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Application of the Multiplicative Dimensional Reduction Method (M-DRM) to a Probabilistic Fracture Mechanics Problem

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ABSTRACT: This paper compares the results from a Monte Carlo simulation (MCS) of crack propagation in cyclically loaded as-received and impact treated welds to similar results obtained using the much more computationally efficient multiplicative dimensional reduction method (M-DRM). The basis of both analyses is a 17-variable strain-based fracture mechanics (SBFM) model, which considers nonlinear material effects as the crack propagates over time. One of the challenges in applying M-DRM to this problem is that difficulties arise as the constant amplitude fatigue limit (CAFL) is approached. A practical approach for dealing with this situation is discussed. In the presented results, it is shown that M-DRM is a viable tool for probabilistic fatigue analysis, producing in normal situations results within a percent of those obtained using MCS in a fraction of the computation time.

1. INTRODUCTION

Accurately predicting the fatigue deterioration of welded components can be a computationally expensive undertaking – in particular, in cases where nonlinear material and variable amplitude (VA) loading effects are significant. For example, recent efforts have been undertaken (e.g. [Walbridge et al. 2012]) to model the fatigue behaviour of welds modified by impact treatment using a strain-based fracture mechanics (SBFM) model that employs up to 17 input parameters. This models cycles the material at various points along the crack path, generating non-linear hysteresis loops. The strain peaks are then used to calculate the crack propagation with each load cycle. When analyses are performed for complex VA loading histories, even a single deterministic analysis can be tedious.

Determining the statistical properties of fatigue performance is important for fatigue design. However, to determine these properties experimentally takes many tests, which is both expensive and time-consuming. Deterministic models, such as SBFM models, can be calibrated to experimental results, but they fail to include the variability of the material properties, geometry, defects, loading, etc., of the physical experiments. This variability is important when dealing with reliability in design.

If a probabilistic analysis is performed using Monte Carlo simulation (MCS), then hundreds of thousands of repetitions of the deterministic analysis may be required. In addition, if an entire structure is analyzed, separate simulations may be required for each potential crack initiation site. If the goal is to compare alternative maintenance strategies for the structure (e.g. sequences of inspection and impact treatment retrofitting at various stages during the service life), then the use of the SBFM model ceases to be a feasible option, despite its advantages in terms of accuracy and sophistication.

The multiplicative dimensional reduction method (M-DRM) is a recently-developed statistical method [Zhang 2013], which varies the input variables using Gaussian weights as initial guesses, while holding the other variables constant. Based on an analysis of the results of a limited number of trials, not only are statistical properties, such as mean and standard deviation, calculated, but primary and global sensitivity analysis can also be performed. The main benefit to this method is a reduction in the number of trials required, For example, for a 3-variable problem, instead of the thousands of trials required for MCS, only 16 are required.

Against this background, the main purpose of this paper is to apply M-DRM to a much more complex practical problem than the simple case studies that were employed in the development of this method – namely, the SBFM analysis of impact treated welds under constant amplitude (CA) fatigue loading. In the following sections, the employed SBFM model and M-DRM method are briefly reviewed. Following this, the results of simulations made using MCS and M-DRM are compared, the challenge of dealing with the constant amplitude fatigue limit (CAFL) is addressed, and the suitability of M-DRM for application to complex problems such as the one studied herein is critically assessed.

2. IMPACT TREATMENT OF WELDS

Impact treatments for improving the fatigue performance of welds, including needle peening (see Figure 1), hammer peening, and high frequency mechanical impact (HFMI) or ultrasonic impact treatment (UIT) work by introducing compressive residual stresses near the surface of the treated weld toe, which slow or arrest crack growth at smaller crack depths.

These treatments can be used on new welds, or for retrofitting existing welds, so long as they do not already have large cracks. Various studies (e.g. [Ghahremani & Walbridge 2011]) have shown that the effects of impact treatments on fatigue performance can be predicted using fracture mechanics models, wherein the residual stresses along the crack path are modified to simulate the effect of the treatment.

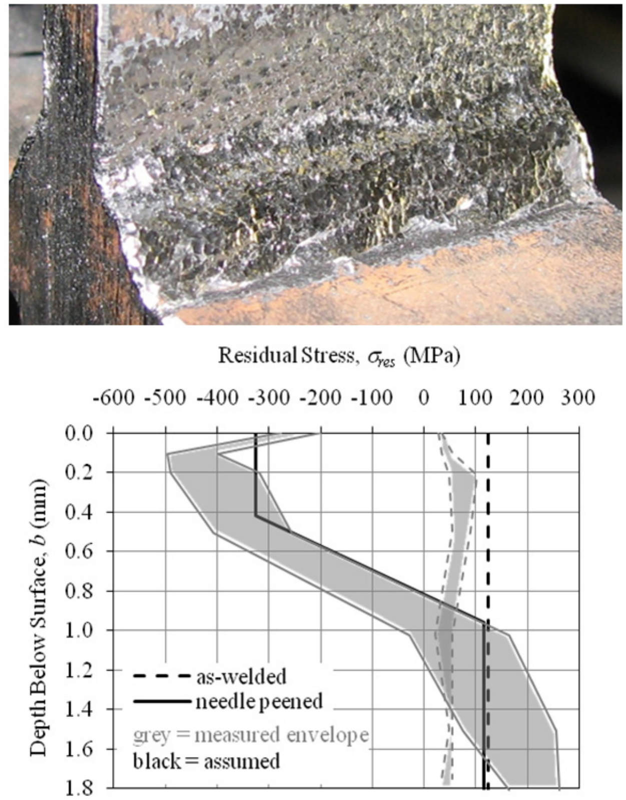

Figure 1: Needle peened weld toe (top) and measured residual stresses due to treatment (bottom).

3. DETERMINISTIC SBFM MODEL

A detailed description of the SBFM model used in this study in its deterministic form is provided in [Ghahremani & Walbridge 2011]. A probabilistic version of the model is described in [Walbridge et al. 2012]. For reasons of brevity, only an overview of the model is provided herein. The basis for the model is the Paris-Erdogan crack growth law, modified to consider crack closure and the presence of a threshold stress intensity factor (SIF) range:

$$N = \int_{a_i}^{a_c} \frac{\mathrm{d}a}{C \cdot \mathrm{MAX}\left(\Delta K_{eff}{}^{m} - \Delta K_{th}{}^{m}, 0\right)} \qquad (1)$$

where a_i and a_c are the initial and critical crack depths and C and m are constants. For the analysis of welds retrofitted using residual stress-based impact treatments, the SBFM model offers advantages over linear elastic fracture mechanics (LEFM) in terms of its ability to model the evolution of the residual stresses under variable amplitude (VA) loading conditions [Ghahremani

& Walbridge 2011]. The main difference between LEFM and SBFM is in the calculation of the SIFs, K. For the SBFM model:

$$K = Y \cdot E \cdot \varepsilon \cdot \sqrt{\pi \cdot a} \tag{2}$$

where ε is the local strain at crack depth, a, and Y is a correction factor to account for the crack shape, free surface, and the finite thickness of the plate. To calculate the stresses and strains for each load cycle, a Ramberg-Osgood material model is used, which requires the cyclic parameters: K' and n'. Strain histories at various depths below the surface of the weld toe and crack closure are considered using established models [Newman 1984, Dowling 2007].

In the probabilistic version of the SBFM model, the input parameters are replaced with statistical variables. This approach has been used previously to perform probabilistic LEFM analysis of as-received and needle peened welds [Walbridge & Nussbaumer 2008]. In the SBFM model, the additional parameters required to model nonlinear material effects include: the elastic modulus, E, the static yield and ultimate strength, σ_y and σ_u, the cyclic material parameters, K' and n', and a parameter, μ, which models the recovery of the crack opening stress following overloads under VA loading.

Given the deterministic model and statistical distributions for the input parameters, MCS can be used to generate a histogram of the number of load cycles to failure, N, for different stress ranges, ΔS. This was done in [Walbridge et al. 2012] for the analysis of a transverse stiffener weld in plate made of generic mild steel.

For the current study, this analysis was repeated for various stress ranges to generate probabilistic S-N curves for as-received and treated welds. CA loading with a stress ratio, $R = \sigma_{min} / \sigma_{max}$, of 0.1 was simulated. The effect of the impact treatment was modelled using the "assumed" nominal residual stress distribution for needle peening in Figure 1. The analysis was performed assuming that the nominal plate thickness and material yield strength are known (T = 25 mm and σ_y = 350 MPa). The assumed statistical variables are described in Table 1.

Table 1: Probabilistic SBFM analysis variables.

Var.	μ	σ	Units	Dist.	Description
a_i	0.15	0.045	mm	LN	initial crack depth
$(a/c)_i$	0.50	0.16	-	LN	initial crack aspect ratio
$VAR(T)$	1.02	0.012	-	N	T = plate thickness
θ_w	39.8	6.9	°	LN	weld toe angle
ρ_w	0.65	0.3	mm	LN	weld toe radius
LN(C)	-29.13	0.55	N, mm	N	Paris law constant
ΔK_{th}	80.0	15.0	N, mm	LN	SIF range threshold
μ	0.002	0.001	-	LN	crack closure parameter
$VAR(E)$	1.04	0.026	-	N	E = elastic modulus
$VAR(\sigma_y)$	1.07	0.053	-	LN	σ_y = yield strength
$VAR(\sigma_u)$	1.0	0.077	-	LN	σ_u = ultimate strength
$VAR(K')$	1.04	0.35	-	LN	K' = Ramberg-Osgood constant
$VAR(n')$	0.92	0.15	-	LN	n' = Ramberg-Osgood constant
$VAR(\sigma_{weld})$	1.0	0.25	-	N	σ_{weld} = welding residual stress
$VAR(\sigma_{pwt})$	1.0	0.20	-	N	σ_{pwt} = peening residual stress
$VAR(\Delta S)$	1.0	0.15	-	N	ΔS = nominal stress range
$VAR(SCF)$	0.93	0.12	-	LN	SCF = stress concentration factor

Note: $VAR()$ denotes multiplier applied to the nominal value.

4. OVERVIEW OF M-DRM

There have been several methods derived for dealing with statistical analysis of multivariate problems, the most common being Monte Carlo simulation (MCS). However, MCS requires significant computational effort. MCS can be optimized using importance sampling, but this still requires many iterations. The first-order reliability method (FORM) [Hasofer & Lind 1974] is an alternative approximate method, but this approach lacks generality when dealing with the response of many variables [Schueller & Pradlwarter 2007]. Similar problems occur for variations of FORM, including the second-order reliability method (SORM), the first-order third-moment reliability method, and the response surface approach [Zhao & Ono 2001].

The method of moments can be used to find an approximate solution to a multivariate problem [Taguchi 1978]. By calculating the first four moments of the response, mean, variance, skewness and kurtosis, the parameters of that distribution can be back-calculated. However, the calculation of moments involves multi-dimensional integrals, which are very complex. Research in the past has looked at efficient evaluation of these integrals using point estimate methods [Taguchi 1978], [Rosenblueth 1981], Taylor series approximation and non-classical orthogonal polynomial approximations [Kennedy & Lennox 2001]. More recently, high-dimensional model representation [Li et al. 2001] and the dimensional reduction method [Rahman & Xu 2004], [Xu & Rahman 2004] have been used, in which the multivariate function is decomposed into orthogonal component functions. The principle of maximum entropy [Jaynes 1957] was introduced to deal with the issue of sensitivity of tail probabilities, but this required significant computational effort in the moment calculations when dealing with a large number of constraints. The recent emergence of fractional moments [Inverardi & Tagliani 2003], [Milev et al. 2012] has made these calculations easier. A fractional moment is a moment of order of real as opposed to whole numbers, which can be combined together to determine information about the resultant distribution.

M-DRM [Zhang 2013] is a combination of fractional moments and the maximum entropy principle, and is a much more efficient method of performing statistical analysis than MCS. The main benefit this method has over MCS is the massive reduction in number of trials needed. Given a problem with three variables, for example, instead of the thousands of iterations required for Monte Carlo, only 16 calculations are required for M-DRM. A second benefit is the additional analysis outputs possible. While MCS only provides the output histogram, M-DRM provides output moments as well as a sensitivity analysis, both primary and global.

M-DRM uses Gaussian quadratures, which are commonly used for multi-dimensional integration. The type of Gaussian quadrature varies depending on the type of distribution, with Gauss-Hermite used for Normal and Lognormal distributions, and Gauss-Laguerre used for Exponential and Weibull. These quadratures are based on the approximation of integrations evaluated at known Gauss points, and the number of points corresponds to the number of orders desired, with five being the most common and the number used in this study.

Table 2 summarizes the Gauss points (z_j) and the Gauss weights (w_j) required by M-DRM.

Table 2: Weights (w_j) and Points (z_j) for 5th order Gaussian quadrature [Balomenos & Pandey 2013]

Gaussian Rule	N	1	2	3	4	5
Gauss-Legendre	w_j	0.24	0.48	0.57	0.48	0.24
	z_j	-0.91	-0.54	0	0.54	0.91
Gauss-Hermite	w_j	0.01	0.22	0.53	0.22	0.01
	z_j	-2.86	-1.36	0	1.36	2.86
Gauss-Laguerre	w_j	0.52	0.40	0.08	0.004	2.3E-5
	z_j	0.26	1.41	3.60	7.09	12.64

Similar to how a normal random variable X can be related to the standard normal random variable Z using the following expression:

$$X = \mu + \sigma \cdot Z \quad (3)$$

with μ equaling the mean value and σ equaling the standard deviation, the Gauss-Hermite points can be obtained as follows:

$$X_j = \mu + \sigma \cdot z_j \quad (4)$$

where z_j is taken from Table 2.

To perform M-DRM, all variables except one are held at their mean value while each variable is varied one at a time according to the five Gaussian quadratures using Equation (4). For example with three variables, denoted as α, β, λ, a total of 16 calculations will need to be performed, five for each of the three variables, and one final calculation with all variables held at the mean value. The first moment for each variable is then calculated as follows:

$$\rho_\alpha = \sum_{i=1}^{5} w_j \cdot R\left(\alpha_i, \beta_0, \lambda_0\right) \quad (5)$$

where w_j are the Gaussian weights according to Table 2, and $R(\alpha_i, \beta_0, \lambda_0)$ is the resultant value achieved by varying α by each Gaussian point according to Equation (4), and holding β and λ at their mean values.

The second moment for each variable is then calculated similarly as follows:

$$\theta_\alpha = \sum_{i=1}^{5} w_j \cdot R\left(\alpha_i, \beta_0, \lambda_0\right)^2 \quad (6)$$

Equations (4)-(6) are explained in Figure 4.

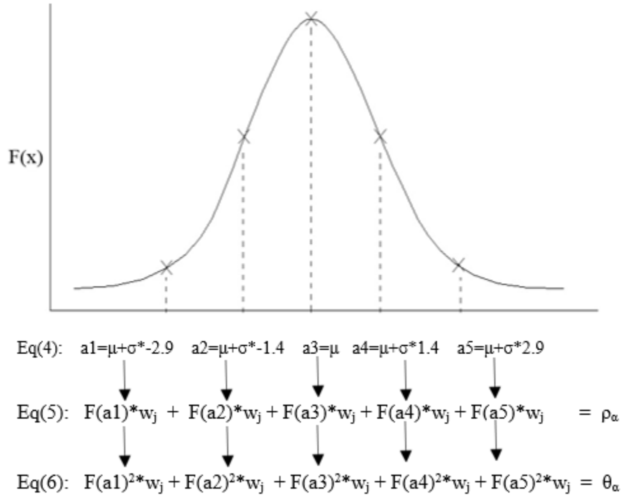

Figure 2: Graphical explanation of M-DRM.

The mean of the result can then be calculated using the following expression:

$$\mu = \rho_{avg}^{(1-n)} \cdot \rho_\alpha \cdot \rho_\beta \cdot \rho_\gamma \tag{7}$$

where n is the number of variables, three in this case, and ρ_{avg} is the result of holding all three variables at their mean. The second moment of the result is calculated by:

$$\theta = \theta_{avg}^{2\cdot(1-n)} \cdot \theta_\alpha \cdot \theta_\beta \cdot \theta_\gamma \tag{8}$$

These are then used to calculate the standard deviation of the result as follows:

$$\sigma = \sqrt{\theta - \mu^2} \tag{9}$$

A sensitivity analysis can also be performed. The primary sensitivity, S, can be calculated for each variable using:

$$S_\alpha = \frac{\dfrac{\theta_\alpha}{(\rho_\alpha)^2} - 1}{\left[\dfrac{\theta_\alpha}{(\rho_\alpha)^2} \cdot \dfrac{\theta_\beta}{(\rho_\beta)^2} \cdot \dfrac{\theta_\gamma}{(\rho_\gamma)^2}\right] - 1} \tag{10}$$

The global sensitivity, Sg, is calculated as:

$$Sg_\alpha = \frac{1 - \dfrac{\theta_\alpha}{(\rho_\alpha)^2}}{1 - \left[\dfrac{\theta_\alpha}{(\rho_\alpha)^2} \cdot \dfrac{\theta_\beta}{(\rho_\beta)^2} \cdot \dfrac{\theta_\gamma}{(\rho_\gamma)^2}\right]} \tag{11}$$

In simple terms, global sensitivity means the contribution of variability (or variance) of one particular random variable to the overall (or global) variance of the model output. The way it is calculated is that we evaluate the conditional variance of the output by keeping all other variables fixed but the one which is in question. This conditional variance is normalized by the global variance of the model output.

5. HANDLING OF INFINITE LIVES

One of the challenges encountered in applying M-DRM to a probabilistic fracture mechanics problem is that infinite fatigue lives are possible when the stress intensity factor range, ΔK_{eff}, drops below the threshold, ΔK_{th}. When the results are shown on an S-N plot (stress range vs. life), the corresponding stress range is the CAFL. When applying M-DRM, if the analyses at any of the Gauss points results in an infinite fatigue life, this can complicate the calculation of the moments of the fatigue life distribution. At higher stress ranges, this is not a problem. However, this issue becomes increasingly significant as the CAFL is approached.

5.1. Infinite fatigue lives in MCS

To analyze the MCS results for any stress range, the 10,000 simulation results were ordered from least to greatest, and the 5,000th result was used to determine the fatigue life associated with a 50% survival probability. Similarly the 95th percentile was determined based on the 500th result. This approach enabled the plotting of characteristic S-N curves even when some of the simulations led to infinite fatigue lives.

5.2. Infinite fatigue lives in M-DRM

As the M-DRM input grid points vary by as much as several times the standard deviation around the mean, the problem of infinite lives occurs at much higher stress ranges than for MCS. However, once the problem occurs it is not as easily handled, as every grid point is needed to determine the moments of the fatigue life distribution at a given stress range.

Figure 3 graphs the first variable whose variation results in infinite lives, ΔK_{th}, vs. the number of cycles to failure for each ΔK_{th}, for two nominal stress ranges, ΔS: 80 MPa and 90 MPa. The dark vertical lines are the input grid points for ΔK_{th}. For ΔS = 90 MPa all grid points result in finite life estimates, but at ΔS = 80 MPa, the last grid point returns an infinite value. Since it is the lower tail of the life distribution that we are most interested in, one possible solution would be to make an estimation for the grid point with an infinite life by extrapolation using the four known points, as shown in Figure 3, accepting that the result will not be accurate for the upper tail. However, eventually so many of the grid points yield infinite lives that the error resulting from this approximation is excessive.

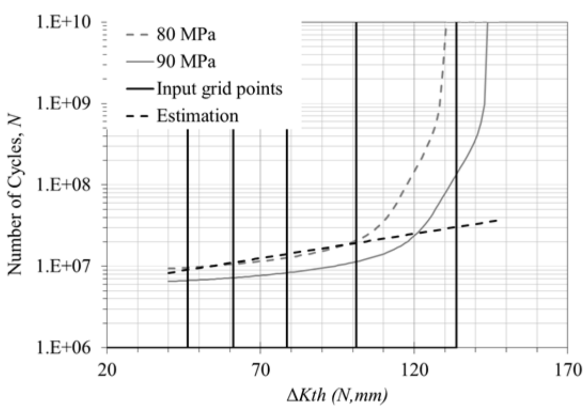

Figure 3: M-DRM results vs. ΔK_{th}.

An alternative solution to this problem is to recast the deterministic SBFM model, so that rather than using it to simulate crack growth until failure, it is used to calculate the CAFL for a given set of input parameters (i.e. the stress range below which there is no crack propagation).

In Figure 4, it is shown how the two analysis types, namely, the finite life analysis, where the SBFM model is used to perform the integration in Equation (1), and finite life analysis, where the stress range, ΔS, is increased until $\Delta K^{eff} = \Delta K_{th}$, can be used to generate probabilistic (50 and 95% survival probability) S-N curves, suitable for fatigue design.

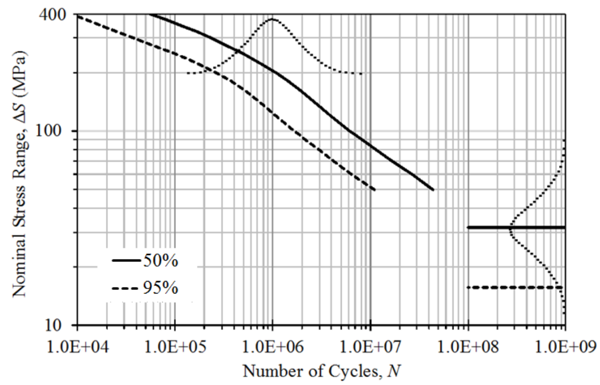

Figure 4: M-DRM finite and infinite life analysis.

6. RESULTS

6.1. Sensitivity Analysis

Using Equation (11), a global sensitivity analysis of the input variables can be performed, as shown in Figure 5. The global sensitivities in this figure are for an as-received weld.

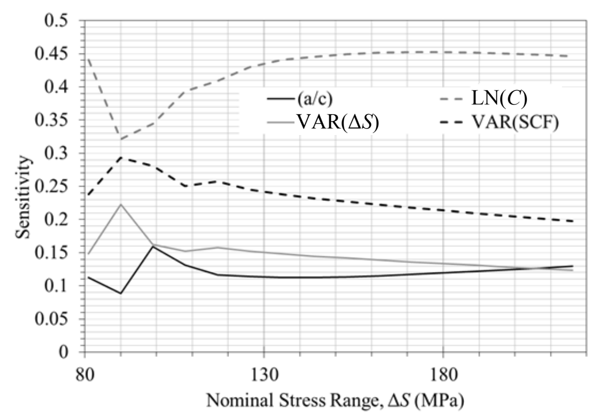

Figure 5: Sensitivity analysis.

The higher the sensitivity, the more significant are the variations in the parameter on the predicted fatigue life, based on a finite life analysis. The sensitivities of the variables not shown in Figure 5 were all negligible. The most sensitive variable at high stress levels, at almost 0.5, is the Paris law constant, LN(C). The reason for the unevenness at low stress ranges is that the extrapolation procedure shown in Figure 3 was used for parameters whose variations resulted in infinite life predictions, starting at 135 MPa.

6.2. Results Comparison

The effect of treatment by needle peening on the fatigue performance of the weld is shown in Figure 6, which compares a weld under constant amplitude (CA) loading in the as-received and needle peened conditions. 50 and 95% survival probability curves are plotted. The results in this figure were obtained by MCS. At high stress ranges (400 MPa) there is little difference in the results, but at lower stress ranges, the benefit of the treatment can be significant.

A comparison between M-DRM and MCS is shown in Figures 7 and 8 for as-received and needle peened welds, respectively. At high stress ranges, M-DRM was found to be accurate in predicting the MCS result within 1%. As the M-DRM starts to result in infinite life grid points, the accuracy decreases slightly, but is still well within 5%. Eventually there are too many infinite grid points, at which point, the finite life analysis is halted. The infinite life analysis is then performed using M-DRM, and the remainder of the curve can be approximated for design by simply extending both lines until they meet.

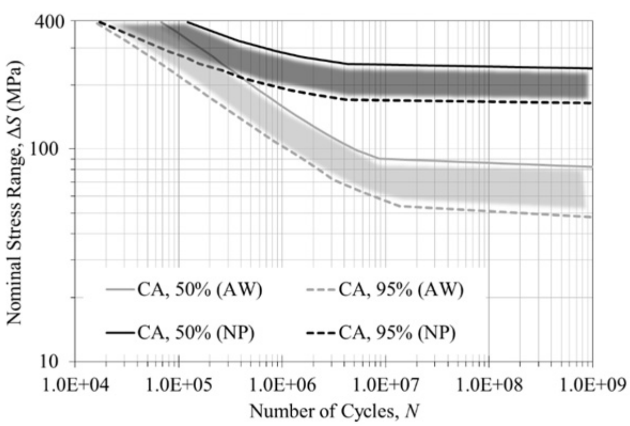

Figure 6: Constant amplitude (CA) loading, as welded (AW) vs. needle peened (NP).

The computational benefits of M-DRM were very obvious when obtaining the data for these figures. The run-time to generate a single S-N curve using MCS several hours to several days, depending on the length of the repeating loading history segment, whereas the M-DRM line could be plotted in several minutes.

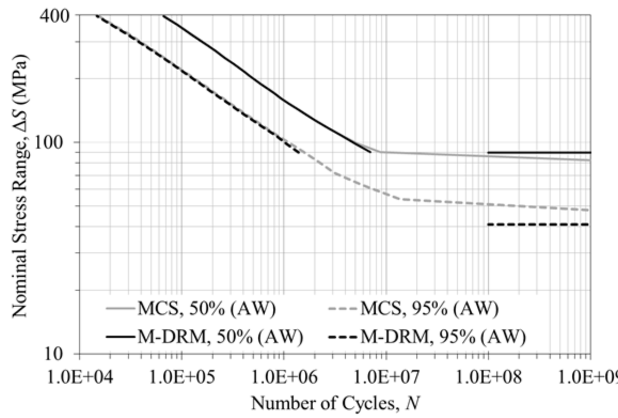

Figure 7: MCS vs. M-DRM (as-welded).

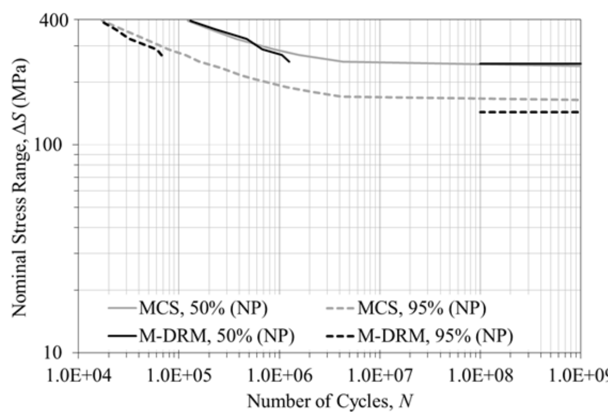

Figure 8: MCS vs. M-DRM (needle peened).

7. CONCLUSIONS

Based on the probabilistic MCS and M-DRM analyses presented in the previous sections, the following conclusions are drawn:

– M-DRM is a viable tool for probabilistic analysis of complex problems, such as the nonlinear fracture mechanics problem studied herein, and requires a fraction of the computational time required for MCS.
– In addition to providing rapid estimates of the output distribution moments, M-DRM enables sensitivity analysis to determine the most significant input parameters.
– While M-DRM analysis does not require trial and error or iteration and can handle a relatively complex nonlinear problem with a large number of statistical variables without difficult, special attention needs to be paid when the output distribution changes

suddenly, as seen in the analyses performed for the current study near the CAFL.

Further research on how to handle discontinuous output distributions in M-DRM is recommended. With regards to the further study of the impact treatments investigated herein, further work is planned to extend the results to VA loading analysis, since the effectiveness of impact treatments such has needle peening has been shown to be dependent on characteristics of the VA loading history, and it is expected that M-DRM will result in even greater time savings for very long VA loading histories.

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