Unifying classical spin models using a quantum formalism

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Motivation

Completeness
The 4D $\mathbb{Z}_2$ lattice gauge theory is complete

Complexity
Approximating the partition function of some models is BQP-complete

Summary
Motivation
Motivation

~ Classical spin models:

- Classical magnetism
- Spin glasses
- Neural networks
- Econophysics
...
Motivation

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- Classical magnetism
- Spin glasses
- Neural networks
- Econophysics
- ...

Toy models to tackle complex systems

Make a simple microscopic model & study the macroscopic behavior
Motivation

Many different kinds of classical spin models

- Different dimensions, defined on complicated graphs...
- Many-body interactions...
Motivation

Many different kinds of classical spin models

- Different dimensions, defined on complicated graphs...
- Many-body interactions...
- Symmetries:

  **Global**: Ising, Potts ...

  \[ H(s) = -J \sum_{(i,j) \in E} s_i s_j \]

  \[ H(s) \quad \text{global flip} \quad H(s') \]

  **Local**: lattice gauge theories

  \[ H(s) = -J \sum_{(i,j,k,l) \in \partial f} s_i s_j s_k s_l \]

  \[ H(s) \quad \text{local flip} \quad H(s') \]
Motivation

Can one relate all these models?

By studying one model, can one learn something of other models?
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By studying one model, can one learn something of other models?

Yes!

Completeness results:

Models with different features can be mapped onto a single model

\[ Z = \sum_{s} e^{-\beta H(s)} \]
Completeness
A model is ‘complete’

Its partition function can specialize
(by tuning its coupling strengths)
to the partition function of any other classical spin model
Completeness of the 2D Ising

Result:

\[ Z_{2D \text{ Ising with } h(J, J')} = Z_{\text{any classical spin model}}(J) \]

\[ \uparrow \]

Ising, Potts, ...

✓ on an arbitrary graph
✓ \(q\)-level systems, any \(q\)
✓ any many-body int.

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- ✓ on an arbitrary graph
- ✓ \( q \)-level systems, any \( q \)
- ✓ any many-body int.

Completeness with real coupl.

~ Result:

- Ising model:

\[ Z_{3D \text{ Ising}}(J, J') = Z_{\text{Ising, any } G(J)} \]

- Analogous for \( q \)-level systems

Completeness with real coupl.

Result:

- Ising model:

\[ Z_{3D \text{ Ising}}(J, J') = Z_{\text{Ising, any } G(J)} \]

same kind of interactions

- Analogous for \( q \)-level systems
Completeness with real coupl.

Result:

- Ising model:
  - $Z_{3D \text{ Ising}}(J, J') = Z_{\text{Ising}, \text{any } G(J)}$
  - $J'$ real ✓
  - larger

- Analogous for $q$-level systems

$\text{tradeoff between 'completeness power' and real parameters?}$
Completeness of the 4D $\mathbb{Z}_2$ LGT

~ Main result:

$Z_{4D\mathbb{Z}_2 \text{LGT}}(J, J') = Z_{\text{any classical spin model}}(J)$

- Abelian discrete
- $\checkmark$ any dimensions
- $\checkmark$ $q$-level systems, any $q$
- $\checkmark$ any many-body int.

Completeness of the 4D $\mathbb{Z}_2$ LGT

~ Idea of the proof:

all $k$-cliques for $k=1,\ldots,n$ with Ising-type int.

4D $\mathbb{Z}_2$ LGT $\rightarrow$ Superclique

real couplings $J = 0, \infty$
Completeness of the 4D $\mathbb{Z}_2$ LGT

~ Idea of the proof:

- lattice gauge theory

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\textbf{~ Idea of the proof:~}

\begin{itemize}
  \item 4D $\mathbb{Z}_2$ LGT \quad \rightarrow \quad \text{Superclique}
  \item \text{Superclique} \quad \leftrightarrow \quad \text{Any Abelian discrete classical spin model}
\end{itemize}

Real couplings $J = 0, \infty$
Completeness of the 4D $\mathbb{Z}_2$ LGT

Idea of the proof:

4D $\mathbb{Z}_2$ LGT $\rightarrow$ Superclique $\rightarrow$ Any Abelian discrete classical spin model $\rightarrow$ non planar

real couplings $J = 0, \infty$

lattice gauge theory

$\{J_1, \ldots, J_{1234}\}$ $\rightarrow$ Potts... on high dim.

$\{J_1, \ldots, J_{1234}\}$ $\rightarrow$ any many-body int....

$\{J_1, \ldots, J_{1234}\}$ $\rightarrow$ Hamiltonian!
Complements of the 4D $\mathbb{Z}_2$ LGT

~ Quantum formulation of Abelian discrete LGTs

- Hamiltonian $H(s) = -\sum_{f \in F} J_f \cos \left[ \frac{2\pi}{q} (s_1 + \ldots + s_k)_{\text{mod } q} \right]$

Partition function: $Z_G(J) = \sum_s e^{-\beta H(s)}$

| $s \rangle_1 := |(s_a + s_b + s_c + s_d)_{\text{mod } q} \rangle_1$

\[ \begin{align*}
&\text{a} & & \text{b} \\
&\text{d} & & \text{c} \\
&1 & & 2 \\
&4 & & 3
\end{align*} \]
Completeness of the 4D $\mathbb{Z}_2$ LGT

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Partition function: \( Z_G(J) = \sum_{s} e^{-\beta H(s)} \)

- State defined on the faces:

\[
|\psi_G\rangle = \sum_{s} \bigotimes_{f \in F} |(s_1 + \cdots + s_k)_{\text{mod} q}\rangle_f
\]

Product state with coefficients: \( |\alpha(J)\rangle = \bigotimes_{f} |\alpha_f(J_f)\rangle \)

\[
|\alpha_f(J_f)\rangle = \sum_{s_c \in \partial f} e^{\beta J_f \cos \left[ \frac{2\pi}{q} (s_1 + \ldots + s_k) \right]} |s_1 + \ldots + s_k\rangle_f
\]
Completeness of the 4D $\mathbb{Z}_2$ LGT

~ Quantum formulation of Abelian discrete LGTs

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\[ Z_G(J) = \langle \alpha(J) | \psi_G \rangle \]
Completeness of the 4D $\mathbb{Z}_2$ LGT

~ Tools to ‘transform’ the model:

- Merge rule:

  - $J_f = \infty$
  - $s_b = s_a + s_c + s_d$

- Deletion rule:

  - $J_f = 0$

- Fixing the spins using the gauge symmetry:

  - No loops!
Completeness of the 4D $\mathbb{Z}_2$ LGT

~ Construction of the superclique

- Construction of many-body Ising-type int.:
Completeness of the 4D $\mathbb{Z}_2$ LGT

Construction of the superclique

- Transportation in the 4D lattice:

Propagation

Replication

Turns

4th dimension required to avoid loops!
Completeness of the 4D $\mathbb{Z}_2$ LGT

~ Construction of the superclique

- Layout of the superclique:

Superclique: complicated interaction pattern with simple interactions
Completeness of the 4D $\mathbb{Z}_2$ LGT

Hamiltonian of superclique

$\Downarrow$

Hamiltonian of any classical spin model

1. General Hamiltonian on $n$ 2-level systems: different $E(s)$ for each $s$

2. Show that one can invert the system of equations

$$\begin{pmatrix}
1 & (-1)^0 & \ldots & (-1)^{0+0+\ldots+0} \\
1 & (-1)^0 & \ldots & (-1)^{0+0+\ldots+1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & (-1)^{1} & \ldots & (-1)^{1+1+\ldots+1}
\end{pmatrix}
\begin{pmatrix} J \\ J_1 \\ \vdots \\ J_{12\ldots n} \end{pmatrix} =
\begin{pmatrix} E(s = (0,0,\ldots,0)) \\ E(s = (0,0,\ldots,1)) \\ \vdots \\ E(s = (1,1,\ldots,1)) \end{pmatrix}$$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

3. All rows are linearly independent, thus the determinant is non zero

4. $q$-level models: encode each $q$-level system into $\lceil \log_2 q \rceil$ 2-level sys.
Completeness of the 4D $\mathbb{Z}_2$ LGT

~ Note: efficient constructions for specific target models
Example: 2D Ising model: linear overhead

Note: 2D Ising can magnetize, but LGT cannot
Completeness of the 4D $\mathbb{Z}_2$ LGT

We have proven that:

$Z_{4D\mathbb{Z}_2\text{LGT}}(J, J') \cong Z_{\text{any classical spin model}}(J)$

- constructive
- global & local symmetries

- real
- larger

$Z_{4D\mathbb{Z}_2\text{LGT}}(J, J') = Z_{\text{any classical spin model}}(J)$

Target Hamiltonian with $M$ terms and $k$-body int: scaling $\text{poly}(M, 2^k)$

- any dimensions
- $q$-level systems, any $q$
- any many-body int.

Result holds approx for continuous models: let $q \to \infty$
Applications of completeness

~ Symmetries of the states \[ Z_G(J) = \frac{\langle \alpha(J)|S|\varphi_G \rangle}{\langle \alpha(J') \rangle} = Z_G(J') \]

~ Mapping models with poly overhead: infer comput. complexity
  
  e.g. 2D Ising with fields \[ \text{poly larger} \] 4D \[ \mathbb{Z}_2 \text{ LGT} \] 
  
  #P-hard \[ \text{poly larger} \] #P-hard

~ Many different universality classes are mapped to a single model

? They should be reproducible in the phase diagram of the complete model

this includes ‘unexplored’ models
Computational complexity
Mapping partition functions to quantum circuits

**Classical spin model**

- Boltzmann weight of each int.
  \[ w^a = e^{-\beta h^a(s_1, s_2)} \]

- Product of interactions
  \[ \prod_a w^a \]

- Left & Right bound. cond.
  \[ L = (s^L_1, \ldots, s^L_n) \]
  \[ R = (s^R_1, \ldots, s^R_n) \]

**Quantum circuit**

- Quantum gate, e.g.
  \[ W^a_{(12)(12)} = \sum e^{-\beta h^a(s_1, s_2)} |s_1, s_2\rangle\langle s_1, s_2| \]

- Contraction of quantum gates = Circuit \( \mathcal{C} \)

- Output & Input of circuit
  \[ |L\rangle = |s^L_1\rangle \ldots |s^L_n\rangle \]
  \[ |R\rangle = |s^R_1\rangle \ldots |s^R_n\rangle \]

- \( Z^{L,R} = \langle L | \mathcal{C} | R \rangle \)

- \( \square PBC \quad Z = \text{Tr} \mathcal{C} \)
- \( \checkmark \) other geometries
Mapping for vertex models

- Particles at the edges
- Interaction at vertex $a$

$$w^a(s) = \sum e^{-\beta h^a(s_is_js ks)}$$

Two-qudit gate

$$W^a_{(ij)(kl)} = \sum e^{-\beta h^a(s_is_js ks)} |s_i, s_j\rangle \langle s_k s_l|$$

**Z**$_{vm}^{L,R}$ = $\langle L|C|R\rangle$
Mapping partition functions to quantum circuits

~ Mapping for edge models

- Particles at the vertices
- Int. at edge in time dir.
  \[ w^{ij} = e^{-\beta h(s_i, s_j)} \]
- Int. at edge in space dir.
  \[ w^{jk} = e^{-\beta h(s_j, s_k)} \]

Single qudit gate
\[ w(i)(j) = \sum e^{-\beta h(s_i, s_j)} |s_i\rangle \langle s_j| \]

Two qudit diagonal gate
\[ w(jk)(jk) = \sum e^{-\beta h(s_j, s_k)} |s_j s_k\rangle \langle s_j s_k| \]

\( Z_{em}^{L,R} = \langle L|C|R \rangle \)
Mapping partition functions to quantum circuits

Mapping for lattice gauge theories

- Particles at the edges
- Int. at face in time dir.
  \[ w^{ij} = e^{-\beta h(s_i,s_j)} \]
- Int. at face in space dir.
  \[ w^{jk} = e^{-\beta h(s_j,s_k)} \]

&

Fixing the temporal gauge

Single qudit gate

\[ w(i)(j) = \sum e^{-\beta h(s_i,s_j)} |s_i\rangle \langle s_j| \]

Four qudit diagonal gate

\[ w(jk)(jk) = \sum e^{-\beta h(s_j,s_k)} |s_j s_k\rangle \langle s_j s_k| \]

\[ Z_{\text{LGT}}^{L,R} = \langle L|C|R \rangle \]

[Diagram showing mappings and quantum gates]
II BQP-completeness results

Main idea:

Model \rightarrow Gates corresp. to that model \rightarrow Show that they form a universal gate set

\[ Z = \langle L|C|R \rangle \]

Approximating that partition function is as hard as simulating arbitrary quantum computation
BQP-completeness results

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**BQP-completeness results**

**Main idea:**
- Model
- Gates correspond to that model
- Show that they form a universal gate set

$$Z = \langle L|C|R \rangle$$

Approximating that partition function is as hard as simulating arbitrary quantum computation

- Prove BQP-completeness of computing $Z$
- Provide a quantum algorithm
\section{BQP-completeness results}

\textbf{Six vertex model}

- (Encoded) universal interaction $U = e^{itH_{\text{ex}}}$ with $H_{\text{ex}} = \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z$

- Encoding $|0\rangle = \frac{1}{2} (|01\rangle - |10\rangle)^\otimes 2$

- Preparation of $|0\rangle|0\rangle \ldots |0\rangle$ from $|R\rangle = |0\rangle|1\rangle|0\rangle|1\rangle \ldots$ possible

- The exchange int. is achieved with the six-vertex model-type gate:

$$W_{(ij)(jk)} = \begin{bmatrix} e^{i2t} & \cos(2t) & i \sin(2t) \\ i \sin(2t) & \cos(2t) & e^{i2t} \end{bmatrix}$$
Ⅱ BQP-completeness results

~ Six vertex model

- (Encoded) universal interaction $U = e^{itH_{ex}}$ with $H_{ex} = \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z$
- Encoding $|0\rangle = \frac{1}{2} (|01\rangle - |10\rangle) \otimes 2$
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    e^{2it} & \cos(2t) & i\sin(2t) \\
    \cos(2t) & i\sin(2t) & \cos(2t) \\
    i\sin(2t) & \cos(2t) & e^{2it}
\end{bmatrix}$$

Approximating the partition function of the six vertex model on a certain complex parameter regime is BQP-complete
## BQP-completeness results

### Potts model

- **Encoding:** \( |0\rangle = |0\rangle |1\rangle \)
  \[ |1\rangle = |1\rangle |2\rangle \]
- **Trivial preparation of** \( |0\rangle \ldots |0\rangle \) **from** \( |R\rangle = |0\rangle |1\rangle \ldots |0\rangle |1\rangle \)
- Each Potts gate is characterized by the pair \( (e^{\beta J_{ii}}, e^{\beta J_{ij}}) \)
- Construct an (encoded) universal gate set:

![Diagram showing single qubit identity, phase gate, two qubit identity, and controlled phase gate](image)

II BQP-completeness results

Example of part of a circuit:

Note distribution of physical and auxiliary qubits
Approximating the partition function of a 2D 3-level Potts with auxiliary qubits on a certain complex parameter regime is BQP-complete
BQP-completeness results

3D $\mathbb{Z}_2$ LGT

- **Encoding:**
  - $|0\rangle = |0\rangle|0\rangle|0\rangle|0\rangle$
  - $|1\rangle = |1\rangle|1\rangle|1\rangle|1\rangle$

- **Trivial preparation of** $|0\rangle \ldots |0\rangle$ **from** $|R\rangle = |0\rangle \ldots |0\rangle$

- **Each** $\mathbb{Z}_2$ **LGT-type gate is characterized by the pair** $(e^{\beta J_{ii}}, e^{\beta J_{i\neq j}})$

- **Construct an (encoded) universal gate set:**
BQP-completeness results

3D $\mathbb{Z}_2$ LGT

Hadamard gate:

Note: many spins fixed by the gauge
BQP-completeness results

3D \( \mathbb{Z}_2 \) LGT

Verify that there are no loops of spins fixed by the gauge:

Two–qubit gate

Single–qubit gate

Two–qubit gate

\( e^{i\varphi \sigma_z \otimes \sigma_z} \)

\( R_z(\alpha + \frac{\pi}{2}) \)

\( R_z(\frac{\pi}{2})H \)

\( R_z(\beta + \frac{\pi}{2}) \)

\( R_z(\frac{\pi}{2})H \)

\( R_z(\gamma) \)

\( e^{i\varphi \sigma_z \otimes \sigma_z} \)

1

2

3

4

5

6

7

8

9

10

11

12

13

\( \otimes \)

all time steps
Ⅱ BQP-completeness results

3D $\mathbb{Z}_2$ LGT

Verify that there are no loops of spins fixed by the gauge:

Approximating the partition function of a 3D $\mathbb{Z}_2$ LGT on a certain complex parameter regime is BQP-complete.
Summary

~ Completeness

\[
Z = \langle \alpha | \psi \rangle
\]

\[
Z_{4DZ_2 LGT}(J, J') = Z_{\text{any classical spin model}}(J)
\]

\[
\begin{align*}
\uparrow & \quad \text{real} \\
\uparrow & \quad \text{Abelian discrete} \\
\checkmark & \quad \text{any dimensions} \\
\checkmark & \quad q\text{-level systems, any } q \\
\checkmark & \quad \text{any many-body int.}
\end{align*}
\]

~ Complexity

\[
Z = \langle L | C | R \rangle
\]

Approximating \( Z \) of Six vertex model Potts model

3D \( \mathbb{Z}_2 \) LGT

is BQP-complete in a certain complex parameter regime
Thank you for your attention!


