A Two-step Density Estimation Method and Its Applications

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ABSTRACT: Probability density evolution method (PDEM) provides a feasible approach for dynamic responses analysis of nonlinear stochastic structures. The key issue of PDEM is to solve a generalized density evolution equation (GDEE). Previously, finite difference method (FDM) is resorted to solve GDEE, thus getting the probability density function (PDF) result directly. In this paper, a two-step method that improves the result of FDM by means of nonlinear wavelet density estimation is proposed. By taking advantage of the property of multi-resolution of wavelet, the proposed method tends to give a more accurate result. An 8-story reinforced concrete shear frame with Bouc-Wen hysteretic restoring forces subjected to seismic excitation is investigated. The result shows that the proposed method performs better than FDM.

1. INTRODUCTION

Deterministic finite element method has been studied in depth, and is widely applied in practical engineering problems to date. On the other hand, there are a lot of inevitable randomness in many aspects of structural analysis, such as material properties, geometric sizes, boundary conditions and applied loads. Variability of these parameters may lead to considerable fluctuation in structural response, and may even result in structural failure. In order to trace the propagation of randomness and measure its influence on structural response, it is necessary to resort to stochastic finite element method (SFEM).

Study on SFEM was initiated in 1970's. Over the past four decades, great developments have been achieved in this field. Available approaches by far can be broadly divided into Monte Carlo method (MCM), Perturbation stochastic finite element method (PSFEM), and orthogonal polynomial expansion method (OPEM). Pioneering work to introduce MCM into stochastic structural analysis was conducted by Shinozuka and Jan (1972). Despite its versatility, MCM is seldom applied to complex structures due to its prohibitive computational consumption. In order to increase the efficiency of MCM, Shinozuka and Deodatis (1988) and Yamazaki et al. (1988) introduced the Neumann expansion technique so that the time-consuming matrix inversion operation in each random sampling is avoided. Hisada and Nakagiri (1981) proposed PSFEM by using stochastic perturbation technique to deal with fluctuation of random parameters. Frangopol et al. (1996) carried out nonlinear static analysis of concrete frame structures and concrete plates by combining PSFEM with concrete constitutive law. First order PSFEM is rather effective on condition that the random parameters are of very small variability. Second order PSFEM has looser limit on the variability of the random parameters, but is of very low efficiency and thus not suitable for practical engineering applications. Moreover, the secular terms problem (Liu et al. (1988)) will result in large error when applied to dynamic problems. Sun (1979) proposed a new class of numerical algorithm to expand the structural response by Hermit orthogonal polynomial when trying to solve random differential equations with random coefficients.

Enlightened by Sun's work, Jensen and Iwan (1991,1992) applied OPEM into structural seismic response analysis. Ghanem and Spanos (1991) suggested to use chaos polynomial as orthogonal basis. By applying the sequential orthogonal decomposition principle in random space, Li (1995a, 1995b) independently derived an extended system method. OPEM is not troubled by the limit on the variability of random parameters and the secular terms problem. However, when the number of random parameters is large, order of the extended system will be extremely higher than that of the original one, thus making the computational consumption almost unbearable.

Recently, Li and Chen (2004,2005) developed a new class of PDEM, where they derived a GDEE based on the principle of preservation of probability. FDM was adopted to obtain the evolution of PDF of the specific dynamic response. In this paper, a two-step density estimation method that improves the result of FDM by means of wavelet density estimation is proposed. An 8-story reinforced concrete shear frame with Bouc-Wen hysteretic restoring forces subjected to seismic excitation is investigated.

2. PROBABILITY DENSITY EVOLUTION METHOD FOR STOCHASTIC STRUCTURAL RESPONSE ANALYSIS

For a general nonlinear multi-degree-of-freedom (MDOF) system, its equation of motion is

$$\boldsymbol{M}(\boldsymbol{\Theta})\ddot{\boldsymbol{X}}+\boldsymbol{C}(\boldsymbol{\Theta})\dot{\boldsymbol{X}}+\boldsymbol{f}(\boldsymbol{\Theta},\boldsymbol{X})=\boldsymbol{F}(\boldsymbol{\Theta},t) \quad (1)$$

where $\boldsymbol{M}$, $\boldsymbol{C}$ are mass and damping matrix of order n, respectively; $\boldsymbol{f}$ is a n-dimensional nonlinear restoring force vector; $\boldsymbol{F}$ is a n-dimensional external excitation vector; $\boldsymbol{\Theta}=(\Theta_1,\cdots,\Theta_s)$ is a s-dimensional random vector that contains all the random factors in both the structural parameters and the external excitation whose joint probability density function is $p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$. Its initial condition is

$$\dot{\boldsymbol{X}}(t_0)=\dot{\boldsymbol{x}}_0, \boldsymbol{X}(t_0)=\boldsymbol{x}_0 \quad (2)$$

It is obvious that Eq. (1) has a formal solution

$$\boldsymbol{X}(t)=\boldsymbol{H}(\boldsymbol{\Theta},t) \quad (3)$$

To consider a state space spanned by $(\boldsymbol{X},\boldsymbol{\Theta})$, according to the principle of preservation of probability, a first order linear partial differential equation can be derived(Li and Chen (2004,2005))

$$\frac{\partial p_{\boldsymbol{X\Theta}}(\boldsymbol{x},\boldsymbol{\theta},t)}{\partial t}+\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\left[p_{\boldsymbol{X\Theta}}(\boldsymbol{x},\boldsymbol{\theta},t)\dot{H}_j(\boldsymbol{\theta},t)\right]=0 \quad (4)$$

By integrating both sides of Eq. (4) with respect to $x_1,\cdots,x_{l-1},x_l,\cdots,x_n$, we can get a one-dimensional partial differential equation

$$\frac{\partial p_{X_l\boldsymbol{\Theta}}(x_l,\boldsymbol{\theta},t)}{\partial t}+\dot{H}_l(\boldsymbol{\theta},t)\frac{\partial p_{X_l\boldsymbol{\Theta}}(x_l,\boldsymbol{\theta},t)}{\partial x_l}=0 \quad (5)$$

where

$$p_{X_l\boldsymbol{\Theta}}(x_l,\boldsymbol{\theta},t)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}p_{\boldsymbol{X\Theta}}(\boldsymbol{x},\boldsymbol{\theta},t)\mathrm{d}x_1\cdots\mathrm{d}x_{l-1}\mathrm{d}x_{l+1}\cdots\mathrm{d}x_n \quad (6)$$

Note that in the above integration procedure, the following boundary condition is used

$$\lim_{x_j\to\pm\infty}p_{\boldsymbol{X\Theta}}(\boldsymbol{x},\boldsymbol{\theta},t)=0;\ j=1,2,\cdots,n \quad (7)$$

For notational convenience, omit the subscript l and we can get

$$\frac{\partial p_{X\boldsymbol{\Theta}}(x,\boldsymbol{\theta},t)}{\partial t}+\dot{H}(\boldsymbol{\theta},t)\frac{\partial p_{X\boldsymbol{\Theta}}(x,\boldsymbol{\theta},t)}{\partial x}=0 \quad (8)$$

with the initial condition

$$p_{X\boldsymbol{\Theta}}(x,\boldsymbol{\theta},t_0)=\delta(x-x_0)p_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) \quad (9)$$

In order to get the final result $p_X(x,t)$, it is necessary to partition the whole random space $\boldsymbol{\Theta}$ uniformly (Li and Chen (2007)) and integrate both sides of Eq. (8) with respect of $\boldsymbol{\theta}$ in each subspace

$$\begin{aligned}&\int_{\Omega_q}\frac{\partial p_{X\boldsymbol{\Theta}}(x,\boldsymbol{\theta},t)}{\partial t}d\boldsymbol{\theta}+\int_{\Omega_q}\dot{H}(\boldsymbol{\theta},t)\frac{\partial p_{X\boldsymbol{\Theta}}(x,\boldsymbol{\theta},t)}{\partial x}d\boldsymbol{\theta}\\&\approx\frac{\partial}{\partial t}p_X^{(q)}(x,t)+\dot{H}(\boldsymbol{\theta}_q,t)\frac{\partial}{\partial x}p_X^{(q)}(x,t)=0\end{aligned} \quad (10)$$

where $\Omega_q\left(q=1,\cdots,N_{sel}\right)$ are mutually disjoint and complementary subspaces of $\boldsymbol{\Theta}$; $\boldsymbol{\theta}_q\left(q=1,\cdots,N_{sel}\right)$ are corresponding representative points. In the same way, Eq. (9) turns to be

$$\begin{aligned} p_X^{(q)}\left(x,t_0\right) &= \int_{\Omega_q} p_{X\boldsymbol{\Theta}}\left(x,\boldsymbol{\theta},t_0\right)d\boldsymbol{\theta} \\ &= \delta\left(x-x_0\right)\int_{\Omega_q} p_{\boldsymbol{\Theta}}\left(\boldsymbol{\theta}\right)d\boldsymbol{\theta} \\ &= P_q\delta\left(x-x_0\right) \end{aligned} \tag{11}$$

where $P_q\left(q=1,\cdots,N_{sel}\right)$ are called assigned probabilities. As long as $p_X^{(q)}\left(x,t\right)\left(q=1,\cdots,N_{sel}\right)$ are obtained, it is easy to further get the final result $p_X\left(x,t\right)$ by summation

$$p_X\left(x,t\right)=\sum_{q=1}^{N_{sel}} p_X^{(q)}\left(x,t\right) \tag{12}$$

3. WAVELET DENSITY ESTIMATION

Wavelets are a new family of localized basis functions with a combination of powerful features, such as orthonormality, locality in time and frequency domains, multi-resolution, different degrees of smoothness and fast implementations. These basis functions can be used to approximate a large class of functions.

Let $\left\{X_i\right\}_{i=1}^n$ be a stationary process and $f\left(x\right)$ the probability density function. By invoking wavelet multi-resolution analysis, $f\left(x\right)$ can be expanded as follows

$$f\left(x\right)=\sum_k c_{j_0,k}\phi_{j_0,k}\left(x\right)+\sum_{j=j_0}^{\infty}\sum_k d_{j,k}\psi_{j,k}\left(x\right) \tag{13}$$

where $\phi\left(x\right)$ and $\psi\left(x\right)$ are Daubechies scaling function and wavelet function, respectively. According to the orthogonality of Daubechies basis functions, we have

$$\begin{aligned} c_{j_0,k} &= \left\langle \phi_{j_0,k}, f\right\rangle = \int_{-\infty}^{\infty}\phi_{j_0,k}\left(x\right)f\left(x\right)dx = E\left[\phi_{j_0,k}\left(X\right)\right] \\ d_{j,k} &= \left\langle \psi_{j,k}, f\right\rangle = \int_{-\infty}^{\infty}\psi_{j,k}\left(x\right)f\left(x\right)dx = E\left[\psi_{j,k}\left(X\right)\right] \end{aligned} \tag{14}$$

Since $f\left(x\right)$ is unknown, the unbiased estimation of each coefficient can be computed as follows

$$\begin{aligned} \hat{c}_{j_0,k} &= \frac{1}{n}\sum_{i=1}^n \phi_{j_0,k}\left(x_i\right) \\ \hat{d}_{j,k} &= \frac{1}{n}\sum_{i=1}^n \psi_{j,k}\left(x_i\right) \end{aligned} \tag{15}$$

It is worth noting that only a small number of coefficients are nonzero since the Daubechies wavelets are compactly supported. A simple approach to estimate $f\left(x\right)$ is to determine an appropriate cutoff resolution j_0 and ignore the wavelet terms in Eq. (13)

$$\hat{f}_l\left(x\right)=\sum_k \hat{c}_{j_0,k}\phi_{j_0,k}\left(x\right) \tag{16}$$

where the subscript l denotes a linear wavelet density estimator. In order to increase the accuracy of the estimator as well as ensure sufficient smoothness, Donoho et al. (1995) proposed a nonlinear wavelet density estimator

$$\hat{f}_n\left(x\right)=\sum_k \hat{c}_{j_0,k}\phi_{j_0,k}\left(x\right)+\sum_{j=j_0}^{j_1}\sum_k \delta\left(\hat{d}_{j,k},\lambda\right)\psi_{j,k}\left(x\right) \tag{17}$$

where j_1 is a cutoff resolution, $\delta\left(\cdot\right)$ is a threshold criterion, and λ is a threshold value. Analytical expressions of hard threshold and soft threshold are as follows, respectively

$$\delta^h\left(d,\lambda\right)=d\mathbf{I}\left(\left|d\right|>\lambda\right),\lambda\geq 0,d\in\boldsymbol{R} \tag{18}$$

$$\delta^s\left(d,\lambda\right)=\left(d-\operatorname{sgn}\left(d\right)\lambda\right)\mathbf{I}\left(\left|d\right|>\lambda\right),\lambda\geq 0,d\in\boldsymbol{R} \tag{19}$$

where $\mathbf{I}\left(\cdot\right)$ is an indicator function and $\operatorname{sgn}\left(\cdot\right)$ is a sign function. Eq. (19) is adopted in this paper due to the fact that soft threshold performs better than hard threshold from the perspective of smoothness. It is proved that theoretically a nonlinear wavelet density estimator has an optimal convergence rate provided that j_1 and λ are properly chosen. Limited to space of the paper, these details are not described here, and interested readers can refer to Vannucci and Vidakovic (1997).

In essence, wavelet density estimators can be classified into orthogonal series estimators. However, since usual orthogonal basis functions (e.g. Fourier series) are global basis functions, it is quite difficult for their corresponding estimators to adapt to the local behavior of the underlying density function. On the contrary, wavelet density estimators can easily describe the local information of the density function without sacrificing the merit of orthogonal estimators. Although kernel density estimators can give satisfactory results for a wide range of problems by choosing appropriate kernel functions and bandwidths, they fail to deal with density functions that contain high-frequency oscillations or discontinuities. Wavelet density estimators, however, show distinct advantage in this respect due to their two independent levels of smoothing. To be specific, the scaling terms provide a global approximation of the density function and the wavelet terms provide a local adjustment. From this point of view, a linear wavelet density estimator is similar to a kernel density estimator with the resolution analogous to the bandwidth, while a nonlinear estimator is to some extent an enhancement.

4. A TWO-STEP DENSITY ESTIMATION METHOD

Since Eq. (10) is a simple first order linear partial differential equation, it is natural to solve it by FDM, which is usually done previously. Computation experiences indicate that, in general, satisfactory results can be obtained in this way. Despite its feasibility and effectiveness, FDM still exposes an obvious shortcoming which results from its fixed finite difference grid. In fact, as for the problem concerned in this paper, the grid size of FDM plays an important role in striking a balance between the accuracy and smoothness of the finite difference result. To be specific, the smaller the grid size, the more accurate the result while the larger the grid size, the smoother the result. Hence, when a grid size is determined beforehand, it is probable that it is appropriate for some instants while larger or smaller for others. In other words, it is almost impossible to determine a grid size that is optimal for all instants. On the other hand, even at a specific instant, a fixed grid size may, in some cases, fail to give a satisfactory result since there may be high-frequency oscillations or discontinuities that cannot be reflected by fixed grid FDM. It is worth noting that nonlinear wavelet density estimation method can to some extent overcome these two difficulties. For one thing, at any instant, it is easy to find an optimal resolution which plays the same role as grid size in FDM; for the other, at a specific instant, the scaling terms in Eq. (17) provide a global approximation of the density function and the wavelet terms provide a local adjustment.

Based on the points discussed above, it is beneficial to further improve the result of FDM by nonlinear wavelet density estimation method. In other words, a two-step density estimation method is suggested, which is to determine a small enough grid size beforehand and solve GDEE by FDM followed by nonlinear wavelet density estimation. It is worth noting that the proposed method is in essence a kind of predictor-corrector algorithm. The specific solving procedure is listed below

1. Select representative points $\boldsymbol{\theta}_q\left(q=1,\cdots,N_{sel}\right)$ that are uniformly distributed in $\boldsymbol{\Theta}$ and calculate the corresponding assigned probabilities P_q;
2. For each representative point, solve Eq. (1) to obtain the corresponding velocity time history of interest $\dot{H}\left(\boldsymbol{\theta}_q,t\right)$;
3. For each $\dot{H}\left(\boldsymbol{\theta}_q,t\right)$, solve Eq. (10) by FDM and get the numerical solution $p_X^{(q)}\left(x_j,t_m\right)$, where $x_j=j\Delta x$ $\left(j=0,\pm1,\cdots\right)$, $t_m=m\Delta t$ $\left(m=0,1,\cdots\right)$, Δx and Δt are grid size along *x*-axis and *t*-axis, respectively;
4. Sum $p_X^{(q)}\left(x_j,t_m\right)\left(q=1,\cdots,N_{sel}\right)$ together
$$p_X\left(x_j,t_m\right)=\sum_{q=1}^{N_{sel}}p_X^{(q)}\left(x_j,t_m\right) \tag{20}$$
5. Substitute $p_X\left(x,t\right)$ into Eq. (14) to get the scaling coefficients and wavelet coefficients;

6. Substitute the scaling coefficients and wavelet coefficients into Eq. (17) to get the final result.

It should be noted that since FDM has given a preliminary PDF, Eq. (14) instead of Eq. (15) is used to calculate the coefficients.

5. NUMERICAL EXAMPLE

In order to demonstrate the feasibility and accuracy of the proposed method, an 8-story reinforced concrete shear frame (see Figure 1) subjected to seismic excitation (see Figure 2) is studied here.

Structural parameters are as follows: the height and cross section dimensions of the columns of the first floor are $h_1 = 4\text{m}$ and $600\text{mm}\times 600\text{mm}$ while those of the columns of other floors are $h_2 = 3\text{m}$ and $500\text{mm}\times 500\text{mm}$; the mass of each floor is $m = 1.1\times 10^5\text{kg}$; the elastic modulus of concrete E is assumed to be a normal random variable with mean value $3\times 10^{10}\text{Pa}$ and coefficient of variation 0.1; the Bouc-Wen model is adopted as the nonlinear shear hysteretic model with parameters $\alpha = 0.01$, $A = 1$, $n = 1$, $\beta = 15$ and $\gamma = 0.15$. El Centro E-W wave with the peak acceleration tuned to $A_p = 4.9\text{m/s}^2$ is taken as the excitation.

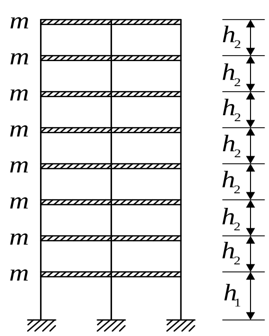

Figure 1: An 8-story stochastic structure

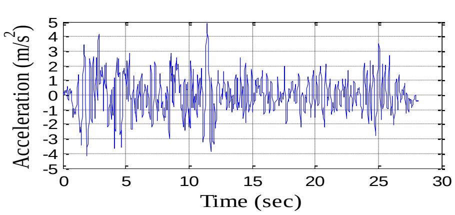

Figure 2: Exciting ground motion

The inter-story displacement of the first floor is selected as the target response. 128 representative points are selected, and both FDM and the proposed method are carried out. In order to compare the results of FDM and the proposed method, MCM with 10,000 simulations is also conducted and its result is used as reference. For notational convenience, the proposed method is abbreviated as DWM.

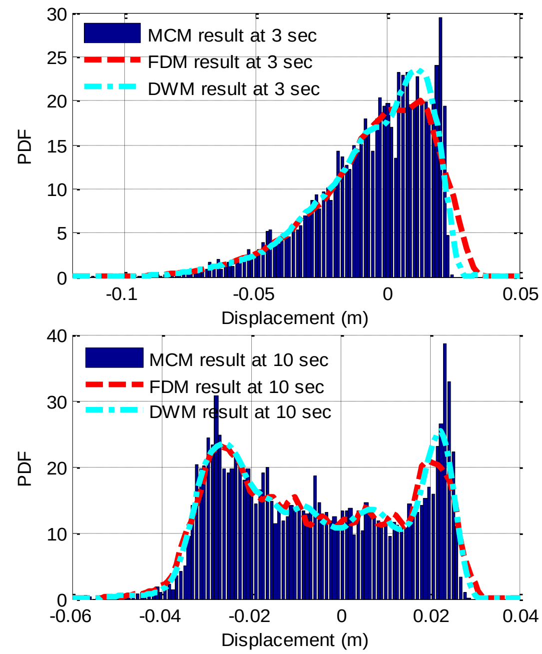

Figure 3: Comparison of PDFs at 3 sec and 10 sec among results evaluated by FDM, DWM and MCM

In Figure 3, PDFs at two different instants evaluated by FDM, DWM and MCM are given. At instant $t = 3\text{sec}$, the PDF is a unimodal function. Although both PDFs fit well with the MCM result on the whole, the DWM result is obviously superior to the FDM result in terms of peak value and right tail. In order to measure the goodness of fit quantitatively, relative entropy is used here. The smaller the relative entropy is, the better the corresponding PDF fits with the reference. The relative entropy between the FDM result and the MCM result is 0.1052, while that between the DWM result and the MCM result is only 0.0582. It is no surprise that this result is consistent with the intuitive judgment. At instant $t = 10\text{sec}$, the PDF turns into a bimodal function.

Similarly, both PDFs fit well with the MCM result on the whole, but the DWM result is better than the FDM result in terms of right peak value and right tail. Besides, the DWM result is less oscillatory than the FDM result in the middle part. Consistently, two relative entropies are 0.0488 and 0.0713, respectively.

6. CONCLUSIONS

GDEE is a governing equation that determines the probability density evolution process of a general stochastic dynamic system. A predictor-corrector strategy that amends the preliminary GDEE result by nonlinear wavelet density estimation method is proposed in this paper. An 8-story reinforced concrete shear frame is studied. The result indicates that the proposed two-step method outperforms FDM. More specifically, the proposed method tends to give PDF result that fits better to the underlying real solution near the peak(s) and in the tail than that by FDM.

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