Single vs Multi-drain Probabilistic Analyses of Soil Consolidation via Prefabricated Vertical Drains

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ABSTRACT: Natural soils are one of the most inherently variable in the ground. Although the significance of inherent soil variability in relation to reliable prediction of consolidation rates of soil deposits has long been realized, there have been few studies that addressed the issue of soil variability for the problem of ground improvement by prefabricated vertical drains (PVDs). Despite showing valuable insights into the impact of soil spatial variability on soil consolidation by PVDs, available stochastic works on this subject are based on a single drain (or unit cell) analysis. In a spatially variable soil, however, the condition of unit cell may be violated. Therefore, in a probabilistic context, it is necessary to assess the feasibility of performing an analysis based on the unit cell concept as compared to the multi-drain analysis. In this study, a rigorous stochastic finite element modeling approach that allows the nature of soil spatial variability to be considered in a quantifiable manner, both for the single and multi-drain cases, is presented. It is shown that with proper input statistics representative of a particular domain of interest, both single and multi-drain analyses yield almost identical results. This study also highlights the importance of proper modeling of soil spatial variability in design of ground improvement by PVDs.

1. INTRODUCTION

The use of prefabricated vertical drains (PVDs) in combination with pre-loading is becoming one of the most commonly used methods for promoting radial drainage and accelerating the time rate of soil consolidation. Natural soils, however, are highly variable in the ground due to the uneven soil micro fabric, geological deposition and stress history. Soil consolidation by PVDs is strongly dependent on several spatially variable soil properties, most significantly is the coefficient of consolidation. The review of relevant literature has indicated that although the significance of inherent soil variability in relation to reliable prediction of consolidation rates in soil deposits has long been realized (Rowe 1972), there have been a few studies (Hong and Shang 1998; Zhou et al. 1999; Bari et al. 2012, 2013; Bari and Shahin 2014) that used stochastic approaches to investigate the problem of ground improvement by PVDs for spatially variable soils. Despite showing valuable insights into the impact of soil spatial variability on soil consolidation, available stochastic works of PVD-improved ground are based on an idealized single-drain (or unit cell) analysis instead of considering the actual full multi-drain situation.

In practice, soil improvement via PVDs typically consists of hundreds of drains installed in square or triangular patterns, with spacing varying between 1–3m. This means that the area being treated including each drain in a numerical analysis can be significantly large and computationally too intensive. In order to reduce the computational effort, full three dimensional (3D) multi-drain system, is usually modeled by considering only a "unit cell" of soil positioned with a central vertical drain, so that the consolidation problem can be analyzed at the unit cell level. In deterministic context, the single-drain ''unit cell'' analysis is often sufficient to investigate the overall consolidation behavior of soil (Indraratna and Redana 2000). However, in a spatially variable soil, the condition of the unit cell may be violated. Therefore, the aim of this paper is to investigate the feasibility of performing an equivalent unit cell analysis to substitute the multi-drain analysis in a probabilistic context.

In order to treat the soil spatial variability in most geotechnical engineering problems, a stochastic computational scheme that combine the finite element (FE) method and Monte Carlo technique is often used (e.g., Fenton and Griffiths 2005; Huang et al. 2010; Bari et al. 2013; Bari and Shahin 2014). The same approach is adopted in the present study which allows the nature of soil spatial variability to be considered in a quantifiable manner, both for the single and multi-drain analyses. The approach involves the development of advanced numerical models that merge the local average subdivision (LAS) technique (Fenton and Vanmarcke 1990) of the random field theory (Vanmarcke 1984) and the finite element method into a Monte Carlo frame work. For the case of PVDs, the overall consolidation is governed by the radial (horizontal) flow of water rather than the vertical flow for the fact that the drainage length in the horizontal direction is much less than that of the vertical direction and the horizontal permeability is often much higher than the vertical permeability (Hansbo 1981). Under such reasoning, the soil consolidation in the current study is considered as a perfectly radial drainage problem where the single-drain influence area is approximated by a square (equivalent to a circular area). The results obtained from both the multi-drain analysis and idealized unit cell model are used to establish probability density functions relating to the degree of consolidation. In the sections that follow, the stochastic finite element Monte Carlo (FEMC) approach is described in some detail followed by detailed demonstration and discussion of the obtained results.

2. STOCHASTIC FINITE ELEMENT MONTE CARLO APPROACH

As indicated earlier, the equivalence between the single and multi-drain cases is examined by employing a stochastic FEMC approach. The procedure of the stochastic FEMC approach is as follows:

1. Create a virtual soil profile that contains realizations of the designated soil properties, allowing the inherent soil spatial variability to be realistically simulated;
2. Incorporate the generated realizations of soil profile into a FE modeling of soil consolidation by PVDs; and
3. Repeat these steps numerous times using the Monte Carlo technique by creating new realizations of virtual soil profile and performing the subsequent FE analysis so that a series of consolidation responses can be obtained from which the statistical distribution parameters of the output quantities can be estimated.

The above steps, as well as the numerical procedures, are described below.

2.1. Simulation of virtual soil profiles

In order to warrant the true influence of soil spatial variability for the problem in hand, virtual soil profiles that allow the rational distributions of the designated spatially variable soil properties across the soil mass need to be generated (based on a predefined probability density function (PDF) and a prescribed spatial correlation function) which can then be implemented in the FEM modeling. Prior to proceeding with this step, it is necessary to identify the soil properties that have the most significant impact on soil consolidation by PVDs so that they can be treated as random fields when creating the virtual soil profiles. As indicated earlier, spatial variability of several soil properties can affect soil consolidation by PVDs. However, as far as the radial drainage is concerned, the coefficient of horizontal consolidation, c_h, is the most significant random soil property affecting the behavior of soil consolidation by PVDs, as indicated by many researchers (e.g., Hong and Shang 1998; Zhou et al. 1999). Accordingly, in the current study, c_h, is considered to be the only spatially variable soil property, while the other soil properties are held constant and treated deterministically so as to reduce the superfluous complexity of the problem.

The spatial variability of c_h is assumed to be characterized by lognormal distribution because the observation obtained from field tests data reported by Chang (1985) suggested that the variation of c_h can be adequately modeled by a lognormal distribution. Based on the random field theory, random fields of a spatially variable soil property with lognormal distribution can be characterized by the soil property mean value, μ, variance, σ^2 (can also be represented by the standard deviation, σ) and correlation length or scale of fluctuation (SOF), θ. The value of θ describes the limits of spatial continuity and can simply be defined as the distance over which a soil property shows considerable correlation between two spatial points. Therefore, a large value of θ indicates strong correlation (i.e., uniform soil property field), whereas a small value of θ implies weak correlation (i.e., erratic soil property field). In this study, the LAS method (Fenton and Vanmarcke 1990) extracted from the random field theory (Vanmarcke 1984) is used to generate 2D random fields of c_h. The LAS algorithm generates realizations of c_h in the form of a grid of cells that are assigned locally averaged values of c_h different from one another across the soil mass, by taking full account of the finite element size in the local averaging process, albeit remained constant within each element of the soil domain.

In the process of simulating the realization of c_h, correlated local averages of standard normal random field $G(x)$ are first generated with zero mean, unit variance and spatial correlation function using the 2D LAS technique. The required lognormally distributed random field of c_h defined by μ_{c_h} and σ_{c_h} is then obtained using the following transformation function (Fenton and Vanmarcke 1990):

$$c_{h_i} = \exp\left\{\mu_{\ln c_h} + \sigma_{\ln c_h} G(x_i)\right\} \tag{1}$$

where, x_i and c_{h_i} are, respectively, the vector containing the coordinates of the center of the ith element and the soil property value assigned to that element; $\mu_{\ln c_h}$ and $\sigma_{\ln c_h}$ are, respectively, the mean and standard deviation of the underlying normally distributed c_h, i.e., ln(c_h). The correlation coefficient between c_h measured at a point x_1 and a second point x_2 is specified by an exponentially decaying spatial correlation function, $\rho(\tau)$, as follows (Fenton and Vanmarcke 1990):

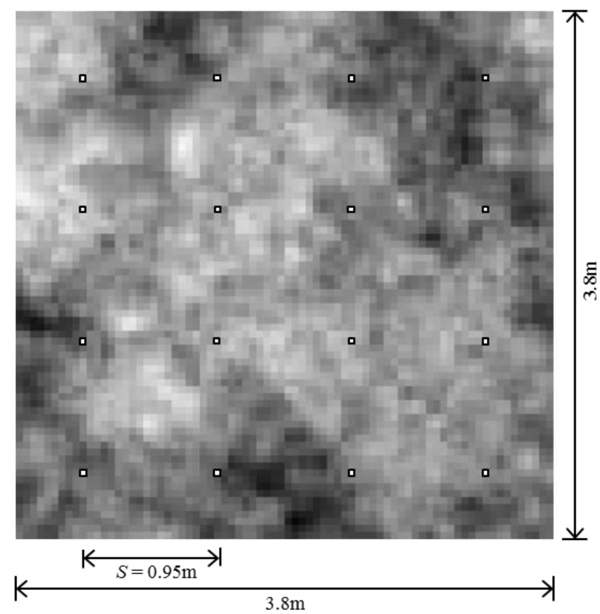
$$\rho(\tau) = \exp\left(-\frac{2|\tau|}{\theta_{c_h}}\right) \tag{2}$$

where, $\tau = |x_1 - x_2|$. It should be noted that the spatial correlation function in Eq. (2) is assumed to be statistically isotropic, i.e., SOF in the x and y directions on a horizontal plane are assumed to be the same (i.e., $\theta_{c_h(x)} = \theta_{c_h(y)} = \theta_{c_h}$). This means that on a horizontal (x-y) plane, c_h is spatially variable but it is spatially constant with infinite SOF in the vertical (z) direction. Although the correlation structures in a naturally occurring soil stratum are usually different in any spatial direction (i.e., anisotropic), the reason for assuming c_h as an isotropic random field is that the correlation structure is more related to the formation process (i.e., layer deposition). Therefore, on a horizontal plane the spatial correlation structure of c_h would have similar SOF in any direction. It is worthy to note that the spatial correlation length is estimated with respect to the underlying normally distributed field, i.e., ln(c_h).

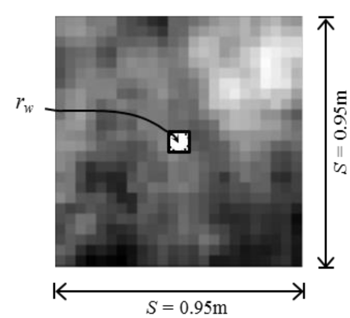
2.2. Finite element modeling incorporating soil spatial variability

The soil profile simulated in the previous step with the specified spatial variation of c_h can now be mapped onto a refined FE mesh. An uncoupled consolidation analysis is then followed. A modified version of the FE computational scheme ‘‘Program 8.6’’ as presented in the book by Smith and Griffiths (2004) is used in this study to carry out all the analyses in which soil consolidation is treated as a 2D uncoupled problem under an axisymmetric condition. Originally “Program 8.6” was for general two (plane) or three dimensional analyses of uncoupled consolidation using an implicit time integration with the ‘‘theta’’ method. The authors of the current paper modified the source code of “Program 8.6” to allow axisymmetric and repetitive Monte-Carlo analyses.

In this study, the potentially complicated multi-drain influence boundary is approximated by assuming a square grid pattern of sixteen drains enclosing an area equivalent to the sum of all single drain influence areas (see Figure 1a). The spacing, *S*, between the drains and the equivalent radius of the drain, r_w, are assumed to be equal to 0.95m and 0.032m, respectively. On the other hand, the spacing, *S*, in the multi-drain analysis represents the side length (*S*) of the square influence area in the single drain “unit cell” analysis (see Figure 1b). It should be noted that, during mandrel installation of PVDs, a disturbed zone (i.e., smear zone) of reduced permeability is produced. However, in the present study, no smear zone is considered bearing in mind that the main characteristics of the stochastic equivalence between the single and multi-drain consolidation needs to be known for the simplest case first, and more complex disparities on the subject is left for future refinement. In addition, for simplicity, the well resistance which may affect the rate of consolidation is also not considered. This is due to the fact that the discharge capacities of most PVDs available in the market are relatively high; hence, the impact of well resistance can be ignored in most practical cases, as suggested by many researchers (e.g., Chu et al. 2004).



(a)



(b)

Figure 1: Realizations of PVD-improved ground: (a) 16 drains in a square grid pattern; (b) single drain in a square geometry

Generally speaking, the more elements used to discretize the domain of the problem, the greater the accuracy of the FE solution. However, a trade-off between accuracy and run-time efficiency is necessary. In the current study, a sensitivity analysis on two different FE meshes with element sizes of 0.05m and 0.025m is conducted. For a certain SOF, two random fields of two selected meshes are generated using the same seed value, and FE analysis is conducted. The results obtained from the two meshes are then compared to see if they are identical, otherwise finer meshes are generated and the previous process is repeated. Several different random seeds and SOFs are tested, for the highest coefficient of variation of c_h considered in this study. It is found that a ratio of SOF to FE size ≥ 2 gives a reasonable precision. Based on this observation and in order to comply with the minimum correlation length used, a more refined mesh with an element size of 0.05m × 0.05m is adopted in the current study. The initial condition for the uncoupled analysis (i.e., no displacement degrees of freedom and only pore pressure degrees of freedom) is such that the excess pore pressure at all nodes (except at the nodes of the drain boundary) is set to be equal to 100kPa, while the excess pore pressure at each node of the drain boundary is set to be equal to zero. After the generation of a given realization and the subsequent implementation of the finite-element analysis of that realization, the corresponding degree of consolidation, $U(t)$, at any consolidation time, t, is calculated based on the excess pore pressure concept with the help of the following expression:

$$U(t) = 1 - \frac{\bar{u}}{u_0} \quad (3)$$

where, u_0 and $\bar{u}$ are the initial uniform and average excess pore water pressures, respectively. It has to be emphasized that the average excess pore pressure ($\bar{u}$) at any time of the consolidation process is calculated by numerically integrating the pore water pressure across the entire area of the mesh and dividing by the total mesh area.

2.3. Repetition of process based on the Monte Carlo technique

By applying the Monte Carlo simulation technique, the process of generating a realization of c_h and the subsequent implementation of the FE analysis is repeated numerous times until an acceptable accuracy of the estimated statistics of $U(t)$ is achieved. It was found that 2000 Monte Carlo simulations are sufficient to yield reasonably reproducible estimate of the first two moments (i.e., mean, μ_U, and standard deviation, σ_U) of $U(t)$. Each simulation of the Monte Carlo process involves the same σ_{c_h} and θ_{c_h} (i.e., standard deviation and SOF of c_h); however, the spatial distribution of c_h varies from one simulation to the next. The obtained $U(t)$ from the suite of 2000 realizations of the Monte Carlo process are collated and μ_U and σ_U of the degree of consolidation over the 2000 simulations are estimated using the method of moments.

3. PARAMETIC STUDIES

Following the stochastic FEMC procedure set out above, parametric studies are performed to investigate the equivalence between the single and multi-drain analyses in terms of μ_U and σ_U of the degree of consolidation. For this purpose, two groups of FEMC analysis are performed. In the first group, the point mean, standard deviation and SOF are assumed to be the same for both the single and multi-drain cases, while in the second group the point statistics are derived based on assumed local average statistics associated with the soil domain of interest. It should be noted that, the random fields are characterized by their point statistics, meaning that μ_{c_h}, σ_{c_h} and θ_{c_h} of c_h are defined at the point level. However, the soil properties are rarely measured at the point level and most engineering measurements concerned with soil properties are performed on samples of some finite volume. Therefore, the measured soil properties are actually locally averaged over the sample volume. The point statistics associated with the local average measurements depends on several factors, namely: (i) the size of the sample over which the measurement represents an average; (ii) the correlation coefficient between all points in the soil domain; and (iii) the type of averaging that the observations represent. The size of the averaging domain, D, is taken into account to compute point statistics of c_h in the second group of analysis. The details of each group analysis and the results obtained are described below.

3.1. Results of parametric studies considering same point statistics for both single and multi-drain cases

The results obtained from the single and multi-drain FEMC analyses employing the same input random field parameters are compared in this section for different combinations of σ_{c_h} and θ_{c_h} while μ_{c_h} is kept at a fixed value equal to 15 m²/ year. It should be noted that σ_{c_h} is presented herein by the normalized coefficient of variation, υ_{c_h}, where $\upsilon_{c_h} = \sigma_{c_h} / \mu_{c_h}$. The following values of υ_{c_h} and θ_{c_h} are considered:

- υ_{c_h} = 25, 50 and 100 (%)
- θ_{c_h} = 0.5, 1.0, 4.0 and 100 (m)

The abovementioned selected range of υ_{c_h} is typical to those reported in the literature (e.g., Beacher and Christian 2003). The SOF is less well-documented, particularly in the horizontal direction. Phoon and Kulhway (1999) reported that the horizontal SOF typically ranges between 3 and 80 m. Accordingly, a wide range of SOF is selected in this study where its minimum and maximum values are specified to be 0.5m and 100m, respectively. A series of FEMC analyses for various combination of υ_{c_h} and θ_{c_h} are performed. The sensitivity of μ_U and σ_U to the statistically defined input data (i.e., υ_{c_h} and θ_{c_h}) is examined in Figures 2−3 by expressing them as functions of the consolidation time t.

The comparison between μ_U derived via the single and multi-drain FEMC simulations is examined in Figure 2. The effect of increasing υ_{c_h} on μ_U at a fixed value of θ_{c_h} = 0.5m is illustrated in Figure 2a, which indicates that μ_U obtained from the single drain case agrees very well with that obtained from the multi-drain counterpart, for all cases of υ_{c_h}. For both cases, μ_U decreases with the increase of υ_{c_h}, and the decreasing rate of μ_U consistently increases with the increase of υ_{c_h}. On the other hand, Figure 2b shows the variation of μ_U as estimated via the single and multi-drain FEMC analyses, for various values of θ_{c_h} and at a fixed value of υ_{c_h} = 50%. In general, it can be observed that even though the results for various θ are drawn in Figure 2b, they are embodied into a single curve, implying that the obtained results at different θ are very close and cannot be distinguished. The virtually identical curves for all θ demonstrate that μ_U is largely independent of θ. This is expected because in principle θ does not affect the local average mean of the process.

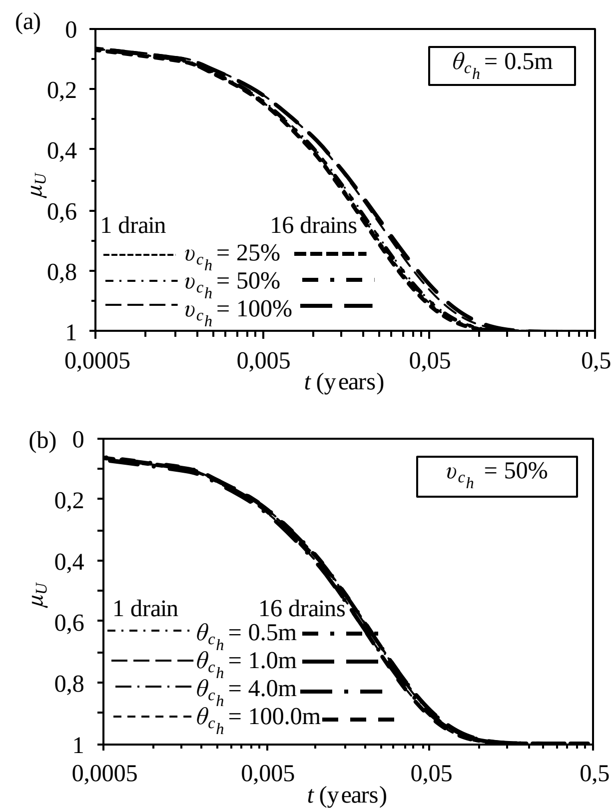


Figure 2. Effect of: (a) υ_{c_h} for θ_{c_h} = 0.5m; (b) θ_{c_h} for υ_{c_h} =50% on μ_U.

The equivalence between the single and multi-drain analyses is further examined via matching the estimated σ_U at different values of υ_{c_h} and θ_{c_h}, as shown in Figure 3. It can be seen that σ_U obtained from the single drain cases is significantly higher than that obtained from the multi-drain cases and the difference in σ_U between the two solutions increases as υ_{c_h} increases (see Figure 3a). This behavior can be explained by noting that the total flow from the vicinity of PVD is effectively an averaging process where high flow rates in some regions are offset by lower flow rates in the other regions. Notice also that, $U(t)$ is determined by averaging the excess pore pressure over the entire mesh. The main impact of the local averaging is to reduce the variance and damp the contribution from the high frequency components. As the averaging domain is significantly smaller for the single drain case compared to the multi-drain domain, there is less variance reduction, resulting in higher σ_U in the single drain case than the multi-drain solution.

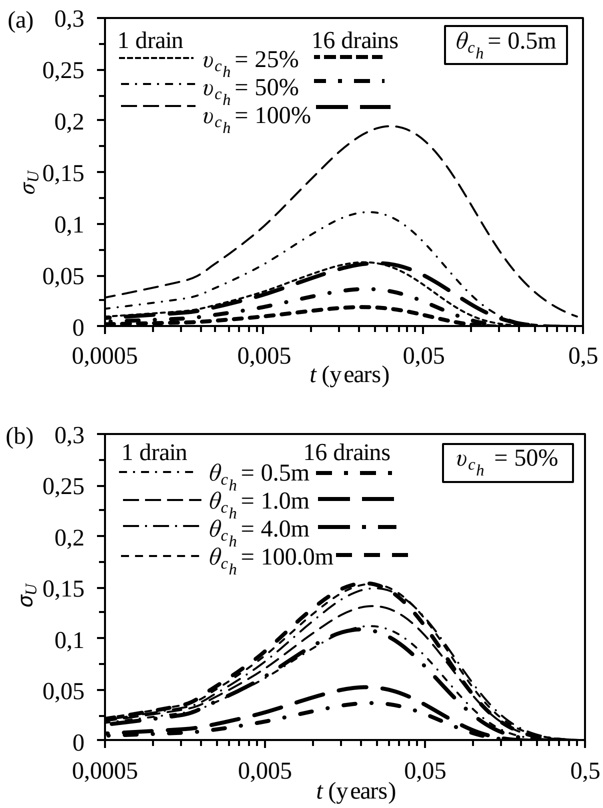


Figure 3.Effect of: (a) υ_{c_h} for θ_{c_h} = 0.5m; (b) θ_{c_h} for υ_{c_h} =50% on σ_U.

The influence of θ_{c_h} on the compliance between the single and multi-drain solutions in terms of σ_U at a fixed value of υ_{c_h} = 50% is shown in Figure 3b. It can be seen that considerable differences in σ_U obtained from the two solutions are found particularly when θ_{c_h} is as low as 4.0. On the other hand, little or no differences in σ_U are found for very high θ_{c_h} (e.g., 100.0m). This is due to the fact that when $\theta_{c_h} >> D$ (where D is the size of the problem), the variance reduction factor $\gamma(D)$ $\rightarrow$1.0 implying no variance reduction. It can be seen that the maximum σ_U occur at an intermediate t, while σ_U is zero at t = 0 and at large t. This behaviour can be explained by noting that $U(t)$ approaches 0 and 1 as t approaches 0 and ∞ regardless of the variability of c_h.

3.2. Results of parametric studies considering same local average statistics for both single and multi-drain cases

As indicated earlier, soil property measurements are generally averages over a volume (or area). In this group of parametric study, it is assumed that the local average statistics of c_h corresponding to an average over the area of the single drain and over the area of the multi-drain are the same. The mean, μ_D, and coefficient of variation, υ_D, of the local average measurement of c_h are assumed to be equal to 15 m^2/ year and 0.2, respectively. The given local average statistics are now needed to be transformed to point statistics for generating the random field of c_h. Assuming that each local average measurement is deemed to be a geometric average, the relationship between the local average statistics and the ideal point mean, μ_{c_h}, and standard deviation, σ_{c_h}, are as follows (Fenton and Griffiths 2008):

$$\mu_{c_h} = \mu_D \exp\left[\ln\left(1+\upsilon_D^2\right)\left\{\frac{1-\gamma(D)}{2\gamma(D)}\right\}\right] \tag{4}$$

$$\sigma_{c_h} = \sqrt{\left(\mu_{c_h}^2 \exp\left[\left\{\frac{\ln\left(1+\upsilon_D^2\right)}{\gamma(D)}\right\}-1\right]\right)} \tag{5}$$

where, $\gamma(D)$ is the variance reduction factor corresponding to the underlying normal random field ln (c_h). Considering the geometry of the single and multi-drain problem, $\gamma(D)$ for various θ_{c_h} is computed numerically from the corresponding variance reduction function of the correlation structure shown in Eq. (2), as presented by Fenton and Griffiths (2008) and summarized in Table 1.

Table 1 Variance reduction factor for various SOF.

θ_{c_h}	Single drain $\gamma(D)$	16 drains $\gamma(D)$
0.5	0.206	0.023
1.0	0.413	0.076
4.0	0.786	0.413
100.0	0.99	0.9613

By substituting the given μ_D, υ_D and computed values of $\gamma(D)$ in Eqs. (4) and (5), μ_{c_h} and σ_{c_h} are computed for both the single and multi-drain problems and the results are shown in Table 2. Employing the computed μ_{c_h} and σ_{c_h} corresponding to each θ_{c_h}, a series of FEMC analyses is performed for both the single and multi-drain cases and the equivalence between μ_U and σ_U obtained from the two solutions are examined (Figure 4).

Table 2 Estimated point mean and standard deviation computed from the given local average statistics.

θ_{c_h}	Single drain		16 drains	
	μ_{c_h}	σ_{c_h}	μ_{c_h}	σ_{c_h}
0.5	16.18	7.41	34.5	73.2
1.0	15.4	4.87	19.04	15.65
4.0	15.08	3.41	15.42	4.87
100.0	15.003	3.01	15.01	3.06

It can be seen from Figure 4 that both μ_U (Figure 4a) and σ_U (Figure 4b) obtained from the single drain analysis agree well with those obtained from the multi-drain analysis, for all cases of θ_{c_h}, despite the slight discrepancy in μ_U and σ_U when θ_{c_h} is as low as 0.5. Overall, the good agreement between the single and multi-drain analyses in terms of μ_U and σ_U indicates that stochastic equivalence between the unit cell and multi-drain solutions can be established by assigning representative parameters for their corresponding domain.

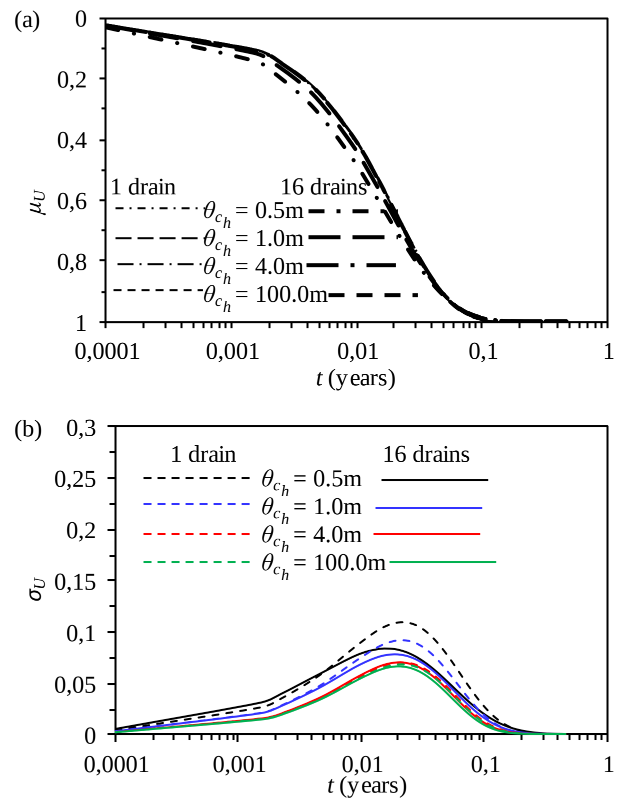


Figure 4. Effect of (a) θ_{c_h} on μ_U; (b) θ_{c_h} on σ_U for different point μ_{c_h} and σ_{c_h} computed from the same given local average statistics.

4. CONCLUSIONS

This paper used the random field theory and finite element modeling to investigate the stochastic equivalence between the single drain "unit cell" and multi-drain solution for PVD-improved ground. The horizontal coefficient of consolidation, c_h, was treated as the only random field and an uncoupled 2D finite element analysis was applied. Two groups of stochastic finite element Monte Carlo (FEMC) analyses were performed. In the first group, the point input statistical parameters were assumed to be the same for both the single and multi-drain cases. It was found that the mean degree of consolidation, μ_U, obtained from the single drain analysis agrees reasonably well with that obtained from the multi-drain counterpart irrespective of the input parameters. However, a considerable difference in σ_U obtained from the two solutions was found except for very high scale of fluctuation. In the second group, it was assumed that the local average statistics of c_h corresponding to an average over the single drain area and over the area of the multi-drain are the same. By computing corresponding point statistics and performing FEMC analysis it was found that both μ_U and σ_U obtained from the single drain analysis agrees very well with those obtained from the multi-drain analysis, for all selected scales of fluctuation. Overall, it was shown that to establish stochastic equivalence between the unit cell and multi-drain analyses, proper input statistics representative to the soil domain of interest need to be used. This study also demonstrated the importance of proper modeling of soil spatial variability in design of ground improvement by PVDs.

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