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Stochastic Renewal Process Models for Life Cycle Cost and Utility Analysis

Mahesh D. Pandey and Z. Wang
Department of Civil and Environmental Engineering, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada

T. Cheng
Technology and Engineering Center for Space Utilization, Chinese Academy of Science, Beijing, China

ABSTRACT: The paper presents a systematic formulation of the life cycle cost and utility analysis based on the theory of stochastic renewal processes. The paper derives integral equations for expected cost and net present value, variance and expected utility over a given period. The proposed formulation can be used to optimize the design and rehabilitation activities to improve the life cycle performance of structures that are vulnerable to external hazards, such as earthquakes, winds storms and floods.

1. INTRODUCTION

The life-cycle cost analysis focuses on the estimation of the total cost of construction, operation, maintenance, decommissioning, and many other activities, over a given time horizon or service period of a structure or facility. In this analysis, one of the most uncertain elements is the cost of repairing and restoring the structural damage caused by external hazards, such as earthquakes, wind storms and floods. Uncertainty in the estimation of repair cost arises from intrinsic uncertainty associated with the occurrence frequency and intensity of a given type of hazard.

As the focus of the performance-based design has shifted to life-cycle cost analysis (LCCA), this topic has become an important area of research (Koduru and Haukaas, 2010). LCCA is also intended to support the decision making regarding improvements in design and retrofitting of structure. For this purpose, a suitable metric of cost is used, such as the expected cost and expected discounted cost (or Net Present Value - NPV). The expected utility approach is also finding application with the purpose of incorporating the risk aversion of the decision maker (Cha and Ellingwood, 2013).

In a technical sense, LCC estimation problem should be analyzed using the theory of compound renewal processes. In structural engineering, a special case of renewal process, the homogeneous Poisson process (HPP), has been traditionally used LCCA. A most notable example is the seismic risk analysis. Although the HPP model greatly simplifies the analytical formulation, it has a downside that it completely masks the real probabilistic structure of the solution. What it means is that the results obtained under the HPP assumption cannot be readily extended to other situations without understanding the theory of the stochastic renewal processes.

The main objective of this paper is to provide a clear exposition of key ideas of the theory of stochastic renewal processes and illustrate their applications to the life-cycle cost analysis. In particular, derivations of the expected value and the variance of the cost, expected NPV, and expected utility are presented for a stochastic renewal process model. Results for HPP model are obtained as a special case of the renewal process. The example of seismic risk analysis is discussed in more detail. An ulterior motive of this study is to help new generation of engineers understand the key concepts of stochastic models for life-cycle cost analysis.

2. STOCHASTIC RENEWAL PROCESS

2.1. Stochastic Point Process

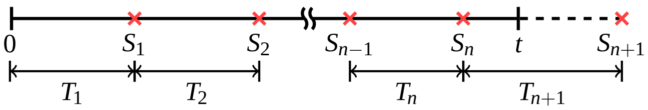

Figure 1: A schematic of the renewal process.

Mathematically, a point process is a strictly increasing sequence of real numbers, $S_1 < S_2 < \cdots$, without a finite limit point, i.e., as $i \to \infty$, $\lim S_i \to \infty$ and $S_0 = 0$. In an engineering sense however, S_i denotes the time of arrival of an i^{th} event (or hazard), as shown in Figure 1.

A point process can be equivalently represented by a sequence of inter-arrival times, $T_1, T_2, \cdots$, with $T_n = S_{n+1} - S_n$. An ordinary *renewal process* is defined as a sequence of non-negative, independent and identically distributed random variables $T_1, T_2, \cdots, T_n$ with a distribution $F_T(t)$.

The arrival time of an n^{th} event, S_n, can be written as a partial sum, $S_n = T_1 + T_2 \cdots + T_n$. The probability distribution of S_n can be derived in principle from an n-fold convolution of distribution, $F_T(t)$, as

$$F_{S_n}(x) = \mathbb{P}[T_1 + T_2 \cdots + T_n \leq x] = F_T^{(n)}(x) \quad (1)$$

This convolution can be evaluated in a sequential manner as

$$F_T^{(n)}(x) = \int_0^t F_T^{(n-1)}(x-y)\,\mathrm{d}F_T(y), \quad (n \geq 2) \quad (2)$$

Note that $\mathrm{d}F_T(y) = f_T(y)\,\mathrm{d}y$ when the probability density of T, $f_T(y)$, exists.

The number of events, $N(t)$, in the time interval $(0,t]$ is referred to as a *counting (or renewal) process* associated with the partial sums S_n, $n \geq 1$, and formally defined as

$$N(t) = \max\{n, S_n \leq t\} \quad (3)$$

Note that $N(t) = n$ is equivalent to the event $S_n \leq t < S_{n+1}$. This analogy is very useful in analyzing functions of the renewal process, as shown later in the paper.

The probability distribution of $N(t)$ can be written as

$$\mathbb{P}[N(t) = n] = \mathbb{P}[S_n \leq t < S_{n+1}] = F_{S_n}(t) - F_{S_{n+1}}(t) \quad (4)$$

The distribution of $N(t)$ is not easy to derive in a general setting. Instead, the expected number of failures is used to characterize the renewal process.

2.1.1. Renewal Function

The renewal function, $\Lambda(t)$, is defined as the expected number of renewals in $(0,t]$, which is computed using an integral equation (Tijms, 2003). The derivation of the integral equation is based on the renewal argument that is extremely useful in analyzing a variety of problems.

The use of a binary indicator function makes it easier to write concise mathematical statements. It is used to test a logical condition in the following way:

$$\mathbf{1}_{\{A\}} = \begin{cases} 1 & \text{only if } A \text{ is true} \\ 0 & \text{otherwise.} \end{cases} \quad (5)$$

From probability theory, it is known that $\mathbb{E}\left[\mathbf{1}_{\{A\}}\right] = \mathbb{P}[A]$. Using the indicator function, the counting process can be written as

$$\begin{aligned} N(t) &= \sum_{i=1}^{\infty} \mathbf{1}_{\{S_i \leq t\}} \quad \text{and} \\ \mathbb{E}[N(t)] = \Lambda(t) &= \sum_{i=1}^{\infty} \mathbb{P}[S_i \leq t] \end{aligned} \quad (6)$$

Eq (6) and Eq. (1) implies that the renewal function can be written as a sum of convolutions:

$$\Lambda(t) = \sum_{i=1}^{\infty} F_T^{(n)}(t) \quad (7)$$

However, this approach is not fruitful, as the computation of higher order convolutions is not an easy task. Therefore, the computation relies on the renewal property of the process.

The analysis starts with breaking the sum given in Eq (6) as

$$\begin{aligned}\Lambda(t) &= \mathbb{P}[S_1 \le t] + \sum_{i=2}^{\infty} \mathbb{P}[S_i \le t] \quad \text{or} \\ &= \mathbb{P}[S_1 \le t] + \sum_{i=1}^{\infty} \mathbb{P}[S_{i+1} \le t] \qquad (8)\end{aligned}$$

The essence of the renewal argument is that after a renewal a (probabilistic) replica of the original renewal process starts again. Noting that $S_1 = T_1$ and $S_{i+1} = T_1 + S_i$, Eq. (8) can be rewritten as

$$\Lambda(t) = \mathbb{P}[T_1 \le t] + \sum_{i=1}^{\infty} \mathbb{P}[T_1 + S_i \le t] \qquad (9)$$

Any i^{th} term in the sum in Eq. (9) can be simplified as

$$\mathbb{P}[T_1 + S_i \le t] = \int_0^t \mathbb{P}[S_i \le t - x]\, \mathrm{d}F_T(x) \qquad (10)$$

Substituting this in Eq. (9) and inter-changing the order of summation and integration leads to

$$\Lambda(t) = \mathbb{P}[T_1 \le t] + \int_0^t \left(\sum_{i=1}^{\infty} \mathbb{P}[S_i \le t - x] \right) \mathrm{d}F_T(x) \qquad (11)$$

Comparing the summation term inside the parentheses with Eq. (6), it is nothing but the expected number of renewals in time interval $(0, t-x]$, which is equal to $\Lambda(t-x)$ by virtue of the renewal argument. This leads to the final result known as the renewal equation (Tijms, 2003):

$$\Lambda(t) == F_T(t) + \int_0^t \Lambda(t-x)\, \mathrm{d}F_T(x). \qquad (12)$$

The renewal rate is defined as the expected number of renewals per unit time given as:

$$\lambda(t) = \frac{\mathrm{d}\Lambda(t)}{\mathrm{dt}} \qquad (13)$$

2.1.2. Marked and Compound Renewal Processes

In addition to the time of occurrence of a hazard, its severity (or intensity) tends to be highly uncertain. Variability in the intensity can be modelled by a random variable, which is referred to as the mark of the renewal process. If a random mark X_i is assigned to the arrival time S_i, then the sequence $\{(S_1, X_1), (S_2, X_2), \cdots\}$ is called a *marked point process*, as shown in Figure 2. This model is useful to evaluate the probability of intensity exceeding a critical limit in a time interval $(0, t)$.

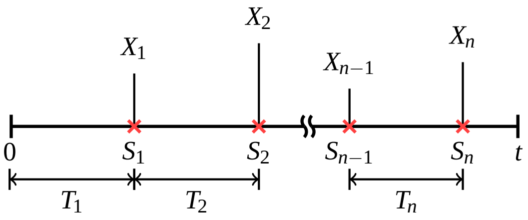

Figure 2: An example of a marked renewal process.

The compound process refers to the cumulative effect of a renewal process. For example, each occurrence of a hazard results in damage costing \$ C to repair the structure. Suppose the repair cost is modelled as a random variable, then total (cumulative) repair cost in the interval $(0, t)$ is given as a random sum:

$$K(t) = \sum_{i=1}^{N(t)} C_i \qquad (14)$$

$K(t)$ is technically referred to as the compound process. Note that $K(t) = 0$ for $t < S_1$. The mean and variance of the compound process are useful in the life cycle cost analysis, as shown later in the paper.

2.2. *Homogeneous Poisson Process*

This is the simplest and most widely used renewal process model in which the time between the occurrence of events is an exponentially distributed random variable with the distribution $F_T(x) = 1 - e^{-\lambda x}$ and the mean of $\mu_T = 1/\lambda$. The distribution of $N(t)$ is explicitly given by the Poisson probability mass function as

$$\mathbb{P}[N(t)) = k] = p_k = \frac{(\lambda\ t)^k\, e^{-\lambda t}}{k!} \qquad (15)$$

The probability of no renewal in $(0, t]$ (e.g., no occurrence of failure) is synonymous with the reliability in time interval, $(0, t]$ as

$$\mathbb{P}[N(t)) = 0] = \mathbb{P}[T_1 > t] = R(t) = e^{-\lambda t}, \qquad (16)$$

which is essentially the complementary distribution of T. The renewal function of HPP is given as

$$\Lambda(t) = \lambda\ t \tag{17}$$

Here, λ is referred to as the failure rate. In summary, the probabilistic structure of the homogeneous Poisson process is completely defined by simple analytical formulas.

2.2.1. *Decomposition and Superposition*

The two most useful properties of the Poisson process are related to the decomposition and superposition defined as follows.

Suppose the seismic events occur with a rate λ, but the probability of structural failure given an event is p, then the structural failure is also a Poisson process with the rate λp. This assumes that the structure is renewed to the original state after each failure, and the repair time is negligible. In fact, a Poisson process can be randomly decomposed into n sub-processes, each with the rate parameters as λp_i, $i = 1, \cdots, n$ and $p_1 + \cdots + p_n = 1$.

The superposition is related to the merger of Poisson processes. Suppose there are n seismic sources (faults) and λ_i is earthquake occurrence rate from an i^{th} source. The occurrence of earthquakes from any of n sources is also a Poisson process with rate being the sum of rates of all the seismic sources, i.e.,

$$\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_n \tag{18}$$

This property is extensively used in probabilistic seismic hazard assessment (PSHA).

2.2.2. *Marked and Compound Poisson Processes*

Randomness in the intensity and occurrence of an external hazard can be modelled as a marked Poisson process. For example the peak ground acceleration (pga) associated with an earthquake event is modelled as a random variable. The marked process is useful in structural reliability analysis over a time interval (or life-cycle).

Given the distribution of the pga, $F_S(s)$, earthquake events exceeding certain level, say s_1, also constitute a Poisson process with the rate λp_1, where $p_1 = 1 - F_S(s_1)$ This is based on the decomposition property of the Poisson process. Based on Eq. (16), the structural reliability is given as $e^{-t\lambda p_1}$.

The compound Poisson process also has a simple analytical structures, which will be discussed in the next Section.

3. LIFE-CYCLE COST ANALYSIS (LCCA)

3.1. *Expected Life-Cycle Cost*

Suppose the occurrence of a hazard costs the owner of a facility C \$, in repairing and restoring the structure. The repair cost is uncertain due to uncertainty associated with the intensity of hazard and other design features. The repair cost has a mean μ_C and standard deviation σ_C. The total cost, $K(t)$, in a time interval $(0, t]$ is a random sum given by Eq. (14). The evaluation of expected cost begins with rewriting it as

$$K(t) = \sum_{i=1}^{\infty} C_i \mathbf{1}_{\{S_i \le t\}} \tag{19}$$

If the cost is assumed to be independent of inter-occurrence time (T) of hazards, then the evaluation of expected cost becomes rather simple as shown below.

$$\begin{aligned} \mathbb{E}\left[K(t)\right] &= \sum_{i=1}^{\infty} \mathbb{E}\left[C_i\right] \mathbb{E}\left[\mathbf{1}_{\{S_i \le t\}}\right] \\ &= \mathbb{E}\left[C\right] \sum_{i=1}^{\infty)} \mathbb{P}\left[S_i \le t\right] \\ \mathbb{E}\left[K(t)\right] &= \mu_C\ \Lambda(t) \quad \text{(from Eq. (6))} \end{aligned} \tag{20}$$

Eq. (20) is a standard result of the compound renewal process (Gallager, 2013).

3.2. *Variance of Life-Cycle Cost*

The evaluation of the variance of LCC begins in a much more systematic way (Cheng and Pandey, 2012). Firstly, second moment of the cost is written using the law of total expectation as

$$\mathbb{E}\left[K^2(t)\right] = \mathbb{E}\left[K^2(t)\mathbf{1}_{\{T_1 \le t\}}\right] + \mathbb{E}\left[K^2(t)\mathbf{1}_{\{T_1 > t\}}\right] \tag{21}$$

In case of $T_1 > t$, i.e., the first occurrence of hazard is beyond the time interval of interest, no cost will accrue, i.e., $K(t) = K^2(t) = 0$.

On the other hand $T_1 \leq t$ means that at least one hazard would occur in the time interval. Therefore, the total cost will be the sum of damage cost due to the first hazard, C_1 at time T_1, and the cost, $K(T_1,t)$, in the remaining time interval. Thus, $K(t) = C_1 + K(T_1,t)$. Using these result,

$$\mathbb{E}\left[K^2(t)\right] = \mathbb{E}\left[\left(C_1 + K(T_1,t)\right)^2 \mathbf{1}_{\{T_1 \leq t\}}\right] \quad (22)$$

In the spirit of the renewal argument that the renewal process starts again after the first renewal at time T_1, the damage cost in the remaining interval can be written as $K(t-T_1)$ which has the same distribution as $K(t)$. Using this argument and expanding the square term in Eq. (22) lead to

$$\begin{aligned}\mathbb{E}\left[K^2(t)\right] = \mathbb{E}\left[C_1^2 \mathbf{1}_{\{T_1 \leq t\}}\right] + \\ \mathbb{E}\left[2C_1 K(t-T_1)\mathbf{1}_{\{T_1 \leq t\}}\right] + \\ \mathbb{E}\left[K^2(t-T_1))^2 \mathbf{1}_{\{T_1 \leq t\}}\right]\end{aligned} \quad (23)$$

Assuming that the C is independent of T, these expectations can be evaluated in the following manner.

$$\mathbb{E}\left[C_1^2 \mathbf{1}_{\{T_1 \leq t\}}\right] = \mathbb{E}\left[C^2\right] F_T(t), \quad (24)$$

$$\begin{aligned}&\mathbb{E}\left[2C_1 K(t-T_1)\mathbf{1}_{\{T_1 \leq t\}}\right] \\ &= 2\ \mu_C \ \textstyle\int_0^t \mathbb{E}\left[K(t-x)\right] \mathrm{d}F_T(x)\end{aligned} \quad (25)$$

and lastly

$$\mathbb{E}\left[K^2(t-T_1))^2 \mathbf{1}_{\{T_1 \leq t\}}\right] = \int_0^t \mathbb{E}\left[K^2(t-x)\right] \mathrm{d}F_T(x) \quad (26)$$

Substituting all these simplified terms in Eq.(23) leads to the following renewal equation:

$$\mathbb{E}\left[K^2(t)\right] = g(t) + \int_0^t \mathbb{E}\left[K^2(t-x)\right] \mathrm{d}F_T(x) \quad (27)$$

where

$$g(t) = \mu_{2C} F_T(t) + 2\mu_C \int_0^t \mathbb{E}\left[K(t-x)\right] \mathrm{d}F_T(x) \quad (28)$$

Note that μ_{2C} is the second moment of the damage cost, and $\mathbb{E}\left[K(t-x)\right]$ is already given by Eq.(20).

In fact, the expected cost can also be derived using the renewal argument in form of the following integral equation:

$$\mathbb{E}\left[K(t)\right] = \mu_C F_T(t) + \int_0^t \mathbb{E}\left[K(t-x)\right] \mathrm{d}F_T(x) \quad (29)$$

Using this result, a simpler expression for $g(t)$ is obtained as

$$g(t) = \left(\mu_{2C} - 2\mu_C^2\right) F_T(t) + 2\mu_C\ \mathbb{E}\left[K(t)\right] \quad (30)$$

Since $g(t)$ is a bounded and integrable function, the renewal equation, Eq.(27) has the following solution:

$$\mathbb{E}\left[K^2(t)\right] = g(t) + \int_0^t g(t-x)\mathrm{d}\Lambda(x)$$

which can also be written as

$$\boxed{\mathbb{E}\left[K^2(t)\right] = g(t) + \int_0^t g(t-x)\lambda(x)\mathrm{d}x} \quad (31)$$

In summary, given the renewal function and the expected life cycle cost, the second moment of the life-cycle cost can be calculated using the above equation. Then the variance can be calculated using the standard relationship as $\left[\mathbb{E}\left[K^2(t)\right] - \left(\mathbb{E}\left[K(t)\right]\right)^2\right]$.

3.3. Discounted Life Cycle Cost

The expected value of the discounted life-cycle cost or (NPV) can be obtained using the renewal argument. Here, the damage cost C_i incurring at time S_i is discounted back to present time, $S_0 = 0$, as $C_i e^{-\rho S_i}$, where ρ is the interest rate. Thus,the total discounted life-cycle cost can be written as

$$K_D(t) = \sum_{i=1}^{\infty} C_i\ e^{-\rho S_i}\ \mathbf{1}_{\{S_i \leq t\}} \quad (32)$$

Its expected value is then written as

$$\begin{aligned}\mathbb{E}\left[K_D(t)\right] &= \mathbb{E}\left[C_i\right] \sum_{i=1}^{\infty} \mathbb{E}\left[e^{-\rho S_i}\ \mathbf{1}_{\{S_i \leq t\}}\right] \\ &= \mu_C \sum_{i=1}^{\infty} \int_0^t e^{-\rho x} \mathrm{d}F_{S_i}(x)\end{aligned} \quad (33)$$

Recall that $F_{S_i}(x) = F_T^{(i)}(x)$ and substituting it in above equation leads to

$$\begin{aligned}\mathbb{E}\left[K_D(t)\right] &= \mu_C \sum_{i=1}^{\infty} \int_0^t e^{-\rho x} \mathrm{d}F_T^{(i)}(x) \\ &= \mu_C \int_0^t e^{-\rho x} \sum \mathrm{d}F_T^{(i)}(x)\end{aligned} \quad (34)$$

Based on Eq. (7), the final result is obtained as

$$\boxed{\mathbb{E}[K_D(t)] = \mu_C \int_0^t e^{-\rho x}\, \mathrm{d}\Lambda(x)} \tag{35}$$

It is interesting to note that even for a general renewal process the expected discounted cost has a fairly simple analytical form.

3.4. *Expected Utility Analysis*

A renewal type integral equation can be derived for a restricted family of the utility function, such as an exponential function of the following form (Cha and Ellingwood, 2013):

$$U(K(t)) = -a\, e^{-\alpha K(t)} \tag{36}$$

For sake of notational simplicity, define the expected utility without any multiplicative constants as

$$\mathbb{E}[U(K(t)))] = \phi(t) = \mathbb{E}\left[e^{-\alpha K(t)}\right] \tag{37}$$

The formulation proceeds in the same way as that for the second moment in Section 3.2.

$$\begin{aligned}\mathbb{E}[U(K(t)] &= \mathbb{E}\left[U(K(t)\mathbf{1}_{\{T_1\le t\}}\right] \\ &\quad + \mathbb{E}\left[U(K(t)\mathbf{1}_{\{T_1> t\}}\right]\end{aligned} \tag{38}$$

The first term can be simplified by invoking the renewal argument as (Cheng et al., 2012)

$$\begin{aligned}\mathbb{E}\left[U(K(t)\mathbf{1}_{\{T_1\le t\}}\right] &= \mathbb{E}\left[e^{-\alpha(C_1+K(t-T_1))}\mathbf{1}_{\{T_1\le t\}}\right] \\ &= \textstyle\int_0^t \mathbb{E}\left[e^{-\alpha K(t-x)}\right]\, \mathbb{E}\left[e^{-\alpha C_1}\right] \mathrm{d}F_T(x)\end{aligned} \tag{39}$$

The second expectation term in Eq.(38) simply turns out to be $\overline{F_T}(t)$, since $K(t) = 0$ when $T_1 > t$. The final solution can be written as an integral equation

$$\phi(t) = \overline{F_T}(t) + \int_0^t \phi(t-x)\ f_\phi(x)\ \mathrm{d}(x) \tag{40}$$

where

$$f_\phi(x) = \mathbb{E}\left[e^{-\alpha C_1}\right] f_T(x) \tag{41}$$

is a defective density.

Eq.(40) is also referred to as a defective renewal equation, and its solution will depend on the probability density of the renewal interval, $f_T(x)$, as well as the density of the cost, C, which is needed to evaluate $\mathbb{E}\left[e^{-\alpha C_1}\right]$.

Since the solution of a defective renewal equation is a more involved task, further analysis of this problem is pursued in a separate investigation.

3.5. *Life-Cycle Cost Analysis: HPP Model*

If the occurrence of a hazard is modelled as an HPP, then the life-cycle cost is equivalent to a compound Poisson process. Using Eq.(20) and noting that the renewal function of HPP is simply $\Lambda(t) = \lambda\ t$, the expected cost can be easily obtained as

$$\mathbb{E}[K(t)] = \mu_C\ \lambda\ t \tag{42}$$

The second moment (or mean square) of the cost can be obtained from Eq.(31) with $\lambda(x) = \lambda$ and $F_T(t) = 1 - e^{-\lambda t}$:

$$\mathbb{E}\left[K^2(t)\right] = \mu_{2C}\ \lambda t + (\mu_C \lambda t)^2 \tag{43}$$

The variance of cost is simply

$$\sigma^2_{K(t)} = \mu_{2C}\ \lambda t \tag{44}$$

The expected discounted cost can be obtained from Eq.(34) as

$$\mathbb{E}[K_D(t)] = \mu_C \int_0^t e^{-\rho x}\, \lambda \mathrm{d}x = \frac{\mu_C \lambda}{\rho}(1 - e^{-\rho t}) \tag{45}$$

This standard formula is commonly used in seismic risk analysis (Liu et al., 2004; Porter et al., 2004).

4. SEISMIC RISK ANALYSIS: DISCUSSION

4.1. *Seismic Hazard Analysis*

Consider a hypothetical site that is vulnerable to damage from earthquakes originating from three seismic sources shown in Figure 3. This Figure also provides occurrence rates, magnitude and depth of seismic sources. Here, the earthquake intensity is quantified as the spectral acceleration S_a at 1 second period of the structure situated at the site.

For details of the probabilistic seismic hazard analysis (PSHA), readers are referred to McGuire (2004). The ground motion prediction equation, a function of earthquake magnitude (M) and distance (R), given by Abrahamson and Silva (1997) is used.

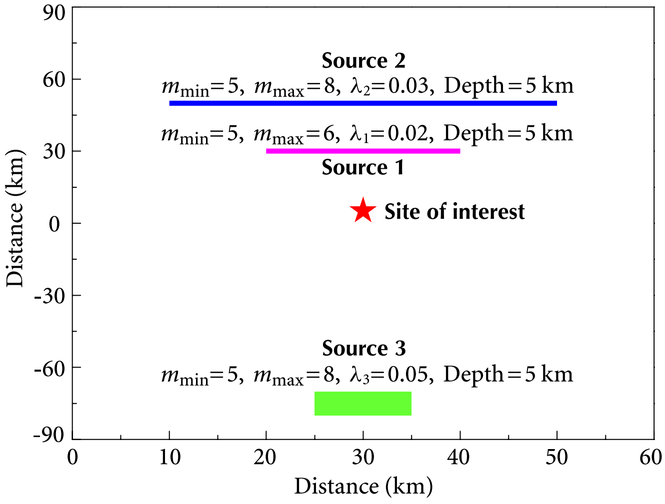

Figure 3: Seismic sources near the site.

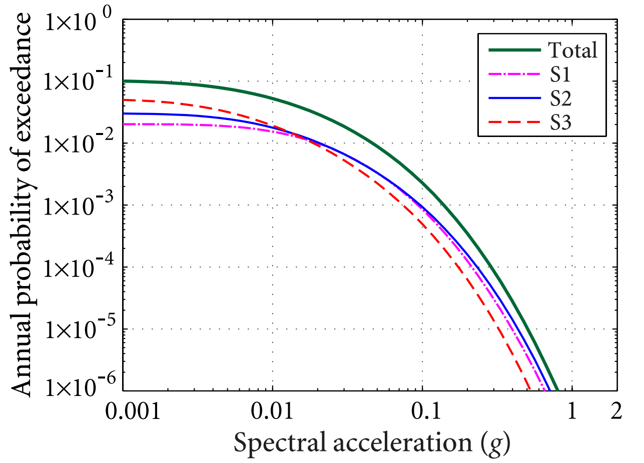

Figure 4: Seismic hazard curves resulting from three sources.

PSHA leads to the complementary distribution of S_a from an i^{th} source as

$$\begin{aligned}\overline{F_i}(s_i) &= \mathbb{P}[S_{ai} > s_i] \\ &= \int_M \int_R \mathbb{P}[S_{ai} > s_i | m, r]\, f_R(r)\, f_M(m) \mathrm{d}r \mathrm{d}m \qquad (46)\end{aligned}$$

Since the earthquake occurrence is modelled as an HPP with rate λ_i per year, earthquakes exceeding an intensity s_i is also an HPP with rate $\lambda_i \overline{F_i}(s_i)$. This result is based on the decomposition property of the Poisson process.

From the superposition principle, the combined earthquake hazard from all the three sources is also a Poisson process with an overall rate of $\lambda = \lambda_1 + \lambda_2 + \lambda_3 = 0.1$ per year. In particular, the occurrence rate of earthquakes exceeding a magnitude s is also given by the superposition principle as

$$\lambda(s) = \lambda_1 \overline{F_1}(s) + \lambda_2 \overline{F_2}(s) + \lambda_3 \overline{F_3}(s) \qquad (47)$$

A plot of $\lambda(s)$ versus s, referred to as the hazard curve, is shown in Figure 4.

The probability of no occurrence of earthquake exceeding the intensity s in $(0,t]$ is equivalent to the cumulative probability that the earthquake intensity does not exceed s in $(0,t]$, which is given as $F_{S_a,t}(s) = e^{-\lambda(s)t}$. On an annual basis, i.e., $t = 1$, this distribution is given as

$$F_{S_a}(s) = e^{-\lambda(s)} \qquad (48)$$

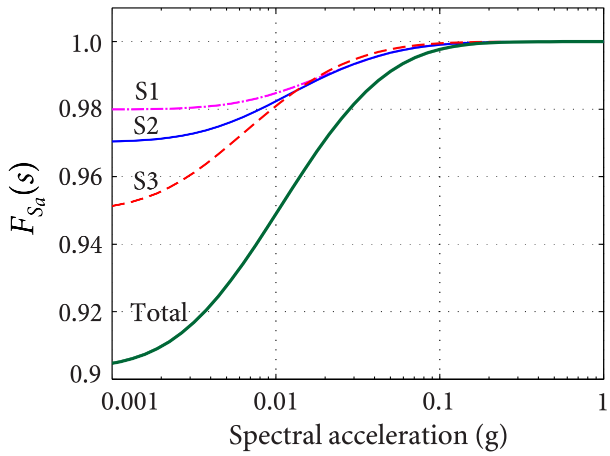

Figure 5: Probability distribution (CDF) of the spectral acceleration (S_a at 1 sec period).

In summary, the earthquake hazard at the site in question is described as a marked Poisson process with the rate of $\lambda = 0.1$ per year and the distribution of mark (S_a) is shown in Figure 5.

4.2. *Life-Cycle Damage Cost*

The damage cost, C, given the occurrence of an earthquake tends to be a function of the extent of structural damage, which in turn depends on the earthquake intensity, S_a. The seismic damage to structure is typically quantified in terms of drift (D), which is related with S_a by a simple empirical relationship, such as the following logarithmic relation (Cornell et al., 2002):

$$\ln D = a + b \ln S_a + \varepsilon \qquad (49)$$

where ε is a normally distributed random variable with mean 0 and standard deviation of β_D.

The damage cost is then related to drift by a similar simple, empirical relation. The overall aim is to derive a conditional distribution of damage cost given S_a, i.e., $F_{C|S_a}(c)$. However, the estimation of the moments of life-cycle cost, $K(t)$, requires only the first two moments of C, as shown by the results given in Section 3.5. This simplification is a result of two assumptions, namely, (1) damage cost is independent of inter-occurrence time, and (2) earthquake occurrences follow the homogeneous Poisson process. What it means is that conditional relationships between the cost, damage and intensity are needed to estimate only the conditional moments of the cost, $\mathbb{E}[C^n|s]$. Given these conditional moments, an n^{th} order moment can be calculated as (Porter et al., 2004):

$$\mathbb{E}[C^n] = \int_{S_a} \mathbb{E}[C^n|s] \; \mathrm{d}F_{S_a}(s)ds \qquad (50)$$

where $F_{S_a}(s)$ is given by Eq.(48). Using the moments of the cost and occurrence rate, formulas given in Section 3.5 can be used to calculate various estimates related to the expected cost

5. CONCLUSIONS

In the life-cycle cost estimation, damage caused by external hazards introduces a great deal of uncertainty due to random nature of the occurrence and intensity of hazards. The marked renewal process serves as a conceptual model of occurrence and intensity of a hazard, whereas the cumulative cost of repairing and restoring the structure can be treated as a compound renewal process. This paper presents systematic derivations of various measures of the life-cycle cost, such as the expected value, variance, expected NPV and expected utility.

In structural engineering, the homogeneous Poisson process model is widely used which has such a simple probabilistic structure that it completely by passes the use of formal theory of renewal process. The paper emphasizes that the understanding of the renewal theory is necessary to analyze life-cycle cost measures in a general setting.

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