Random Vibration of Arbitrarily Supported Single-Span Beams Subject to Random Moving Loads

Colin C. Caprani
Lecturer, Dept. of Civil Engineering, Monash University, Melbourne, Australia

ABSTRACT: This paper presents a framework to assess the random vibration response of single-span beams. The beams are arbitrarily supported and subject to a single random moving load of constant velocity. A General Modal Precise Integration Method (GMPIM) is proposed to carry out numerical integration of the problem. This method is extremely efficient and accurate. It is coupled with the Pseduo-Excitation Method (PEM) to arrive at the statistical description of the random vibration response. The approach uses the power spectral density (PSD) of the moving load to arrive at the PSD and root mean square of the response. The approach is far more computationally efficient than the alternative Monte Carlo simulation using Newmark-β integration (for example). As such, this work should find value for those estimating the random vibration of beams subject to random moving loads.

1. INTRODUCTION

1.1. Background

The vibration of beams under moving loads is a common problem in many engineering fields. From bridges to computer hard drives, it is vital to assess the level of vibrations expected so that adequate performance can be assured.

Recently, the research in the area has evolved significantly. From consideration of the deterministic problem, attention has moved to the random vibration response under random moving loads. Frýba (1976) considered the vibration response of a simply-supported beam under constant velocity moving random load of various closed-form power spectral densities. Zibdeh et al. (1995) also considered a simply-supported beam but under varying velocity moving Poissonian loads. Zibdeh and Rachwitz (1996) extend this to consider arbitrarily supported beams. Abu-Hilal (2003) considers the vibration of an arbitrarily supported beam under a varying-velocity load with white noise spectral density, which is a reasonable approximation to broadband type excitation. However, in many fields of application, excitation can be arbitrary and narrow-banded.

1.2. Contribution

It is clear from the literature that there is a need for a method that addresses the random vibration of arbitrarily supported beams under moving loads with narrowband, or arbitrary spectral density. While Monte Carlo simulation of the problem is feasible, due to the computational intensity an alternative method is desirable.

In this work, a numerical procedure is introduced which can determine the vibration response at an arbitrary point on arbitrarily supported single span beam when subject to a moving random point load of constant velocity but with arbitrary spectral density. A numerical procedure is adopted, since the achievement of closed-form solutions for arbitrary load spectral densities is difficult.

The numerical procedure developed is a fast and computationally efficient alternative to Monte Carlo simulation. The pseudo-excitation method (PEM) is used to determine the response statistics. As part of the method, many time-domain calculations of the vibration response to a single harmonic moving force are required. Therefore a very efficient means of computing such response is adopted, namely a development of the precise integration method (PIM).

2. PSEDUO-EXCITATION METHOD

2.1. Background

The Pseudo Excitation Method (PEM) was introduced by Lin (1992) and others (Lin et al. 1994) and is explained in a number of publications (Lin 2004; Lin et al. 1997; Zhong 2004). It has strong computational advantages in comparison to other methods and is very well suited to systems with a large number of degrees of freedom that may have closely spaced frequencies. This is typical of real bridges. PIM and PEM have been used together as an accurate and efficient means of establishing stochastic response to stationary and non-stationary loads (Lin et al. 1997).

Caprani (2014c) explains the theoretical background to the PEM, and this is not repeated here for brevity. Instead, the application of the method is explained in relation to the problem of footbridge vibration.

2.2. Application of PEM

The procedure to calculate the non-stationary response due to a moving load with arbitrary spectral density using PEM is illustrated in Figure 1, and explained below:

1. Establish the spectral density of the pedestrian forcing function.
2. Divide the force spectrum into N frequency points of width Δf. For broadband excitation $N = 100$ is reported to be satisfactory (Lin 2004). For the narrowband excitation studied here a dense discretization of 100 steps per unit frequency is used here.
3. For each of the $j = 0,\ldots,N-1$ frequencies:.
   - Determine the spectral density $S_{XX}\left(j\Delta f\right)$
   - Apply the pseudo excitation $\tilde{x}\left(t, j\Delta f\right) = \sqrt{S_{XX}\left(j\Delta f\right)}\exp\left(ij\Delta ft\right)$ to the beam as a moving point load of constant velocity v. The real and imaginary parts of the pseudo excitation ($\tilde{x} = \tilde{x}_R + i\tilde{x}_I$) are applied separately to obtain the real and imaginary pseudo responses, which are then combined to find the complex pseudo response ($\tilde{y} = \tilde{y}_R + i\tilde{y}_I$). In this work, the very efficient GMPIM is used to get the response at $k = 0,\ldots,M-1$ time or distance steps, though any suitable integration can be used.
   - The spectrum $S_{YY}\left(t_k, j\Delta f\right)$ of response when the load is at point $x_k = vt_k$ is found from:

$$S_{YY}\left(\omega, t\right) = \tilde{y}\left(t,\omega\right)\tilde{y}^*\left(t,\omega\right) \tag{1}$$

in which the asterisk denotes the complex conjugate.

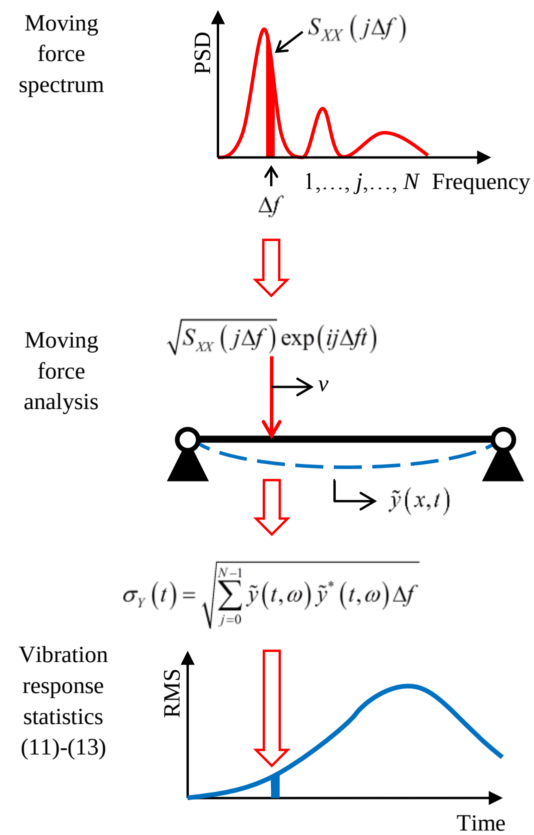

Figure 1: Illustration of the algorithm for footbridge vibration response from an arbitrary footfall force spectrum.

4. The mean-square of the response is found at each position of the load, k, or time, t, by integration of the response spectrum, for example:

$$E\left[Y^2(t)\right] = \sum_{j=0}^{N-1} S_{YY}\left(t_k, j\Delta f\right)\Delta f \qquad (2)$$

This is repeated for all time steps. For a zero mean process, such as acceleration at mid-span, the mean square and variance are equal, and so the root-mean-square (RMS) or standard deviation is:

$$\sigma_Y(t) = \sqrt{E\left[Y^2(t)\right]} \qquad (3)$$

This methodology relates to single moving load only, where a constant velocity is assumed during the traverse.

3. GENERAL MODAL PRECISE INTEGRATION METHOD

3.1. Precise Integration Method (PIM)

Zhong and Williams (1994) introduced the Precise Integration Method which allows calculation of vibration response to extreme accuracy, even for very long time steps, which can be multiples of the structure's first period, making the method extremely efficient. The accuracy comes from precise calculation of a transition matrix by iteration and closed-form solutions to loads that vary during the time step according to a known expression.

The basic equations for a multi-degree of freedom model of a structure are:

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{F}(t) \qquad (4)$$

with initial conditions: $\mathbf{x}(0) = \mathbf{x}_0; \dot{\mathbf{x}}(0) = \dot{\mathbf{x}}_0$. Writing this is state-space form we have:

$$\dot{\mathbf{v}} = \mathbf{H}\mathbf{v} + \mathbf{r} \qquad (5)$$

Where:

$$\mathbf{v} = \begin{Bmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \end{Bmatrix}; \mathbf{H} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix}; \mathbf{r} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{F} \end{Bmatrix} \qquad (6)$$

The general solution to (4) is:

$$\begin{aligned} \mathbf{v}(t) &= \exp(t\mathbf{H})\mathbf{v}_0 \\ &\quad + \exp(t\mathbf{H})\int_0^t \exp\left[-\tau\mathbf{H}\right]\mathbf{r}(\tau)\,d\tau \end{aligned} \qquad (7)$$

For calculation, the time is divided into steps of length Δt. The *k*th time step occurs at $t_k = k\Delta t$. Re-writing (7) for the time period between steps *k* and *k*+1, and introducing notation for the transition matrix $\mathbf{T}(t) \equiv \exp[t\mathbf{H}]$ gives:

$$\mathbf{v}(t_{k+1}) = \mathbf{T}(\Delta t)\mathbf{v}(t_k) + \int_{t_k}^{t_{k+1}} \mathbf{T}(t_{k+1} - \tau)\mathbf{r}(\tau)\,d\tau \qquad (8)$$

The second term on the right hand side is the particular integral and depends on the loading through the time step.

3.2. Modal PIM

Recently Caprani (2013) developed a Modal PIM (MPIM) in which the particular solution for (8) is found in closed form for the modal force caused by a moving load on a simply-supported beam. Each mode in this system can be written in the form of (5) as follows:

$$\begin{Bmatrix} \dot{q} \\ \ddot{q} \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\xi\omega \end{bmatrix} \begin{Bmatrix} q \\ \dot{q} \end{Bmatrix} + \begin{Bmatrix} 0 \\ Q(t) \end{Bmatrix} \qquad (9)$$

In which q is the modal coordinate, ξ its damping ratio and ω its natural frequency. In the case of a moving varying point load $F(t)$ with constant velocity, *v*, The modal force, $Q(t)$, is given by:

$$Q = \frac{F(t)\phi(vt)}{M} \qquad (10)$$

where M is the modal mass and $\phi(x)$ is the mode shape. Hence, the particular solution in (5) is given by:

$$\int \mathbf{T}(t-\tau)\mathbf{r}(\tau)\,d\tau = \frac{1}{\omega_d}\int_0^t \mathbf{B}(\tau)\,d\tau \qquad (11)$$

Where, writing $\phi \equiv \omega_d(t-\tau)$ and the damped natural frequency, $\omega_d = \omega\sqrt{1-\xi^2}$:

$$\mathbf{B}(\tau) = e^{-\xi\omega(t-\tau)} \begin{Bmatrix} \sin\phi \\ \omega_d \cos\phi - \xi\omega\sin\phi \end{Bmatrix} Q_i(t) \qquad (12)$$

In (10), if $F(t)$ is described by a Fourier series of frequencies α_j and phases φ_j , and we consider mode i of a simply-supported beam where $\phi(x) = \sin i\pi x/L$ and η_j is the amplitude of the jth force component, then:

$$Q_i = \sum_{j=0}^{n} \eta_j \sin\beta_i t \cos(\alpha_j t + \varphi_j) \qquad (13)$$

where $\beta_i = \omega_i v$. Caprani (2014a) gives the particular solution for (11) with (13).

3.3. General Modal PIM (GMPIM)

One of the contributions of this work is to extend the MPIM method beyond simply-supported beams to consider arbitrarily-supported beams. For such beams, the mode shape is described by (Weaver et al. 1990):

$$\begin{aligned}\phi(x) = C_1 \cos\omega x + C_2 \sin\omega x \\ + C_3 \cosh\omega x + C_4 \sinh\omega x\end{aligned} \qquad (14)$$

then the modal force (for mode i) for each force harmonic, j, at time t is:

$$\begin{aligned} Q_{i,j}(t) = C_1^{ij} \cos(\alpha_j t + \varphi_j)\sin\beta_i t \\ + C_2^{ij} \cos(\alpha_j t + \varphi_j)\cos\beta_i t \\ + C_3^{ij} \cos(\alpha_j t + \varphi_j)\sinh\beta_i t \\ + C_4^{ij} \cos(\alpha_j t + \varphi_j)\cosh\beta_i t \end{aligned} \qquad (15)$$

in which:

$$C_k^{ij} = C_k \frac{\eta_j}{M_i}; \quad k = 1,\ldots,4 \qquad (16)$$

For this problem, the particular solution is expressed as:

$$\int_{t_k}^{t_{k+1}} \mathbf{T}(t_{k+1}-\tau)\mathbf{r}(\tau)\,d\tau = \mathbf{v}_p(t_{k+1}) - \mathbf{T}(\Delta t)\mathbf{v}_p(t_k) \qquad (17)$$

where $\mathbf{v}_p(t)$ accounts for all forces acting and so for mode i is given by:

$$\mathbf{v}_p(t,i) = \sum_{j=0}^{N}\sum_{k=1}^{4} C_k^{ij} \mathbf{v}_{p,k}(t,i,j) \qquad (18)$$

and where the particular solutions for each term are:

$$\begin{aligned} \mathbf{v}_{p,1}(t,i,j) = \mathbf{v}_{11}(\Omega^+)\sin(\Omega^+ t + \varphi_j) \\ -\mathbf{v}_{21}(\Omega^+)\cos(\Omega^+ t + \varphi_j) \\ -\mathbf{v}_{11}(\Omega^-)\sin(\Omega^- t + \varphi_j) \\ +\mathbf{v}_{21}(\Omega^-)\cos(\Omega^- t + \varphi_j) \end{aligned} \qquad (19)$$

$$\begin{aligned} \mathbf{v}_{p,2}(t,i,j) = \mathbf{v}_{21}(\Omega^+)\sin(\Omega^+ t + \varphi_j) \\ +\mathbf{v}_{11}(\Omega^+)\cos(\Omega^+ t + \varphi_j) \\ +\mathbf{v}_{21}(\Omega^-)\sin(\Omega^- t + \varphi_j) \\ +\mathbf{v}_{11}(\Omega^-)\cos(\Omega^- t + \varphi_j) \end{aligned} \qquad (20)$$

$$\begin{aligned} \mathbf{v}_{p,3}(t,i,j) = \mathbf{v}_{12}^{sh}(t)\sin(\alpha_j t + \phi_j) \\ +\mathbf{v}_{12}^{sh}(t)\cos(\alpha_j t + \phi_j) \end{aligned} \qquad (21)$$

$$\begin{aligned} \mathbf{v}_{p,4}(t,i,j) = \mathbf{v}_{12}^{ch}(t)\sin(\alpha_j t + \phi_j) \\ +\mathbf{v}_{22}^{ch}(t)\cos(\alpha_j t + \phi_j) \end{aligned} \qquad (22)$$

In these, for the circular terms (19) and (20):

$$\mathbf{v}_{11}(\Omega) = \frac{\gamma_1(\Omega)}{2} \begin{Bmatrix} -(\Omega^2 - \omega_i^2) \\ 2\xi_i\omega_i\Omega^2 \end{Bmatrix} \qquad (23)$$

$$\mathbf{v}_{21}(\Omega) = \frac{\gamma_1(\Omega)}{2} \begin{Bmatrix} 2\xi_i\omega_i\Omega \\ \Omega(\Omega^2 - \omega_i^2) \end{Bmatrix} \qquad (24)$$

where:

$$\gamma_1^{-1}(\Omega) = (2\xi_i\omega_i)^2(\Omega^2 - 1) + (\Omega^2 - \omega_i^2)^2 \qquad (25)$$

and

$$\Omega^{-} = \alpha_j - \beta_i; \quad \Omega^{+} = \alpha_j + \beta_i \tag{26}$$

And for the hyperbolic terms, (21) and (22):

$$\mathbf{v}_{12}^{sh}(t,i,j) = \frac{1}{2}\left[e^{\beta t}\mathbf{v}_1(\beta_i) - e^{-\beta t}\mathbf{v}_1(-\beta_i)\right] \tag{27}$$

$$\mathbf{v}_{22}^{sh}(t,i,j) = \frac{1}{2}\left[e^{\beta t}\mathbf{v}_2(\beta_i) - e^{-\beta t}\mathbf{v}_2(-\beta_i)\right] \tag{28}$$

$$\mathbf{v}_{12}^{ch}(t,i,j) = \frac{1}{2}\left[e^{\beta t}\mathbf{v}_1(\beta_i) + e^{-\beta t}\mathbf{v}_1(-\beta_i)\right] \tag{29}$$

$$\mathbf{v}_{22}^{ch}(t,i,j) = \frac{1}{2}\left[e^{\beta t}\mathbf{v}_2(\beta_i) + e^{-\beta t}\mathbf{v}_2(-\beta_i)\right] \tag{30}$$

where:

$$\mathbf{v}_1(b) = \gamma_2(b)\begin{Bmatrix} \alpha_j\left[b + (b + 2\xi_i\omega_i)\right] \\ \alpha_j\left[(\alpha_j^2 + b^2) - \omega_i^2\right] \end{Bmatrix} \tag{31}$$

$$\mathbf{v}_2(b) = \gamma_2(b)\begin{Bmatrix} b(b + 2\xi_i\omega_i) - (\alpha_j^2 - \omega_i^2) \\ (b + 2\xi_i\omega_i)(\alpha_j^2 + b^2) + \omega_i^2 b \end{Bmatrix} \tag{32}$$

and

$$\begin{aligned}\gamma_2^{-1}(b) = &\left[b^2 + (\alpha_j^2 - \omega_i^2)\right] \\ &\times\left[(\alpha_j^2 - \omega_i^2) + (b + 2\xi_i\omega_i)^2\right] \\ &+\left[\omega_i^2(2b + 2\xi_i\omega_i)^2\right]\end{aligned} \tag{33}$$

Once the modal responses have been determined using the above procedure, the physical acceleration response, $\ddot{u}$, at a point of interest on the beam, x*, can be found from:

$$\ddot{u}(x^*,t) = \sum_{i=1}^{m}\phi_i(x^*)\ddot{q}_i(t) \tag{34}$$

4. EXAMPLE APPLICATION

For this work, an example application of the proposed GMPIM-PEM methodology is made. The random vibration response of a bridge subject to moving human walking force with a prescribed spectral density is determined.

4.1. Bridge

A 50 m long beam with 500 kg/m linear mass is used (Caprani et al. 2012). It has natural frequency of 2.00 Hz and a representative damping ratio for all modes of 0.5% (Heinemeyer et al. 2009). The pedestrian traverses the bridge at a constant velocity of 1.25 m/s with fundamental pacing frequency of $f_p = 2$ Hz, matching the bridge as an example of a critical case. The pedestrian weight is taken to be 800 N. A moving force model is used, and the structure response is found using the GMPIM as described above in the PEM algorithm.

The bridge support conditions are shown in Figure 2. The rotational stiffness at the supports varies from simply-supported to fully-fixed. Ten such support stiffnesses, *k*, are considered, logarithmically evenly spaced. The normalized support stiffness, *k'*, used are given by:

$$k' = \frac{kL}{EI} = 10^p \quad p = -1,\ldots,3 \tag{35}$$

Since the increasingly stiff rotational restraints alter the natural frequency of the bridge, it is not reasonable to compare vibration responses, since only those bridges with frequencies nearest the force, $f_p = 2$ Hz, will be most excited. Therefore, for each support stiffness considered, the flexural rigidity (*EI*) of the beam is altered so that the fundamental frequency remains at 2.0 Hz, giving results that are directly comparable. The mass is not altered so as to maintain the same energy dissipation in each simulation.

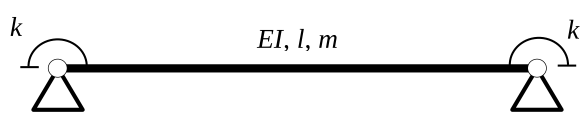

Figure 2: Considered beam with rotational spring supports.

4.2. Modal Properties

The modal properties for the arbitrarily supported beam are determined using the dynamic stiffness matrix and Wittrick-Williams algorithm (Wittrick and Williams 1971). The modal mass is estimated using a closed form solution, and the mode shapes are mass normalized so that the integral $\int_0^1 \phi_i^2(\xi)\, d\xi = 1$. This process will be further explained in a coming paper. In this work, all computations are carried out using the first 5 modes of all the beams considered.

4.3. Pedestrian Walking Force Spectrum

The dynamic load factors (DLFs) of Brownjohn et al. (2004) are used as the basis of the spectral model. The simple spectral model proposed by Caprani (2014b) is used here and is given by:

$$S_{XX}(f) = \sum_{i=1}^{n} \phi\left(f; \mu_j, \sigma_j\right) \frac{\eta_j^2}{2} \tag{36}$$

where $\phi(f)$ is the normal probability density function with mean of the jth harmonic $\mu_j = jf_p$ and the standard deviation found from an assumed coefficient of variation (CoV), $\mu_j = \sigma_j \times$CoV. The model is illustrated in Figure 3. Numerically, a discrete number of frequency intervals are used of width Δf. The force spectrum from a single pedestrian, or a population of pedestrians, can be matched by calibrating the CoV (Caprani 2014b). A CoV of 1.0% gives similar results to Brownjohn et al. (2004) and this is the value adopted here.

4.4. Results

The root mean square (RMS) response of the bridges (of different rotational support stiffnesses) are determined at midspan. Figure 4 shows the results for the extreme cases of no rotational stiffness (simply-supported) and full rotational stiffness (fixed-fixed beam), along with two intermediate cases of partial fixity.

The complete range of results obtained for the 10 increments of support rotational restraint are presented in Figure 5. An alternative view of this surface is given as a contour plot in Figure 6.

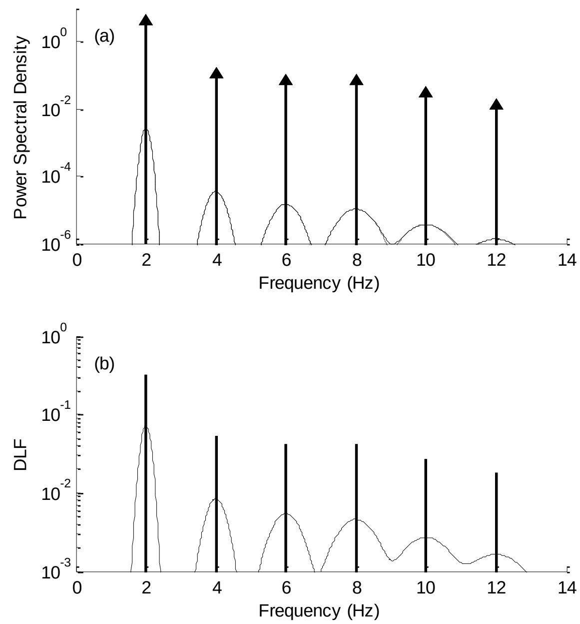

Figure 3: Simple spectral model for f_p = 2 Hz showing a 5% CoV for clarity (note that 1% is used in this paper): (a) idealized walking and normally-distributed discrete spectral densities of equal energy; (b) idealized and distributed discrete DLF spectrum, transformed from (a).

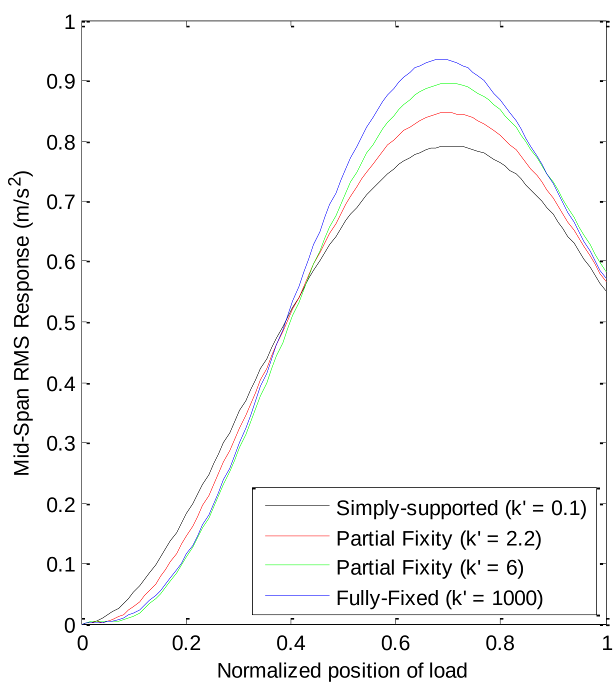

Figure 4: Vibration response of 2 Hz beams of varying rotational support fixities, subject to a constant velocity pedestrian walking force spectrum.

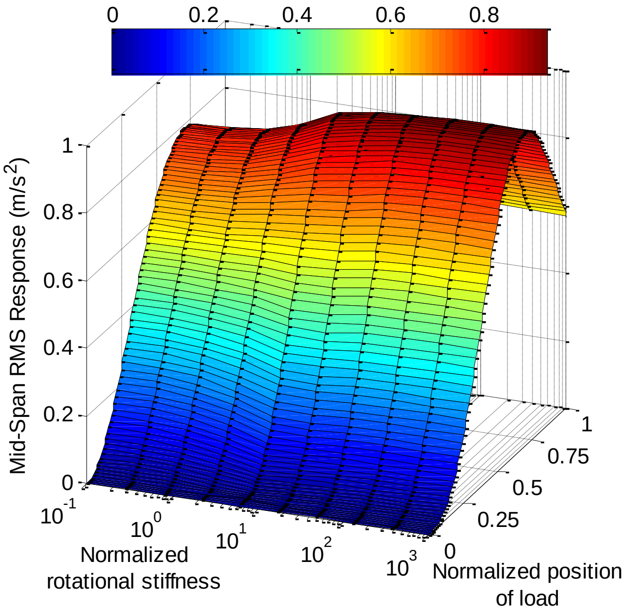

Figure 5: Complete suite of responses obtained for the RMS vibration response of the 2 Hz beams.

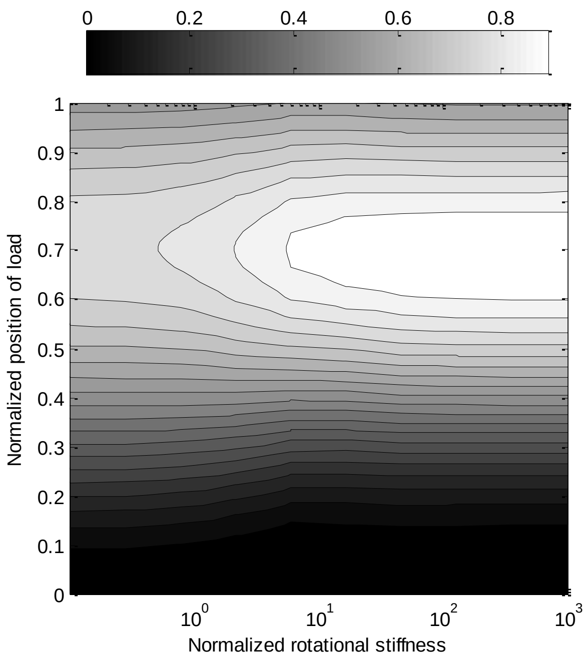

Figure 6: Alternative contour plot of the RMS vibration responses obtained for the 2 Hz beams.

4.5. Discussion

It is interesting to note from the presented results that, once it is ensured that all the beams, regardless of stiffness, have a natural frequency of 2 Hz, the mid-span vibration response increases for increasing rotational restraint. This is due to the higher mode ordinates towards the centre of the beam, once the mode shapes have been mass-normalized. To illustrate this point, Figure 7 shows the first two mode shapes for the extreme cases of simply-supported and fixed-fixed beams. From this figure it can be seen that the mode shape ordinates lead to the increasing response as the rotational fixity at the supports increases. It must be recalled here that the beam first natural frequencies are the same for both cases.

5. CONCLUSIONS

This work outlines a means of establishing the response spectrum of arbitrarily supported beams loaded by a moving force of arbitrary spectrum without recourse to computationally expensive Monte Carlo simulations. This vastly improved speed is due to a new general modal precise integration method when coupled with the pseudo excitation method. Problems that were computationally difficult previously, can now be addressed in short time. Consequently, the method may have application in structural control situations, where close to real-time evaluation of potential response spectra may be required.

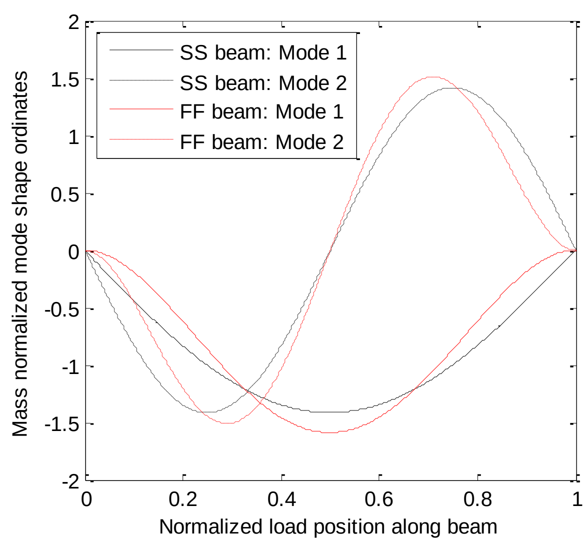

Figure 7: Mass normalized mode shapes for simply-supported (SS) and fixed-fixed (FF) beams.

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