12th International Conference on Applications of Statistics and Probability in Civil Engineering, ICASP12
Vancouver, Canada, July 12-15, 2015

The Moment Equation Closure Method Revisited through the Use of Complex Fractional Moments

Gioacchino Alotta
PhD student, DICAM, Univ. of Palermo, Palermo, Italy

Christian Bucher
Professor, Faculty of Civil Engineering, Vienna Univ. of Technology

Alberto Di Matteo
PhD student, DICAM, Univ. of Palermo, Palermo, Italy

Mario Di Paola
Professor, DICAM, Univ. of Palermo, Palermo, Italy

Antonina Pirrotta
Associate Professor, DICAM, Univ. of Palermo, Palermo, Italy

ABSTRACT: In this paper the solution of the Fokker Planck (FPK) equation in terms of (complex) fractional moments is presented. It is shown that by using concepts coming from fractional calculus, complex Mellin transform and related ones the probability density function response of nonlinear systems may be written in discretized form in terms of complex fractional moment not requiring a closure scheme.

Excitations such as ground motion, wind turbulence, sea waves, surface roughness, blasts and impacts loads being stochastic processes induce that structural responses are stochastic processes too. Thus, the analyst is concerned with the problem of the response statistical characterization. However, to consider a model closer to reality a nonlinear system has to be considered, then a complete statistical characterization of the response may be performed by solving the Fokker–Planck–Kolmogorov (FPK) equation, a partial differential equation whose solution is the joint probability density function (PDF) of the response variables (Lin and Cai, 1995). Unfortunately, the FPK equation admits analytical solution in very few cases, for this reason we resort to numerical methods. Among the numerical approaches, more attractive, from a computational point of view, is the moment equation (ME) approach, in which the response statistical characterization is given by the response moments or by other quantities related to the former such as cumulants or quasi-moments (Stratanovich, 1997; Ibrahim, 1985). This method consists of writing differential equations for the response statistical moments of any order. However, when dealing with nonlinear systems, a serious problem arises in the ME approach, the entire system is hierarchic in the sense that the equations for the moments of a fixed order, say K, contain moments of order higher than K. In this way, the ME form an infinite hierarchy. In order to overcome this difficulty, the so-called closure methods were born. The key idea is to express the response PDF as a Edgeworth or Gram-Charlier series, truncating it at a certain term. The coefficients of the above mentioned series can be written as functions of the response central moments or of the response cumulants or of the response quasi-moments. Thus, neglecting the terms beyond a given order is equivalent to make central moments or cumulants or quasi-moments zero, which makes the ME solvable. The moments of order larger than K are expressed in terms of moments of order equal or lower to K by means of the relationships that are obtained by putting the above cited quantities equal to zero.

Recently in Di Paola (2014) it has been introduced the complex fractional moments (CFM) through which the FPK equation has been converted, returning a simple method to perform a PDF response function; in Di Matteo et al. (2014) and in Alotta and Di Paola (2015) the method has been applied successfully also for the case of Kolmogorov-Feller equation (Poissonian white noises) and for fractional FPK equation (α-stable white noises), respectively. In this paper it will be explored the useful tool of the complex fractional moments to overcome this moment closure procedure.

1. SERIES FORM OF PDF THROUGH COMPLEX FRACTIONAL MOMENTS

Starting from the equation of motion of a nonlinear half oscillator in the form, consider the scalar stochastic process $X(t)$ satisfying the stochastic differential equation

$$\begin{cases} \dot{X} = f(X,t) + W(t) \\ X(0) = X_0 \end{cases} \tag{1}$$

Where $W(t)$ is a Gaussian white noise with zero mean and correlation function $E[W(t)W(t+\tau)] = 2\pi S_0 \delta(t) = q\delta(t)$ with S_0 being the spectral density of $W(t)$. Moreover $f(X,t)$ is a nonlinear function of the process $X(t)$ and X_0 is the random variable with assigned PDF in zero $p_X(x,0) = \bar{p}(x)$.

The equation ruling the evolution of the PDF of the response process $X(t)$ is the so-called Fokker-Planck (FP) equation that may be written as

$$\begin{cases} \dfrac{\partial p_X(x,t)}{\partial t} = -\dfrac{\partial}{\partial x}\left(f(x,t)\, p_X(x,t)\right) + \dfrac{\partial^2}{\partial x^2}\left(p_X(x,t)\right) \\ p_X(x,0) = \bar{p}_X(x) \end{cases} \tag{2}$$

For simplicity sake's, assume that $f(x,t) = -f(-x,t)$, and the initial condition $X(0)$ has a symmetric PDF. In these conditions the PDF of $X(t)$ is symmetric at every time instant t. The case of non-symmetric PDF of the response may be solved following Di Paola and Pinnola (2012).

Introducing the Mellin transform of the PDF $p_X(x,t)$, labeled as $\mathcal{M}\{p_X(x,t);\gamma\}$ or $\mathcal{M}_{p_X}(\gamma-1,t)$ defined in the form

$$\mathcal{M}\{p_X(x,t);\gamma\} = \int_0^\infty p(x)\, x^{\gamma-1} dx = \mathcal{M}_{p_X}(\gamma-1,t); \quad \gamma = \rho + i\eta \tag{3}$$

where i is the imaginary unit, $p_X(x,t)$ is defined in the range $0 \le x < \infty$ as aforementioned and $\rho,\eta \in \mathbb{R}$. Further, ρ belongs to the Fundamental Strip (FS) of the Mellin transform (Di Paola, 2014) that is $-p < \rho < -q$ where p, q are the order of zeros at $x = 0$ and $x = \infty$, respectively.

From formula (3) two salient observations come out:

i) The $\mathcal{M}_{p_X}(\gamma-1,t)$ is related to moments of the type $E[|X(t)|^{\gamma-1}]$, ($E[\bullet]$ means ensemble average) called complex fractional moments (CFM) as well, in particular

$$\mathcal{M}_{p_X}(\gamma-1,t) = \int_0^\infty p(x)\, x^{\gamma-1} dx = \frac{1}{2} E[|X(t)|^{\gamma-1}] \tag{4}$$

ii) The PDF $p_X(x,t)$ is returned by the inverse Mellin transform as

$$p_X(x,t)=\frac{1}{2\pi}\int_{\eta=-\infty}^{\infty}\mathcal{M}_{p_X}(\gamma-1,t)\,x^{-\gamma}d\eta \quad (5)$$

It is worth underscoring that the integration is performed along the imaginary axis, while ρ remains fixed.

The latter consideration is of fundamental importance because, since $\mathcal{M}_{p_X}(\gamma-1,t)$ is holomorphic in the FS going to zero as η diverges, it follows that Eq. (5) may be discretized in the form

$$p_X(x,t)\cong\frac{\Delta\eta}{2\pi}x^{-\rho}\sum_{k=-m}^{m}\mathcal{M}_{p_X}(\gamma_k-1,t)\,x^{-ik\pi/b} \quad (6)$$

where $b=\dfrac{\pi}{\Delta\eta}$, $\gamma_k=\rho+ik\Delta\eta$, $\Delta\eta$ is the discretization step along the imaginary axis, $m\Delta\eta=\bar{\eta}$ is a cut-off value selected in such a way that terms of higher order than $\bar{\eta}$ do not produce sensible variations on $p_X(x,t)$.

Notice that the presence of the factor $x^{-\rho}$ in Eq. (6) ensures us that the trend of the approximation in Eq. (6) as $x\to\infty$ goes to zero, no matter the type of distribution.

From Eq. (6) we realize that by knowing the CFMs the whole PDF is performed (unless the value in zero). To aim at this, it needs to write the evolution equation in terms of CFMs obtained by multiplying the FP equation by $x^{\gamma-1}$ and integrating in the range $0\div\infty$:

$$\begin{aligned}\dot{\mathcal{M}}_{p_X}(\gamma-1,t)&=-\left[x^{\gamma-1}f(x,t)\,p_X(x,t)\right]_0^{\infty}+\\&+(\gamma-1)\int_0^{\infty}x^{\gamma-2}f(x,t)\,p_X(x,t)\,dx+\\&+\frac{q}{2}\left[\frac{\partial p_X(x,t)}{\partial x}x^{\gamma-1}\right]_0^{\infty}-\frac{q}{2}(\gamma-1)\left[x^{\gamma-2}p_X(x,t)\right]_0^{\infty}+\\&+\frac{q}{2}(\gamma-1)(\gamma-2)\int_0^{\infty}x^{\gamma-3}p_X(x,t)\,dx\end{aligned} \quad (7)$$

If $X(t)$ is stable in distribution and moments up to the third order are stable, by selecting $\rho>2$, he first, the third and the fourth term in Eq. (7), coming from integration by part, vanish the Eq. (7) reverts to

$$\begin{aligned}\dot{\mathcal{M}}_{p_X}(\gamma-1,t)&=(\gamma-1)\int_0^{\infty}x^{\gamma-2}f(x,t)\,p_X(x,t)\,dx+\\&+\frac{q}{2}(\gamma-1)(\gamma-2)\mathcal{M}_{p_X}(\gamma-3,t)\\\mathcal{M}_{p_X}(\gamma-1,0)&=\int_0^{\infty}x^{\gamma-1}\bar{p}_X(x)\,dx\end{aligned} \quad (8.a,b)$$

Being Eq.(8b) the known initial condition.

Particularizing this equation for $\gamma_k=\rho+ik\Delta\eta$ with $k=-m\div m$ we get a system of (2m+1) equations in terms of (2m+1) variables say $\mathcal{M}_{p_X}(\gamma_k-1,t)$ useful to find out the PDF in series form through Eq.(6).

A close observation of system (8a) reveals the need of evaluation of CFM with respect different values of ρ, in fact, at least for linear system we have both CFMs $\mathcal{M}_{p_X}(\gamma-1,t)$ and $\mathcal{M}_{p_X}(\gamma-3,t)$.

Recently Di Paola (2014) showed that, because the PDF representation given in Eq. (6) does not depend on the value of the selected ρ (provided that ρ belongs to the fundamental strip), then there is a relationship between $\mathcal{M}_{p_X}(\rho_1-1+i\eta)$ and $\mathcal{M}_{p_X}(\rho_2-1+i\eta)$. That is if we know the fractional moments of $p_X(x)$ for a certain value of ρ say ρ_2, then we may evaluate any other CFM in a different value of ρ say ρ_1, provided that both ρ_1 and ρ_2 belong to the fundamental strip of the Mellin transform.

Such a relationship is

$$\mathcal{M}_p\left(\gamma_s^{(1)}-1\right)=\sum_{k=-m}^{m}\mathcal{M}_p\left(\gamma_k^{(2)}-1\right)a_{ks}(\Delta\rho);\quad s=-m,\ldots,0,\ldots,m \quad (9)$$

where $\Delta\rho=\rho_2-\rho_1\,(\rho_2>\rho_1)$, and

$$a_{ks}(\Delta\rho)=\frac{\sin\left[\pi(k-s)-ib\Delta\rho\right]}{\Delta\eta(k-s)-ib\Delta\rho} \quad (10)$$

It is worth underscoring that the evaluation of CFMs in different values of ρ does not require a closure leading to neglecting statistics of higher order, but from Eq. (9) it is apparent that once the Mellin transform of a given function is evaluated for a given value of ρ, say ρ_2, we may evaluate the Mellin transform of the given function for any other value of ρ inside the fundamental strip.

2. PDF THROUGH CFMS VS INTEGER MOMENTS

The expression of PDF in terms of CFMs reminds that one in terms of cumulants of integer order j K_j in the form at steady state condition as (Ibrahim, 1985)

$$p_x(x) \cong \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]\left(1+\sum_{j=3}^{m}\frac{K_j(-1)^j}{j!\sigma^j}H_j\left(\frac{x-\mu}{\sigma^2}\right)\right) \quad (11)$$

being σ the standard deviation, μ the mean, $H_j(x)$ the probabilistic Hermite polynomials and K_j the cumulants of order j which are related to the integer moments through the following relation

$$E[X^j] = K_j + \sum_{k=1}^{j-1} \frac{(r-1)!}{j!(r-1-j)!} K_{k-j} E[X^k] \quad (12)$$

With the above expression the PDF of the system response is approximated with the Gram-Charlier series. However as it is well known, such a series can be inconsistent with probability theory, e.g. negative probabilities may result. Moreover another problem related to this expression is the *jth*-order hierarchy truncation method.

In fact the cumulants K_j are written once all integer moments $E\left[X(t)^j\right]$ are known solving the following system of differential equation

$$\dot{E}\left[X(t)^j\right] = (j)\int_0^\infty x^{k-1} f(x,t)\, p_X(x,t)\, dx + \\ +\frac{q}{2}(j)(j-1)E\left[X(t)^{j-2}\right] \quad (13)$$

Such a strategy belongs to the moment equation (ME) approach, proposed in 1980 (Dover, 1980) as an alternative method to Monte Carlo approach. If on one hand the ME method requires much less computation involving the solution of a system of coupled deterministic ordinary differential equations, on the other hand the disadvantage of the ME is that, unless for linear systems or special case of nonlinear ones, the differential equations for moments of a given order will contain terms involving higher-order moments leading to an infinite hierarchy of coupled equations requiring a closure scheme-procedure. Then, the *jth*-order hierarchy truncation will require approximations for the *(j+1)th*- and *(j+2)th*- order moments.

At this point, some important remarks come out:

i) Although the system equation (12) is very similar to system (8a) (setting $(\gamma-1)$ =j) the hierarchy problem is not arisen in the latter case.

ii) At first glance the fact that it requires the evaluation of CFMs in different values of ρ may mislead, but if one thinks that the same requirement occurs for linear systems it will be clear that this is not a closure scheme procedure.

3. NUMERICAL APPLICATION

Let the nonlinear function $f(X,t)$ in Eq. (1) be given in the form $f(X,t) = -c_1 X - c_2 |X|^\beta \operatorname{sgn}(X)$ with $\beta > 0$. Further let the assigned PDF in zero be $\overline{p}(x) = \delta(x)$, that is the system is quiescent in $t = 0$. In order to show the accuracy of the proposed approach, the case of a bimodal PDF is considered. Thus, let $c_1 < 0$, $c_2 > 0$ and $\beta = 3$

(quartic oscillator). Note that in this case the steady state PDF is known in closed form as

$$p_X(x,\infty)=\upsilon\exp\left[\frac{1}{2q}\left(x^2-\frac{x^4}{2}\right)\right] \qquad (14)$$

in which υ is a normalization constant such that $\int_0^\infty p_X(x,\infty)\,dx=1/2$.

As far as the Gram-Charlier series expansion in Eq. (11) is concerned, the equation of integers moments for the steady state case can be particularized as

$$-kc_1E[X^k]-kc_2E[X^{k-1+\beta}]+\frac{q}{2}k(k-1)E[X^{k-2}]=0 \quad (14)$$

Note that this equation cannot be solved since an infinite order hierarchy problem appears. However the aforementioned issue can be circumvented expressing integer moments in terms of cumulants through Eq. (12) and considering equal to zero cumulants of order $n>\tilde{m}$ with $\tilde{m}$ arbitrary.

In Fig. 1 comparison among the exact steady state PDF and the PDF obtained through Eq. (11) is reported, for the case $c_1=-0.5$ and $c_2=0.5$, considering two different values of $\tilde{m}$.

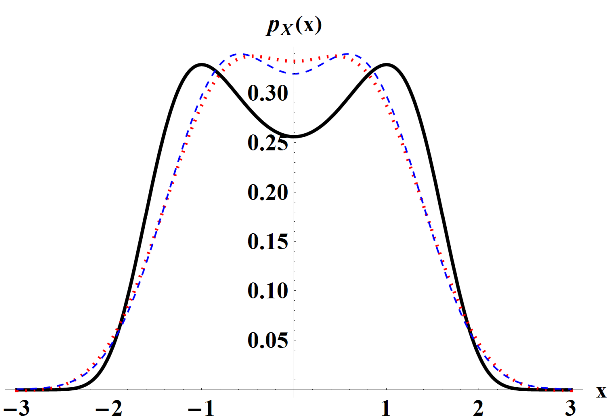

Figure 1: Comparison among the Exact steady state PDF (black line) and Gram-Charlier series expansion with 8 cumulants (red dotted line) and 10 cumulants (blue dashed line).

As it can be observed from this figure, as the number of cumulants increases, the Gram-Charlier expansion does not lead to the exact solution and even considering 10 cumulants the approximated PDF is rather different from the exact steady state solution $p_X(x,\infty)$.

On the other hand, as far as the series form of the PDF through CFMs is concerned, for the system under consideration the equation ruling the evolution of the CFMs is explicitly given as

$$\begin{aligned}&\dot{\mathcal{M}}_{p_X}(\gamma-1,t)=-c_1(\gamma-1)\mathcal{M}_{p_X}(\gamma-1,t)+\\&-c_2(\gamma-1)\mathcal{M}_{p_X}(\gamma+\beta-2,t)+\\&+\frac{q}{2}(\gamma-1)(\gamma-2)\mathcal{M}_{p_X}(\gamma-3,t)\end{aligned} \qquad (15)$$

in which CFMs $\mathcal{M}_{p_X}(\gamma+\beta-2,t)$ and $\mathcal{M}_{p_X}(\gamma-3,t)$ can be easily evaluated through the following relations

$$\mathcal{M}_{p_X}(\gamma+\beta-2,t)=\sum_{k=-m}^{m}\mathcal{M}_p(\gamma_k-1)a_{ks}(1-\beta) \quad (16)$$

$$\mathcal{M}_{p_X}(\gamma-3,t)=\sum_{k=-m}^{m}\mathcal{M}_p(\gamma_k-1)a_{ks}(2) \quad (17)$$

By inserting these equations in Eq. (15) particularized for $\gamma=\gamma_s$ $(s=-m,\cdots,0,\ldots,m)$ we get a set of complex ordinary differential equations in the unknown $\mathcal{M}_{p_X}(\gamma_s-1,t)$. Note that if the system of differential equations is directly implemented into a computer program the solution is incorrect because additional information is needed, that is the area of the PDF in the interval $0\div\infty$ is $1/2$. This constraint may be easily enforced taking into account Eq. (6), leading to

$$\frac{1}{2b}\sum_{k=-m}^{m}\mathcal{M}_{p_X}(\gamma_k-1,t)\int_0^\infty x^{-\gamma_k}dx=\frac{1}{2} \qquad (18)$$

which yields

$$\mathcal{M}_{p_X}(\gamma_0-1,t)=b-\sum_{\substack{k=-m\\k\neq0}}^{m}\mathcal{M}_p(\gamma_k-1,t) \qquad (19)$$

By inserting this condition into Eq. (15) and taking into account Eqs. (16) and (17) particularized for $s=-m,\cdots,-1,1,\ldots,m$ we get a set of *2m+1* differential equations ruling the evolution of the CFM. Eq. (15), particularized for $\gamma_s=\rho+is\Delta\eta$ is rewritten in the form

$$\begin{aligned}&\dot{\mathcal{M}}_{p_X}(\gamma_s-1,t)=-c_1(\gamma_s-1)\mathcal{M}_{p_X}(\gamma_s-1,t)+\\&-c_2(\gamma_s-1)\sum_{k=-m}^{m}\mathcal{M}_{p_X}(\gamma_k-1)a_{ks}(1-\beta)+\\&+\frac{q}{2}(\gamma_s-1)(\gamma_s-2)\sum_{k=-m}^{m}\mathcal{M}_{p_X}(\gamma_k-1)a_{ks}(2);\\&s=-m,\cdots,-1,1,\ldots,m\end{aligned}\tag{20}$$

Equations (19) and (20) constitute a set of linear coupled ordinary differential equations in the unknown $\mathcal{M}_{p_X}(\gamma_s-1,t)$ that may be easily solved considering the initial condition.

Once solution of the previous equation is obtained, the evolution of the PDF can be restored through Eq. (6).

In Fig. 2 the evolution of the system response PDF is reported for various time instants vis-à-vis the exact steady state solution. In this case a value of $\Delta\eta=0.5$ and $m=140$ have been chosen for solution in terms of CFMs. Note that, even if the value *m* of CFMs is much bigger than the number of cumulants chosen $\tilde{m}$, computational time is comparable for the two approaches.

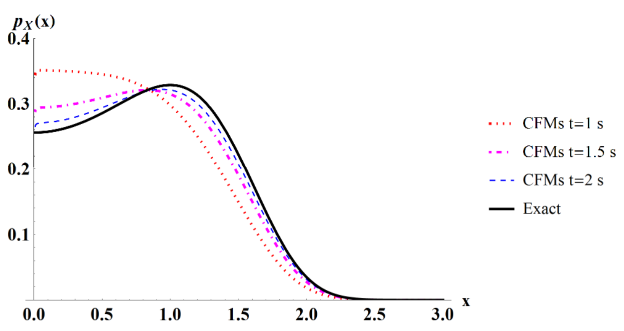

Figure 2: Evolution of the response PDF.

Finally, in order to show the accuracy of the proposed approach with respect to the closure method, in Fig. 3 solution obtained through CFMs is contrasted with the Gram-Charlier expansion for the steady state case in Eq. (14)

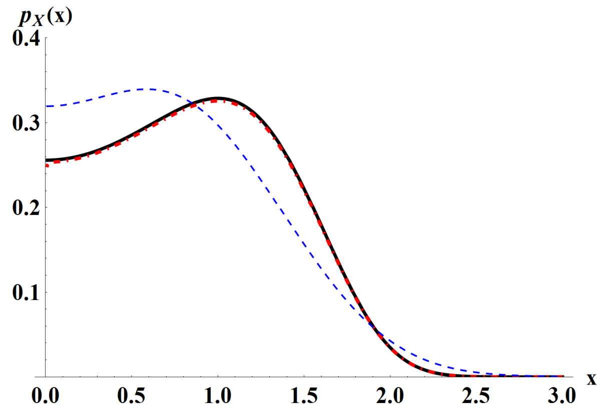

Figure 3: Comparison among Exact steady state solution (Black line), CFMs (Red dashed line) and Gram-Charlier Expansion for 10 cumulants (Blue dashed line).

4. CONCLUSIONS

It is apparent that dealing with nonlinear systems that is considering, for instance, the entire system is hierarchic in the sense that the equations for the moments of a fixed order, say *j*, contain moments of order higher than *j*, or simply the MEs form an infinite hierarchy.

In order to overcome this difficulty, the so called closure methods were born. The key idea is to express the response PDF as a Edgeworth or Gram-Charlier series, truncating it at a certain term. The coefficients of the above mentioned series can be written as functions of the response central moments or of the response cumulants or of the response quasi-moments. Thus, neglecting the terms beyond a given order is equivalent to make central moments or cumulants or quasi-moments zero, which makes the ME solvable. The moments of order higher than *j* are expressed in terms of moments of order equal or lower to *j* by means of the relationships that are obtained by putting the above cited quantities equal to zero. But, as aforementioned this implies an approximation of the *(j+1)th-* and *(j+2)th-* order moments, while by expressing the PDF in terms of fractional moments no closure scheme is required leading to accurate results as shown in the numerical application.

5. REFERENCES

Lin, Y.K., and Cai, G.Q., (1995).
"Probabilistic structural dynamics: advanced theory and applications." New York: McGraw Hill.

Stratonovich, R.L., (1967).
"Topics in the theory of random noise", 1–2. New York: Gordon and Breach;.

Ibrahim, R.A., (1985).
"Parametric random vibrations", New York: John Wiley & Sons.

Di Paola, M., (2014).
"Fokker Plank equation solved in terms of complex fractional moments" *Probabilistic Engineering Mechanics,* 38, 70-76.

Di Matteo, A., Di Paola, M., Pirrotta, A., (2014).
"Probabilistic characterization of nonlinear systems under Poisson white noise via complex fractional moments" *Nonlinear Dynamics,* 77(3), 729-738 DOI: 10.1007/s11071-014-1333-1

Alotta, G., and Di Paola, M., (2015).
"Probabilistic characterization of nonlinear systems under α-stable white noise via complex fractional moments" *Physica A: Statistical Mechanics and its Applications,* 420, 265-276.

Di Paola, M., and Pinnola, F.P. (2012).
"Riesz fractional integrals and complex fractional moments for the probabilistic characterization of random variables" *Probabilistic Engineering Mechanics*, 29, 149-156.

Bover, D.C.C., (1978).
"Moment equation methods for nonlinear stochastic system" *Journal of Mathematical Analysis and Applications,* 65, 306–20.