Fault-tolerant quantum computation with cluster states

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Our setting

- 2D/3D: Nearest-neighbor translation-invariant interaction.
- High fault-tolerance threshold
Our setting
Fault-tolerant quantum computation

Task:

- Maintain the quantum speedup in the presence of decoherence.

Solution:

Fault-tolerance theorem*: If for a universal quantum computer the noise per elementary operation is below a constant non-zero error threshold $\epsilon$ then arbitrarily long quantum computations can be performed efficiently with arbitrary accuracy.

Talk outline

1. Universal cluster state computation.
   - The scheme: computation by local measurements
   - Cluster states: creation, definition, experiment

2. General introduction to quantum error-correction

3. Making cluster state computation fault-tolerant
Part I:
Cluster state quantum computation
1.1 Cluster state quantum computation

- Universal computational resource: cluster state.
- Information written onto the cluster, processed and read out by one-qubit measurements only.

1.2 Cluster states - creation

1. Prepare product state $\bigotimes_{a \in \mathcal{C}} \frac{|0\rangle_a + |1\rangle_a}{\sqrt{2}}$ on $d$-dimensional qubit lattice $\mathcal{C}$.

2. Apply the Ising interaction for a fixed time $T$:

$$U_{\text{Ising}} = e^{-i \frac{gT}{\hbar} \sum_{\langle i, j \rangle} \sigma_z^{(i)} \sigma_z^{(j)}}$$

with

$$\frac{gT}{\hbar} = \frac{\pi}{4}.$$

- Interaction time $T$ independent of cluster size.
1.2 Cluster states - simple examples

|ψ⟩_2 = |0⟩_1|+⟩_2 + |1⟩_1|−⟩_2

Bell state

|ψ⟩_3 = |+⟩_1|0⟩_2|+⟩_3 + |−⟩_1|1⟩_2|−⟩_3

GHZ-state

|ψ⟩_4 = |0⟩_1|+⟩_2|0⟩_3|+⟩_4 + |0⟩_1|−⟩_2|1⟩_3|−⟩_4 + |1⟩_1|−⟩_2|0⟩_3|+⟩_4 + |1⟩_1|+⟩_2|1⟩_3|−⟩_4

Number of terms exponential in number of qubits!
1.2 Cluster states - definition

A cluster state $|\phi\rangle_C$ on a cluster $C$ is the single common eigenstate of the stabilizer operators $\{K_a\}$,

$$K_a|\phi\rangle_C = |\phi\rangle_C, \forall a \in C,$$

with

$$K_a = X_a \bigotimes_{b \in N(a)} Z_b, \quad \forall a \in C. \quad (1)$$

Therein, $b \in N(a)$ if $a,b$ are spatial next neighbors in $C$. 
1.2 Cluster states - experiment

Cold atoms in optical lattices [1,2]

The $QC_C$ with photons [3].

Part II:
Introduction to quantum error correction

... take a break from cluster states
2.1 Quantum vs. classical bits

\[ |\Psi\rangle = \alpha |0\rangle + \beta |1\rangle \]

<table>
<thead>
<tr>
<th>quantum bit</th>
<th>classical bit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Measurement affects state</td>
<td>Mmnt does not affect state</td>
</tr>
<tr>
<td>Set of states continuous</td>
<td>Set of states discrete</td>
</tr>
</tbody>
</table>

Despite the differences:

- **Quantum error-correction (QEC) is possible.**
- **QEC is based on classical error correction.**
2.2 Starting point: Classical EC

An example: the repetition code.

<table>
<thead>
<tr>
<th>bit</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>encoded state (0)</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>erroneous states</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>encoded state (1)</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Individually read out all bits & Perform majority vote

- Procedure on \(n\)-bit code corrects \(\lfloor \frac{n-1}{2} \rfloor\) errors.
- Error-correction procedure learns encoded state.
### 2.2 Starting point: Classical EC

Same effect without state measurement: Read out parities only.

<table>
<thead>
<tr>
<th>bit</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>encoded state 0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>erroneous states</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>encoded state 1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

- Syndrome only reveals error, not encoded state:
  \[
  Sy(c) = 0, \ \forall \ \text{codewords } c.
  \]

\[
Sy(E \oplus c) \equiv Sy(E). \quad (2)
\]

*Learning the state is not crucial for classical error-correction.*


2.3 How Quantum Error Correction works

Classical-to-quantum dictionary:

\[ c \in \{000, 111\} \rightarrow |\Psi\rangle = \alpha|000\rangle + \beta|111\rangle \]

Errors: bit flip \[ \rightarrow \] spin & phase flips \( \sigma_x, \sigma_y, \sigma_z \)

Parity check matrix

\[
\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 1
\end{pmatrix}
\]

\[ \rightarrow \] stabilizer operators

\( Z_1 \otimes Z_2, Z_2 \otimes Z_3 \)

Syndrome \[ \rightarrow \] Measured eigenvalues of stabilizer operators.
2.3 How Quantum Error Correction works

- Repeated measurement of the stabilizer operators, and conditional correction.

- Correctable errors *anti-commute* with at least one stabilizer operator → error-syndrome.

- Syndrome informs about an error, not the encoded state.
Emergence of the error threshold

Fault-tolerance theorem: For a universal quantum computer, an error per gate $< \epsilon$ is effectively as good as zero error.
So far...

• Have explained the basics of quantum error-correction.

• Have ignored:
  
  – Errors introduced by error-correction itself.

  – Computation.

... but that can be fixed
Part III:
Fault-tolerant quantum computation with 3D cluster states
Part III outline

3.1 Topological quantum error-correction with 3D cluster states

3.2 Topological quantum gates

3.3 Fault-tolerance threshold, overhead scaling, mapping to 2D
### Known threshold values

<table>
<thead>
<tr>
<th></th>
<th>geometric constraint</th>
<th>2D</th>
<th>1D</th>
</tr>
</thead>
<tbody>
<tr>
<td>no constraint</td>
<td></td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>[1] 0.03, est.</td>
<td></td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>[2] $10^{-3}$, est.</td>
<td></td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>[3] $10^{-4}$, est.</td>
<td></td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>[4] $10^{-5}$, bd.</td>
<td></td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>[5] $7 \cdot 10^{-3}$, est.</td>
<td></td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>[6] $2 \cdot 10^{-5}$, bd.</td>
<td></td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>[7] $10^{-8}$, bd.</td>
<td></td>
<td>10</td>
<td>10</td>
</tr>
</tbody>
</table>

- Error sources:
  - $|\oplus\rangle$-Preparation, $\Lambda(Z)$-gates, Hadamard gates, measurement.

Main idea

3D cluster state = fault-tolerant substrate

Gates from non-trivial cluster topology
Three cluster regions:

\( V \) (Vacuum), \( D \) (Defect) and \( S \) (Singular qubits).

- Qubits \( q \in V \): local \( X \)-measurements,
- Qubits \( q \in D \): local \( Z \)-measurements,
- Qubits \( q \in S \): local measurements of \( X \pm Y \).

**Microscopic view**

Cluster edges

Elementary cell of $\mathcal{L}$

- **qubit location**: (even, odd, odd) - face of $\mathcal{L}$,
- **qubit location**: (odd, odd, even) - edge of $\mathcal{L}$,
- **syndrome location**: (odd, odd, odd) - cube of $\mathcal{L}$,
- **syndrome location**: (even, even, even) - site of $\mathcal{L}$. 
Lattice duality $\mathcal{L} \leftrightarrow \overline{\mathcal{L}}$

Translation by vector $(1, 1, 1)^T$:
- Cluster $\mathcal{C}$ invariant,
- $\mathcal{L}$ (primal) $\rightarrow$ $\overline{\mathcal{L}}$ (dual).

\begin{align*}
&\text{face of } \mathcal{L} \rightarrow \text{edge of } \overline{\mathcal{L}}, \\
&\text{edge of } \mathcal{L} \rightarrow \text{face of } \overline{\mathcal{L}}, \\
&\text{site of } \mathcal{L} \rightarrow \text{cube of } \overline{\mathcal{L}}, \\
&\text{cube of } \mathcal{L} \rightarrow \text{site of } \overline{\mathcal{L}},
\end{align*}

Many objects in this scheme exist as ‘primal’ and ‘dual’.
Part III.1:
Quantum Error-correction in 3D cluster states
3.1 Measuring the cluster state stabilizer

$K_1 = X_1Z_2Z_3Z_4Z_5$

But ...
3.1 Measuring the cluster state stabilizer

\[ K_1 = X_1 Z_2 Z_3 Z_4 Z_5 \]

\[
\begin{align*}
\lambda_{X,1} & \quad \lambda_{Z,2} & \quad \lambda_{Z,3} & \quad \lambda_{Z,4} & \quad \lambda_{Z,5} = +1. \\
\pm 1 & \quad \pm 1 & \quad \pm 1 & \quad \pm 1 & \quad \pm 1
\end{align*}
\]

Measure eigenvalue of \( K_1 \) by local measurements on qubits 1 - 5.

But ...
3.1 Measuring the cluster state stabilizer

But ... all measurements in cluster region $V$ are in the $X$-basis.

Are there stabilizer elements that we can measure by local $X$-measurements only?

Criterion:

$$K_J = \bigotimes_{a \in J} X_a.$$
3.1 Measuring the cluster state stabilizer

Criterion: 

\[ K_J = \bigotimes_{a \in J} X_a. \]

Such stabilizer elements exist!

Example:

\[ X_1 X_2 X_3 X_4 X_5 X_6 = K_1 K_2 K_3 K_4 K_5 K_6 \]

Correlation of measured eigenvalues:

\[ \lambda_{X,1} \lambda_{X,2} \lambda_{X,3} \lambda_{X,4} \lambda_{X,5} \lambda_{X,6} = +1, \text{ if no error.} \]

\[ \pm 1 \pm 1 \pm 1 \pm 1 \pm 1 \pm 1 \]
3.1 Measuring the cluster state stabilizer

\[ \lambda_{X,1}\lambda_{X,2}\lambda_{X,3}\lambda_{X,4}\lambda_{X,5}\lambda_{X,6} = -1 \] indicates an error.

- One bit of error syndrome per lattice cell.
3.1 Measuring the cluster state stabilizer

- Each error leaves characteristic signature in the syndrome.
- Identify error by that syndrome.

Z-error on face qubit yields non-trivial syndrome on adjacent cells.
3.1 Geometry and topology

Error syndrome supported on closed surface located on dual edge multiple edges = chain
3.1 Geometry and topology

- An error chain $Z(\overline{e})$ is detected by a syndrome $\text{Sy}(f)$ if $e$ and $f$ intersect an even number of times.
- Intersection number is a topological invariant.
3.1 Geometry and topology

- Homologically equivalent error chains have the same effect on the computation:

\[ \overline{e}_2 = \overline{e}_1 + \partial \overline{f} \rightarrow Z(\overline{e}_2) \equiv Z(\overline{e}_1). \]

- Only need to identify the homology class of the error.
3.1 Topological error-correction

- Topological error-correction in 3D cluster states described by *Random plaquette $Z_2$-gauge model* (RPGM) [1].

- FT quantum memory with toric code described by RPGM as well [1].

3.1 Phase diagram of the RPGM

Map error correction to statistical mechanics:

- Have an error budget of 3%.

Part III.2:
Topological quantum gates
3.2 Encoded quantum gates

- Local $Z$-measurements remove the qubits in region $D$ from the cluster.
- Remaining cluster has non-trivial topology.
3.2 Encoded quantum gates

Surface perpendicular to “time” supports a quantum code
3.2 Surface codes

- Storage capacity of the code depends upon the topology of the code surface.
3.2 The surface code

Surface codes are stabilizer codes associated with 2D lattices. Only the homology class of an error chain matters.

\[ |\psi\rangle = A_s |\psi\rangle = B_p |\psi\rangle, \ \forall |\psi\rangle \in \mathcal{H}_C, \forall s, p. \]  

(4)

- Surface codes are stabilizer codes associated with 2D lattices.
- Only the homology class of an error chain matters.

3.2 The surface code

Non-correctable error: small weight-distance away from non-trivial cycle.
3.2 Surface code on plane with holes

- There are two types of holes: primal and dual.
- A pair of same-type holes constitutes a qubit.
3.2 Surface code on plane with holes

Surface code with boundary:

- $X$-chain cannot end in primal hole, can end in dual hole.
- $Z$-chain can end in primal hole, cannot end in dual hole.
3.2 Encoded quantum gates

Defect $D$ = worldline of hole.
3.2 Encoded quantum gates

Topological quantum gates are encoded in the way worldlines of primal and dual holes are braided.
### 3.2 A CNOT-gate

- Propagation relation: $X_c \rightarrow X_c \otimes X_t$.

- Remaining prop rel $Z_c \rightarrow Z_c$, $X_t \rightarrow X_t$, $Z_t \rightarrow Z_c \otimes Z_t$ for CNOT derived analogously.
3.2 Topological quantum gates

control

CNOT

target

Out

Z-prep.

Out

Z-meas.

In

X-prep.

Out

X-meas.

In
3.2 Universal gate set

- Need one non-Clifford element: fault-tolerant preparation of \( |A\rangle := \frac{X+Y}{\sqrt{2}} |A\rangle \).

- FT prep. of \( |A\rangle \) provided through realization of \textit{magic state distillation}\(^*\).

Part III.3:
Threshold and Overhead Scaling
3.3 Sequential cluster state creation

How large or small can the depth \( d \) be?
3.3 Sequential cluster state creation

How large?

Limitation:

- Temporal order of measurements $\rightarrow$ accumulating memory error.
3.3 Sequential cluster state creation

How small?

- $d \geq 1$.
  - $d = 1$: mapping to circuit model, $d \geq 2$: cluster state.
### 3.3 Error model

- **Locality:** Errors are associated with the elementary gates of a quantum computer. Errors act where the gates act.

- **Independence:** Errors associated with different gates are stochastically independent.

- **Probabilistic error-model:** The elementary errors are probabilistic Pauli-flips $\sigma_x, \sigma_y, \sigma_z$ on all qubits.

All these assumptions can be relaxed.

3.3 Error Model

- Error sources:

<table>
<thead>
<tr>
<th>2D ((d = 1))</th>
<th>3D ((d \geq 2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. (</td>
<td>+\rangle)-preparation</td>
</tr>
<tr>
<td>2. cPhase-gates</td>
<td>2. cPhase-gates</td>
</tr>
<tr>
<td>3. Hadamard-gates</td>
<td>3. Memory error</td>
</tr>
<tr>
<td>4. Local measurement</td>
<td>4. Local measurement</td>
</tr>
</tbody>
</table>

- Every quantum operation has same error \(p\).

- Instant classical processing.
3.3 Fault-tolerance threshold

Topological threshold in cluster region $V$:

\[ p_c = 7.5 \times 10^{-3} \ (2D), \]
\[ p_c = 6.8 \times 10^{-3} \ (3D). \]  \hfill (5)

Purification threshold for fault-tolerant $|A\rangle$-preparation:

\[ p_c = 3.7 \times 10^{-2}. \]  \hfill (6)

Topological EC sets the overall threshold.
3.3 Fault-tolerance threshold

Numerical estimate of the fault-tolerance threshold in 2D.
3.3 Overhead and Robustness

- Denote by $S$ ($S'$) the bare (encoded) size of a quantum circuit. Then, for the described method:

$$S' \sim S \log^3 S.$$  \hspace{1cm} (7)

- The threshold is robust against variations in the error model such as higher weight elementary errors, long-distance errors.
3.3 Overhead in absolute terms

Operational cost of a fault-tolerant gate, at 1/3 threshold.
Summary

- The 3D cluster state has error-correction built-in
- Encoded gates by topology
- High error threshold of 0.7%

Reading: