The traditional notion of noncontextuality in quantum theory
Traditional notion of noncontextuality

A given vector may appear in many different measurements

\[ |\psi_1\rangle \quad |\psi_2\rangle \quad |\psi_3\rangle \]

\[ \chi_1(\lambda) \quad \chi_2(\lambda) \quad \chi_3(\lambda) \]

\[ |\psi'_1\rangle \quad |\psi'_2\rangle \quad |\psi'_3\rangle \]

\[ \chi_1'(\lambda) \quad \chi_2'(\lambda) \quad \chi_3'(\lambda) \]

The traditional notion of noncontextuality:
Every vector is associated with the same \( \chi(\lambda) \) regardless of how it is measured (i.e. the context)
The traditional notion of noncontextuality:
For every $\lambda$, every basis of vectors receives a 0-1 valuation, wherein exactly one element is assigned the value 1 (corresponding to the outcome that would occur for $\lambda$), and every vector is assigned the same value regardless of the basis it is considered a part (i.e. the context).
The traditional notion of noncontextuality:
For every $\lambda$, every basis of vectors receives a 0-1 valuation, wherein exactly one element is assigned the value 1 (corresponding to the outcome that would occur for $\lambda$), and every vector is assigned the same value regardless of the basis it is considered a part (i.e. the context).
Bell-Kochen-Specker theorem: A noncontextual hidden variable model of quantum theory for Hilbert spaces of dimension 3 or greater is impossible.
Example: The CEGA algebraic 18 ray proof in 4d:

Each of the 18 rays appears twice in the following list

\[
\begin{align*}
0,0,0,1 & 0,0,0,1 & 1,-1,1,-1 & 1,-1,1,-1 & 0,0,1,0 & 1,-1,-1,1 & 1,1,-1,1 & 1,1,-1,1 & 1,1,1,-1 \\
0,0,1,0 & 0,1,0,0 & 1,-1,-1,1 & 1,1,1,1 & 0,1,0,0 & 1,1,1,1 & 1,1,1,-1 & -1,1,1,1 & -1,1,1,1 \\
1,1,0,0 & 1,0,1,0 & 1,1,0,0 & 1,0,-1,0 & 1,0,0,1 & 1,0,0,-1 & 1,-1,0,0 & 1,0,1,0 & 1,0,0,1 \\
1,-1,0,0 & 1,0,-1,0 & 0,0,1,1 & 0,1,0,-1 & 1,0,0,-1 & 0,1,-1,0 & 0,0,1,1 & 0,1,0,-1 & 0,1,-1,0 \\
\end{align*}
\]

In each of the 9 columns, one ray is assigned 1, the other three 0
Therefore, 9 rays must be assigned 1

But each ray appears twice and so there must be an even number of rays assigned 1

CONTRADICTION!
Example: The CEGA algebraic 18 ray proof in 4d:

Each of the 18 rays appears twice in the following list:

\[
\begin{align*}
0,0,0,1 & 0,0,0,1 & 1,-1,1,-1 & 1,-1,1,-1 & 0,0,1,0 & 1,-1,-1,1 & 1,1,-1,1 & 1,1,-1,1 & 1,1,1,-1 \\
0,0,1,0 & 0,1,0,0 & 1,-1,-1,1 & 1,1,1,1 & 0,1,0,0 & 1,1,1,1 & 1,1,1,-1 & -1,1,1,1 & -1,1,1,1 \\
1,1,0,0 & 1,0,1,0 & 1,1,0,0 & 1,0,-1,0 & 1,0,0,1 & 1,0,0,-1 & 1,-1,0,0 & 1,0,1,0 & 1,0,0,1 \\
1,-1,0,0 & 1,0,-1,0 & 0,0,1,1 & 0,1,0,-1 & 1,0,0,-1 & 0,1,-1,0 & 0,0,1,1 & 0,1,0,-1 & 0,1,-1,0
\end{align*}
\]

In each of the 9 columns, one ray is assigned 1, the other three 0.
Therefore, 9 rays must be assigned 1.
But each ray appears twice.

CONTRADICTION!
Example: Kochen and Specker’s original algebraic 117 ray proof in 3d
Example: Clifton’s state-specific 8 ray proof in 3d

\[ |\psi\rangle \quad \longrightarrow \quad \chi_{|\psi\rangle}(\lambda) = 1 \]
\[ |\psi\rangle \quad \longrightarrow \quad \chi_{|\psi\rangle}(\lambda) = 0 \]

\[ |1\rangle + |2\rangle + |3\rangle \]
\[ |1\rangle + |2\rangle - |3\rangle \]

CONTRADICTION!
The traditional notion of noncontextuality:
For every $\lambda$, every projector $P$ is assigned a value 0 or 1 regardless of how it is measured (i.e. the context)

$$v(P) = 0 \text{ or } 1 \quad \text{for all } P$$

Every measurement has some outcome
$$v(I) = 1$$

Coarse-graining of a measurement implies a coarse-graining of the value (because it is just post-processing)
$$v(\sum_k P_k) = \sum_k v(P_k)$$
Example: Bell’s proof in 3d based on Gleason’s theorem

Consider a function on projectors $P \mapsto \omega(P)$, satisfying:
1) $0 \leq \omega(P) \leq 1$ for all $P$
2) $\omega(I) = 1$
3) $\omega(\sum_k P_k) = \sum_k \omega(P_k)$

**Gleason’s theorem:** For $\dim(\mathcal{H}) \geq 3$,

$$\omega(P) = \text{Tr}(\rho P)$$

where $\rho$ is a density operator ($\rho \geq 0$, $\text{Tr}(\rho) = 1$).

But there is no $\rho$ such that $\omega(P) = 0$ or $1$ for all $P$
(Any given $\rho$ can only achieve a 0-1 valuation on a single basis)

**CONTRADICTION**
The traditional notion of noncontextuality:

For Hermitian operators A, B, C satisfying

\[ [A, B] = 0 \quad [A, C] = 0 \quad [B, C] \neq 0 \]

the value assigned to A should be independent of whether it is measured together with B or together with C (i.e. the context)

Measure A = measure projectors onto eigenspaces of A, \( \{P_a\} \)

Measure A with B
= measure projectors onto joint eigenspaces of A and B, \( \{P_{ab}\} \)
then coarse-grain over B outcome \( P_a = \sum_b P_{ab} \)

Measure A with C
= measure projectors onto joint eigenspaces of A and C, \( \{P_{ac}\} \)
Then coarse-grain over C outcome \( P_a = \sum_b P_{ac} \)

\( v(P_a) \) is independent of context

Therefore \( v(A) \) is independent of context
Functional relationships among commuting Hermitian operators must be respected by their values

If \( f(L, M, N, \ldots) = 0 \)

then \( f(v(L), v(M), v(N), \ldots) = 0 \)

Proof: the possible sets of eigenvalues one can simultaneously assign to \( L, M, N, \ldots \) are specified by their joint eigenstates. By acting the first equation on each of the joint eigenstates, we get the second.
Example: Mermin’s magic square proof in 4d

\[
\begin{array}{ccc|c}
X_1 & X_2 & X_1X_2 & I \\
Y_2 & Y_1 & Y_1Y_2 & I \\
X_1Y_2 & Y_1X_2 & Z_1Z_2 & I \\
I & I & i & I \\
\end{array}
\]

\[
\begin{align*}
v(X_1) v(X_2) v(X_1X_2) &= 1 \\
v(Y_1) v(Y_2) v(Y_1Y_2) &= 1 \\
v(X_1Y_2) v(Y_1X_2) v(Z_1Z_2) &= 1 \\
v(X_1) v(Y_2) v(X_1Y_2) &= 1 \\
v(Y_1) v(X_2) v(Y_1X_2) &= 1 \\
v(X_1X_2) v(Y_1Y_2) v(Z_1Z_2) &= -1 \\
\end{align*}
\]

Product of LHSs = +1
Product of RHSs = -1

CONTRADICTION
Aside: Local determinism is an instance of traditional noncontextuality where the context is remote

\[ S^A_A - I^B \] is either measured with \[ I^A - S^B_b \]
or with \[ I^A - S^B_{b'} \]

Recall traditional noncontextuality:

For Hermitian operators A, B, C satisfying

\[ [A, B] = 0 \quad [A, C] = 0 \quad [B, C] \neq 0 \]

the value assigned to A should be independent of whether it is measured together with B or together with C (i.e. the context)

Therefore \( v(S^A_a) \) is the same for the two contexts

This is local determinism

Every proof of the impossibility of a locally deterministic model is a proof of the impossibility of a traditional noncontextual model
Aside: Traditional noncontextuality can sometimes be justified by local causality

\[ j\psi_{AB} = \frac{1}{2} \sum_{i=1}^{4} j\bar{i}_A j\bar{i}_B \]

Perfect correlation when same mmt is made on both wings + local causality
\[ \rightarrow \text{Traditional noncontextual hidden variable model for mmts on one wing} \]

CONTRADICTION!
The generalized notion of noncontextuality
Problems with the traditional definition of noncontextuality:
- applies only to sharp measurements
- applies only to deterministic hidden variable models
- applies only to models of quantum theory

A better notion of noncontextuality would determine
- whether any given theory admits a noncontextual model
- whether any given experimental data can be explained by a noncontextual model
A realist model of an operational theory

\[ p(k|P, M) = \int d\lambda \, \xi_{M,k}(\lambda) \, \mu_P(\lambda) \]
Generalized definition of noncontextuality:

A realist model of an operational theory is noncontextual if

Operational equivalence of two experimental procedures \[\rightarrow\] Equivalent representations in the realist model
Operational equivalence classes

$P$ is equivalent to $P'$ if

$\forall M \forall k : \quad p(k|P, M) = p(k|P', M)$
Difference of Equivalence class
Example from quantum theory

Different density op’s
Example from quantum theory

\[
\frac{1}{2} I = \frac{1}{2} |0\rangle \langle 0| + \frac{1}{2} |1\rangle \langle 1|
\]

\[
\frac{1}{2} I = \frac{1}{2} |+\rangle \langle +| + \frac{1}{2} |\rangle \langle -|
\]
Example from quantum theory

\[ \frac{1}{2} I = \text{Tr}_B \left[ \frac{1}{\sqrt{2}} (|0\rangle \langle 0| + |1\rangle \langle 1|) \right] \]

\[ \frac{1}{2} I = \text{Tr}_B \left[ \frac{1}{\sqrt{2}} (|0\rangle \langle +| + |1\rangle \langle -|) \right] \]
Preparation noncontextual model

$\mu(\lambda)$
Preparation contextual model

\[ \mu_{P_4}(\lambda) \]

\[ \mu_{P_1}(\lambda) \]
Definition of preparation noncontextual model:

\[ \forall M : p(k|P, M) = p(k|P', M) \]

\[ \rightarrow p(\lambda|P) = p(\lambda|P') \]
(a) Some states of a qubit

(b) A preparation noncontextual model of these (RWS, PRA 75, 032110, 2007)

(c) A preparation contextual model of these (Kochen-Specker, 1967)
Difference of context
Example from quantum theory
Example from quantum theory
Example from quantum theory
Example from quantum theory

\[
E = \frac{n}{4} i h^j + (1 - q) \frac{1}{2} I
\]

\[
E = \frac{1}{2} j \sigma j + \frac{1}{2} j + i h + j
\]
Measurement noncontextual model
Measurement contextual model

\[ \xi_{M,1}(\lambda) \]
\[ \xi_{M,2}(\lambda) \]
universal noncontextuality
= noncontextuality for preparations and measurements
Generalized noncontextuality in quantum theory
Defining noncontextuality in quantum theory

**Preparation Noncontextuality in QT**

If $P, P' \rightarrow \rho$ then $\mu_P(\lambda) = \mu_{P'}(\lambda) = \mu_\rho(\lambda)$
Defining noncontextuality in quantum theory

Measurement Noncontextuality in QT

if \( M, M' \rightarrow \{ E_k \} \) then \( \xi_{M,k}(\lambda) = \xi_{M',k}(\lambda) = \xi_{E_k}(\lambda) \)
Preparation-based proof of contextuality

(i.e. of the impossibility of a noncontextual realist model of quantum theory)
Important features of realist models

Let $P \leftrightarrow \mu(\lambda)$
$P' \leftrightarrow \mu'(\lambda)$

Representing one-shot distinguishability:
If $P$ and $P'$ are distinguishable with certainty
then $\mu(\lambda) \mu'(\lambda) = 0$

Representing convex combination:
If $P'' = P$ with prob. $p$ and $P'$ with prob. $1 - p$
Then $\mu''(\lambda) = p \mu(\lambda) + (1 - p) \mu'(\lambda)$
Proof based on finite construction in 2d

\[ P_a \leftrightarrow \psi_a = (1, 0) \]
\[ P_A \leftrightarrow \psi_A = (0, 1) \]
\[ P_b \leftrightarrow \psi_b = (1/2, \sqrt{3}/2) \]
\[ P_B \leftrightarrow \psi_B = (\sqrt{3}/2, -1/2) \]
\[ P_c \leftrightarrow \psi_c = (1/2, -\sqrt{3}/2) \]
\[ P_C \leftrightarrow \psi_C = (\sqrt{3}/2, 1/2) \]
Proof based on finite construction in 2d

\[
\begin{align*}
P_a &\leftrightarrow \sigma_a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\
P_A &\leftrightarrow \sigma_A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\
P_b &\leftrightarrow \sigma_b = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \sqrt{3} \\ \frac{1}{4} \sqrt{3} & \frac{3}{4} \end{pmatrix} \\
P_B &\leftrightarrow \sigma_B = \begin{pmatrix} \frac{3}{4} & -\frac{1}{4} \sqrt{3} \\ -\frac{1}{4} \sqrt{3} & \frac{1}{4} \end{pmatrix} \\
P_c &\leftrightarrow \sigma_c = \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} \sqrt{3} \\ -\frac{1}{4} \sqrt{3} & \frac{3}{4} \end{pmatrix} \\
P_C &\leftrightarrow \sigma_C = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \sqrt{3} \\ \frac{1}{4} \sqrt{3} & \frac{1}{4} \end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
\sigma_a \sigma_A &= 0 \\
\sigma_b \sigma_B &= 0 \\
\sigma_c \sigma_C &= 0
\end{align*}
\]

\(P_a\) and \(P_A\) are distinguishable with certainty
\(P_b\) and \(P_B\) are distinguishable with certainty
\(P_c\) and \(P_C\) are distinguishable with certainty

\[
\begin{align*}
\mu_a(\lambda) \mu_A(\lambda) &= 0 \\
\mu_b(\lambda) \mu_B(\lambda) &= 0 \\
\mu_c(\lambda) \mu_C(\lambda) &= 0
\end{align*}
\]
\[ P_{aA} \equiv P_a \text{ and } P_A \text{ with prob. } 1/2 \text{ each} \]
\[ P_{bB} \equiv P_b \text{ and } P_B \text{ with prob. } 1/2 \text{ each} \]
\[ P_{cC} \equiv P_c \text{ and } P_C \text{ with prob. } 1/2 \text{ each} \]
\[ P_{abc} \equiv P_a, P_b, \text{ and } P_c \text{ with prob. } 1/3 \text{ each} \]
\[ P_{ABC} \equiv P_A, P_B, \text{ and } P_C \text{ with prob. } 1/3 \text{ each} \]

\[ \begin{align*}
\mu_{aA}(\lambda) &= \frac{1}{2} \mu_a(\lambda) + \frac{1}{2} \mu_A(\lambda) \\
\mu_{bB}(\lambda) &= \frac{1}{2} \mu_b(\lambda) + \frac{1}{2} \mu_B(\lambda) \\
\mu_{cC}(\lambda) &= \frac{1}{2} \mu_c(\lambda) + \frac{1}{2} \mu_C(\lambda) \\
\mu_{abc}(\lambda) &= \frac{1}{3} \mu_a(\lambda) + \frac{1}{3} \mu_b(\lambda) + \frac{1}{3} \mu_c(\lambda) \\
\mu_{ABC}(\lambda) &= \frac{1}{3} \mu_A(\lambda) + \frac{1}{3} \mu_B(\lambda) + \frac{1}{3} \mu_C(\lambda)
\end{align*} \]
\[
I/2 = \frac{1}{2}\sigma_a + \frac{1}{2}\sigma_A
\]
\[
= \frac{1}{2}\sigma_b + \frac{1}{2}\sigma_B
\]
\[
= \frac{1}{2}\sigma_c + \frac{1}{2}\sigma_C
\]
\[
= \frac{1}{3}\sigma_a + \frac{1}{3}\sigma_b + \frac{1}{3}\sigma_c
\]
\[
= \frac{1}{3}\sigma_A + \frac{1}{3}\sigma_B + \frac{1}{3}\sigma_C.
\]

\[
\mathcal{P}_{aA} \cong \mathcal{P}_{bB} \cong \mathcal{P}_{cC}
\]
\[
\cong \mathcal{P}_{abc} \cong \mathcal{P}_{ABC}
\]

By preparation noncontextuality

\[
\mu_{aA}(\lambda) = \mu_{bB}(\lambda) = \mu_{cC}(\lambda)
\]
\[
= \mu_{abc}(\lambda) = \mu_{ABC}(\lambda)
\]
\[
\equiv \nu(\lambda)
\]

\[
\nu(\lambda) = \frac{1}{2}\mu_a(\lambda) + \frac{1}{2}\mu_A(\lambda)
\]
\[
= \frac{1}{2}\mu_b(\lambda) + \frac{1}{2}\mu_B(\lambda)
\]
\[
= \frac{1}{2}\mu_c(\lambda) + \frac{1}{2}\mu_C(\lambda)
\]
\[
= \frac{1}{3}\mu_a(\lambda) + \frac{1}{3}\mu_b(\lambda) + \frac{1}{3}\mu_c(\lambda)
\]
\[
= \frac{1}{3}\mu_A(\lambda) + \frac{1}{3}\mu_B(\lambda) + \frac{1}{3}\mu_C(\lambda).
\]
Our task is to find 
\( \mu_a(\lambda), \mu_A(\lambda), \mu_b(\lambda), \mu_B(\lambda), \mu_c(\lambda), \mu_C(\lambda), \) and \( \nu(\lambda) \) such that

\[
\begin{align*}
\mu_a(\lambda) \mu_A(\lambda) &= 0 \\
\mu_b(\lambda) \mu_B(\lambda) &= 0 \\
\mu_c(\lambda) \mu_C(\lambda) &= 0
\end{align*}
\]

i.e., paralleling the quantum structure:

\[
\begin{align*}
\sigma_a \sigma_A &= 0 \\
\sigma_b \sigma_B &= 0 \\
\sigma_c \sigma_C &= 0
\end{align*}
\]

\[
\begin{align*}
\nu(\lambda) &= \frac{1}{2} \mu_a(\lambda) + \frac{1}{2} \mu_A(\lambda) \\
&= \frac{1}{2} \mu_b(\lambda) + \frac{1}{2} \mu_B(\lambda) \\
&= \frac{1}{2} \mu_c(\lambda) + \frac{1}{2} \mu_C(\lambda) \\
&= \frac{1}{3} \mu_a(\lambda) + \frac{1}{3} \mu_b(\lambda) + \frac{1}{3} \mu_c(\lambda) \\
&= \frac{1}{3} \mu_A(\lambda) + \frac{1}{3} \mu_B(\lambda) + \frac{1}{3} \mu_C(\lambda).
\end{align*}
\]

\[
\begin{align*}
I/2 &= \frac{1}{2} \sigma_a + \frac{1}{2} \sigma_A \\
&= \frac{1}{2} \sigma_b + \frac{1}{2} \sigma_B \\
&= \frac{1}{2} \sigma_c + \frac{1}{2} \sigma_C \\
&= \frac{1}{3} \sigma_a + \frac{1}{3} \sigma_b + \frac{1}{3} \sigma_c \\
&= \frac{1}{3} \sigma_A + \frac{1}{3} \sigma_B + \frac{1}{3} \sigma_C.
\end{align*}
\]
Our task is to find \( \mu_a(\lambda), \mu_A(\lambda), \mu_b(\lambda), \mu_B(\lambda), \mu_c(\lambda), \mu_C(\lambda) \), and \( \nu(\lambda) \) such that

\[
\begin{align*}
\mu_a(\lambda) \mu_A(\lambda) &= 0 \\
\mu_b(\lambda) \mu_B(\lambda) &= 0 \\
\mu_c(\lambda) \mu_C(\lambda) &= 0
\end{align*}
\]

Consider \( \lambda' \) such that \( \nu(\lambda') \neq 0 \)

From decompositions (1)-(3)

\[
\begin{align*}
\mu_a(\lambda') &= 0 \text{ or } 2\nu(\lambda') \\
\mu_b(\lambda') &= 0 \text{ or } 2\nu(\lambda') \\
\mu_c(\lambda') &= 0 \text{ or } 2\nu(\lambda')
\end{align*}
\]

But then the RHS of decomposition (4) is

\[
0, \frac{2}{3} \nu(\lambda'), \frac{4}{3} \nu(\lambda'), 2\nu(\lambda') \neq \nu(\lambda')
\]

CONTRADICTION
Example: A “reverse” Gleason theorem for all dimensions

Consider a function on density operators $\rho \mapsto f(\rho)$, satisfying:

1) $0 \leq f(\rho) \leq 1$ for all $\rho$

2) $f(\sum_k w_k \rho_k) = \sum_k w_k f(\rho_k)$ where $0 \leq w_k \leq 1$ and $\sum_k w_k = 1$.

The ”reverse” Gleason’s theorem:

$$f(\rho) = Tr (E \rho)$$

for some effect $E$ (i.e. $0 \leq E \leq I$).
Suppose $\rho \leftrightarrow \mu_\rho(\lambda) \quad \text{preparation noncontextuality}$

$\mu_\rho(\lambda) \geq 0$

$\mu_\rho(\lambda)$ is convex-linear in $\rho$

$\boxed{\mu_\rho(\lambda) = \text{Tr}(\rho E_\lambda) \quad \text{for some effect} \ E_\lambda}$

Recall: If $\rho_1 \rho_2 = 0$, then $\mu_{\rho_1}(\lambda) \mu_{\rho_2}(\lambda) = 0$

If one knew $\lambda$, one could retrodict with certainty which state was prepared from an orthogonal basis, for any basis. There is no effect such that finding it would allow one to achieve such a retrodiction.

CONTRADICTION
Aside: justifying preparation noncontextuality by local causality

\[
I/2 \quad = \quad \frac{1}{2}\sigma_a + \frac{1}{2}\sigma_A \\
= \quad \frac{1}{2}\sigma_b + \frac{1}{2}\sigma_B \\
= \quad \frac{1}{2}\sigma_c + \frac{1}{2}\sigma_C \\
= \quad \frac{1}{3}\sigma_a + \frac{1}{3}\sigma_b + \frac{1}{3}\sigma_c \\
= \quad \frac{1}{3}\sigma_A + \frac{1}{3}\sigma_B + \frac{1}{3}\sigma_C. 
\]

By preparation noncontextuality

\[
\nu(\lambda) \quad = \quad \frac{1}{2}\mu_a(\lambda) + \frac{1}{2}\mu_A(\lambda) \\
= \quad \frac{1}{2}\mu_b(\lambda) + \frac{1}{2}\mu_B(\lambda) \\
= \quad \frac{1}{2}\mu_c(\lambda) + \frac{1}{2}\mu_C(\lambda) \\
= \quad \frac{1}{3}\mu_a(\lambda) + \frac{1}{3}\mu_b(\lambda) + \frac{1}{3}\mu_c(\lambda) \\
= \quad \frac{1}{3}\mu_A(\lambda) + \frac{1}{3}\mu_B(\lambda) + \frac{1}{3}\mu_C(\lambda). 
\]

PNC for \( I/2 \) can be justified by local causality

But PNC for \( \sigma_x \) cannot be justified by local causality
Also,

Any bipartite Bell-type proof of nonlocality $\rightarrow$ proof of preparation contextuality

(proof due to Jon Barrett)
Measurement contextuality

New definition versus traditional definition
How to formulate the traditional notion of noncontextuality:
This is equivalent to assuming:

\[ \psi_1 \rangle \langle \psi_1, I - |\psi_1\rangle\langle \psi_1| \]

\[ \psi_2 \rangle \text{ and } |\psi_3\rangle \text{ measure coarse-grain } \]

\[ \psi_1' \rangle \langle \psi_1, I - |\psi_1\rangle\langle \psi_1| \]

\[ \psi_2' \rangle \text{ and } |\psi_3'\rangle \text{ measure coarse-grain } \]
But recall that the most general representation was

\[ \{ P_k \} \rightarrow M \]

\[ \xi_{P_1}(\lambda) \quad \xi_{P_2}(\lambda) \quad \xi_{P_3}(\lambda) \]

Therefore:

traditional notion of noncontextuality = revised notion of noncontextuality for sharp measurements

and
outcome determinism for sharp measurements
So, the new definition of noncontextuality is not simply a generalization of the traditional notion.

For sharp measurements, it is a revision of the traditional notion.
Local determinism:  
We ask: Does the outcome depend on space-like separated events (in addition to local settings and \( \lambda \))?

Local causality:  
We ask: Does the probability of the outcome depend on space-like separated events (in addition to local settings and \( \lambda \))?

__________________________

Traditional notion of measurement noncontextuality:  
We ask: Does the outcome depend on the measurement context (in addition to the observable and \( \lambda \))?

The revised notion of measurement noncontextuality:  
We ask: Does the probability of the outcome depend on the measurement context (in addition to the observable and \( \lambda \))?

Noncontextuality and determinism are separate issues
No-go theorems for previous notion are not necessarily no-go theorems for the new notion!

In face of contradiction, could give up ODSM
However, one can prove that

![Diagram showing preparation, noncontextuality, and outcome determinism for sharp measurements]

\[ \mu_{I/3}(\lambda) = \frac{1}{3} \mu_{\psi_1}(\lambda) + \frac{1}{3} \mu_{\psi_2}(\lambda) + \frac{1}{3} \mu_{\psi_3}(\lambda) \]

\[ \mu_{I/3}(\lambda) = p \mu_\psi(\lambda) + \ldots \]
We’ve established that preparation noncontextuality implies outcome determinism for sharp measurements.

Therefore:

measurement noncontextuality and preparation noncontextuality imply measurement noncontextuality and outcome determinism for sharp measurements.
We’ve established that

preparation noncontextuality → outcome determinism for sharp measurements

Therefore:

measurement noncontextuality
and
preparation noncontextuality

→

Traditional notion of noncontextuality

no-go theorems for the traditional notion of noncontextuality can be salvaged as no-go theorems for the generalized notion

... and there are many new proofs
Measurement-based proof of contextuality

(i.e. of the impossibility of a noncontextual realist model of quantum theory)
Proof of contextuality for unsharp measurements in 2d

By definition

- \( M_a \leftrightarrow \{ \Pi_a, \Pi_A \} \)
- \( M_b \leftrightarrow \{ \Pi_b, \Pi_B \} \)
- \( M_c \leftrightarrow \{ \Pi_c, \Pi_C \} \)

\( \Pi_x \) projects onto \( \psi_x \)

- \( \Pi_a + \Pi_A = I \)
- \( \Pi_b + \Pi_B = I \)
- \( \Pi_c + \Pi_C = I \)

- \( \Pi_a \Pi_A = 0 \)
- \( \Pi_b \Pi_B = 0 \)
- \( \Pi_c \Pi_C = 0 \)

\( \Pi_A \)

- \( M_a \leftrightarrow \{ \chi_a(\lambda), \chi_A(\lambda) \} \)
- \( M_b \leftrightarrow \{ \chi_b(\lambda), \chi_B(\lambda) \} \)
- \( M_c \leftrightarrow \{ \chi_c(\lambda), \chi_C(\lambda) \} \)

By definition

- \( \chi_a(\lambda) + \chi_A(\lambda) = 1 \)
- \( \chi_b(\lambda) + \chi_B(\lambda) = 1 \)
- \( \chi_c(\lambda) + \chi_C(\lambda) = 1 \)

By outcome determinism for sharp measurements

- \( \chi_a(\lambda)\chi_A(\lambda) = 0 \)
- \( \chi_b(\lambda)\chi_B(\lambda) = 0 \)
- \( \chi_c(\lambda)\chi_C(\lambda) = 0 \)

Thus, \( \{ \chi_x(\lambda), \chi_X(\lambda) \} \) = \{0, 1\} or \{1, 0\} for every \( \lambda \).
M ≡ implement one of Mₐ, Mₐ and Mₙ with prob. 1/3 each, register only whether first or second outcome occurred

\[
\begin{align*}
M & \leftrightarrow \left\{ \frac{1}{3} \Pi_a + \frac{1}{3} \Pi_b + \frac{1}{3} \Pi_c, \frac{1}{3} \Pi_A + \frac{1}{3} \Pi_B + \frac{1}{3} \Pi_C \right\} = \left\{ \frac{1}{2} I, \frac{1}{2} I \right\} \\
\tilde{M} & \leftrightarrow \left\{ \frac{1}{3} \chi_a(\lambda) + \frac{1}{3} \chi_b(\lambda) + \frac{1}{3} \chi_c(\lambda), \frac{1}{3} \chi_A(\lambda) + \frac{1}{3} \chi_B(\lambda) + \frac{1}{3} \chi_C(\lambda) \right\}
\end{align*}
\]

\(\tilde{M} \equiv\) ignore the system, flip a fair coin

\[
\begin{align*}
\tilde{M} & \leftrightarrow \left\{ \frac{1}{2} I, \frac{1}{2} I \right\} \\
\tilde{M} & \leftrightarrow \left\{ \frac{1}{2}, \frac{1}{2} \right\}
\end{align*}
\]

By the assumption of measurement noncontextuality

\[
\begin{align*}
M \simeq \tilde{M} & \rightarrow \left\{ \frac{1}{3} \chi_a + \frac{1}{3} \chi_b + \frac{1}{3} \chi_c, \frac{1}{3} \chi_A + \frac{1}{3} \chi_B + \frac{1}{3} \chi_C \right\} = \left\{ \frac{1}{2}, \frac{1}{2} \right\}
\end{align*}
\]

But \(\{0, 1\}, \{\frac{1}{3}, \frac{2}{3}\}, \{1, 0\}, \{\frac{2}{3}, \frac{1}{3}\} \neq \left\{ \frac{1}{2}, \frac{1}{2} \right\}\)

CONTRADICTION
Example: A variant of Busch’s generalization of Gleason to 2d


Consider a function on effects
$E \mapsto \omega(E)$, satisfying:
1) $0 \leq \omega(E) \leq 1$ for all $E$
2) $\omega(I) = 1$
3) $\omega(\sum_k w_k E_k) = \sum_k w_k \omega(E_k)$

**Generalized Gleason’s theorem:** For all $\text{dim}(\mathcal{H})$,

$$\omega(E) = Tr(\rho E')$$

where $\rho$ is a density operator
($\rho \geq 0$, $Tr(\rho) = 1$).
$E \rightarrow \xi_E(\lambda)$  Measurement noncontextuality

$\xi_E(\lambda)$ is convex linear in $E$

$\xi_E(\lambda)$ considered as a function of $E$ satisfies the conditions of the generalized Gleason’s theorem

\[
\xi_E(\lambda) = \text{Tr}(\rho_\lambda E)
\]

for some density operator $\rho_\lambda$

By outcome determinism for sharp measurements

$\xi_P(\lambda) = 0$ or $1$  for all projectors $P$

But there is no $\rho$ such that $\text{Tr}(\rho P)=0$ or $1$ for all $P$

(Any given $\rho$ can only achieve a 0-1 valuation on a single basis)

CONTRADICTION
The mystery of contextuality

There is a tension between

1) the dependence of representation on certain details of the experimental procedure

and

2) the independence of outcome statistics on those details of the experimental procedure
Noncontextuality and the characterization of classicality
Classicality as non-negativity

Continuous Wigner function for a harmonic oscillator

Common slogan:
A quantum state is nonclassical if it has a negative Wigner representation

|0⟩

Better to ask whether a quantum experiment admits of a classical explanation

|4⟩

Negativity is not necessary for nonclassicality: the nonclassicality could reveal itself in the negativity of the representation of the measurement rather than the state

|α⟩ + | − α⟩

Negativity is not sufficient for nonclassicality: When considering possibilities for a classical explanation, we need to look at representations other than that of Wigner

From: qis.ucalgary.ca/quantech/wiggalery.html
Quasi-probability representations of QM:

States

\[ \rho \leftrightarrow \mu_\rho(\lambda) \]

\[ \mu_\rho : \Lambda \rightarrow \mathbb{R} \]

\[ \int \mu_\rho(\lambda) d\lambda = 1 \]

Measurements

\[ \{ E_k \} \leftrightarrow \{ \xi_{E_k}(\lambda) \} \]

\[ \xi_{E_k} : \Lambda \rightarrow \mathbb{R} \]

\[ \sum_k \xi_{E_k}(\lambda) = 1 \]

\[ \text{Tr}[\rho E_k] = \int d\lambda \, \mu_\rho(\lambda) \, \xi_{E_k}(\lambda) \]

Examples:

- Wigner representation
- discrete Wigner representation  
  (e.g. Wootters, quant-ph/0306135)
- Q representation of quantum optics
- P representation of quantum optics
- Hardy-type formulation of QM using fiducial measurements
- Hardy-type formulation of QM using fiducial preparations
- ...

Quasi-probability representations of QM:

States

\[ \rho \leftrightarrow \mu_\rho(\lambda) \]

\[ \mu_\rho : \Lambda \rightarrow \mathbb{R} \]

\[ \int \mu_\rho(\lambda) d\lambda = 1 \]

Measurements

\[ \{E_k\} \leftrightarrow \{\xi_{E_k}(\lambda)\} \]

\[ \xi_{E_k} : \Lambda \rightarrow \mathbb{R} \]

\[ \sum_k \xi_{E_k}(\lambda) = 1 \]

\[ \text{Tr}[\rho E_k] = \int d\lambda \mu_\rho(\lambda) \xi_{E_k}(\lambda) \]

This provides a classical explanation if and only if

- \[ \mu_\rho(\lambda) \geq 0 \] for all \( \rho \)
- \[ \xi_{E_k}(\lambda) \geq 0 \] for all \( \{E_k\} \)

Classicality from nonnegativity, take II:

A quantum experiment is nonclassical if it fails to admit a quasi-probability representation that is nonnegative for all states and measurements.
Quasi-probability representations of QM:

States

\[ \rho \leftrightarrow \mu_\rho(\lambda) \]
\[ \mu_\rho : \Lambda \rightarrow \mathbb{R} \]
\[ \int \mu_\rho(\lambda) d\lambda = 1 \]

Measurements

\[ \{ E_k \} \leftrightarrow \{ \xi_{E_k}(\lambda) \} \]
\[ \xi_{E_k} : \Lambda \rightarrow \mathbb{R} \]
\[ \sum_k \xi_{E_k}(\lambda) = 1 \]

\[ \text{Tr}[\rho E_k] = \int d\lambda \; \mu_\rho(\lambda) \xi_{E_k}(\lambda) \]

This provides a classical explanation if and only if

\[ \mu_\rho(\lambda) \geq 0 \]
for all \( \rho \)

\[ \xi_{E_k}(\lambda) \geq 0 \]
for all \( \{ E_k \} \)

Nonnegative quasi-probability representation of QM = Noncontextual ontological model of QM

Equivalent notions of classicality
Noncontextuality inequalities and applications of contextuality
Quantum Spellcraft

Based on noncontextuality-inequality violation

Parity-oblivious multiplexing
RS, Buzacott, Keehn, Toner, Pryde, PRL 102, 010401 (2009)

Computational advantages?
Raussendorf, arXiv:0907.5449
Anders and Browne, Phys. Rev. Lett. 102, 050502 (2009)

Secure key distribution?
Horodecki, Pawlowski, Bourennane, arXiv:1002.2410
Why isn’t the world *more* contextual?
The game of parity-oblivious multiplexing

Alice and Bob win if \( b = x_y \)

The catch: no information about parity \( (x_0 \oplus x_1) \) can be conveyed!
Theorem: For all theories admitting a preparation noncontextual model

\[ p(b=x_y) \leq 3/4 \]

A “noncontextuality inequality”

RWS, Buzacott, Keehn, Toner, Pryde, PRL 102, 010402 (2009)