CLOSE RANGE PHOTOGRAMMETRIC SYSTEMS
AND THEIR APPLICATIONS IN OPHTHALMOLOGY

by

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Abstract

Photogrammetry as a measuring tool has been applied mainly in topographic mapping, although from the very beginning of its development there have been a sufficient number of attempts to apply photogrammetry in various fields of science.

However with the exception of the utilization of photogrammetry in architecture, criminology and investigation of traffic accidents, which has been a standard procedure in many European countries, all other applications have remained in the experimental stage.

There are many reasons for the fact that non-topographic photogrammetry has not obtained general acceptance. The methods, instruments and ample potential of photogrammetry are practically unknown to scientists. Fortunately developments in recent years have been changing the situation slightly. A rapid increase of interest in the application of photogrammetry in various branches of science cannot be satiated with metric cameras only. Standard amateur and professional cameras, television systems, holography, x-rays and many other "nonconventional" photogrammetric systems have been serving as non-metric data acquisition systems. Very concentrated investigations in numerous photogrammetric centres all over the world are now underway to evaluate the quality of non-metric data acquisition systems.

A special problem in the determination of very accurate measurements from photographs taken by non-metric cameras
represents the camera calibration. The standard laboratory methods used for metric cameras are not very suitable for non-metric cameras because of their unstable parameters of interior orientation. This thesis includes a great variety of approaches in the camera calibration describing and assessing many methods that are used or suggested by various scientists all over the world.

In the restitution of photographs taken by non-metric cameras using standard existing plotting instruments photogrammeters face the very serious problem of rather significant and irregular radial and decentering distortions which cannot be easily eliminated. Another problem is the plotting instruments which do not have sufficient range of principal distance. Plotting in such cases must be performed in an affine model with an exaggerated principal distance and vertical scale.

All these problems can be avoided by the application of analytical plotters. The analytical approach is especially advantageous in the most general case of close-range photogrammetry where the elements of interior and exterior orientation as well as the calibration parameters of the cameras are simultaneously determined with the object space coordinates. It can only be hoped that in coming years instrument manufacturers will be able to produce a small and inexpensive stereocomparator with automatic coordinate registration.
The field of ophthalmology is particularly suitable for photogrammetry. The eye as an object of research has very specific properties which make almost any measurements by conventional methods extremely difficult. The fundamental problem of measurements is the mobility of the living eye. To solve that problem photogrammetry may be the ideal measuring tool. The last part of this thesis deals with small number of attempts to utilize the great potential of photogrammetry in ophthalmology showing that without the combined efforts of the medical profession and photogrammetrists no success can be achieved.
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Kresho Frankich
CHAPTER I
HISTORICAL DEVELOPMENT OF NONTOPOGRAPHIC PHOTOGRAMMETRY

According to the "Manual of Photogrammetry" published by the American Society of Photogrammetry, "photogrammetry is the science or art of obtaining reliable measurements by means of photography."* Under measurements we include the geometrical characteristics of the photographed objects, such as size, form and dimensions. Photogrammetry has been utilized in many different fields, but its most important applications are in geodesy and surveying in the production of various kinds of topographic maps. During its development photogrammetry has, however, passed the bounds of topographic surveys and has been applied in many other branches of science, such as archeology, geology, medicine, balistics, architecture, criminology, astronomy, microscopy, and many others. In recent years a new concept of close-up photogrammetry has been introduced, in which the objects are small, and their photographs are taken at relatively short distances from the camera. Under nontopographic photogrammetry we generally understand the applications of photogrammetry in all fields other than topographic mapping.

The historical development of nontopographic photogrammetry progressed parallel to the development of photogrammetry in general. It is usually understood that the history of photogrammetry begins with the history of photography, although this is correct only to a certain extent. The basis of the whole photogrammetric science is central projection and its laws.

*[4]
and since the French mathematician Lambert (1728-1777) in his book "About Free Perspective" of 1759, discussed the formation of the perspective image without horizontal axis, we can consider the beginning of photogrammetry to be earlier than the invention of photography. A more obvious beginning of the science can be assigned to a Frenchman, Beaumetz-Baupres, who in 1808 made two perspective drawings of a terrain from two different stations by means of "camera clara." From these drawings and knowing the distance between the two stations he made a topographic plan of Cape St. Cruz.

However the most important factor in the development was the invention of photography by Daguerre and Niépse in 1839.

Daguerre presented their new photographic process before the French Academy of Sciences. They had perfected a very sophisticated photographic method using silver iodide as a light sensitive material. The old problem of "fixing" the picture was solved by the application of a chemical known as sodium thiosulfate which dissolves light-sensitive silver compounds before they have been transformed into a visible image but not afterward. Thus, they exposed a plate and before any other light struck the picture bathed it in sodium thiosulfate to halt further action by light. The life of daguerreotype photography was not very long. Only a few weeks after Daguerre announced his invention the English scientist Henry Fox Talbot presented to the Royal Institution of Great Britain his negative-positive system, which is basically the system we still use.
To the invention of the photographic process another very important invention must be added. In 1846 a French chemist by name of Louis Menard discovered that cellulose nitrate would dissolve in a mixture of ether and alcohol to produce a highly viscous liquid that dried into a hard, colourless transparent film. He called the substance "collodion." The idea of using collodion as a photographic emulsion was first advanced by Robert Bingham, a British chemist in 1850.

Very soon after the discovery of photography Arago and Gay-Lussac pointed out that between the photography of terrain and the terrain itself there exists a pure perspective relation, which might initiate the application of photographs to mapping purposes. They suggested that photography could be substituted for ground surveys wherever terrain was inaccessible.

The main credit for the introduction of photogrammetry belongs to an officer in the Engineer Corps of the French Army Aimé Laussedat. He is known today as the "Father of Photogrammetry." He constructed the first usable cameras in 1851. Using two photographs he made a few maps by means of photographs taken from a balloon. The balloon photography was finally abandoned since it was difficult to expose a sufficient number of photographs from a single station because of problems in orientation of the balloon. Laussedat's remaining research was concerned with terrestrial photogrammetry. Photographs were taken with the first phototheodolite which was a combination of
camera and theodolite. In 1898 Laussedat finalized his research of many years with a book in which he described photogrammetric instruments and methods for purposes of making topographic maps. The book "Recherches sur les instruments, les méthodes et le dessin topographiques" is still regarded as a very valuable book, because the main principles laid down and explained are still in use.

Laussedat's ideas did not find a great response in France. Other European and North American countries, however, immediately realized the importance and applicability of photogrammetry and led to its tremendous development in Austria, Germany, Italy, Russia and Canada.

Edouard Gaston Deville introduced photogrammetry to Canada. As Surveyor-General of Canada he started a photogrammetric survey for mapping purposes in mountainous terrain in 1887 and in 1889 he published his historical book "Photographic Surveying," in the preface of which he wrote: "The ordinary methods of topographical surveying were too slow and expensive for the purpose; rapid surveys based on a triangulation and on sketches were tried and proved ineffectual, then photography was resorted to and the results have been all that could be desired."* In only the two seasons of 1893 and 1894 the Canadian-Alaskan Boundary Commission covered some 14,000 square miles using photogrammetry. Deville was an extremely intelligent scientist and he realised very soon that the apparent simplicity of photographic surveying

*[14].
was a delusion. Professor E.H. Thompson in the Deville Memorial Lecture* in 1965 wrote: "After eight years' experience of photographic surveying, Deville asked himself why such an apparently advantageous method should be accepted so reluctantly by the surveying profession generally. He was writing in 1895; if he were writing today he might still find his question not entirely out of place." Deville realized that the advantages of photogrammetry must be emphasized, but also that its difficulties must not be minimized. In 1902 Deville made an additional contribution to the science publishing a paper in the Transactions of the Royal Society of Canada entitled "On the Use of the Wheatstone Stereoscope in Photographic Surveying." In the history of photogrammetry this was the first description of an automatic plotting instrument. The original drawing of Deville's instrument is shown in Fig. 1-1.

Fig. 1-1. Deville's automatic plotting instrument

*[70]
In 1858 Meydenbauer, a German architect, utilized photogrammetric surveying for nontopographic purposes for the first time by obtaining reliable measurements of inaccessible details on historical buildings from two photographs. In his article of 1896 "Das Denkmalarchiv und seine Herstellung durch das Messbildverfahren"* he suggested the foundation of a special photoarchive of monuments for the purpose of their maintenance and restoration. This idea was later realized by The International Archive of Photogrammetry (Quatrième congres international de photogrammétrie tenu à Paris. Proces-verbaux des séances des commissions, 1936, pp. 311-313: Doležal, E.: "Über photogrammetrische Denkmalarchive "). Meydenbauer was also the first scientist who used the word "photogrammetry" in one of his many papers of 1893.

The end of the nineteenth century was characterized by the development of terrestrial photogrammetry, which was applied in many mapping projects. The phototheodolite invented by Laussedat obtained in its various constructions the shape which has remained until today. This is obvious from Fig. 1-2 and Fig. 1-3 in which three older phototheodolites are shown.

Using phototheodolites, Jordan and Remelé in 1874 made a map of the Libyan oasis Dachel at a scale of 1:5000.** At the same time S. Finsterwalder plotted Alpine glaciers by means

*Archive of historical buildings and its foundations by means of photography.

**[40]
Fig. 1-2. Phototheodolite of Italian Army from 1889

of photographs. In 1898 he also published a book entitled "Die geometrischen Grundlagen der Photogrammetrie" (Fundamental Geometry of Photogrammetry). In that work he solved the problem of determining the position of the two camera stations independently of terrain measurements from four points identified on both photographs.
At the turn of the century a research scientist of Karl Zeiss-Jena, Pulfrich, designed the first modern photogrammetric instrument, a stereocomparator (Fig. 1-4) which used the principles of stereophotogrammetry and a floating mark. Independently of Pulfrich two members of the Geographic Institute of Vienna, A. von Hübl and E. von Orel constructed a stereocomparator and a stereoautograph respectively. These instruments
meant a revolution in photogrammetry and practically solved the 
main problems of terrestrial photogrammetry.

The International Society of Photogrammetry was established 
in 1910 with Dr. Edward Doležal as its first president. Three 
years later the society had its first congress in Vienna.

The construction of the airplane spurred the development of 
aerial photogrammetry. Between the First and the Second World 
Wars Hugershof invented Multiplex, with the anaglyphic principle. 
During this period the great majority of stereo instruments such as 
the Stereoplanigraph, Autograph, Stereotopograph, Photocartograph, 
Aerocartograph, Stereophot, Stereotop and many others were 
invented and constructed. All these instruments were based on

Fig. 1-4. Pulfrich’s stereocomparator
the optical or mechanical solution of the fundamental photogrammetric problem of obtaining reliable measurements and finally maps from aerial photographs. The problem was solved by Otto von Gruber. He solved the problem, at first, numerically utilizing the method of least squares in his famous book "Doppelpunkt Einschaltung im Raum." He then suggested his optical-mechanical method. Under the condition that the interior orientation of photographs is known, the exterior orientation may be solved in two parts, first the establishment of relative orientation and second, the establishment of absolute orientation. Relative orientation yields the model of terrain on eliminating y-parallax. Absolute orientation will orient the model in space with respect to the vertical after scale is brought to a desired value.

Nontopographic photogrammetry also experienced a sudden development between the two World Wars. Meydenbauer's idea of utilizing photographic surveying in architecture was fully developed. The International Archive of Photogrammetry already had a great collection of photographs of almost all important European buildings and historical monuments. Even today this is not a forgotten branch of photogrammetry as is obvious from [20]. Fig. 1-5 represents the plotting of contour lines of the interior of the Dome of Santa Maria del Fiore in Florence. The plotting also indicates the various fractures existing in the structure and the erosion of frescos. The significance of the work of the International Archive can be easily realized
from the fact that after the Second World War the majority of buildings destroyed during the war were reconstructed by means of photographs taken by the International Archive of Photogrammetry.

Fig. 1-5

A number of buildings were reconstructed from simple photographs and post cards taken by amateurs and later evaluated in photogrammetric plotters. In 1931 K. Schwidensky, a well known
photogrammetrist wrote a dissertation about the application of stereophotogrammetry in architecture. (K. Schwidefsky: "Über die Anwendung der Stereophotogrammetrie auf Architekturvermessung").* He particularly emphasized the applicability of photogrammetry to measurements of deformations and movements of buildings.

Archaeology is also a field where photogrammetry was used to a great extent. Archaeologists very soon realized that some objects could become visible on air photographs, although it was almost impossible to discover them on the ground. During the first World War Theodor Wiegand took many pictures from an aircraft over Syria and Palestine for archaeological purposes. After the war he evaluated the pictures and published them in 1920.

For the methodical organization of archaeological photogrammetry we are particularly indebted to R.P. Poidebard (French) and O.G.S. Crawford (English). The former researched the archaeological monuments of Syria (R.P. Poidebard: "Photographie aérienne et archeologie. Recherches en Steppe syrienne - 1925-1931," Bulletin de photogrammetrie Paris - 1932), and latter the prehistoric monuments in England. The Surveying Institute of Berlin in 1909 measured historical monuments of Greece. Dr. E. Doležal, the first president of the International Society of Photogrammetry reported about the whole procedure in "Aufnahme der Baudenkmäler Griechenlands durch die Königliche Messbild-

*"The application of stereophotogrammetry in architectural surveying"
In the investigation of characteristics of waves and their movements photogrammetry played a very significant part. W. Laas indicated as early as 1906 in his article "Die Messung von Meereswellen und ihre Bedeutung für den Schiffsbau"** the possibility of serious application of photography to the purpose of wave measurements. Russian scientist W. Dmitrevsky published in Leningrad in 1927 a small book "Photogrammetric measurements of sea waves." This work was followed by German A. Schumacher, who in 1928 made known the results of photogrammetric evaluation of waves during the German expedition under the title: "Die stereophotogrammetrischen Wellen aufnahmen der deutschen atlantischen Expedition."***

**"Surveying of Greek historical monuments by the Royal Surveying Office to Berlin"

***"Measurements of sea waves and their Importance for Ship Construction"

***"Stereophotographs of Waves from the German Atlantic Expedition"
Aufnahme: Deutsche Seewarte
Hamburg.

Auswertung: Institut für Photogrammetrie an der
Techn. Hochschule
Berlin.
Prof. Dr.-Ing. Laumann.

Bildpaar: Nr. 235.
Mittlere Neigung der Basis: + 15,5.
Mittlere Neigung der Aufnahmeebene: w. + 15,5.
Basislänge: 244,5 m
Basishöhe: 26,5 m.
Um negative Zahlen zu vermeiden, beziehen
sich die Höhen auf eine 36,50 m unter dem
linken Aufnahmepunkt liegenden Ausgangsebene.
Schiffsspurabstand: 1,66 m (0,30 m)

Am Stereoplanographen ausgewertet
im August 1935
von Opl. Ing. F. Mazar.

Fig. 1-6
A typical plotting of waves is shown in Fig. 1-6 taken from Lacmann's book.* Another survey of waves that is historically important was accomplished by Frenchman Ph. Jarre from a pier in the harbour of Algiers in 1935.**

A very interesting contribution of photogrammetry was also made in meteorology. In 1891 the International Meteorological Congress in Munich decided that the years 1896-97 would be the International Years of Clouds to initiate extensive research of the atmosphere. Even before this decision Strachey and Whipple made many photogrammetric observations of clouds at Kew observatory, but later many photogrammetrists participated in the international research. World famous photogrammetrist C. Koppe wrote an article about the whole work under the title "Photogrammetrie und internationale Wolkenmessung" - Zeitschrift für Vermessungs Wesen, 1898.***

About 1930 R. Finsterwalder initiated the application of photogrammetry to measurements of glacier movements in the Swiss Alps, and in doing so he solved several very important problems in glacial measurements. First, danger as a very important factor in physical surveying of glaciers was reduced to a minimum. In the majority of cases year-round observations did not create any difficulties and therefore seasonal changes of velocity of glaciers could be determined. The main advantage of the photogrammetric method was that it was not restricted in observations to a single point, as had been the case in classical measurements.

* [46]
** Ph. Jarre: "Etude photogrammetrique de la houle aux abords de la jetee de Mustapha dans la rade d'Alger." (Bulletin de Photogrammetric 1935)
*** "Photogrammetry and International Measurements of Clouds."
but that observations over a much larger area gave more reliable and more objective results.

Engineering, in general, offered a great variety of problems that were solved by photogrammetric methods. It is beyond the scope of this thesis to go into a detailed description of these methods. It might be of general interest to state that deformations occupied the minds of many photogrammetrists. They evaluated slow and fast deformations, two and three-dimensional deformations, deformations of buildings, movements of towers and dams, deformations of bridges caused by side winds, loads of traffic, sinking of pillars and other factors. In the International Archive of Photogrammetry, volume IV of 1913-1914 J. Pantoflicek describes the stereophotographic measurement of small movements in his article: "Stereophotographisches Messen Kleiner Bewegungen." At present there is ample evidence that photogrammetry is being used for the determination of deformations.

As can be seen from the applications of non-topographic photogrammetry already touched upon, photogrammetry is a particularly valuable method when normal classical surveying procedures, which must be performed on some objects or events, are too dangerous, time consuming, difficult or complicated. Non-topographic photogrammetry has demonstrated special value in surveying of extremely slow or extremely fast events, and of objects that would be deformed if they were measured by traditional means. For these reasons it could be widely used in

*"Stereophotographic measurement of small movements"

**[56] and [69]
anthropology, ethnology and medicine. Many physiological investigations and measurements on humans can be made directly on photographs. This is also important in ethnology, in studies of characteristics of races and to some extent in zoology. Although photogrammetry offers easy and simple solutions to many problems in medicine it has not yet established a place in medicine which it deserves. However, there is hope that with simplifications of photogrammetric procedures, medicine will realize the potential of photogrammetry and will adapt some of the solutions. During the first World War Dr. W. Exner suggested that mutilated limbs of soldiers should be stereoscopically photographed and then photogrammetrically measured for the construction of the best prosthesis. In the war the suggestion has never been realized. One of the first applications of stereophotogrammetry was the determination of deformations of the spine during pregnancy. It has been known for a long time that the spine deforms and after childbirth recovers a certain amount but never to its initial position. By means of stereoscopic pictures the amount of permanent deformation was ascertained. Photogrammetry was also used to establish the ratio and correlation between specific diseases and the surface area involved by determining the area, or volumes of parts of the body.

To nontopographic photogrammetry must be added microphotogrammetry and x-ray photogrammetry. Microphotogrammetry is not very suitable for stereophotogrammetry since the images
obtained by microscopes have insufficient depth of field. The
problems of microphotogrammetry are described in the article by
M. Zeller: "Die Mikrophotogrammetrie" from the Photogrammetric
Institute of T.H. Zurich in 1938.

X-ray photogrammetry is even today not used for precise
measurement because of its poor resolution, although it has
been used for the determination of locations of some opaque
bodies. The main problem of x-ray photogrammetry is the fact
that shadow images are not the result of a central projection.
X-ray photographs would be a representation of objects in
central projection only if the source of x-rays was physi­
cally represented by a point. Another problem is the unsharp­
ness of the x-ray image. A measured object must be of
high contrast if a high degree of accuracy is required.
Experiments are constantly being conducted to increase the
definition of x-ray photography and if they succeed they will,
naturally, increase the applicability of x-ray photogrammetry
in more accurate measurements.

Last but not least is the application of photogrammetry
to criminology for the registration of factual findings. The
feasibility of photogrammetry for those purposes was discovered
very early, a fact which can be concluded from two articles by
F. Eichberg. The first article was published in 1911 under
the title "Die Photogrammetrie bei kriminalistischen Tatbest­
andsaufnahmen"* and the second from 1913 was "Ein neuer Apparat

*"Photogrammetry in criminalistic factual findings"
fur kriminalistische Tatbestandsaufnahmen."** In later years many articles were written about the application of photogrammetry to police service. One of them by C. Sannie and L. Amy appeared in Bulletin de Photogrammetrie (1934) under the title "La photographic métrique sur les lieux de crime." Some companies even produced special stereoscopic cameras with a fixed base which could be instantly used, like the DK-120 of Zeiss-Aerotopograph. However, the first photographs for the registration of factual findings were taken in a single camera invented by Eichberg. To enable measurements in three dimensions to be taken he added to the camera a special grid in central projection. The optical axis of the camera was always horizontal and the camera was approximately 1.5 metres above usually horizontal ground. Since the focal length of the objective was a known quantity and since the grid, which at first consisted of very thin steel wires that were installed in the focal plane and later of a plane-parallel glass plate with engraved lines, was always represented on a photograph, reliable measurements in three dimensions could have been taken directly from the photograph.

After the second World War the development of photogrammetry was influenced by the rapid development of electronic computers. They opened the new field of numerical and analytical photogrammetry. The problem of long and tedious computation was the main hindrance to the utilization of numerical photogrammetry before the appear-

**"A new apparatus for criminalistic factual findings"
ance of computers. Today this ceases to be a problem. Further instruments were improved and particularly large improvement in negative material has been achieved. This improvement greatly influences the quality of final results. New photographic emulsions combine very high sensitivity with fairly good resolving power.

How far numerical photogrammetry will establish itself in the field of nontopographic photogrammetry remains to be seen. There are already a few attempts in that direction in architectural photogrammetry, where numerical methods have been applied in the critical interpretation of an architectural monument by means of statistical analysis.

At the end of this historical introduction of nontopographic photogrammetry it is important to emphasize that the analytical development of the observed data from photographs presents few limitations but there is undeniably a limit of absolute accuracy which is defined by the qualities of the photographic images.
CHAPTER II
CLOSE-RANGE PHOTOGRAMMETRIC SYSTEMS

INTRODUCTION

According to the name, close-range photogrammetry is a branch of nontopographic photogrammetry which involves relatively short object distances. The maximum object distance is not defined. Some photogrammetrists set the maximum range of close-range photogrammetry at about 300 metres*, some others are limited to much shorter distances. K. Schwidefsky thinks that the limits of close range photogrammetry may be fixed at those distances where the photographic range begins for the usual topographic cameras which are focused to infinity.** The writer disagrees with any rigid limits and thinks that the measuring field of close-range photogrammetry should remain flexible and open to facilitate all kinds of solutions of problems that may occur in nontopographic photogrammetry. The terms "close-range photogrammetry" and "nontopographic photogrammetry" are interchangeably used and are associated with the application of photogrammetric measurements to all other fields except to the field of topographic mapping. There should be no difference between them.

The history of nontopographic photogrammetry displays a wide spectrum of scientific problems which can be solved by photogrammetric methods. With the exception of the utilization of photogrammetry in architecture, criminology and investigation of traffic accidents, which has been a standard procedure in many European countries, all other applications have remained in the experimental stage. "Even though such experiments have

*[18]
**[65]
proven the potentiality and usefulness of photogrammetry as a measuring tool in these disciplines, the widespread application of this technique in most of these fields has not gained general acceptance. The main reason for this situation seems to be the rather high degree of heterogeneity in conditions and requirements of the various applications. In general, each case in close-range photogrammetry has to be considered a special case requiring a different and perhaps unique application of photogrammetric techniques and instrumentation."* The manufacturers of photogrammetric cameras cannot produce a sufficiently large variety of metric cameras and restitution instruments at a sufficiently low price to cover rather different needs of nontopographic photogrammetry.

There is also another equally important reason that close-range photogrammetry has not obtained general acceptance. The methods, instruments and ample potentiality of photogrammetry are practically unknown to scientists. In the great majority of countries modern photogrammetry, as a subject of academic studies, has remained the exclusive property of geodesy and surveying. However, developments in recent years are changing the situation slightly. From an economical point of view, today, the cost of photogrammetric surveying, although still very high, is very often more acceptable than the costs of surveying by conventional methods. Particularly when operating conveniences of photogrammetry in some difficult cases are compared to those of conventional survey the merits of the former often become obvious.

*[18]
The great advances of analytical photogrammetry have not been used to a large extent in nontopographic photogrammetry. However, they will become an inevitable tool when simple non-metric cameras are introduced into precise photogrammetry.

DATA ACQUISITION SYSTEMS

Cameras that are used in nontopographic photogrammetry can be classified into two main categories: metric and non-metric cameras.

Metric cameras are specially developed for photogrammetric purposes. The great majority of these cameras have elements of "interior orientation" as fixed values that cannot be changed. An object which is photographed must be at such a distance from the camera that the object is "in focus." Metric cameras include phototheodolites and stereometric cameras.

The phototheodolites represent a combination of camera and theodolite. They generally operate with glass plates and are designed for a focus at infinity, although some of them have undergone subsequent modifications and can use either film or plate magazines. Every photogrammetric survey, regarding the choice of camera is influenced by two main factors: the object size shown in one photograph and its reduction to the photograph scale. Given object distances are often controlled by circumstances of site and for economical reasons, or to avoid the bridging of several models photogrammetrists are forced to use a wide angle lens camera. In the last ten years manufacturers
have shown a tendency to use more and more wide angle lenses with short focal distances. Some new metric cameras can even focus an object at close range by varying the principal distance. Those that have a fixed principle distance are limited within the depth of focus by the focal length and the aperture of the lens.

Stereometric cameras are fixed to base of a known length. Therefore the minimum and maximum operational object distances, within the required accuracy limits, are defined by the fixed geometry of the stereometric camera system. The first stereometric cameras appeared about 1907 when Thiele, in Russia,
Fig. 2.2. Stereometric Camera

experimented with his stereopanoramograph. At about the same
time Ranza, in Italy, and Boulade, in France performed similar
experiments. Between the two World Wars many types of these
cameras were offered by several manufacturers. Zeiss had the
DK 40 and the DK 120 (f = 55 mm), Wild the C4 and C12 (f = 90 mm)
which after the second World War were redesigned and appeared
as the C 120 and the C 40 cameras (f = 64 mm). Askania manu­
factures the DMK 100/1318 which can be used either as a stereo­
metric camera or as a phototheodolite, Galileo has the Bi-Camera
(f = 150 mm), Zeiss (Oberkochen) today offers the SMK 40 and
the SMK 120, and Sokkisha has also two stereometric cameras, the
SKB-45 and the SKB-100. These are, by no means, all of the
cameras in use. There are many others. Some can change the length of the base line, some others incorporate additional degrees of freedom by introducing convergence between the optical axes of the two cameras or tilt of the optical axes.

There is no doubt that metric cameras newly developed by such established factories as Zeiss, Wild, the Galileo are well suited to many applications. However, they have their demerits and they cannot solve many problems. Since the great majority of metric cameras use glass plates as an emulsion holder and therefore have no great problems with differential shrinkage, they are very heavy and bulky and cannot be used when photographs must be taken in very short time intervals. They must always have a very stable platform and can hardly be used for exposures taken vertically downwards. A special disadvantage of most metric cameras is their inability to exchange lenses.

A rapid increase of interest in the application of photogrammetry in various branches of science cannot be satiated with only metric cameras. Standard amateur and professional cameras, television systems, holography, x-rays and many other "non-conventional" photographic systems have been serving as non-metric data acquisition systems. Very concentrated investigations in numerous photogrammetric centres all over the world are now underway to evaluate the quality of non-metric data acquisition systems. The idea of using relatively cheap cameras for photogrammetric purposes is, naturally, not new. Several attempts have been made to modify non-metric cameras by introducing
fiducial marks and stabilizing the interior orientation, but these changes have proved to be very expensive, especially if the accuracy of metric cameras is to be achieved. The results of resent investigations show that there is a definite place in photogrammetry for non-metric cameras, particularly when the requirements in accuracy of measurements are not too high. The main problem to be solved is to devise a reliable and simple method of calibrating of cameras. The standard laboratory methods used for metric cameras are not very suitable for non-metric because of their unstable parameters of interior orientation. In general, the unknown parameters of the interior and exterior orientation have to be determined for each individual picture by the methods that will be explained later.

The application of non-metric cameras for data acquisition systems in photogrammetry may open the door of photogrammetry to many engineers and scientists in a great variety of fields and may enable them to make use of the technical and economical advantages of photogrammetry. All indications are that non-metric cameras will play an important role in future expansion of close-range photogrammetry and in its general acceptance as a measuring tool in a wide spectrum of disciplines and fields. Although experimental research has so far concentrated in better non-metric cameras (Hasselbland, Rolleiflex SL, Robot, Linhof Technika, etc.) and proven their photogrammetric worthiness, it is anticipated that less elaborate cameras can be used, particularly in applications with medium and low accuracy
requirements."*

**CALIBRATION OF CAMERAS**

"Camera calibration is a process whereby the individual characteristics of the mapping or charting camera are determined. These include the geometric constants known as the inner, or interior, orientation elements and the image quality of the cartographic lens."**

To obtain size, form and dimensions of an object from a photograph, the photograph must be correctly oriented. The orientation is a geometrical condition determined by the orientation elements. The orientation elements of a single photograph can be classified into two groups: elements of interior and elements of exterior orientation.

The elements of interior orientation in turn, can be referred to a camera or to a photograph. A bundle of light rays from a point source which enters the entrance pupil of the objective optical system leaves the exit pupil (centre of projection) and forms an image in the plane of photographic emulsion. This image is more or less blurred because of imperfections of the optical system. After later development and fixing, the latent image is transformed to a negative image. It is obvious that the negative image will also be blurred, but its shape and density distribution will differ from those of the latent image. The change depends upon "the characteristics of the photographic

* [44]
** [4]
material, the spectral composition of the light, the contrast between source and background, and the processing method applied.* From these considerations, and knowing that in restitution the observer decides where the "middle" of blur is for plotting purposes and therefore introduces a certain personal error, it follows that the position of the image is rather an undetermined concept. It must be substituted by fictitious image points, i.e. the points at which the measuring mark is placed when evaluating the photograph.

The purpose of interior orientation is the reconstruction of the object bundle. In dealing with the object bundle we consider an infinite number of object rays which have the centre of the entrance pupil as their common projection centre. The object bundle must be distinguished from the image bundle, which has the centre of exit pupil as its projection centre. The considered object and image rays correspond to the principal rays and in an ideal case the perspective centres" are conjugate axial points whose characteristics are that the principal rays, passing through the perspective centre in the object space, emerge from the perspective centre in the image space parallel to their original direction."

To obtain the undeformed model from two overlapping photographs taken at two different stations two operations must be performed: (a) reconstruction of object bundles of both photographs and

* [57]
** [4]
(b) reconstruction of their relative positions

For the latter the original distance between the perspective centres of the object bundles is naturally reduced by a scale factor.

Therefore it can be concluded that the interior orientation reconstructs the object bundles and not necessarily the image bundles. Only in the case of Porro-Koppe-principle where the geometrical and optical properties of the camera and the restitution projector are identical the interior orientation includes also the reconstruction of the image bundles. However, the Porro-Koppe-principle, to the writer's knowledge, has not been used in close-range photogrammetry.

The elements of interior orientation are normally determined by three types of definitions: geometrical, physical and factual. The geometrical definition treats photography as a result of a simple central projection disregarding all deformations. The physical definition includes all optical deformations of the camera assuming the rotational symmetry of deformations around the principal point. The factual definition comprises a general case. The parameters of the factual interior orientation are determined by the calibration procedure and application of the method of least squares for the evaluation of obtained results.

Since the object bundle has a spacial character (it is determined in three dimensional space) and the corresponding images are recorded on the two-dimensional photographic plane,
The reconstruction of object bundles from two-dimensional images requires a perspective centre. According to the physical definition the perspective centre is the centre of the exit pupil. The principal point is then defined as the foot of the perpendicular from the perspective centre to the picture plane. This is a geometrical definition because it assumes a lens. In the factual definition object circles are projected as some distorted curved lines and the calibrated principal point is defined as a point with minimum asymmetry. W. Roos uses a very simple definition of the principal point which corresponds to the applied calibration method. "The principle point is the trace on the photograph of a ray of light through the centre of the entrance pupil, which is perpendicular in the object space to the picture plane."*

The camera constant, $C_K$, is a factor which connects

![Diagram](image)

*Fig. 2-3*
angular value $\tau$ and the corresponding linear distance $r^1$ (see Fig. 2-3).

$$r^1 = C_K F(\tau) \quad (2.1)$$

Under the assumption of distortion-free optical system and when the photographed objects are at infinity the camera constant becomes the focal length of the objective. According to the geometrical definition the camera constant is the distance between the perspective centre and the principle point.

If the object bundle is shifted parallel to itself from the entrance pupil to the exit pupil as its projection centre and the rays are produced to the intersection with the plane of photograph, then these intersection points, in general, will not be identical to the corresponding images on the photograph. The position of the perspective centre may be moved but normally

![Diagram](image-url)
it is impossible to find a position of the perspective centre such that the reconstructed object bundle from images of points on the photograph is identical to the original object bundle. The difference in location between the intersection points and the image points represents linear distortion. It is obvious that the distortion has no absolute, or constant value for all points but depends upon the position of a ray bundle with respect to the photograph. Since restitution instruments use a central projection to reconstruct the object bundle special measures must be considered to eliminate or at least to minimize the influence of the distortion. Various methods that have been applied and all use the shift of the image points to positions such that the distortion is neutralized. The shift can be performed optically or mechanically. Optical methods use either a special lens with the required amount of distortion or a distortion compensating glassplate. The mechanical solution is achieved by varying the camera constant.

As a conclusion, it can be said, that interior orientation defines a position of the photograph relative to the projection centre and is given by the position of the principal point on the photograph and the camera constant, according to the geometrical definition.

The determination of the position of the principal point can be made when two conditions are fulfilled. First, there must be a sufficient number of image points in the plane of the photograph, and, second, the original shape of the object bundles
must be known. The determination is then reduced to a resection problem with an additional condition: the remaining differences in location between the image points and the points of intersection of the plane of photograph and the rays produced of the object bundles must be symmetrical about the principal point. For practical purposes one ray of the object bundles which passes close to the centre of the photograph is chosen as the initial ray. The ray intersects the photograph at right angle in its corresponding image point. If now some value for the camera constant is assumed (an approximate value of the focal length), then a provisional position of the object bundles is defined. Naturally other rays intersect the photograph in points that are not identical to their corresponding images but they are relatively close. The differences in positions between image points and the corresponding intersection points measured in a rectangular coordinate system of the image plane represent provisional distortions.

\[
\begin{align*}
\Delta'x & = x - x' \\
\Delta'y & = y - y',
\end{align*}
\]

(2.2)

where \(x'\) and \(y'\) are the coordinates of intersection points. Since the choice of the camera constant and the provisional principal point is relatively good, only small, differential changes in position of the perspective centre must be determined by the method of least squares. These differential changes consist of three shifts \(dx, dy, dz\), and three rotations \(d\alpha, d\beta\)
and $d\gamma$ about $x$, $y$ and $z$-axis respectively. Introducing these changes the provisional distortions $\Delta'x$ and $\Delta'y$ will create new values denoting the new values by $\Delta x$ and $\Delta y$, the approximate value of the camera constant by $h$, and applying Otto von Gruber's well known differential formulae,* two equations are obtained

$$\Delta x = \Delta'x - dx - \frac{x}{h} \, dz + \frac{xy}{h} \, d\alpha - h(1 + \frac{x^2}{h^2}) \, d\beta - yd\gamma$$

$$\Delta y = \Delta'y - dy - \frac{y}{h} \, dz + h(1 + \frac{y^2}{h^2}) \, d\alpha - \frac{xy}{h} \, d\beta + xd\gamma$$

\[2.3\]

*[28], or [24]
R. Roelofs* restricts the whole computation to only two rows of points along the x-axis and the y-axis assuming that the tangential distortion is zero. Then equations (2.3) for points along the x-axis become

\[ \Delta x = \Delta'x - dx - \frac{x}{h} dz - h(1 + \frac{x^2}{h^2})d\beta \]  
\[ \Delta y = 0, \]  

and for points along the y-axis

\[ \Delta x = 0 \]  
\[ \Delta y = \Delta'y - dy - \frac{y}{h} dz + h(1 + \frac{y^2}{h^2})d\alpha \]

The coordinates of the principal point with respect to the changed position of the perspective centre are dx and dy. If now the condition of symmetrical distortions is to be satisfied, pairs of points on the x-axis symmetrical with respect to the principal point are considered. For any pair of symmetrical points \( x_i \) and \( x_j \) there are two equations in accordance with (2.4)

\[ \Delta x_i = \Delta'x_i - dx - \frac{x_i}{h} dz - h(1 + \frac{x_i^2}{h^2})d\beta \]

and

\[ \Delta x_j = \Delta'x_j + dx - \frac{x_i}{h} dz + h(1 + \frac{x_i^2}{h^2})d\beta \]

If now distortions \( \Delta x_i \) and \( \Delta x_j \) are to be symmetrical, the difference between them will be zero.

* [57]
\[ \Delta'x_i - \Delta'x_j - 2dx - 2h(1 + \frac{x_i^2}{h^2})d\beta = 0 \]  

(2.6)

Since the approximate value of the camera constant is directly eliminated from further computations, it has no influence on the symmetry of distortion. However it does influence the amount of distortion. Therefore a camera constant must be determined such that the distortion is distributed over the whole photograph. Then that camera constant is called: calibrated focal length.

The exterior orientation defines the position of the camera in an object coordinate system. The perspective centre is determined by rectangular coordinates \( x_o, y_o, z_o \). The additional elements of the exterior orientation are three rotation angles \( \omega, \phi, \kappa \) about \( x, y, \) and \( z \)-axes respectively. The purpose of the exterior orientation is to orient the reconstructed object bundles with respect to the plotted orientation points in the same relative position that the original object bundles occupied in the respect to the original ground points. In terrestrial photogrammetry the elements of exterior orientation are determined by some conventional survey procedures. In a real photogrammetry they are indirectly obtained by absolute orientation.

In close range photogrammetry the parameters of exterior orientation are sometimes determined in the same way as in terrestrial photogrammetry. However in the great majority of cases, particularly when non-metric cameras are used, the
parameters of the exterior orientation are determined simultaneously with the elements of interior orientation by a calibration procedure.

Laboratory and field calibration procedures for aerial cameras are well established and known. These procedures are not particularly suitable for close-range cameras, since the use of collimators, multi-collimators, goniometers, etc... assumes camera focusing to infinity. For the same reasons stellar methods are also not applicable. In the great majority of practical cases cameras used in close-range photogrammetry are focussed to some finite distance or they have variable focus.

The need for a standard method of calibration of close-range cameras (metric as well as non-metric) led to the development of relatively large number of procedures recommended by various authors and institutions. Almost all have one thing in common: they are performed under normal working conditions and therefore yield realistic calibration results.

From these methods five characteristic approaches used in close-range camera calibration have been selected and they will be described in detail.

HALLERT'S GRID METHOD*

Hallert's grid method is used to determine the parameters of interior orientation and the distortion of cameras and

*[25]
projectors. The image of a very accurate grid projected through the lens of a photogrammetric instrument in the normal position \((\phi = \omega = \kappa = 0)\) is evaluated. The coordinates of the projected grid are measured in the coordinate system of the object or model space of the instrument. "The discrepancies between the measured machine coordinates and the corresponding enlarged grid coordinates are assumed to depend upon the errors in the inner and outer orientation of the projector and the accidental errors of the measurements."*

![Diagram](image)

Fig. 2-6

If redundant observations are made, the adjustment of the parameters of exterior orientation \((d x_o, d y_o, d z_o, d \phi, d \omega, d \kappa)\)

* [25]
is performed by the method of least squares.

The measured coordinates are compared with the grid coordinates multiplied by scale factor \( h/f \). Then the discrepancies in the measured coordinates are

\[
\begin{align*}
dx &= x - x' \frac{h}{f}, \\
dy &= y - y' \frac{h}{f}
\end{align*}
\]

Small changes of the parameters of exterior orientation cause changes of the rectangular coordinates of projected points and these changes are determined by well known differential formulae derived by von Gruber:

\[
\begin{align*}
dy &= dy_o + \frac{y}{h} \frac{dz_o}{d\omega} + x \frac{dx_k}{d\phi} + h \left( 1 + \frac{y^2}{h^2} \right) d\omega, \\
dx &= dx_o + \frac{x}{h} \frac{dz_o}{d\omega} - y \frac{dx_k}{d\phi} + h \left( 1 + \frac{x^2}{h^2} \right) d\phi + \frac{x y}{h} d\omega.
\end{align*}
\]

If instead of machine coordinates the image coordinates are applied \((x = x' \frac{h}{f}; y = y' \frac{h}{f})\) two similar equations are obtained.

\[
\begin{align*}
dy &= dy_o + \frac{y'}{f} \frac{dz_o}{d\omega} + x' \frac{dx_k}{d\phi} + h \left( 1 + \frac{y'^2}{f^2} \right) d\omega, \\
dx &= dx_o + \frac{x'}{f} \frac{dz_o}{d\omega} - y' \frac{dx_k}{d\phi} + h \left( 1 + \frac{x'^2}{f^2} \right) d\phi + \frac{x' y'}{f^2} d\omega
\end{align*}
\]

From these equations the observation equations result.

\[
\begin{align*}
v_y &= dy_o + \frac{y'}{f} \frac{dz_o}{d\omega} + x' \frac{dx_k}{d\phi} + h \left( 1 + \frac{y'^2}{f^2} \right) d\omega - dy, \\
v_x &= dx_o + \frac{x'}{f} \frac{dz_o}{d\omega} - y' \frac{dx_k}{d\phi} + h \left( 1 + \frac{x'^2}{f^2} \right) d\phi + h \frac{x' y'}{f^2} d\omega - dx.
\end{align*}
\]
The same final equations are obtained if instead of the projector, a camera is calibrated. The photograph of the grid is taken from a point vertically above the centre point of the grid and with the negative plane accurately parallel to the object plane. A two dimensional test field contains a great number of grid points or targets arranged in a concentric circular pattern. The adjustment is performed independently for points on a single circle. The number of points on each circle can vary. Hallert suggested two combinations of five and nine-point adjustments. The five-point adjustment includes the centre of the grid (C) and four stations on a circle. The nine-point adjustment includes eight points on a circle and the
centre. It is obvious that the nine-point adjustment is considerably stronger than the corresponding five-point combination. Therefore the nine-point combination may be applied for larger circles, and the five-point adjustment will suffice for smaller radii.

If the corrections to the elements of orientation are to be determined equations (2.10) must be changed to the form

\[ \begin{align*}
  v_y &= -dy_o - \frac{y'Q}{f} dz_o - x'h f d\kappa - \frac{h x'y'}{f^2} d\phi - h(1 + \frac{y'^2}{f^2}) d\omega - dy \\
  v_x &= -dx_o - \frac{x'Q}{f} dz_o + y'h f d\kappa - h(1 + \frac{x'^2}{f^2}) d\phi - \frac{h x'y'}{f^2} d\omega - dx
\end{align*} \] (2.11)

Taking, for example, a circle of radius R containing four stations 1, 2, 3, 4 and the centre (C) for the five-point adjustment (see Fig. 2-7), the coefficients of the normal equations are obtained from the observation equations knowing the rectangular coordinates of the five stations.

<table>
<thead>
<tr>
<th>Stations:</th>
<th>C</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Latitude (Y'):</td>
<td>o</td>
<td>a</td>
<td>-a</td>
<td>-a</td>
<td>a</td>
</tr>
<tr>
<td>Departure (X'):</td>
<td>o</td>
<td>a</td>
<td>a</td>
<td>-a</td>
<td>-a</td>
</tr>
</tbody>
</table>

\[ a = R \cos 45^\circ = \]

\[ R \sin 45^\circ = \]
| Points | $dx$ | $dy$ | $dz$ | $ae$ | $ab$ | $ac$ | $ad$ | $ae$ | $af$ | $bb$ | $bc$ | $bd$ | $be$ | $f$ | $bl$ | $cc$ | $cd$ | $ce$ | $cf$ | $cl$ | $dd$ | $ee$ | $ef$ | $el$ | $ff$ | $tk$ |
|--------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|
| $C$    | $-1$ |       |      | $d_x$ | $d_y$ | $d_z$ | $a+1$ | $d_y$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ | $a_7$ | $a_8$ | $a_9$ | $a_{10}$ | $a_{11}$ | $a_{12}$ | $a_{13}$ | $a_{14}$ | $a_{15}$ | $a_{16}$ | $a_{17}$ | $a_{18}$ | $a_{19}$ | $a_{20}$ |
| $V_2$  | $-1$ |       |      | $d_x$ | $d_y$ | $d_z$ | $a+1$ | $d_y$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ | $a_7$ | $a_8$ | $a_9$ | $a_{10}$ | $a_{11}$ | $a_{12}$ | $a_{13}$ | $a_{14}$ | $a_{15}$ | $a_{16}$ | $a_{17}$ | $a_{18}$ | $a_{19}$ | $a_{20}$ |
| $V_3$  | $-1$ |       |      | $d_x$ | $d_y$ | $d_z$ | $a+1$ | $d_y$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ | $a_7$ | $a_8$ | $a_9$ | $a_{10}$ | $a_{11}$ | $a_{12}$ | $a_{13}$ | $a_{14}$ | $a_{15}$ | $a_{16}$ | $a_{17}$ | $a_{18}$ | $a_{19}$ | $a_{20}$ |
| $V_4$  | $-1$ |       |      | $d_x$ | $d_y$ | $d_z$ | $a+1$ | $d_y$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ | $a_7$ | $a_8$ | $a_9$ | $a_{10}$ | $a_{11}$ | $a_{12}$ | $a_{13}$ | $a_{14}$ | $a_{15}$ | $a_{16}$ | $a_{17}$ | $a_{18}$ | $a_{19}$ | $a_{20}$ |
| Sums   | $=5$ | $0$  | $0$  | $5+120$ | $0$  | $0$  | $0$  | $0$  | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ | $a_7$ | $a_8$ | $a_9$ | $a_{10}$ | $a_{11}$ | $a_{12}$ | $a_{13}$ | $a_{14}$ | $a_{15}$ | $a_{16}$ | $a_{17}$ | $a_{18}$ | $a_{19}$ | $a_{20}$ |
\[
\begin{align*}
\text{[dl]} &= a(dy_1+dy_2-dy_3-dy_4-dx_1+dx_2+dx_3-dx_4) = a N_5_2 \\
\text{[el]} &= [dx] + \frac{a^2}{f^2}(dy_1-dy_2+dy_3-dy_4+dx_1+dx_2+dx_3+dx_4) = [dx] + \frac{a^2}{f^2} N_5_3 \\
\text{[fl]} &= [dy] + \frac{a^2}{f^2}(dy_1+dy_2+dy_3+dy_4-dx_1-dx_2+dx_3-dx_4) = [dy] + \frac{a^2}{f^2} N_5_4 
\end{align*}
\]

The normal equations will then be:

\[
\begin{align*}
5 dx_o + h(5 + \frac{4a^2}{f^2})d\phi + [dx] &= 0 \\
5 dy_o + h(5 + \frac{4a^2}{f^2})d\omega + [dy] &= 0 \\
8 \frac{a^2}{f} dz_o + a N_5_1 &= 0 \\
8 \frac{a^2}{f} d\kappa + a N_5_2 &= 0 \\
(5 + \frac{4a^2}{f^2})dx_o + h(5 + \frac{8a^2}{f^2} + \frac{8a^4}{f^4})d\phi + [dx] + \frac{a^2}{f^2} N_5_3 &= 0 \\
(5 + \frac{4a^2}{f^2})d\omega + h(5 + \frac{8a^2}{f^2} + \frac{8a^4}{f^4})d\omega + [dy] + \frac{a^2}{f^2} N_5_4 &= 0 
\end{align*}
\]

Since many terms of the normal equations are zero standard solutions of the normal equations such as Gauss-Dolittle, Cholesky, Gerasimov, Banakiewicz and some others are not the shortest methods. In this case, probably, the best method is to solve the linear equations directly. From the first and fifth normal equations unknowns \(dx_o\) and \(d\phi\) are obtained by the elimination method. The first normal equation is multiplied by the factor \(-\frac{1}{5}(5 + \frac{4a^2}{f^2})\) and is added to the fifth equation.
The unknown \(dx_o\) is obtained when the last equation for \(d\phi\) is substituted into the first normal equation.

\[
5 \ dx_o = - [dx] - h(5 + \frac{4a^2}{f^2}) \frac{f^2}{a^2h} (\frac{[dx]}{6} - \frac{5}{24} N_{53})
\]

\[
5 \ dx_o = - [dx] - [dx] \frac{5f^2}{6a^2} - \frac{2}{3} [dx] + \frac{25f^2 N_{53}}{24a^2} + \frac{20}{24} N_{53}
\]

\[
5 \ dx_o = - [dx] \frac{10a^2 + 5f^2}{6a^2} + N_{53} \frac{25f^2 + 20a^2}{24a^2}
\]

And from here

\[
dx_o = - [dx] \frac{2a^2 + f^2}{6a^2} + N_{53} \frac{5f^2 + 4a^2}{24a^2} \quad (2.13)
\]

In the same manner, taking the second and sixth normal equations the next two unknowns \(dy_o\) and \(d\omega\) are determined.

\[
d\omega = \frac{f^2}{a^2h} (\frac{[dy]}{6} - \frac{5}{24} N_{53}) \quad (2.14)
\]
\[ dy_0 = - [dy] \frac{2a^2 + f^2}{6a^2} + N_5 + \frac{5f^2 + 4a^2}{24a^2} \] 

(2.15)

The last two unknowns \( dz_0 \) and \( d\kappa \) are derived directly from the third and fourth normal equations respectively.

\[ dz_0 = - \frac{f}{8a} N_{51} \] 

(2.16)

\[ d\kappa = - \frac{f}{8ah} N_{52} \] 

(2.17)

"If the points are chosen so that they within each combination have the same distances from the centre point, the radial distortion will result in changes of \( dz_0 \) between the different combinations."* Naturally, the whole adjustment procedure is based on the assumption that the approximate values of the elements of orientation are close to the real values. In the case in which the corrections to the approximations are large quantities, differential equations (2.10) are no longer strictly valid, since in their derivation by Taylor's series all terms of second or higher order were disregarded as being practically insignificant.

In a similar way the formulae for the nine-point adjustment combination may be derived. The final expressions for the corrections to the elements of orientation, the standard deviation of unit weight and weight and correlation numbers may be found as in [26].

Hallert's procedure has been applied to a great number of practical adjustments in all kinds of perspective imaging. The applications to x-rays apparatus, microscopes, televisions and

*[25]
a variety of metric and non-metric cameras are described in many of his papers published in international surveying and photogrammetric periodicals.*

**JACOBI'S METHOD FOR NON-METRIC CAMERAS**

Jacobi's method can be used for metric but is particularly suitable for non-metric cameras. The mechanical system of a non-metric camera needs no modifications and can be left untouched since the elements of interior and exterior orientations are simultaneously determined for every single photograph. This is possible only if a calibration system of known points is photographed together with the measured object. The great majority of non-metric cameras have variable focus objectives and some of them also have exchangeable lenses. "To overcome these stabilization problems, the optical axis of the lens is introduced. The exterior orientation is defined from the orientation of the optical axis. The radial lens distortion is defined from the same optical axis. The picture plane may not be perpendicular to the optical axis but the introduction of 2 angles, \( \alpha \) and \( \beta \) will describe the difference in direction between the optical axis and the picture normal so that the camera geometry of a flexible camera can be sufficiently described analytically."** (See Fig. 2-8)

The lens distortion is analytically defined by means of a series and the number of terms defines degree of sophistication.

*\[26\], \[25\], \[29\], \[27\]

**\[39\]
Jacobi in his work used only the radial distortion expressed by the well-known polynomial

\[ dr = a_3 r^3 + a_5 r^5 + a_7 r^7 \]

naturally, this simple formula cannot theoretically satisfy all lenses but the distortion of most quality lenses can be expressed by this equation. The parameters of the lens distortion \((a_3, a_5, a_7, a_9, a_{11} \ldots)\) are to a great extent correlated, but as long as only the first three terms are used the correlation is at a minimum and can be tolerated. The number of control points in a calibration system is another important factor of correlation. "Especially in applying a calibration system containing only the minimum number of control points, it might happen that the iterative optimization procedure does not converge."*

In the actual calibration procedure all parameters are

*[5]
seldom determined for every photograph since some of them are not altered from photograph to photograph. The most convenient way is to determine these parameters once and for all in a laboratory test field, and then the normal calibration procedure will determine only the unstable parameters for every single photograph. One of the stable parameters is the radial distortion of the lens. In some cameras the housing and focal plane are so rigid and stable that the angles $\alpha$ and $\beta$ can be given the fixed value of zero.

In general the calibration provides the six elements of exterior orientation $(X_o, Y_o, Z_o, \omega, \phi, \kappa)$, the five elements of interior orientation $(X_o', Y_o', c, a, \beta)$, and the three parameters of the radial distortion $(a_3, a_5, a_7)$. To find a unique solution of the fourteen unknowns at least 5 well distributed control points and their measured images must be known. In a practical case of calibration a larger number of control points is taken and then by the utilization of the method of least squares the most probable values of the unknown parameters are determined. The number of unknowns is reduced when the parameters of the radial distortion are determined beforehand using a three-dimensional laboratory test field. In this case they are referred to the optical axis of the lens and not as usual to the principal point of the photograph. With the exception of the lens distortion, all parameters of interior as well as of exterior orientation are determined analytically for each photograph. Jacobi's solution is basically very simple
and involves four consecutive coordinate transformations. Figure 2-9, 2-10, and 2-11 display the analytical relations between an object and its photographed image graphically. The diagrams may help to understand and explain the elements involved in the coordinate transformations.

Fig. 2-9

The first transformation is a general spatial transformation of a point from the geodetic or calibration coordinate system \((X_p, Y_p, Z_p)\) to the camera coordinate system \((x_c, y_c, z_c)\) where the z-axis of the camera coordinate system is identical with the optical axis of the lens. The transformation is normally expressed in matrix notation by the following formula
\[
\begin{bmatrix}
    x_c \\
    y_c \\
    z_c
\end{bmatrix} =
\begin{bmatrix}
    a_{11} & a_{12} & a_{13} \\
    a_{21} & a_{22} & a_{23} \\
    a_{31} & a_{32} & a_{33}
\end{bmatrix}
\begin{bmatrix}
    x_p - x_o \\
    y_p - y_o \\
    z_p - z_o
\end{bmatrix},
\]

where the \( a \)-matrix is a known function of \( \omega, \phi \) and \( \kappa \).

\[
A = \begin{bmatrix}
    -\cos\phi\cos\kappa + \sin\phi\sin\omega\sin\kappa & -\cos\omega\sin\kappa & \sin\phi\cos\kappa + \cos\phi\sin\omega\sin\kappa \\
    -\cos\phi\sin\kappa - \sin\phi\sin\omega\cos\kappa & \cos\omega\cos\kappa & \sin\phi\sin\kappa - \cos\phi\sin\omega\cos\kappa \\
    \sin\phi\cos\omega & \sin\omega & \cos\phi\cos\omega
\end{bmatrix}
\]

The second transformation "performs a perspective projection from a camera-located system upon a plane in the distance \( c \) from the perspective centre."** This plane is normal to the optical axis. The origin for both coordinate systems is the perspective centre, \( O_c \).

*\[4\]
**\[39\]
The resulting coordinates $x'_c$ and $y'_c$ are referred to a coordinate system with origin at the principal point.

The third transformation transforms the last coordinates upon a plane perpendicular to the prolongation of the optical axis. Physically this is non-existing plane. The uncorrected
image coordinates in this plane are denoted by \( x'_d \) and \( y'_d \) and the corrected coordinates by \( x'_a \) and \( y'_a \) where

\[
x'_a = x'_d \left[ a_3 \left( (x'_d)^2 + (y'_d)^2 \right) + a_5 \left( (x'_d)^2 + (y'_d)^2 \right)^2 + a_7 \left( (x'_d)^2 + (y'_d)^2 \right)^3 + 1 \right]
\]

\[
y'_a = y'_d \left[ a_3 \left( (x'_d)^2 + (y'_d)^2 \right) + a_5 \left( (x'_d)^2 + (y'_d)^2 \right)^2 + a_7 \left( (x'_d)^2 + (y'_d)^2 \right)^3 + 1 \right]
\]

\[
z'_a = c_d
\]

The first two transformations treat the whole problem as a purely geometrical central projection with the distortion-free camera lens while the third transformation determines the radial distortion in the auxiliary plane by the application of the appropriate polynomials.

The fourth and last transformation brings the corrected intermediate coordinates into the adopted image coordinate system by rotations \( \alpha \) and \( \beta \) around the perspective centre.

The transformations from the comparator coordinate system \((x', y')\) into a two dimensional geodetic system are well described in [15]. The transformations are made in five steps and are given by the final formula in matrix notation

\[
P = D_2^t \begin{bmatrix} Z_c & \frac{C_d}{C} \end{bmatrix} E \begin{bmatrix} C_d \end{bmatrix} (K - H) - P_o, \tag{2.22}
\]

where symbols \( P, D_2^t, E, K, H, \) and \( P_o \) are abbreviations of the following matrices:
\[
P = \begin{bmatrix} X_P \\ Y_P \\ 0 \end{bmatrix}
\]

\[
D_2^t = \begin{bmatrix}
\cos \phi \cos \kappa & \sin \omega \sin \phi \cos \kappa - \sin \kappa \cos \omega & \cos \omega \sin \phi \cos \kappa + \sin \omega \sin \kappa \\
\cos \phi \sin \kappa & \sin \omega \sin \phi \sin \kappa + \cos \omega \cos \kappa & \cos \omega \cos \phi \sin \kappa - \sin \omega \sin \kappa \\
-\sin \phi & \sin \omega \cos \phi & \cos \omega \cos \phi
\end{bmatrix}
\]

\[
E = \begin{bmatrix}
e_p & 0 & 0 \\
0 & e_p & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

where \( e_p = 1 - a_3 [(x'_d)^2 + (y'_d)^2] - a_5 [(x'_d)^2 + (y'_d)^2]^2 - a_7 [(x'_d)^2 + (y'_d)^2]^3 \) (2.26)

\[
K = \begin{bmatrix} x' \\ y' \\ C \end{bmatrix} \quad \text{coordinates in the comparative system} \quad (2.27)
\]

\[
H = \begin{bmatrix} x_o \\ y_o \\ 0 \end{bmatrix} \quad \text{coordinates of the principal point of symmetry} \quad (2.28)
\]

\[
P_o = \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} \quad \text{coordinates in the camera system} \quad (2.29)
\]

If the elements of interior and exterior orientation are known, any point in the calibration net can be transformed into
the image plane by means of these matrices. Orientation elements, however, are known only to their approximations and the mathematical transformations are made at first using these approximate values. Use of the latter will naturally lead to a certain amount of error between computed and measured image coordinates. Under the assumption that there are more observations than unknowns \( n > u \) the most probable values of the orientation elements are determined by the method of least squares.

\[
v^t v = \min
\]

(2.30)

The above condition of least squares is satisfied when

\[
\frac{\partial (v^t v)}{\partial x_j} = 0,
\]

(2.31)

where \( x_j \) represents the unknowns. The derivation yields the coefficients of observation equations

\[
a_{ij} = \frac{\partial F_i}{\partial x_j} = \frac{\partial V_i}{\partial x_j},
\]

(2.32)

where \( F_i \) is a functional relation between unknowns and observations. The corresponding matrix of the coefficients of observation equation will then be

\[
A = \begin{bmatrix}
a_1 & b_1 & c_1 & \ldots & u_1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_n & b_n & c_n & \ldots & u_n
\end{bmatrix}
\]

(2.33)
The normalization of the observation equations is performed by the following procedure

\[
N = \begin{bmatrix}
[aa][ab][ac] \ldots [au] \\
[ab][bb][bc] \ldots [bu] \\
\vdots \\
[au][bu][cu] \ldots [uu]
\end{bmatrix} = A^tA
\tag{2.34}
\]

Differences between computed and measured values are usually denoted by \( l_i \), and then the matrix of absolute terms is equal to

\[
 l = \begin{bmatrix}
l_1 \\
l_2 \\
l_3 \\
\vdots \\
l_n
\end{bmatrix}
\tag{2.35}
\]

and

\[
 n = \begin{bmatrix}
[a1] \\
[bl] \\
\vdots \\
[u1]
\end{bmatrix} = A^t l
\tag{2.37}
\]

The system of normal equations given by the expression \( Nx + n = 0 \) whose number is equal to the number of unknowns is solved and yields the corrections for the approximations of the orientation elements.

\[
x_j = -N^{-1}n
\tag{2.38}
\]
\[ x_j = x_{jo} + x_j', \]  \hspace{1cm} (2.39)

where \( x_{jo} \) is an approximation and \( x_j \) is its correction.

Although the theoretical adjustment of a calibration is relatively simple, the practical procedure faces a very serious problem, namely the correlation between the unknowns and the approximation of the orientation parameters. Some approximation, particularly those of the interior orientation are very close to the real values, while the elements of exterior orientation can be very far from the actual values of the unknowns. The correlation and the inaccuracy of the exterior orientation elements will not allow iterative adjustments to converge.

Jacobi suggested a method which, according to his articles [38] and [39], has shown quite satisfactory results. In his words in [39] "this insufficiency of the adjustment can be overcome by letting all the orientation elements that have good approximate values appear as constants, while at first the orientation elements with poor approximated values appear as unknowns."

Under the assumption that the principal point is in the middle of the photograph and the lens is distortion free, the camera constant is taken from the distance setting of the lens and the elements of exterior orientation are adjusted. The adjustment is usually performed by three or four iterative adjustments which normally yield relatively good results of the parameters of exterior orientation. "In the next step, the exterior orientation \( c, x_o' \) and \( y_o' \) are all incorporated as unknowns, the
distortion parameters as well as the angles $\alpha$ and $\beta$ are given constant values of zero. In this way the approximated values of all the unknowns are encumbered with the same error, and the correlation between $\omega$ and $y_o'$, $\phi$ and $x_o'$ will not disturb the adjustment."* The final step is the introduction of the radial distortion parameters as unknowns in the adjustment. Since all other elements have very good approximations from previous adjustments, only one or two iterations usually suffice and normally that ends up the adjustment. Very seldom is the fourth step used to determine angles $\alpha$ and $\beta$ since that adjustment makes no significant change in the already adjusted elements of orientation.

Although Jacobi's method has received an international acceptance and has been applied by various institutions all over the world there are still some questions left unanswered by this theory and practice: to what degree of accuracy the approximations of parameters of exterior orientations must really be known, what is the influence of correlation of the unknowns, and finally, what is the optimum number and distribution of calibration control points? When these questions receive definite answers the method will come close to the perfect solution of the calibration problem.

**BROWN'S ANALYTICAL PLUMB LINE METHOD**

This method, unlike other previously described methods, is concerned only with the determination of radial and decentering lens distortions, while the remaining elements of interior orientation.

*[39]*
orientation must be predetermined by some other method. Brown used the well known fact that radial distortion is a function of object distance, and therefore for any focal setting, the corresponding distortion correction must theoretically be determined by a camera calibration procedure. However, in practical photogrammetry it is sufficient to know radial and decentering lens distortions for only two distinct focal settings. The distortion for any other setting can be mathematically computed applying Magill's formula

$$\text{dr}_s = \text{dr}_\infty - m_s \text{dr}_\infty,$$  \hspace{1cm} (2.40)

where \(\text{dr}_s\) is distortion for focus on object plane at distance \(s\) from the camera, \(\text{dr}_\infty\) is the distortion of lens focusing at infinity, \(\text{dr}_\infty\) is the distortion of lens for inverted infinite focus, and \(m_s\) is the magnification of the lens for the object plane at distance \(s\). The magnification \(m_s\) is obtained from the formula

$$m_s = \frac{f}{s - f}$$ \hspace{1cm} (2.41)

From the original Magill's formula, which is for close range photogrammetry of rather small practical value Brown developed a more convenient formula. If distortion \(\text{dr}_{s_1}\) and \(\text{dr}_{s_2}\) for two object planes at distances \(s_1\) and \(s_2\) from the camera are known and if they are substituted into equation (2.40) two expressions will result with new unknowns \(\text{dr}_\infty\) and \(\text{dr}_\infty\) thus
\[ dr_{s_1} = dr_{\infty} - \frac{f}{s_1 - f} \, dr_{\infty} \]

\[ dr_{s_2} = dr_{\infty} - \frac{f}{s_2 - f} \, dr_{\infty} \]

(2.41)

The solution of the two equations yields the values of the unknown quantities.

\[ dr_{\infty} = \frac{(dr_{s_1} - dr_{s_2})(s_2 - f)(s_1 - f)}{f(s_1 - s_2)} \]

(2.42)

\[ dr_{\infty} = dr_{s_1} + \frac{(dr_{s_1} - dr_{s_2})(s_2 - f)(s_1 - f)}{s_1 - s_2} \]

(2.43)

If the last two equations are now substituted into Magill's original formula (2.40) the distortion for an arbitrary object distance \( s \) is obtained.

\[ dr_s = dr_{s_1} + \frac{(dr_{s_1} - dr_{s_2})(s_2 - f)}{s_1 - s_2} - \frac{f}{s - f} \left( \frac{dr_{s_1} - dr_{s_2}}{s_1 - s_2} \right) \frac{(s_2 - f)(s_1 - f)}{f(s_1 - s)} \]

\[ dr_s = dr_{s_1} + dr_{s_1} \frac{s_2 - f}{s_1 - s_2} - dr_{s_1} \frac{(s_2 - f)(s_1 - f)}{(s - f)(s_1 - s_2)} - dr_{s_2} \frac{s_2 - f}{s_1 - s_2} + \frac{(s_2 - f)(s_1 - f)}{(s - f)(s_1 - s_2)} \]

\[ dr_s = dr_{s_1} \left[ 1 + \frac{s_2 - f}{s_1 - s_2} \left( 1 - \frac{s_1 - f}{s - f} \right) \right] - dr_{s_2} \frac{s_2 - f}{s_1 - s_2} \left( 1 - \frac{s_1 - f}{s - f} \right) \]

(2.44)

The final formula can be written in the following form:

\[ dr_s = (1 + \alpha_s)dr_{s_1} - \alpha_s dr_{s_2} \]

where
Therefore, when distortions for two distinctive object planes are known one can compute the distortion for any object plane at distance \( s \) from the camera. The known radial distortions can be expressed by previously known polynomial for the radial component of the total distortion

\[
dr = a_3 r^3 + a_5 r^5 + a_7 r^7
\]

and

\[
dr = a_3 s r^3 + a_5 s r^5 + a_7 s r^7
\]

or for the general case

\[
dr = a_3 s r^3 + a_5 s r^5 + a_7 s r^7
\]

The unknown coefficients, \( a_{3s}, a_{5s}, a_{7s} \) in the last equation can be obtained by the application of equation (2.44).

\[
a_{3s} = (1 + \alpha_s)a_{3s1} - \alpha_s a_{3s2}
\]

\[
a_{5s} = (1 + \alpha_s)a_{5s1} - \alpha_s a_{5s2}
\]

(2.47)

\[
a_{7s} = (1 + \alpha_s)a_{7s1} - \alpha_s a_{7s2}
\]

The distortion function \( dr_s \) of a normal lens is usually a function of only the first term of the polynomial. Other terms can be disregarded as practically insignificant. They become
important in the case of lenses for aerial photogrammetry which are made with a very small distortion over the usable field. "When higher order terms are insignificant for a given lens, equation (2.44) has a consequence of special importance to some applications, it implies the existence of an object plane distance for which distortion is zero."*

Knowing the radial distortions at two distances $s_1 = 2f$ and $s_2 = \infty$ one can compute the distance $s$ for which the distortion defined only by the first term of the polynomial will be zero throughout the usable field. Equation (2.44) will then be

$$(1 + \alpha_s)dr_{2f} - \alpha_s dr_\infty = 0,$$

or from which

$$\alpha_s = \frac{dr_{2f}}{dr_\infty - dr_{2f}} \quad (2.48)$$

Equation (2.45) which defines $\alpha_s$ can be also written in the following form

$$\alpha_s = \frac{(1 - \frac{f}{s_2})(s - s_1)}{(1 - \frac{s_1}{s_2})(s - f)} \quad (2.49)$$

which after the introduction of the corresponding values for $s_1$ and $s_2$ becomes

$$\alpha_s = \frac{s - 2f}{f - s} \quad (2.50)$$

Equating expressions (2.48) and (2.50) and then rearranging *\[10\]
the equation for the distance $s$ at which the distortion is zero is determined as follows:

$$\frac{s - 2f}{f - s} = \frac{dr_{2f}}{dr_{\infty} - dr_{2f}}$$

$$s(1 + \frac{dr_{2f}}{dr_{\infty} - dr_{2f}}) = f(2 + \frac{dr_{2f}}{dr_{\infty} - dr_{2f}})$$

$$\frac{dr_{\infty}}{dr_{\infty} - dr_{2f}} = f \frac{2dr_{\infty} - dr_{2f}}{dr_{\infty} - dr_{2f}}$$

$$s = f(2 - \frac{dr_{2f}}{dr_{\infty}}) \quad (2.51)$$

When an object is in a plane at distance $s$ from the camera its image will be distortion free. This naturally implies only that points are in the object plane. All other image points that are outside the plane are still sharp due to the depth of the field as a function of the focal length, and the aperture will be affected by distortions. For all practical purposes in the case of spacial objects Magill's formula in its original form cannot satisfy the requirements. "What is needed, then, is a further extension of Magill's formula to account for the variation of distortion for points distributed throughout the photographic field."* To solve the problem Brown used simple geometrical ratios from Fig. 2-12, the polynomials of the radial distortion and the Gaussian form of the thin-lens equation.

* [10]
Points 0, P and Q are in the image plane for a lens focussed on an object plane at distance s from the camera. Points 0', P', and Q' are in an image plane for a lens focussed on an object plane at distance s'.

From similar triangles COP and CO'P' we obtain

\[ \frac{r'}{r} = \frac{C_s'}{C_s} \]

or

\[ r' = \frac{C_s'}{C_s} r \]  

(2.52)

The radial distortion \( dr_s \), is computed by the previously known polynomial
\[
\text{dr}_s = a'_3(r')^3 + a'_5(r')^5 + a'_7(r')^7.
\]

When the value of \( r' \) from equation (2.52) is substituted in the last expression we have

\[
\text{dr}_s = a'_3\left(\frac{C_{s'}}{C_s}\right)^3 r^3 + a'_5\left(\frac{C_{s'}}{C_s}\right)^5 r^5 + a'_7\left(\frac{C_{s'}}{C_s}\right)^7 r^7 \tag{2.53}
\]

From Fig. 2-12 it is obvious that

\[
\text{dr}_{ss'} = \frac{C_s}{C_{s'}} \text{dr}_s',
\]

or

\[
\text{dr}_{ss'} = a'_3\left(\frac{C_{s'}}{C_s}\right)^2 r^3 + a'_5\left(\frac{C_{s'}}{C_s}\right)^4 r^5 + a'_7\left(\frac{C_{s'}}{C_s}\right)^6 r^7 \tag{2.54}
\]

The Gaussian equations of the thin lens applied to the two image planes give

\[
\frac{1}{s} + \frac{1}{C_s} = \frac{1}{f}
\]

and

\[
\frac{1}{s'} + \frac{1}{C_s} = \frac{1}{f}
\]

Rearranged, the last two equations yield in turn

\[
\frac{1}{C_s} = \frac{s' - f}{fs'}
\]

and

\[
\frac{1}{C_s} = \frac{s - f}{fs}
\]
Now dividing the second equation by the first, the required ratio \( \frac{C_s'}{C_s} \) for equation (2.54) is obtained.

\[
\frac{C_s'}{C_s} = \frac{(s-f)s'}{(s'-f)s} \tag{2.55}
\]

When distortion functions \( (\delta r_s) \) are determined by the calibration process then the correction, according to Brown in [10], is performed in four steps.

(a) Distance \( s' \) is first computed from the approximate coordinates \( x, y, z \) of the photographed point applying photogrammetric intersection.

(b) Using (2.49) and later (2.47) coefficients \( a_{1s'}, a_{2s'}, \) and \( a_{3s'} \), are determined.

(c) The ratio \( \frac{C_s'}{C_s} \) is then computed by (2.55). This ratio in conjunction with the coefficients of polynomials yields the radial distortion at the observed radial distance \( r \).

(d) With known values of the radial distortions the observed image coordinates \( (x, y) \) are corrected by the following amounts.

\[
\delta x = \frac{x}{r} \delta r_{ss'},
\]

and

\[
\delta y = \frac{y}{r} \delta r_{ss'}.
\]
It is advisable to use an iterative computation process until the required degree of accuracy of the final results is obtained.

The determination of radial distortions for two different settings of focused lens is performed by the plumb line method. For the use of Magill's formula two values of radial distortions must be known, naturally, for two different distances. "It requires that distortion coefficients be precalibrated for one object plane $s_2$ (usually, $s_2 = \infty$) and regards as unknown the distortion coefficients for the particular object plane on which the camera is focused."* The numerical reduction needs no absolute control points but if there are some known distances in object space and the geometry of photographs is highly convergent, Brown's method can determine the coordinates of the principal point and the principal distance. When distances are not given, a pre-established value of the principal distance is used in computations.

For projects in which two different cameras are used for moving objects, the distortion coefficients of each station will have to be determined by the plumb-line method. "This method involves photographing a set of plumb lines arrayed in the desired object plane and exploits the fact that, in the absence of distortion, the central projection of a straight line is itself a straight line. Systematic deviations of the images of plumb lines from straight lines thus provide a measure of

*[10]
distortion if properly reduced.*

The non-distorted image of a plumb line in the coordinate system of the photograph can be expressed in the following form:

\[ x' \sin \theta + y' \cos \theta = \rho, \]

(2.57)

where \( \rho \) is the perpendicular distance of the line from the origin and \( \theta \) is the bearing of the distance (See Fig. 2-13). Since the applied lenses are not distortion-free plumb lines will be represented by some curved lines, when corrected for radial and decentering distortion will represent points of a straight line. This fact can be mathematically expressed by the following two equations

\[ x' = x + \bar{x} (a_3 r^2 + a_5 r^4 + a_7 r^6) + \]
\[ + [b_1 (r^2 + 2\bar{x}^2) + 2b_2 \bar{x} \bar{y}] [1 + b_3 r^2 + ...] \]  
(2.58)

\[ y' = y + \bar{y} (a_3 r^2 + a_5 r^4 + a_7 r^6) + \]
\[ + [2b_1 \bar{x} \bar{y} + b_2 (r^2 + 2\bar{y}^2)] [1 + b_3 r^2 + ...], \]

where

\[ \bar{x} = x - x_p \]
\[ \bar{y} = y - y_p \]

(2.59)

\[ r = \sqrt{(x - x_p)^2 + (y - y_p)^2} \]

Factors \( a_3, a_5 \) and \( a_7 \) are the coefficients of the radial

*[10]
distortion and \( b_1, b_2 \) and \( b_3 \) are the coefficients of the
decentering distortion.

\[
\begin{align*}
\text{Fig. 2-13}
\end{align*}
\]

When the image coordinates \( x' \) and \( y' \) of the \( j \)-th point on
the \( i \)-th line are substituted into (2.58), from thence into
(2.59) and finally into (2.57), the following type of observation
equations is obtained.

\[
f(x_{ij}', y_{ij}'; x_P, y_P, a_3, a_5, a_7, b_1, b_2, b_3; \theta_i, \rho_i) = 0 \quad (2.60)
\]

The number of observation equations is equal to the number
of measured points. The number of normal equations is equal to
the number of unknowns, which in case of the \( m \) used lines is
\( 8 + 2m \) (eight unknowns of interior orientation \( x_P, y_P, a_3, a_5,
a_7, b_1, b_3, b_5 \) and a pair of unknowns \( \theta_i, \rho_i \) for each of the
lines). If the number of observations exceeds the number of unknowns a least squares adjustment which yields the most probable values of the unknown parameters can be performed. As in the majority of cases in adjustments by indirect observations, approximations of the required quantities are determined first and then the adjustment provides small corrections to the approximations.

\[ x_{ij} = x_{ij}^0 + v_{x_{ij}} \]
\[ y_{ij} = y_{ij}^0 + v_{y_{ij}} \]

where \( x_{ij}^0 \) and \( y_{ij}^0 \) are measured coordinates and \( v_{x_{ij}} \) and \( v_{y_{ij}} \) are the corresponding residuals.

\[ x_p = x_p^0 + \delta x_p \]
\[ y_p = y_p^0 + \delta y_p \]
\[ a_3 = a_3^0 + \delta a_3 \]
\[ a_5 = a_5^0 + \delta a_5 \]
\[ a_7 = a_7^0 + \delta a_7 \]
\[ b_1 = b_1^0 + \delta b_1 \]
\[ b_2 = b_2^0 + \delta b_2 \]
\[ b_3 = b_3^0 + \delta b_3 \]
\[ \theta_i = \theta_i^0 + \delta \theta_i \]
\[ \rho_i = \rho_i^0 + \delta \rho_i \]
where \( ^0 \) values are approximations and \( \delta \)'s are corrections obtained by the adjustment.

Substituting expressions (2.61) and (2.62) into the original observation equations and expanding the result by Taylor's series, the final form of the observation equations in matrix notation is obtained:

\[
v = Ax + \lambda, \quad (2.63)
\]

where

\[
v = \begin{bmatrix}
v_{xi}\n v_{yi}\n v_x\n v_y\n \vdots\n v_{\theta_i}\n v_{\rho_i}
\end{bmatrix}
\]

\[
A = \begin{bmatrix}
\left(\frac{\partial f}{\partial x_i}\right)_1 & \left(\frac{\partial f}{\partial y_i}\right)_1 & \left(\frac{\partial f}{\partial x_p}\right)_1 & \left(\frac{\partial f}{\partial y_p}\right)_1 & \left(\frac{\partial f}{\partial a_1}\right)_1 & \cdots & \left(\frac{\partial f}{\partial \theta_i}\right)_1 & \left(\frac{\partial f}{\partial \rho_i}\right)_1 \\
\left(\frac{\partial f}{\partial x_i}\right)_2 & \left(\frac{\partial f}{\partial y_i}\right)_2 & \left(\frac{\partial f}{\partial x_p}\right)_2 & \left(\frac{\partial f}{\partial y_p}\right)_2 & \left(\frac{\partial f}{\partial a_2}\right)_2 & \cdots & \left(\frac{\partial f}{\partial \theta_i}\right)_2 & \left(\frac{\partial f}{\partial \rho_i}\right)_2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\left(\frac{\partial f}{\partial x_i}\right)_n & \left(\frac{\partial f}{\partial y_i}\right)_n & \left(\frac{\partial f}{\partial x_p}\right)_n & \left(\frac{\partial f}{\partial y_p}\right)_n & \left(\frac{\partial f}{\partial a_3}\right)_n & \cdots & \left(\frac{\partial f}{\partial \theta_i}\right)_n & \left(\frac{\partial f}{\partial \rho_i}\right)_n
\end{bmatrix}, \quad (2.65)
\]
Quantities \( l_i \) are the values of the function (2.60), for approximations (2.61) and (2.62).

Further procedure is identical with the already described method of normalizing of the observation equations and solving of normal equation in Jacobi's method of calibration.

\[
N = \mathbf{A}^t \mathbf{A} \\
n = \mathbf{A}^t \boldsymbol{l} \\
x = - N^{-1} n
\] 

(2.68)

As Brown states "the recoverability of \( x_p \) and \( y_p \) in the plumb-line calibration method depends directly on the magnitude
of the radial distortion; the greater the distortion, the better the recovery of $x_p, y_p$."

Although Brown calls his method the plumb-line calibration method the author cannot completely accept it as a calibration method since it does not determine the camera constant. Compared to Jacobi's method it has one definite advantage. Brown's method needs no special surveying methods to determine the coordinates of targets in the object space. It requires only $n$ number of plumb lines. The method is attractive in its observational simplicity. The disadvantage compared to Jacobi's method is that it does not determine the elements of exterior orientation.

Recent research by Youssef Abdel-Aziz at the University of Illinois also showed that Brown's extension of Magill's formula gives good results for symmetrical lens distortion only for those points which are inside the range of the two known principal distances. The results obtained for points outside the range are not good. Abdel-Aziz developed a new version of the extended Magill's formula and apparently achieved a much higher accuracy than that obtained by D.C. Brown.

**ABDEL-AZIZ'S CALIBRATION METHOD**

In a way similar to Brown's Abdel-Aziz investigates the geometrical characteristics of parallel lines in a central

* [10]
** [1]
projection to determine the principal point, the principal
distance and the radial distortions of a photograph. However
he does not use plumb-lines as test objects but uses a set of
parallel lines which intersect at right angles in a plane and
four thin wires (pins) perpendicular to the plane.

Two oblique photographs are taken of the test field with
a change in tilt of approximately 90 degrees.

Neither of the optical axes should be collinear with or
parallel to the directions of lines in the plane of the test
object.

The position of the principal point is determined as the
intersection of two loci of the principal point. These loci
are the lines perpendicular to the vanishing lines (vv) and passing through the nadir points. The nadir point is obtained using the radial displacements of the thin pins which are perpendicular to the plane of the test object. One locus of the principal point is determined using each oblique photograph and then the intersection of the two loci is the principal point. This procedure is obvious from diagrams 2-15, 2-16 and 2-17.

Fig. 2-15
Fig. 2-16. Construction of the nadir point

\[ C = \sqrt{ab} \]  \hspace{1cm} (2.69)
Although Abdel-Aziz's method is very simple the author cannot see that it has any particular practical value. Compared to other methods already described its simplicity is its only merit.

**ABDEL-AZIZ-KARARA CALIBRATION METHOD**

This method of calibration was recently developed at the University of Illinois. The method has a very interesting and significant characteristic, since for data reduction the classical elements of interior orientation (principal point and camera constant) are not used. "The proposed method involves a direct linear transformation from comparator coordinates into object space coordinates. In a sense, it is a simultaneous solution for two transformations. Since the image coordinate system is not involved in the approach, fiducial marks are not needed. Furthermore, the method is a direct solution and does not involve initial approximations for the unknown parameters of inner and outer orientation of the camera.*

The two mentioned transformations are:

1. transformation from comparator coordinates into image coordinates, and

2. transformation from image coordinates into object-space coordinates.

Since the method does not involve fiducial marks it is particularly suitable for reduction of data obtained by non-metric

*[2]
cameras, which rarely have fiducial marks.

The first transformation from comparator coordinates into image coordinates in analytical photogrammetry is done by the following formulae

\[
\begin{align*}
\bar{x} &= a_1 + a_2 x + a_3 y \\
\bar{y} &= a_4 + a_5 x + a_6 y,
\end{align*}
\]

where \( \bar{x}, \bar{y} \) are image coordinates and \( x, y \) are comparator coordinates. The second transformation is performed by the formula

\[
\begin{bmatrix}
\bar{x} \\
\bar{y} \\
-c
\end{bmatrix} = \lambda \begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix} \begin{bmatrix}
X - X_o \\
Y - Y_o \\
Z - Z_o
\end{bmatrix},
\]

where \( X, Y, Z \) are object space coordinates, \( X_o, Y_o, Z_o \) are object space coordinates of the exposure station, \( \lambda \) is a scale factor and \( a_{ij} \) are coefficients of spacial transformation.

Equation (2.71) in matrix form can be also expressed in form of two equations

\[
\begin{align*}
\bar{x} + c \frac{a_{11}(X-X_o) + a_{12}(Y-Y_o) + a_{13}(Z-Z_o)}{a_{31}(X-X_o) + a_{32}(Y-Y_o) + a_{33}(Z-Z_o)} &= 0 \\
\bar{y} + c \frac{a_{21}(X-X_o) + a_{22}(Y-Y_o) + a_{23}(Z-Z_o)}{a_{31}(X-X_o) + a_{32}(Y-Y_o) + a_{33}(Z-Z_o)} &= 0
\end{align*}
\]

The expressions for image coordinates \((\bar{x}, \bar{y})\) from equations (2.70) are now substituted into equation (2.72).
To eliminate $y$ from the last two equations the first equation is multiplied by $a_6$, the second by $-a_3$ and the two are added.

\[
(a_1a_6 - a_3a_4) + (a_2a_6 - a_3a_5)x + 
\frac{(a_6a_{11} - a_3a_{21})(X-X_0) + (a_6a_{12} - a_3a_{22})(Y-Y_0) + (a_6a_{13} - a_3a_{23})(Z-Z_0)}{a_{31}(X-X_0) + a_{32}(Y-Y_0) + a_{33}(Z-Z_0)} c = 0
\]

In the same manner $x$ can be eliminated if the first equation of (2.73) is multiplied by $a_5$ and the second by $-a_2$ and the two are added.

\[
(a_1a_5 - a_2a_4) + (a_3a_5 - a_2a_6)y + 
\frac{(a_5a_{11} - a_2a_{21})(X-X_0) + (a_5a_{12} - a_2a_{22})(Y-Y_0) + (a_5a_{13} - a_2a_{23})(Z-Z_0)}{a_{31}(X-X_0) + a_{32}(Y-Y_0) + a_{33}(Z-Z_0)} c = 0
\]

The last two equations obtained by the elimination of $y$ and $x$ respectively can be simplified by introduction of new symbols to the following form

\[
d_1 + d_2x + \frac{b_1X + b_2Y + b_3Z + b_4}{b_9X + b_{10}Y + b_{11}Z + b_{12}} = 0
\]

\[
d_3 + d_4y + \frac{b_5X + b_6Y + b_7Z + b_8}{b_9X + b_{10}Y + b_{11}Z + b_{12}} = 0
\]

where
\[ d_1 = a_1 a_6 - a_3 a_4 \]
\[ d_2 = a_2 a_6 - a_3 a_5 \]
\[ d_3 = a_1 a_5 - a_2 a_4 \]
\[ d_4 = a_3 a_5 - a_2 a_6 \]
\[ b_1 = c (a_6 a_{11} - a_3 a_{21}) \]
\[ b_2 = c (a_6 a_{12} - a_3 a_{22}) \]
\[ b_3 = c (a_6 a_{13} - a_3 a_{23}) \]
\[ b_4 = -(b_1 X_O + b_2 Y_O + b_3 Z_O) \]
\[ b_5 = a_5 a_{11} - a_2 a_{21} \]
\[ b_6 = a_5 a_{12} - a_2 a_{22} \]
\[ b_7 = a_5 a_{13} - a_2 a_{23} \]
\[ b_8 = -(b_5 X_O + b_6 Y_O + b_7 Z_O) \]
\[ b_9 = a_{31} \]
\[ b_{10} = a_{32} \]
\[ b_{11} = a_{33} \]
\[ b_{12} = -(a_{31} X_O + a_{32} Y_O + a_{33} Z_O) \]

When terms \( d_1 \) and \( d_3 \) in equations (2.74) are put over the denominator of the third terms the equations become

\[ d_2 x + \frac{(b_1 + b_9 d_1) X + (b_2 + b_1 b_6 d_1) Y + (b_3 + b_1 b_1 d_1) Z + (b_4 + b_1 b_2 d_1)}{b_9 X + b_1 b_2 Y + b_1 b_3 Z + b_{12}} = 0 \]
\[ d_4 y + \frac{(b_5 + b_9 d_3) X + (b_6 + b_1 b_6 d_3) Y + (b_7 + b_1 b_1 d_3) Z + (b_8 + b_1 b_2 d_3)}{b_9 X + b_1 b_2 Y + b_1 b_3 Z + b_{12}} = 0 \]

or applying new abbreviated notation

\[ x + \frac{b_1^* X + b_2^* Y + b_3^* Z + b_4^*}{b_9 X + b_1 b_2 Y + b_1 b_3 Z + b_{12}} = 0 \]
\[ y + \frac{b_5^* X + b_6^* Y + b_7^* Z + b_8^*}{b_9 X + b_1 b_2 Y + b_1 b_3 Z + b_{12}} = 0 \]
where

\[ b_1^* = \frac{1}{d_2} (b_1 + b_9 d_1) \]
\[ b_2^* = \frac{1}{d_2} (b_2 + b_10 d_1) \]
\[ b_3^* = \frac{1}{d_3} (b_3 + b_{11} d_1) \]
\[ b_4^* = \frac{1}{d_4} (b_4 + b_{12} d_1) \]
\[ b_5^* = \frac{1}{d_4} (b_5 + b_9 d_3) \]
\[ b_6^* = \frac{1}{d_4} (b_6 + b_{10} d_3) \]
\[ b_7^* = \frac{1}{d_4} (b_7 + b_{11} d_3) \]
\[ b_8^* = \frac{1}{d_4} (b_8 + b_{12} d_3) \]

If the numerators and denominators in equations (2.76) are divided by \( b_{12} \) we obtain

\[
\begin{align*}
X + \frac{b_1^*}{b_{12}} \frac{X}{b_9} + \frac{b_2^*}{b_{12}} \frac{Y}{b_10} + \frac{b_3^*}{b_{12}} \frac{Z}{b_11} + \frac{b_4^*}{b_{12}} &= 0 \\
Y + \frac{b_5^*}{b_{12}} \frac{X}{b_9} + \frac{b_6^*}{b_{12}} \frac{Y}{b_10} + \frac{b_7^*}{b_{12}} \frac{Z}{b_11} + \frac{b_8^*}{b_{12}} &= 0
\end{align*}
\]

or with new symbols

\[
\begin{align*}
X + \frac{\lambda_1 X + \lambda_2 Y + \lambda_3 Z + \lambda_4}{\lambda_9 X + \lambda_{10} Y + \lambda_{11} Z + 1} &= 0 \\
Y + \frac{\lambda_5 X + \lambda_6 Y + \lambda_7 Z + \lambda_8}{\lambda_9 X + \lambda_{10} Y + \lambda_{11} Z + 1} &= 0
\end{align*}
\]
The last equations are the fundamental equations for Abdel Aziz-Karara method. They can be slightly simplified by selecting the image coordinate system in such a way that the coordinate axes are parallel to the axes of the comparator's coordinate system and so that its origin is in the principal point. In this case the coefficient $a_5$ in the second of equations (2.70) becomes zero.

\[ \bar{x} = a_1 + a_2 x + a_3 y \]
\[ \bar{y} = a_4 + a_6 y \]  

(2.79)

This will naturally also simplify equations (2.73)

\[ a_1 + a_2 x + a_3 y - c \frac{a_{11}(X-X_Q) + a_{12}(Y-Y_Q) + a_{13}(Z-Z_Q)}{a_{31}(X-X_Q) + a_{32}(Y-Y_Q) + a_{33}(Z-Z_Q)} = 0 \]
\[ a_4 + a_6 y - c \frac{a_{21}(X-X_Q) + a_{22}(Y-Y_Q) + a_{23}(Z-Z_Q)}{a_{31}(X-X_Q) + a_{32}(Y-Y_Q) + a_{33}(Z-Z_Q)} = 0 \]

(2.80)

Equations (2.80) have 12 unknowns which are linearly dependent. The number of unknowns can be reduced to 11 by substituting

\[ c_x = \frac{c}{a_2} \text{ and } c_y = \frac{c}{a_6} \]

Then equations (2.80) become

\[ \bar{a}_1 + x + \bar{a}_2 y - c \frac{a_{11}(X-X_Q) + a_{12}(Y-Y_Q) + a_{13}(Z-Z_Q)}{a_{31}(X-X_Q) + a_{32}(Y-Y_Q) + a_{33}(Z-Z_Q)} = 0 \]
\[ \bar{a}_3 + y - c \frac{a_{21}(X-X_Q) + a_{22}(Y-Y_Q) + a_{23}(Z-Z_Q)}{a_{31}(X-X_Q) + a_{32}(Y-Y_Q) + a_{33}(Z-Z_Q)} = 0 \]  

(2.81)
These equations are the original observation equations but they cannot be applied in least square adjustment because they are not linear. However, they can be made linear by expanding them in Taylor's series and neglecting terms of second or higher order as sufficiently small and practically insignificant. The proposed method of Abdel Aziz-Karara does not use the conventional collinearity approach because it requires the approximations of unknowns. Abdel Aziz and Karara expand equations (2.78) in the Taylor's series and obtain

\[ v_x + x\lambda_1 + y\lambda_2 + z\lambda_3 + l_4 + x\lambda_9 + x\lambda_{10} + xZ\lambda_{11} + x = 0 \]

\[ v_y + x\lambda_5 + y\lambda_6 + z\lambda_7 + l_8 + x\lambda_{9} + y\lambda_{10} + yZ\lambda_{11} + y = 0 \]

Compared with conventional methods the proposed method has two clear advantages. It does not contain errors due to iteration criteria and is not influenced by neglecting of second and higher order terms in linearization of the observation equations.

The method is particularly suitable for non-metric cameras without fiducial marks. Since there are altogether eleven unknowns, six well distributed points will give a unique solution. Redundant measurements make the adjustment of required quantities possible and lead to the most probable values according to the theory of least squares.
CONCLUSION ON CALIBRATION OF CLOSE-RANGE CAMERAS

Close-range photogrammetry as a relatively new branch of the science of photogrammetry has to meet the requirements of a great variety of special applications. This need is particularly obvious with respect to the used cameras. Since available commercial metric cameras cannot completely satisfy the needs, non-metric cameras are used to enlarge the field of application of close-range photogrammetry. The calibration of non-metric cameras must also include some checks of conditions that are normally assumed to be fulfilled in commercial cameras. These conditions are the stability of the interior orientation under different exposure set-ups, stability of radial and decentering distortion, the perpendicularity of the optical axis and the image plane, and flatness of film. To avoid the instability of camera calibration parameters, some recent approaches combine the processes of data acquisition and calibration using the same exposure.

In its fundamental concept the calibration of camera is a space resection problem. Since a great majority of calibration methods applies to cameras with almost perfectly vertical optical axis, there must, in the author's opinion, be good correlation between the camera constant and the \( z_0 \) coordinate of the camera station when they are simultaneously determined by calibration. This problem was touched upon by some German photogrammetrists* but needs more investigation.

*Dohler, Gelhaus, Linkwitz
Another extremely important question is the number and distribution of control points in the test field which will lead to the most economical and best solution of calibration parameters.

**DATA REDUCTION SYSTEMS**

Data reduction instruments can generally be classified into two main categories:

(1) Analogue plotters

(2) Analytical plotters

**ANALOGUE PLOTTERS**

The task of an analogue plotter is to convert two conjugate central projections into a single orthogonal projection. They consist of three basic parts:

(a) projection system,

(b) system which determines the space intersection of the corresponding rays, and

(c) viewing system

Projection systems can be: optical, mechanical, or optical-mechanical.

Restitution instruments with optical projection systems consist of two or more projectors which project the corresponding conjugate bundle of rays and the intersection of rays creates a space model. This principle was already suggested by Scheimp-
flug and the first instrument of that type was build by Gasser using dichromatic anaglyphic (red-green) projection as well as the alternate blinking system to determine the intersection points of the corresponding rays. Although these instruments are technically very simple, from economical reasons very feasible, and are used in photogrammetric compilation more than any other type of instruments, the author does not think that they can be applied to a greater extent in close-range photogrammetry. Their main limitation is a very narrow range of principal distance. In addition they take no care of distortion and assume that photography is made by distortion-free lenses. The exceptions are instruments with Porro-Koppe principle.

Fig. 2-18. Principle of optical projection system
In the case of mechanical projection systems the intersection of conjugate rays is realized as the mechanical intersection of two space rods. The distortion, particularly with wide angle lenses, can be practically compensated by mechanical or optical means. Typical examples of these restitution instruments are Santoni's Stereocartograph and Wild's Autographs A5, A6, A7 and A8.

Fig. 2-19. Principle of mechanical projection system

In instruments with optical-mechanical projection systems, a combination of optical and mechanical systems is used. Thus the main requirements of optical and mechanical systems are
avoided. An example is Hugershoff's Aerocartograph.

Fig. 2-20. Principle of optical-mechanical projection systems

It is beyond the scope of this thesis to describe in detail restitution instruments for close-range photogrammetry. There are actually no special instruments and the data compilation is performed with existing instruments of terrestrial and aerial photogrammetry which are well described in most text books of photogrammetry. When metric cameras are used these instruments have no great problems in the evaluation of photographs. However the photographs obtained by non-metric cameras are largely influenced by rather significant and irregular radial and de-centering distortion. Standard mechanical and optical systems
cannot completely compensate for the influence of distortions and for higher requirements of accuracy only analytical plotters provide satisfactory results. It is also important to note that standard plotting instruments basically do not have sufficient range of principal distance to evaluate photography taken with sterometric or non-metric cameras. Plotting in such cases must be performed in an affine model with an exaggerated principal distance and vertical scale. Generally this technique is not well known to photogrammetrists. Also since this is the only method by analogue approach for reducing of photographs taken by metric and non-metric cameras with shorter focal lenses it is worthwhile describing the method in detail, particularly since the author could find very little about the method in English language.

All conventional analogue stereo restitution instruments have changed very little in their conception from the beginning until today. They were all based on the idea of Scheimpflug that the restitution can only be correctly performed if the object bundles are reconstructed. Although this widely adapted idea restricted the development of photogrammetry in other directions it brought the first practical successes in aerial photogrammetry. A great majority of "todays" restitution instruments were designed when aerial cameras had an angle of view of 60 degrees. The mechanical and optical construction of these instruments were made according to the used focal lengths of used cameras and according to acceptable accuracies. When
in the mid 1930's, R. Richter in Jena constructed a camera objective of 80 degrees a new era started. The old instruments could not accommodate wide angle photography and only after the Second World War various companies started the production of stereo instruments for shorter focal lengths. All these instruments were, naturally, built according to the idea of Scheimpflug with the reconstruction of object bundles. However, it was realized that conventional instruments with normal concept could not be further developed. A limit was set at a lens of about 110 degrees. Even for these super wide lenses restitution instruments have to be specially redesigned. When principal distances of cameras used for aerial mapping become shorter than the principal distances of plotting instruments the need for affine restitution arises. In these cases the restitution can be accomplished by different principal distances, giving improper inner orientation.

The idea was conceived in Russia. At first Russian photogrammetrists tried to solve the problem by conventional restitution instruments applying special working methods. This solution could not fulfill the requirements in accuracy and another solution had to be found. For some strange reasons the theoretical ideas and conceptions in Russia were not as rigid as in Western Europe and America. That fact gave Russian photogrammetrists a certain advantage which proved to be very fruitful and resulted in quite a number of universal plotting instruments. Under the term "universal plotting instrument" they refer to the application
of photography made using cameras of various focal lengths. Some of these instruments, for example, the Stereograph SD of Drobyshev and the Stereoprojector SPR-2 of Romanovski are well described in two publications which appeared outside Russia.* The principal distance of an affine restitution instrument \((C_a)\) is larger than the camera constant \((C_k)\), when photography is made with smaller focal length, and therefore the restitution bundle of rays becomes narrower and creates a very desirable effect. The angle between the space rods in plotters with mechanical projection system can be kept within reasonable limits. Naturally, it should not be forgotten that the scale in the z-direction is deformed and the amount of deformation is a function of the ratio of the camera constant and the principal distance of the restitution instrument.

The affine restitution can be also performed with used standard universal plotting instruments. An example with the Stereometrograph – Zeiss, Jena will help to understand the general procedure. The basis of the theory is presented in Figure 2-21. A wide angle lens was used for photography and therefore the principal distance of the restitution instrument is larger than the camera constant. The resulting tilt of the affine model \(\tilde{t}\) also becomes larger than the tilt \(t\) of the original photograph. This fact is obvious from the diagram. \(B_I\) and \(B_{II}\) represent planes of the original photographs, where \(\tilde{B}_I\) and \(\tilde{B}_{II}\) are the elevated positions of photographs used. The corresponding projection centres of the original and affine model are \(O_I, O_{II}\)

*\[4\], [66]
and $\bar{O}_I$, $\bar{O}_{II}$ respectively. The original photographs are elevated in such a way that the rays coming from the affine model coincide with the original positions in a horizontal plane.

Fig. 2-21. Elevated affine model

The derivation of basic geometrical relations can be obtained from Figure 2-22.
The tilt of the elevated photograph depends upon the tilt of the original photograph and an affine factor $k$.

$$\tan \bar{t} = k \tan t$$ \hspace{1cm} (2.83)

The affine factor is defined as the ratio of the vertical scale $m_h$ to the planimetric scale $m_1$.

$$k = \frac{m_h}{m_1}$$ \hspace{1cm} (2.84)

From the last diagram it can be seen that $(ON) = k(ON) = k \frac{C_k}{\cos t}$. The principal distance of the affine photograph will then be
\[ C_a = (O\bar{N}) \cos \bar{t} = k C_k \frac{\cos \bar{t}}{\cos t}, \quad (2.85) \]

where \( C_k \) is the camera constant. From the last expression it is obvious that the value of \( C_a \) depends upon the affine factor and the tilt. From the diagram it is also obvious that the principal point \( H \) of the original photograph is displaced for distance \( d \) from the principal point in the image plane \( \bar{B} \).

\[
d = (\bar{M}N) - (\bar{HN}) \tag{2.86}
\]

\[
(\bar{M}N) = C_a \tan \bar{t} \tag{2.87}
\]

\[
(\bar{HN}) \cos \bar{t} = (HN) \cos t,
\]

or

\[
(\bar{HN}) = (HN) \frac{\cos \bar{t}}{\cos t} = C_k \tan t \frac{\cos \bar{t}}{\cos t} \tag{2.88}
\]

Substituting expressions (2.88) and (2.87) into equation (2.86) and bearing in mind that \( \tan \bar{t} = k \tan t \) and \( C_a = k C_k \frac{\cos \bar{t}}{\cos t} \), the final formula for distance \( d \) is obtained

\[
d = C_k \tan t \left( k^2 \frac{\cos \bar{t}}{\cos t} - \frac{\cos \bar{t}}{\cos t} \right), \quad (2.89)
\]

which can be further simplified for the case when the angle of tilt is a very small quantity.

\[
d = \frac{C_k}{k} \tan \bar{t} \left( k^2 - 1 \right) \tag{2.90}
\]

This decentering of the photograph is in the Stereometrograph automatically accomplished. To simplify the procedure the tilt is divided into two components \( \phi \) and \( \omega \). At first \( \phi \) is introduced
for the decentering $d_x$ of the image holder $d_x$:

$$d_x = \frac{a}{b} \frac{c_k}{k^2 - 1},$$  \hspace{1cm} (2.91)

where the meaning of quantities $a$ and $b$ is obvious from Figure 2-23.

![Diagram](image)

Fig. 2-23

Sliding mechanism $A$ can be shifted along the image holder and along a scale which is subdivided in units of $C$.

The practical procedure consists of the determination of the affine factor $k = \frac{a}{c_k}$ approximately, which is followed by
the decentering of the image holder according to equation (2.91). The decentering is performed manually and by iterations. No considerations are given to the introduced errors of the affine model which result from the difference between equations (2.91) and (2.89), and from a non-linear scale enlargement along the image holder.

This method does not give perfect results. Nobody can expect that from an affine restitution, although the method could be further elaborated. The necessity of the close-range photogrammetry systems may initiate further elaboration. As long as the general accuracy of the output is greater than the errors of the affine restitution the geometrical and mathematical approximations can be considered to be valid.

**ANALYTICAL PLOTTERS**

Analytical plotters use, instead of optical, mechanical or optical-mechanical projection systems, a mathematical projection system, which describes the relationship between points and lines in various coordinate systems. There are basically two coordinate systems which are employed in analytical photogrammetry. They are: the image coordinate system and the object space coordinate system. The mathematical formulae which connect the two systems have already been given and explained in conjunction with the calibration of cameras (see Jacobi's method).

The need for analytical photogrammetry in close-range
photogrammetry systems arises when the accuracy required is too high to be satisfied by an analogue restitution instrument. Various sources of systematic errors cannot be eliminated by a conventional restitution instrument. The Manual of Photogrammetry* states that "the justification for analytical photogrammetry is found in those applications in which the concept of a simple central-perspective projection is no longer adequate to describe the physical characteristics of the record."

Analytical photogrammetry has a general meaning and the same formulae are valid for terrestrial, aerial, ballistic and non-topographic photogrammetry.

Analytical plotters fundamentally consist of two parts: a digital computer and a comparator. The first part is responsible for the numerical elaboration of "the simultaneous restitution of the orientation of any number of photographic records and the reconstruction of the three-dimensional space by the intersection of corresponding rays."* The comparator serves to obtain the coordinates of photograph images. Comparators can be mono or stereo comparators. Monocomparators measure the image coordinates of points identified on a single photograph. Stereocomparators permit simultaneous measurements of identified points on both photographs, but each photograph has an independent and separate coordinate system.

There are quite a number of comparators on the market and their detailed description can be obtained from text books in

* [4]
photogrammetry or directly from manufacturers. There is no need to repeat these descriptions in this thesis.

As far as non-topographic photogrammetry is concerned it is important to emphasize that analytical plotters basically have no limitations as to the type and orientation of a camera, as long as the bundle of rays can be expressed as a mathematical model which characterizes the geometric and dynamic properties of the particular data acquisition system.

The analytical approach is especially advantageous in the most general case of close-range photogrammetry where the elements of interior and exterior orientation as well as the calibration parameters of the cameras are simultaneously determined with the object space coordinates. The general trend in close-range photogrammetry is toward increased application of analytical restitution methods. Solving very sophisticated mathematical models these methods reached extremely high accuracies. It can only be hoped that instrument manufacturers in coming years will be able to produce a small and inexpensive stereocomparator with automatic coordinate registration which will yield sufficient accuracy to satisfy the very high requirements of close-range photogrammetry.
CHAPTER III
APPLICATION OF PHOTOGRAMMETRY IN OPHTHALMOLOGY

INTRODUCTION

The application of close-range photogrammetry in ophthalmology as in all other non-topographic fields has gained very little acceptance. There have been, however, quite a number of attempts to introduce the conveniences of photogrammetry for measuring of inside or outside parts of the eye. In the great majority of cases these attempts remained in the experimental stage. Many scientists recognized a considerable potential for growth of the application of photogrammetry in ophthalmology, but until now photogrammetry has not developed effective means of reaching and communicating with the large number of potential users. Another problem not less significant is the ignorance of people in the medical profession of photogrammetry as a measuring tool for their disciplines. However, in recent years there have been sufficient indications that the situation is undergoing significant changes. On the one hand it may be hoped that very soon photogrammetrists will modify methods and instruments to accommodate the potential users. On the other hand it is hoped that these potential users will obtain an effective education in photogrammetry such that they will be able to use rather unique photogrammetric techniques and instruments.

Some ophthalmological institutions in the world have a very advanced research centres where photogrammetry has became a normal tool. Probably the best example is the Helmholtz
Moscow Research Institute for Eye Diseases. Under the supervision of Dr. L.S. Urmakher this Institute had developed by the early fifties a number of measuring techniques which are now standard in Russia. They even developed a special plotter for visual biomicroscopic measurements of the eye.

The Department of Experimental Ophthalmology at the University Eye Clinic in Lund, Sweden, in conjunction with the Division of Photogrammetry at the Royal Institute of Technology in Stockholm developed several photogrammetric methods for exophthalmometry, for the determination of the pupillary aqueous flow in the living human eye and for measurements of the apparent size of structures in the anterior chamber.

The Ophthalmic Research Institute of Australia with the assistance of the Department of Lands and Survey of the Victoria State Government produced the first stereophotogrammetric analysis of the human fundus oculi in 1969.

Last but not least is the research at the Department of Ophthalmology at the University of British Columbia in evaluation of the cup of the optic nervehead for a study in chronic simple glaucoma.

The eye as an object of research has very specific properties which make almost any measurement by conventional methods extremely difficult. The fundamental problem of measurements is the mobility of the living eye which makes direct measurements
practically impossible regardless of instruments, apparatus and methods that are applied. The additional problem is the heterogeneity of requirements with respect to every individual element of the eye and its pathological changes.

Basically the problems can be classified into three major groups depending on the part of the eye that is measured:

(a) measurements of the front of the eye
(b) measurements of the optical system, and
(c) measurement of the retina

A complete photogrammetric procedure is seldom performed. Very often plotting is unnecessary since sufficient information can be obtained directly by stereoscopic investigation or by spot measurements sometimes relative comparison of two photographs will suffice and absolute measurements are unnecessary. The absolute measurements to a very high degree of accuracy inside the eye are impossible by photogrammetric methods.

Although the last factor largely reduces the applicability of photogrammetry there are still enough problems which can be easily solved by applying the proper photogrammetric approach.

**MEASUREMENTS OF THE FRONT OF THE EYE**

In this group of problems the author includes stereophotogrammetric exophthalmometry, determination of the diameter of the cornea, measurements of tumourous growths upon the eye, investigation and differential diagnostics of internal neoplasms
and measurements of the radius of curvature of the sclera (see Fig. 3-1).

(a) **Stereophotogrammetric exophthalmometry**

Determination of the position of the ocular bulbs is divided into the determination of the position of a single eye or both eyes with respect to the orbits by means of exophthalmometry, and the determination of the position of the eyes relative to each other (pupillar distance).

![Diagram of the normal eye](image_url)

**Fig. 3-1. Schematic representation of the normal eye**

The first problem in clinical praxis normally is solved by means of exophthalmometer, an instrument whose original construction was made over a hundred years ago by Cohn in Breslau.
At present the most commonly used type is Hertel's mirror exophthalmometer, although there are some improved versions, like Davanger's exophthalmometer.

In 1968 Dr. E.O. Backlund from the Department of Neurosurgery, Karolinska Sjukhuset in Stockholm suggested a new stereophotogrammetric exophthalmometry.

The stereophotography of the object was made with a pair of Nikon cameras with frames 24 x 36 mm. They were mounted on a rigid metal base and connected with a stero prism.

Exposed stereophotographs were restituted by various methods depending on the degree of accuracy required. For an extremely high accuracy a stereocomparator was used and by means of analytical solution the systematic errors due to the inaccuracy of calibration parameters were eliminated. For lesser accuracy an analogue approach was applied disregarding the systematic errors of calibration. The cross-section before and after a decompression operation were recorded numerically or graphically with a standard deviation of about 0.5 mm for differences between the sections. According to E.O. Backlund the precision of Hertel's exophthalmometer cannot be compared to that of stereophotogrammetry.

\[ dx \]

Fig. 3-2*

*the cross-section and the description are taken from [7]
"The section shows the circular shape of the eyes, the nose etc. One section (dotted line) is recorded before and the other after a right-sided decompression operation for exophthalmos. The new position of the eyeball and a slight postoperative edema over the nose can be seen."

The conclusion about the method can be taken directly from [68]: "The advantages of a stereophotogrammetric exophthalmometry are the photographic documentation, the accuracy, the irrespective of the examiner and the comfort to the patients compared to Hertel's method. The drawbacks of the method are the tedious procedures of evaluation and the complex instrumentation required for the same procedures."

(b) Measurements of the radius of curvature of the sclera

There are two photogrammetric methods, as far as the author knows, which are at present used to determine the radius of curvature of the sclera of a living eye. The first method of Dencks-Rzymkowski** was developed in Bown, Germany by 1940.

The photographs were taken in a stereocamera with a format of 6 x 13 cm and a focal length of 9 cm. The necessary base of photography was 6.5 cm. To allow the absolute measurements on the restituted model a small millimeter scale is photographed together with the sclera. The stereophotogrammetric evaluations are performed directly on negatives by means of a mirror stereoscope and graphical measurements of parallax difference. Any

*the cross-section and the description are taken from [7]

**[60]
deviation from a regular pattern of contour lines can be easily detected directly on the plotted model.

Fig. 3-3** The horizontal and vertical cross-section of the sclera

The second method was developed at the Helmholtz Moscow Research Institute for Eye Diseases. The radius of curvature of the sclera was determined by means of a stereopheric net. Simultaneously comparing a stereopair of the eye and a stereo net of similar radius the differences between the two surfaces can be easily detected. From the differences in elevation the radii of curvature are computed. To simplify the computation they prepared a special set of tables from which the radii
can be directly obtained from the differences in elevation.

Other problems in the application of photogrammetry for measurements of the front of the eye are very similar from a technical point of view. From a stereopair taken by stereo cameras measurements are made on a restitution instrument. There are sometimes some specific modifications in the approach because of the special character of the problem involved or by the required accuracy. For example, to study a tumourous growth upon the eye the photographs are taken periodically and photogrammetric measurements determine indirectly the rate of change to an accuracy of ±0.02 mm. To achieve this high accuracy in absolute units "... a small brass object consisting of six steps, each of 0.4 mm depth, was photographed stereoscopically and the plotted measurements were compared with those obtained by physical measurement."*

Another specific problem was the photogrammetric determination of the pupillary aqueous flow in the living eye.** The recording camera consisted of a corneal microscope with a multi-slit projector and a movie camera mounted at fixed angles. A very detailed description of the method and the results obtained can be found in [37]. From a photogrammetric point of view it is of no particular value.

* [53]
** [37]
MEASUREMENTS OF THE OPTICAL SYSTEM

Under the optical system the author considers the anterior chamber and the lens of the eye. It is impossible to make any direct measurements on the optical system of the living eye and photogrammetry practically remains as the only method which will provide reasonably good results.

To obtain the elements of the eye's optical system for diagnostic purposes of various forms of glaucoma, the depth of the anterior chamber must be known. Photographic recording is normally performed by a stereo slit camera. From the stereopairs the measurements are then obtained by some restitution instrument. The required quantity is the distance between the vertex of the cornea and the vertex of the front surface of the crystalline lens.

All absolute measurements inside the eye are influenced by the curved surface of cornea. To investigate the amount of systematic error. The Swedish team of S. Henriksson, O. Holm and C.E.T. Krakau* used an optical model which closely simulated the real situation. The model was spherical like the outer corneal surface and its refractive index was the same (n = 4/3) as that of the cornea and the anterior chamber.

To follow a refracted light ray a rectangular cartesian coordinate system with the origin in the centre of the sphere was used. The orientation of the coordinate axes was such that *[33]
the y-axis was parallel to the ray outside the sphere.

Point P on the outside surface of the cornea where the ray path is refracted is known from its coordinates. Instead of using pure rectangular or polar coordinates the authors* used a mixture such that point P was defined by $\theta$, $r$ and $y$, where $r$ is the perpendicular distance of point P from the y-axis of the coordinate system. The refracted ray intersects the xy coordinate plane in point $P'$ defined by coordinates $\theta$, $\xi_0$, 0.

A well known formula for refraction determines a ratio between angles of incidence ($\alpha$), refraction ($\beta$) and refractive index ($n$).

$$n = \frac{\sin \alpha}{\sin \beta} \quad (3.1)$$

*[33]
\[
\sin \alpha = \frac{r}{R} \quad (3.2)
\]
\[
\tan (\alpha - \beta) = \frac{r - \xi_0}{(R^2 - r^2)^{1/2}} \quad (3.3)
\]

Combining equations (3.2) and (3.1) we obtain

\[
\sin \beta = \frac{1}{n} \frac{r}{R} \quad (3.4)
\]

The unknown quantity is value \( \xi_0 \). To compute this value as a function of \( r, R, \) and \( n \) let us write equation (3.3) in the following form

\[
\xi_0 = r - (R^2 - r^2)^{1/2} \tan (\alpha - \beta) \quad (3.5)
\]

Since

\[
\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta} = \frac{\sin \alpha - \sin \beta}{\cos \alpha - \cos \beta} = \frac{\sin \alpha \cos \beta - \cos \alpha \sin \beta}{\cos \alpha \cos \beta + \sin \alpha \sin \beta}
\]

\[
- \frac{\sin \beta \cos \alpha}{\sin \alpha \sin \beta} = \frac{\sin \alpha \sqrt{1 - \sin^2 \beta} - \sin \beta \sqrt{1 - \sin^2 \alpha}}{\sqrt{(1 - \sin^2 \beta)(1 - \sin^2 \alpha)} + \sin \alpha \sin \beta},
\]

and substituting in the last expression the values for \( \sin \alpha \) and \( \sin \beta \) from equations (3.2) and (3.4) we obtain

\[
\tan(\alpha - \beta) = \frac{\frac{r}{R} \sqrt{1 - \frac{r^2}{n^2 R^2} - \frac{r}{n R} \sqrt{1 - \frac{r^2}{R^2}}}}{\sqrt{(1 - \frac{r^2}{n^2 R^2})(1 - \frac{r^2}{R^2}) + \frac{1}{n} \frac{r^2}{R^2}}} = \frac{r \sqrt{n^2 R^2 - r^2} - r^2 - r \sqrt{R^2 - r^2}}{\sqrt{(n^2 R^2 - r^2)(R^2 - r^2)} + r^2} \quad (3.6)
\]

When the last expression for \( \tan (\alpha - \beta) \) is substituted
into equation (3.5) and \( r \) is put over the denominator of (3.6) we obtain

\[
\xi_0 = \frac{r\sqrt{(n^2R^2 - r^2)(R^2 - r^2)} + r^3 - r\sqrt{(n^2R^2 - r^2)(R^2 - r^2)}}{\sqrt{(n^2R^2 - r^2)(R^2 - r^2)} + r^2} + r(R^2 - r^2)
\]

or finally

\[
\xi_0 = \frac{rR^2}{\sqrt{(n^2R^2 - r^2)(R^2 - r^2)} + r^2}
\]  

(3.7)

Instead of taking a single ray let us now consider a bundle of parallel rays outside the eye which are parallel to the \( y \)-axis of the coordinate system. These rays will intersect the sphere at various distances \( d \) from the \( yz \)-plane, where

\[
d = r \cos \theta
\]  

(3.8)

After refraction the same rays will be at decreased distances from the \( yz \) plane at \( y = 0 \) by the amount of \( \zeta \), where

\[
\zeta = (r - \xi_0) \cos \theta
\]  

(3.9)

Planes which contain light rays at distances \( d_1, d_2, d_3, \ldots \) from the \( yz \)-plane are parallel outside the sphere but because of the refraction are not parallel inside the sphere. Therefore the rays which emerge from the anterior chamber to the cornea and later to the camera will diverge by an amount that cannot be neglected. Compared with the distance to the nodal point of the slit camera the size of the anterior chamber is small and therefore the light rays at the nodal point can be considered
Fig. 3-5. Projection of a refracted ray in the xy-plane as being parallel. Denoting the orthogonal projections of $r$ and $\xi$ on the xy-plane as $r'$ and $\xi'$ we obtain

$$r' = r \cos \theta$$
$$\xi' = \xi \cos \theta$$

(3.10)

or

$$\tan \gamma' = \frac{(r - \xi_{o}) \cos \theta}{\sqrt{R^2 - r^2}}$$

An optical section through the anterior chamber is introduced by the XY plane where the Y axis makes an angle $\nu$ with the $y$-axis of the coordinate system of the light rays and the $z$ and $\xi$-axes coincide.
From Figure 3-5 it is obvious that

\[ \frac{PR}{QR} = \frac{PA}{AB} = \frac{PC}{CD}, \]

or

\[ \frac{\sqrt{R^2 - r^2}}{r^\prime - \xi_0} = \frac{\sqrt{R^2 - r^2 - \xi'_1 \cot \nu}}{r^\prime - \xi'_1} = \frac{\sqrt{R^2 - r^2 - \xi'_1 \cot \nu - d \sin(\nu - \gamma')}}{r^\prime - \xi'_2} \cos \gamma' \]

To determine the scale ratio at a point on the sphere and in the plane of the optical section the quotient of the areas is computed. A differential part of the surface area of the sphere is given by the expression

\[ dA = rdr \, d\theta, \quad (3.13) \]

and of the corresponding part of the optical section by

\[ dA' = \frac{\xi_2 \, d\xi_2 \, d\theta}{\cos(90^\circ - \nu)} \]

Thus, the scale ratio becomes

\[ Q = \frac{rdr \, \sin \nu}{\xi_2 \, d\xi_2} \]

The authors* tabulated values for the scale ratio expressing it as a function of \( r \) only where \( r \) is directly measured on photographs. The second variable \( \theta \) was varied in steps of 5 degrees from 0° to 25°, where \( r \) is given for every 0.5 mm and \( d \) for -2 mm, 0, and +2 mm. "When estimating the effect on \( Q \) at a displacement along the \( x \)-axis (i.e. parallel to the iris plane) we have to take the simultaneous change of \( r \) into consideration."*

*[33]
The whole discussion was concerned with displacements of objects inside the anterior chamber. Just how realistic the results obtained are is difficult to say. All derived formulae were based on the assumption that the cornea is a sphere. The author could not find in literature what kind of deviations the cornea can have from the idealized spherical surface nor what the range of the refractive index in the cornea and the anterior chamber is. These questions must be answered before we can refer to measurements inside the living eye as the absolute measurements.

MEASUREMENTS OF THE RETINA

The observations and photography of the retina are made through the pupil. The whole process of observations of parts of the retina is relatively new. In year 1850 the famous German ophthalmologist and physicist Hermann von Helmholtz concluded that, according to the principle of the reversibility of the light path, light will traverse the same route through an optical instrument (the eye can be considered as an optical instrument) from one end to the other. Therefore the light which comes from a luminous body through the pupil to the retina will return in exactly the same way back to the luminous body. If an observer could insert his eye between the source of light and the illuminated retina he would be able to get some of the reflected light. Naturally, this cannot be done without some auxiliary apparatus which will prevent the illuminating light from being intercepted.
Helmholtz solved the problem by inserting three plane-parallel glass plates in the direction of observation under an angle which enabled the light of a luminous body to be reflected inside the living eye. The principle is obvious from Figure 3-6.
Helmholtz also solved the second problem by obtaining a sharp image of retina. The reflected light from the retina is refracted by the eye lens, anterior chamber and cornea and has various ray paths. When the focal point of the dioptric system is in the cornea (as with emmetropic individuals) the light rays leave the eye as parallel rays. When the focal point is farther back (myopic persons) the exit rays converge, and in the case of hyperopic individuals the rays diverge. To eliminate these effects Helmholtz introduced a very weak concave lens which could be moved along the optical axis and therefore was able to bring "in focus" the image of the retina for any living eye. The whole apparatus was called ophthalmoscope.

The discovery of Helmholtz was further developed. Many scientists contributed to various improved designs of ophthalmoscopes. In 1861 Girand-Jeulon succeeded in constructing the first stereoscopic ophthalmoscope. It has since been changed several times and the latest design is the Shepens-Binocularophthalmoscope. Its principle can be obtained from Figure 3-7.

Photography combined with an electronic flash unit made it possible to obtain measurements of the retina. Cameras especially designed for these purposes are called fundus cameras. Such cameras are mostly used for routine fundus photography in hospitals, ophthalmological practice and research centres, but are supplemented by auxiliary accessories and can be used for many other research projects. Among these accessories a stéréo-
Fig. 3-7

separator for successive stereophotography of the ocular fundus is of particular interest. A great majority of ophthalmological institutions in the Western World use the Zeiss-fundus camera with various kinds of stereoseparators. These instruments are all well described in the ophthalmological literature. The author found a very interesting version of the fundus camera designed by the Helmholtz Moscow Research Institute (see Fig. 3-8).

Light rays coming from the observed eye (1) are reflected by two mirrors (2) which make two separate images. The two individual bundles of rays then pass through a plane-parallel plate (3) which corrects eventual distortions of the lens (4). The bundles of rays are then reflected by two symmetrical
Fig. 3-8

mirrors (5) onto film holders (6). The sharp images of the object are obtained by an additional optical system (view-range-finder) which is not shown on the diagram.

Unfortunately fundus photographs in restitution instruments cannot be evaluated in absolute units to a very high degree of accuracy. The magnification of the focussed image in the film plane is due to two factors which depend on the eye optical system and the camera optical system. The magnification is usually expressed by the following formula

$$\beta = \frac{f_c}{f_e},$$

(3.16)

where \(f_c\) is the focal length of the camera and \(f_e\) is the focal length of the observed eye. The second factor is generally known
only approximately." As long as it is impossible to determine
the focal length of the eye with sufficient accuracy, no magnifi-
cation can be given for measuring purposes, and we have to
content ourselves with indicating the angle which a certain
object subtends on the fundus. Thus 1° in the eye is roughly
equivalent to 0.75 mm on the film, or 1 mm on the film to 1°20'
in the eye. This relationship is a function of the refractive
and axial ametropia of the patient's eye."

The stereophotogrammetric measurements of the fundus were
initiated at the Ophthalmic Research Institute of Australia in
1968 to obtain two dimensional or three dimensional quantative
measurements of structures and to time events in the fundus.
The first part of the research project was done using classical
restitution instruments (A-5 and A-8 from Wild) and the output
was in small contour-line representations of the fundus. From
the contour-lines profiles were graphically constructed. The
second section of the project was entirely evaluated on an
analytical plotter. For this purpose they used Nistri-Bendix
analytical plotter based on the concepts of Helava of the
National Research Council of Canada. Dr. G. Crock described
in detail the whole operation. "The operator placed the stereo-
pairs on the film carriage plates and first found the optical
centre of the film by measuring the diameter of the fundus image
on the negative. He then entered the photo base measurement
into the analytical plotter which would automatically compute
the perspective centres. All relief displacements were then

*[51]
radial from that point. Other data such as the camera focal length, the object distance, the camera base and magnification were then entered. Next he chose six points near vessel crossings, well outside the centre of interest. These points were coded and given X.Y.Z. coordinates for model levelling and absolute orientation subsequently by the I.B.M. 7044 computer."

The author finds this description somewhat confusing and ambiguous. In the first instance G. Crock explains nothing about the accuracy of data entered into the analytical plotter. The uncertainty of the magnification is not even mentioned at all. Secondly, he talks about X, Y, Z coordinates of the points near vessel crossings which are used for the absolute orientation. Therefore these coordinates are not model coordinates but coordinates in some absolute coordinate system. There is not a single word about how these coordinates can be determined to a higher degree of accuracy. The author consulted several ophthalmologists but not a single one could tell him what the limits of deviations from the standard eye are. There are, however, several dimensions that are relatively standardized but they certainly cannot insure a very precise absolute orientation, or all measurements obtained by photogrammetric methods of the fundus are much less reliable than can normally be expected from photogrammetry.

The Department of Ophthalmology at the University of British Columbia initiated a similar project to the above Australian

*[12]
research. The main task was to determine whether photogrammetry can be applied to obtain reliable quantitative measurements of the cup of the optic nerve and by evaluation of measurements to determine the rate of progression of glaucomatous damage. A Zeiss fundus camera was used for stereophotography on 35 mm KX 135 Kodachrome film. The camera had fiducial marks attached additionally. The first stereophotographs were taken using the Allen stereoseparator with a base of 2.25 mm. To avoid movements of the fundus as a photographed object, instantaneous stereophotographs were taken by a special twin prism mounted in front of the objective so as to insure a constant base separation. The restitution of photographs was performed on the Wild A8 plotter at the British Columbia Institute of Technology by an experienced photogrammetrist. A detailed description of the method and results can be found in [61]. Again, as in the article by G. Crock it can be found that "absolute orientation was achieved by marking and recording 3 points at retinal bifurcation or arteriovenous crossing on the photographs which could subsequently be used as reference points."* The "absolute orientation" must have some other meaning in this context, because according to "Manual of Photogrammetry," third edition, volume II page 614, the following definition can be found. "The stereoscopic model formed by the completion of relative orientation has an undetermined relationship to the horizontal and vertical datums. The adjustment of the relatively oriented model, to make it conform in scale and in horizontal and vertical position with the datums of the map sheet, is called absolute orientation, *[61]
and generally follows relative orientation."

From the stereoscopic model in the Wild A8, contour lines of the optic nervehead and then four cross-sections directly from the contour lines were drawn. The areas of cross-sections were determined by the method of graphical integration (planimeter).

Saheb, Drance and Nelson in [61] finished with the statistical analysis of achieved results. As factors which influence the accuracy of the final results they took photography, photogrammetric plotting and the construction of the profile and its planimetry. Using the same model the contour-line plotting was repeated several times. In the same manner cross-sections were plotted and areas were determined repeatedly. Naturally, the actual accuracy of the method remained unknown since all repetitions were performed with the same instruments and under the same conditions. Eventual systematic errors influence all results equally and therefore the standard deviations obtained characterize precision and not accuracy of the method. The explanation for the standard deviation of photography is not presented but the author assumes that the value was directly obtained from the manufacturer of the film used. This value is, naturally, influenced by the grain of the film, stability of emulsion holder and the image-side resolving power of the fundus camera.

The research at the University of British Columbia has
Fig. 3-9
Stereogram and plotting of the fundus
shown that "stereophotogrammetry may now be an additional tool in evaluating and following glaucoma patients and might be particularly valuable in following the progress of ocular hypertensives, patients unreliable on visual field testing, and the congenital glaucomas."*

**SUMMARY AND CONCLUSION**

All the described problems in quantitative ophthalmology show that there is undoubtedly a place for photogrammetry as a measuring tool. Unfortunately, they also show that the application of close-range photogrammetry is almost at the very beginning as a recognized measuring device in ophthalmology. The attempts which have been made to solve particular problems were always concerned only about particular problems disregarding a universal solution to all questions. Each particular camera, plotting instrument and method applied was simplified to the very maximum so as to serve the problem tackled. They might have found the best and easiest solution but this solution only proved that photogrammetry can be helpful and can be used. In the great majority of cases the application of photogrammetry remained on in the experimental stage. Without the combined efforts of medical profession and photogrammetrists no success can be achieved. Potential users in medicine must be persuaded by photogrammetrists that many of their problems can be solved much faster, more easily and more accurately. It should be demonstrated to them that photogrammetry includes a very fast

*[61]*
procedure of registering facts by photography, that photogrammetry can also measure and evaluate very fast changing events, that photogrammetry offers a means to measure an object without having to touch it physically, that non-visible light rays can be employed to achieve special effects, and that photogrammetry can measure objects under the microscope. It should not be forgotten that photogrammetric measurements usually require very complicated and expensive instruments for photography and restitution, that to operate economically a project must be of a certain minimum size, and that measurements cannot be directly obtained as in classical procedures.

In order to make full use of photogrammetry, the author suggests that some central photogrammetric station should be established. This central station can then serve many hospitals and medical institutions. Only in this manner can the expenditure of sufficient funds for photogrammetric instruments be justified. At the same time such a central station could carry on extensive research into initiating the production of inexpensive cameras and plotters which can be applied generally and not just for a particular problem. As long as there is only a limited number of problems in clinical practice which are solved by photogrammetry today, the expensive universal plotter is hardly justifiable.

Another problem not less important is the education of the medical profession in photogrammetry. There is quite a number of potential users who have never heard of photogrammetry at all.
This fact might be the major hindrance for a quicker success. Let us hope that the considerable flexibility of close-range photogrammetric systems will be recognized by potential users as well as by photogrammetrists.
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