Graph Theory in Quantum Information

Lecture 1

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Finite field $\mathbb{F}$ $|\mathbb{F}| = q$

Affine plane over $\mathbb{F}$

(x,y) $x,y \in \mathbb{F}$

pt

$[a,b]$

lines

$(x,y) \in [a,b]$ $\Leftrightarrow$ $y = ax+b$

(x, y) —— $[a, b]$

(x, y) is on $[a, b]$

$(0, \mathbb{F})$ $[0, b]$

$(1, \mathbb{F})$ $[1, b]$

$\vdots$ $\vdots$

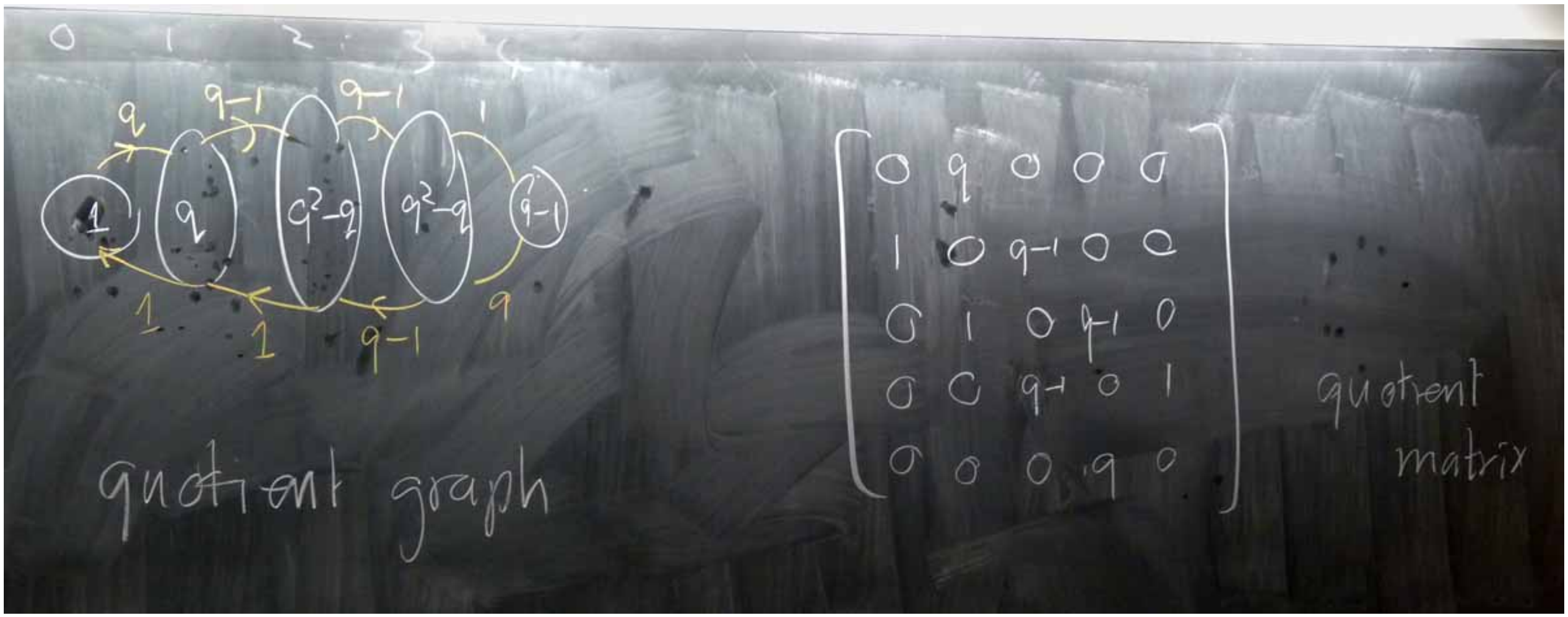
$([illegible], \mathbb{F})$

bipartite

regular of valency q

diameter is ~~three~~ four

antipodal


0 1 2 3 4
q
q−1
q−1
1
1
q
q²−q
q²−q
q−1
1
1
q−1
q
quotient graph
0 q 0 0 0
1 0 q−1 0 0
0 1 0 q−1 0
0 0 q−1 0 1
0 0 0 q 0
quotient matrix

Hadamard matrices

$H^T H = nI$ $\quad$ H is $n \times n$, ± 1

$H = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ $\quad$ $\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$ $\quad$ $H \otimes K$

Had $\to$ Graph on $4n$ vertices

$$\begin{array}{ll} a_1^+ & b_1^+ \\ a_1^- & b_1^{-1} \\ \vdots & \vdots \\ a_n^+ & b_n^+ \\ a_n^- & b_n^{-1} \end{array}$$

$$\left.\begin{array}{l} a_i^+ \text{ --- } b_j^+ \\ a_i^- \text{ --- } b_j^{-1} \end{array}\right\} \Leftrightarrow H_{i,j} = 1$$

$$\begin{array}{ll} a_i^+ & b_j^+ \\ a_i^- & b_j^- \end{array}$$

bipartite

regular of valency n

diameter is four

antipodal

Lines in $\mathbb{R}^d$ & $\mathbb{C}^d$

lines $l_1, \dots l_m$

represent l_i by a unit vector that spans it.

$\rightarrow x_1, \dots, x_m$

If x_i spans ℓ_i, let

$$P_i = x_i x_i^* = x_i x_i^\dagger = |x_i\rangle\langle x_i|$$

Angles

$$\langle x_i, x_j\rangle\langle x_j, x_i\rangle = |\langle x_i, x_j\rangle|^2.$$

$$\begin{aligned}
&\mathrm{tr}(P_i P_j) \\
&= \mathrm{tr}(x_i x_i^* x_j x_j^*) \\
&= \mathrm{tr}(x_j^* x_i\, x_i^* x_j) \\
&= \langle x_j, x_i\rangle\langle x_i, x_j\rangle
\end{aligned}$$

$$
\begin{aligned}
\|P_i - P_j\|^2 &= \langle P_i - P_j, P_i - P_j \rangle \\
&= \langle P_i, P_i \rangle + \langle P_j, P_j \rangle - \langle P_i, P_j \rangle - \langle P_j, P_i \rangle \\
&= 2 - 2 \langle P_i, P_j \rangle
\end{aligned}
$$

SIC-povms

- In $\mathbb{C}^d$ an equiangular set of lines has size at most d^2

  If equality holds, we get SIC-POVMs

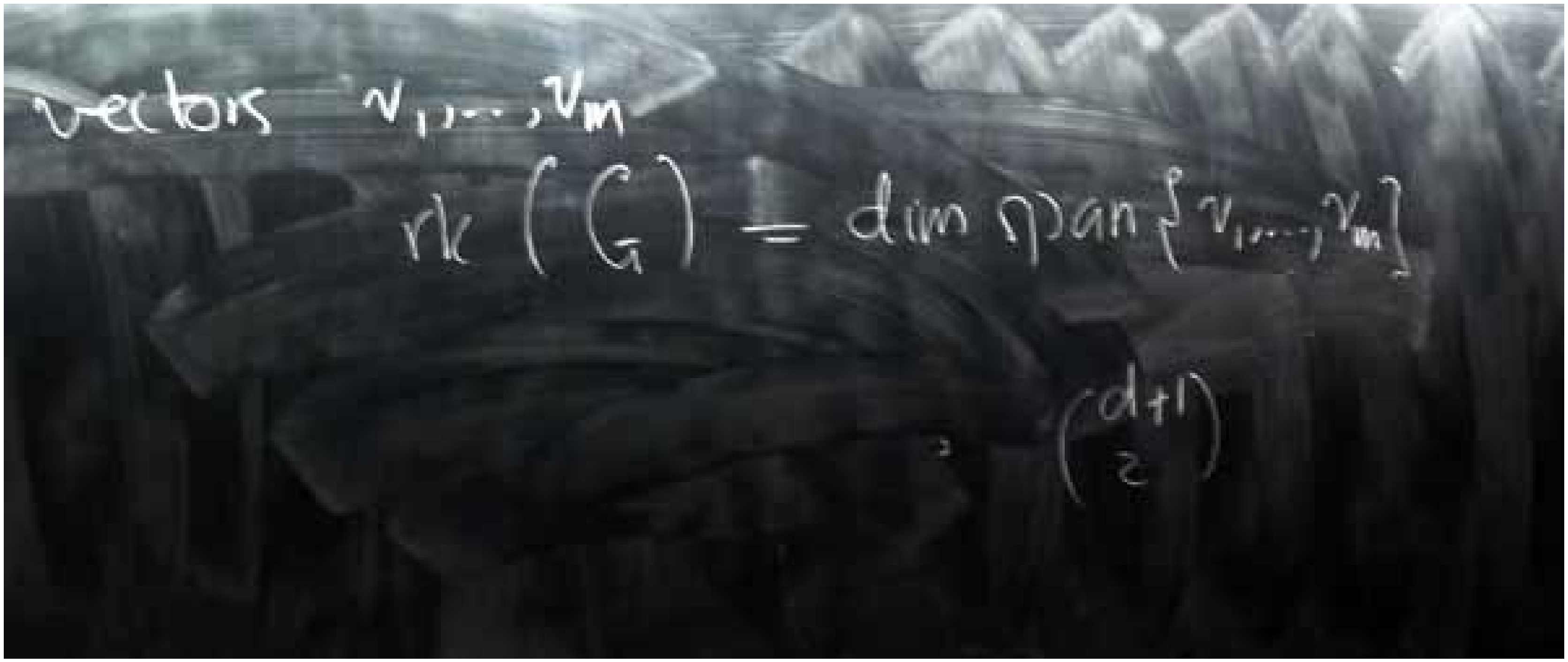
Mutually unbiased bases (mubs) $d+1$

Basic tools

$$G = \underbrace{\left(\langle P_i, P_j \rangle \right)}_{m \times m} \qquad \text{Gram matrix}$$

SIC.povms

$$G = \begin{bmatrix} 1 & & \alpha^2 \\ & \ddots & \\ \alpha^2 & & 1 \end{bmatrix} \qquad \alpha^2 < 1 \qquad (1-\alpha^2) I + \begin{bmatrix} \alpha^2 \end{bmatrix}$$


vectors $v_1, \dots, v_m$
$rk(G) = \dim \operatorname{span}\{v_1, \dots, v_m\}$
$\binom{d+1}{2}$