CLASSICAL SIMULATIONS
OF QUANTUM COMPUTATION

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Motivation

- The quantum computational speed-up is much more subtle than originally believed
- Turns out to be very difficult to build quantum algorithms for natural problems which exponentially outperform classical computers
- There exist non-trivial quantum computations that provably can be simulated classically in poly-time
Motivation

So:

What are the essential ingredients responsible for quantum computational power?

What is the relationship between $P$ and $BQP$?

Is there any problem where quantum computers provably outperform classical ones?

Understanding such questions may:

- give insight into fundamental difference between classical and quantum physics

- expose where we should look for new quantum algorithmic primitives
Outline

- Classical simulation
- Gottesman-Knill i.e. Clifford circuits
- Valiant i.e. Matchgates circuits
- Tensor contraction methods
CLASSICAL SIMULATION OF Q.C.
What is classical simulation (ctd)?

- weak classical simulation: sample once from $\mathcal{S}_{\pi \nu \Psi}$ in $\text{poly}(n)$ time on classical computer

Remarks:
- weak simulation (obviously) more natural notion
- E.g., circuits that can be weakly simulated but \underline{not} strongly
  [cf. workshop talk]
- strong simulation implies weak simulation
What is classical simulation of QC?

- quantum computation: \( 1\text{0}^N \rightarrow \psi \rightarrow 1\text{0}^N \) 
  - poly-size q. circuit
  - measure \( h \) qubits in \( \langle 0 \rangle, \langle 1 \rangle \) basis

  - output = \( k \)-bit string \( x = (x_1, \ldots, x_k) \) \( x_i \in \{0, 1\} \)
  - with probability \( p(x) = \langle 0^N e^+ | x \rangle | 10^N \rangle \langle 10^N | \)

- strong classical simulation:
  - evaluate \( x \rightarrow p(x) \) up to \( m \) bits in \( \text{poly}(m,n) \) time on classical computer
  - evaluate all marginal distributions of \( \{ p(x) \} \) up to \( m \) bits in \( \text{poly}(m,n) \) time on class. comp!

  i.e. \( S \subseteq \{1, \ldots, k\} \) arbitrary subset; bit string \( y = (y_i : i \in S) \)
  then \( p_S(y) = \text{prob. of measuring } y \text{ when performing measurement on qubits in } S \)
  then \( y \rightarrow p_S(y) \) up to \( m \) bits in \( \text{poly}(m,n) \) time \( \forall S \)
Strong simulation implies weak simulation

- \( p(x_1, \ldots, x_k) \) prob. distribution on \( k \) bits
- Marginals: \( S \leq \{x_1, \ldots, x_k\} \) \( y = (y_i : i \in S) \) \( p_S(y) = p(y) = \sum_{z} p(y, z) \) all bits outside \( S \)

Suppose \( \forall S, \forall y : p(y) \) can be computed eff. \( \Rightarrow \) possible to sample from \( \{ p(x_1, \ldots, x_k) \} \)

- Proof: take \( k = 4 \) i.e. goal is to sample from \( p(x_1, x_2, x_3, x_4) \)
- Central ingredient: conditional probabilities e.g. \( p(x_2, x_3, x_4) = \frac{p(x_2, x_3, x_4)}{p(x_2, x_3)} \) can be computed eff. also!

\[
p(x_1, x_2, x_3, x_4) = p(x_4) \cdot \frac{p(x_2, x_4)}{p(x_2)} \cdot \frac{p(x_3, x_2, x_4)}{p(x_3, x_2)} \cdot \frac{p(x_1, x_2, x_3, x_4)}{p(x_1, x_2, x_3)}
\]

\[
= p(x_4) \cdot p(x_2 | x_4) \cdot p(x_3 | x_2, x_4) \cdot p(x_1 | x_2, x_3, x_4)
\]

- Algorithm:
  - Sample from \( \{ p(x_4 = 0), p(x_4 = 1) \} \) \( \rightarrow \) outcome \( x_1 \)
  - Sample from \( \{ p(x_2 = 0 | x_3, x_4), p(x_2 = 1 | x_3, x_4) \} \) \( \rightarrow \) outcome \( x_2 \)
  - Sample from \( \{ p(x_3 = 0), p(x_3 = 1) \} \) \( \rightarrow \) outcome \( x_3 \)
  - ... 

Then total prob of obtaining \( x_1 x_2 x_3 x_4 \) is precisely \( \otimes \) procedure is efficient!
THE GOTTESMAN - KWILL THEOREM
Gottesman-Knill theorem

. (non-precise version :) Every quantum circuit composed of CNOT, H
   and PHASE can be simulated classically efficiently
   \( \text{diag}(i, i) \)

[ henceforth : Clifford circuit \( = \) composed of CNOT, H, PHASE ]
Gottesman-Knill Thm

- 1st key example of nontrivial class of simulatable q. circuits
- Clifford operations: central in QIT
- Conceptual importance of GK: provides insight in power of QC
- Practical importance: Clifford operations alone do not yield good quantum algorithms
Outline of this chapter

- 1st variant of GK theorem + proof ("Heisenberg picture")
- 2nd variant of GK theorem + proof ("Schrödinger picture")
- Remarks
GK theorem, 1st variant

. **GK theorem ①**: Consider poly-size Clifford circuit acting on arbitrary n-qubit product input, followed by standard basis measurement of first qubit; denote output probabilities \( p_0 \) & \( p_1 = 1 - p_0 \). Then \( p_0 \) and \( p_1 \) can be computed classically up to \( m \) bits in \( \text{poly}(m,n) \) time.

. **Note**: arbitrary product state as input + single-qubit measurement

. **Note**: strong simulation
Proof of 1st variant

- Main ingredient: CNOT, \( H \) and PHASE are Clifford operations
  - Pauli operation \( \mathcal{P} = p_n \otimes \ldots \otimes p_1 \quad p_i \in \{1, X, Y, Z\} \)
  - Clifford operation \( \mathcal{C} \): for every Pauli \( \mathcal{P} \) there exists Pauli \( \mathcal{P}' \) such that \( \mathcal{C}^+ \mathcal{P} \mathcal{C} = \pm \mathcal{P}' \)
    
    [important class of operations, cf. vast literature]

- Easy to check: CNOT, \( H \) and PHASE are Clifford (Exercise!)

- If \( \mathcal{C} \) and \( \mathcal{C}' \) are Clifford then so is \( \mathcal{C} \mathcal{C}' \)
Proof of 1st variant, ctd

. Now consider poly-size Clifford circuit $E$ + product input $|1\alpha\rangle = |1\alpha_1\rangle \otimes \ldots \otimes |1\alpha_n\rangle$

. Then $p_0 = \langle 2|E^+|0\rangle |0\alpha_1 1\rangle |1\alpha\rangle$

. $p_1 = \langle 2|E^+|1\rangle |1\alpha_1 1\rangle |1\alpha\rangle$

. So $p_0 - p_1 = \langle 2|E^+ |1\alpha_1 1\rangle |1\alpha\rangle$

. $Z \otimes 1 =$ Pauli operation so $E^+ (Z \otimes 1) E = \pm p$ for some Pauli $\tau$

. Moreover, $\pm p$ can be computed efficiently classically!

. So $p_0 - p_1 = \pm \langle 2| \tau |1\rangle$ where $\tau = \prod p =$ product operator

. $\langle 1\rangle =$ product state

. This yields efficient classical computation of $p_0 - p_1$; hence also of $p_0$ and $p_1$
GK theorem, 2nd variant

- **GK theorem)** Consider poly-size Clifford circuit acting on \( |0\rangle^n \), followed by standard basis measurement off all qubits. Then it is possible to sample classically in poly-time from distribution \( \text{Prob}(x) = \langle x | \mathbf{E} | 0 \rangle^n |^2 \quad x \in \{0, 1\}^n \)

- **Note:** input = \( |0\rangle^n \); Thm still true for any standard basis input but not for any product input!

- **Note:** Thm still true for measurement of subset of qubits

- **Note:** here weak simulation; but strong simulation also possible
Proof of 2nd variant

Main ingredient: consider states of the following form:

\[ |\Psi\rangle = \sum_{x \in A} l(x) (-1)^{q(x)} |x\rangle \equiv |14(A, q, l)\rangle \]

where:
- \[ A = \{ Ru + t : u \in \mathbb{Z}_2^k \} \] affine subspace of \( \mathbb{Z}_2^n \)
- \( q(x) = x^T B x + b^T x \) quadratic form over \( \mathbb{Z}_2 \)
- \( l(x) = a^T x \) linear function over \( \mathbb{Z}_2 \)

if \( U \in \{ \text{CNOT}, H, \text{PHASE} \} \) then \( U |14(A, q, l)\rangle = |14(A', q', l')\rangle \)

for some \( A', q', l' \)

AND update \( (A, q, l) \to (A', q', l') \) is efficient!

[Proof: straightforward] [Exercise!]

Proof of 2nd variant, ctd.

- Now consider Clifford circuit \( E \) and input \( |\mathcal{E}\rangle \).
- Note: \( |\mathcal{E}\rangle \) corresponds to \( |\text{Y}(A, q, 1)\rangle \) with \( A = 0.5 \), \( q = 0 \), \( l = 0 \) i.e. trivial instance.
- Therefore: state of quantum register has the form \( |\text{Y}(A, q, 1)\rangle \) throughout entire computation and each update is efficient.
- Thus we may efficiently compute final state \( |\text{Y}(A, q, 1)\rangle_{\text{final}} \).
Proof of 2nd variant, etc.

- Final state: $$|\psi_{\text{final}}\rangle = \sum_{x \in A} (i)^{q(x)} i^{l(x)} |x\rangle$$

- Now consider $$\langle 10, 11 \rangle \$$ measurement of qubits
  
  - Outcome is independent of q and x!
  
  - Measurement yields uniformly random x in A

- Classical simulation = trivial:
  
  $$A = \{ R_{u+t} : u \in \mathbb{Z}_2^k \}$$

  so just generate random bit string u and apply R; then $$x := R_u$$ is random element in A

[Exercise: show that strong simulation is also possible]
Final remark:

\[ E(10^n = 14) = \sum_{x \in A} \langle 1 \rangle^{g(x)} \cdot i^{l(x)} \cdot |x\rangle \]

Suggests alternative way to prepare \(14\rangle\)

Highly simplified circuit that prepares same output state

IMPORTANT: no "destructive" interference
GT variant ① versus ②

① focus on observables  ⇔  ② focus on states

"Heisenberg picture"  "Schrödinger picture"

* crucial ingredient in both proofs: Clifford operations preserve certain closed family of states/observables

* Once this family is identified, proof is simple
  but this may be non-trivial
Clifford circuits may generate complex entangled states.

E.g. Cluster / graph states

\[ G = (V, E) \]

\[ 1^n \otimes \prod_{ab \in E} CZ_{ab} |+\rangle^n \]

Graph states may be highly entangled: e.g. \( G_{2D} = \) 2D grid

\[ |G_{2D}\rangle = \text{2D cluster state} \]

Thus, high degrees of entanglement are not sufficient for quantum speed-up!
Characterizing the power of Clifford circuits

- Clifford operations are not universal for q. computation; but what is their power?
- Power of Clifford circuits is equivalent to power of classical circuits composed of CNOT and NOT
  - i.e., presence of H gate is essentially irrelevant (cf. also §)
  - Associated complexity class: \( \oplus L \) ("parity-L")
  - \( \oplus L \subseteq P \); inclusion probably strict but no proof
- Thus, Clifford circuits are (probably) not even universal for classical computation
VALIANT'S THEOREM
Valiant's theorem

- Matchgate $G$: 2-qubit unitary gate of the form
  
  $G = \begin{bmatrix} a & b \\ x & y \\ z & t \\ c & d \end{bmatrix}$ with $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\begin{bmatrix} x & y \\ z & t \end{bmatrix} \in SU(2)$

- (Valiant's thm; non-precise version:) Every quantum circuit composed of matchgates acting on nearest-neighbor qubits can be simulated classically efficiently

  [henceforth: Matchgate circuit $\equiv$ composed of n.n. matchgates]
Valiant's theorem

- End key example of simulatable quantum circuits
- Conceptual importance: Matchgates highlight fragile gap between classical & quantum computation
- Practical importance: Matchgates describe time evolution of important class of physical systems i.e. non-interacting fermions spin chains such as Ising, XY model
- Matchgates are interdisciplinary:
  - Computer science ("holographic algorithms")
  - Math (graph theory & planar matchings)
Outline of this chapter

- Matchgates, Clifford algebra's, non-interacting fermions
- Proof of Valiant's thm
- Achieving universal QC by adding SWAP
- Remarks
Clifford algebra's and quadratic Hamiltonians

- Jordan-Wigner operators on n qubits
  \[ c_1 = X \otimes 1 \otimes \cdots \otimes 1 \]
  \[ c_2 = Y \otimes 1 \otimes \cdots \otimes 1 \]
  \[ c_3 = Z \otimes X \otimes \cdots \otimes 1 \]
  \[ c_{2k-1} = Z \otimes \cdots \otimes Z \otimes X \otimes 1 \otimes \cdots \otimes 1 \]
  \[ c_{2k} = Z \otimes \cdots \otimes Z \otimes Y \otimes 1 \otimes \cdots \otimes 1 \]

- Anti-commutation relations
  \[ \{ c_\mu, c_\nu \} = 2 \delta_{\mu\nu} \]
  \[ \mu, \nu = 1, \ldots, 2n \] [exercise: check!]

- Quadratic Hamiltonian
  \[ H = i \sum_{\mu, \nu} h_{\mu\nu} c_\mu^\dagger c_\nu \]
  \( (h_{\mu\nu}) \) real + antisymmetric \( 2n \times 2n \) matrix

  - Important class of Hamiltonians
  - Describes all non-interacting fermionic systems
    \[ a_k := \frac{1}{2} (c_{2k-1} + i c_{2k}) \]
    [check!]
    \[ \{ a_i, a_j \} = 0 \] and \[ \{ a_i, a_j^\dagger \} = \delta_{ij} \]

  - Describes some 1D spin systems e.g. Ising model
    \[ H = -J \sum X_k X_{k+1} - h \sum Z_k \]
    \[ X_k X_{k+1} \propto c_k c_\ell c_{2(\ell+1)-1} \]
    \[ Z_k \propto c_\ell c_k c_\ell \]
Matchgates and quadratic Hamiltonians

- Recall: Matchgate

\[
G(A,B) := \begin{bmatrix}
a & b \\
x & y \\
z & t \\
c & d \\
\end{bmatrix}
\]

\[
A = \begin{bmatrix} a & b \\
                c & d \end{bmatrix}, \quad B = \begin{bmatrix} x & y \\
                  z & t \end{bmatrix}
\]

\[
A, B \in SU(2)
\]

- Consider n-qubit system,

Then every nearest-neighbor matchgate \( G(A,B)_{k,k+1} \) is exponential of quadratic Hamiltonian!

E.g. Matchgate acting on qubit 1 and 2

\[
G(A,B) = e^{iH} \quad \text{with} \quad H = \begin{bmatrix}
k & \ell \\
-p & q \\
-q & -p \\
\ell & -k \\
\end{bmatrix}
\]

\[
e \in \text{Span}\left(\{Z_{a1}, Y_{aX}, Y_{aY}, X_{aX}, X_{aY}, 1, 2\}\right)
\]

\[
-1, 1, i, -i, c_2, i c_3, i c_4, -i c_3, i c_4, -i c_2
\]
Matchgates and quadratic Hamiltonians

- Important:
  - Nearest-neighbor
  - $A, B \in SU(2)$ \{ crucial conditions \}
Time evolutions of quadratic Hamiltonians

Theorem: consider quadratic Hamiltonian $H$ and $U := e^{iH}$. Then $U^c_p U = \sum \xi_{p\nu} \xi_{p\nu}$ with $R = (R_{p\nu}) \in SO(2n)$

Proof: write $\xi_p(t) := e^{-iHt} \xi_p e^{iHt}$

Then $\frac{d\xi_p(t)}{dt} = i \left[ H, \xi_p(t) \right]$ \quad \Rightarrow \quad \text{since } H \text{ commutes with } e^{iHt} \text{ one has}$

\[ i \left[ H, \xi_p(t) \right] = i e^{iHt} \left[ H, \xi_p \right] e^{iHt} \]

\text{using commutation relations of } \xi_p \text{'s, one has}

\[ i \left[ H, \xi_p(t) \right] = \sum \xi_{p\nu} \xi_{p\nu}(t) \]

\begin{align*}
\sum & \xi_{p\nu} \xi_{p\nu}(t) \\
& = \sum \xi_{p\nu}(t)
\end{align*}
Proof (ctd): thus \[ \frac{d}{dt} \psi_{\mu}(t) = \sum_{\nu} h_{\mu\nu} \psi_{\nu}(t) \]

+ initial condition \( \psi_{\mu}(0) = \psi_{\mu} \)

unique solution: \( \psi_{\mu}(t) = \sum_{\nu} R(t)_{\mu\nu} \psi_{\nu} \) with \( R(t) = e^{iht} \)

Exercise: check !

Remark: 
- \( t = 1 \) special case: \( \psi_{\mu}(t=1) = U^+ \psi_{\mu} U \)
  thus \( U^+ \psi_{\mu} U = \sum_{\nu} R(1)_{\mu\nu} \psi_{\nu} \)

- \( R = R(1) = e^{i\hbar} \in SO(2n) \) since \( h \) antisymmetric + real
Proof of Valiant's theorem

Valiant: Consider poly-size nearest-neighbor matchgate circuit $E$ acting on arbitrary $n$-qubit product input $|\psi\rangle$, followed by standard basis measurement of first qubit; denote $p_0$ and $p_1$ as before. Then $p_0$ and $p_1$ can be computed classically up to $m$ bits in poly$(m,n)$ time.

Note: arbitrary product input + single-qubit measurement

Note: strong simulation
Proof of Valiant's thm

. As before: $p_0 - p_1 = i \langle d | e^t Z \sigma_1 \sigma_0 \ldots \sigma_1 e | d \rangle$

. $Z \sigma_1 \sigma_0 \ldots \sigma_1 = -i c_1 c_2$ [I recall $c_1 = x \sigma_1 \ldots \sigma_1$ and $c_2 = y \sigma_1 \ldots \sigma_1$]

. $C = U_N \ldots U_1$ where each $U_i$ is n.n. matchgate

so $U_i \leftrightarrow R_i \in SO(2n)$ such that $U_i^t c_{\alpha} U_i = \sum_v [R_i]_{\mu \nu} c_\nu$

let $R := R_N \ldots R_1$; then $C^t c_{\alpha} C = \sum_v R_{\mu \nu} c_\nu + \text{computing } R \text{ is efficient!}$

. $p_0 - p_1 = -i \langle d | e^t c_{\alpha} c_{\alpha} e | d \rangle = -i \langle d \left[ e^t c_{\alpha} e \right] \left[ e^t c_{\alpha} e \right] | d \rangle$

$= -i \sum_{\mu, \nu} R_{\mu \nu} \langle d | c_\mu c_\nu | d \rangle$ ($\mu, \nu = 2, \ldots, 2n$)

. Sum contains $(2n)^2 = \text{poly}(n)$ terms; $R_{\mu \nu}$ and $R_{\nu \mu}$ are eff computable;
also $\langle d | c_\mu c_\nu | d \rangle$ eff computable as $c_\mu c_\nu = \text{product operator}$.

$|d\rangle = \text{product state}$
Similarities GK & Valiant

- Compare Valiant to variant 0 of GK
- As in case of GK: Matchgate circuits preserve closed (algebraic) framework via quadratic Hamiltonians
Importance of nearest-neighbor + SU(2) conditions

- It is known that $G(A,B)$ gates, with $A, B \in SU(2)$, acting on arbitrary qubit pairs are universal for QC!

- Equivalently, nearest-neighbor $G(A,B) + \text{SWAP}$ gate are universal

- Stronger: n.n $G(A,B) + \text{next-n.n.} G(A,B)$ are also universal!

- Alternatively: $G(A,B)$ gates acting on n.n. with $A, B \in U(2)$ are also universal:

$$\text{SWAP} = \begin{bmatrix} 1 & \ldots & \ldots & \ldots \\ \ldots & 0 & 1 & \ldots \\ \ldots & \ldots & \ldots & 1 \\ \ldots & \ldots & \ldots & \ldots \end{bmatrix} = G(\mathbb{I}, X) \quad \text{where } X \in U(2)$$
Matchgates and \textit{log-space} quantum computation

- \textit{E} equivalence between \textit{n}-qubit matchgate circuits and arbitrary quantum circuits acting on $\log(n)$ qubits

\[ E \xrightarrow{\text{log-space classical comp}} E' \]

- \textit{E} log circuit acting on \textit{n} qubits, composed of \textit{N} gates
- Standard q. circuit acting on $O(\log n)$ qubits composed of $O(N \log n)$ gates

\[ \checkmark \]

\textit{E} and \textit{E'} are equivalent i.e. have same output distribution \forall \rho, \rho'

- Reverse arrow $E' \xrightarrow{\text{}} E$ also holds

\textit{Idea of proof:} regard $R \in SO(2n)$ as q. computation on $O(\log n)$ qubits
TENSOR CONTRACTION
METHODS
Tensor contraction methods

- Classical simulation based on structural properties of certain q. circuits
- Conceptual importance: highlights role of entanglement as necessary ingredient for quantum speed-up
- Practical importance: used in major investigations of strongly correlated systems (viz. MPS, PEPS, MERA, ...)

Outline of this chapter

- Tensor networks
- Example: nearest-neighbor constant-depth circuits
- Role of entanglement
Tensor networks

- Tensor $A_{ijk}$ for $i, j, k = a, \ldots, d$ of $d^3$ complex numbers
  
  $[\text{rank} = 3; \text{dimension} = d]$  

- Graphically:

  ![Graphical representation of tensor](image)

  [often without labels]

- Example:
  - Vector $v = (v_i : i = a, \ldots, d)$
    
    ![Vector representation](image)

  - Matrix $M = (M_{ij} : i, j = a, \ldots, d)$
    
    ![Matrix representation](image)

- Contraction:

  $\sum_{\ell=1}^{d} A_{ijk\ell} B_{\ell mn} = C_{ijkmn}$

  ![Contraction diagram](image)

  $\text{rank}(A) = 4$ \quad $\text{rank}(B) = 2$

  $\text{rank}(C) = 4 - 1 + 3 - 1 = 5$
Tensor networks

1. Tensor network = collection of tensors, contracted with each other at certain indices, according to some graphical pattern [edge = contraction]

Examples:

- Matrix element of a product of \( N \) by \( d \times d \) matrices

\[
A_{ij}^{(a)} : \ ij = 1, \ldots, d \quad \text{Matrices ; } j = 1, \ldots, N
\]

\[
a^\top A^{(a)} \cdots A^{(N)} b = a^{(1)} A^{(a)} A^{(b)} A^{(c)} b
\]

- Contraction of rank 3 tensors w.r.t. cycle graph

\[
A_{ijk}^{(a)} : \ ij, k = 1, \ldots, d \quad \text{a = 1, 2, 3}
\]

\[
\sum_{jkl} A_{ijk}^{(a)} A_{jkl}^{(b)} A_{cjk}^{(c)} =
\]

\[
A^{(a)} A^{(b)} A^{(c)}
\]
Tensor networks

- Central problem: given network of $N$ tensors $A^{(k)}$ of dimension $d = \text{const}$, how hard is it to contract this network?
  \[ \Rightarrow \text{efficient if contraction possible in poly}(N) \text{ time} \]

- Examples:
  - $\Box\Box\Box\Box\Box\Box = \text{matrix product} = \text{efficient}$
  - $\text{Tree tensor network} = \text{efficient}$
    
    [exercise! Tip: contract from bottom to top]
  - $\text{2D grid} \not\Rightarrow \text{efficient (\#P hard)}$

- General result: Every tensor network with is sufficiently "tree-like" can be contracted efficiently (\$\Rightarrow\$ notion of TREE-WIDTH)
Example: n.n. constant-depth circuits

Theorem: Consider an n-qubit depth-d quantum circuit \( C \) composed of [potentially non-unitary] \( e \)-qubit gates. The input state is any product state \( |x\rangle \). Let \( A = A_1 \otimes \cdots \otimes A_n \) be any product operator. Then \( \langle x | e^\dagger A e | x \rangle \) can be computed classically up to \( m \) bits in \( \text{poly}(m, n) \) time.
Proof: [Sketch]

\[\langle a | e^T A e | 12 \rangle = \]

1d

\[\langle a | \rightarrow \]

\[\text{I replace } [\ ] \text{ by } \text{[ ]} \]

\[\rightarrow \]

\[O(d) \]

\[O(n)\]
Proof (ctd)

\[ O(d) \]

\[ O(n) \]

\[ \begin{array}{cccc}
    i & j & k & \ldots \\
    & n & r & s \\
\end{array} = x \\
\begin{array}{cccc}
    a & b & c & d \\
    & e & f & g \\
\end{array} = y \\

Thus \( \xi \) has rank 2 and dimension = constant (i.e. \( O(d^4) \))

\[ \rightarrow \text{contraction is now similar to matrix product of } O(n) \text{ DxD matrices with } D \text{ constant!} \]
Role of entanglement

- n.n. constant-depth circuits cannot produce much entanglement, in the following sense:
  - 14) multiquubit state; bipartition $(A, B)$
  $$\chi_{A,B}(14) = \min \{ r : 14 = \sum_{k=1}^{r} 14_k \otimes 14_k \}$$  
  \[ \text{schmidt rank} \]
  - take $14 = \epsilon |12\rangle$ with $\epsilon$ n.n. constant-depth
  take bipartition $1...m \mid m+1...n$ of $n$-qubit system
  then $\chi_{A,B}(14) \leq \text{Constant}$ \ [Exercise!]

- This is general feature of tree-like tensor networks / circuits: they cannot generate much "schmidt-rank-type" entanglement

- Reverse statements are also true: "Bounded schmidt-rank-type entanglement implies classical simulation" i.e. entanglement as necessary ingredient for Q. speed-up
THANK YOU!