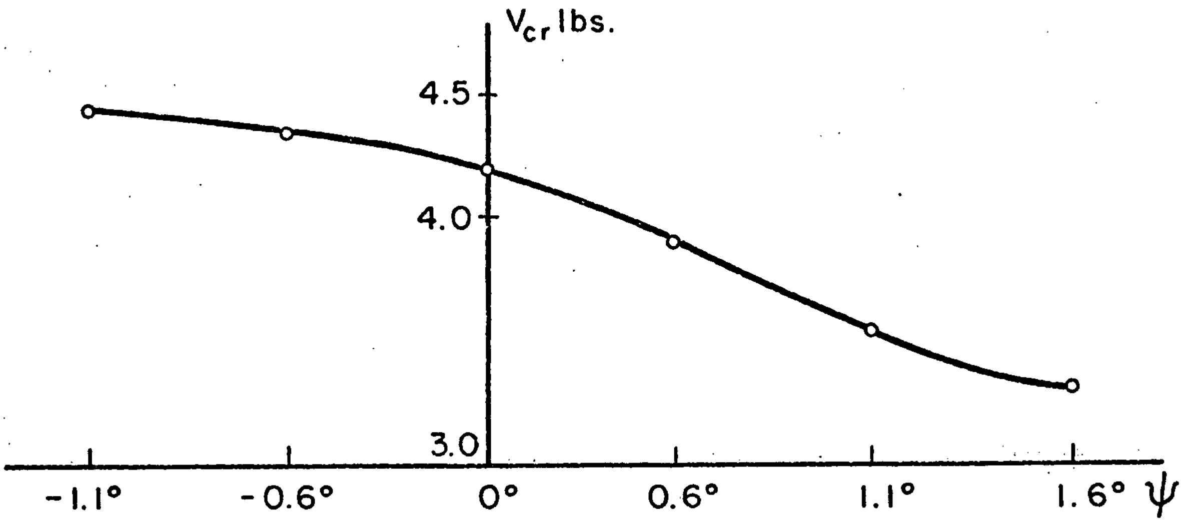
$I_p = 0.951 \times 10^{-2}$ in^4

$K_{13} = -\Gamma = -0.901 \times 10^{-4}$ in^6, $J = 0.28 \times 10^{-5}$ in^4

$N_7/E = 0.793 \times 10^{-4}$ in^6 , $N_8/E = 0.81 \times 10^{-3}$ in^5

$N_9/E = -0.218 \times 10^{-2}$ in^5





For this particular example, the values of E , G and L are

$$E = 10 \times 10^6 \text{ psi.}$$

$$G = 3.76 \times 10^6 \text{ psi.}$$

$$L = 18.5 \text{ in.}$$

There are several points of interest about the curve shown above. One is the unsymmetry of the critical load about the origin and the other is the sensitivity of the critical load to the angle ψ . The dependency of the critical load on the angle ψ is not unexpected, as the channel section shown previously has critical loads that depend upon which axis and direction is loaded. The example shown above however is very sensitive to ψ , and this may be due to its extreme flexibility in torsion.

In general the solution of $[K(\delta)]\delta = f$ for an amplification problem requires more iterations to converge to a δ vector than a bifurcation type problem. This may cause difficulties if the section is very flexible and is loaded near critical, as a large number of iterations may be required to arrive at a deflected shape. In some cases, particularly when one deflection in the direction of buckling becomes large, a study of both deflections and determinants is required to determine a critical load. However, this is not always the case, as there are many well behaved amplification problems.

CHAPTER 7

DISCUSSION

There are several points of detail that should be discussed, but were not mentioned in the main body of the thesis, as it was felt that introduction at a previous stage would be a digression from the main purpose of the thesis, that is, the assembly of a stiffness matrix.

7:1 Small Rotation Theory

If the assumptions that were applied in this thesis of $u'v' \ll v'$ etc. were ignored, then the rotations could not be treated as vectors. This would have the result of introducing terms such as $u'v'x$ or $u'\beta y$ into the equation for axial displacement w. These terms are non-linear in the rotations and the way they appear in the equation for w depends upon which order the rotations are taken. This order dependancy could be removed by changing the co-ordinate system to one of Euler angles, but Euler angles are not as straightforward to use as the system chosen. The end result would be that no matter how the non-linear terms $u'\beta y$ etc. in w were treated, they would complicate w and immensely complicate the strain-deflection and stress-strain equations. It is for this reason that the constraints on the angles were maintained, as it greatly reduces the complexity of the resulting equations while still allowing a reasonably large field of validity.

7:2 Secondary Stresses

The secondary, or plate-bending stresses need only be considered for sections of two certain types. The first type is typified by the thin rectangular section, as it requires the plate-bending stresses in order to have any stiffness at all in the weak plane. The second type of cross-section is typified by the thin cruciform or angle section, where any warping resistance must come from the secondary stresses. In this case, neglect of the secondary stresses does not cause a zero torsional stiffness, as plate torsional resistance would still be in effect. However, the mathematical solution technique uses the quantity λ^2 which is the ratio of plate torsion stiffness to warping torsional stiffness. If the warping is taken as zero, λ^2 becomes infinite and the equations break down.

Sections such as channels, wide flanges, etc. which have substantial membrane stiffness contributions from all possible modes need not consider the secondary stresses. It is therefore recommended for sections of this type that the constants K_2, K_4, K_9, K_7, K_{16}, K_{17}, K_{18}, K_{19}, K_{20}, K_{21}, K_{22}, K_{23}, K_{24} be ommitted in order to ease calculations.

7:3 Constants

In the evaluation of the constants of Equations (78) there are several points to be noted.

First, placement of the origin of s at any extremity gives a non-zero $\bar{\omega}_1$ constant. It is possible to place the origin of s at a point where $\bar{\omega}_1$ will equal zero. Doing this allows K_{13}, or Γ, the warping constant, to be written as $K_{13} = \int_0^{\bar{s}} \omega^2 \, ds$ rather than $\int_0^{\bar{s}} (\omega - \bar{\omega}_1)^2 ds$ as done in this development. They both have the same numerical value if their origin in the x, y plane is common. This means that tables listing Γ (or K_{13}) may be used without having to consider the position of $s = 0$ as long as the origin of x, y is the shear centre.

The placement of s at an extremity in this development causes some problems with axial loads. Since they are assumed applied at $w_0 = \delta_1$

or δ_2 , which is taken to be at $s=0$, any axial load automatically applies moments and bi-moments due to its eccentricity from the shear centre. Therefore, if an axial load is to be applied at any other point on the cross-section, compensating moments and bi-moments must be applied to place the axial resultant in the required position.

Secondly, as was mentioned previously, the calculation of $\int_0^s r\,ds$ involves a sign convention. It is considered positive if the swept area $r\,ds$ is clockwise, negative if counterclockwise. If $r = x\sin\phi + y\cos\phi$ is integrated instead, the signs are automatically accounted for. Defining r in terms of x, y and ϕ also allows easier evaluation of r in terms of cross-section properties.

Thirdly, the section constants in Equations (79) were developed for a cross-section which does not branch. In branched systems, a closer look must be taken at the equilibrium equation:

$$q_0 = \int_0^s T_1'\,ds \tag{15}$$

This was written for the element shown in Figure 5 wherein q_0 acted only on two edges. At a branch, there are three or more edges and a set of q_0's , q_{0_1} , q_{0_2} , q_{0_3} , etc. acting one to an edge. These q_0's and their derivatives must be in equilibrium with $\overline{T}_1$ and $\overline{T}_1'$. Since Equation (15) was not developed for this situation an extension is necessary for general branched systems. This can be done by using a series of equations similar to Equation (15),

$$q_{0i} = \underline{q_{0i}} + \int_{s_B}^{s} \overline{T}'\,ds$$

where q_{0i} is the shear flow between branches and $\underline{q_{0i}}$ is the shear flow just past the branch at s_B . In this equation $\underline{q_{0i}}$ is unknown. When an equation such as this has been written for each element of s between branches, they may be assembled and the unknown $\underline{q_{oi}}$ evaluated using the following conditions:

(a) The $\underline{q_{oi}}$ must be in equilibrium at each branch point.

(b) The q_{oi} must be zero at free edges.

These conditions then determine what value q_{oi} will be at any point s.

Having to use this approach has its effect on the constants, as they were integrated over an unbranched system. This can be seen, for example, in some of the terms for the torque T. One of the contributing terms is $\int_0^{\bar{s}} q_o r\,ds$. Expanding q_o gives terms of the type

$$\int_0^{\bar{s}} r \int_0^s t(\omega_1 - \bar{\omega}_1)\,ds\,ds = K_{13}$$

amongst others. The portion $\int_0^s t(\omega_1 - \bar{\omega}_1)\,ds$ represents part of q_o as determined by $q_o = \int_0^s \bar{T}'\,ds$, an equilibrium equation. This no longer holds for a branched system. The other two integrals involved in K_{13}, $\int_0^{\bar{s}} (q_o r)\,ds$ and $\omega_1 = \int_0^s r\,ds$ are geometric integrals and are unaffected by branches. This means that constants K_i arising from consideration of the shear flows q_o will have to be calculated using $q_{oi} = \underline{q_{oi}} + \int_{s_B}^{s} \bar{T}_1'\,ds$ and a stepwise integration procedure. This means that

$$K_{13} = \int_0^{\bar{s}} r \int_0^s t(\omega_1 - \bar{\omega}_1)\,ds\,ds = \int_0^s r\,ds \int_0^s t(\omega_1 - \bar{\omega}_1)\,ds \Big|_0^{\bar{s}} - \int_0^{\bar{s}} t(\omega_1 - \bar{\omega}_1) \int_0^s r\,ds\,ds$$

$$= -\int_0^{\bar{s}} t(\omega_1 - \bar{\omega}_1)\omega_1\,ds = -\int_0^{\bar{s}} t(\omega_1 - \bar{\omega}_1)^2\,ds$$

may not be valid for all sections, as it is based on an unbranched system.

However, for a two branched system, the following constants, I_{yc}, I_{xc}, K_{13}, which are a result of q_o terms, can be shown to take the same form as an unbranched system.

The inclusion of I_{yc} and I_{xc} in the list of constants necessary to be investigated arises from the fact that there are two types of integrals defining I_{yc} and I_{xc}. The first type is due to the stresses σ and takes the form

$$\int_0^{\bar{s}} (\bar{y} - y) y t \, ds = \Phi_{xc}$$

This integral is unaffected by branching in the cross-section, as it is based on σ, not q_o. The second type is due to q_o terms and takes the form

$$\int_0^{\bar{s}} \sin\phi \int_0^s t(\bar{y} - y) \, ds \, ds = -\Phi_{xc}$$

for an unbranched section. Since the internal integral is a measure of q_o, it must be modified to take account of equilibrium at branches. It is this Φ_{xc} that was investigated for branched systems. Of course, from knowledge of linear beam behaviour, it is known that the constant giving shear values is Φ_{xc}, so it would be expected that no matter how branched the system is, or what co-ordinate system is used, the value that should arise from the integral is Φ_{xc}. However, this is not immediately clear from the integral

$$\int_0^{\bar{s}} \sin\phi \int_0^s t(\bar{y} - y) \, ds \, ds$$

if it is written in the form necessary to handle branched systems. The implication of this discussion seems to be that the co-ordinate system s is not a particularly good one for branched systems, and only leads to complications in the calculations which in the end give an expected result.

It may be that a more judicious choice of co-ordinate system is possible for the calculation of K_{13} and the Φ_{xc}, Φ_{yc} due to q_o, once the nature of these constants is understood from the development and role q_o plays in calculating them. It might then be possible to see that in general, no matter how many branches, the constants have the same form as the unbranched system, as is most likely the case.

Fourthly, the I_{xc}, I_{yc} values should be calculated about the centroidal axis, all the others about the axis through the shear centre.

Finally, in some cases a general formula for some of the constants should be derived first, as using numbers directly may result in accuracy problems.

7:4 Loads

The lateral loads are assumed to act at the shear centre. The axial loads are taken to act at the point on the cross-section $S=O$. If the lateral loads are applied at any other position in the cross-section, for example the top flange of an I beam, extra terms will have to be introduced to account for this effect during deformation. This is not difficult to do, but has not been done in this development.

7:5 Differential Equations

If the differential equations (78) are compared to those of Oden, Timoshenko, Vlassov, etc., it will be seen that Equations (78) are much more general, as they do not require the use of linear values of V, M etc. in their evaluation. They are non-linear differential equations and they include the possibility that V, M etc. may change due to deformed geometry. This may be of importance, for instance, with the presence of axial forces near critical in one plane, as they may magnify the moments tending to cause lateral buckling. This possibility is ignored in the usual linear differential equations of buckling.

There are several methods of solving the differential equation using approximate methods. A Galerkin method might be used, or any one of several variations of an iteration technique. The iteration techniques differ as to what is treated as the lefthand side of the equation and what is treated as the righthand side. In this development, the lefthand side was taken to be the linear structure equations, as it was felt that this achieved the most even handed treated of the non-linear structure terms and also kept the equations simple. An alternative method is to place some of the non-linear structure terms (which may be linear in the differential equation) on the lefthand side, but this has the effect of emphasizing some non-linear structure terms compared to the others, and complicates the solution calculations.

Unfortunately, the technique used in this thesis does not take full advantage of the non-linearity of the equations, as using one iteration is equivalent to using linear forces in the differential equation solution. Either two iterations would be required, or one of the alterantive forms of iteration used, to adequately cover the non-linear interactions in the equations.

Finally, since the iteration technique only approximates the correct solution of the equilibrium equations, there is no reason for the resulting displacements to give equilibrium forces. Of course, the forces at a joint will be in equilibrium, but an individual member may not be.

7:6 Symmetry and Conservativeness

The moments, when defined as maintaining their line of action, are non-conservative when acting on free edge boundary conditions. This will produce a non-symmetric member matrix, [10], as each member is developed under free boundary conditions. However, when the individual matrices are added up into a structure matrix, it is the boundary conditions on the structure which determine the non-conservativeness, and in most cases they render the system conservative and the matrix symmetric. An example of an exception is a cantilever under applied end moment, with the moment defined as in this thesis as maintaining its direction of action.

If any non-conservative problems are envisaged, the load vector may have to be modified to become a function of the displacements. Of course, indirectly the structure matrix is modified by taking the parts of the load vector that become functions of δ and transferring them to the other side of the equation where they can be placed in the structure matrix. However, if the loading is non-conservative, the approach used here is in general inadequate as a dynamic approach is best for the most general solutions [10]. It is interesting though that no mention of conservative forces was necessary to develop the equations in this work. The only limitations on these equations, aside from the restrictions on rotations and torques, is the assumption of a static, elastic solution.

7:7 Approximations for Small Terms

In some of the approximations used to neglect terms, the fact that the length of the beam was large compared to certain terms was utilized. However, the question of the validity of these approximations for the elements arises, since the element length may be quite short. This is not a real problem, however, as the elements need only be able to duplicate the actual structure. If, for example, the effects were included in the element because the approximations about L were invalid, it would make no difference to the analysis of the large overall structure as the new terms would all fall out, because they are insignificant in affecting the behaviour of the of the large structure.

For instance, shear deflection behaviour is not usually included in ordinary linear beam element stiffness matrices unless short sections of beam were going to be studied. If shear behaviour was placed in these elements and a long thin beam analysed, it would be seen that these terms contribute nothing to the analysis, as for a long thin beam, shear deflection effects are trivial.

It can therefore be concluded that the elements need not be complete in themselves, but only must be able to duplicate the required behaviour of the actual physical problem.

7:8 Point of Action of Axial Load

In this development, the axial load is taken to act at the free edge of the cross-section where $S=0$, whereas the shear forces and moments are taken to act through and about an axis system through the shear centre. This means there are two points of interest at each end of the section, rather than the usual one point. While this is different to the usual beam analysis where only one point serves to define all the deflections, it should not be viewed with alarm. There are many instances where two or more points of reference have an advantage over the usual one reference point. For example, a doubly symmetric wide flange can be represented two ways. The first and most usual method is to concentrate all the forces and deflections at the centroid. The second method is to use two reference points, one at the intersection of the top flange and the web, one at the intersection of the lower flange and the web. Using these two points of reference, all the pertinent deflections of the section can be described. However, an important advantage has been gained. It is now possible to specify separate boundary conditions for each flange, which is not possible using the centroid as the reference point. It can therefore be stated that the use of two reference points greatly enlarges the type of boundary conditions which may be easily handled. This discussion was introduced only to indicate that while two reference points for a beam may not be common, they may sometimes be easily introduced with significant advantage.

In this thesis there are reasons other than increased usefulness for using two reference points. Some physical feel for these reasons can be gained from the following example.

Postulate a wide flange beam with an axial load acting at a point A, which is located along a line perpendicular to the web and passing through the centroid and shear centre. This is illustrated in Figure 12.

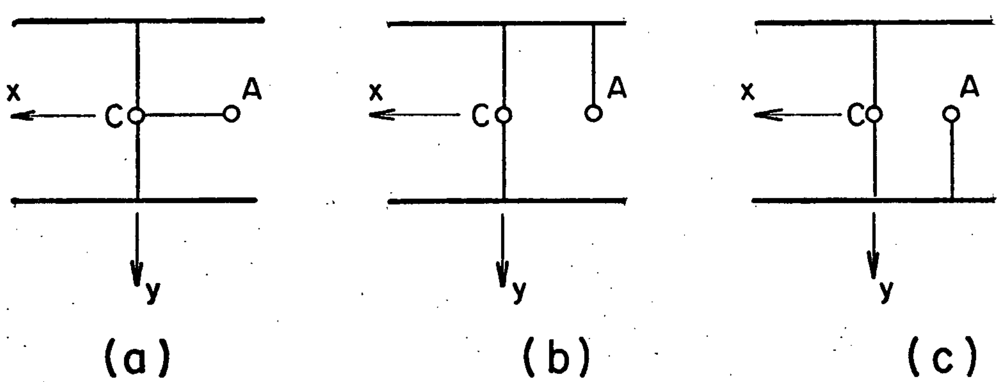


FIG. 12 THREE METHODS OF CONNECTION FOR ECCENTRIC AXIAL LOAD FOR I SECTION.

Three different types of connection are illustrated in Figure 12. In Figure 12a, the load is passed to the section from A by a rigid arm connected to the web. Figure 12b and 12c show rigid connections to the upper and lower flanges respectively.

Now, for translations and rotations where plane sections remain plane, the work done by the axial force at A is the same for the three connections shown. However, for a pure warping deformation, which is characterized by the flanges rotating about the y axis but in opposite directions, the work done by an axial load at A is different for all three cases. In Figure 12a, the work is zero, and in Figure 12b, the work is non-zero and of opposite sign to the work of Figure 12c.

Recalling that the work done by the applied loads during a warping displacement is defined as the bi-moment, it can be clearly seen that the axial force at A exerts three different bi-moments on the section in Figure 12, depending on how it is connected. This implies that it is not enough to specify the position of the axial load; its method of connection to the cross-section must also be specified. This means that placing the axial point of reference at the shear centre would be almost meaningless because any eccentric loads would be referenced to the method and point of connection, not to the shear centre. For the general case, where the shear centre may not even be in the cross-section, it becomes totally meaningless to specify the shear centre as the point of reference for axial load and deflection, as the first thing that must be done is to specify some kind of connection from the axial load to the beam, which implies corrective bi-moments. It is for these reasons that the point of reference for axial terms was taken to be the arbitrary point $S=O$. This also gives some computational advantages.

If the axial load is not at $S=O$, then moments must be applied to correct the resultant, and these moments are calculated by the eccentricity of the axial load from $s=0$, not from the shear centre. This is because the matrix takes into account and expects that the axial load is at the point $s=0$. These corrective moments may be functions of β, the torsional rotation, since the β rotations affect the eccentricity of the axial load about $s=0$, unless the joint at which the axial load is applied is restrained torsionally.

The calculation of the corrective bi-moment for an eccentric axial load cannot be done using equivalent force resultants, as the bi-moment has no resultant. Instead, the corrective value must be found using energy. For instance, the extra work of warping Q due to the eccentric load may be found by

$$Q = P w_{\beta'}$$

where $w_{\beta'}$ is the axial displacement due to warping only of the point of action of P with respect to the axial displacement of $s=0$. The term $w_{\beta'}$ is found from Equation (7) when all deflections other than β' are zero. This gives

$$w_{\beta'} = \beta' \int_0^s r = \beta' \omega_1$$

Clearly, the corrective bi-moment is

$$P\omega_1$$

Once again, since ω_1 is only meaningful for points on the cross-section, it illustrates that P must in some way be connected to some point on the cross-section.

The value of ω_1 in $P\omega_1$ is calculated by integrating to the point s at which P is connected to the cross-section. The connection itself may cause P to be above or below the point of connection to the cross-section. If this is the case, then extra terms accounting for this will have to be added to $P\omega_1$. These terms will depend on the type of connection used. If P is eccentric but on the cross-section, then $P\omega_1$ is the only corrective term needed.

CHAPTER 8

CONCLUSIONS

In this thesis, a set of non-linear force deflection relations were developed using a set of common engineering assumptions consistently applied. A discussion of the size of the terms that were discarded on the basis of these assumptions is given. These force-deflection equations, when placed in the overall beam equations of equilibrium, yield a set of differential equations of equilibrium in terms of the displacements. These equations are similar in form to those developed in Timoshenko and Gere [7], Bleich [3], Oden [6] and Vlassov [9], except that the equations herein are coupled non-linear equations, and so more general. In this sense, the equations are new.

The solution of the differential equations was attained by employing an iteration, or successive approximation method. This produces a solution to the differential equations, in terms of the boundary conditions, which must be placed back into the non-linear force-deflection relations and evaluated at the boundaries to yield a stiffness matrix.

In this sense, the matrix is new, as it is the solution of a more general set of non-linear differential equations than usual and has been developed using an alternative and quite different method than the more common and well developed energy methods.

This stiffness matrix is non-linear in the δ terms and is difficult to solve directly. A secant matrix approach involving several iterations was used, although there are other alternative methods of solving non-linear equations which could be used. The matrix was used to analyse several classical problems and the agreement was good in all cases.

Once the matrix has been derived for a member, it is possible to apply various transformations which allow the member matrix to handle eccentric connections and eccentric loads. This in turn allows the building of arbitrary space structures with unusual joint and load conditions. This step, however, was not taken in this thesis.

There are several areas of investigation that are open for future work.

First, alternative solutions of the differential equation may be looked at. The Galerkin method could be used, or several alternative iteration techniques could be employed.

Secondly, the differential equations could be extended to include the effect of shaft buckling, which was neglected in this thesis.

Thirdly, a better technique than the secant matrix approach for the solution of $K\delta = f$ might be employed.

Finally, the relative effects of the various physical constants on the critical load might be studied.

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APPENDIX I

THE MATRIX

Since Equations (118) are non-linear in δ , there are several ways of removing a δ vector to form the equation $[K(\delta)]\delta = f$.

For instance, the term $N u'' v'$ may be grouped as

$$(N u'') v'$$

or

$$(N v') u''$$

The former grouping is of the form $M v'$, and is the form that would have been obtained if the equations were linear differential equations involving linear values of M, V, P in the co-efficients. This method of grouping will be used, and this will be used as the rule to factor out the δ . For example, in $N u'' v'$, the δ associated with v' will be taken outside as δ .

Once this has been decided, it is next convenient to represent the matrix as a sum of four sub-matrices $K = K_1 + K_2 + K_3 + K_4$. The linear structure terms are K_1 and K_2, whereas K_3 and K_4 are the non-linear terms. Expansion of Equations (124) under these conditions yields the following:

$$
K_1 = \frac{AE}{L} \times
\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline
1 & -1 & & \bar{\bar{y}} & & -\bar{\bar{y}} & & -\bar{\bar{x}} & & \bar{\bar{x}} & & \bar{\omega}_1 & & -\bar{\omega}_1 \\
\hline
 & 1 & & -\bar{\bar{y}} & & \bar{\bar{y}} & & \bar{\bar{x}} & & -\bar{\bar{x}} & & -\bar{\omega}_1 & & \bar{\omega}_1 \\
\hline
 & & & & & & & & & & & & & \\
\hline
 & & & \bar{\bar{y}}^2 & & -\bar{\bar{y}}^2 & & -\bar{\bar{y}}\bar{\bar{x}} & & \bar{\bar{y}}\bar{\bar{x}} & & \bar{\omega}_1\bar{\bar{y}} & & -\bar{\omega}_1\bar{\bar{y}} \\
\hline
 & & & & & & & & & & & & & \\
\hline
 & & & & & \bar{\bar{y}}^2 & & \bar{\bar{y}}\bar{\bar{x}} & & -\bar{\bar{y}}\bar{\bar{x}} & & -\bar{\omega}_1\bar{\bar{y}} & & \bar{\omega}_1\bar{\bar{y}} \\
\hline
 & & & & & & & & & & & & & \\
\hline
 & & & & & & & \bar{\bar{x}}^2 & & -\bar{\bar{x}}^2 & & -\bar{\omega}_1\bar{\bar{x}} & & \bar{\omega}_1 x \\
\hline
 & & & & & & & & & & & & & \\
\hline
 & & \text{Symmetric} & & & & & & & \bar{\bar{x}}^2 & & +\bar{\omega}_1\bar{\bar{x}} & & -\bar{\omega}_1\bar{\bar{x}} \\
\hline
 & & & & & & & & & & & -\frac{N_4 L}{AE} & & \\
\hline
 & & & & & & & & & & & \bar{\omega}_1^2 & & -\bar{\omega}_1^2 \\
\hline
 & & & & & & & & & & & & & \frac{N_4 L}{AE} \\
\hline
 & & & & & & & & & & & & & \bar{\omega}_1^2 \\
\hline
\end{array}
$$

$$\bar{\bar{x}} = \bar{x} - x_o , \quad \bar{\bar{y}} = \bar{y} - y_o$$

$K_2 =$

	3 4 5 6
3	$-N_1 \frac{1}{L^2} \left(\dddot{\phi}_0 \, [\bar{\Phi}_1^1]^{-1} \right)$
4	$-N_1 \frac{1}{L} \left(\ddot{\phi}_0 \, [\bar{\Phi}_1^1]^{-1} \right)$
5	$+N_1 \frac{1}{L^2} \left(\dddot{\phi}_L \, [\bar{\Phi}_1^1]^{-1} \right)$
6	$+N_1 \frac{1}{L} \left(\ddot{\phi}_L \, [\bar{\Phi}_1^1]^{-1} \right)$

	7 8 9 10
7	$N_2 \frac{1}{L^2} \left(\dddot{\phi}_0 \, [\bar{\Phi}_1^2]^{-1} \right)$
8	$-N_2 \frac{1}{L} \left(\ddot{\phi}_0 \, [\bar{\Phi}_1^2]^{-1} \right)$
9	$-N_2 \frac{1}{L^2} \left(\dddot{\phi}_L \, [\bar{\Phi}_1^2]^{-1} \right)$
10	$+N_2 \frac{1}{L} \left(\ddot{\phi}_L \, [\bar{\Phi}_1^2]^{-1} \right)$

	11 12 13 14
11	$-N_3 \frac{1}{L^3} \left(\dddot{\theta}_0 \, [\bar{\Phi}_1^3]^{-1} \right)$
12	$+N_3 \frac{1}{L^2} \left(\ddot{\theta}_0 \, [\bar{\Phi}_1^3]^{-1} \right)$
13	$+N_3 \frac{1}{L^3} \left(\dddot{\theta}_L \, [\bar{\Phi}_1^3]^{-1} \right)$
14	$-N_3 \frac{1}{L^2} \left(\ddot{\theta}_L \, [\bar{\Phi}_1^3]^{-1} \right)$

$K_3 =$

$$\begin{array}{c|c|}
 & 3 \qquad\quad 4 \qquad\quad 5 \qquad\quad 6 \\ \hline
3 & -N_1 \frac{1}{L^2}\left[\dddot{f}^{\nu}_{1,0} - \dddot{\Phi}_0 \left[\bar{\Phi}^1_2\right]^{-1} f^1_1 \right] \\ \hline
4 & -N_1 \frac{1}{L}\left[\ddot{f}^{\nu}_{1,0} - \ddot{\Phi}_0 \left[\bar{\Phi}^1_2\right]^{-1} f^1_1 \right] \\ \hline
5 & +N_1 \frac{1}{L^2}\left[\dddot{f}^{\nu}_{1,L} - \dddot{\Phi}_L \left[\bar{\Phi}^1_2\right]^{-1} f^1_1 \right] \\ \hline
6 & +N_1 \frac{1}{L}\left[\ddot{f}^{\nu}_{1,L} - \ddot{\Phi}_L \left[\bar{\Phi}^1_2\right]^{-1} f^1_1 \right] \\ \hline
\end{array}$$

$$\begin{array}{c|c|}
 & 7 \qquad\quad 8 \qquad\quad 9 \qquad\quad 10 \\ \hline
7 & +N_2 \frac{1}{L^2}\left[\dddot{f}^{\mu}_{1,0} - \dddot{\Phi}_0 \left[\bar{\Phi}^2_2\right]^{-1} f^2_1 \right] \\ \hline
8 & -N_2 \frac{1}{L}\left[\ddot{f}^{\mu}_{1,0} - \ddot{\Phi}_0 \left[\bar{\Phi}^2_2\right]^{-1} f^2_1 \right] \\ \hline
9 & -N_2 \frac{1}{L^2}\left[\dddot{f}^{\mu}_{1,L} - \dddot{\Phi}_L \left[\bar{\Phi}^2_2\right]^{-1} f^2_1 \right] \\ \hline
10 & +N_2 \frac{1}{L}\left[\ddot{f}^{\mu}_{1,L} - \ddot{\Phi}_L \left[\bar{\Phi}^2_2\right]^{-1} f^2_1 \right] \\ \hline
\end{array}$$

$$\begin{array}{c|c|}
 & 11 \qquad\quad 12 \qquad\quad 13 \qquad\quad 14 \\ \hline
11 & -N_3 \frac{1}{L^3}\left[\dddot{f}^{\beta}_{3,0} - \dddot{\Theta}_0 \left[\bar{\Phi}^3_2\right]^{-1} f^3_3 \right] \\ \hline
12 & +N_3 \frac{1}{L^2}\left[\ddot{f}^{\beta}_{3,0} - \ddot{\Theta}_0 \left[\bar{\Phi}^3_2\right]^{-1} f^3_3 \right] \\ \hline
13 & +N_3 \frac{1}{L^3}\left[\dddot{f}^{\beta}_{3,L} - \dddot{\Theta}_L \left[\bar{\Phi}^3_2\right]^{-1} f^3_3 \right] \\ \hline
14 & -N_3 \frac{1}{L^2}\left[\ddot{f}^{\beta}_{3,L} - \ddot{\Theta}_L \left[\bar{\Phi}^3_2\right]^{-1} f^3_3 \right] \\ \hline
\end{array}$$

$K_{3_{cont}} =$

<table>
<tr><th></th><th>11</th><th>12</th><th>13</th><th>14</th></tr>
<tr><td>3</td><td colspan="4">$-N_1 \frac{1}{L^2}\left[\dddot{f}^{\nu}_{2,0} - \dddot{\phi}_0 [\bar{\phi}^1_2]^{-1} f^1_2\right]$</td></tr>
<tr><td>4</td><td colspan="4">$-N_1 \frac{1}{L}\left[\ddot{f}^{\nu}_{2,0} - \ddot{\phi}_0 [\bar{\phi}^1_2]^{-1} f^1_2\right]$</td></tr>
<tr><td>5</td><td colspan="4">$+N_1 \frac{1}{L^2}\left[\dddot{f}^{\nu}_{2,L} - \dddot{\phi}_L [\bar{\phi}^1_2]^{-1} f^1_2\right]$</td></tr>
<tr><td>6</td><td colspan="4">$+N_1 \frac{1}{L}\left[\ddot{f}^{\nu}_{2,L} - \ddot{\phi}_L [\bar{\phi}^1_2]^{-1} f^1_2\right]$</td></tr>
</table>

<table>
<tr><th></th><th>11</th><th>12</th><th>13</th><th>14</th></tr>
<tr><td>7</td><td colspan="4">$N_2 \frac{1}{L^2}\left[\dddot{f}^{\mu}_{2,0} - \dddot{\phi}_0 [\bar{\phi}^2_2]^{-1} f^2_2\right]$</td></tr>
<tr><td>8</td><td colspan="4">$-N_2 \frac{1}{L}\left[\ddot{f}^{\mu}_{2,0} - \ddot{\phi}_0 [\bar{\phi}^2_2]^{-1} f^2_2\right]$</td></tr>
<tr><td>9</td><td colspan="4">$-N_2 \frac{1}{L^2}\left[\dddot{f}^{\mu}_{2,L} - \dddot{\phi}_L [\bar{\phi}^2_2]^{-1} f^2_2\right]$</td></tr>
<tr><td>10</td><td colspan="4">$+N_2 \frac{1}{L}\left[\ddot{f}^{\mu}_{2,L} - \ddot{\phi}_L [\bar{\phi}^2_2]^{-1} f^2_2\right]$</td></tr>
</table>

<table>
<tr><th></th><th>3</th><th>4</th><th>5</th><th>6</th></tr>
<tr><td>11</td><td colspan="4">$-N_3 \frac{1}{L^3}\left[\dddot{f}^{\beta}_{2,0} - \dddot{\Theta}_0 [\bar{\phi}^3_2]^{-1} f^3_2\right]$</td></tr>
<tr><td>12</td><td colspan="4">$+N_3 \frac{1}{L^2}\left[\ddot{f}^{\beta}_{2,0} - \ddot{\Theta}_0 [\bar{\phi}^3_2]^{-1} f^3_2\right]$</td></tr>
<tr><td>13</td><td colspan="4">$+N_3 \frac{1}{L^3}\left[\dddot{f}^{\beta}_{2,L} - \dddot{\Theta}_L [\bar{\phi}^3_2]^{-1} f^3_2\right]$</td></tr>
<tr><td>14</td><td colspan="4">$-N_3 \frac{1}{L^2}\left[\ddot{f}^{\beta}_{2,L} - \ddot{\Theta}_L [\bar{\phi}^3_2]^{-1} f^3_2\right]$</td></tr>
</table>

$K_{3\,\text{cont}} =$

	7	8	9	10
11	$-N_3 \frac{1}{L^3}\left[\dddot{f}^{\,\beta}_{1,0} - \dddot{\Theta}_0 \left[\bar{\Phi}^3_2\right]^{-1} f^3_1\right]$			
12	$+N_3 \frac{1}{L^2}\left[\ddot{f}^{\,\beta}_{1,0} - \ddot{\Theta}_0 \left[\bar{\Phi}^3_2\right]^{-1} f^3_1\right]$			
13	$+N_3 \frac{1}{L^3}\left[\dddot{f}^{\,\beta}_{1,L} - \dddot{\Theta}_L \left[\bar{\Phi}^3_2\right]^{-1} f^3_1\right]$			
14	$-N_3 \frac{1}{L^2}\left[\ddot{f}^{\,\beta}_{1,L} - \ddot{\Theta}_L \left[\bar{\Phi}^3_2\right]^{-1} f^3_1\right]$			

$K_4 =$

$(3,4) = +P$ $\qquad$ 3 = row, 4 = column

$(5,6) = -P$

$(7,8) = -P$

$(9,10) = +P$

$$(11,4) = +P\bar{x} + N_6 \frac{1}{L}\left[\ddot{\phi}_0 [\bar{\phi}^2_1]^{-1}\delta^2 + \ddot{f}^{u}_{1,0}\delta^2 + \ddot{f}^{u}_{2,0}\delta^3 - \ddot{\phi}_0 [\bar{\phi}^2_2]^{-1}[f^2_1\delta^2 + f^2_2\delta^3]\right]$$

$$(13,6) = -P\bar{x} - N_6 \frac{1}{L}\left[\ddot{\phi}_L [\bar{\phi}^2_1]^{-1}\delta^2 + \ddot{f}^{u}_{1,L} + \ddot{f}^{u}_{2,L}\delta^3 - \ddot{\phi}_L [\bar{\phi}^2_2]^{-1}[f^2_1\delta^2 + f^2_2\delta^3]\right]$$

$$(11,8) = +P\bar{y} - N_5 \frac{1}{L}\left[\ddot{\phi}_0 [\bar{\phi}^1_1]^{-1}\delta^1 + \ddot{f}^{v}_{1,0}\delta^1 + \ddot{f}^{v}_{2,0}\delta^3 - \ddot{\phi}_0 [\bar{\phi}^1_2]^{-1}[f^1_1\delta^1 + f^1_2\delta^3]\right]$$

$$(13,10) = -P\bar{y} + N_5 \frac{1}{L}\left[\ddot{\phi}_L [\bar{\phi}^1_1]^{-1}\delta^1 + \ddot{f}^{v}_{1,L}\delta^1 + \ddot{f}^{v}_{2,L}\delta^3 - \ddot{\phi}_L [\bar{\phi}^1_2]^{-1}[f^1_1\delta^1 + f^1_2\delta^3]\right]$$

$$(3,11) = N_2 \frac{1}{L^2}\left[\dddot{\phi}_0 [\bar{\phi}^2_1]^{-1}\delta^2 + \dddot{f}^{u}_{1,0}\delta^2 + \dddot{f}^{u}_{2,0}\delta^3 - \dddot{\phi}_0 [\bar{\phi}^2_2]^{-1}[f^2_1\delta^2 + f^2_2\delta^3]\right]$$

$$(3,12) = -P\bar{x} + N_2 \frac{1}{L}\left[\ddot{\phi}_0 [\bar{\phi}^2_1]^{-1}\delta^2 + \ddot{f}^{u}_{1,0}\delta^2 + \ddot{f}^{u}_{2,0}\delta^3 - \ddot{\phi}_0 [\bar{\phi}^2_2]^{-1}[f^2_1\delta^2 + f^2_2\delta^3]\right]$$

$$(4,11) = -P(\bar{x} - x_0) + N_2 \frac{1}{L}\left[\ddot{\phi}_0 [\bar{\phi}^2_1]^{-1}\delta^2 + \ddot{f}^{u}_{1,0}\delta^2 + \ddot{f}^{u}_{2,0}\delta^3 - \ddot{\phi}_0 [\bar{\phi}^2_2]^{-1}[f^2_1\delta^2 + f^2_2\delta^3]\right]$$

$$(5,13) = -N_2 \frac{1}{L^2}\left[\dddot{\phi}_2 [\bar{\phi}^2_1]^{-1}\delta^2 + \dddot{f}^{u}_{1,L}\delta^2 + \dddot{f}^{u}_{2,L}\delta^3 - \dddot{\phi}_L [\bar{\phi}^2_2]^{-1}[f^2_1\delta^2 + f^2_2\delta^3]\right]$$

$$(5,14) = +P\bar{x} - N_2 \frac{1}{L}\left[\ddot{\phi}_L [\bar{\phi}^2_1]^{-1}\delta^2 + \ddot{f}^{u}_{1,L}\delta^2 + \ddot{f}^{u}_{2,L}\delta^3 - \ddot{\phi}_L [\bar{\phi}^2_2]^{-1}[f^2_1\delta^2 + f^2_2\delta^3]\right]$$

$$(6,13) = +P(\bar{x} - x_0) - N_2 \frac{1}{L}\left[\ddot{\phi}_L [\bar{\phi}^2_1]^{-1}\delta^2 + \ddot{f}^{u}_{1,L}\delta^2 + \ddot{f}^{u}_{2,L}\delta^3 - \ddot{\phi}_L [\bar{\phi}^2_2]^{-1}[f^2_1\delta^2 + f^2_2\delta^3]\right]$$

$$(7,11) = +N_1 \frac{1}{L^2}\left[\dddot{\Phi}_0 [\bar{\Phi}_1^1]^{-1}\delta^1 + \dddot{f}^{\nu}_{1,0}\delta^1 + \dddot{f}^{\nu}_{2,0}\delta^3 - \dddot{\Phi}_0 [\bar{\Phi}_2^1]^{-1}\left[f_1^1\delta^1 + f_2^1\delta^3\right]\right]$$

$$(7,12) = +P\bar{y} + N_1 \frac{1}{L}\left[\ddot{\Phi}_0 [\bar{\Phi}_1^1]^{-1}\delta^1 + \ddot{f}^{\nu}_{1,0}\delta^1 + \ddot{f}^{\nu}_{2,0}\delta^3 - \ddot{\Phi}_0 [\bar{\Phi}_2^1]^{-1}\left[f_1^1\delta^1 + f_2^1\delta^3\right]\right]$$

$$(8,11) = -P(\bar{y} - y_0) + N_1 \frac{1}{L}\left[\ddot{\Phi}_0 [\bar{\Phi}_1^1]^{-1}\delta^1 + \ddot{f}^{\nu}_{1,0}\delta^1 + \ddot{f}^{\nu}_{2,0}\delta^3 - \ddot{\Phi}_0 [\bar{\Phi}_2^1]^{-1}\left[f_1^1\delta^1 + f_2^1\delta^3\right]\right]$$

$$(9,13) = -N_1 \frac{1}{L^2}\left[\dddot{\Phi}_L [\bar{\Phi}_1^1]^{-1}\delta^1 + \dddot{f}^{\nu}_{1,L}\delta^1 + \dddot{f}^{\nu}_{2,L}\delta^3 - \dddot{\Phi}_L [\bar{\Phi}_1^1]^{-1}\left[f_1^1\delta^1 + f_2^1\delta^3\right]\right]$$

$$(9,14) = -P\bar{y} - N_1 \frac{1}{L}\left[\ddot{\Phi}_L [\bar{\Phi}_1^1]^{-1}\delta^1 + \ddot{f}^{\nu}_{1,L}\delta^1 + \ddot{f}^{\nu}_{2,L}\delta^3 - \ddot{\Phi}_L [\bar{\Phi}_2^1]^{-1}\left[f_1^1\delta^1 + f_2^1\delta^3\right]\right]$$

$$(10,13) = +P(\bar{y} - y_0) + N_1 \frac{1}{L}\left[\ddot{\Phi}_L [\bar{\Phi}_1^1]^{-1}\delta^1 + \ddot{f}^{\nu}_{1,L}\delta^1 + \ddot{f}^{\nu}_{2,L}\delta^3 - \ddot{\Phi}_L [\bar{\Phi}_2^1]^{-1}\left[f_1^1\delta^1 + f_2^1\delta^3\right]\right]$$

$$(11,12) = \frac{P}{A}I_p - N_9 \frac{1}{L}\left[\ddot{\Phi}_0 [\bar{\Phi}_1^1]^{-1}\delta^1 + \ddot{f}^{\nu}_{1,0}\delta^1 + \ddot{f}^{\nu}_{2,0}\delta^3 - \ddot{\Phi}_0 [\bar{\Phi}_2^1]^{-1}\left[f_1^1\delta^1 + f_2^1\delta^3\right]\right]$$

$$- N_8 \frac{1}{L}\left[\ddot{\Phi}_0 [\bar{\Phi}_1^2]^{-1}\delta^2 + \ddot{f}^{\mu}_{1,0}\delta^2 + \ddot{f}^{\mu}_{2,0}\delta^3 - \ddot{\Phi}_0 [\bar{\Phi}_2^2]^{-1}\left[f_1^2\delta^2 + f_2^2\delta^3\right]\right]$$

$$- N_7 \frac{1}{L^2}\left[\ddot{\Theta}_0 [\bar{\Phi}_1^3]^{-1}\delta^3 + \ddot{f}^{\beta}_{1,0}\delta^2 + \ddot{f}^{\beta}_{2,0}\delta^1 + \ddot{f}^{\beta}_{3,0}\delta^3 - \ddot{\Theta}_0 [\bar{\Phi}_2^3]^{-1}\left[f_1^3\delta^2 + f_2^3\delta^1 + f_3^3\delta^3\right]\right]$$

$$(13,14) = -\frac{P}{A}I_p + N_9 \frac{1}{L}\left[\ddot{\Phi}_L [\bar{\Phi}_1^1]^{-1}\delta^1 + \ddot{f}^{\nu}_{1,L}\delta^1 + \ddot{f}^{\nu}_{2,L}\delta^3 - \ddot{\Phi}_L [\bar{\Phi}_2^1]^{-1}\left[f_1^1\delta^1 + f_2^1\delta^3\right]\right]$$

$$+ N_8 \frac{1}{L}\left[\ddot{\Phi}_L [\bar{\Phi}_1^2]^{-1}\delta^2 + \ddot{f}^{\mu}_{1,L}\delta^2 + \ddot{f}^{\mu}_{2,L}\delta^3 - \ddot{\Phi}_L [\bar{\Phi}_2^2]^{-1}\left[f_1^2\delta^2 + f_2^2\delta^3\right]\right]$$

$$+ N_7 \frac{1}{L^2}\left[\ddot{\Theta}_L [\bar{\Phi}_2^3]^{-1}\delta^3 + \ddot{f}^{\beta}_{1,L}\delta^2 + \ddot{f}^{\beta}_{2,L}\delta^1 + \ddot{f}^{\beta}_{3,L}\delta^3 - \ddot{\Theta}_L [\bar{\Phi}_2^3]^{-1}\left[f_1^3\delta^2 + f_2^3\delta^1 + f_3^3\delta^3\right]\right]$$

The addition of K_1, K_2, K_3 and K_4 gives the K matrix in

$$\{f\} = [K(\delta)]\{\delta\}$$

There are a few relations which ease the calculations of K. For instance

$$[\bar{\Phi}_1^1]^{-1} = [\bar{\Phi}_2^1]^{-1}$$

$$[\bar{\Phi}_1^2]^{-1} = [\bar{\Phi}_2^2]^{-1}$$

$$[\bar{\Phi}_1^3]^{-1} = [\bar{\Phi}_2^3]^{-1}$$

Also, $$[\bar{\Phi}_1^1]^{-1}[T] = [\bar{\Phi}_1^2]^{-1}$$

where $$[T] = [T]^{-1} = \begin{vmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{vmatrix}$$

Many of the terms are repetitive, so that one calculation suffices to handle several terms. This still leaves a formidable amount of work in assembling K. Appendix II, however, can be used to ease the calculation of the individual terms.

APPENDIX II

TERMS NEEDED IN MEMBER MATRIX

This appendix contains the particular solutions to the first iteration of the differential equations, as well as the evaluated terms necessary to construct the matrix. The particular solution follows.

$$\lambda^2 = -\frac{N_4 L^2}{N_3}$$

$$[\tilde{\tilde{\Phi}}] = \left[\bar{z}^5/20 \,,\; \bar{z}^4/12 \,,\; \bar{z}^3/6 \,,\; \bar{z}^2/2 \right]$$

$$[\tilde{\tilde{\Theta}}] = \left[\frac{1}{\lambda^2}\sinh\lambda\bar{z} \,,\; \frac{1}{\lambda^2}\cosh\lambda\bar{z} \,,\; \bar{z}^3/6 \,,\; \bar{z}^2/2 \right]$$

$$[\widetilde{\widetilde{\ddot{\Phi}^T\Theta}}] =$$

$$\begin{bmatrix} \frac{6}{\lambda}\left(\frac{\bar{z}}{\lambda}\sinh\lambda\bar{z} - \frac{2}{\lambda^2}\cosh\lambda\bar{z}\right), & \frac{6}{\lambda}\left(\frac{\bar{z}}{\lambda}\cosh\lambda\bar{z} - \frac{2}{\lambda^2}\sinh\lambda\bar{z}\right), & \bar{z}^4/2, & \bar{z}^3 \\ \frac{2}{\lambda^2}\sinh\lambda\bar{z}, & \frac{2}{\lambda^2}\cosh\lambda\bar{z}, & \bar{z}^3/3, & \bar{z}^2 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{bmatrix}$$

$$[\overset{\vee\vee}{\ddot{\Phi}^{T}\dot{\Phi}}] =$$

$$\begin{bmatrix} 18\left(-\frac{\bar{z}^4}{4\lambda^2} - \frac{3\bar{z}^2}{\lambda^4}\right), & 12\left(-\frac{\bar{z}^3}{3\lambda^2} - \frac{2\bar{z}}{\lambda^4}\right), & 6\left(-\frac{\bar{z}^2}{2\lambda^2}\right), & 0 \\ 6\left(-\frac{\bar{z}^3}{3\lambda^2} - \frac{2\bar{z}}{\lambda^4}\right), & 4\left(-\frac{\bar{z}^2}{2\lambda^2}\right), & 2\left(-\frac{\bar{z}}{\lambda^2}\right), & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{bmatrix}$$

$$[\overset{\vee\vee}{\dot{\Phi}}] = \left[3\left(-\frac{\bar{z}^3}{3\lambda^2} - \frac{2\bar{z}}{\lambda^4}\right),\ 2\left(-\frac{\bar{z}^2}{2\lambda^2}\right),\ -\frac{\bar{z}}{\lambda^2},\ 0\right]$$

$$[\overset{\vee\vee}{\Phi}] = \left[\left(-\frac{\bar{z}^4}{4\lambda^2} - \frac{3\bar{z}^2}{\lambda^4}\right),\ \left(-\frac{\bar{z}^3}{3\lambda^2} - \frac{2\bar{z}}{\lambda^4}\right),\ -\frac{\bar{z}^2}{2\lambda^2},\ -\frac{\bar{z}}{\lambda^2}\right]$$

$$[\overset{\vee\vee}{\ddot{\Theta}^{T}\dot{\Theta}}] =$$

$$\begin{bmatrix} \lambda^3\left(\frac{1}{6\lambda^2}\sinh^2\lambda\bar{z}\right), & \lambda^3\left(\frac{1}{6\lambda^3}\sinh\lambda\bar{z}\cosh\lambda\bar{z} + \frac{\bar{z}}{2\lambda^2}\right), & \lambda^2\left(\frac{1}{2\lambda^2}\bar{z}\sinh\lambda\bar{z}\right), & 0 \\ \lambda^3\left(\frac{1}{6\lambda^3}\sinh\lambda\bar{z}\cosh\lambda\bar{z} - \frac{\bar{z}}{2\lambda^2}\right., & \lambda^3\left(\frac{1}{6\lambda^2}\sinh^2\lambda\bar{z}\right), & \lambda^2\left(\frac{1}{2\lambda^2}\bar{z}\cosh\lambda\bar{z}\right), & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{bmatrix}$$

$$\left[\ddot{\Phi}^T \dot{\Theta}\right] =$$

$$\begin{bmatrix}
6\lambda\left(\frac{1}{4\lambda^2}\bar{z}^2\cosh\lambda\bar{z} - \frac{3}{4\lambda^3}\bar{z}\sinh\lambda\bar{z}\right), & 6\lambda\left(\frac{1}{4\lambda^2}\bar{z}^2\sinh\lambda\bar{z} - \frac{3}{4\lambda^3}\bar{z}\cosh\lambda\bar{z}\right), & 6\left(-\frac{\bar{z}^2}{2\lambda^2}\right), & 0 \\
2\lambda\left(\frac{1}{2\lambda^2}\bar{z}\cosh\lambda\bar{z}\right), & 2\lambda\left(\frac{1}{2\lambda^2}\bar{z}\sinh\lambda\bar{z}\right), & 2\left(-\frac{\bar{z}}{\lambda^2}\right), & 0 \\
0, & 0, & 0, & 0 \\
0, & 0, & 0, & 0
\end{bmatrix}$$

$$\left[\dot{\Theta}\right] = \left[\lambda\left(\frac{1}{2\lambda^2}\bar{z}\cosh\lambda\bar{z}\right),\ \lambda\left(\frac{1}{2\lambda^2}\bar{z}\sinh\lambda\bar{z}\right),\ -\bar{z}/\lambda^2,\ 0\right]$$

The remaining terms are those necessary to build the matrix K.

$[\bar{\Phi}^1_1] = [\bar{\Phi}^1_2] =$

0	0	0	L
0	0	−1	0
L	L	L	L
−3	−2	−1	0

$[\bar{\Phi}^2_1] = [\bar{\Phi}^2_2] =$

0	0	0	L
0	0	1	0
L	L	L	L
3	2	1	0

$[\bar{\Phi}^3_1] = [\bar{\Phi}^3_2] =$

0	1	0	1
λ/L	0	$1/L$	0
$\sinh\lambda$	$\cosh\lambda$	1	1
$(\frac{\lambda}{L})\cosh\lambda$	$(\frac{\lambda}{L})\sinh\lambda$	$1/L$	0

$$\overset{\circ\circ\circ}{\phi}_{0} = (6, 0, 0, 0)$$

$$\overset{\circ\circ}{\phi}_{0} = (0, 2, 0, 0)$$

$$\overset{\circ\circ\circ}{\phi}_{L} = (6, 0, 0, 0)$$

$$\overset{\circ\circ}{\phi}_{L} = (6, 2, 0, 0)$$

$$\dddot{\theta}_{0} = (\lambda^{3}, 0, 0, 0)$$

$$\ddot{\theta}_{0} = (0, \lambda^{2}, 0, 0)$$

$$\dddot{\theta}_{L} = (\lambda^{3}\cosh\lambda, \lambda^{3}\sinh\lambda, 0, 0)$$

$$\ddot{\theta}_{L} = (\lambda^{2}\sinh\lambda, \lambda^{2}\cosh\lambda, 0, 0)$$

$$\overset{\circ\circ\circ}{f}{}^{\nu}_{1,0} = M_{1}(0, 0, 1, 0)[\bar{\Phi}^{1}_{1}]^{-1}$$

$$\overset{\circ\circ}{f}{}^{\nu}_{1,0} = M_{1}(0, 0, 0, 1)[\bar{\Phi}^{1}_{1}]^{-1}$$

$$\overset{\circ\circ\circ}{f}{}^{\nu}_{1,L} = M_{1}(3, 2, 1, 0)[\bar{\Phi}^{1}_{1}]^{-1}$$

$$\overset{\circ\circ}{f}{}^{\nu}_{1,L} = M_{1}(1, 1, 1, 1)[\bar{\Phi}^{1}_{1}]^{-1}$$

$$\overset{\circ\circ\circ}{f}{}^{\mu}_{1,0} = M_{4}(0, 0, 1, 0)[\bar{\Phi}^{2}_{1}]^{-1}$$

$$\overset{\circ\circ}{f}{}^{\mu}_{1,0} = M_{4}(0, 0, 0, 1)[\bar{\Phi}^{2}_{1}]^{-1}$$

$$\dddot{f}^{u}_{1,L} = M_4(3, 2, 1, 0)[\bar{\Phi}^2_i]^{-1}$$

$$\ddot{f}^{u}_{1,L} = M_4(1, 1, 1, 1)[\bar{\Phi}^2_i]^{-1}$$

$$\dddot{f}^{v}_{2,0} = M_2(\lambda, 0, 1, 0)[\bar{\Phi}^3_i]^{-1}$$

$$+ M_3[S^2]^T([\bar{\Phi}^2_i]^{-1})^T \begin{bmatrix} 0, & 6, & 0, & 6 \\ 2\lambda, & 0, & 2, & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{bmatrix} [\bar{\Phi}^3_i]^{-1}$$

$$\ddot{f}^{v}_{2,0} = M_2(0, 1, 0, 1)[\bar{\Phi}^3_i]^{-1}$$

$$+ M_3[S^2]^T([\bar{\Phi}^2_i]^{-1})^T \begin{bmatrix} 0, & 0, & 0, & 0 \\ 0, & 2, & 0, & 2 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{bmatrix} [\bar{\Phi}^3_i]^{-1}$$

$$\dddot{f}^{v}_{2,L} = M_2(\lambda\cosh\lambda, \lambda\sinh\lambda, 1, 0)[\bar{\Phi}^3_i]^{-1}$$

$$+ M_3[S^2]^T([\bar{\Phi}^2_i]^{-1})^T \begin{bmatrix} 6\lambda\cosh\lambda + 6\sinh\lambda, & 6\lambda\sinh\lambda + 6\cosh\lambda, & 12, & 6 \\ 2\lambda\cosh\lambda, & 2\lambda\sinh\lambda, & 2, & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{bmatrix} [\bar{\Phi}^3_i]^{-1}$$

$$\ddot{f}^{v}_{2,L} = M_2(\sinh\lambda,\ \cosh\lambda,\ 1,\ 1)\,[\bar{\Phi}^3_i]^{-1}$$

$$+ M_3 [S^2]^T \left([\bar{\Phi}^2_i]^{-1}\right)^T \begin{bmatrix} 6\sinh\lambda, & 6\cosh\lambda, & 6, & 6 \\ 2\sinh\lambda, & 2\cosh\lambda, & 2, & 2 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{bmatrix} [\bar{\Phi}^3_i]^{-1}$$

$$\dddot{f}^{u}_{2,0} = M_5(\lambda, 0, 1, 0)\,[\bar{\Phi}^3_i]^{-1}$$

$$+ M_6 [S^1]^T \left([\bar{\Phi}^1_i]^{-1}\right)^T \begin{bmatrix} 0, & 6, & 0, & 6 \\ 2\lambda, & 0, & 2, & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{bmatrix} [\bar{\Phi}^3_i]^{-1}$$

$$\ddot{f}^{u}_{2,0} = M_5(0, 1, 0, 1)\,[\bar{\Phi}^3_i]^{-1}$$

$$+ M_6 [S^1]^T \left([\bar{\Phi}^1_i]^{-1}\right)^T \begin{bmatrix} 0, & 0, & 0, & 0 \\ 0, & 2, & 0, & 2 \\ 6, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{bmatrix} [\bar{\Phi}^3_i]^{-1}$$

$$\dddot{f}^{u}_{2,L} = M_5(\lambda\cosh\lambda,\ \lambda\sinh\lambda,\ 1,\ 0)\,[\bar{\Phi}^3_i]^{-1}$$

$$+M_6[S']^T([\bar{\Phi}^1_i]^{-1})^T \begin{bmatrix} 6\lambda\cosh\lambda+6\sinh\lambda, & 6\lambda\sinh\lambda+6\cosh\lambda, & 12, & 6 \\ 2\lambda\cosh\lambda, & 2\lambda\sinh\lambda, & 2, & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{bmatrix} [\bar{\Phi}^3_i]^{-1}$$

$$\ddot{f}^{u}_{2,L} = M_5(\sinh\lambda,\ \cosh\lambda,\ 1,\ 1)\,[\bar{\Phi}^3_i]^{-1}$$

$$+M_6[S']^T([\bar{\Phi}^1_i]^{-1})^T \begin{bmatrix} 6\sinh\lambda, & 6\cosh\lambda, & 6, & 6 \\ 2\sinh\lambda, & 2\cosh\lambda, & 2, & 2 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{bmatrix} [\bar{\Phi}^3_i]^{-1}$$

$$\dddot{f}^{\,\beta}_{1,0} = -M_8\left(-6/\lambda^2,\ 0,\ 0,\ 0\right)[\bar{\Phi}^{2}_{i}]^{-1}$$

$$+\ M_9\left(0,\ -2/\lambda^2,\ 0,\ 0\right)[\bar{\Phi}^{2}_{i}]^{-1}$$

$$-M_7[\delta']^T\left([\bar{\Phi}^{1}_{i}]^{-1}\right)^T \begin{bmatrix} 0 & -24/\lambda^2 & 0 & 0 \\ -12/\lambda^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} [\bar{\Phi}^{2}_{i}]^{-1}$$

$$\ddot{f}^{\,\beta}_{1,0} = -M_8\left(0,\ -2/\lambda^2,\ 0,\ 0\right)[\bar{\Phi}^{2}_{i}]^{-1}$$

$$+\ M_9\left(-6/\lambda^4,\ 0,\ -1/\lambda^2,\ 0\right)[\bar{\Phi}^{2}_{i}]^{-1}$$

$$-M_7[\delta']^T\left([\bar{\Phi}^{1}_{i}]^{-1}\right)^T \begin{bmatrix} -108/\lambda^4 & 0 & -6/\lambda^2 & 0 \\ 0 & -4/\lambda^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} [\bar{\Phi}^{2}_{i}]^{-1}$$

$$\dddot{f}^{\,\beta}_{1,2} = -M_8\left(-6/\lambda^2,\ 0,\ 0,\ 0\right)[\bar{\Phi}^{2}_{i}]^{-1}$$

$$+\ M_9\left(-6/\lambda^2,\ -2/\lambda^2,\ 0,\ 0\right)[\bar{\Phi}^{2}_{i}]^{-1}$$

$$-M_7[\delta']^T\left([\bar{\Phi}^{1}_{i}]^{-1}\right)^T \begin{bmatrix} -108/\lambda^2 & -24/\lambda^2 & 0 & 0 \\ -12/\lambda^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} [\bar{\Phi}^{2}_{i}]^{-1}$$

$$\ddot{f}^{\beta}_{1,L} = -M_8\left(-6/\lambda^2,\ -2/\lambda^2,\ 0,\ 0\right)[\bar{\Phi}^2_1]^{-1}$$

$$+M_9\left(-3/\lambda^2 - 6/\lambda^4,\ -2/\lambda^2,\ -1/\lambda^2,\ 0\right)[\bar{\Phi}^2_1]^{-1}$$

$$-M_7[\delta^1]^T([\bar{\Phi}^1_1]^{-1})^T \begin{bmatrix} -\frac{54}{\lambda^2} - \frac{108}{\lambda^4}, & -24/\lambda^2, & -6/\lambda^2, & 0 \\ -12/\lambda^2, & 0, & -4/\lambda^2, & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{bmatrix} [\bar{\Phi}^2_1]^{-1}$$

$$\dddot{f}^{\beta}_{2,0} = -M_{11}\left(-6/\lambda^2,\ 0,\ 0,\ 0\right)[\bar{\Phi}^1_1]^{-1}$$

$$+M_{12}\left(0,\ -2/\lambda^2,\ 0,\ 0\right)[\bar{\Phi}^1_1]^{-1}$$

$$-M_{10}[\delta^2]^T([\bar{\Phi}^2_1]^{-1})^T \begin{bmatrix} 0, & -24/\lambda^2, & 0, & 0 \\ -12/\lambda^2, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{bmatrix} [\bar{\Phi}^1_1]^{-1}$$

$$\ddot{f}^{\beta}_{2,0} = -M_{11}\left(0,\ -2/\lambda^2,\ 0,\ 0\right)[\bar{\Phi}^1_1]^{-1}$$

$$+M_{12}\left(-6/\lambda^4,\ 0,\ -1/\lambda^4,\ 0\right)[\bar{\Phi}^1_1]^{-1}$$

$$-M_{10}[\delta^2]^T([\bar{\Phi}^2_1]^{-1})^T \begin{bmatrix} -108/\lambda^4, & 0, & -6/\lambda^2, & 0 \\ 0, & -4/\lambda^2, & 0, & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{bmatrix} [\bar{\Phi}^1_1]^{-1}$$

$$\dddot{f}^{\beta}_{2,L} = -M_{11}\left(-6/\lambda^2, 0, 0, 0\right)[\bar{\phi}^1_1]^{-1}$$

$$+ M_{12}\left(-6/\lambda^2, -2/\lambda^2, 0, 0\right)[\bar{\phi}^1_1]^{-1}$$

$$-M_{10}[S^2]^T\left([\bar{\phi}^2_1]^{-1}\right)^T \begin{bmatrix} -104/\lambda^2, & -24/\lambda^2, & 0, & 0 \\ -12/\lambda^2, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{bmatrix} [\bar{\phi}^1_1]^{-1}$$

$$\ddot{f}^{\beta}_{2,L} = -M_{11}\left(-6/\lambda^2, -2/\lambda^2, 0, 0\right)[\bar{\phi}^1_1]^{-1}$$

$$+ M_{12}\left(-3/\lambda^2 - 6/\lambda^4, -2/\lambda^2, -1/\lambda^2, 0\right)[\bar{\phi}^1_1]^{-1}$$

$$-M_{10}[S^2]^T\left([\bar{\phi}^2_1]^{-1}\right)^T \begin{bmatrix} -\frac{54}{\lambda^2} - \frac{108}{\lambda^4}, & -\frac{24}{\lambda^2}, & -\frac{6}{\lambda^2}, & 0 \\ -12/\lambda^2, & -4/\lambda^2, & 0, & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{bmatrix} [\bar{\phi}^1_1]^{-1}$$

$$\dddot{f}^{\beta}_{3,0} = -M_{16}\left(3\lambda/2,\ 0,\ 0,\ 0\right)[\bar{\varphi}^{3}_{i}]^{-1}$$

$$-M_{13}[\delta^{3}]^{T}\left([\bar{\varphi}^{3}_{i}]^{-1}\right)^{T}\begin{bmatrix} 0, & \frac{4\lambda^{3}}{6}, & 0, & 0 \\ \frac{4\lambda^{3}}{6}, & 0, & \frac{3\lambda^{2}}{2}, & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{bmatrix}[\bar{\varphi}^{3}_{i}]^{-1}$$

$$-M_{14}[\delta^{2}]^{T}\left([\bar{\varphi}^{2}_{i}]^{-1}\right)^{T}\begin{bmatrix} 0, & -\frac{18}{4}, & 0, & 0 \\ 3\lambda, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{bmatrix}[\bar{\varphi}^{3}_{i}]^{-1}$$

$$-M_{15}[\delta^{1}]^{T}\left([\bar{\varphi}^{1}_{i}]^{-1}\right)^{T}\begin{bmatrix} 0, & -\frac{18}{4}, & 0, & 0 \\ 3\lambda, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{bmatrix}[\bar{\varphi}^{3}_{i}]^{-1}$$

$$\ddot{\mathring{f}}{}^{\beta}_{3,0} = -M_{16}\,(0,\ 1,\ 0,\ 0)\,[\bar{\Phi}^3_1]^{-1}$$

$$-M_{13}\,[\delta^3]^T\left([\bar{\Phi}^3_1]^{-1}\right)^T \begin{bmatrix} \lambda^2/3 , & 0 , & \lambda , & 0 \\ 0 , & \lambda^3/3 , & 0 , & 0 \\ 0 , & 0 , & 0 , & 0 \\ 0 , & 0 , & 0 , & 0 \end{bmatrix} [\bar{\Phi}^3_1]^{-1}$$

$$-M_{14}\,[\delta^2]^T\left([\bar{\Phi}^2_1]^{-1}\right)^T \begin{bmatrix} -6/\lambda , & 0 , & -6/\lambda^2 , & 0 \\ 0 , & 2 , & 0 , & 0 \\ 0 , & 0 , & 0 , & 0 \\ 0 , & 0 , & 0 , & 0 \end{bmatrix} [\bar{\Phi}^3_1]^{-1}$$

$$-M_{15}\,[\delta^1]^T\left([\bar{\Phi}^1_1]^{-1}\right)^T \begin{bmatrix} -6/\lambda , & 0 , & -6/\lambda^2 , & 0 \\ 0 , & 2 , & 0 , & 0 \\ 0 , & 0 , & 0 , & 0 \\ 0 , & 0 , & 0 , & 0 \end{bmatrix} [\bar{\Phi}^3_1]^{-1}$$

$$\dddot{f}^{\,B}_{3,L} = -M_{16}\left(\tfrac{1}{2\lambda}\left(3\lambda^2\cosh\lambda + \lambda^3\sinh\lambda\right),\ \tfrac{1}{2\lambda}\left(3\lambda^2\sinh\lambda + \lambda^3\cosh\lambda\right),\ 0,\ 0\right)[\bar{\Phi}^3_{\,i}]^{-1}$$

$$-M_{13}[\delta^3]^T\left([\bar{\Phi}^3_{\,i}]^{-1}\right)^T \begin{bmatrix} \left(\frac{\lambda^4}{3}\right)4\sinh\lambda\cosh\lambda & , \frac{\lambda^3}{6}\left(4\sinh^2\lambda + 4\cosh^2\lambda\right) & , \frac{1}{2}\left(3\lambda^2\sinh\lambda + \lambda^3\cosh\lambda\right) & , 0 \\ \frac{\lambda^3}{6}\left(4\sinh^2\lambda + 4\cosh^2\lambda\right) & , \frac{\lambda^4}{3}\left(4\sinh\lambda\cosh\lambda\right) & , \frac{1}{2}\left(3\lambda^2\cosh\lambda + \lambda^3\sinh\lambda\right) & , 0 \\ 0 & , 0 & , 0 & , 0 \\ 0 & , 0 & , 0 & , 0 \end{bmatrix}[\bar{\Phi}^3_{\,i}]^{-1}$$

$$-M_{14}[\delta^2]^T\left([\bar{\Phi}^2_{\,i}]^{-1}\right)^T \begin{bmatrix} 6\lambda\left(\left(\frac{3}{4}\right)\cosh\lambda + \left(\frac{1}{4}\right)\sinh\lambda - \left(\frac{3}{4\lambda}\right)\sinh\lambda\right) & , 6\lambda\left(\left(\frac{3}{4}\right)\sinh\lambda + \left(\frac{1}{4}\right)\cosh\lambda - \left(\frac{3}{4\lambda}\right)\cosh\lambda\right) & , 0 & , 0 \\ 2\lambda\left(\frac{1}{2\lambda^2}\right)\left(3\lambda^2\cosh\lambda + \lambda^3\sinh\lambda\right) & , 2\lambda\left(\frac{1}{2\lambda^2}\right)\left(3\lambda^2\sinh\lambda + \lambda^3\cosh\lambda\right) & , 0 & , 0 \\ 0 & , 0 & , 0 & , 0 \\ 0 & , 0 & , 0 & , 0 \end{bmatrix}[\bar{\Phi}^3_{\,i}]^{-1}$$

$$-M_{15}[\delta^1]^T\left([\bar{\Phi}^1_{\,i}]^{-1}\right)^T \begin{bmatrix} 6\lambda\left(\left(\frac{3}{4}\right)\cosh\lambda + \left(\frac{1}{4}\right)\sinh\lambda - \left(\frac{3}{4\lambda}\right)\sinh\lambda\right) & , 6\lambda\left(\left(\frac{3}{4}\right)\sinh\lambda + \left(\frac{1}{4}\right)\cosh\lambda - \left(\frac{3}{4\lambda}\right)\cosh\lambda\right) & , 0 & , 0 \\ 2\lambda\left(\frac{1}{2\lambda^2}\right)\left(3\lambda^2\cosh\lambda + \lambda^3\sinh\lambda\right) & , 2\lambda\left(\frac{1}{2\lambda^2}\right)\left(3\lambda^2\sinh\lambda + \lambda^3\cosh\lambda\right) & , 0 & , 0 \\ 0 & , 0 & , 0 & , 0 \\ 0 & , 0 & , 0 & , 0 \end{bmatrix}[\bar{\Phi}^3_{\,i}]^{-1}$$

$$\ddot{f}^{\,\beta}_{3,i} = -M_{16}\left(\tfrac{1}{2\lambda}\left(2\lambda\sinh\lambda+\lambda^2\cosh\lambda\right),\ \tfrac{1}{2\lambda}\left(2\lambda\cosh\lambda+\lambda^2\sinh\lambda\right),\ 0,\ 0\right)\left[\bar{\Phi}^{3}_{;i}\right]^{-1}$$

$$-M_{13}\left[\delta^3\right]^T\left(\left[\bar{\Phi}^{3}_{;i}\right]^{-1}\right)^T \begin{bmatrix} \frac{\lambda^3}{3}\left(\sinh^2\lambda+\cosh^2\lambda\right), & \frac{\lambda^2}{6}\left(4\sinh\lambda\cosh\lambda\right), & \frac{1}{2}\left(2\lambda\cosh\lambda+\lambda^2\sinh\lambda\right), & 0 \\ \frac{\lambda^2}{6}\left(4\sinh\lambda\cosh\lambda\right), & \frac{\lambda^3}{3}\left(\sinh^2\lambda+\cosh^2\lambda\right), & \frac{1}{2}\left(2\lambda\sinh\lambda+\lambda^2\cosh\lambda\right), & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{bmatrix}\left[\bar{\Phi}^{3}_{;i}\right]^{-1}$$

$$-M_{14}\left[\delta^2\right]^T\left(\left[\bar{\Phi}^{2}_{;i}\right]^{-1}\right)^T \begin{bmatrix} 6\lambda\left(\left(\frac{1}{4\lambda}\right)\sinh\lambda-\left(\frac{1}{\lambda^2}\right)\cosh\lambda+\left(\frac{1}{4}\right)\cosh\lambda\right), & 6\lambda\left(\left(\frac{1}{4\lambda}\right)\cosh\lambda-\left(\frac{1}{\lambda^2}\right)\sinh\lambda+\left(\frac{1}{4}\right)\sinh\lambda\right), & \frac{-6}{\lambda^2}, & 0 \\ 2\lambda\left(\frac{1}{2\lambda^2}\right)\left(2\lambda\sinh\lambda+\lambda^2\cosh\lambda\right), & 2\lambda\left(\frac{1}{2\lambda^2}\right)\left(2\lambda\cosh\lambda+\lambda^2\sinh\lambda\right), & 0, & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{bmatrix}\left[\bar{\Phi}^{3}_{;i}\right]^{-1}$$

$$-M_{15}\left[\delta^1\right]^T\left(\left[\bar{\Phi}^{1}_{;i}\right]^{-1}\right)^T \begin{bmatrix} 6\lambda\left(\left(\frac{1}{4\lambda}\right)\sinh\lambda-\left(\frac{1}{\lambda^2}\right)\cosh\lambda+\left(\frac{1}{4}\right)\cosh\lambda\right), & 6\lambda\left(\left(\frac{1}{4\lambda}\right)\cosh\lambda-\left(\frac{1}{\lambda^2}\right)\sinh\lambda+\left(\frac{1}{4}\right)\sinh\lambda\right), & -\frac{6}{\lambda^2}, & 0 \\ 2\lambda\left(\frac{1}{2\lambda^2}\right)\left(2\lambda\sinh\lambda+\lambda^2\cosh\lambda\right), & 2\lambda\left(\frac{1}{2\lambda^2}\right)\left(2\lambda\cosh\lambda+\lambda^2\sinh\lambda\right), & 0, & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{bmatrix}\left[\bar{\Phi}^{3}_{;i}\right]^{-1}$$

The f_1^1, f_2^1, f_2^2 etc. are 4×4 matrices. Since each row of these matrices may be a very complicated expression, it will be convenient to break them up. In the following pages the f_1^1, f_1^2, f_2^2 etc. will be written in the form

$$f_1^1 = \begin{matrix} f_1^1(1) \\ f_1^1(2) \\ f_1^1(3) \\ f_1^1(4) \end{matrix} \qquad f_2^2 = \begin{matrix} f_2^2(1) \\ f_2^2(2) \\ f_2^2(3) \\ f_2^2(4) \end{matrix} \qquad \text{etc.}$$

where $f_2^2(i)$ indicates the i^{th} row of the f_2^2 matrix.

f_1'

$$f_1'(1) = LM_1(0, 0, 0, 0)[\bar{\Phi}_1^1]^{-1}$$

$$f_1'(2) = -M_1(0, 0, 0, 0)[\bar{\Phi}_1^1]^{-1}$$

$$f_1'(3) = LM_1(1/20, 1/12, 1/6, 1/2)[\bar{\Phi}_1^1]^{-1}$$

$$f_1'(4) = -M_1(1/4, 1/3, 1/2, 1)[\bar{\Phi}_1^1]^{-1}$$

f_2'

$$f_2'(1) = LM_2(0, 1/\lambda^2, 0, 0)[\bar{\Phi}_1^3]^{-1}$$

$$+ LM_3[\delta^2]^T([\bar{\Phi}_1^2]^{-1})^T \begin{bmatrix} -\frac{12}{\lambda^3}, & 0, & 0, & 0 \\ 0, & 2/\lambda^2, & 0, & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{bmatrix} [\bar{\Phi}_1^3]^{-1}$$

$$f_2'(2) = -M_2(1/\lambda, 0, 0, 0)[\bar{\Phi}_1^3]^{-1}$$

$$- M_3[\delta^2]^T([\bar{\Phi}_1^2]^{-1})^T \begin{bmatrix} 0, & -6/\lambda^2, & 0, & 0 \\ 2/\lambda, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{bmatrix} [\bar{\Phi}_1^3]^{-1}$$

$$f_2^1(3) = LM_2\left(\sinh\lambda/\lambda^2,\ \cosh\lambda/\lambda^2,\ 1/6,\ 1/2\right)[\bar{\Phi}_1^3]^{-1}$$

$$+LM_3[S^2]^T([\bar{\Phi}_1^2]^{-1})^T \begin{bmatrix} \frac{6}{\lambda}\left(\frac{\sinh\lambda}{\lambda}-\frac{2\cosh\lambda}{\lambda^2}\right), & \frac{6}{\lambda}\left(\frac{\cosh\lambda}{\lambda}-\frac{2\sinh\lambda}{\lambda^2}\right), & 1/2, & 1 \\ 2\sinh\lambda/\lambda^2, & 2\cosh\lambda/\lambda^2, & 1/3, & 1 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{bmatrix} [\bar{\Phi}_1^3]^{-1}$$

$$f_2^1(4) = -M_2\left(\cosh\lambda/\lambda\ \ \sinh\lambda/\lambda,\ 1/2,\ 1\right)[\bar{\Phi}_1^3]^{-1}$$

$$-M_3[S^2]^T([\bar{\Phi}_1^2]^{-1})^T \begin{bmatrix} 6\left(\frac{\cosh\lambda}{\lambda}-\frac{\sinh\lambda}{\lambda^2}\right), & 6\left(\frac{\sinh\lambda}{\lambda}-\frac{\cosh\lambda}{\lambda^2}\right), & 2, & 3 \\ 2\cosh\lambda/\lambda, & 2\sinh\lambda/\lambda, & 1, & 2 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{bmatrix} [\bar{\Phi}_1^3]^{-1}$$

f_1^2

$$f_1^2(1) = LM_4(0, 0, 0, 0)[\bar{\Phi}_1^2]^{-1}$$

$$f_1^2(2) = -M_4(0, 0, 0, 0)[\bar{\Phi}_1^2]^{-1}$$

$$f_1^2(3) = LM_4(1/20,\ 1/12,\ 1/6,\ 1/2)[\bar{\Phi}_1^2]^{-1}$$

$$f_1^2(4) = -M_4(1/4,\ 1/3,\ 1/2,\ 1)[\bar{\Phi}_1^2]^{-1}$$

f_2^2 $\quad f_2^2(1) = LM_5(0, 1/\lambda^2, 0, 0)[\bar{\Phi}_i^3]^{-1}$

$$+ LM_6[\delta']^T([\bar{\Phi}_i^3]^{-1})^T \begin{bmatrix} -\frac{12}{\lambda^3}, & 0, & 0, & 0 \\ 0, & 2/\lambda^2, & 0, & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{bmatrix} [\bar{\Phi}_i^3]^{-1}$$

$f_2^2(2) = +M_5(1/\lambda, 0, 0, 0)[\bar{\Phi}_i^3]^{-1}$

$$+M_6[\delta']^T([\bar{\Phi}_i^3]^{-1})^T \begin{bmatrix} 0, & -6/\lambda^2, & 0, & 0 \\ 2/\lambda, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{bmatrix} [\bar{\Phi}_i^3]^{-1}$$

$f_2^2(3) = LM_5(\sinh\lambda/\lambda^2, \cosh\lambda/\lambda^2, 1/6, 1/2)[\bar{\Phi}_i^3]^{-1}$

$$+LM_6[\delta']^T([\bar{\Phi}_i^3]^{-1})^T \begin{bmatrix} \frac{6}{\lambda}\left(\frac{\sinh\lambda}{\lambda} - \frac{2\cosh\lambda}{\lambda^2}\right), & \frac{6}{\lambda}\left(\frac{\cosh\lambda}{\lambda} - \frac{2\sinh\lambda}{\lambda^2}\right), & \frac{1}{2}, & 1 \\ 2\sinh\lambda/\lambda^2, & 2\cosh\lambda/\lambda^2, & 1/3, & 1 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{bmatrix} [\bar{\Phi}_i^3]^{-1}$$

$$f_2^2(4) = +M_5\left(\cosh\lambda/\lambda,\ \sinh\lambda/\lambda,\ 1/2,\ 1\right)[\bar{\Phi}_i^3]^{-1}$$

$$+M_6\,[\delta']^T\left([\bar{\Phi}_i^1]^{-1}\right)^T \begin{bmatrix} 6\left(\frac{\cosh\lambda}{\lambda}-\frac{\sinh\lambda}{\lambda^2}\right), & 6\left(\frac{\sinh\lambda}{\lambda}-\frac{\cosh\lambda}{\lambda^2}\right), & 2, & 3 \\ 2\cosh\lambda/\lambda, & 2\sinh\lambda/\lambda, & 1, & 2 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{bmatrix} [\bar{\Phi}_i^3]^{-1}$$

f_1^3 $\quad$ $$f_1^3(1) = -M_8\,(0,0,0,0)[\bar{\Phi}_1^2]^{-1}$$

$$+ M_9\,(0,0,0,0)[\bar{\Phi}_1^2]^{-1}$$

$$-M_7[\delta']^T([\bar{\Phi}_1]^{-1})^T \begin{bmatrix} 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{bmatrix} [\bar{\Phi}_1^2]^{-1}$$

$$f_1^3(2) = -\frac{M_8}{L}\left(-6/\lambda^4,\ 0,\ -1/\lambda^2,\ 0\right)[\bar{\Phi}_1^2]^{-1}$$

$$+ \frac{M_9}{L}\left(0,\ -2/\lambda^4,\ 0,\ -1/\lambda^2\right)[\bar{\Phi}_1^2]^{-1}$$

$$-\frac{M_7}{L}[\delta']^T([\bar{\Phi}_1]^{-1})^T \begin{bmatrix} 0, & -24/\lambda^2, & 0, & 0 \\ -12/\lambda^4, & 0, & -2/\lambda^2, & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{bmatrix} [\bar{\Phi}_1^2]^{-1}$$

$$f_1^3(3) = -M_8\left(3\left(-\frac{1}{3\lambda^2}-\frac{2}{\lambda^4}\right),\ 2\left(-\frac{1}{2\lambda^2}\right),\ -\frac{1}{\lambda^2},\ 0\right)[\bar{\Phi}_1^2]^{-1}$$

$$+\,M_4\left(\left(-\frac{1}{4\lambda^2}-\frac{3}{\lambda^4}\right),\ \left(-\frac{1}{3\lambda^2}-\frac{2}{\lambda^4}\right),\ -\frac{1}{2\lambda^2},\ -\frac{1}{\lambda^2}\right)[\bar{\Phi}_1^2]^{-1}$$

$$-M_7[S']^T\left([\bar{\Phi}_1^1]^{-1}\right)^T\begin{bmatrix} 18\left(-\frac{1}{4\lambda^2}-\frac{3}{\lambda^4}\right), & 12\left(-\frac{1}{3\lambda^2}-\frac{2}{\lambda^4}\right), & -\frac{6}{2\lambda^2}, & 0 \\ 6\left(-\frac{1}{3\lambda^2}-\frac{2}{\lambda^4}\right), & -4/2\lambda^2, & -2/\lambda^2, & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{bmatrix}[\bar{\Phi}_1^2]^{-1}$$

$$f_1^3(4) = -\frac{M_8}{L}\left(3\left(-\frac{1}{\lambda^2}-\frac{2}{\lambda^4}\right),\ -\frac{2}{\lambda^2},\ -\frac{1}{\lambda^2},\ 0\right)[\bar{\Phi}_1^2]^{-1}$$

$$+\,\frac{M_4}{L}\left(\left(-\frac{1}{\lambda^2}-\frac{6}{\lambda^4}\right),\ \left(-\frac{1}{\lambda^2}-\frac{2}{\lambda^4}\right),\ -1/\lambda^2,\ -1/\lambda^2\right)[\bar{\Phi}_1^2]^{-1}$$

$$-\frac{M_7}{L}[S']^T\left([\bar{\Phi}_1^1]^{-1}\right)^T\begin{bmatrix} 18\left(-1/\lambda^2-6/\lambda^4\right), & 12\left(-1/\lambda^2-2/\lambda^4\right), & -6/\lambda^2, & 0 \\ 6\left(-1/\lambda^2-2/\lambda^4\right), & -4/\lambda^2, & -2/\lambda^2, & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{bmatrix}[\bar{\Phi}_1^2]^{-1}$$

f_2^3 $\quad f_2^3(1) = -M_{11}\,(0,0,0,0)\,[\bar{\Phi}_1^1]^{-1}$

$$+\dot{M}_{12}\,(0,0,0,0)\,[\bar{\Phi}_1^1]^{-1}$$

$$-M_{10}\,[\delta^2]^T([\bar{\Phi}_1^2]^{-1})^T \begin{bmatrix} 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{bmatrix} [\bar{\Phi}_1^1]^{-1}$$

$$f_2^3(2) = -\frac{M_{11}}{L}\left(-6/\lambda^4,\ 0,\ -1/\lambda^2,\ 0\right)[\bar{\Phi}_1^1]^{-1}$$

$$+\frac{M_{12}}{L}\left(0,\ -2/\lambda^4,\ 0,\ -1/\lambda^2\right)[\bar{\Phi}_1^1]^{-1}$$

$$-\frac{M_{10}}{L}\,[\delta^2]^T([\bar{\Phi}_1^2]^{-1})^T \begin{bmatrix} 0, & -24/\lambda^4, & 0, & 0 \\ -12/\lambda^4, & 0, & -2/\lambda^2, & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{bmatrix} [\bar{\Phi}_1^1]^{-1}$$

$$f_2^3(3) = -M_{11}\left(3\left(-\frac{1}{3\lambda^2}-\frac{2}{\lambda^4}\right),\ -1/\lambda^2,\ -1/\lambda^2,\ 0\right)[\bar{\Phi}_1^1]^{-1}$$

$$+M_{12}\left(\left(-\frac{1}{4\lambda^2}-\frac{3}{\lambda^4}\right),\ \left(-\frac{1}{3\lambda^2}-\frac{2}{\lambda^4}\right),\ -1/2\lambda^2,\ -1/\lambda^2\right)[\bar{\Phi}_1^1]^{-1}$$

$$-M_{10}[\delta^2]^T([\bar{\Phi}_1^2]^{-1})^T\begin{bmatrix} 18\left(-\frac{1}{4\lambda^2}-\frac{3}{\lambda^4}\right), & 12\left(-\frac{1}{3\lambda^2}-\frac{2}{\lambda^4}\right), & -\frac{6}{2\lambda^2}, & 0 \\ 6\left(-\frac{1}{3\lambda^2}-\frac{2}{\lambda^4}\right), & -4/2\lambda^2, & -2/\lambda^2, & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{bmatrix}[\bar{\Phi}_1^1]^{-1}$$

$$f_2^3(4) = -\frac{M_{11}}{L}\left(3\left(-1/\lambda^2-2/\lambda^4\right),\ -2/\lambda^2,\ -1/\lambda^2,\ 0\right)[\bar{\Phi}_1^1]^{-1}$$

$$+\frac{M_{12}}{L}\left(\left(-1/\lambda^2-6/\lambda^4\right),\ \left(-1/\lambda^2-2/\lambda^4\right),\ -1/\lambda^2,\ -1/\lambda^2\right)[\bar{\Phi}_1^1]^{-1}$$

$$-\frac{M_{10}}{L}[\delta^2]^T([\bar{\Phi}_1^2]^{-1})^T\begin{bmatrix} 18\left(-\frac{1}{\lambda^2}-\frac{6}{\lambda^4}\right), & 12\left(-\frac{1}{\lambda^2}-\frac{2}{\lambda^4}\right), & -\frac{6}{\lambda^2}, & 0 \\ 6\left(-1/\lambda^2-2/\lambda^4\right), & 4\left(-1/\lambda^2\right), & -2/\lambda^2, & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{bmatrix}[\bar{\Phi}_1^1]^{-1}$$

f_3^3

$$f_3^3(1) = -M_{16}\,(0,0,0,0)\,[\bar{\Phi}_1^3]^{-1}$$

$$-M_{13}\,[\delta^3]^T\left([\bar{\Phi}_1^3]^{-1}\right)^T \begin{bmatrix} 0,&0,&0,&0\\ 0,&0,&0,&0\\ 0,&0,&0,&0\\ 0,&0,&0,&0 \end{bmatrix} [\bar{\Phi}_1^3]^{-1}$$

$$-M_{14}\,[\delta^2]^T\left([\bar{\Phi}_1^2]^{-1}\right)^T \begin{bmatrix} 0,&0,&0,&0\\ 0,&0,&0,&0\\ 0,&0,&0,&0\\ 0,&0,&0,&0 \end{bmatrix} [\bar{\Phi}_1^3]^{-1}$$

$$-M_{15}\,[\delta^1]^T\left([\bar{\Phi}_1^1]^{-1}\right)^T \begin{bmatrix} 0,&0,&0,&0\\ 0,&0,&0,&0\\ 0,&0,&0,&0\\ 0,&0,&0,&0 \end{bmatrix} [\bar{\Phi}_1^3]^{-1}$$

$$f_3^3(2) = -\frac{M_{16}}{L}\left(1/2\lambda,\ 0,\ -1/\lambda^2,\ 0\right)[\bar{\Phi}_1^3]^{-1}$$

$$-\frac{M_{13}}{L}[\delta^3]^T\left([\bar{\Phi}_1^3]^{-1}\right)^T \begin{bmatrix} 0, & \frac{\lambda}{6}+\frac{\lambda}{2}, & 0, & 0 \\ \frac{\lambda}{6}-\frac{\lambda}{2}, & 0, & 1/2, & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{bmatrix} [\bar{\Phi}_1^3]^{-1}$$

$$-\frac{M_{14}}{L}[\delta^2]^T\left([\bar{\Phi}_1^2]^{-1}\right)^T \begin{bmatrix} 0, & -\frac{18}{4\lambda^2}, & 0, & 0 \\ 1/\lambda, & 0, & -2/\lambda^2, & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{bmatrix} [\bar{\Phi}_1^3]^{-1}$$

$$-\frac{M_{15}}{L}[\delta^1]^T\left([\bar{\Phi}_1^1]^{-1}\right)^T \begin{bmatrix} 0, & -\frac{18}{4\lambda^2}, & 0, & 0 \\ 1/\lambda, & 0, & -2/\lambda^2, & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{bmatrix} [\bar{\Phi}_1^3]^{-1}$$

$$f^3_3(3) = -M_{16}\left(\lambda\left(\frac{1}{2\lambda^2}\right)\cosh\lambda,\ \lambda\left(\frac{1}{2\lambda^2}\right)\sinh\lambda,\ -1/\lambda^2,\ 0\right)[\bar{\Phi}^3_1]^{-1}$$

$$-M_{13}[\delta^3]^T([\bar{\Phi}^3_1]^{-1})^T \begin{bmatrix} \left(\frac{\lambda}{6}\right)\sinh^2\lambda, & \left(\frac{1}{6}\right)\sinh\lambda\cosh\lambda+\frac{\lambda}{2}, & \left(\frac{1}{2}\right)\sinh\lambda, & 0 \\ \left(\frac{1}{6}\right)\sinh\lambda\cosh\lambda-\frac{\lambda}{2}, & \left(\frac{\lambda}{6}\right)\sinh^2\lambda, & \left(\frac{1}{2}\right)\cosh\lambda, & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{bmatrix} [\bar{\Phi}^3_1]^{-1}$$

$$-M_{14}[\delta^2]^T([\bar{\Phi}^2_1]^{-1})^T \begin{bmatrix} 6\lambda\left(\left(\frac{1}{4\lambda^2}\right)\cosh\lambda-\left(\frac{3}{4\lambda^3}\right)\sinh\lambda\right), & 6\lambda\left(\left(\frac{1}{4\lambda^2}\right)\sinh\lambda-\left(\frac{3}{4\lambda^3}\right)\cosh\lambda\right), & -3/\lambda^2, & 0 \\ \left(\frac{1}{\lambda}\right)\cosh\lambda, & \left(\frac{1}{\lambda}\right)\sinh\lambda, & -2/\lambda^2, & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{bmatrix} [\bar{\Phi}^3_1]^{-1}$$

$$-M_{15}[\delta^1]^T([\bar{\Phi}^2_1]^{-1})^T \begin{bmatrix} 6\lambda\left(\left(\frac{1}{4\lambda^2}\right)\cosh\lambda-\left(\frac{3}{4\lambda^3}\right)\sinh\lambda\right), & 6\lambda\left(\left(\frac{1}{4\lambda^2}\right)\sinh\lambda-\left(\frac{3}{4\lambda^3}\right)\cosh\lambda\right), & -3/\lambda^2, & 0 \\ \left(\frac{1}{\lambda}\right)\cosh\lambda, & \left(\frac{1}{\lambda}\right)\sinh\lambda, & -2/\lambda^2, & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{bmatrix} [\bar{\Phi}^3_1]^{-1}$$

$$f_3^3(4) = -\frac{M_{16}}{L}\left(\frac{1}{2\lambda}(\lambda\sinh\lambda + \cosh\lambda),\ \frac{1}{2\lambda}(\lambda\cosh\lambda + \sinh\lambda),\ -1/\lambda^2,\ 0\right)[\bar{\Phi}_1^3]^{-1}$$

$$-\frac{M_{13}}{L}[\delta^3]^T([\bar{\Phi}_1^3]^{-1})^T \begin{bmatrix} \left(\frac{\lambda^2}{3}\right)\sinh\lambda\cosh\lambda, & \frac{\lambda}{6}(\cosh^2\lambda + \sinh^2\lambda) + \frac{\lambda}{2}, & \frac{1}{2}(\sinh\lambda + \cosh\lambda(\lambda)), & 0 \\ \frac{\lambda}{6}(\sinh^2\lambda + \cosh^2\lambda) - \frac{\lambda}{2}, & \left(\frac{\lambda^2}{3}\right)\sinh\lambda\cosh\lambda, & \frac{1}{2}(\cosh\lambda + \lambda\sinh\lambda), & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{bmatrix} [\bar{\Phi}_1^3]^{-1}$$

$$-\frac{M_{14}}{L}[\delta^2]^T([\bar{\Phi}_1^2]^{-1})^T \begin{bmatrix} 6\lambda\left(\left(-\frac{1}{4\lambda^2}\right)\cosh\lambda + \left(\frac{1}{4\lambda}\right)\sinh\lambda + \left(\frac{3}{4\lambda^3}\right)\sinh\lambda\right), & 6\lambda\left(\left(-\frac{1}{4\lambda^2}\right)\sinh\lambda + \left(\frac{\lambda}{4\lambda^2}\right)\cosh\lambda - \left(\frac{3}{4\lambda^3}\right)\cosh\lambda\right), & -\frac{6}{\lambda^2}, & 0 \\ \frac{1}{\lambda}(\lambda\sinh\lambda + \cosh\lambda), & \frac{1}{\lambda}(\lambda\cosh\lambda + \sinh\lambda), & -\frac{2}{\lambda^2}, & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{bmatrix} [\bar{\Phi}_1^3]^{-1}$$

$$-\frac{M_{16}}{L}[\delta^1]^T([\bar{\Phi}_1^1]^{-1})^T \begin{bmatrix} 6\lambda\left(\left(-\frac{1}{4\lambda^2}\right)\cosh\lambda + \left(\frac{1}{4\lambda}\right)\sinh\lambda - \left(\frac{3}{4\lambda^3}\right)\sinh\lambda\right), & 6\lambda\left(\left(-\frac{1}{4\lambda^2}\right)\sinh\lambda + \left(\frac{1}{4\lambda}\right)\cosh\lambda - \left(\frac{3}{4\lambda^3}\right)\cosh\lambda\right), & -\frac{6}{\lambda^2}, & 0 \\ \frac{1}{\lambda}(\lambda\sinh\lambda + \cosh\lambda), & \frac{1}{\lambda}(\lambda\cosh\lambda + \sinh\lambda), & -\frac{2}{\lambda^2}, & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{bmatrix} [\bar{\Phi}_1^3]^{-1}$$

A study of the terms in the appendices reveals that there are many repetitive matrices involved in the calculations. In many cases, only the insertion of a different constant and δ makes one large group of matrices different from another. This can be used to advantage in construction of the member matrix.

First of all, construct sub-routines which can handle the matrix operations which are constantly repeating, such as

$$\text{matrix} \times \text{matrix}$$

$$\text{matrix} \times \text{vector}$$

$$\text{vector} \times \text{matrix}$$

etc.

Using these sub-routines, those matrix products which are constant throughout the solution can be evaluated and stored.

Once into the iteration technique, the sub-routines can be re-used to evaluate the required terms using the deflections δ and the stored constant matrix.

This technique allows the computer to do all the multiplications of the matrices involved.