Nanomechanics Based Theory of Size Effect on Strength, Lifetime and Residual Strength Distributions of Quasibrittle Failure: A Review

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ABSTRACT: The paper reviews a series of studies at Northwestern University which led to the establishment of a theory of probability distributions of short-time strength, residual strength after static preload and lifetime of structures made of quasibrittle materials such as concrete, fiber composites and tough ceramics. The theory is based on the frequency of probability of interatomic bond breaks on the atomic scale and on the multi-scale transition of power-law probability tail. The conclusion is that if the failure is not perfectly brittle, the probability distribution of strength and lifetime is a graft of Gaussian and Weibull distributions and varies from nearly Gaussian at the scale of one RVE to Weibullian for very large structures consisting of many RVEs. As a consequence, the safety factors should depend on structure size. Numerous experimental comparisons and computational simulations are given.

1. INTRODUCTION

In most engineering applications such as bridges, dams, ships, aircraft and microelectronic components, it is essential for the design to ensure a very low failure probability such as 10^{-6} throughout the lifetime (Bazant and Pang (2006)). Therefore, the cdf of the structure must be known up to the very tail region, which must be established theoretically since such small probabilities are beyond direct experimental verification. The type of cdf of strength for perfectly ductile structures must be Gaussian (from the central limit theorem), whereas for perfectly brittle structures, it must be Weibullian (from the weakest link model with infinite links). However this is more complicated for quasibrittle materials, which represent heterogeneous materials characterized by brittle constituents that are not negligible compared to structural dimensions e.g. concrete, fiber composites, tough ceramics, rocks,

and many more (Bazant and Planas (1998)). These behave as ductile when small and brittle when large, thus making the type of cdf a function of the structure size.

The type of cdf of strength and of static lifetime for quasibrittle structures, was mathematically established from atomistic scale arguments based on nano-scale cracks propagating by many small, activation energy-controlled, random breaks of atomic bonds in the nanostructure (Bazant and Pang (2007); J.-L. Le and Bazant (2011)). It was shown that a quasibrittle structure (of positive geometry) must be modeled by a finite (rather than infinite) weakest-link model, and that the cdf of structural strength as well as lifetime varies from nearly Gaussian to Weibullian as a function of structure size and shape. Excellent agreement with experimentally obtained distributions was demonstrated.

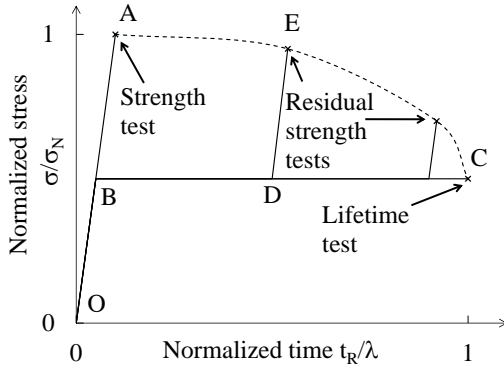In this paper, the theory is briefly reviewed and

Figure 1: Schematic of various load histories

extended to the probabilistic distributions of residual strength after a period of sustained load. Knowing the statistics of residual strength is important for meaningful estimates of safety factors by taking into account the strength degradation of the structure depending on the load history and duration. It is also important to better estimate the remaining service life of structures, for which maintenance design is a primary concern such as modern large aircraft made of load bearing quasibrittle composites.

2. THEORETICAL FORMULATION

The nano-mechanical derivation of the cdf of RVE strength as well as lifetime under static and cyclic loads is based on the fact that failure probability can be exactly predicted only on the atomic scale because the bond breakage process is quasi-stationary, which means that the probability must be exactly equal to the frequency (Kramer's rule). To derive the statistics of residual strength of an RVE, it is first noted that the crack growth rate on the atomic scale must follow a power law of applied stress with the exponent of 2. Equating the time rates of energy disssipations on the RVE and on the atomic level explains why Evans' law for sub-critical macrocrack growth has a much higher exponent, typically about 10 for concrete and 30 for tough ceramics (Evans (1972); M. D. Thouless and Evans (1983); Evans and Fu (1984)).

Using Evans' law to integrate the failure probability contributions over time yielded a simple relation be-tween the strength and static lifetime statistics (J.-L. Le and Bazant (2011)) – assuming the mechanisms of crack growth in a strength test and

a static lifetime test are the same. The argument is extended here to the statistics of residual strength.

2.1. Relation between structural strength, static lifetime and static residual strength

Evans's law for subcritical crack growth under sus-tained load reads

$$\dot{a} = A\, e^{-Q_0/kT} K_1{}^n \qquad (1)$$

Where a is the crack length, $\dot{a} = da/dt$, t=time, A=material constant, Q_0 = activation energy, k=Boltzmann constant and T = absolute temperature. The stress intensity factor is denoted as K_1 where the subscript 1 indicates the RVE level. So, we have $K_1 = \sigma\sqrt{l_0}k_1(\alpha)$ where $\sigma = F/l_0^2$ = nominal stress, l_0 = RVE size, $\alpha = a_1/l_0$ = relative crack length and k_1 = dimensionless stress intensity factor. Accordingly, the above equation becomes:

$$\dot{a} = Ae^{-Q_0/kT}\sigma^n l_0^{n/2}k_1{}^n(\alpha) \qquad (2)$$

Consider now the different load histories illustrated in Fig. 1. The load history O-A corresponds to the strength test, O-B-C to a static lifetime test and O-B-D-E to a residual strength test. Integration over load history O-B-D-E provides:

$$\frac{1}{r}\int_0^{\sigma_0}\sigma^n d\sigma + \int_{t_0}^{t_R}\sigma_0^n dt + \frac{1}{r}\int_{\sigma_0}^{\sigma_R}\sigma^n d\sigma =$$
$$e^{Q_0/kT}\int_{\alpha_0}^{\alpha_c}\frac{1}{Al_0^{\frac{n-2}{2}}k_1{}^n(\alpha)}d\alpha \qquad (3)$$

By a similar integration of load histories O-A and O-B-C and appropriate substitution, one gets a very simple correlation between σ_N, λ, and σ_R as

$$\sigma_R = [\sigma_N{}^{n+1} - \sigma_0^n(n+1)(rt_R - \sigma_0)]^{\frac{1}{n+1}} \qquad (4)$$

This is the equation for the degradation of the residual strength as a function of two independent (deterministic) variables, applied load and time of sustained load application. This equation also represents a link between the short-time strength and the residual strength.
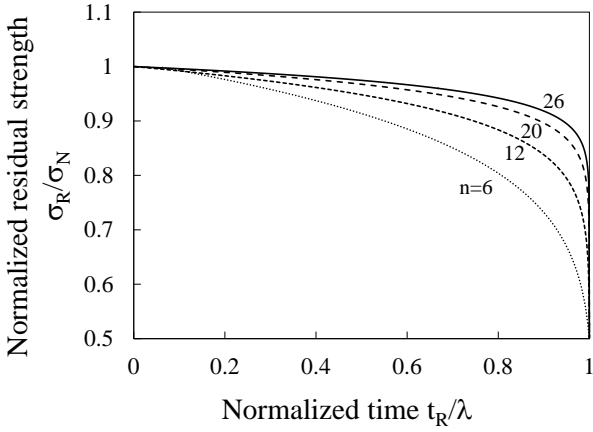
Figure 2: Predicted degradation curves for various values of static crack growth exponent n

2.2. Analysis of residual strength degradation for one RVE

We now proceed to analyze the effect of the exponent of the crack propagation law (Eq. 1) on the residual strength degradation. Fig. 2 shows the degradation in strength of one RVE under static load for various values of n for applied load $\sigma_0 = 0.5\sigma_N$. The time of load application, normalized with respect to the lifetime, is shown on the horizontal axis. σ_N is assumed to be unity and the loading rate is taken as 0.5 MPa/second. It is seen that the rate of strength degradation is negligible initially but progressively increases and the most rapid degradation is seen in the end. This effect is seen to be more pronounced for higher values of n. Based on this observation, the degradation curve could be roughly divided in two regions, one of relatively slow degradation and one of rapid degradation – the distinction being more pronounced for higher values of n. This study reveals the usefulness of Eq. 4 since for given load parameters and crack growth exponent, one may determine the portion of lifetime for which the strength degradation is negligible.

2.3. Formulation of statistics of residual strength for one RVE

The analysis of inter-atomic bond breaks and multiscale transitions to the RVE has shown that the strength of one RVE must have a Gaussian distribution transitioning to a power law in the tail of

probability within the range of 10^{-4} to 10^{-3} (J.-L. Le and Bazant (2011)). Starting from the cdf of strength, it is now possible to determine the cdf of residual strength for one RVE by means of Eq. 4. This yields (M. Salviato and Bazant (2014)) :

$$P_{(\sigma_R)} = 1 - \exp[-(\langle \sigma_R^{n+1} + \sigma_A \rangle / s_R)^{\overline{m}}] \quad (5)$$

for $\sigma_0 \leq \sigma_R < \sigma_{R,gr}$, and

$$P_{1,R} = P_{gr} + \frac{r_f}{\sqrt{2\pi}\delta_G} \int_{\sigma_{R,gr}}^{(\sigma_R^{n+1}+\sigma_A)^{\frac{1}{n+1}}} e^{-(\sigma'-\mu_G)^2/2\delta_G^2} d\sigma' \quad (6)$$

for $\sigma_R \geq \sigma_{R,gr} > \sigma_0$

Note that in above eqs, $\sigma_A = \sigma_0^n(n+1)(rt_R - \sigma_0)$, $\sigma_{R,gr} = (\sigma_{N,gr}^{n+1} - \sigma_A)^{1/(n+1)}$, while for the parameters $s_R = s_0^{n+1}$, $\overline{m} = m/(n+1)$; $P_{1,R}$ represents the probability of failure of one RVE under an overload, and $P_{1,R}(\sigma_0)$ represents the probability of failure of one RVE before the overload is applied. Note that only the part of the cdf where the residual strength is defined, i.e. where $\sigma_R \geq \sigma_0$ is considered.

Unlike the strength distribution, the residual strength cdf of one RVE does not have a pure Weibull tail. It is noteworthy that Eq. 5 describes a three parameter Weibull distribution in the variable σ_R^{n+1}, which has a finite threshold. Although it was proved that there can be no finite threshold in the distribution of strength (J.-L. Le and Bazant (2011)), the same does not hold true for the residual strength. The existence of a threshold value, σ_A in the cdf stems from the fact that some specimens could fail already during the period of sustained preload, which excludes them from the statistics of the overload. These are the specimens for which $\lambda < t_R$ or $\sigma_N < \sigma_0$.

2.4. Formulation of residual strength cdf for structures of any size

Once the cdf of residual strength related to one RVE is found, the cdf of failure of a structure of any size and geometry can be determined by means of the weakest link theory. The general applicability of this theory for brittle, ductile or quasi-brittle

structures is guaranteed by the definition of RVE itself and the fact that failure is considered to occur at macro-crack initiation. One RVE is defined as the smallest part of the structure whose failure causes the failure of the entire structure. Thus, the RVE statistically represents a link (the failing RVE is the weakest link) and the structure can be statistically treated as a chain.

Similar to the definition of nominal strength, we define the nominal applied stress, $\sigma_0 = c_n P / bD$ or $c_n P / D^2$ for two- or three-dimensional scaling, where P = applied load. Then, by applying the joint probability theorem to the survival probabilities, the residual strength distribution of the structure can be expressed as:

$$P_{f,R} = 1 - \prod_{i=1}^{N} \{1 - P_{1,R}[\langle \sigma_0 s(\mathbf{x}_i) \rangle, t_R, \sigma_R]\} \quad (7)$$

where $s(\mathbf{x})$ = dimensionless stress field and $\mathbf{x}$ is the position vector. Similar to the chain model for the cdf of structural strength, the residual strength of the ith RVE is here assumed to be governed by the maximum average principal stress $\sigma_0 s(\mathbf{x}_i)$ within the RVE, which is valid provided that the other principal stresses are fully statistically correlated.

3. RESULTS AND DISCUSSION

3.1. *Optimum fits of strength and residual strength histograms of borosilicate glass*

In this section, we determine the parameters of the distribution by fitting strength histograms and then we use them to predict the cdf of residual strength of borosilicate glasses. The predictions are compared to experiments by (Sglavo and Renzi (1999)). Figure 3a to 3d show the experimentally observed strength and residual strength histograms plotted in the Weibull scale. All the data considered were determined by conducting, in deionized water, four-point bend tests of borosilicate glass rods with a nominal diameter of 3 mm and length of 100 mm. The loading rate was set to about 60 MPa/s and different sustained load durations were used. Since glass is a brittle material and its RVE size is very small compared to the tested specimen size, the distribution of strength is virtually indistinguishable

from the Weibull distribution, as can be seen in Fig 3. By the optimum fitting of strength and residual strength, a Weibull modulus m of about 6 and a value of n of about 30 have been estimated. The fit predicted by the statistical formulation, shown by the solid line curves, is seen to be in good agreement with the experimental results. Except for the one hour case, all the other plots show the deviation of the residual strength distribution from the strength distribution to reach the probability value. It should be emphasized that, despite the scatter and a low number of data, all the residual strength distributions are predicted using the same set of parameters.

3.2. *Optimum fit of strength histograms and prediction of lifetime and mean residual strength for unidirectional glass/epoxy composites*

The methodology of the previous section is now pursued for the strength, lifetime and residual strength data on unidirectional glass-epoxy composites re-ported by (Hahn and Kim (1975)). Each specimen analyzed consisted of 8 unidirectional plies. 71 specimens were tested to obtain the strength and lifetime distributions. A constant sustained load 758 MPa was applied for all the lifetime tests. Fig. 4a shows the fit of strength histograms by means of the grafted Gauss-Weibull distribution in the Weibull scale. This fit shows a kink in the curve corresponding to the transition from Weibull to Gaussian distribution. A value of m equal to 56 and a value of n equal to 27 are estimated by least-square optimum fitting. Now that the required parameters of the distribution have been identified, the theory is applied to predict the mean residual strength and compare it to the experimental data. The comparison is made only for the mean since the number of available data is not sufficient to study the entire cdf. The resulting cdf of residual strength is then used to compute the mean values. The results are shown in Figure 4b for the different initial overloads and durations considered. Note that the predictions agree with the experiments, the difference being always less than 7%. The agreement provides another support for the present theory.
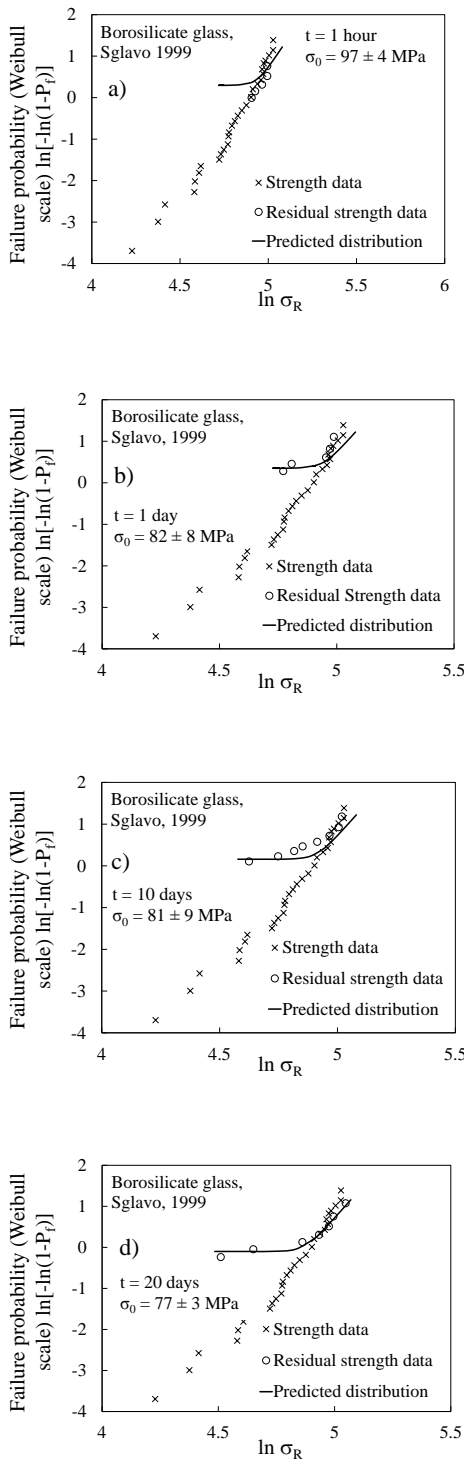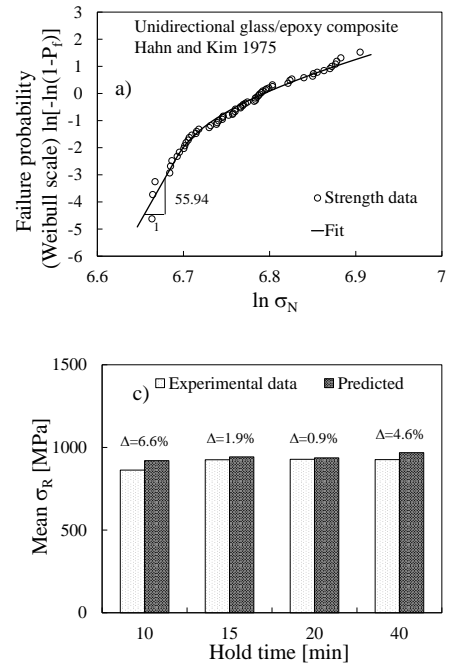
Figure 4: (a) Optimum Gauss Weibull fit of strength histogram (b) Comparison of predicted mean residual strength for unidirectional glass/epoxy composite (Hahn and Kim, 1975)

3.3. Size effect on mean residual strength

A more severe check on the theory would be to test the size effect on the mean lifetime and residual strength. However, no such test data seem to be available in the literature. It is nevertheless interesting to predict the size effect on the mean residual strength integrating Eq. 7. Figure 5 shows the calculated size effect on the mean residual strength of 99.6% Al_2O_3 . The set of parameters of the distribution is determined from the strength and lifetime histograms reported in (Fett and Munz (1991)). An applied load $\sigma_0 = 0.78\overline{\sigma}_N$ is considered. Different times of load application are used, as reported in the figure, depending on the mean strength, i.e., $rt_R = \beta\overline{\sigma}_N$. Note that, for a given t_R, the mean residual strength shows a similar trend as the strength and lifetime for the large size limit. In fact, the means tend to a straight line with the same slope as the mean strength.

It is impossible to obtain closed-form analytical expressions for the mean residual strength. However, sufficiently accurate analytical formulas can be derived by asymptotic matching. The size effect
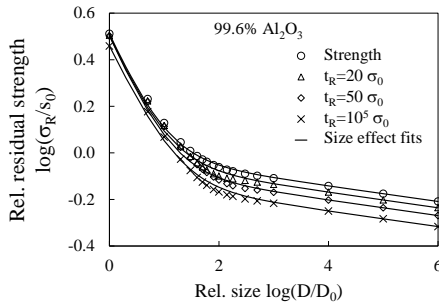

Figure 3: Optimum fits of residual strength histograms for borosilicate glass Hold times : (a) 1 hour (b) 1 day (c) 10 days and (d) 20 days

Figure 5: Calculated size effect curves on the mean strength residual strength at different hold times for 99.6% Al_2O_3

can reasonably be approximated by the equation:

$$\overline{\sigma}_R = \left[\frac{M_a}{D} + \left(\frac{M_b}{D} \right)^{\eta/m} \right]^{1/\eta} \qquad (8)$$

where m is the Weibull modulus of the cdf of strength and M_a, M_b and η can be derived by matching three asymptotic conditions:

1. $[\overline{\sigma}_R]_{D \to l_0}$
2. $[d\overline{\sigma}_R/dD]_{D \to l_0}$ and
3. $[\overline{\sigma}_R D^{1/m}]_{D \to \infty}$

As can be noted from Figure 5, the approximation given by Eq. 8 is rather good for all the different times of load application. In deriving the foregoing result, the two ratios, i.e., the applied load to strength and the hold time to lifetime, were kept constant across the sizes. It is trivial to note however that if the absolute value of the applied load or the hold time, or both, are kept constant, the size effect will of course be much stronger. However, in this case, the mean residual strength does not resemble the strength curve and it cannot be described by Eq. 8.

4. CONCLUSIONS

- A theory for predicting the probabilistic distributions of residual strength after a period of static load has been developed and validated against test data. An important practical merit of the present theory combined with predecessors (Bazant and Pang (2006, 2007); J.-L. Le and Bazant (2011)) is that it provides a way to determine the strength, residual strength and

lifetime distributions without any histogram testing.

- The rate of degradation of strength under a constant static load is not constant. Initially it is very slow and in the end very rapid. This effect is more pronounced for higher static crack growth exponents.

- The cdf of residual strength of quasibrittle materials may be closely approximated by a graft of Gaussian and Weibull distributions. In the left tail, the distribution is a three parameter Weibull distribution in the variable . Unlike the cdf's of strength and lifetime, the cdf of residual strength has a finite threshold, albeit often very small.

- The finiteness of the threshold is explained by the fact some specimens may fail during the sustained static preload and are thus excluded from the statistics of overload.

- An expression for the size effect on the residual strength is derived using asymptotic matching. It is shown that the size effect on the residual strength is as strong as the size effect on strength.

- Good agreement with the existing test data on glass-epoxy composites and on borosilicate and soda-lime silicate glasses is demonstrated.

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