A phase space reconstruction method for getting instantaneous probability density function of nonlinear stochastic systems

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ABSTRACT: Dynamic response analysis of non-linear structures involving random parameters has been an important and challenging problem for a long time. A variety of theoretical and numerical methods have been studied for engineering purpose. As a newly developed method, the probability density evolution method (PDEM) is capable of capturing the instantaneous probability density function (PDF) of stochastic dynamic responses of structures. In this paper, a new method named "phase space reconstruction method" (PSRM) is introduced. By this method, the instantaneous PDF of dynamic responses of highly non-linear systems can be obtained effectively. Some examples, including a Riccati oscillator, a SDOF oscillator, and a Van der Pol oscillator, are investigated. The results demonstrate the applicability and accuracy of PSRM in getting the solution of the generalized density evolution equation (GDEE) .

Dynamic response analysis of structures is inherently uncertain due to the uncontrollability of many factors in engineering practice, such as the material properties, the external loads and the boundary conditions. As a consequence, great efforts have been devoted to the quantification of uncertainty in dynamic response analysis of engineering structures.

For structures with random parameters, after several decades of development, the available approaches can be broadly classified into three categories, i.e., the Monte Carlo simulation method (Shinozuka 1972), the random perturbation technique (Kleiber and Hien 1992) and the orthogonal polynomials expansion method (Ghanem and Spanos 1990). However, it seems that there is still no approach that can satisfy the requirements of both accuracy and efficiency in the analysis of non-linear stochastic structures. In addition, all the approaches mentioned above just consider the moment information such as the mean and the standard deviation which could not completely capture the time-varying property of the probability density function (PDF). As a newly developed method, the probability density evolution method (PDEM) proposed by Li and Chen (Li J 2006; Li and Chen 2009)2009) is capable of capturing the instantaneous probability density function (PDF) of dynamic responses of structures. Because it is difficult to obtain the analytical solution of the probability density evolution equation (PDEE), numerical algorithms such as the finite difference method is more practical for applications.

In this paper, a more accurate, efficient method named the "phase space reconstruction method" (PSRM) is presented for the solution of PDEM. By the proposed method, the probabilistic solutions of three typical SDOF oscillators are obtained. The comparisons between analytical solutions and the PSRM demonstrate the accuracy, efficiency and convenience of the proposed method.

1. THE GENERALIZED DENSITY EVOLUTION EQUATION

Without loss of generality, a stochastic dynamical system can be expressed as

$$\dot{\mathbf{X}}(t) = \mathbf{\Phi}(\mathbf{X}, \mathbf{\Theta}, t) \tag{1}$$

with the initial condition

$$\mathbf{X}(t)\big|_{t=0} = \mathbf{x}_0 \tag{2}$$

where $\mathbf{X} = (X_1, X_2, \cdots, X_N)^{\mathrm{T}}$ is the state vector consisting of N components $X_i(t)$, $\mathbf{\Phi} = (\Phi_1, \Phi_2, \cdots, \Phi_N)^{\mathrm{T}}$ the dynamic operator, $\mathbf{\Theta} = (\Theta_1, \Theta_2, \cdots, \Theta_s)^{\mathrm{T}}$ the s-dimensional vector with the joint probability density function (PDF) $p_{\mathbf{\Theta}}(\mathbf{\theta})$, $\mathbf{x}_0 = (x_{0,1}, x_{0,2}, \cdots, x_{0,s})^{\mathrm{T}}$ the initial value vector with the PDF $p_{\mathbf{X}_0}(\mathbf{x})$.

It is well known that, under certain regularity conditions, the solution of Eq. (1) and Eq. (2) exists and is unique. This solution is expressible in the form

$$\begin{aligned} &\mathbf{X}(t) = \mathbf{H}(\mathbf{\Theta}, t) \quad \text{or} \\ &X_j(t) = H_j(\mathbf{\Theta}, t), \quad j = 1, 2, \cdots, N \end{aligned} \tag{3}$$

Then, according to the principle of preservation of probability, there exists the generalized density evolution equation (Li and Chen 2009)

$$\frac{\partial p_{\mathbf{X\Theta}}(\mathbf{x}, \mathbf{\theta}, t)}{\partial t} + \sum_{j=1}^{N} \dot{X}_j(\mathbf{\theta}, t) \cdot \frac{\partial p_{\mathbf{X\Theta}}(\mathbf{x}, \mathbf{\theta}, t)}{\partial x_j} = 0 \tag{4}$$

with the initial condition

$$p_{\mathbf{X\Theta}}(\mathbf{x}, \mathbf{\theta}, t) = \delta(\mathbf{x} - \mathbf{x}_0) p_{\mathbf{\Theta}}(\mathbf{\theta}) \tag{5}$$

As a partial differential equation, the analytical solution of Eq. (4) is hard to achieve, but its numerical solution is usually available (Li and Chen 2009).

2. PHASE SPACE RECONSTRUCTION METHOD

Note that the generalized density evolution equation (Eq.(4) and (5)) is a first order quasi-linear partial differential equation. The method of characteristics is advantageously used to obtain the analytical solution. Actually, the formal solution of Eq. (4) and Eq. (5) can be written as

$$p_{\mathbf{X\Theta}}(\mathbf{x}, \mathbf{\theta}, t) = \delta(\mathbf{x} - \mathbf{H}(\mathbf{\theta}, t)) p_{\mathbf{\Theta}}(\mathbf{\theta}) \tag{6}$$

Let

$$\mathbf{G}(\mathbf{\theta}, \mathbf{x}, t) = \mathbf{x} - \mathbf{H}(\mathbf{\theta}, t) \tag{7}$$

then Eq. (6) becomes

$$p_{\mathbf{X\Theta}}(\mathbf{x}, \mathbf{\theta}, t) = \delta[\mathbf{G}(\mathbf{\theta}, \mathbf{x}, t)] p_{\mathbf{\Theta}}(\mathbf{\theta}) \tag{8}$$

For $a \neq 0$, there is

$$\delta(a\mathbf{x}) = \frac{1}{|a|} \delta(\mathbf{x}) \tag{9}$$

More generally, the delta function of $g(\mathbf{x})$ is given by (Kanwal 2004)

$$\delta[g(\mathbf{x})] = \sum_i \frac{\delta(\mathbf{x} - \mathbf{x}_i)}{|g'(\mathbf{x}_i)|} \tag{10}$$

where $\mathbf{x}_i$ are the roots of $g(\mathbf{x})$. Substituting Eq. (10) in the formal solution of the PDEE (8) leads to

$$\delta[\mathbf{G}(\mathbf{\theta}, \mathbf{x}, t)] = \sum_{i=1}^{N_{\mathrm{sol}}} \frac{\delta(\mathbf{\theta} - \tilde{\mathbf{\theta}}_i(\mathbf{x}, t))}{\left| \frac{\partial \mathbf{G}(\mathbf{\theta}, \mathbf{x}, t)}{\partial \mathbf{\theta}} \Big|_{\mathbf{\theta} = \tilde{\mathbf{\theta}}_i(\mathbf{x}, t)} \right|} \tag{11}$$

where $\tilde{\mathbf{\theta}}_i(\mathbf{x}, t)$ are the root lines of $\mathbf{G}(\mathbf{\theta}, \mathbf{x}, t) = 0$, N_{sol} is the number of the root lines of $\mathbf{G}(\mathbf{\theta}, \mathbf{x}, t) = 0$.

The Dirac delta function has the fundamental property that

$$\int_{-\infty}^{\infty} f(\mathbf{x}) \delta(\mathbf{x} - a) \mathrm{d}\mathbf{x} = f(a) \tag{12}$$

Combing Eqs. (8), (11), and (12) and taking the integral of $\mathbf{\theta}$ will obtain

$$\begin{aligned}
&p_{\mathbf{X}}(\mathbf{x},t)\\
&=\int_{\Omega_\Theta} p_{\mathbf{X}\Theta}(\mathbf{x},\theta,t)\mathrm{d}\theta\\
&=\int_{\Omega_\Theta} \delta[\mathbf{G}(\theta,\mathbf{x},t)]p_\Theta(\theta)\mathrm{d}\theta\\
&=\int_{\Omega_\Theta}\sum_{i=1}^{N_{\mathrm{sol}}}\frac{\delta(\theta-\tilde{\theta}_i(\mathbf{x},t))}{\left|\frac{\partial\mathbf{G}(\theta,\mathbf{x},t)}{\partial\theta}\Big|_{\theta=\tilde{\theta}_i(\mathbf{x},t)}\right|}p_\Theta(\theta)\mathrm{d}\theta\\
&=\sum_{i=1}^{N_{\mathrm{sol}}}\frac{p_\Theta(\tilde{\theta}_i(\mathbf{x},t))}{\left|\frac{\partial\mathbf{G}(\theta,\mathbf{x},t)}{\partial\theta}\Big|_{\theta=\tilde{\theta}_i(\mathbf{x},t)}\right|}H[\Omega_{\Theta_i}]
\end{aligned}\tag{13}$$

where $H(\cdot)$ is Heaviside's function, Ω_{Θ_i} is the regions of integration.

3. CASE STUDY

In order to check the validity of the proposed method, three oscillators are studied.

3.1.1. A Riccati oscillator

The equation of this oscillator reads

$$\dot{X}+\theta\cdot X^2-X=0\tag{14}$$

with the initial condition $X(0)=1$. The random variable θ follows the standard normal distribution with PDF

$$p_\Theta(\theta)=\frac{1}{\sqrt{2\pi}}\exp(-\frac{\theta^2}{2})\tag{15}$$

The analytical solution of Eq. (14) is

$$X(\theta,t)=\frac{\exp(t)}{1+\theta[\exp(t)-1]}\tag{16}$$

Let

$$\begin{aligned}G(\theta,x,t)&=x-X(\theta,t)\\&=x-\frac{\exp(t)}{1+\theta[\exp(t)-1]}\end{aligned}\tag{17}$$

When $t=1$, there is

$$\begin{aligned}G(\theta,x)&=x-X(\theta)\\&=x-\frac{\exp(1)}{1+\theta[\exp(1)-1]}\end{aligned}\tag{18}$$

The curve of $G(\theta,x)=0$ when $t=1$ are shown in Figures 1.

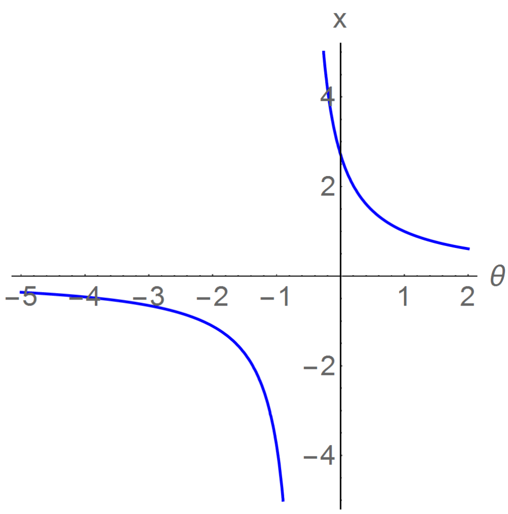

Figure 1: $X-\theta$ curve satisfying $G(\theta,x)=0$ when $t=1$

According to Eq. (17) and Figure 1, we have

$$\begin{aligned}
&\delta[G(\theta,x)]\\
&=\frac{\delta(\theta-\tilde{\theta}_1(x))}{\left|\frac{\partial G(\theta,x)}{\partial\theta}\Big|_{\theta=\tilde{\theta}_1(x)}\right|}\\
&=\frac{\delta(\theta-\frac{\exp(1)-x}{x[\exp(1)-1]})}{\left|\frac{\exp(1)[\exp(1)-1]}{\{1+\theta[\exp(1)-1]\}^2}\Big|_{\theta=\tilde{\theta}_1(x)}\right|}\\
&=\frac{\delta(\theta-\frac{\exp(1)-x}{x[\exp(1)-1]})}{\left|\frac{\exp(1)[\exp(1)-1]}{\left\{1+\frac{\exp(1)-x}{x[\exp(1)-1]}[\exp(1)-1]\right\}^2}\right|}
\end{aligned}\tag{19}$$

Then Eq. (13) can be rewritten in the form

$$
\begin{aligned}
& p_X(x,t=1) \\
&= p_{X_1}(x) \\
&= \frac{p_\Theta(\tilde{\theta}_1(x))}{\left|\frac{\partial G(\theta,x)}{\partial \theta}\Big|_{\theta=\tilde{\theta}_1(x)}\right|} H[\Omega_{\Theta_1}] \\
&= \frac{p_\Theta\left(\frac{\exp(1)-x}{x[\exp(1)-1]}\right)}{\left\| \frac{\exp(1)[\exp(1)-1]}{\left\{1+\frac{\exp(1)-x}{x[\exp(1)-1]}[\exp(1)-1]\right\}^2} \right\|} \\
& \cdot H[-\infty \; to \; \infty] \\
&= \frac{\frac{1}{\sqrt{2\pi}} \exp\{-\frac{\left[\frac{\exp(1)-x}{x[\exp(1)-1]}\right]^2}{2}\}}{\left\| \frac{\exp(1)[\exp(1)-1]}{\left\{1+\frac{\exp(1)-x}{x[\exp(1)-1]}[\exp(1)-1]\right\}^2} \right\|}
\end{aligned} \tag{20}
$$

The comparison between the result of proposed method and analytical solution is shown in Figure 2. It shows the results by the proposed method accord almost exactly with the analytical solution.

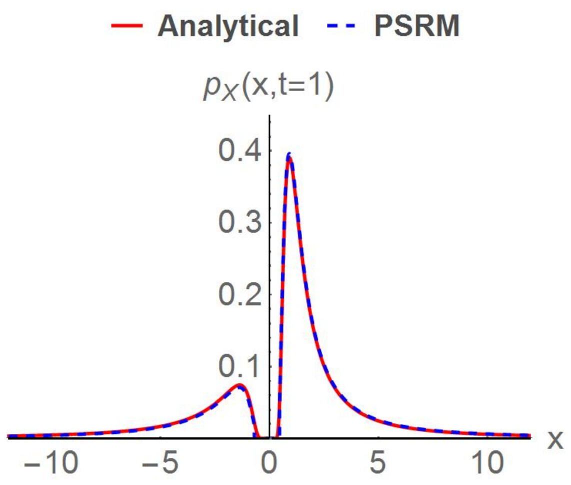

Figure 2: Comparison between the theoretical solution and PSRM result for the Riccati system

3.1.2. A SDOF oscillator

The equation of this oscillator reads

$$\ddot{X} + \omega^2 X = 0 \tag{21}$$

with the initial condition $X(0) = 0.1, \dot{X}(0) = 0$.

The random variable ω is uniformly distributed over the interval $[5\pi/4, 7\pi/4]$. The analytical solution of the problem is (Li and Chen 2009)

$$X(\omega,t) = 0.1\cos(\omega t) \tag{22}$$

The theoretical results of the system and the solution of the proposed method are shown in Figure 3. Clearly, the two curves accords almost exactly.

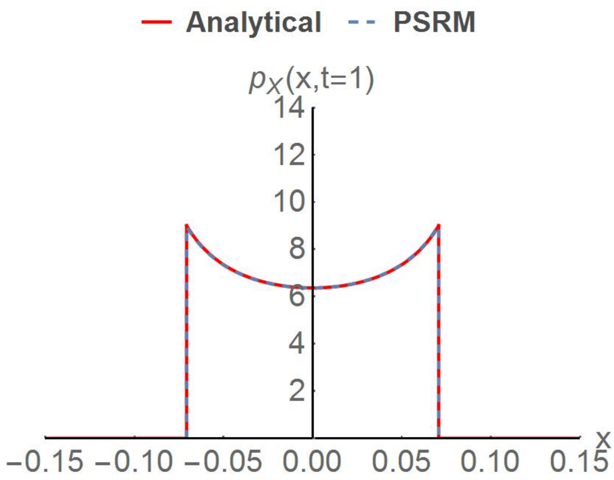

Figure 3: Comparison between the theoretical solution and PSRM for the SDOF system

3.1.3. A Van der Pol oscillator

The equation of this oscillator reads

$$\ddot{X} - (1 - X^2)\dot{X} + X = \theta \cdot \cos(1.1t) \tag{23}$$

with the initial condition $X(0) = 0, \dot{X}(0) = 1$.

The random variable θ is uniformly distributed over the interval $[0,5]$. 10^5 times of Monte Carlo simulation are carried on to verify the proposed method. The comparison between the Monte Carlo simulation result and the PSRM solution is shown in Figure 4. Again, it is seen that the proposed method is of high accuracy.

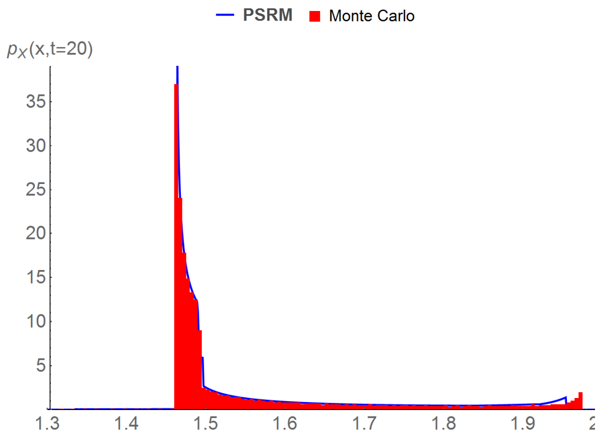

Figure 4: Comparison between the Monte Carlo simulation and PSRM for the Van der Pol system

4. CONCLUSIONS

The phase space reconstruction method (PSRM) is proposed to solve the generalized density evolution equation (GDEE). Three SDOF systems with random parameters, including a Riccati oscillator, a linear SDOF system and a Van der Pol oscillator, are studied as examples. The results are compared with the analytical solution or the Monte Carlo simulation results, verifying the accuracy of the proposed method.

The results show that, through the proposed method, the instantaneous PDF is obtainable with high accuracy and efficiency. More complex multi-dimensional stochastic systems with multiple basic random variables are to be studied in the future investigations.

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