On the Use of Spatially Averaged Shear Strength for the Bearing Capacity of a Shallow Foundation

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ABSTRACT: This study examines the validity for representing the mobilized shear strength as the spatial average over a prescribed soil volume. A shallow foundation problem is adopted to demonstrate this. The approach is simple. Two sets of field finite element (FEM) analyses are taken. The first set considers a spatially variable soil mass whose shear strength is simulated by a random field. The bearing capacity simulated by this first set of FEM is the actual (reference) capacity. The same random field is averaged over a prescribed volume of interest to obtain the mobilized value. The second set of FEM then considers a homogeneous soil mass whose shear strength is equal to this mobilized value. The bearing capacity simulated by this second set of FEM is then compared to the reference value. The comparison will be made on two levels. Level I compares the statistics of the two sets of capacities, whereas Level II compares the two sets of capacities on the 1:1 line. Based on these numerical studies, it is observed that the two sets of capacities are at most equal in "distribution", but not "almost everywhere".

1. INTRODUCTION

Soil-structure interaction occurs over a finite volume of soil (influence zone). For a spatially variable soil mass, it is natural to examine if an equivalent homogeneous soil mass exist that can reproduce the same response statistics. It is equally natural to assume that the governing soil parameter in this homogeneous soil mass is the spatial average (Vanmarcke 1977, 1984) over the influence zone. Fenton and Griffiths (2005) studied the settlement of a footing on a three-dimensional (3D) spatially variable soil mass with this practical objective in mind. They found that the settlement can be effectively simulated by considering the geometric average of the elastic modulus random field within a prescribed volume under the footing. Honjo and Otake (2013) studied the capacity of a footing on a two-dimensional (2D) spatially variable soil mass. They found that the capacity for the footing can be effectively simulated by considering the spatial average of the shear strength random field within a prescribed area under the footing.

In contrast to the spatial averaging over a prescribed region, Ching and Phoon (2013) found that the shear strength of a laboratory test specimen can NOT be effectively simulated by considering spatial averaging over any prescribed area or curve. Instead, they found that the shear strength can be effectively simulated by considering the average over the critical slip curve. Note that the critical difference here is that the critical slip curve is not a prescribed curve, but an emergent curve that depends on the random field realization. Hu and Ching (2015) also found that the active lateral force for a retaining wall can NOT be effectively simulated by considering the spatial average over any prescribed area or line.

Fenton and Griffiths (2005) and Honjo and Otake (2013) focused on the global response of a footing (settlement, capacity). Ching and Phoon (2013) and Hu and Ching (2015) focused at a more local level on the strength mobilized along an emergent critical slip curve. It is difficult to explain why these mechanical responses, which appear similar, would produce diametrically opposite conclusions. There is a strong practical motivation to examine the general validity of spatial averaging, because it is obviously easier to carry out reliability-based design using a random variable (spatial average) than a random field. The purpose of this study is to examine in more detail under what conditions would spatial averaging over some prescribed region be sufficient to reproduce the response statistics arising from a spatially variable field. A shallow foundation problem is adopted to demonstrate this. The spatially variable field is restricted to the undrained shear strength in this paper. The method adopted by this paper is straightforward. Two sets of finite element method (FEM) analyses will be conducted. The first set considers a spatially variable soil mass whose shear strength is simulated by a random field. The outcome of this first set of random field finite element method (RFEM) is the actual bearing capacity. It is the reference of the spatially variable soil mass. The same random field is then averaged over a prescribed area to obtain the spatial average. The second set of FEM then considers a homogeneous soil mass whose shear strength is equal to the spatial average. Then, the outcome of this second set of FEM will be referred to as the spatial average bearing capacity. It is then compared to the actual bearing capacity simulated by the RFEM. The comparison will be made on the following two levels: Level I compares the probability distributions of the two sets of responses, whereas Level II compares the two sets of responses on the 1:1 line.

2. RANDOM FIELD AND ITS SIMULATION

In this study, the only random soil property is the soil shear strength (τ_f). The shear strength $\tau_f(x,z)$ at a point in the soil mass is simulated by a random field, where x and z are respectively the horizontal and vertical coordinates. The friction angle is taken to be 0^o for simplicity, i.e., $\tau_f(x,z) = s_u(x,z)$, where s_u is the undrained shear strength. The shear strength $\tau_f(x,z)$ is simulated as a stationary lognormal random field with inherent mean = μ and inherent standard deviation = σ. The coefficient of variation (COV) of this random field is equal to σ/μ. A stationary lognormal random field can be simulated by taking exponential of a stationary Gaussian random field. To define the correlation structure in $\tau_f(x,z)$ between two locations with horizontal distance = Δx and vertical distance = Δz, the single exponential auto-correlation model is considered (Vanmarcke 1977, 1984):

$$\rho(\Delta x,\Delta z) = \exp\left(-2\,|\Delta x|/\delta_x - 2\,|\Delta z|/\delta_z\right) \quad (1)$$

where δ_x and δ_z are the horizontal and vertical scales of fluctuation (SOFs), respectively. Jha and Ching (2013) developed the Fourier series method (FSM) for simulating stationary normal random fields (point process). A 2D stationary normal random field W(x,z) over a domain of size $L_x \times L_z$ can be simulated by

$$W(x,z) = \mu + \mathrm{Re}\left[\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\left(a_{mn} + ib_{mn}\right)\exp\left(\frac{i2m\pi x}{L_x} + \frac{i2n\pi z}{L_z}\right)\right] \quad (2)$$

where Re[.] denotes the real part of the enclosed complex number; a_{mn} and b_{mn} are independent zero-mean normal random variables with variance σ_{mn}^2 given by (Jha and Ching 2013)

$$\sigma_{mn}^2 = \frac{\sigma^2}{q_x q_z}\left[\frac{1-\exp(-q_x)(-1)^m}{1+m^2\pi^2/q_x^2}\right]\times\left[\frac{1-\exp(-q_z)(-1)^n}{1+n^2\pi^2/q_z^2}\right] \quad (3)$$

where $q_x = L_x/\delta_x$ and $q_z = L_z/\delta_z$. Besides simulating the point process of a normal random field, the FSM is also able to directly simulate the spatial average of the normal random field over a prescribed rectangular region in 2D (Jha and Ching 2013).

3. RANDOM FIELD FINITE ELEMENT MODEL

This study compares the actual response with the spatial average response for a shallow foundation. The RFEM model for this particular problem is described below.

The ultimate bearing capacity (q_u) of a shallow foundation can be simulated by the RFEM (Fenton and Griffiths 2003; Honjo and Otake 2013). Figure 1a shows the model employed in the RFEM. A strip footing (width B = 3 m) is modeled as a rigid plate with a rough base. The unit weight of the soil is equal to 20 kN/m^3, the Young's modulus is equal to 40 MN/m^2, the Poisson ratio is 0.3, and the friction angle $\phi = 0^o$. The undrained shear strength $\tau_f(x,z)$ is simulated as a stationary lognormal random field. The $\ln(\tau_f)$ for each element is taken to be the cell average of the $\ln[\tau_f(x,z)]$ random field over that element. In the RFEM, the first step is the geostatic equilibrium to build up the in-situ geostatic stress field, and the second step is to apply a vertical load up to the point of non-convergence. The histories of the bearing stress of the footing and the vertical settlement at the center of the footing are recorded. The actual bearing capacity, denoted by q_u^m, is defined as the bearing stress when the settlement reaches 0.02B.

Three spatial averages are considered: (a) averaging over the rectangular area of size $L_x \times L_z$ = 2B × 0.7B (Figure 1b), suggested by Honjo and Otake (2013) for cohesive soil; (b) averaging over a larger rectangular area of size $L_x \times L_z$ = 3B × 1B (Figure 1c), close to the Prandtl-type plastic zone; and (c) averaging over an even larger rectangular area of size $L_x \times L_z$ = 3B × 5B (Figure 1d). Similarly, the averages of the τ_f values for all elements in the areas are first simulated, and a homogeneous FEM is simulated to obtain the spatial average responses. The three spatial average responses are denoted by q_u^{RA1}, q_u^{RA2}, and q_u^{RA3}, respectively.

4. ACTUAL VERSUS SPATIAL AVERAGE RESPONSES

The comparison between the actual (q_u^m) and spatial average responses (q_u^{RA1}, q_u^{RA2}, and q_u^{RA3}) can be conducted on the following two levels (the number of random field realizations = 100 for all cases):

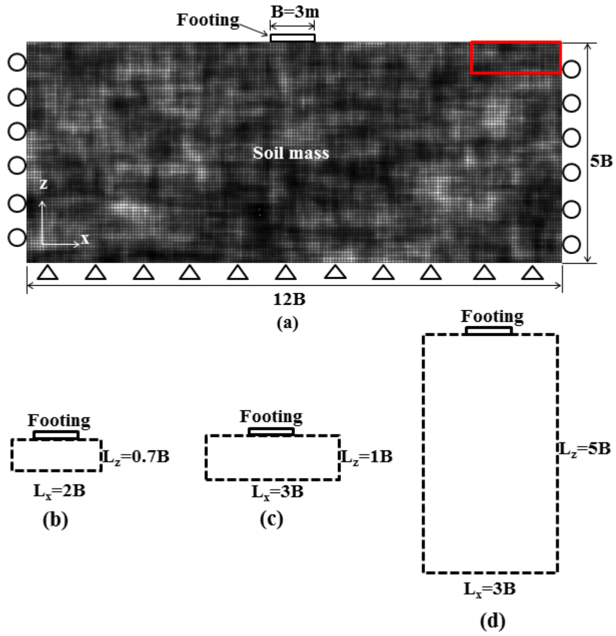

Figure 1: (a) RFEM model and the averaging areas: (b) 2B × 0.7B, (c) 3B × 1B, and (d) 3B × 5B.

Level I – Comparison between the probability distributions of the actual and spatial average responses. The quantile-quantile (Q-Q) plot between the actual (from the RFEM) and spatial average responses (from the homogeneous FEM) will be used to compare the probability distributions. The good-of-fit is checked by the Kolmogorov-Smirnov test (K-S test). Based on the resulting p-values, the K-S test can determine whether the null hypothesis (H_0) that the two sets of responses are identically distributed or not. If the p-value is less than 0.05 (significance level), H_0 can be rejected at the customary 95% level of confidence. Otherwise, there is insufficient evidence to reject H_0. In the Level I comparison, two statistics are also computed: (a) the ratio in the mean values r_μ = (mean of the spatial average response)/(mean of the actual response) and (b) the ratio in the COVs r_{COV} = (COV of the spatial average response)/(COV of the actual response). If the two probability distributions are identical, $r_\mu = r_{COV} = 1$.

Level II – Comparison between the actual and spatial average responses on the 1:1 line. This level of comparison can identify whether the two sets of responses are equal to each other or not. The root mean square (RMS) of the normalized distance to the 1:1 line is used to quantify the deviation to the 1:1 line: RMS = $[(d_1^2 + d_2^2 +\ldots + d_n^2)/n]^{0.5}$, where d_i = (spatial average response – actual response)/(actual response) is the normalized distance for the i-th data point to the 1:1 line. RMS equals zero if and only if the two sets of responses lie exactly on the 1:1 line. Level II comparison is more strict than the Level I. If the two sets of responses lie on the 1:1 line, they must have identical probability distributions – the perfect fulfillment of Level II implies the perfect fulfillment of Level I. However, the converse is not true.

For the Level I comparison, Figure 2a and 2b show the Q-Q plots between the q_u^m and q_u^{RA1} samples for two cases with B = 3 m, μ = 20 kN/m^2, COV = 0.3 and isotropic SOF (δ/B = 0.5 and 300). As mentioned previously, q_u^{RA1} refers to the spatial average response for the rectangular area of size $L_x \times L_z$ = 2B $\times$ 0.7B (see Figure 1b), and q_u^m is the actual response. For the case with δ/B = 300, the large p-value for the K-S test (Figure 2b) indicates that the null hypothesis H_0 that the two sets of responses are identically distributed cannot be rejected. The ratio in the mean value (r_μ) and the ratio in the COV (r_{COV}) are both very close to 1. The Level II comparison for the case with δ/B = 300 further shows that the two sets of responses lie very close to the 1:1 line (RMS = 0.07) (Figure 2d). This result is reasonable because the case with δ/B = 300 is close to a homogeneous case.

However, for the case with δ/B = 0.5, the small p-value (nearly zero) in the Level I comparison (Figure 2a) indicates that H_0 can be rejected (i.e., q_u^m and q_u^{RA1} are not identically distributed). Because r_μ = 1.06, q_u^{RA1} has a mean value that is 1.06 – 1 = 6% larger than the mean of q_u^m (q_u^{RA1} is unconservative). The Level II comparison (Figure 2c) further shows that q_u^m and q_u^{RA1} are quite different: they are not close to the 1:1 line (RMS = 0.22).

Based on the results of the above Level I and Level II comparisons, it can be concluded that the spatial average is effective to characterize the shear strength spatial variability for δ/B = 300 but not effective for δ/B = 0.5. Moreover, it is on the unconservative side for δ/B = 0.5.

4.1. Isotropic cases

More detailed comparison results for isotropic cases ($\delta_x = \delta_z = \delta$) are present below. Consider the case with B = 3 m, μ = 20 kN/m^2, COV = 0.3 and isotropic SOF (δ/B = 0.1, 0.5, 2, 10, 50, 300). As mentioned previously, three spatial averages (q_u^{RA1}, q_u^{RA2}, q_u^{RA3}) are considered (see Figure 1b~d). Table 1 shows the comparison results. For the three cases with δ/B $\geq$ 10, H_0 cannot be rejected in the Level I comparison. For all cases, r_μ is close to 1, although some slight deviations from 1 are observed for the three cases with δ/B $\leq$ 2. r_{COV} can be noticeably less than 1 for the four cases with δ/B $\leq$ 10. Even though the Level I comparison indicates that (q_u^{RA1}, q_u^{RA2}, q_u^{RA3}) and q_u^m are roughly identically distributed for the three cases with δ/B $\geq$ 10, the large RMS values in the Level II comparison indicate that the (q_u^{RA1}, q_u^{RA2}, q_u^{RA3}) versus q_u^m data are in general not close to the 1:1 line. This is illustrated in Figure 3, which shows the q_u^{RA1} versus q_u^m plot for the case with δ/B = 10. It is clear that although q_u^{RA1} and q_u^m are roughly identically distributed (Figure 3a), they are not close to the 1:1 line (Figure 3b). That is, these cases fulfill the Level I requirement but do not fulfill the Level II requirement. The only case that fulfills both requirements is the one with δ/B = 300.

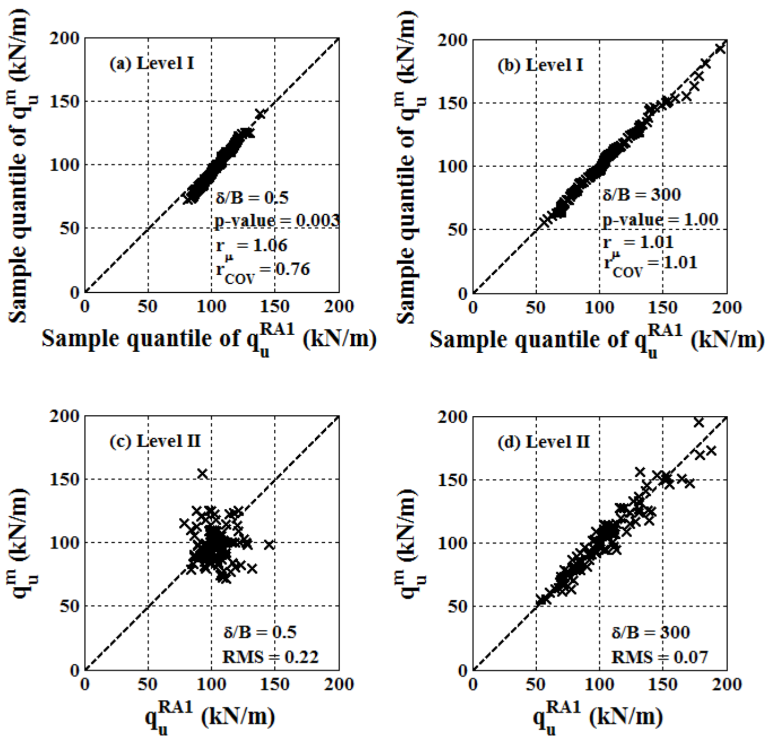

Figure 2: Level I and II comparisons between the actual and spatial average responses.

Table 1: Level I and II comparison results for various isotropic δ/B

δ/B	Level I			Level II
(a) q_u^m v.s. q_u^{RA1}				
δ/B	p-value	r_μ	r_{COV}	RMS
0.1	0.00 (reject)	1.05	0.43	0.08
0.5	0.003 (reject)	1.06	0.76	0.22
2	0.005 (reject)	1.07	0.89	0.33
10	0.19 (do not reject)	0.99	0.74	0.33
50	0.89 (do not reject)	1.01	1.00	0.20
300	1.00 (do not reject)	1.01	1.01	0.07
(b) q_u^m v.s. q_u^{RA2}				
δ/B	p-value	r_μ	r_{COV}	RMS
0.1	0.00 (reject)	1.05	0.25	0.08
0.5	0.00 (reject)	1.06	0.54	0.20
2	0.003 (reject)	1.06	0.78	0.31
10	0.14 (do not reject)	0.99	0.72	0.31
50	0.89 (do not reject)	1.01	0.99	0.20
300	0.99 (do not reject)	1.01	1.01	0.07
(c) q_u^m v.s. q_u^{RA3}				
δ/B	Level I			Level II
	p-value	r_μ	r_{COV}	RMS
0.1	0.00 (reject)	1.03	0.14	0.07
0.5	0.00 (reject)	1.04	0.25	0.16
2	0.003 (reject)	1.02	0.48	0.23
10	0.26 (do not reject)	1.00	0.76	0.25
50	0.89 (do not reject)	0.98	0.95	0.13
300	1.01 (do not reject)	1.01	0.99	0.05

In summary, for the Level I point of view, (q_u^{RA1}, q_u^{RA2}, q_u^{RA3}) are acceptable representations (p-value > 0.05) for q_u^m for the cases with $\delta/B \geq 10$ because they are roughly identically distributed. However, the only case that fulfills both Levels I and II requirements is the one with $\delta/B = 300$.

4.2. Anisotropic cases

The above results focus on isotropic random fields. It is interesting to compare the actual and spatial average responses under anisotropic random fields ($\delta_x \neq \delta_z$), which are common in reality. Detailed comparison results are present below.

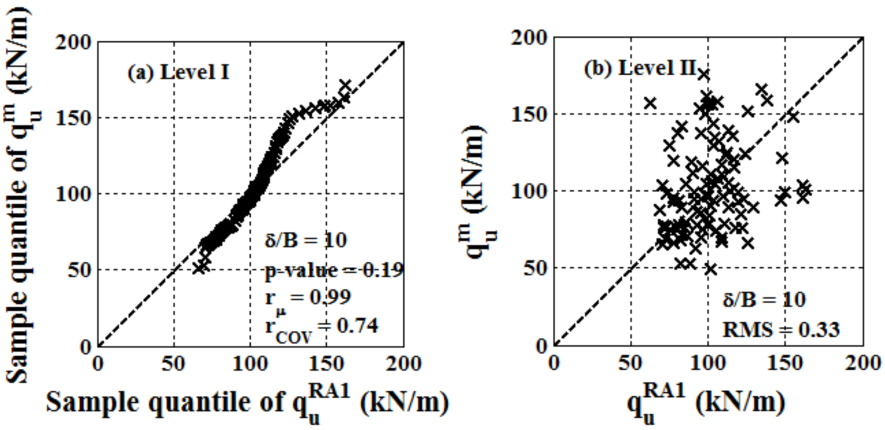

Figure 3: Level I and II comparisons between the actual and spatial average responses for $\delta/B = 10$.

Consider the case with B = 3 m, $\mu = 20$ kN/m^2, COV = 0.3 and anisotropic SOF ($\delta_x/B = \infty$ and $\delta_z/B = 0.1, 0.5, 2, 10, 50, 300$). Table 2 shows the comparison results. For the four cases with $\delta_z/B \geq 2$, H_0 cannot be rejected in the Level I comparison. For all cases, r_μ is close to 1, although some slight deviations from 1 are observed for the three cases with $\delta_z/B \leq 2$. r_{COV} can be noticeably less than 1 for the three cases with $\delta_z/B \leq 2$. Even though the Level I comparison indicates that (q_u^{RA1}, q_u^{RA2}, q_u^{RA3}) and q_u^m are roughly identically distributed for the four cases with $\delta_z/B \geq 2$, the large RMS values in the Level II comparison indicate that the (q_u^{RA1}, q_u^{RA2}, q_u^{RA3}) versus q_u^m data are in general not close to the 1:1 line. That is, these cases fulfill the Level I requirement but do not fulfill the Level II requirement. The only case that fulfills both requirements is the one with $\delta_z/B = 300$.

In summary, for the Level I point of view, (q_u^{RA1}, q_u^{RA2}, q_u^{RA3}) are acceptable representations (p-value > 0.05) for q_u^m for the cases with $\delta_z/B \geq 2$ because they are roughly identically distributed. However, the only case that fulfills both Levels I and II requirements is the one with $\delta_z/B = 300$.

4.3. Discussions

The results in Tables 1 and 2 show that (q_u^{RA1}, q_u^{RA2}, q_u^{RA3}) do not significantly outperform one another. In terms of Level I view, their p-values are comparable, and in terms of Level II, their RMSs are also comparable. r_μ values are all close to 1, meaning that (q_u^{RA1}, q_u^{RA2}, q_u^{RA3}) are all nearly unbiased. However, r_{COV} values are noticeably less than 1 when δ/B is small, meaning that the variabilities for (q_u^{RA1}, q_u^{RA2}, q_u^{RA3}) are less than that for q_u^m. Among them, the r_{COV} values for q_u^{RA1} are closer to 1. Therefore, it is concluded that q_u^{RA1} (averaging over $L_x \times L_z = 2B \times 0.7B$; Figure 1b) has the best match to the actual response q_u^m. This conclusion is consistent to the one made by Honjo and Otake (2013). Table 3 shows the q_u^{RA1} cases that fulfill the Level I or Level II requirement. A case fulfills the Level I requirement if the actual and spatial average responses are (roughly) identically distributed (p-value > 0.05). It fulfills the Level II requirement if the actual and spatial average responses are close to the 1:1 line (RMS < 0.1). In general, fulfilling anisotropic cases are more than fulfilling isotropic cases. This means that the spatial average capacity can better represent the actual capacity in anisotropic cases than in isotropic cases. For isotropic cases, cases with $\delta/B \geq 10$ fulfill Level I, but the only case fulfills both Levels I and II is $\delta/B = 300$. For anisotropic cases, cases with $\delta_z/B \geq 2$ fulfill Level I, but the only case fulfills both Levels I and II is still $\delta_z/B = 300$. It is worth noting that the isotropic case with $\delta/B = 0.1$ fulfills Level II but not Level I.

Table 2: Level I and II comparison results for $\delta_x/B = \infty$ and various δ_z/B.

(a) q_u^m v.s. q_u^{RA1}				
δ_z/B	Level I			Level II
	p-value	r_μ	r_{COV}	RMS
0.1	0.00 (reject)	1.08	0.67	0.14
0.5	0.01 (reject)	1.07	0.86	0.25
2	0.34 (do not reject)	1.03	0.92	0.26
10	0.79 (do not reject)	1.02	0.97	0.20
50	0.96 (do not reject)	1.00	1.01	0.07
300	0.99 (do not reject)	1.00	1.00	0.02
(b) q_u^m v.s. q_u^{RA2}				
δ_z/B	Level I			Level II
	p-value	r_μ	r_{COV}	RMS
0.1	0.00 (reject)	1.08	0.50	0.12
0.5	0.00 (reject)	1.06	0.69	0.20
2	0.19 (do not reject)	1.03	0.80	0.24
10	0.56 (do not reject)	1.02	0.93	0.18
50	0.96 (do not reject)	1.00	1.00	0.07
300	0.99 (do not reject)	1.00	1.00	0.02
(c) q_u^m v.s. q_u^{RA3}				
δ_z/B	Level I			Level II
	p-value	r_μ	r_{COV}	RMS
0.1	0.00 (reject)	1.05	0.50	0.11
0.5	0.04 (reject)	1.04	0.80	0.19
2	0.13 (do not reject)	0.97	0.92	0.16
10	0.77 (do not reject)	1.01	1.14	0.15
50	1.00 (do not reject)	0.99	0.97	0.07
300	0.97 (do not reject)	0.98	1.01	0.04

The results in Table 3 imply the following conclusions. If the goal is to maintain the correct *probability distribution* of the actual response, the spatial average of the spatially variable shear strength over a prescribed region may be an acceptable representation for the reality (spatially variable field) as long as both δ_x/B and δ_z/B are sufficiently large (e.g., > 2). However, if the goal is to obtain the correct *value* of the actual response, the spatial average of the spatially variable shear strength is acceptable only when both δ_x/B and δ_z/B are very large (close to homogeneity).

Based on the above observations, we must point out that the Level I requirement is necessary but not sufficient. For a spatial average to be an accurate representation for the spatially variable shear strength, the Level I requirement must be fulfilled. However, if a spatial average fulfills the Level I requirement, it does not mean that the spatial average is an accurate representation. Consider an inappropriate spatial average and isotropic case with $\delta/B = 2$. The spatial average is taken over a rectangular region that is remote from the foundation (see the red rectangular region in Figure 1a) to obtain q_u^{RA4}. Figure 4a shows the Q-Q plot for q_u^{RA4} and q_u^m: q_u^{RA4} and q_u^m are roughly identically distributed. However, it is inappropriate to assert that such a spatial average is an accurate representation, because the prescribed rectangular region is remote and can hardly affect the actual failure of the shallow foundation. The Level II comparison is more sensible for this case (see Figure 4b): q_u^{RA4} and q_u^m are not close to the 1:1 line. In fact, they are independent of each other. In our opinion, the Level II comparison is more suitable and meaningful than the Level I comparison. However, if we stick to the Level II comparison, q_u^{RA4} is an accurate representation for q_u^m only for the isotropic case with $\delta/B = 300$.

Table 3: Cases that fulfill the Level I and Level II requirements.

Case	Fulfill Level I	Fulfill Level II
Isotropic ($\delta_x = \delta_z = \delta$)	δ/B = 10, 50, 300	δ/B = 0.1, 300
Anisotropic ($\delta_x \neq \delta_z$)	δ_z/B = 2, 10, 50, 300	δ_z/B = 50, 300

5. CONCLUSIONS

The purpose of this study is to examine in more detail under what conditions would spatial averaging over some prescribed region be sufficient to reproduce the response statistics arising from a spatially varying field. The answer is: it depends. If the goal is to maintain the probability distribution of the actual response (the Level I requirement), the answer is probably "Yes, as long as the scale of fluctuation is large enough". Here "large enough" means the scale of fluctuation should not be smaller than 2 times of the width of the shallow foundation. If the goal is to maintain the correct value of the actual response (the Level II requirement), the answer is probably "No, unless both of the horizontal and vertical scales of fluctuation are very large (close to homogeneity)". However, we argue that the Level II requirement is more sensible than Level I, because the Level I requirement is necessary but not sufficient.

This study is preliminary, but it has revealed complications to the attractive concept of smearing a spatial varying field into a homogeneous spatial average over a prescribed domain. The applicability of smearing merits more research, given its potential usefulness in reliability-based design.

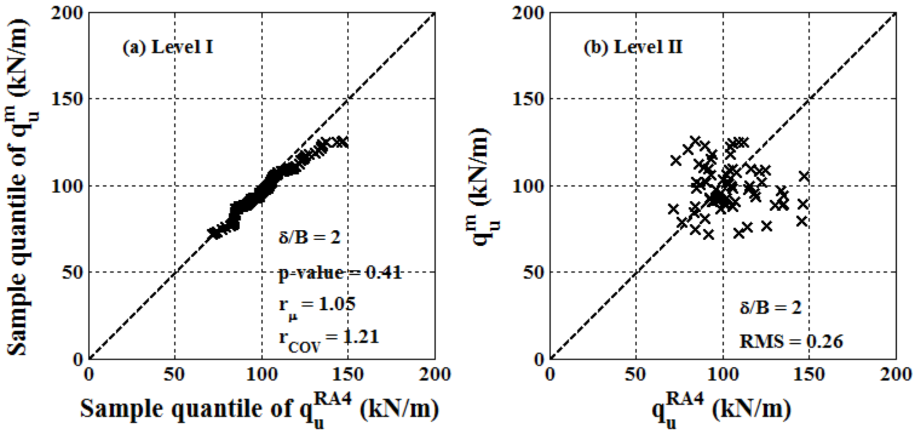

Figure 4: Comparisons between the actual and spatial average response (average over the red rectangular in Figure 1a and isotropic case with δ/B = 2)

6. ACKNOWLEDGEMENTS

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