Quantum Error Correction
Lecture 4

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University of British Columbia, July 18, 2010
Fault-Tolerance:

- Perform operations on states encoded in a QECC without losing protection against errors.
- Control error propagation
For a stabilizer code:

$N(5)/5$ maps codewords to different codewords.

Tensor products of single-qubit operations — no error propagation.

$N(5)/5$ can be used to implement logical Pauli group in a fault-tolerant way.

$N(5)/5 \cong P_k$
For a full FT protocol:

- FT gates (a universal set)
- Prepare encoded states in a FT way
- FT measurement
- FT error correction
7-qubit code:

\[ |\bar{0}\rangle = \sum_{c} x |0\rangle \]
\[ |\bar{1}\rangle = \sum_{c} x |1\rangle \]

Logical $X$: $\otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \ottimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \ottimes \otimes \otimes \otimes \otimes \ottimes \otimes \otimes \otimes \otimes \ottimes \otimes \otimes \otimes \otimes \ottimes \otimes \otimes \otimes \otimes \ottimes \otimes \otimes \otimes \otimes \ottimes \otimes \otimes \otimes \otimes \ottimes \otimes \otimes \otimes \ottimes \otimes \otimes \otimes \ottimes \otimes \otimes \otimes \ottimes \otimes \otimes \otimes \ottimes \otimes \otimes \otimes \ottimes \otimes \otimes \otimes \ottimes \otimes \otimes \otimes \ottimes \otimes \otimes \otimes \ottimes \otimes \otimes \ottimes \otimes \otimes \ottimes \otimes \otimes \ottimes \otimes \otimes \ottimes \otimes \otimes \ottimes \ottimes \otimes \otimes \ottimes \otimes \otimes \ottimes \otimes \otimes \ottimes \ottimes \otimes \otimes \ottimes \otimes \otimes \ottimes \ottimes \otimes \otimes \ottimes \otimes \otimes \ottimes \ottimes \otimes \otimes \ottimes \otimes \otimes \ottimes \ottimes \ottimes \otimes \ottimes \otimes \otimes \ottimes \ottimes \otimes \otimes \ottimes \otimes \otimes \ottimes \ottimes \otimes \otimes \ottimes \otimes \otimes \ottimes \ottimes \otimes \otimes \ottimes \otimes \ottimes \otimes \ottimes \otimes \ottimes \otimes \otimes \ottimes \ottimes \otimes \otimes \ottimes \otimes \otimes \ottimes \ottimes \otimes \otimes \ottimes \otimes \otimes \ottimes \ottimes \otimes \otimes \ottimes \otimes \ottimes \otimes \ottimes \otimes \ottimes \otimes \otimes \ottimes \ottimes \otimes \otimes \ottimes \otimes \otimes \ottimes \ottimes \otimes \otimes \ottimes \otimes \ottimes \otimes \ottimes \otimes \ottimes \ot提供商笑话文本。
Transversal gates

Interacts $i$th qubit of one block with $i$th qubit of second block

Logical CNOT:

Errors propagate - but not within a block.
Quantum Teleportation:

Alice

\[ |1\rangle \]

Bob

\[ p = I, X, Y, Z \]

Bell measurement:

Projector on

\[ |10\rangle \pm |11\rangle \]

\[ |01\rangle \pm |10\rangle \]
Encoded Ancilla

\[ |00\rangle \langle 11| = \left( |00\rangle \langle 00| + |11\rangle \langle 11| \right) \]

Encoded Teleport using this ancilla – provided we can perform \( UU^+ \) then can do gate \( U \).

FT
FT EC:

(Non-FT EC: \( |+\rangle \rightarrow \text{eigenvalue of } Z \otimes Z \otimes Z \ldots \))

cat state:
\( |0\ldots 0\rangle + |1\ldots 1\rangle \)

data

- Create cat state FT
- Repeat measurement with new cat state.
Imagine error rate $p$ per gate or time step

\[ \begin{array}{c}
\text{EC} & \text{EC} \\
\text{EC} & \text{EC}
\end{array} \]

...many physical gates

$A$ in total

If there are two errors in here, then we have a logical error

If there's 0 or 1 errors, we are OK

\[
\text{Prob. (2 errors)} = \left( \frac{1}{2} \right)^2 p
\]

If this is less than $p$, then the logical gates are more reliable than unencoded gates

\[
\left( \frac{1}{2} \right)^2 p < p \Rightarrow p < \frac{1}{(A)^2} = p_T
\]

\[
\frac{1}{2} \leq p_T = \frac{p}{q_T}
\]
Threshold Theorem:

If $p < p_T$, then we can perform arbitrarily long quantum computations with polylogarithmic overhead.

Proof: Use concatenated codes - encode code state multiple times using a QECC

$p \rightarrow (p/p_T)^2 \rightarrow (p/p_T)^4 \rightarrow (p/p_T)^8 \rightarrow \ldots \rightarrow (p/p_T)^2^L$ with $L$ levels.

Overhead $c^L$ physical qubits per logical qubit.

$T$ logical gates $\Rightarrow$ logical error rate $\sim 1/T \Rightarrow L = \log \log T \leq \text{overhead poly}(\log T)$.
Value of $p_T$:

Best proofs $p_T \approx 10^{-3}$

Simulations $p_T \approx 5\%$